STATS 531 HW 2

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Contents

Question 2.1

Consider the AR(1) model, $X_n = \phi X_{n-1} + \epsilon_n$, where $\{\epsilon_n\}$ is white noise with variance σ^2 and $-1 < \phi < 1$. We assume the process is stationary, i.e. it is initialized with a random draw from its stationary distibution.

A

The autocovariance function is:

$$\begin{aligned}
\gamma_h &= \text{Cov}(X_n, X_{n+h}) \\
&= \text{Cov}(X_n, \phi X_{n+h-1} + \epsilon_{n+h}) \\
&= \text{Cov}(X_n, \phi X_{n+h-1}) + \text{Cov}(X_n, \epsilon_{n+h}) \\
&= \phi\gamma_{h-1} + \underbrace{\text{Cov}(X_n, \epsilon_{n+h})}_{=0} \\
&= \phi\gamma_{h-1}
\end{aligned}$$

Because we want explicit A and λ such that $\gamma_h = A\lambda^h$, we derive the initial condition by finding γ_0:

$$\begin{aligned}
\gamma_0 &= \text{Cov}(X_n, X_n) \\
&= \text{Var}(X_n) \\
&= \text{Var}(\phi X_{n-1} + \epsilon_n) \\
&= \phi^2 \text{Var}(X_n) + \text{Var}(\epsilon_n) \\
&= \phi^2\gamma_0 + \sigma^2
\end{aligned}$$

This implies that $\gamma_0(1-\phi^2) = \sigma^2$. Therefore, we have $\gamma_0 = \frac{\sigma^2}{1-\phi^2}$. Now, we may solve the recurrence relation:

$$\gamma_h = \phi\gamma_{h-1} = \phi(\phi\gamma_{h-2}) = \phi^3\gamma_{h-3} = \cdots = \phi^h\gamma_0 = \phi^h \frac{\sigma^2}{1-\phi^2}$$

Therefore, we have $\gamma_h = A\lambda^h$ where $\lambda = \phi$ and $A = \frac{\sigma^2}{1-\phi^2}$.

B

From real analysis (or calculus) we know that a Taylor series representation of $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$, converging for $|x| < 1$. Therefore, we have

$$\frac{1}{1-\phi x} = \sum_{n=0}^{\infty} (\phi x)^n$$

for $|\phi x| < 1$.

From the design of our AR(1) model, and using the backshift operator B, we have:

$$\begin{aligned}
\epsilon_n &= X_n - \phi X_{n-1} = X_n \\
&= X_n - \phi B X_n \\
&= (1-\phi B) X_n.
\end{aligned}$$

Therefore, we have $X_n = (1-\phi B)^{-1}\epsilon_n$. Using a Taylor series representation of $(1-\phi B)^{-1}$, we obtain the MA(∞) representation of the AR(1) model:

$$\begin{aligned}
X_n &= \left(\sum_{i=0}^{\infty} (\phi B)^i\right) \epsilon_n \\
&= (B^0 + \phi B + \phi^2 B^2 + \cdots)\epsilon_n \\
&= B^0 \epsilon_n + \phi B \epsilon_n + \phi^2 B^2 \epsilon_n + \cdots \\
&= \epsilon_n + \phi \epsilon_{n-1} + \phi^2 \epsilon_{n-2} + \cdots \\
&= \sum_{k=0}^{\infty} \phi^k \epsilon_{n-k}.
\end{aligned}$$

Then we apply the general formula for the autocovariance function:

$$\begin{aligned}
\gamma_h &= \mathrm{Cov}\left(X_n, X_{n+h}\right) \\
&= \mathrm{Cov}\left(\sum_{j=0}^{\infty} \phi^j \epsilon_{n-j}, \sum_{k=0}^{\infty} \phi^k \epsilon_{n+h-k}\right) \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \phi^j \phi^k \cdot \qquad \underbrace{\mathrm{Cov}\left(\epsilon_{n-j}, \epsilon_{n+h-k}\right)}_{\neq 0 \text{ when } n-j=n+h-k, \text{ i.e. } k=j+h} \\
&= \sum_{j=0}^{\infty} \sum_{k=j+h}^{\infty} \phi^j \phi^k \mathrm{Cov}(\epsilon_{n-j}, \epsilon_{n+h-k}) \\
&= \sum_{j=0}^{\infty} \phi^j \phi^{j+h} \mathrm{Cov}(\epsilon_{n-j}, \epsilon_{n-j}) \\
&= \sum_{j=0}^{\infty} \phi^j \phi^{j+h} \sigma^2 \\
&= \sigma^2 \phi^h \sum_{j=0}^{\infty} (\phi^2)^j
\end{aligned}$$

