BASIC CONTROL SYSTEMS

05 POLES AND ZEROS

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WHERE STUDENTS MATTER

TRANSFER FUNCTIONS

- Transfer functions can be written as:

$$\frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots.. + b_2 s^2 + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots.. + a_2 s^2 + a_1 s + a_0}$$

All coefficients a_n and b_m are real.

- Or as:

$$\frac{Y(s)}{X(s)} = \frac{b_m}{a_n} \cdot \frac{(s-z_1)(s-z_2)\ldots..(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)\ldots..(s-p_{p-1})(s-p_n)}$$

- Which is the same as:

$$H(s) = k_{pz} \frac{(s-z_1)(s-z_2)\ldots..(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)\ldots..(s-p_{p-1})(s-p_n)}$$

Additional gain

POLES AND ZEROS

Theorem 5.1. The Fundamental Theorem of Algebra
Let $f(x) = \sum_{n=0}^{k} a_n x^n$ be a non-constant polynomial and $a_n \in \mathbb{C}$, then there exist a unique factorization such that:

$$f(x) = \sum_{n=0}^{k} a_n x^n = r_0 \prod_{i=1}^{k} (x - r_i)$$

This fundamental theorem of algebra enables us to obtain a unique decomposition of a irreducible rational polynomial transfer function.

Definition 5.1. Poles
The value(s) of s such that the denominator $D(s) = 0$

Definition 5.2. Zeros
The value(s) of s such that the numerator $N(s) = 0$

These guarantees: ***the poles and zeros are either real or in complex conjugate pairs.***

Poles and zeros

- The zeros of a transfer function (z) are the values of the Laplace transform variable *s* that causes the transfer function to become <u>zero</u>

  → numerator

- The poles of a transfer function (p) are the values of the Laplace transform variable s that causes the transfer function to become <u>infinite</u>

  → denominator

$$TF = \frac{Numerator}{Denominator}$$

Counting

Naming

AN EXAMPLE

Input: x, Output: y,
Assume 0 initial conditions.
Given an ODE:

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}t^2} + 2\frac{\mathrm{dy}}{\mathrm{d}t} - 8y = 3\frac{\mathrm{dx}}{\mathrm{d}t} + 1\mathrm{x}$$

We do the Laplace transform:

$$s^2Y + 2sY - 8Y = 3sX + 1X$$

Define transfer function H:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s+1}{s^2+2s-8} = 3\frac{s+\frac{1}{3}}{(s+4)(s-2)}$$

IDENTIFYING POLES AND ZEROS

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{3}\frac{s+\frac{1}{3}}{(s+4)(s-2)}$$

According to the definitions:

Gain K	$\frac{1}{3}$
Zeros z	$-\frac{1}{3}$
Poles p	$-4, +2$

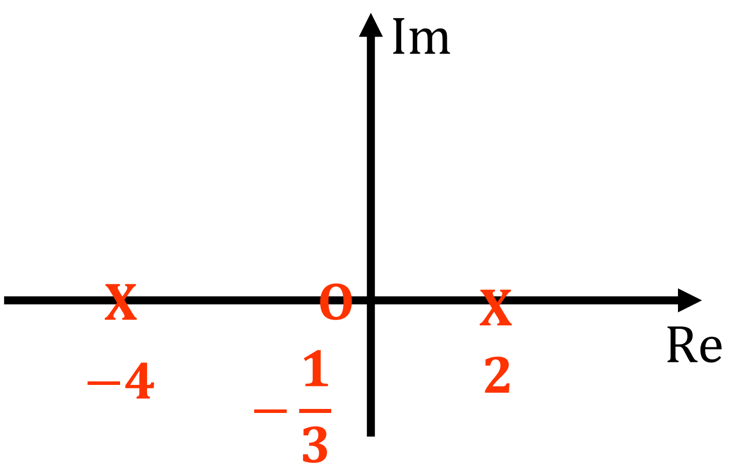
Obviously, when s = -4 or 2 (POLE), we have $H(s) \to \infty$

Obviously, when s = $-\frac{1}{3}$ (ZERO), we have $H(s) \to 0$

DRAWING POLES AND ZEROS IN THE COMPLEX PLANE

Components	Values	
Gain K	$\frac{1}{3}$	We don't draw this here.
Zeros $s = z$	$-\frac{1}{3}$	X
Poles $s = p$	$-4, +2$	O

***s*-domain**



ADDITIONAL PROPERTY OF POLES AND ZEROS

We are modelling causal linear systems in the real world.

ADDITIONAL PROPERTY OF POLES AND ZEROS

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This simple sentence tells us a lot!

ADDITIONAL PROPERTY OF POLES AND ZEROS

We are modelling causal linear systems in the real world.

Number of zeros never more than number of poles

The system can be modeled by a *linear inhomogeneous* ODE

All coefficients are real

The poles and zeros with non-zero imaginary components always comes in **conjugate pairs**.

For all poles and zeros, if there exist a pole/zero $\sigma + j\omega$ with $\omega \neq 0$, there must exist another pole/zero which is $\sigma - j\omega$ with $\omega \neq 0$ (the complex conjugate).

CONTINUING OUR EXAMPLE

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s+1}{s^2+2s-8} = 3\frac{s+\frac{1}{3}}{(s+4)(s-2)}$$

$$H(s) = \frac{\frac{11}{6}}{(s+4)} + \frac{\frac{7}{6}}{(s-2)}$$

Inverse Laplace transform:

$$\mathrm{h}(t) = \frac{1}{6}\left(11e^{-4t} + 7e^{2t}\right)$$

TRANSFORM -> DECOMPOSE

What did we just do?

The process function h(t) can be decomposed to the summation of linearly independent exponentials: $Ce^{\lambda t}$

In fact, with Laplace transform, we can decompose any *linear system* into linearly independent exponentials:

$$h(t) = \sum Ce^{\lambda t}$$

POLES ARE CRUCIAL

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The process function h(t) can be decomposed to the summation of linearly independent exponentials: $Ce^{\lambda t}$

In fact, with Laplace transform, we can decompose any *linear system* into linearly independent exponentials:

$$h(t) = \sum Ce^{\lambda t}$$

λ correspond to the poles of the transfer function.

POLES ARE CRUCIAL

λ correspond to the poles of the transfer function. So,

$$h(t) = \sum Ce^{\lambda t} = \sum Ce^{\sigma t}e^{j\omega t}$$

σ - determines the decay(if stable) of the output signal

ω – determines the oscillation of the output signal

RECALL EXAMPLE

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s+1}{s^2+2s-8} = 3\frac{s+\frac{1}{3}}{(s+4)(s-2)}$$

$$H(s) = \frac{\frac{11}{6}}{(s+4)} + \frac{\frac{7}{6}}{(s-2)}$$

$\boldsymbol{pole: -4}$ $\boldsymbol{pole: \ 2}$

Inverse Laplace transform:

$$\mathrm{h}(t) = \frac{1}{6}\left(11e^{-4t} + 7e^{2t}\right)$$

STABILITY

Is $h(t)$ stable?

