BASIC CONTROL SYSTEMS

05 POLES AND ZEROS

HANSHU YU

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WHERE STUDENTS MATTER

POLES AND ZEROS

Theorem **The Fundamental Theorem of Algebra**
Let $f(x) = \sum_{n=0}^{k} a_n x^n$ be a non-constant polynomial and $a_n \in \mathbb{C}$, then there exist a unique factorization such that:

$$f(x) = \sum_{n=0}^{k} a_n x^n = r_0 \prod_{i=1}^{k} (x - r_i)$$

This fundamental theorem of algebra enables us to obtain a unique decomposition of a irreducible rational polynomial transfer function.

$$\frac{Numerator}{Denominator}$$

Definition **Poles**
The value(s) of s such that the denominator $D(s) = 0$

Definition **Zeros**
The value(s) of s such that the numerator $N(s) = 0$

These guarantees: ***the poles and zeros are either real or in complex conjugate pairs.***

TRANSFER FUNCTIONS

- Transfer functions can be written as:

$$\frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots\ldots + b_2 s^2 + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots\ldots + a_2 s^2 + a_1 s + a_0}$$

All coefficients a_n and b_m are real.

- Or as:

$$\frac{Y(s)}{X(s)} = \frac{b_m}{a_n} \cdot \frac{(s - z_1)(s - z_2)\ldots\ldots(s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2)\ldots\ldots(s - p_{p-1})(s - p_n)}$$

- Which is the same as:

$$H(s) = k_{pz} \frac{(s - z_1)(s - z_2)\ldots\ldots(s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2)\ldots\ldots(s - p_{p-1})(s - p_n)}$$

Additional gain

AN EXAMPLE

Input: x, Output: y,
Assume 0 initial conditions.
Given an ODE:

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}\mathrm{t}^2} + 2\frac{\mathrm{dy}}{\mathrm{d}t} - 8y = 3\frac{\mathrm{dx}}{\mathrm{d}t} + 1\mathrm{x}$$

We do the Laplace transform:

$$s^2Y + 2sY \ - 8Y = 3sX + 1X$$

Define transfer function H:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s+1}{s^2+2s-8} = 3\frac{s+\frac{1}{3}}{(s+4)(s-2)}$$

IDENTIFYING POLES AND ZEROS

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{3}\frac{s+\frac{1}{3}}{(s+4)(s-2)}$$

According to the definitions:

Gain K	$\frac{1}{3}$
Zeros z	$-\frac{1}{3}$
Poles p	$-4, +2$

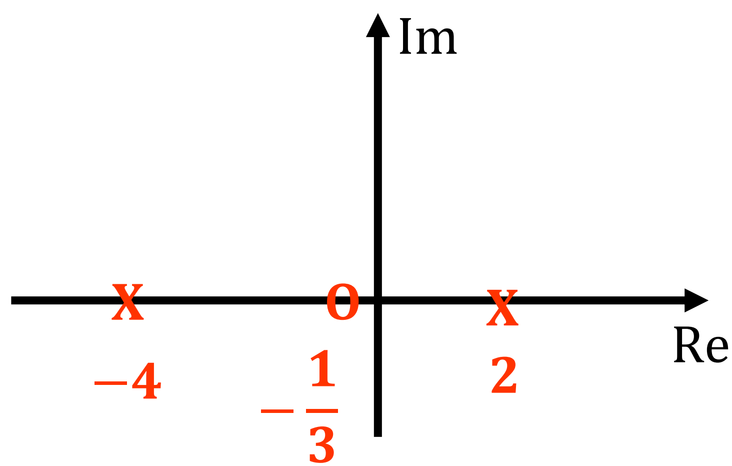
Obviously, when s = -4 or 2 (POLE), we have $H(s) \to \infty$

Obviously, when s = $-\frac{1}{3}$ (ZERO), we have $H(s) \to 0$

DRAWING POLES AND ZEROS IN THE COMPLEX PLANE

Components	Values	
Gain K	$\frac{1}{3}$	We don't draw this here.
Zeros $s = z$	$-\frac{1}{3}$	X
Poles $s = p$	$-4, +2$	O

***s*-domain**



ADDITIONAL PROPERTY OF POLES AND ZEROS

We are modelling causal linear systems in the real world.

ADDITIONAL PROPERTY OF POLES AND ZEROS

We are modelling causal linear systems in the real world.

This simple sentence tells us a lot!

ADDITIONAL PROPERTY OF POLES AND ZEROS

We are modelling causal linear systems in the real world.

Number of zeros never more than number of poles

The system can be modeled by a *linear inhomogeneous* ODE

All coefficients are real

The poles and zeros with non-zero imaginary components always comes in **conjugate pairs**.

For all poles and zeros, if there exist a pole/zero $\sigma + j\omega$ with $\omega \neq 0$, there must exist another pole/zero which is $\sigma - j\omega$ with $\omega \neq 0$ (the complex conjugate).

CONTINUING OUR EXAMPLE

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s+1}{s^2+2s-8} = 3\,\frac{s+\frac{1}{3}}{(s+4)(s-2)}$$

$$H(s) = \frac{\frac{11}{6}}{(s+4)} + \frac{\frac{7}{6}}{(s-2)}$$

Inverse Laplace transform:

$$\mathrm{h}(t) = \frac{1}{6}\,(11e^{-4t} + 7e^{2t})$$

TRANSFORM -> DECOMPOSE

What did we just do?

The process function h(t) can be decomposed to the summation of linearly independent exponentials: $Ce^{\lambda t}$

In fact, with Laplace transform, we can decompose any *linear system* into linearly independent exponentials:

$$h(t) = \sum Ce^{\lambda t}$$

POLES ARE CRUCIAL

What did we just do?

The process function h(t) can be decomposed to the summation of linearly independent exponentials: $Ce^{\lambda t}$

In fact, with Laplace transform, we can decompose any *linear system* into linearly independent exponentials:

$$h(t) = \sum Ce^{\lambda t}$$

λ correspond to the poles of the transfer function.

POLES ARE CRUCIAL

λ correspond to the **poles** of the transfer function. So,

$$h(t) = \sum Ce^{\lambda t} = \sum Ce^{\sigma t}e^{j\omega t}$$

σ - determines the decay(if stable) of the output signal
ω – determines the oscillation of the output signal

POLES ARE CRUCIAL – RECALL EXAMPLE

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s+1}{s^2+2s-8} = 3\,\frac{s+\frac{1}{3}}{(s+4)(s-2)}$$

$$H(s) = \frac{\frac{11}{6}}{(s+4)} + \frac{\frac{7}{6}}{(s-2)}$$

$\boldsymbol{pole: -4}$ $\boldsymbol{pole: \ 2}$

Inverse Laplace transform:

$$\mathrm{h}(t) = \frac{1}{6}\left(11e^{-4t} + 7e^{2t}\right)$$

STABILITY

Is $h(t)$ stable?

$$h(t) = \frac{1}{6}\,(11e^{-4t} + 7e^{2t})$$

STABILITY

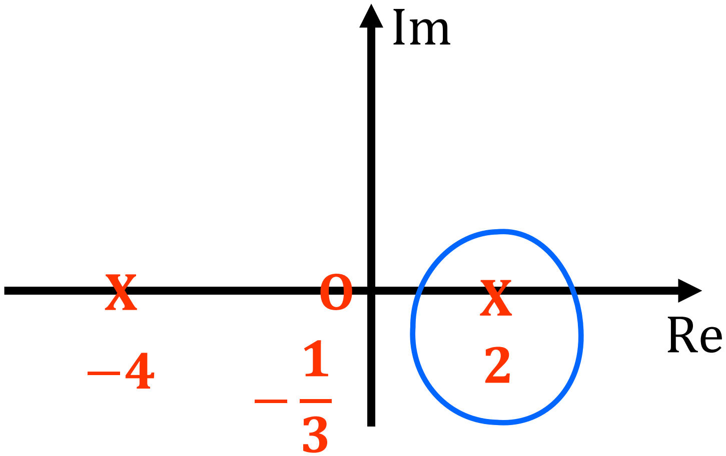
Is $h(t)$ stable?

$$h(t) = \frac{1}{6}\left(11e^{-4t} + 7e^{2t}\right)$$

Obviously not, if we look at $h(t)$ as $t \to \infty$:

$$\lim_{t\to\infty} \frac{1}{6}\left(11e^{-4t} + 7e^{2t}\right) = 0 + \infty$$

So not stable!