Since $|\phi| < 1$ by design, we have $|\phi^2| < 1$. Then, it follows that

$$\gamma_h = \phi^h \frac{\sigma^2}{1-\phi^2}.$$

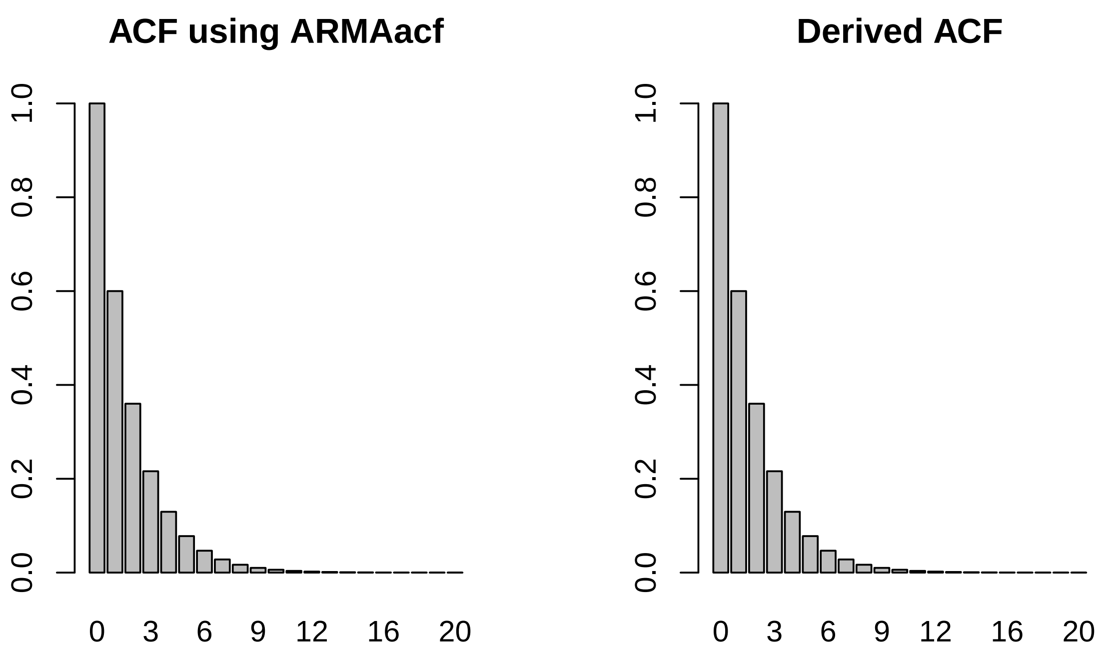
C

From our previous work, we can derive our autocorrelation function:

$$\rho_h = \frac{\gamma_h}{\gamma_0} = \frac{\phi^h \frac{\sigma^2}{1-\phi^2}}{\frac{\sigma^2}{1-\phi^2}} = \phi^h.$$

In the code below, we compare our formula above with the `ARMAacf` function. We find that these produce the same autocorrelations.

```
set.seed(123456789)
par(mfrow=c(1,2))
barplot(ARMAacf(ar=c(0.6),lag.max=20),main='ACF using ARMAacf')
barplot(.6^(0:20), main='Derived ACF',names.arg=c(0:20))
```




```
#Number of autocorrelations that are different
num_diff = sum(round(ARMAacf(ar=c(0.6),lag.max=20),5) != round(.6^(0:20),5))
num_diff
```

```
## [1] 0
```

Question 2.2

We compute the autocovariance function of the random walk model, i.e. $X_n = X_{n-1} + \epsilon_n$ where $\{\epsilon_n\}$ is white noise with variance σ^2 and $X_0 = 0$. If we suppose $m > n$, then we have:

$$\gamma_{mn} = \text{Cov}\left(X_m, X_n\right) = \text{Cov}\left(\sum_{k=1}^{n} \epsilon_k, \sum_{j=1}^{m} \epsilon_j\right) = \sum_{k=1}^{n} \text{Var}(\epsilon_k) = n\sigma^2.$$

If we let $m \leq n$ then we obtain

$$\gamma_{mn} = \text{Cov}\left(X_m, X_n\right) = \text{Cov}\left(\sum_{k=1}^{n} \epsilon_k, \sum_{j=1}^{m} \epsilon_j\right) = \sum_{j=1}^{m} \text{Var}(\epsilon_j) = m\sigma^2.$$

Then the autocovariance function is $\gamma_{mn} = \min(m, n)\sigma^2$.

Sources

- 531W16 HW2 Solutions for 2.1 A & B used to check final answers. In addition, GSI Joonha Park helped elaborate on independence between ϵ_n and $X_1, X_2, \ldots, X_{n-1}$.
- 531W16 HW2 Solutions for 2.2 used to check final answer.

Please Explain

- For problem 2.1 B, we use a Taylor expansion for the function $(1 - \phi B)^{-1}$. But isn't that only valid if $|\phi B| < 1$? I.e. doesn't this only work if we're guaranteed $|\phi B(X_i)| < 1$ for all i (where $B(\cdot)$ is the B operator applied to X_i)?