9 Dihedral groups and some properties of group actions (10/09)

Example 9.1 (Dihedral groups)**.** Fix $n \geqslant 3$. Let X be a regular n-gon with vertices labeled as $\{1, \ldots, n\}$ sitting in $\mathbf{R}^2$ centered at the origin. Let $D_{2n} \subseteq S_n$ be the set of permutations of the vertex set $\{1, \ldots, n\}$ consisting of those which can be achieved by a rigid motion of X in $\mathbf{R}^3$ returning X bijectively to itself. Among these, we single out two. Let r denote the permutation obtained by counterclockwise rotation about the origin by $\frac{2\pi i}{n}$. Let s denote the reflection across the line between 1 and the origin. Geometrically, we see that $rs = sr^{-1}$. This implies that the elements $\{e, r, r^2, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}\}$ form a subgroup of S_n. Indeed,

$$(s^a r^b)(s^c r^d) = s^a s^c r^{(-1)^c b} r^d = s^{a+c} r^{(-1)^c b + d} = s^e r^f,$$

where $e \equiv a + c \mod 2$ is in $\{0, 1\}$ and $f \equiv (-1)^c b + d \mod 2$ is in $\{0, \ldots, n\}$. We claim that this is all of D_{2n} and hence that D_{2n} is a subgroup of S_n, known as the **dihedral group of order** $2n$. To see this, suppose that $x \in D_{2n}$. We want to show that x is in the list of $2n$ elements above. We can compose with a rotation and assume that x sends the vertex 1 to itself. Then, since it arises from a rigid motion of $\mathbf{R}^3$, we must have that x sends either 2 to itself and $n-1$ to itself, or it sends 2 to $n-1$ and $n-1$ to 2. In the first case, it must be the identity. In the second case, it must be the reflection s.

Remark 9.2. Let $n = 4$ and consider the dihedral group D_8 of order 8. Let s denote the reflection across the diagonal through 1 and 3 and let s' denote the reflection across the diagonal through 2 and 4. Then, ss' has cycle decomposition $(1\,3)(2\,4)$. But, so does r^2. So, $ss' = r^2$.

9.1 Exercises

Exercise 9.1. Make a list of all elements of D_8, their orders, and a cycle decomposition for each (with respect to the action above of D_8 on $\{1, 2, 3, 4\}$).

Exercise 9.2. Find the lattice of subgroups of D_8.

Exercise 9.3. Find the lattice of subgroups of D_{10}.