D KNOWLEDGE RECAP: ORDINARY DIFFERENTIAL EQUATIONS (ODE)

ODEs are equations that involve ordinary derivatives. This recap section is made to be example-based such that the reader can quickly recall the techniques they have learned.

The 1st example

$$\frac{dx}{dy} = \cos(t)$$

Well this is perhaps the most straight forward one to solve, we can easily get:

$$x = \sin(t) + C\,, \quad \text{C is a constant}\,.$$

Of course if you are dealing with:

$$\frac{dx}{dy} = 0$$

then,

$$x = C\,, \quad \text{C is a constant}\,.$$

The 2nd example

$$\frac{dx}{dt} = \sin(t) + t^2$$

To find $x(t)$, we first integrate both sides of the equation:

$$\int \frac{dx}{dt}\,dt = \int \sin(t) + t^2\,dt$$

We can also discover $x(t)$ by straight forward computation of the indefinite integral:

$$x = -\cos(t) + \frac{t^3}{3} + C$$

Well C is a constant, although from the differentiation's point of view, a constant is not interesting at all but in many application scenarios we still would like to know the constant. We need initial conditions, for instance $x(t_0) = K$. Then,

$$\int_{t_0}^{t} \frac{dx}{d\tau} d\tau = \int_{t_0}^{t} \sin\,\tau + \tau^2 d\tau$$

We may find:

$$x(t) = -\cos\,t + \frac{t^3}{3} + \cos\,t_0 - \frac{t_0^3}{3} + K$$

The 3rd example

$$\frac{\mathrm{d}\,x(t)}{\mathrm{d}\,t} = mx(t) + n\,, \quad \text{with } m \neq 0\,.$$

We can find $x(t)$ by separating variables to two sides of the equation and integrate both sides.

$$\frac{\frac{\mathrm{d}\,x(t)}{\mathrm{d}\,t}}{mx(t)+n} = 1$$
$$\frac{\mathrm{d}\,x(t)}{mx(t)+n} = 1\,\mathrm{d}\,t$$

We completed the separation and now we integrate both sides to find $x(t)$,

$$\int \frac{1}{mx(t)+n}\,\mathrm{d}\,x = \int 1\,\mathrm{d}\,t$$
$$\frac{1}{m}\log|mx(t)+n| + C_l = t + C_r$$
$$|mx(t)+n| = e^{mt}e^{mC_r - mC_l}$$
$$x(t) = \frac{e^{mC_r - mC_l}}{m}e^{mt} - \frac{n}{m}$$
$$x(t) = Ce^{mt} - \frac{n}{m}$$

The 4th example

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + x(t) = t^2$$

Well, we see that there is already a separation of variables, we attempt to integrate both sides,

$$\int \frac{\mathrm{d}\,x}{\mathrm{d}\,t} + x(t)\,\mathrm{d}\,t = \int t^2\,\mathrm{d}\,t$$

The right part of the equation is easily approachable but the left part of the equation looks terrible! When we recall the Lebniz product rule for derivatives:

$$\frac{\mathrm{d}\,x\mathrm{d}\,y}{\mathrm{d}\,t} = \frac{\mathrm{d}\,x}{\mathrm{d}\,t}y(t) + x(t)\frac{\mathrm{d}\,x}{\mathrm{d}\,t}$$

If there is a function $\gamma(t)$ such that:

$$\frac{\mathrm{d}\,\gamma(t)}{\mathrm{d}\,t} = \gamma(t)\,.$$

Then we could utilize this property of $\gamma(t)$ and multiply it to both sides of our ODE, such that on the left hand side of the equation we may construct the right hand side of the product rule:

$$\frac{\mathrm{d}\,x\gamma}{\mathrm{d}\,t} = \frac{\mathrm{d}\,x}{\mathrm{d}\,t}\gamma(t) + x(t)\frac{\mathrm{d}\,\gamma}{\mathrm{d}\,t} = \gamma(t)t^2$$

Luckily, we have such a $\gamma(t)$:

$$\gamma(t) = \mathbf{e^t} = \frac{\mathrm{d}\,\gamma}{\mathrm{d}\,t}$$

Thus, our ODE becomes:

$$e^t \frac{dx}{dt} + x(t)e^t = e^t t^2$$

We integrate both sides now:

$$\int e^t \frac{dx}{dt} + x(t)e^t \, dt = \int e^t t^2 \, dt$$

Using the Lebniz product rule for the left hand side and utilize the integration by parts formula

$$\int x(t)y'(t) \, dt = x(t)y(t) - \int x'(t)y(t) \, dt$$

to the right hand side (twice), we will come to:

$$e^t x(t) = e^t(t^2 - 2t + 2) + C, \quad \text{where } C \text{ is a constant.}$$

Then we find:

$$x(t) = t^2 - 2t + 2 + Ce^- t$$