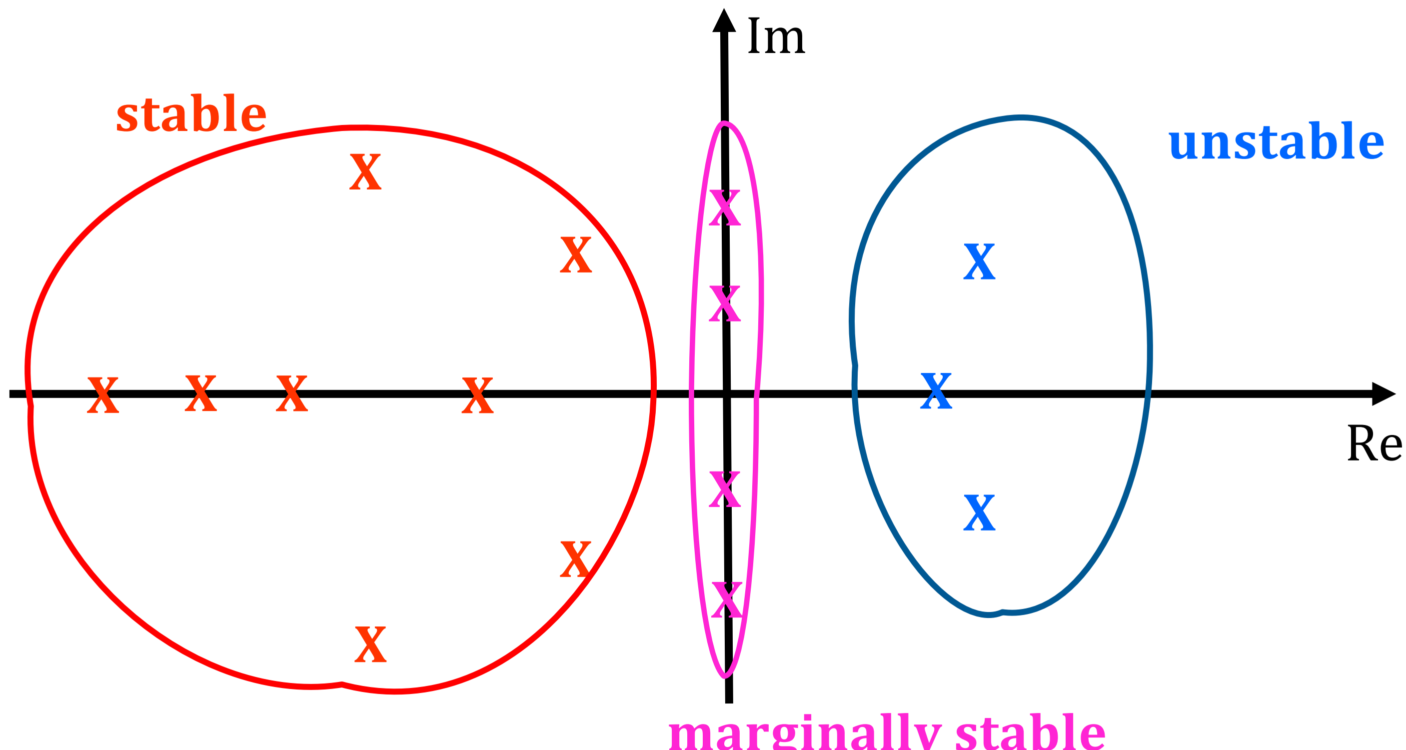


STABILITY CRITERIA

Left Half Plane

All poles should be in the <u>open LHP</u> of the s-plane.

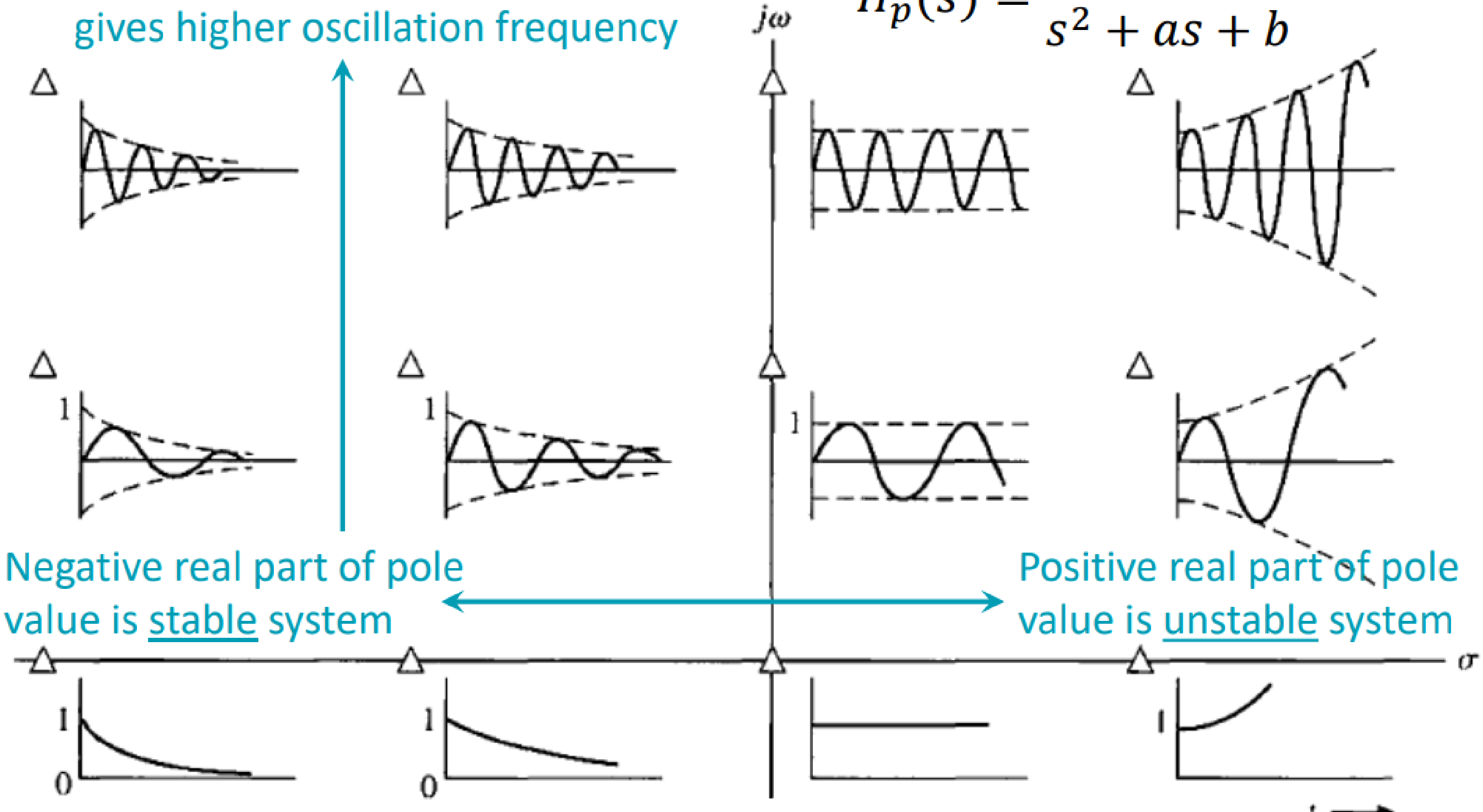
$$\text{iff } \forall\, \mathrm{Re}(p) < 0, \text{stable!}$$



