Northwestern University

Real Analysis

MATH 321-2

uniform continuity go brrrr

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1 The Riemann Stieljis Integral

Remark 1.1. For this section, let there be a standing assumption that f is bounded.

Definition 1.2. Let $[a, b]$ be a given interval. A **partition** P of $[a, b]$ is a finite set of points $\{x_0, x_1, \dots, x_n\}$ where

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b.$$

We will adopt the following notation: $\Delta x_i = x_i - x_{i-1}$. Now, let P be any partition of $[a, b]$. We put

1. $M_i = \sup f(x) \quad (x_{i-1} \leq x \leq x_i)$,
2. $m_i = \inf f(x) \quad (x_{i-1} \leq x \leq x_i)$,
3. $U(P, f) = \sum_{i=1}^n M_i \Delta x_i$,
4. $L(P, f) = \sum_{i=1}^n m_i \Delta x_i$,

and finally obtain the *upper* and *lower Riemann integrals* of f over $[a, b]$:

1. $\overline{\int_a^b} f \, dx = \inf_{P \in \mathscr{P}} U(P, f)$,
2. $\underline{\int_a^b} f \, dx = \sup_{P \in \mathscr{P}} L(P, f)$

where $\mathscr{P}$ is the set of all partitions P of $[a, b]$.

Lemma 1.3

The set $\{U(P, f) \mid P \in \mathscr{P}\}$ is bounded below.

Proof. Since f is bounded, $f(x) \geq m$ for all $x \in [a, b]$. Notice that

$$U(P, f) = \sum_{i=1}^n \sup_{x \in [x_{i-1}, x_i]} f(x) \, \Delta x_i \geq \sum_{i=1}^n m \cdot \Delta x_i = m(b - a).$$

□

Definition 1.4. We say that f is **Riemann-integrable** and write $f \in \mathscr{R}([a, b])$ if

$$\overline{\int_a^b} f \, dx = \underline{\int_a^b} f \, dx.$$

Remark 1.5. Notice that $L(P, f)$ and $U(P, f)$ are bounded by $m(b - a)$ and $M(b - a)$, where $m \leq f(x) \leq M$ for all $x \in [a, b]$. In other words, the upper and lower integrals exist for *every* bounded real function.

Definition 1.6. Let α be nondecreasing (monotonically increasing) function on $[a, b]$. We write $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. (Clearly, $\Delta \alpha_i \geq 0$. We put

1. $U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta \alpha_i$,
2. $L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta \alpha_i$.

where M_i, m_i have the same meaning as in Definition 1.2 and we define

$$\overline{\int_a^b} f\, d\alpha = \inf_{P \in \mathscr{P}} U(P, f, \alpha) \tag{1}$$

and

$$\underline{\int_a^b} f\, d\alpha = \sup_{P \in \mathscr{P}} L(P, f, \alpha). \tag{2}$$

If (1) and (2) are equal, then we say f is *integrable with respect to α over* $[a, b]$, written $f \in \mathscr{R}(\alpha)$, and notate their common value, known as the **Riemann-Stieltjes integral** as

$$\int_a^b f\, d\alpha.$$

Question 1.7. When is $f \in \mathscr{R}(\alpha)$

It may be helpful to rephrase the question to ask when f is *not* in $\mathscr{R}(\alpha)$.

- Nonexample: The function
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$
is *not* in $\mathscr{R}([a, b])$. Notice that for any partition P, $U(P, f) = 1 \neq 0 = L(P, f)$.

So whenever $\inf_{P \in \mathscr{P}} U(P, f, \alpha)$ is *strictly* greater than $\sup_{P \in \mathscr{P}} L(P, f, \alpha)$, we know $f \notin \mathscr{R}(\alpha)$.

Definition 1.8. For partitions $P, Q \in \mathscr{P}$,

1. If $Q \supset P$, we say Q is a **refinement** of P.
2. We call $P^* = P \cup Q$ a **common refinement**.

Lemma 1.9

If $Q \supset P$, then $U(Q, f, \alpha) \leq U(P, f, \alpha)$ and $L(Q, f, \alpha) \geq L(P, f, \alpha)$.

