STATS 531 HW 2

Elliott Evans

1/29/2018

Contents

Question 2.1

Consider the AR(1) model, $X_n = \phi X_{n-1} + \epsilon_n$, where $\{\epsilon_n\}$ is white noise with variacen σ^2 and $-1 < \phi < 1$. We assume the process is stationary, i.e. it is initialized with a random draw from its stationary distibution.

A

The autocovariance function is:

$$\begin{aligned}
\gamma_h &= \text{Cov}(X_n, X_{n+h}) \\
&= \text{Cov}(X_n, \phi X_{n+h-1} + \epsilon_{n+h}) \\
&= \text{Cov}(X_n, \phi X_{n+h-1}) + \text{Cov}(X_n, \epsilon_{n+h}) \\
&= \phi\gamma_{h-1} + \underbrace{\text{Cov}(X_n, \epsilon_{n+h})}_{=0} \\
&= \phi\gamma_{h-1}
\end{aligned}$$

Because we want explicit A and λ such that $\gamma_h = A\lambda^h$, we derive the initial condition by finding γ_0:

$$\begin{aligned}
\gamma_0 &= \text{Cov}(X_n, X_n) \\
&= \text{Var}(X_n) \\
&= \text{Var}(\phi X_{n-1} + \epsilon_n) \\
&= \phi^2\text{Var}(X_n) + \text{Var}(\epsilon_n) \\
&= \phi^2\gamma_0 + \sigma^2
\end{aligned}$$

This implies that $\gamma_0(1-\phi^2) = \sigma^2$. Therefore, we have $\gamma_0 = \frac{\sigma^2}{1-\phi^2}$. Now, we may solve the recurrence relation:

$$\gamma_h = \phi\gamma_{h-1} = \phi(\phi\gamma_{h-2}) = \phi^3\gamma_{h-3} = \cdots = \phi^h\gamma_0 = \phi^h\frac{\sigma^2}{1-\phi^2}$$

Therefore, we have $\gamma_h = A\lambda^h$ where $\lambda = \phi$ and $A = \frac{\sigma^2}{1-\phi^2}$.

Sources

- 531W16 HW2 Solution for 2.1 A used to confirm that white noise error terms are in fact independent of the random $X_{1:N}$, then used to check final answers.

Please Explain

- For question 2.1 A, clearly the white noise error terms ϵ_i are independent of $X_{1:N}$. Is that always the case for ARMA models? Or is this implied from the error terms being **white noise**?