POLES AND ZEROS

$$H_p(s) = \frac{b}{s^2 + as + b}$$

Larger imaginary part of pole value gives higher oscillation frequency



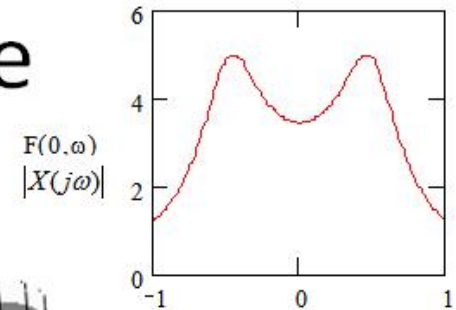
Negative real part of pole value is stable system

Positive real part of pole value is unstable system

Zero real part of pole value is marginally stable system

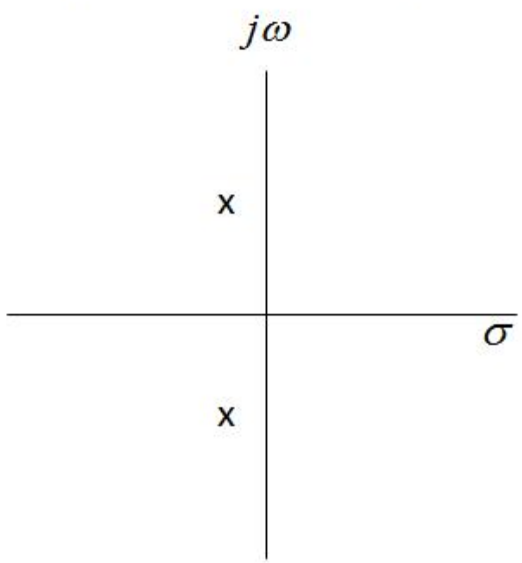
Poles and zeros: Why we care!

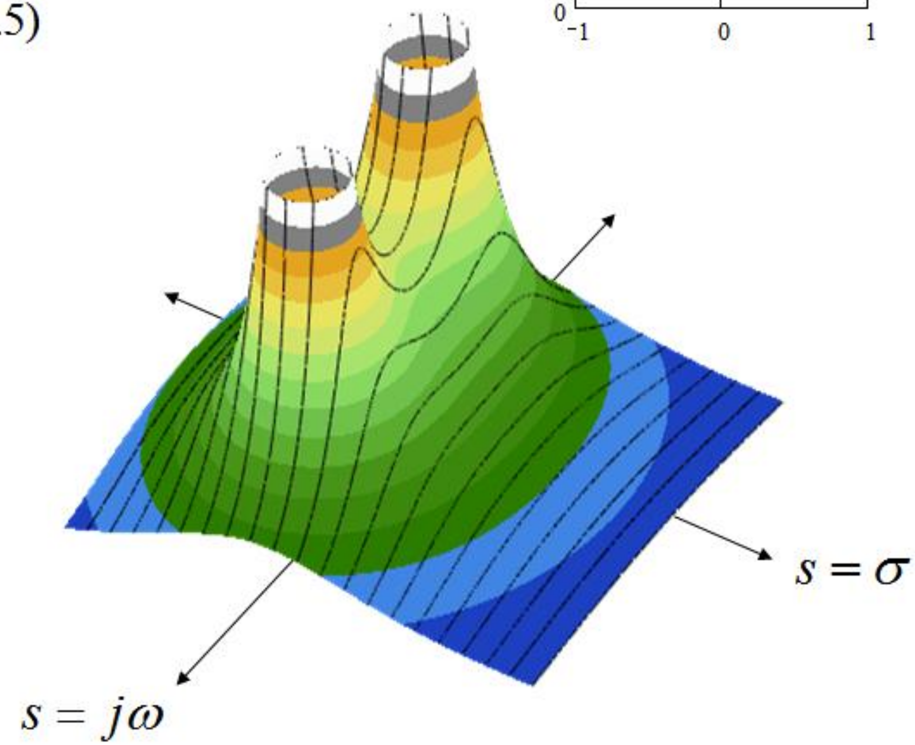
Filter Example



$$X(s)=\frac{1}{(s+.2+j.5)(s+.2-j.5)}$$

(poles at $s=-.2\pm j.5$)

