Computational Modeling HW 1

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1.4: Trapezoidal method coefficients

$$\sum_{j=0}^{r} \alpha_j V_{n+j} = h \sum_{j=0}^{r} \beta_j f(t_{n+j}, V_{n+j}) \tag{1}$$

The trapezoidal method is this:

$$V_{n+1} = V_n + h\Big(\frac{1}{2} f(V_n, t_n) + \frac{1}{2} f(V_{n+1}, t_{n+1})\Big) \tag{2}$$

By examining these two equations, we can see that:

$$\begin{aligned}
\alpha_0 &= -1 & \beta_0 &= \frac{1}{2} \quad (3)\\
\alpha_1 &= 0 & \beta_1 &= 1 \quad (4)\\
\alpha_2 &= 0 & \beta_2 &= 1 \quad (5)\\
&... & &... \quad (6)\\
\alpha_{r-1} &= 0 & \beta_{r-1} &= 1 \quad (7)\\
\alpha_r &= 1 & \beta_1 &= \frac{1}{2} \quad (8)\\
& & & \quad (9)
\end{aligned}$$

$$V_{n+1} = V_n + h\Big(\frac{1}{2} f(V_n, t_n) + \frac{1}{2} f(V_{n+1}, t_{n+1})\Big) \tag{10}$$

If we let our function f take the following form: $f(x, t) = \alpha x$ where $x(0) = x_0$, we can see that:

$$\begin{aligned}
V_{n+1} &= V_n + \frac{h}{2}\Big(f(V_n, t_n) + f(V_{n+1}, t_{n+1})\Big) && (11)\\
V_{n+1} &= V_n + \frac{h}{2}\Big(\alpha V_n + \alpha V_{n+1}\Big) && (12)\\
V_{n+1} &= V_n + \frac{\alpha h V_n}{2} + \frac{\alpha h V_{n+1}}{2} && (13)\\
V_{n+1}\Big(1 - \frac{\alpha h}{2}\Big) &= V_n\Big(1 + \frac{\alpha h}{2}\Big) && (14)\\
V_{n+1} &= V_n \frac{1 + \frac{\alpha h}{2}}{1 - \frac{\alpha h}{2}} && (15)\\
V_{n+1} &= V_n \frac{2 + \alpha h}{2 - \alpha h} && (16)\\
V_{n+1} &= V_0 \Big(\frac{2 + \alpha h}{2 - \alpha h}\Big)^{n+1} && (17)\\
& && (18)
\end{aligned}$$

From this we can see that as long as the following condition is satisfied, we will have a converging solution:

$$\begin{aligned}
|2 + \alpha h| \leq |2 - \alpha h| && (19)\\
&& (20)
\end{aligned}$$

1.9: Order of Accuracy for Trapezoidal and Euler Methods

We want to analytically represent the error terms for the Euler and Trapezoidal methods and show how they depend on our timestep, h.

First we solve for the behavior of our function using a Taylor Series. Let $x(t)$ be our function and $v(t) = \frac{dx(t)}{dt}$. Let $N = \frac{t}{h}$, where t is the current time and h the timestep. Finally, let $x(t_0) = 0$.

$$x(t) = \sum_{i=0}^{i=N-1} \int_{t_i}^{t_{i+1}} x'(t)dt = \sum_{i=0}^{i=N-1} \int_{t_i}^{t_{i+1}} v(t)dt \tag{21}$$

$$v(t) = v(t_i) + (t - t_i)v'(t_i) + (t - t_i)^2 v''(t_i) + ... \tag{22}$$

$$\int_{t_i}^{t_{i+1}} v(t)dt \approx \int_{t_i}^{t_{i+1}} v(t_i) + (t - t_i)v'(t_i) + \frac{1}{2}(t - t_i)^2 v''(t_i)dt \tag{23}$$

$$\tag{24}$$

We now do a u-substitution, where $S = \frac{t-t_i}{t_{i+1}-t_i}$ and $h = t_{i+1} - t_i$:

$$\int_{t_i}^{t_{i+1}} v(t)dt \approx h \int_{i}^{1} v(t_i) + hSv'(t_i) + \frac{1}{2}h^2 S^2 v''(t_i)dS \tag{25}$$

