STATS 531 HW 1

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Question 1.1

Let $X_{1:N}$ be a covariance stationary time series model with autocovariance function $\gamma_h = \text{Cov}(X_n, X_{n+h})$ and constant mean function $\mu_n = \mu$. Considering the sample mean as an estimator of μ,

$$\hat{\mu}(x_{1:N}) = \frac{1}{N}\sum_{n=1}^{N} x_n,$$

we derive the equation $\text{Var}(\hat{\mu}(X_{1:N})) = \frac{1}{N}\gamma_0 + \frac{2}{N^2}\sum_{h=1}^{N-1}(N-h)\gamma_h$.

Proof. Noting that the time series is covariance stationary, we have

$$\begin{aligned}
\text{Var}(\hat{\mu}(X_{1:N})) &= \text{Var}\left(\frac{1}{N}\sum_{n=1}^{N} X_n\right) \\
&= \frac{1}{N^2}\sum_{i,j=1}^{N} \text{Cov}\left(X_i, X_j\right) \\
&= \frac{1}{N^2}\sum_{i=1}^{N} \text{Var}(X_i) + \frac{1}{N^2}\sum_{i\neq j}^{N} \text{Cov}(X_i, X_j).
\end{aligned}$$

Now, we observe that

$$\gamma_0 = \text{Cov}(X_n, X_{n+0}) = \text{Var}(X_n)$$

for all $n \in \{1, 2, \ldots, N\}$. Thus,

$$\begin{aligned}
\text{Var}(\hat{\mu}(X_{1:N})) &= \frac{1}{N^2}(N\gamma_0) + \frac{1}{N^2}\sum_{i\neq j}^{N} \text{Cov}(X_i, X_j) \\
&= \frac{1}{N}\gamma_0 + \frac{2}{N^2}\sum_{i<j}^{N} \text{Cov}(X_i, X_j)
\end{aligned}$$

Now, we may observe that

$$\begin{aligned}\sum_{i<j}^{N}\text{Cov}(X_i,X_j) &= (\gamma_1+\gamma_2+\cdots+\gamma_{N-1})+(\gamma_1+\gamma_2+\cdots+\gamma_{N-2})+\cdots+(\gamma_1+\gamma_2)+(\gamma_1)\\ &= (N-1)\gamma_1+(N-2)\gamma_2+(N-3)\gamma_3+\cdots+2\gamma_{N-2}+\gamma_{N-1}\\ &= \sum_{h=1}^{N-1}(N-h)\gamma_h\end{aligned}$$

Thus, we have that

$$\text{Var}(\hat{\mu}(X_{1:N})) = \frac{1}{N}\gamma_0 + \frac{2}{N^2}\sum_{h=1}^{N-1}(N-h)\gamma_h.$$

Question 1.2 A

Suppose the null hypothesis holds, i.e. that $X_{1:N}$ are iid random variables with mean 0 and variance σ_X^2. Now, we note that the sample autocovariance and sample variance functions are $\hat{\gamma}_h = \frac{1}{N}\sum_{n=1}^{N-h}(x_n-\hat{\mu}_n)(x_{n+h}-\hat{\mu}_{n+h})$ and $\hat{\gamma}_0 = \frac{1}{N}\sum_{n=1}^{N}(x_n-\hat{\mu}_n)^2$. The sample autocorrelation is $\hat{\rho}_h = \frac{\hat{\gamma}_h}{\hat{\gamma}_0}$.

Since $X_{1:N}$ are iid mean 0 random variables, we know $\hat{\mu}$ converges in distribution to 0. Thus, we use $\hat{\mu}\equiv 0$ as our mean estimator. We define define U and V in terms of the random sample autocorrelation:

$$\hat{\rho}_h = \frac{\sum_{n=1}^{N-h}X_nX_{n+h}}{\sum_{n=1}^{N}X_n^2} = \frac{U}{V}$$

where $U=\sum_{n=1}^{N-h}X_nX_{n+h}$ and $V=\sum_{n=1}^{N}X_n^2$. Now, we define a nonlinear function $g((U,V))=\frac{U}{V}$. Using the multivariate version of the delta method, we have that

$$\hat{\rho}_h = g\left(\begin{bmatrix}U\\V\end{bmatrix}\right) \approx g\left(\begin{bmatrix}\mu_U\\\mu_V\end{bmatrix}\right) + \nabla g\left(\begin{bmatrix}\mu_U\\\mu_V\end{bmatrix}\right)^T\left(\begin{bmatrix}U\\V\end{bmatrix}-\begin{bmatrix}\mu_U\\\mu_V\end{bmatrix}\right)$$