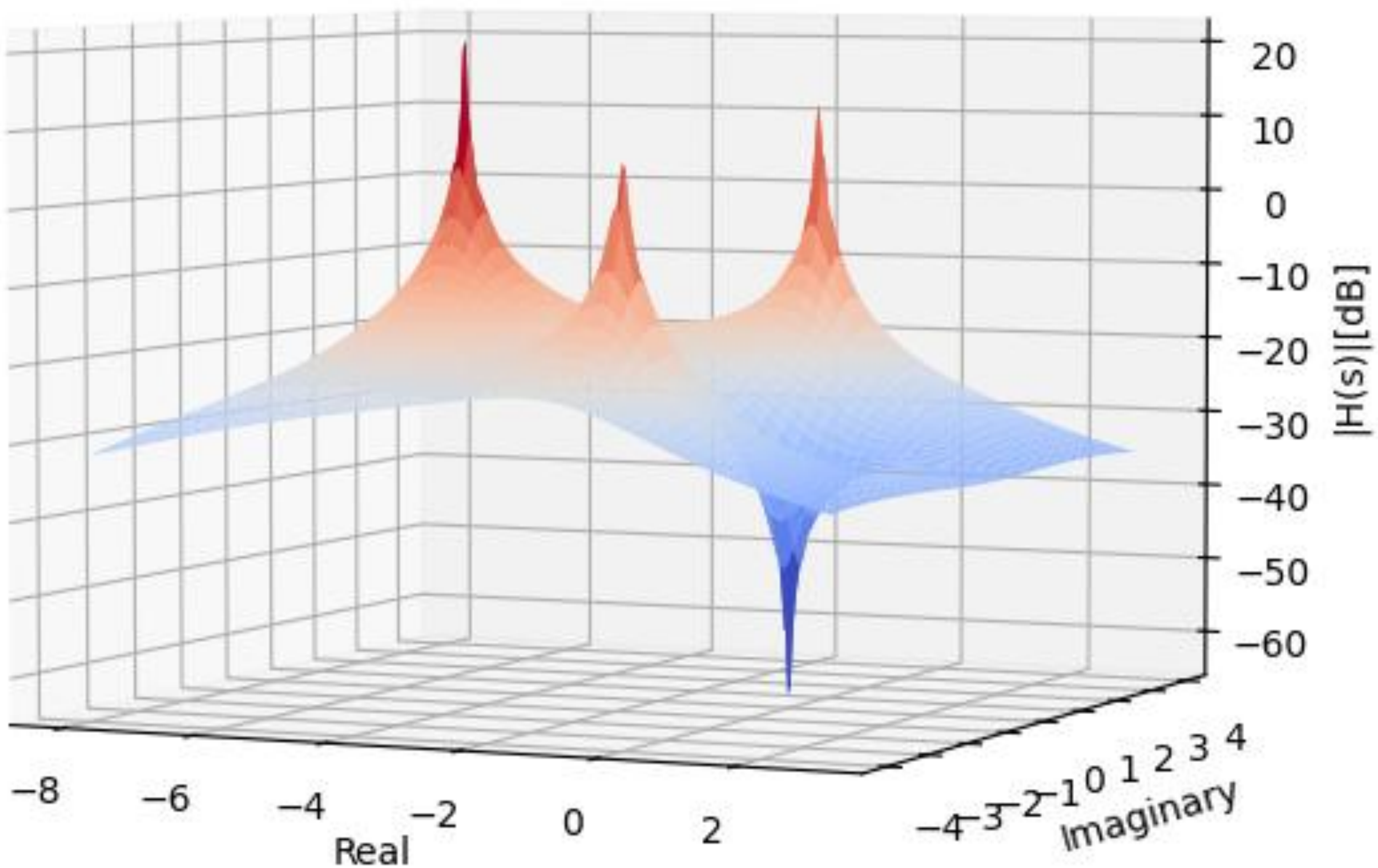




Poles and zeros: Why we care!

$$\frac{s}{(s+5)(s^2+2s+7)}$$

(Visualization in log scale)



SUMMARY

Transfer function:

$$H(s) = \frac{Y(s)}{X(s)}$$

Poles:

$s = p$ such that $X(s = p) = 0$, where $|H(s)| \to \infty$

Zeros:

$s = z$ such that $Y(s = z) = 0$, where $|H(s)| \to 0$

Stability criteria:

all poles in the open LHP

HOMEWORK

Stage ONE exercises:

- Problem 1
- Problem 4

SELF-READING

WHERE STUDENTS MATTER

Determination of the poles and zeros

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1 (both for zeros and poles!)
  - Examples:

Coefficient in the numerator is 0.1

$$G(s) = \frac{0{,}1s + 1}{s^2 + 7s + 12} =$$

Denominator coefficient of the highest power already is 1

Determination of the poles and zeros

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1 (both for zeros and poles!)
  - Examples:

$$G(s) = \frac{0{,}1s+1}{s^2+7s+12} = \frac{0{,}1(s+10)}{s^2+7s+12} =$$

Determination of the poles and zeros

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1 (both for zeros and poles!)
  - Examples:

$$G(s) = \frac{0{,}1s+1}{s^2+7s+12} = \frac{0{,}1(s+10)}{s^2+7s+12} = \frac{1}{10}\cdot\frac{s+10}{s^2+7s+12}$$

Determination of the poles and zeros

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  - Examples:

$$G(s)=\frac{0{,}1s+1}{s^2+7s+12}=\frac{0{,}1(s+10)}{s^2+7s+12}=\frac{1}{10}\cdot\frac{s+10}{s^2+7s+12}$$

$$G(s)=\frac{3s+30}{5s^2+15s+250}=$$

Determination of the poles and zeros

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1 (both for zeros and poles!)
  - Examples:

$$G(s)=\frac{0{,}1s+1}{s^2+7s+12}=\frac{0{,}1(s+10)}{s^2+7s+12}=\frac{1}{10}\cdot\frac{s+10}{s^2+7s+12}$$

$$G(s)=\frac{3s+30}{5s^2+15s+250}=\frac{3(s+10)}{5(s^2+3s+50)}=\frac{3}{5}\cdot\frac{s+10}{s^2+3s+50}$$

Poles and zeros construction steps

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1
- Step 2: rewrite the transfer function in its base parts – a constant, poles and zeros

$$G(s)=\frac{1}{10}\cdot\frac{s+10}{s^2+7s+12}=\frac{1}{10}\cdot\frac{s+10}{(s+3)(s+4)}$$

Poles and zeros construction steps

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1
- Step 2: rewrite the transfer function in its base parts – a constant, poles and zeros

$$G(s) = \frac{1}{10} \cdot \frac{s+10}{s^2+7s+12} = \frac{1}{10} \cdot \frac{s+10}{(s+3)(s+4)}$$

$$G(s) = \frac{3}{5} \cdot \frac{s+10}{s^2+3s+50} =$$

Sometimes the solution is complex
→ results in two complex poles

Poles and zeros construction steps

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1
- Step 2: rewrite the transfer function in its base parts – a constant, poles and zeros

$$G(s)=\frac{1}{10}\cdot\frac{s+10}{s^2+7s+12}=\frac{1}{10}\cdot\frac{s+10}{(s+3)(s+4)}$$

$$G(s)=\frac{3}{5}\cdot\frac{s+10}{s^2+3s+50}=\frac{3}{5}\cdot\frac{s+10}{(s+\frac{3}{2}+\frac{13{,}8}{2}j)(s+\frac{3}{2}-\frac{13{,}8}{2}j)}$$

Sometimes the solution is complex
→ results in two complex poles

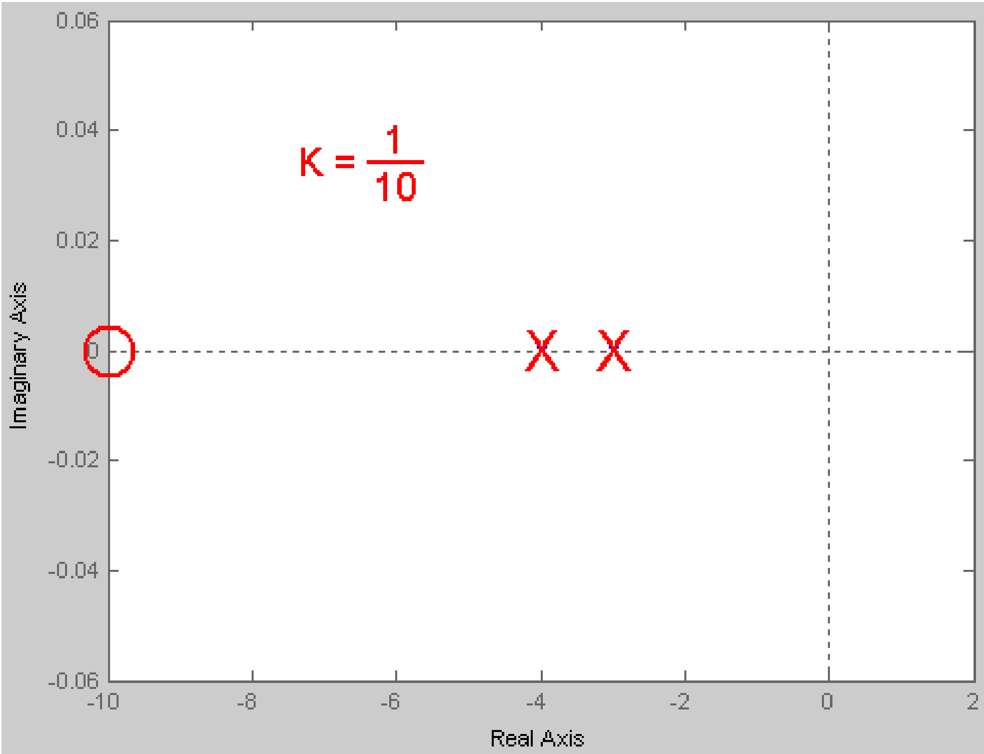
Poles and zeros construction steps

- Step 1: rewrite the transfer function in such a way that the coefficient of the highest power becomes 1
- Step 2: rewrite the transfer function in its base parts – a constant, poles and zeros
- Step 3: draw the poles and zeros in the (complex) s-plane; the constant is mentioned separately as K