Proof. Let $Q = P \cup \{x_0, \ldots, x_k\}$. If $k = 0$, the conclusion obviously holds. Now, suppose $k \in \mathbb{N}$ and $U(Q, f, \alpha) \leq U(P, f, \alpha)$, and let P^* contain just one more point than P, x^*, where $x_{i-1} < x^* < x_i$. Write $w_1 = \sup_{x_{i-1} \leq x \leq x^*} f(x)$ and $w_2 = \sup_{x^* \leq x \leq x_i} f(x)$. Notice $w_1, w_2 \leq M_i$ where $M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$. Then

$$\begin{aligned} U(P, f, \alpha) - U(P^*, f, \alpha) &= M_i[\alpha(x_i) - \alpha(x_{i-1}) - w_1[\alpha(x^*) - \alpha(x_{i-1})] - w_2[\alpha(x_i) - \alpha(x^*)] \\ &= (M_i - w_1)[\alpha(x^*) - \alpha(x_{i-1})] + (M_i - w_2)[\alpha(x_i) - \alpha(x^*)] \\ &\geq 0. \end{aligned}$$

The proof for the lower integrals is the same. □

Remark 1.10. Notice that for any partitions P_1, P_2, it follows with the common refinement $P^* = P_1 \cup P_2$ that

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha).$$

Corollary 1.11

$$\underline{\int_a^b} f\, d\alpha \le \overline{\int_a^b} f\, d\alpha.$$

We now arrive at a useful lemma relating integrability to being able to find partitions that allow the distance between upper and lower integrals to be arbitrarily small:

Lemma 1.12

$f \in \mathscr{R}(\alpha)$ if, and only if, $\forall \epsilon > 0$, there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

Proof. Let $f \in \mathscr{R}$. Then there exists a partition P_1 such that $0 \le U(P_1, f, \alpha) - \overline{\int_a^b} f\, d\alpha < \epsilon/2$. Similarly, there exists P_2 such that $0 \le \underline{\int_a^b} f\, d\alpha - L(P_2, f, \alpha) < \epsilon/2$. (Notice that $f \in \mathscr{R}(\alpha)$, so $\overline{\int} f\, d\alpha = \underline{\int} f\, d\alpha = \int f\, d\alpha$.) Let $P = P_1 \cup P_2$ be the common refinement of P_1 and P_2. Then

$$U(P, f, \alpha) \le U(P_1, f, \alpha) < \int f\, d\alpha + \epsilon/2 < L(P_2, f, \alpha) + \epsilon \le L(P, f, \alpha) + \epsilon.$$

Now assume the converse. Recall that $\overline{\int} f\, d\alpha \le U(P, f, \alpha)$ and $\underline{\int} f\, d\alpha \ge L(P, f, \alpha)$ for any partition P. Let $\epsilon > 0$. Then there exists a partition P such that

$$0 \le \overline{\int} f\, d\alpha - \underline{\int} f\, d\alpha \le U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

$\square$

Now, let's introduce a bit of notation to make our lives easier. We can write that f is continuous on a metric space X as $f \in \mathscr{C}(X)$. Furthermore, we can improve upon our notation of integrability to write $f \in \mathscr{R}(\alpha, S)$ to mean that f is integrable on with respect to α over S.

Theorem 1.13

Let $f \in \mathscr{C}([a, b])$. Then $f \in \mathscr{R}(\alpha, [a, b])$.

Proof. Notice $[a, b]$ is compact. Thus f is uniformly continuous on $[a, b]$, so for $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{\alpha(b) - \alpha(a)}$ whenever $|x - y| < \delta$. Now, pick a partition P (with n elements) such that $\Delta x_j < \delta$ for all j. Then

$$\begin{aligned}
\overline{\int} f\, d\alpha - \underline{\int} f\, d\alpha \le U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{j=1}^n \sup_{I_j} f\, \Delta x_j - \sum_{j=1}^n \inf_{I_j} f\, \Delta x_j \\
&= \sum_{j=1}^n \left(\sup_{I_j} f - \inf_{I_j} f \right) \Delta x_j \\
&< \sum_{j=1}^n \frac{\epsilon}{\alpha(b) - \beta(a)} \Delta x_j \\
&= \frac{\epsilon}{\alpha(b) - \alpha(a)} (\alpha(b) - \alpha(a)) = \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, the proof is complete. $\square$

> **Remark 1.14.** It turns out, we need not require that f is continuous on the entire interval; it suffices for f to be continuous *except at finitely many points*, with α continuous where f is not! Zoo wee mama!

Theorem 1.15 (The cooler Daniel)
If f is continuous at except finitely many points and α is continuous at the points of f's discontinuity, then $f \in \mathscr{R}(\alpha)$.

Proof. Let f be continuous except at finitely many points, say $\{x_0, \ldots, x_n\}$. Because f is continuous at except *finitely* many points, we can let $M = |f|$.

Since the set of discontinuities $S = \{x_0, \ldots, x_n\}$ is finite, α is uniformly continuous on S. It follows from the triangle inequality and the monotone increasing property of α that for all $\epsilon > 0$ there exists a $\delta > 0$ such that $\alpha(x_j + \delta) - \alpha(x_j - \delta) < \epsilon$.