$$h(t) = \frac{1}{6}\left(11e^{-4t} + 7e^{2t}\right)$$

STABILITY

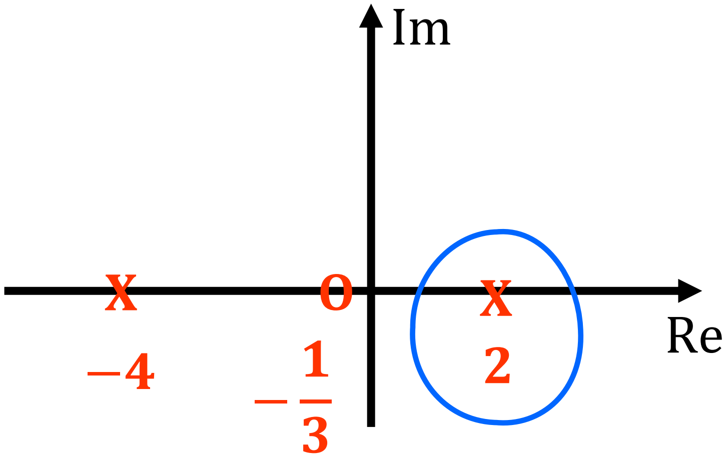
Is $h(t)$ stable?

$$h(t) = \frac{1}{6}\left(11e^{-4t} + 7e^{2t}\right)$$

Obviously not, if we look at $h(t)$ as $t \to \infty$:

$$\lim_{t\to\infty} \frac{1}{6}\left(11e^{-4t} + 7e^{2t}\right) = 0 + \infty$$

So not stable!

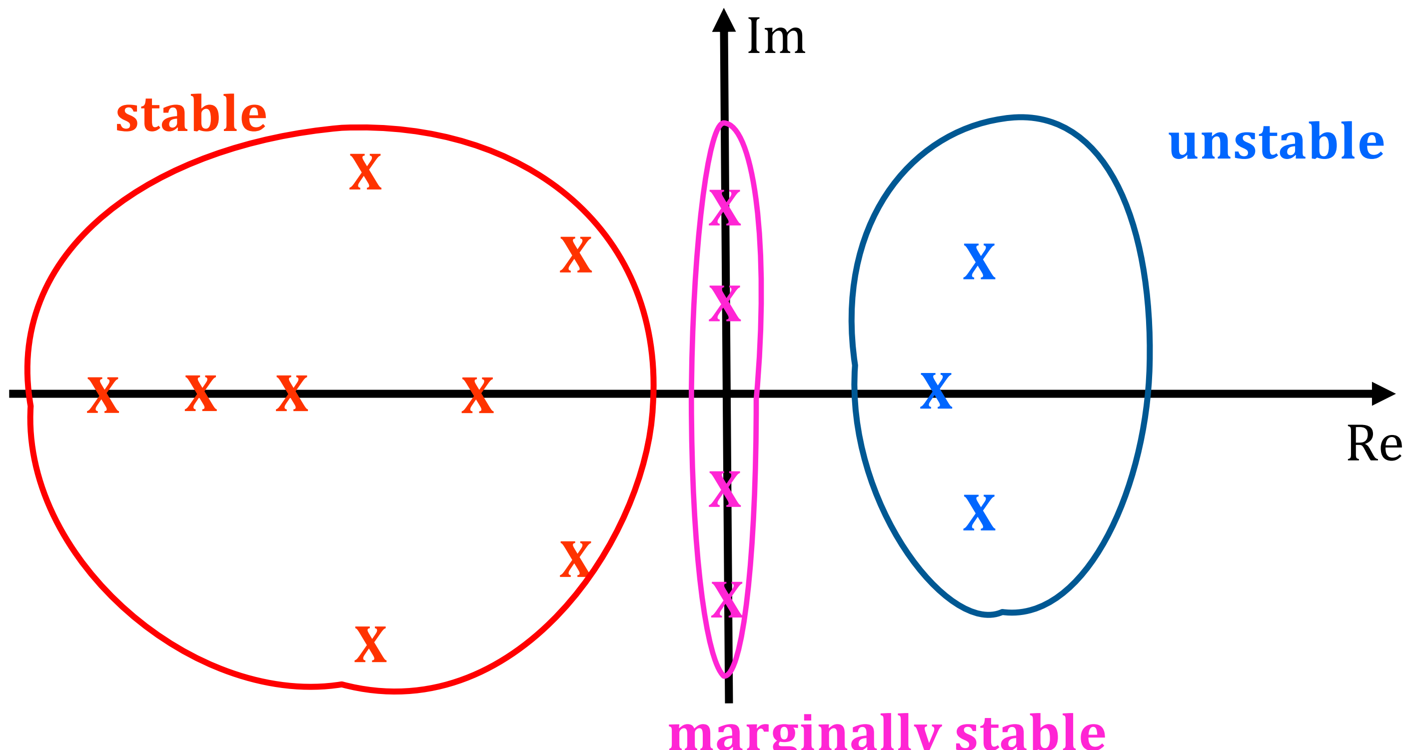


STABILITY CRITERIA

Left Half Plane

All poles should be in the open LHP of the s-plane.

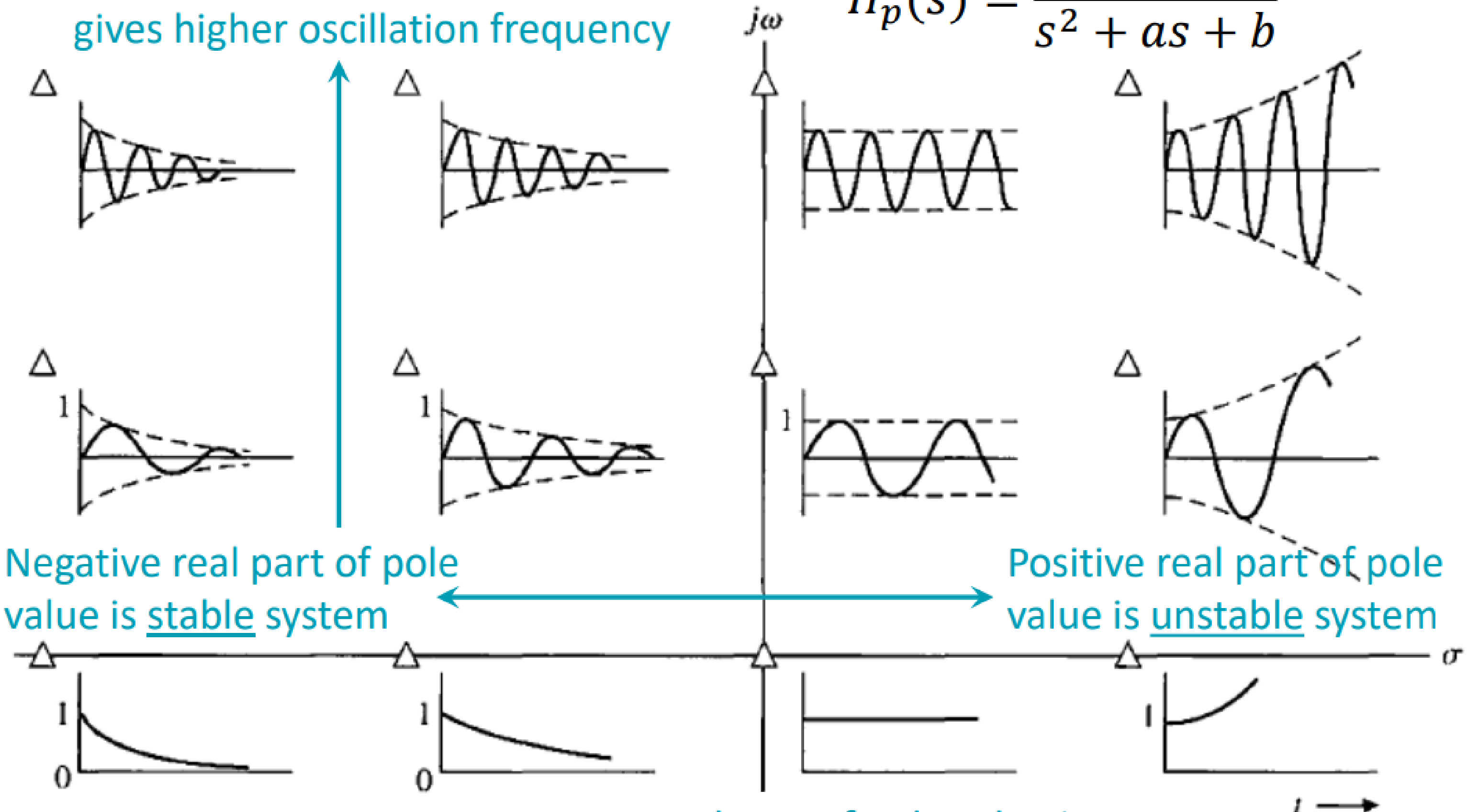
$$\textbf{iff } \forall \ \mathrm{Re}(p) < 0, \textbf{stable}!$$