Poles and zeros example

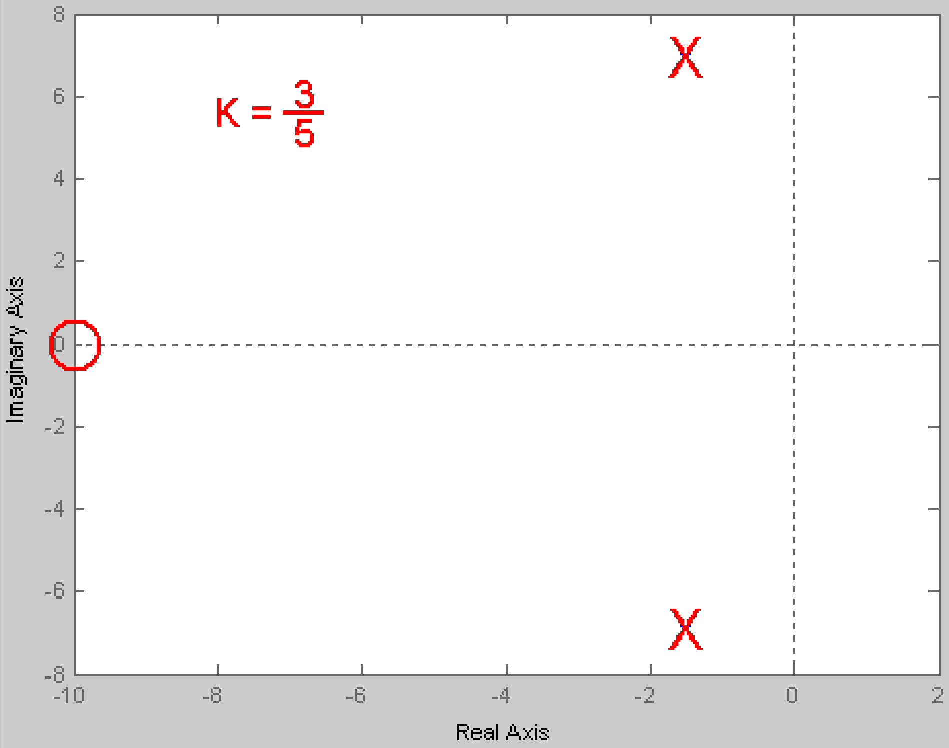
- Step 3: draw the poles and zeros in the (complex) s-plane; the constant is mentioned separately as K

$$G(s) = \frac{1}{10} \cdot \frac{s+10}{(s+3)(s+4)}$$



Poles and zeros example

$$G(s) = \frac{3}{5} \cdot \frac{s+10}{(s+\frac{3}{2}+\frac{13,8}{2}j)(s+\frac{3}{2}-\frac{13,8}{2}j)}$$



Poles and zeros exercises

Draw the poles and zeros in the s-plane for:

1. $$H(s)=\frac{25s+3}{4s^2+9s+2}$$

2. $$H(s)=\frac{3s+4}{s^2+6s+8}$$

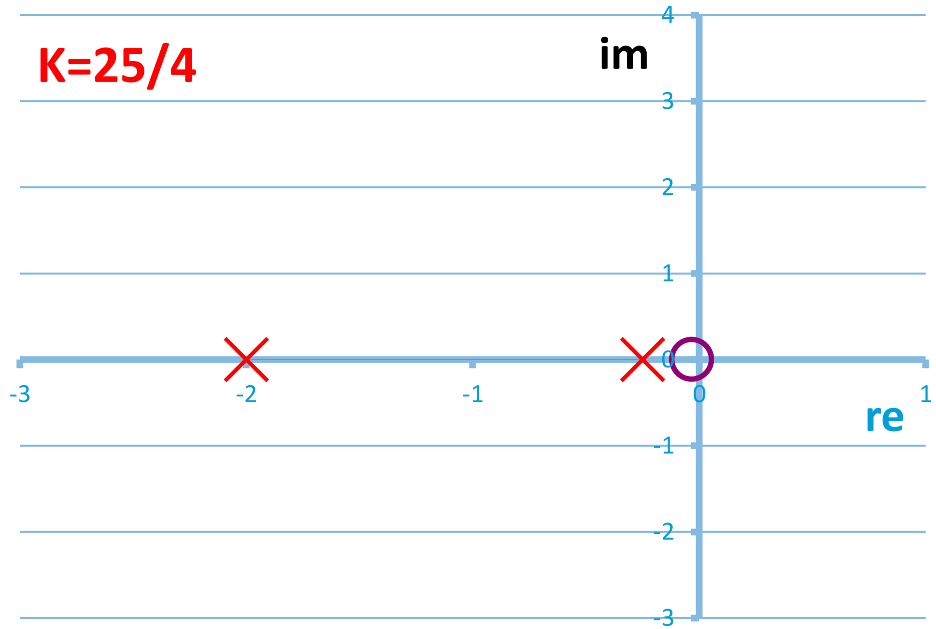
3. $$H(s)=\frac{2s+1}{s^2+4s+8}$$

Poles and zeros exercises

- Draw the poles and zeros in the s-plane for:

1. $$H(s) = \frac{25s+3}{4s^2+9s+2}$$

- zero: -3/25
- poles: -1/4 and -2
- K = 25/4

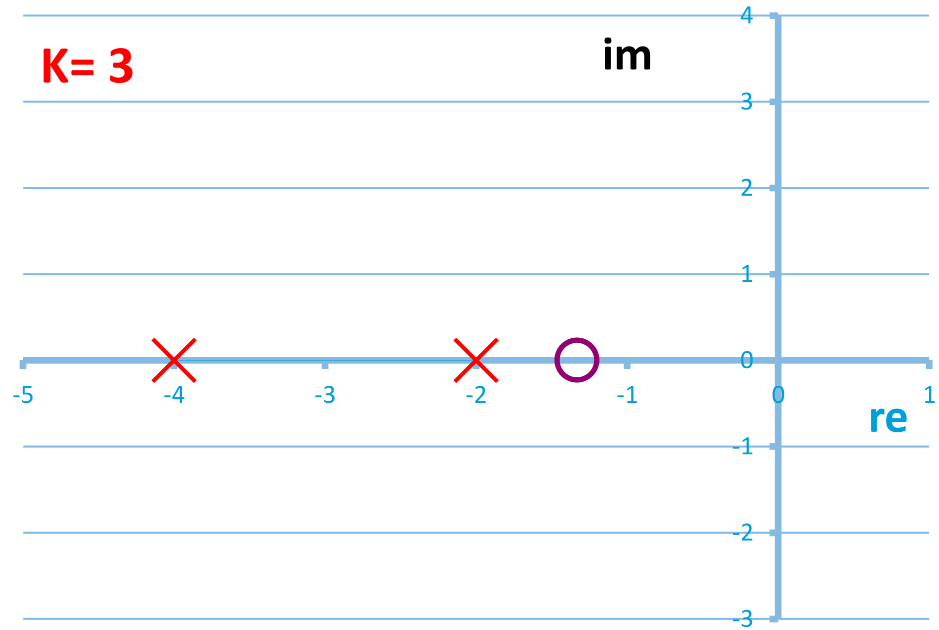


Poles and zeros exercises

- Draw the poles and zeros in the s-plane for:

2. $$H(s) = \frac{3s+4}{s^2+6s+8}$$

- zeros: -4/3
- poles: -2 and -4
- K = 3

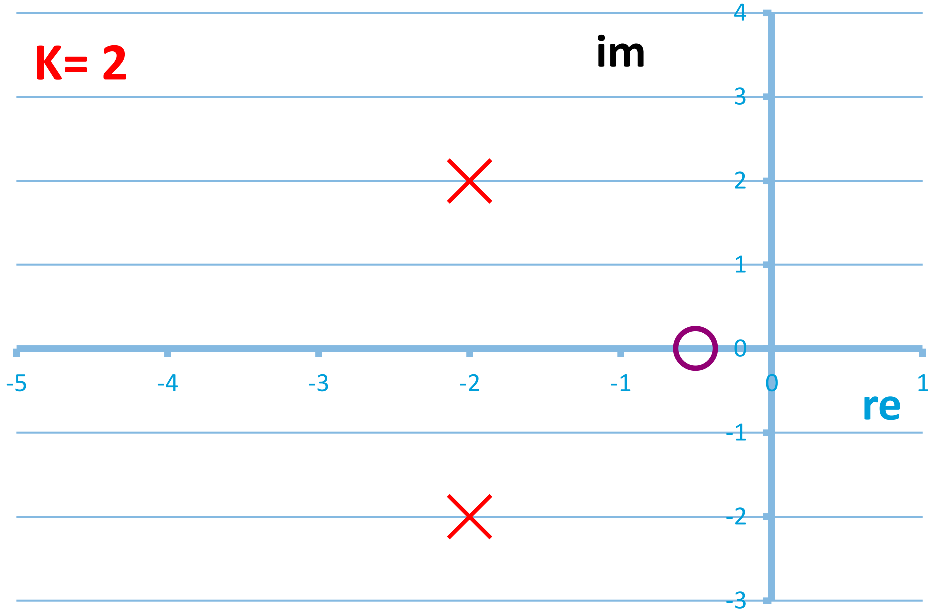


Poles and zeros exercises

- Draw the poles and zeros in the s-plane for:

3. $$H(s)=\frac{2s+1}{s^2+4s+8}$$

- zeros: -1/2
- poles: -2+2j
  and -2-2j
- K = 2



Poles and zeros exercises

4. Draw in the s-plane the poles and zeros of the transfer function H(s) = X(s)/F(s) and:

$$\frac{d^4x(t)}{dt^4}+2\frac{d^3x(t)}{dt^3}+2\frac{d^2x(t)}{dt^2}=\frac{df(t)}{dt}+f(t)$$

All values at time = 0 are zero

(so x’’’(0)=x’’(0)=0, etc.).

Poles and zeros exercises

- Draw the poles and zeros in the s-plane for:

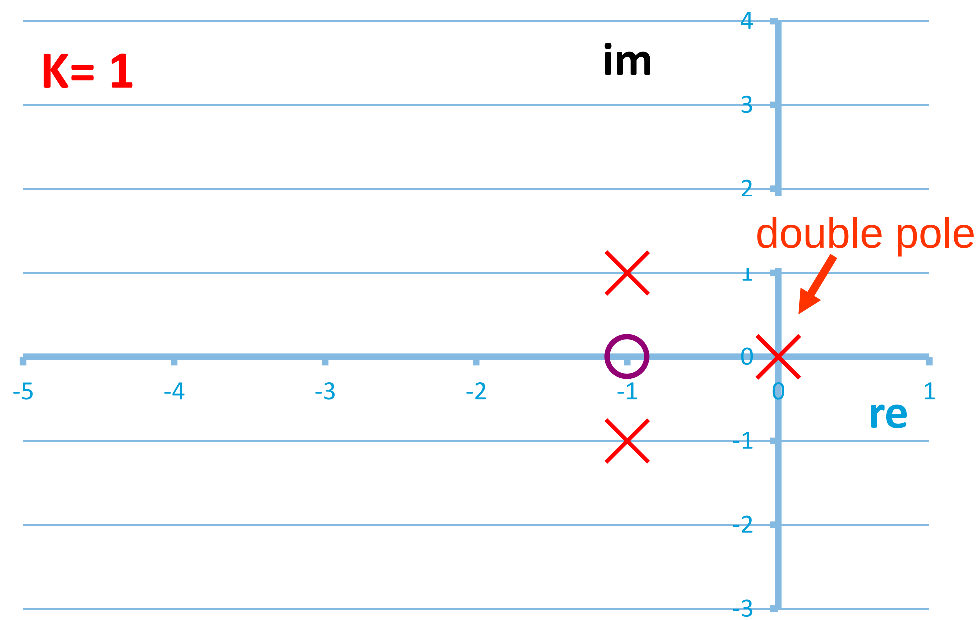
4. $$\frac{d^4x(t)}{dt^4} + 2\frac{d^3x(t)}{dt^3} + 2\frac{d^2x(t)}{dt^2} = \frac{df(t)}{dt} + f(t)$$

Laplace → $s^4 + 2s^3 + 2s^2 = s + 1$

Transfer function:

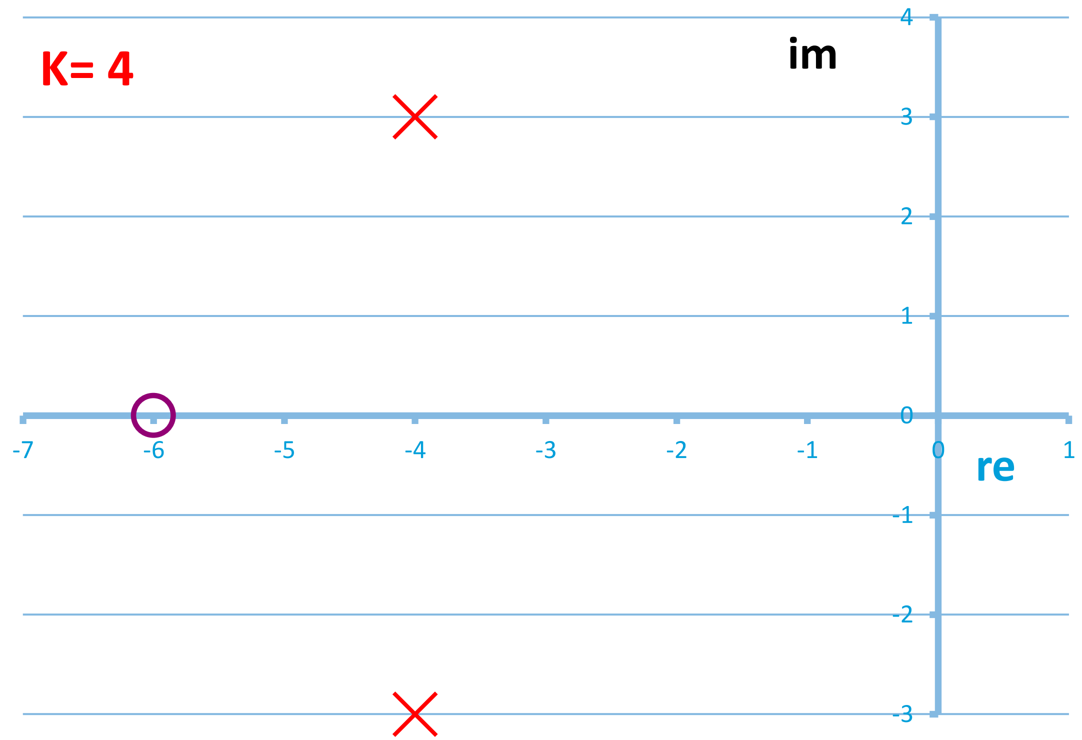
$$H_s = \frac{s+1}{s^4 + 2s^3 + 2s^2}$$

$$= \frac{s+1}{s * s(s+1+j)(s+1-j)}$$



Poles and zeros exercises

5. Find the differential equation for:

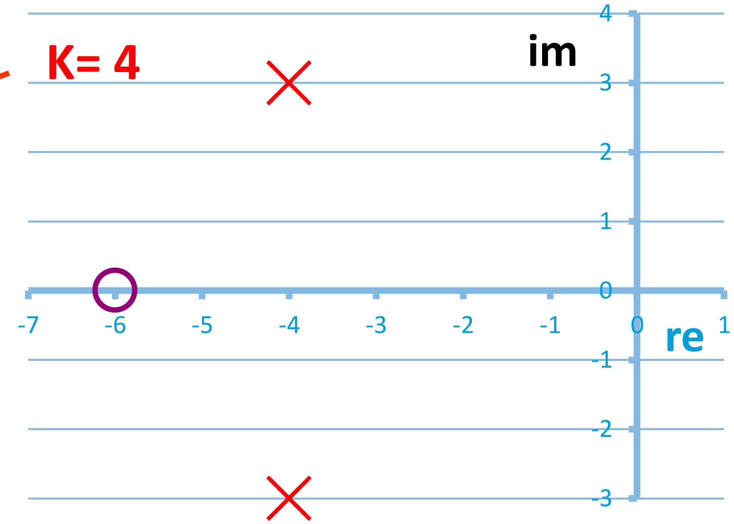


Assume that the initial conditions are zero.

Input is $x(t)$ and output is $y(t)$.

Poles and zeros exercises

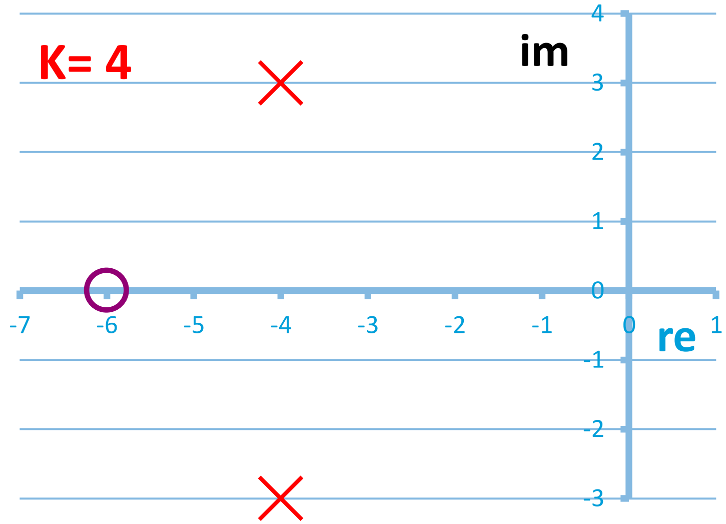
5. Find the differential equation



$$\frac{Y(s)}{X(s)} = 4\frac{s+6}{(s+4-3i)(s+4+3i)}$$

Poles and zeros exercises

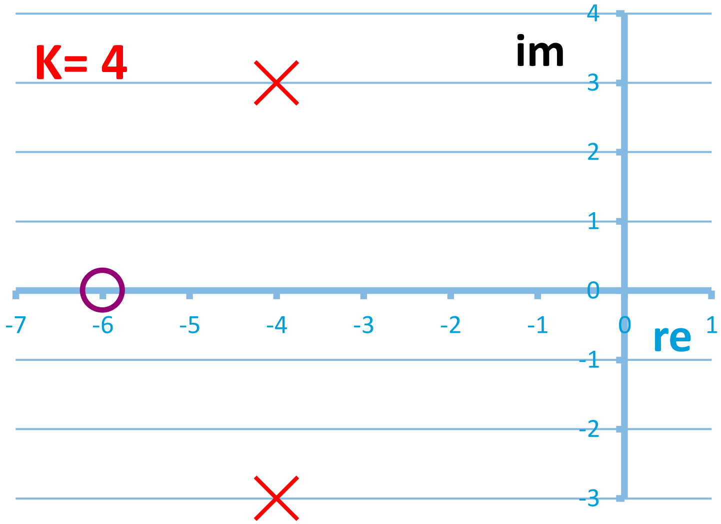
5. Find the differential equation



$$\frac{Y(s)}{X(s)} = 4\frac{s+6}{(s+4-3i)(s+4+3i)} = \frac{4s+24}{s^2+8s+25}$$

Poles and zeros exercises

5. Find the differential equation



$$\frac{Y(s)}{X(s)} = 4\frac{s+6}{(s+4-3i)(s+4+3i)} = \frac{4s+24}{s^2+8s+25}$$

$$\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 25y = 4\frac{dx(t)}{dt} + 24x(t)$$

Poles and zeros exercises

6. Find the differential equation for:

Assume that the initial conditions are zero.

Input is $x(t)$ and output is $y(t)$.