We can always choose δ to be smaller, so without loss of generality, assume the set of $[x_j - \delta, x_j + \delta]$ is disjoint. Let $F = [a, b] \setminus \bigcup_{i=1}^{n} (x_j - \delta, x_j + \delta)$. F is compact, so for all $\epsilon > 0$, there exists $\delta' > 0$ such that $|f(u) - f(v)| < \epsilon$ for all $u, v \in F$ where $|u - v| < \delta'$.

We can now partition F into intervals I_j with $\Delta x_j < \delta'$. Let $J_i = [x_i - \delta, x_i + \delta]$. We can now partition $[a, b]$ into a partition P consisting of the I_i's and J_i's. Then

$$\begin{aligned}
\overline{\int} f\, d\alpha - \underline{\int} f\, d\alpha &\leq \sum_j \sup_{I_j} f\, \Delta x_j - \sum_j \inf_{I_j} f\, \Delta x_j + \sum_j \sup_{J_j} f\, \Delta x_j - \sum_j \inf_{J_j} f\, d\alpha \\
&= \sum_j \left(\sup_{I_j} f - \inf_{I_j} f \right) \Delta x_j + \sum_j \left(\sup_{J_j} f - \inf_{J_j} f \right), \Delta x_j \\
&\leq \epsilon \sum_j \Delta x_j + \sum_j 2M\epsilon \\
&= K\epsilon,
\end{aligned}$$

where $K \in \mathbb{R}$. $\square$

> **Remark 1.16.** What if we want to compose functions? Will their composition be integrable? Well it turns out that if the inner function is integrable, then the outer function being continuous on the range of the inner function is sufficient for integrability of the composition.

Theorem 1.17 (Integrability of composition of functions)
If f takes values in $[m, M]$ on $[a, b]$, $f \in \mathscr{R}(\alpha, [a, b])$, and ϕ continuous on $[m, M]$, then $\phi \circ f \in \mathscr{R}(\alpha, [a, b])$.

The proof for this theorem is pretty funny, so hang on.

Proof. ϕ is uniformly continuous on $[m, M]$ (why?) so for some $\epsilon > 0$ there exists a $\delta < \epsilon$ such that $|\phi(u) - \phi(v)| < \epsilon$ whenever $|u - v| < \delta$. Note that if we find a sufficiently small δ, then any value less than δ also works so we can restrict ourselves to only working with $\delta < \epsilon$. It turns out, this restriction will become very useful later on!

Since $f \in \mathscr{R}(\alpha)$, it follows from **Lemma 1.12** that there exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2.$$

For each $j = 1, \ldots, n$ (where $n = |P|$), if $\sup_{I_j} f - \inf_{I_j} f < \delta$, place $j \in A$. Otherwise place $j \in B$.

1. If $j \in A$, then $|\phi(f(x)) - \phi(f(y))| < \epsilon$, $x, y \in I_j$.
2. If $j \in B$, then $\sup_{I_j}(\phi \circ f) - \inf_{I_j}(\phi \circ f) \leq 2 \sup_{[m,M]} |\phi|$, and let's notate $K = \sup_{[m,M]} |\phi|$. But $U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$, so

$$\sum_{j \in B} \delta \Delta \alpha_j \leq \sum \left(\sup_{I_j} f - \inf_{I_j} f \right) \Delta x_j \leq \delta^2.$$

Dividing both sides by δ, we get $\sum_{j \in B} \Delta \alpha_j < \delta$.

Thus

$$\begin{aligned} \overline{\int} \phi \circ f \, d\alpha - \underline{\int} \phi \circ f \, d\alpha &\leq U(P, \phi \circ f, \alpha) - L(P, \phi \circ f, \alpha) \\ &\leq \sum_{j=1}^{n} \epsilon \Delta \alpha_j + \sum_{j \in B} 2K \Delta \alpha_j \\ &\leq \epsilon(\alpha(b) - \alpha(a)) + 2K\delta \\ &< \epsilon(\alpha(b) - \alpha(a) + 2K). \end{aligned}$$

□

Remark 1.18. Note that we (stupidly, in the words of Jared Wunsch,) overcount in the third-to-last line of the extended equation; summing over *all* j instead of just $j \in A$.

Remark 1.19. You've probably caught on to the style of proving a function is integrable: find a partition such that the difference $U - L$ is bounded above by an arbitrary ϵ.

We will now explore the properties of the integral, which pretty much agree with the intuition of someone who studied linear algebra and multivariate calculus with Aaron Peterson in MATH 291 @ Northwestern University:

1. The integral is *linear* over $\mathbb{R}$;
2. If a function bounds another from above, then the integral of the first will bound the integral of the second from above;
3. We can split integrals by an intermediate bound;
4. If the magnitude of a function is bounded by a finite number M, then the magnitude of the integral of that function will by bounded by the product of M and the width of the integral's bounds.
5. The sum of functions integrable with respect to different "clock speeds" is integrable with respect to the sum of their individual clock speeds. (Really pushing the metaphore here..)