POLES AND ZEROS

Larger imaginary part of pole value gives higher oscillation frequency

$$H_p(s) = \frac{b}{s^2 + as + b}$$



Negative real part of pole value is <u>stable</u> system

Positive real part of pole value is <u>unstable</u> system

Zero real part of pole value is <u>marginally stable</u> system

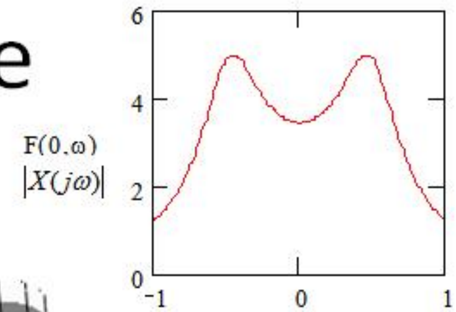
Poles and zeros

Why are the poles and zeros important?

1. The location of the poles indicates directly if a system is stable. Stability is always a major requirement for a controlled system.
2. The dynamic behaviour can be deduced AND specified by the location of the poles and zeros. This is always related to a FIRST or SECOND order system:
    1. Time constant
    2. For example: overshoot and settling time

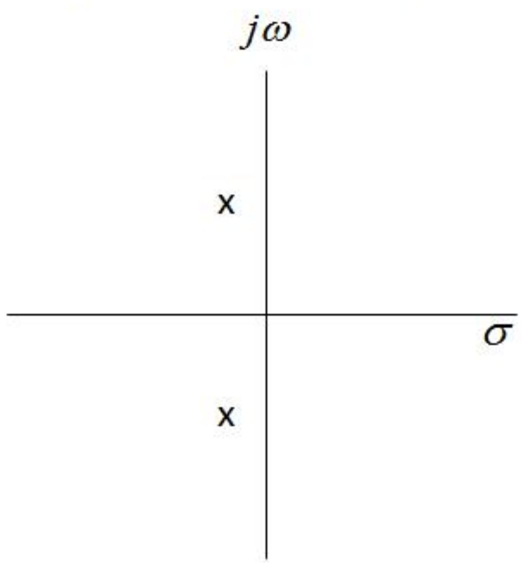
Poles and zeros: Why we care!

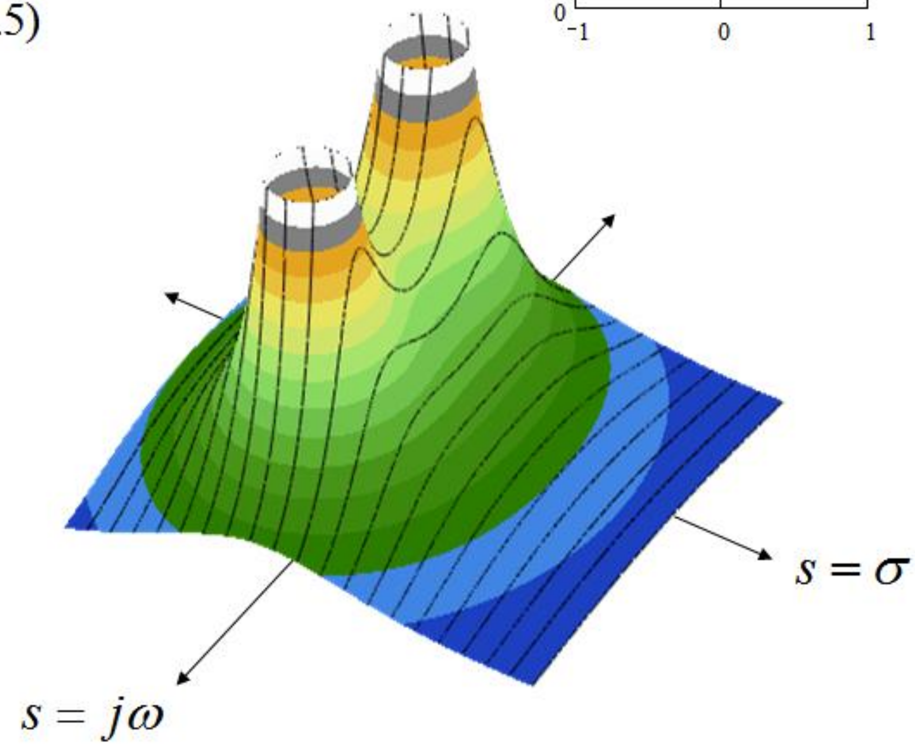
Filter Example



$$X(s) = \frac{1}{(s+.2+j.5)(s+.2-j.5)}$$

(poles at $s = -.2 \pm j.5$)