Poles and zeros exercise

6. Find the differential equation

$$\frac{Y(s)}{X(s)} = 2\frac{(s+1)(s+3)}{(s+2-3i)(s+2+3i)s}$$

Poles and zeros exercise

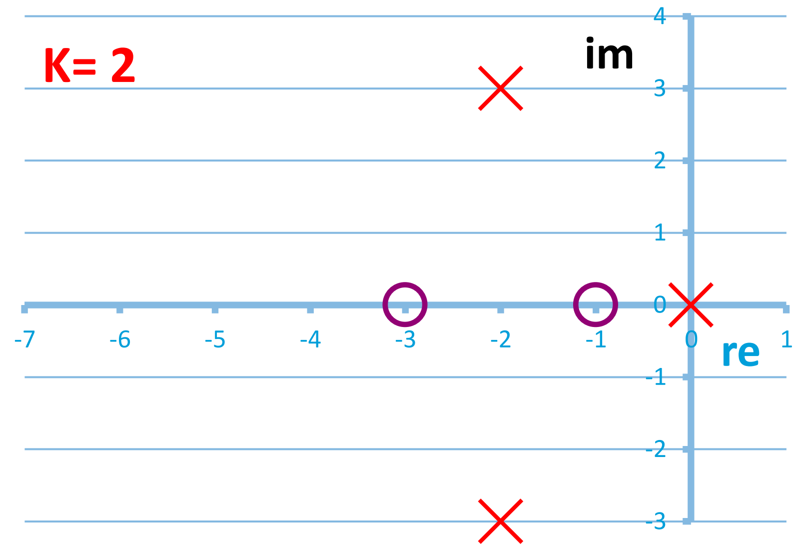
6. Find the differential equation

$$\frac{Y(s)}{X(s)} = 2\frac{(s+1)(s+3)}{(s+2-3i)(s+2+3i)s}$$

$$\frac{Y(s)}{X(s)} = \frac{2s^2+8s+6}{s^3+4s^2+13s}$$

Poles and zeros exercise

6. Find the differential equation



$$\frac{Y(s)}{X(s)} = 2\frac{(s+1)(s+3)}{(s+2-3i)(s+2+3i)s}$$

$$\frac{Y(s)}{X(s)} = \frac{2s^2+8s+6}{s^3+4s^2+13s}$$

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 13\frac{dy(t)}{dt} = 2\frac{d^2x(t)}{dt^2} + 8\frac{dx(t)}{dt} + 6x(t)$$

Poles and zeros exercise

7. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 15y(t) = 5\frac{dx(t)}{dt} + 10x(t)$$

$$x(t) = 2t$$

Poles and zeros exercises

7. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2}+8\frac{dy(t)}{dt}+15y(t)=5\frac{dx(t)}{dt}+10x(t)$$

$$x(t)=2t$$

Laplace Transform

Poles and zeros exercises

7. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2}+8\frac{dy(t)}{dt}+15y(t)=5\frac{dx(t)}{dt}+10x(t) \Rightarrow H(s)=\frac{5s+10}{s^2+8s+15}$$

$$x(t)=2t \Rightarrow X(s)=\frac{2}{s^2}$$

H(s) = Y(s)/X(s) → Y(s)=H(s)·X(s)

Poles and zeros exercises

7. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2}+8\frac{dy(t)}{dt}+15y(t)=5\frac{dx(t)}{dt}+10x(t) \Rightarrow H(s)=\frac{5s+10}{s^2+8s+15}$$

$$x(t)=2t \Rightarrow X(s)=\frac{2}{s^2}$$

$$Y(s)=\frac{10s+20}{s^2(s^2+8s+15)}$$

$$Y(s)=10\frac{s+2}{s^2(s+3)(s+5)}$$

Poles and zeros exercises

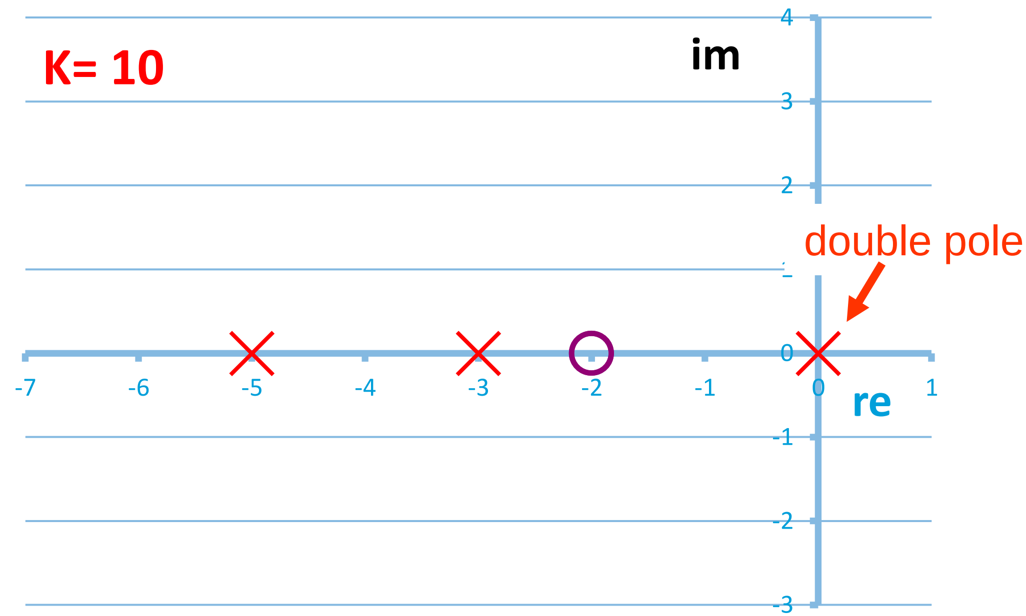
7. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2}+8\frac{dy(t)}{dt}+15y(t)=5\frac{dx(t)}{dt}+10x(t) \Rightarrow H(s)=\frac{5s+10}{s^2+8s+15}$$

$$x(t)=2t \Rightarrow X(s)=\frac{2}{s^2}$$

$$Y(s)=\frac{10s+20}{s^2(s^2+8s+15)}$$

$$Y(s)=10\frac{s+2}{s^2(s+3)(s+5)}$$



Poles and zeros exercises

8. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = 3\frac{dx(t)}{dt} + 18x(t)$$

$$x(t) = 5\cos(3t)$$

Poles and zeros exercises

8. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = 3\frac{dx(t)}{dt} + 18x(t)$$

$$x(t) = 5\cos(3t)$$

Laplace Transform

Poles and zeros exercises

8. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+4y(t)=3\frac{dx(t)}{dt}+18x(t) \Rightarrow H(s)=\frac{3s+18}{s^2+5s+4}$$

$$x(t)=5\cos(3t) \Rightarrow X(s)=5\frac{s}{s^2+9}$$

Laplace Transform

Poles and zeros exercises

8. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+4y(t)=3\frac{dx(t)}{dt}+18x(t) \Rightarrow H(s)=\frac{3s+18}{s^2+5s+4}$$

$$x(t)=5\cos(3t) \Rightarrow X(s)=5\frac{s}{s^2+9}$$

Laplace Transform

$$Y(s)=\frac{5s(3s+18)}{(s^2+9)(s^2+5s+4)}$$

Poles and zeros exercises

8. Draw the poles and zeros in the s-plane for the combination:

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+4y(t)=3\frac{dx(t)}{dt}+18x(t) \Rightarrow H(s)=\frac{3s+18}{s^2+5s+4}$$

$$x(t)=5\cos(3t) \Rightarrow X(s)=5\frac{s}{s^2+9}$$

Laplace Transform

$$Y(s)=\frac{5s(3s+18)}{(s^2+9)(s^2+5s+4)}$$

$$Y(s)=15\frac{s(s+6)}{(s+1)(s+4)(s+3j)(s-3j)}$$

Poles and zeros exercises

8. Draw the poles and zeros in the s-plane for the combination:

$$Y(s) = 15\frac{s(s+6)}{(s+1)(s+4)(s+3j)(s-3j)}$$

Matlab commands

$$H_s = \frac{(s+7)}{s(s+5)(s+15)}$$

Define a system:

You can use:

```
>> sys=zpk(-7,[0 -5 -15],1);
```

or

```
>> sys=tf([1 7],[1 20 75 0]);
```

Another option is

```
>> s=tf('s');
>> sys= (s+7)/(s*(s+5)*(s+15));
```

Look at location of poles and zeros

```
>> pzmap(sys)
>> ltiview(sys)
```