$$\int_{t_i}^{t_{i+1}} v(t)dt \approx h\big(v(t_i)S + \frac{hS^2}{2}v'(t_i) + \frac{h^2 S^3}{6}v''(t_i)\big)|_0^1 \tag{26}$$

$$\int_{t_i}^{t_{i+1}} v(t)dt \approx h\big(v(t_i) + \frac{h}{2}v'(t_i) + \frac{h^2}{6}v''(t_i)\big) \tag{27}$$

$$\int_{t_i}^{t_{i+1}} v(t)dt \approx hv(t_i) + \frac{h^2}{2}v'(t_i) + \frac{h^3}{6}v''(t_i) \tag{28}$$

$$\tag{29}$$

This gives us a fairly exact value for $\int_{t_i}^{t_{i+1}} v(t)dt$. We now need to compare this with the approximations that our two methods make. For Euler's method, $\int_{t_i}^{t_{i+1}} v(t)dt = hf(t_i)$. For the Trapezoidal method, $\int_{t_i}^{t_{i+1}} v(t)dt = \frac{h}{2}\Big(f(t_{i+1}) + f(t_i)\Big)$. We will first plug in the Euler's method approximation to the above expression to find its error term.

$$\int_{t_i}^{t_{i+1}} v(t)dt = hv(t_i) \tag{30}$$

$$\int_{t_i}^{t_{i+1}} v(t)dt = hv(t_i) + \frac{h^2}{2}v'(t_i) + \frac{h^3}{6}v''(t_i) \tag{31}$$

$$hv(t_i) = hv(t_i) + \frac{h^2}{2}v'(t_i) + \frac{h^3}{6}v''(t_i) \tag{32}$$

$$-\frac{h}{3}v''(t_i) = v'(t_i) \tag{33}$$

$$\tag{34}$$

We now substitute this back into our original result:

$$\int_{t_i}^{t_{i+1}} v(t)dt = hv(t_i) + \frac{h^2}{2}v'(t_i) + \frac{h^3}{6}v''(t_i) \tag{35}$$

$$\int_{t_i}^{t_{i+1}} v(t)dt = hv(t_i) + \frac{h^2}{2}(-\frac{h}{3}v''(t_i)) + \frac{h^3}{6}v''(t_i) \tag{36}$$

$$\int_{t_i}^{t_{i+1}} v(t)dt = hv(t_i) + -\frac{h^3}{4}(v''(t_i)) + \frac{h^3}{6}v''(t_i) \tag{37}$$

$$\int_{t_i}^{t_{i+1}} v(t)dt = hv(t_i) + -\frac{h^3}{12}(v''(t_i)) \tag{38}$$

$$\tag{39}$$

As we can see, the error for the forward Euler's method is $-\frac{h^3}{12}(v''(t_i))$ per timestep. We now plug this result into our summed equation to find the total error for all timesteps:

$$x(t) = \sum_{i=0}^{i=N-1} \int_{t_i}^{t_{i+1}} x'(t)dt = \sum_{i=0}^{i=N-1} \int_{t_i}^{t_{i+1}} v(t)dt \tag{40}$$

$$x(t) = \sum_{i=0}^{i=N-1} \int_{t_i}^{t_{i+1}} x'(t)dt = \sum_{i=0}^{i=N-1} hv(t_i) + -\frac{h^3}{12}(v''(t_i)) \tag{41}$$

$$x(t) = \sum_{i=0}^{i=N-1} \int_{t_i}^{t_{i+1}} x'(t)dt = \sum_{i=0}^{i=N-1} hv(t_i) + -\frac{h^3}{12}(v''(t_i)) \tag{42}$$

$$\tag{43}$$

We only care about the error term, so we will focus on that. We substitute N with $\frac{t-t_0}{h}$ and replace $v''(t_i)$ with a term that represents the average value of $v''(t_i)$ over the whole sum: v''_{ave}.