where $\mu_U=\mathbb{E}(U)$ and $\mu_V=\mathbb{E}(V)$. Expanding this, we have

$$\begin{aligned}\hat{\rho}_h &\approx \frac{\mu_U}{\mu_V} + \left[\tfrac{\partial}{\partial U}\tfrac{U}{V} \quad \tfrac{\partial}{\partial V}\tfrac{U}{V}\right]_{\mu_V,\mu_U}\cdot\begin{bmatrix}U-\mu_U\\V-\mu_V\end{bmatrix}\\ &= \frac{\mu_U}{\mu_V} + (U-\mu_U)\frac{\partial}{\partial U}\frac{U}{V}\bigg|_{\mu_V,\mu_U} + (V-\mu_V)\frac{\partial}{\partial V}\frac{U}{V}\bigg|_{\mu_V,\mu_U}\\ &= \frac{\mu_U}{\mu_V} + (U-\mu_U)\frac{1}{\mu_V} + (V-\mu_V)\left(-\frac{\mu_U}{\mu_V^2}\right)\\ &= \frac{U}{\mu_V} - (V-\mu_V)\left(\frac{\mu_U}{\mu_V^2}\right)\end{aligned}$$

We observe that $\mu_V=\mathbb{E}(V)=\mathbb{E}\left(\sum_{n=1}^{N}X_n^2\right)=N\mathbb{E}(X_i^2)=N\sigma_X^2$ since $\mathbb{E}(X_i)=0$. Similarly, $\mu_U=\mathbb{E}(U)=\mathbb{E}\left(\sum_{n=1}^{N-h}X_nX_{n+h}\right)=\sum_{n=1}^{N-h}\mathbb{E}(X_n)\mathbb{E}(X_{n+h})=0$. Therefore,

$$\hat{\rho}_h \approx \frac{U}{N\sigma_X^2} - (V-N\sigma_X^2)\cdot 0 = \frac{U}{N\sigma_X^2}.$$

Then it follows that

$$\text{Var}(\hat{\rho}_h) \approx \text{Var}\left(\frac{U}{N\sigma_X^2}\right) = \frac{1}{N^2\sigma_X^4}\text{Var}(U).$$

Now, it suffices to find $\text{Var}(U)$:

$$\begin{aligned}
\text{Var}(U) &= \text{Var}\left(\sum_{n=1}^{N-h} X_n X_{n+h}\right) \\
&= \mathbb{E}\left[\left(\sum_{n=1}^{N-h} X_n X_{n+h}\right)^2\right] - \underbrace{\mathbb{E}^2\left[\sum_{n=1}^{N-h} X_n X_{n+h}\right]}_{=0} \\
&= \mathbb{E}\left[\sum_{n=1}^{N-h} X_n^2 X_{n+h}^2 + 2\sum_{j=1}^{N-h}\sum_{i=1}^{j-1} X_j X_{j+h} X_i X_{i+h}\right] \\
&= (N-h)\mathbb{E}(X_n^2)\mathbb{E}(X_{n+h}^2) + 0 \\
&= (N-h)\left[\mathbb{E}(X_n^2)\right]^2 \\
&= (N-h)\sigma_X^4.
\end{aligned}$$

Then a reasonable estimate of the asymptotic variance of $\hat{\rho}_h$ is:

$$\text{Var}(\hat{\rho}_h) \approx \frac{1}{N^2\sigma_X^4}(N-h)\sigma_X^4 = \frac{N-h}{N^2} = \frac{1}{N} - \frac{h}{N^2}.$$

Then for small h relative to a large sample size N, $\text{Var}(\hat{\rho}_h) \approx \frac{1}{N}$ is a reasonable approximation of the asymptotic variance (and therefore $1/\sqrt{N}$ a reasonable approximation of the asymptotic standard deviation) of the sample autocorrelation under the null hypothesis.

Question 1.2 B

For some random 95% confidence interval (where randomness is inherent in the CI being a function of the data $X_1, X_2, \ldots, X_N$), there should be 0.95 probability of the true parameter falling in the random confidence interval. Once the data is observed, there is now either a probability of 0 or 1 that the interval covers the parameter.

However, the interval $\left[-\frac{1}{N}, \frac{1}{N}\right]$, determined from the previous problem, is not a function of the data. Therefore it always exhibits a coverage probability of 0 or 1 for any parameter before the data is observed. Therefore, the dashed lines are not a "typical" confidence interval.

Sources

- Wikipedia entries for delta method and variance.
- Wolframalpha entry on power sums.
- 531 W16 HW1 Solution for Q1.2 A Used to note that $\hat{\mu} \equiv 0$ and that multivariate delta method should be used to proceed. Then problem was solved without solutions and checked with solutions. Solutions also used to check Q1.2 B.

Please Explain

- If a time series model for $Y_{1:N}$ is covariance stationary, that implies that the covariance between two points only depends on the time between these two points. Does that always imply $\text{Var}(Y_1) = \text{Var}(Y_2) = \cdots = \text{Var}(Y_N)$?
- Is it mathematically rigorous to let $\hat{\mu} \equiv 0$ throughout problem 1.2 A? Clearly it converges in distribution to 0 since $X_{1:N}$ are iid with mean 0...