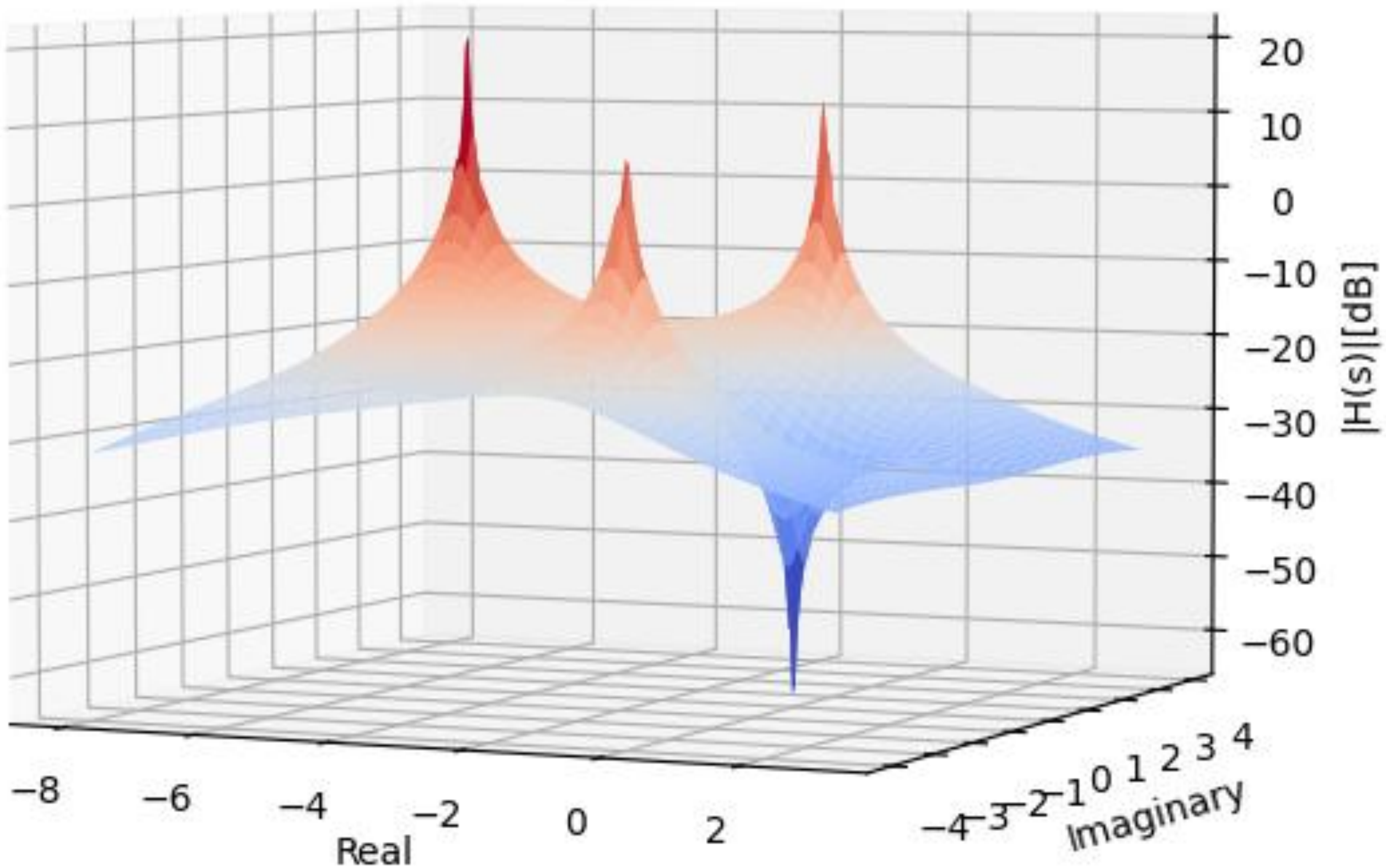




Poles and zeros: Why we care!

$$\frac{s}{(s+5)(s^2+2s+7)}$$

(Visualization in log scale)



SUMMARY

Transfer function:

$$H(s) = \frac{Y(s)}{X(s)}$$

Poles:

$$s = p \text{ such that } X(s = p) = 0, \text{ where } |H(s)| \to \infty$$

Zeros:

$$s = z \text{ such that } Y(s = z) = 0 \text{ , where } |H(s)| \to 0$$

Stability criteria:

all poles in the open LHP

SELF-READING

WHERE STUDENTS MATTER

Determination of the poles and zeros

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1 (both for zeros and poles!)
  - Examples:

Coefficient in the numerator is 0.1

$$G(s) = \frac{0{,}1s + 1}{s^2 + 7s + 12} =$$

Denominator coefficient of the highest power already is 1

Determination of the poles and zeros

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1 (both for zeros and poles!)
  - Examples:

$$G(s) = \frac{0{,}1s + 1}{s^2 + 7s + 12} = \frac{0{,}1(s + 10)}{s^2 + 7s + 12} =$$

Determination of the poles and zeros

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1 (both for zeros and poles!)
  - Examples:

$$G(s)=\frac{0{,}1s+1}{s^2+7s+12}=\frac{0{,}1(s+10)}{s^2+7s+12}=\frac{1}{10}\cdot\frac{s+10}{s^2+7s+12}$$

Determination of the poles and zeros

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  - Examples:

$$G(s) = \frac{0{,}1s + 1}{s^2 + 7s + 12} = \frac{0{,}1(s + 10)}{s^2 + 7s + 12} = \frac{1}{10} \cdot \frac{s + 10}{s^2 + 7s + 12}$$

$$G(s) = \frac{3s + 30}{5s^2 + 15s + 250} =$$

Determination of the poles and zeros

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1 (both for zeros and poles!)
  - Examples:

$$G(s)=\frac{0{,}1s+1}{s^2+7s+12}=\frac{0{,}1(s+10)}{s^2+7s+12}=\frac{1}{10}\cdot\frac{s+10}{s^2+7s+12}$$

$$G(s)=\frac{3s+30}{5s^2+15s+250}=\frac{3(s+10)}{5(s^2+3s+50)}=\frac{3}{5}\cdot\frac{s+10}{s^2+3s+50}$$

Poles and zeros construction steps

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1
- Step 2: rewrite the transfer function in its base parts – a constant, poles and zeros

$$G(s) = \frac{1}{10} \cdot \frac{s+10}{s^2+7s+12} = \frac{1}{10} \cdot \frac{s+10}{(s+3)(s+4)}$$

Poles and zeros construction steps

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- Step 2: rewrite the transfer function in its base parts – a constant, poles and zeros

$$G(s) = \frac{1}{10} \cdot \frac{s+10}{s^2+7s+12} = \frac{1}{10} \cdot \frac{s+10}{(s+3)(s+4)}$$

$$G(s) = \frac{3}{5} \cdot \frac{s+10}{s^2+3s+50} =$$

Sometimes the solution is complex
→ results in two complex poles

Poles and zeros construction steps

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1
- Step 2: rewrite the transfer function in its base parts – a constant, poles and zeros

$$G(s)=\frac{1}{10}\cdot\frac{s+10}{s^2+7s+12}=\frac{1}{10}\cdot\frac{s+10}{(s+3)(s+4)}$$

$$G(s)=\frac{3}{5}\cdot\frac{s+10}{s^2+3s+50}=\frac{3}{5}\cdot\frac{s+10}{(s+\frac{3}{2}+\frac{13{,}8}{2}j)(s+\frac{3}{2}-\frac{13{,}8}{2}j)}$$

