Elliott Evans
BIOSTAT 653
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Homework

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\begin{aligned}
\gamma_h &= \mathrm{Cov}(X_n, X_{n+h}) \\
&= \mathrm{Cov}(X_n, \phi X_{n+h-1} + \epsilon_{n+h}) \\
&= \mathrm{Cov}(X_n, \phi X_{n+h-1}) + \mathrm{Cov}(X_n, \epsilon_{n+h}) \\
&= \phi\gamma_{h-1} + \underbrace{\mathrm{Cov}(X_n, \epsilon_{n+h})}_{=0} \\
&= \phi\gamma_{h-1}
\end{aligned}
$$

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\begin{aligned}
\gamma_0 &= \mathrm{Cov}(X_n, X_n) \\
&= \mathrm{Var}(X_n) \\
&= \mathrm{Var}(\phi X_{n-1} + \epsilon_n) \\
&= \phi^2 \mathrm{Var}(X_n) + \mathrm{Var}(\epsilon_n) \\
&= \phi^2\gamma_0 + \sigma^2
\end{aligned}
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\begin{aligned}
\gamma_h &= \phi\gamma_{h-1} \\
&= \phi(\phi\gamma_{h-2}) \\
&= \phi^3\gamma_{h-3} \\
&\vdots \\
&= \phi^h\gamma_0 \\
&= \phi^h\frac{\sigma^2}{1-\phi^2}
\end{aligned}
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\gamma_h = \phi\gamma_{h-1} = \phi(\phi\gamma_{h-2}) = \phi^3\gamma_{h-3} \cdots = \phi^h\gamma_0 = \phi^h\frac{\sigma^2}{1-\phi^2}
$$

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\begin{aligned}
\epsilon_n &= X_n - \phi X_{n-1} = X_n \\
&= X_n - \phi B X_n \\
&= (1 - \phi B)X_n.
\end{aligned}
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\begin{aligned}
X_n &= \left(\sum_{i=0}^{\infty}(\phi B)^i\right)\epsilon_n \\
&= (B^0 + \phi B + \phi^2 B^2 + \cdots)\epsilon_n
\end{aligned}
$$

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\begin{aligned}
&= B^0\epsilon_n + \phi B\epsilon_n + \phi^2 B^2\epsilon_n + \cdots \\
&= \epsilon_n + \phi\epsilon_{n-1} + \phi^2\epsilon_{n-2} + \cdots \\
&= \sum_{k=0}^{\infty} \phi^k \epsilon_{n-k}
\end{aligned}
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\begin{aligned}
\gamma_h &= \operatorname{Cov}\left(X_n, X_{n+h}\right) \\
&= \operatorname{Cov}\left(\sum_{j=0}^{\infty} \phi^j \epsilon_{n-j}, \sum_{k=0}^{\infty} \phi^k \epsilon_{n+h-k}\right) \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \phi^j \phi^k \cdot \underbrace{\operatorname{Cov}\left(\epsilon_{n-j}, \epsilon_{n+h-k}\right)}_{\neq 0 \text{ when } n-j=n+h-k, \text{ i.e. } k=j+h} \\
&= \sum_{j=0}^{\infty}\sum_{k=j+h}^{\infty} \phi^j \phi^k \operatorname{Cov}(\epsilon_{n-j}, \epsilon_{n+h-k}) \\
&= \sum_{j=0}^{\infty} \phi^j \phi^{j+h} \operatorname{Cov}(\epsilon_{n-j}, \epsilon_{n-j}) \\
&= \sum_{j=0}^{\infty} \phi^j \phi^{j+h} \sigma^2 \\
&= \sigma^2 \phi^h \sum_{j=0}^{\infty} (\phi^2)^j
\end{aligned}
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\gamma_{mn} = \operatorname{Cov}\left(X_m, X_n\right) = \operatorname{Cov}\left(\sum_{k=1}^{n} \epsilon_k, \sum_{j=1}^{m} \epsilon_j\right) = \sum_{k=1}^{n} \operatorname{Var}(\epsilon_k) = n\sigma^2.
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