$$E = \sum_{i=0}^{i=N-1} -\frac{h^3}{12}(v''(t_i)) \tag{44}$$

$$E = -N\frac{h^3}{12}v''_{ave} \tag{45}$$

$$E = -\frac{t-t_0}{h}\frac{h^3}{12}v''_{ave} \tag{46}$$

$$E = -(t-t_0)\frac{h^2}{12}v''_{ave} \tag{47}$$

$$\tag{48}$$

We do the same thing for the Trapezoidal method. We first find the error term:

$$\int_{t_i}^{t_{i+1}} v(t)dt = hv(t_i) \tag{49}$$

$$\int_{t_i}^{t_{i+1}} v(t)dt = hv(t_i) + \frac{h^2}{2}v'(t_i) + \frac{h^3}{4}v''(t_i) \tag{50}$$

$$\frac{h}{2}\Big(v(t_{i+1}) + v(t_i)\Big) = hv(t_i) + \frac{h^2}{2}v'(t_i) + \frac{h^3}{4}v''(t_i) \tag{51}$$

$$\frac{h}{2}\Big(-v(t_{i+1}) + v(t_i)\Big) - \frac{h^3}{4}v''(t_i) = \frac{h^2}{2}v'(t_i) \tag{52}$$

$$\frac{1}{h}\Big(-v(t_{i+1}) + v(t_i)\Big) - \frac{h}{2}v''(t_i) = v'(t_i) \tag{53}$$

$$\tag{54}$$

We then substitute this back into our original result:

$$\int_{t_i}^{t_{i+1}} v(t)dt = hv(t_i) + \frac{h^2}{2}v'(t_i) + \frac{h^3}{6}v''(t_i) \tag{55}$$

$$\int_{t_i}^{t_{i+1}} v(t)dt = hv(t_i) + \frac{h^2}{2}\Big(\frac{1}{h}\Big(v(t_{i+1}) - v(t_i)\Big) - \frac{h}{4}v''(t_i)\Big) + \frac{h^3}{6}v''(t_i) \tag{56}$$

$$\int_{t_i}^{t_{i+1}} v(t)dt = \frac{h}{2}\big(v(t_{i+1}) + v(t_i)\big) - \frac{h^3}{4}v''(t_i) + \frac{h^3}{6}v''(t_i) \tag{57}$$

$$\int_{t_i}^{t_{i+1}} v(t)dt = \frac{h}{2}\big(v(t_{i+1}) + v(t_i)\big) - \frac{h^3}{12}v''(t_i) \tag{58}$$

$$\tag{59}$$

From this we can see that the error term for the trapezoidal rule is $-\frac{h^3}{12}v''(t_i)$ per timestep. We then sum over all timesteps as above:

$$x(t) = \sum_{i=0}^{i=N-1}\int_{t_i}^{t_{i+1}} x'(t)dt = \sum_{i=0}^{i=N-1}\int_{t_i}^{t_{i+1}} v(t)dt \tag{60}$$

$$x(t) = \sum_{i=0}^{i=N-1}\int_{t_i}^{t_{i+1}} x'(t)dt = \sum_{i=0}^{i=N-1}\frac{h}{2}\big(v(t_{i+1}) + v(t_i)\big) - \frac{h^3}{12}v''(t_i) \tag{61}$$

$$x(t) = \sum_{i=0}^{i=N-1}\int_{t_i}^{t_{i+1}} x'(t)dt = \sum_{i=0}^{i=N-1}\frac{h}{2}\big(v(t_{i+1}) + v(t_i)\big) + \sum_{i=0}^{i=N-1} -\frac{h^3}{12}v''(t_i) \tag{62}$$

$$\tag{63}$$

We finally solve for the error:

$$E = \sum_{i=0}^{i=N-1} -\frac{h^3}{12}(v''(t_i)) \tag{64}$$

$$E = -N\frac{h^3}{12}v''_{ave} \tag{65}$$

$$E = -\frac{t - t_0}{h}\frac{h^3}{12}v''_{ave} \tag{66}$$

$$E = -(t - t_0)\frac{h^2}{12}v''_{ave} \tag{67}$$

$$\tag{68}$$

From this we can see that the Trapezoidal method's error is proportional to h^2.