Sometimes the solution is complex
→ results in two complex poles

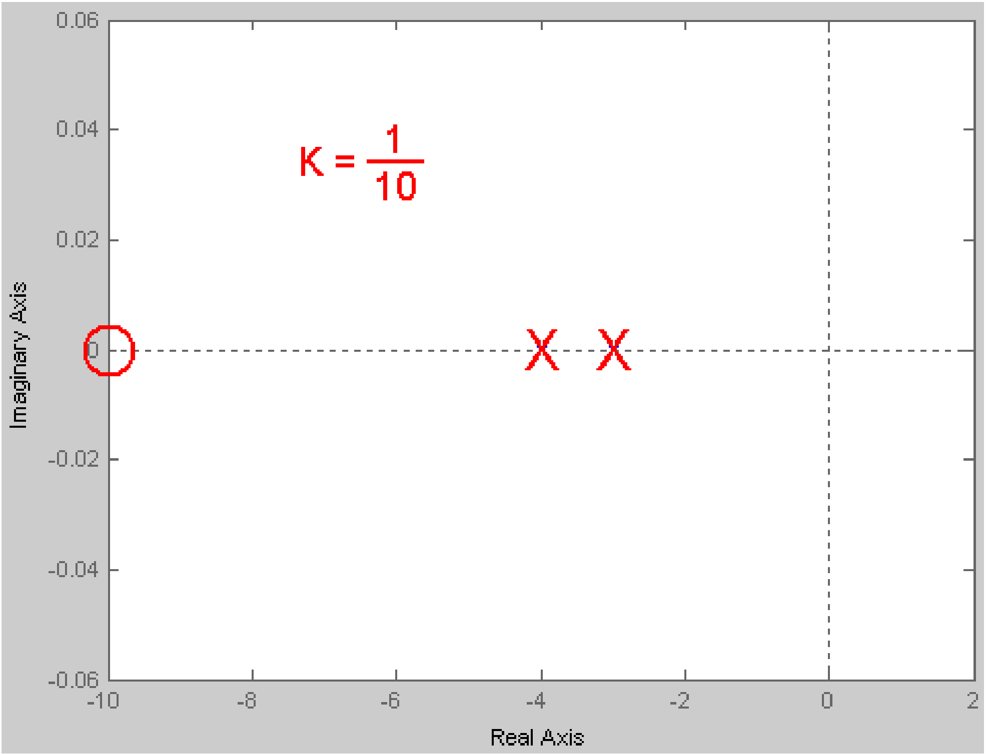
Poles and zeros construction steps

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1
- Step 2: rewrite the transfer function in its base parts – a constant, poles and zeros
- Step 3: draw the poles and zeros in the (complex) s-plane; the constant is mentioned separately as K

Poles and zeros example

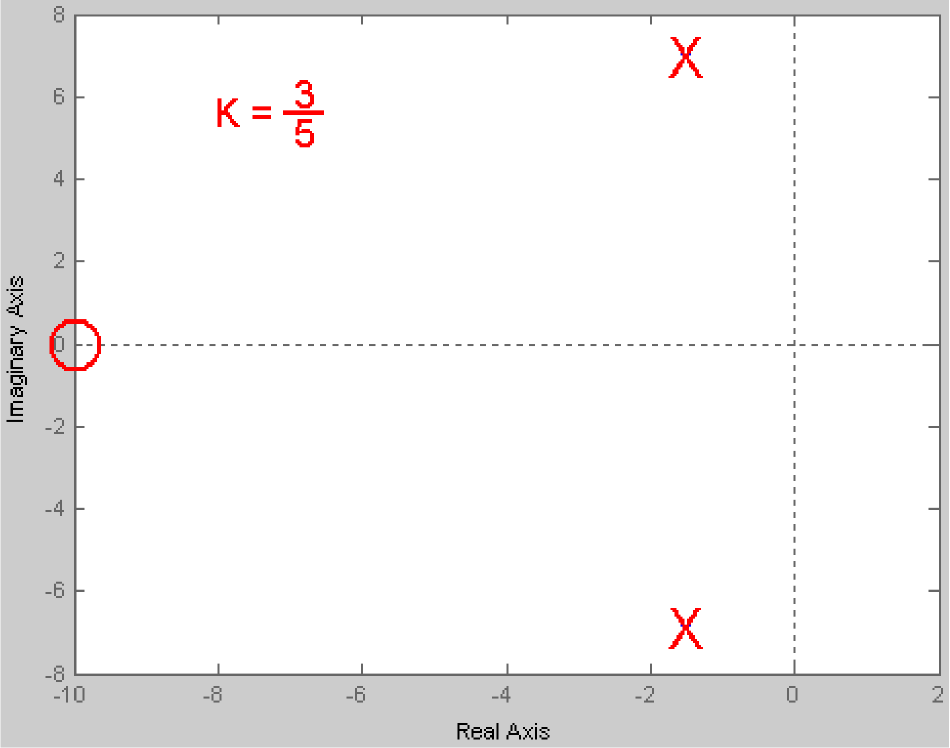
- Step 3: draw the poles and zeros in the (complex) s-plane; the constant is mentioned separately as K

$$G(s) = \frac{1}{10} \cdot \frac{s+10}{(s+3)(s+4)}$$



Poles and zeros example

$$G(s)=\frac{3}{5}\cdot\frac{s+10}{(s+\frac{3}{2}+\frac{13{,}8}{2}j)(s+\frac{3}{2}-\frac{13{,}8}{2}j)}$$



Poles and zeros exercises

Draw the poles and zeros in the s-plane for:

1. $$H(s)=\frac{25s+3}{4s^2+9s+2}$$

2. $$H(s)=\frac{3s+4}{s^2+6s+8}$$

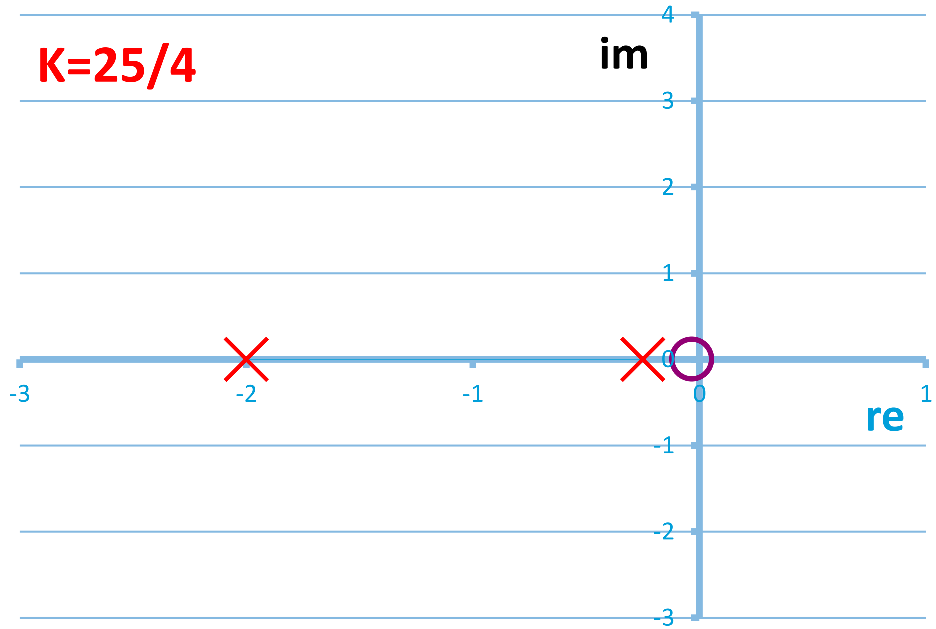
3. $$H(s)=\frac{2s+1}{s^2+4s+8}$$

Poles and zeros exercises

- Draw the poles and zeros in the s-plane for:

1. $$H(s) = \frac{25s+3}{4s^2+9s+2}$$

- zero: -3/25
- poles: -1/4 and -2
- K = 25/4

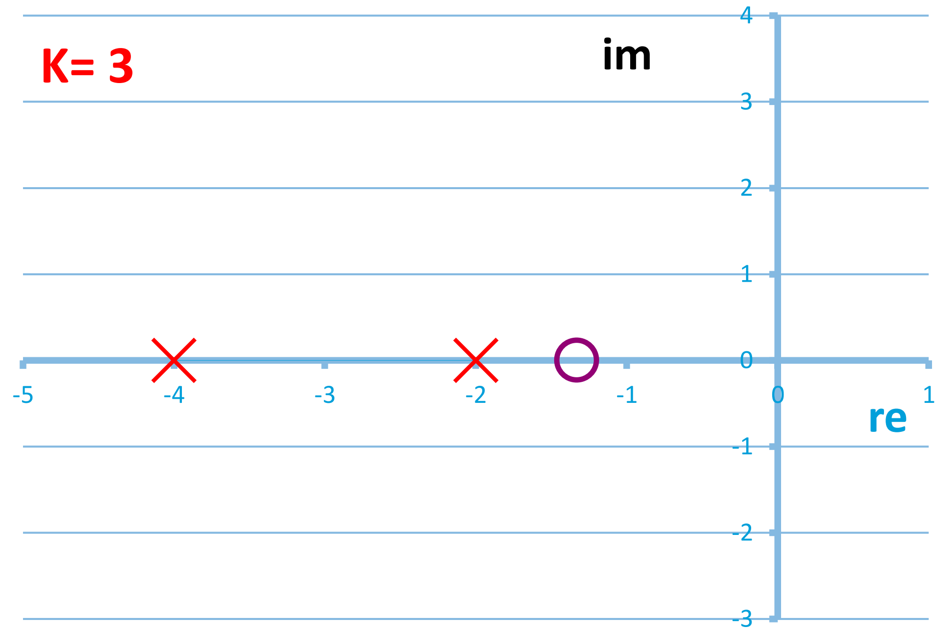


Poles and zeros exercises

- Draw the poles and zeros in the s-plane for:

2. $$H(s) = \frac{3s+4}{s^2+6s+8}$$

- zeros: -4/3
- poles: -2 and -4
- K = 3

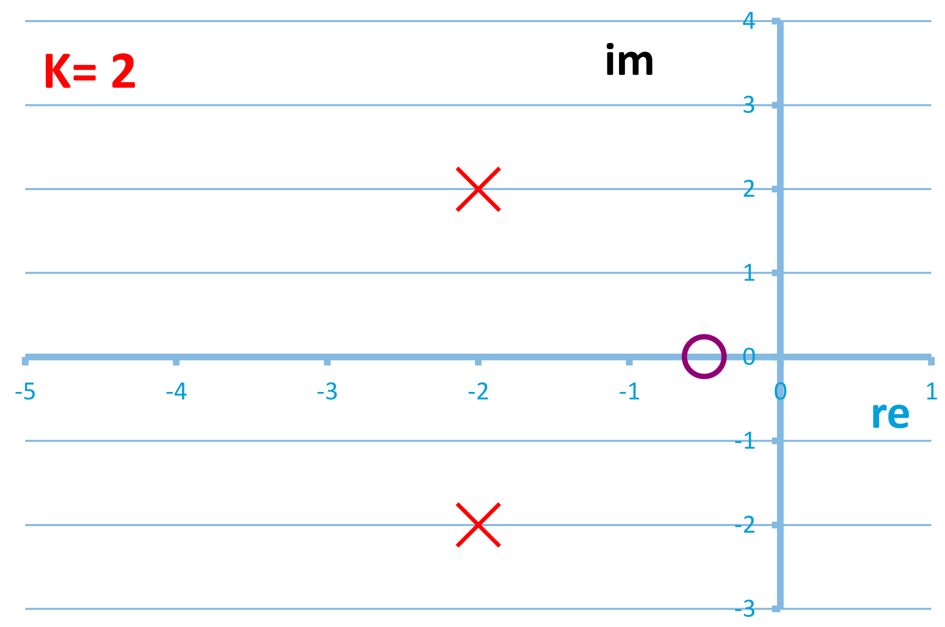


Poles and zeros exercises

- Draw the poles and zeros in the s-plane for:

3. $$H(s) = \frac{2s+1}{s^2+4s+8}$$

- zeros: -1/2
- poles: -2+2j
  and -2-2j
- K = 2



Poles and zeros exercises

4. Draw in the s-plane the poles and zeros of the transfer function H(s) = X(s)/F(s) and:

$$\frac{d^4x(t)}{dt^4}+2\frac{d^3x(t)}{dt^3}+2\frac{d^2x(t)}{dt^2}=\frac{df(t)}{dt}+f(t)$$

All values at time = 0 are zero

(so x’’’(0)=x’’(0)=0, etc.).

Poles and zeros exercises

- Draw the poles and zeros in the s-plane for:

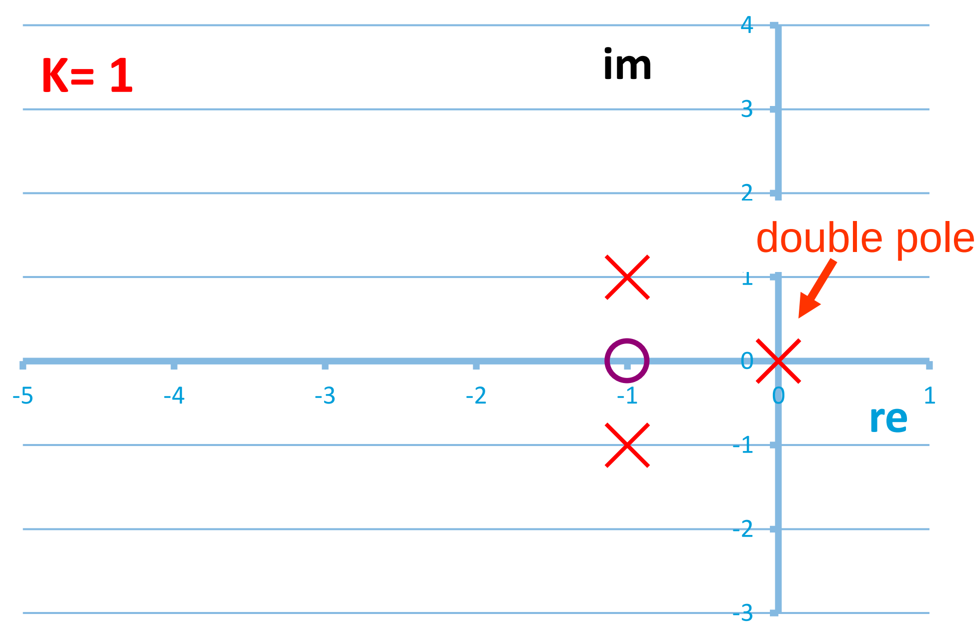
4.
$$\frac{d^4x(t)}{dt^4} + 2\frac{d^3x(t)}{dt^3} + 2\frac{d^2x(t)}{dt^2} = \frac{df(t)}{dt} + f(t)$$

Laplace → $s^4 + 2s^3 + 2s^2 = s + 1$

Transfer function:

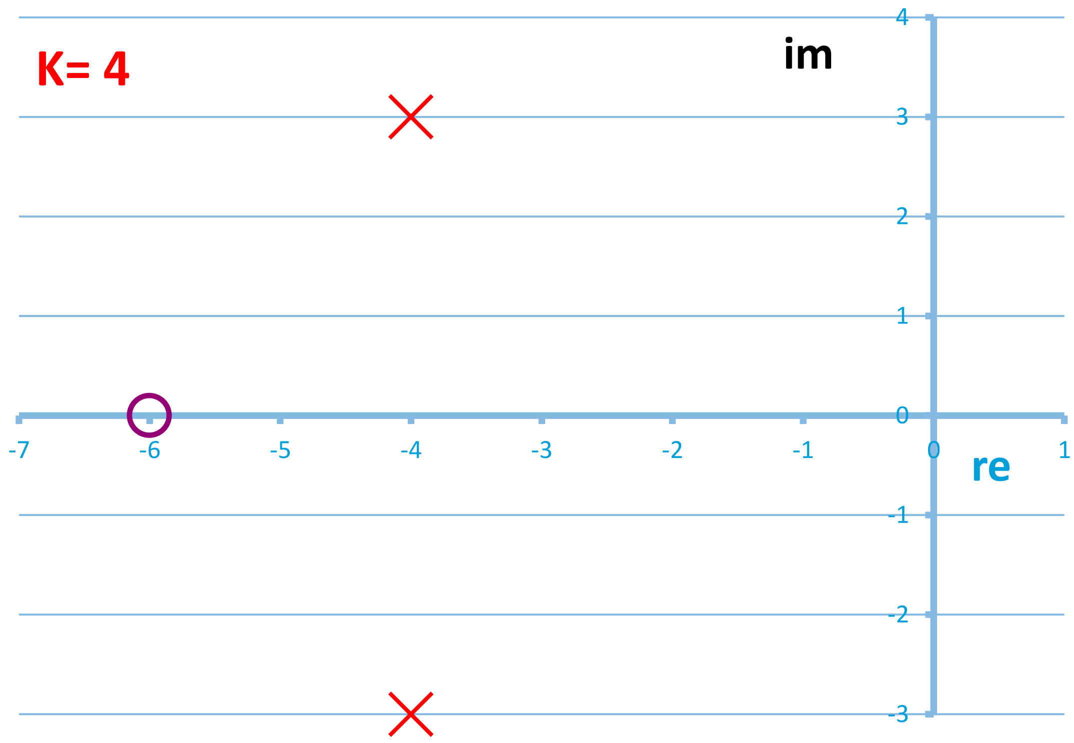
$$H_s = \frac{s+1}{s^4 + 2s^3 + 2s^2}$$

$$= \frac{s+1}{s * s(s+1+j)(s+1-j)}$$



Poles and zeros exercises

5. Find the differential equation for:

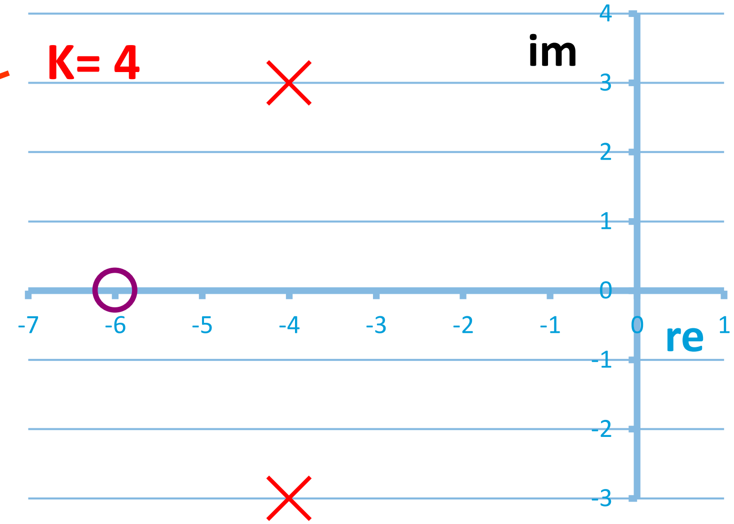


Assume that the initial conditions are zero.

Input is $x(t)$ and output is $y(t)$.

Poles and zeros exercises

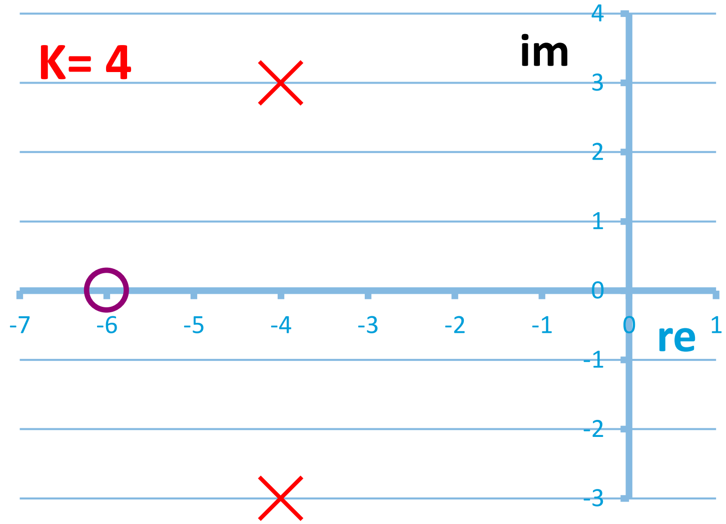
5. Find the differential equation



$$\frac{Y(s)}{X(s)} = 4\frac{s+6}{(s+4-3i)(s+4+3i)}$$

Poles and zeros exercises

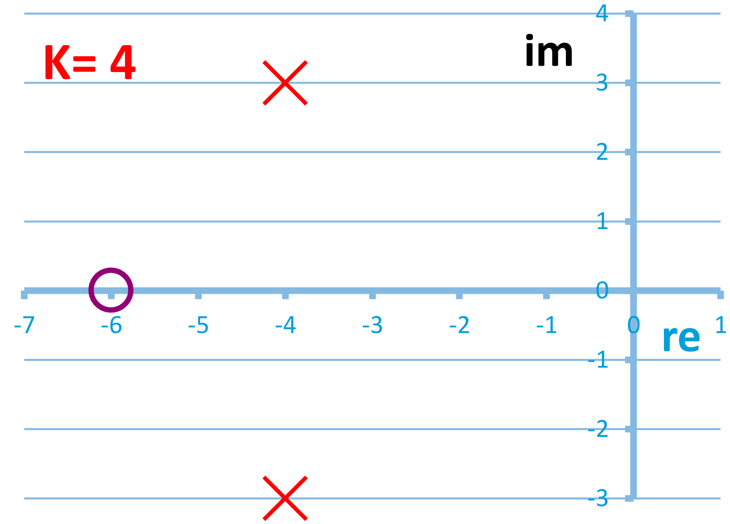
5. Find the differential equation



$$\frac{Y(s)}{X(s)} = 4\frac{s+6}{(s+4-3i)(s+4+3i)} = \frac{4s+24}{s^2+8s+25}$$

Poles and zeros exercises

5. Find the differential equation



$$\frac{Y(s)}{X(s)} = 4\frac{s+6}{(s+4-3i)(s+4+3i)} = \frac{4s+24}{s^2+8s+25}$$

$$\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 25y = 4\frac{dx(t)}{dt} + 24x(t)$$

Poles and zeros exercises

6. Find the differential equation for:

Assume that the initial conditions are zero.

Input is $x(t)$ and output is $y(t)$.

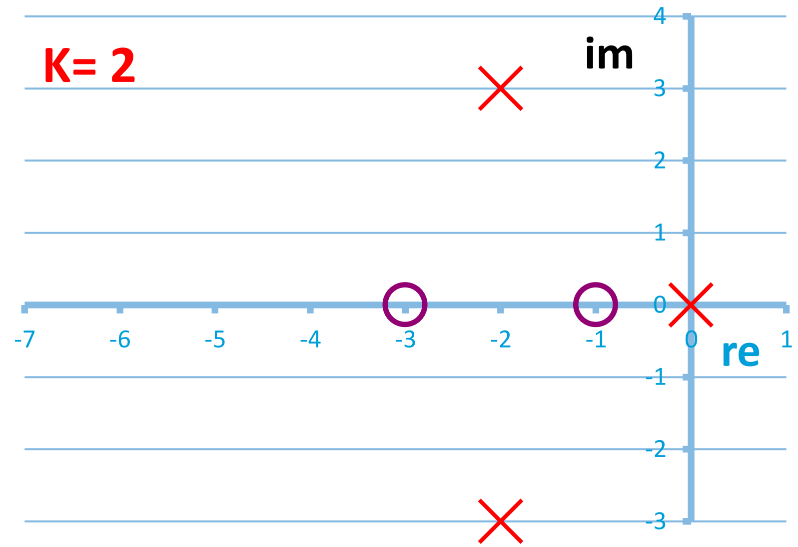
Poles and zeros exercise

6. ## Find the differential equation

$$\frac{Y(s)}{X(s)} = 2\frac{(s+1)(s+3)}{(s+2-3i)(s+2+3i)s}$$

Poles and zeros exercise

6. Find the differential equation



$$\frac{Y(s)}{X(s)} = 2\frac{(s+1)(s+3)}{(s+2-3i)(s+2+3i)s}$$

$$\frac{Y(s)}{X(s)} = \frac{2s^2+8s+6}{s^3+4s^2+13s}$$

Poles and zeros exercise

6. Find the differential equation

$$\frac{Y(s)}{X(s)} = 2\frac{(s+1)(s+3)}{(s+2-3i)(s+2+3i)s}$$

$$\frac{Y(s)}{X(s)} = \frac{2s^2+8s+6}{s^3+4s^2+13s}$$

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 13\frac{dy(t)}{dt} = 2\frac{d^2x(t)}{dt^2} + 8\frac{dx(t)}{dt} + 6x(t)$$

Poles and zeros exercise

7. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 15y(t) = 5\frac{dx(t)}{dt} + 10x(t)$$

$$x(t) = 2t$$

Poles and zeros exercises

7. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 15y(t) = 5\frac{dx(t)}{dt} + 10x(t) \Rightarrow$$

$$x(t) = 2t \Rightarrow$$

Laplace Transform

Poles and zeros exercises

7. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2}+8\frac{dy(t)}{dt}+15y(t)=5\frac{dx(t)}{dt}+10x(t) \Rightarrow H(s)=\frac{5s+10}{s^2+8s+15}$$

$$x(t)=2t \Rightarrow X(s)=\frac{2}{s^2}$$

H(s) = Y(s)/X(s) → Y(s)=H(s)·X(s)

Poles and zeros exercises

7. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2}+8\frac{dy(t)}{dt}+15y(t)=5\frac{dx(t)}{dt}+10x(t) \Rightarrow H(s)=\frac{5s+10}{s^2+8s+15}$$

$$x(t)=2t \Rightarrow X(s)=\frac{2}{s^2}$$

$$Y(s)=\frac{10s+20}{s^2(s^2+8s+15)}$$

$$Y(s)=10\frac{s+2}{s^2(s+3)(s+5)}$$

Poles and zeros exercises

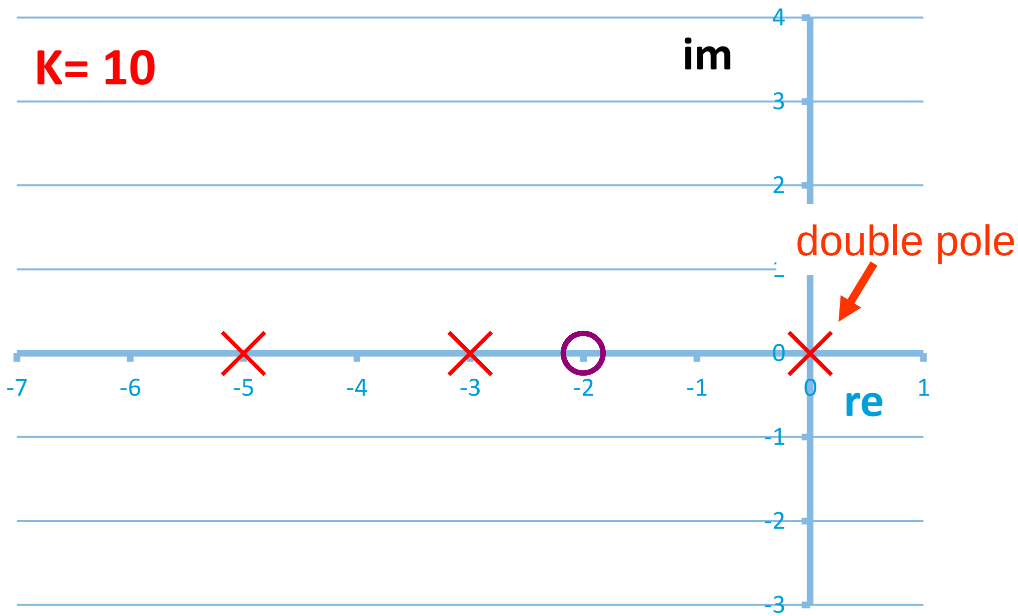
7. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2}+8\frac{dy(t)}{dt}+15y(t)=5\frac{dx(t)}{dt}+10x(t) \Rightarrow H(s)=\frac{5s+10}{s^2+8s+15}$$

$$x(t)=2t \Rightarrow X(s)=\frac{2}{s^2}$$

$$Y(s)=\frac{10s+20}{s^2(s^2+8s+15)}$$

$$Y(s)=10\frac{s+2}{s^2(s+3)(s+5)}$$



Poles and zeros exercises

8. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+4y(t)=3\frac{dx(t)}{dt}+18x(t)$$

$$x(t)=5\cos(3t)$$

Poles and zeros exercises

8. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = 3\frac{dx(t)}{dt} + 18x(t)$$

$$x(t) = 5\cos(3t)$$

Laplace Transform

Poles and zeros exercises

8. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+4y(t)=3\frac{dx(t)}{dt}+18x(t) \quad \Rightarrow \quad H(s)=\frac{3s+18}{s^2+5s+4}$$

$$x(t)=5\cos(3t) \quad \Rightarrow \quad X(s)=5\frac{s}{s^2+9}$$

Laplace Transform

Poles and zeros exercises

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$$Y(s)=15\frac{s(s+6)}{(s+1)(s+4)(s+3j)(s-3j)}$$

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Matlab commands

$$H_s = \frac{(s+7)}{s(s+5)(s+15)}$$

Define a system:

You can use:

>> sys=zpk(-7,[0 -5 -15],1);

or

>> sys=tf([1 7],[1 20 75 0]);

Another option is

>> s=tf('s');

>> sys= (s+7)/(s*(s+5)*(s+15));

Look at location of poles and zeros

>> pzmap(sys)

>> ltiview(sys)

RECAP

Transfer function:

$$H(s) = \frac{Y(s)}{X(s)}$$

Poles:

$s = p$ such that $X(s = p) = 0$, where $|H(s)| \to \infty$

Zeros:

$s = z$ such that $Y(s = z) = 0$, where $|H(s)| \to 0$

Stability criteria:

all poles in the open LHP

RECAP

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