Theorem 1.20 (Rudin 6.12)

1. If $f_1, f_2 \in \mathscr{R}(\alpha)$ then $f_1 + f_2 \in \mathscr{R}(\alpha)$, $cf \in \mathscr{R}(\alpha)$ for every $c \in \mathbb{R}$, and
$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha \quad \text{and} \quad \int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha.$$
2. If $f_1(x) \leq f_2(x)$ on $[a, b]$, then
$$\int_a^b f_1\, d\alpha \leq \int_a^b f_2\, d\alpha.$$
3. If $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and $a < c < b$, then $f \in \mathscr{R}(\alpha)$ on $[a, c]$ and $[c, b]$, and
$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha.$$
4. If $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and if $|f(x)| \leq M$ on $[a, b]$, then
$$\left|\int_a^b f\, d\alpha\right| \leq M[\alpha(b) - \alpha(a)];$$
5. If $f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$, then $f \in \mathscr{R}(\alpha_1 + \alpha_2)$ and
$$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2;$$
If $f \in \mathscr{R}(\alpha)$ and $c \in \mathbb{R}^+$, then $f \in \mathscr{R}(c\alpha)$ and
$$\int_a^b f\, d(c\alpha) = c\int_a^b f\, d\alpha.$$

Proof. The proofs for each part are very similar, so we will only prove (1). However Wunsch messed up here so we'll skip this for now. A proof is in Rudin if you really want to read it. ☐

The previous theorem (**Rudin 6.12**) gives us a lot of power to determine the integrability of functions; we just need to be adept at manipulating expressions into sums and compositions of continuous functions. Thankfully, $x \to x^2$ is continuous and we have a useful identity to translate multiplication into addition:

A useful identity: $xy = \frac{1}{4}\left((x+y)^2 - (x-y)^2\right)$.

Theorem 1.21

Let $f, g \in \mathscr{R}(\alpha)$. Then

1. $fg \in \mathscr{R}(\alpha)$,
2. $|f| \in \mathscr{R}(\alpha)$, and
3. $\left|\int_a^b f\, d\alpha\right| \leq \int_a^b |f|\, d\alpha$.

Proof. Notice that $f \pm g \in \mathscr{R}(\alpha)$, so $(f \pm g)^2 \in \mathscr{R}(\alpha)$. Then

$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right) \in \mathscr{R}(\alpha).$$

Since $u \to u^2$ is continuous, $|f| \in \mathscr{R}(\alpha)$. Finally, there exists a $c = \pm 1$ where

$$\left|\int f\, d\alpha\right| = c\int f\, d\alpha = \int cf\, d\alpha \leq \int |f|\, d\alpha.$$

□

Example 1.22 (Heaviside Function)

We define the **Heaviside Function** as

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 1 \end{cases}$$

If $a < 0 < b$ and f is continuous at $x = 0$, then $f \in \mathscr{R}([a,b], H)$ and $\int_a^b f\, dH = f(0)$.

Proof. Again, we choose a funny partition that will result in some clean shit: Let $P = \{x_0, x_1, x_2, x_3\}$, where $x_0 = a, x_1 = 0, x_3 = b$, and $x_2 \in (0, b)$. Then

$$\begin{aligned} U(P, f, H) &= \sup_{[a,0]} f \cdot (H(0) - H(a)) + \sup_{[0,x_2]} f \cdot (H(x_2) - H(0)) + \sup_{[x_2,b]} f \cdot (H(b) - H(x_2)) \\ &= \sup_{[a,0]} f \cdot (0-0) + \sup_{[0,x_2]} f \cdot (1-0) + \sup_{[x_2,b]} f \cdot (1-1) \\ &= \sup_{[0,x_2]} f. \end{aligned}$$

Similarly, $L(P, f, H) = \inf_{[0,x_2]} f$. Letting x_2 approach 0 from the right, notice that $U(P, f, H) \to f(0)^+$ and $L(P, f, H) \to f(0)^-$. So $\int_a^b f\, dH = 0$. □

Corollary 1.23 (Basically Heaviside, with **linearity!**)

Let $\alpha = \sum_{j=1}^N c_j H(x - s_j)$, $s \in [a, b]$, and $f \in \mathscr{C}([a, b])$. Then

$$\int_a^b f\, d\alpha = \sum_{i=1}^N c_j f(s_j)$$

Proof. Immediate by Theorem 1.20. □

Remark 1.24. Rudin extends α to be an infinite sum, but we don't need to get that crazy here...

Theorem 1.25

Say α' exists for all $x \in [a, b]$, α' is bounded, and f is *Riemann*-integrable (i.e. $f \in \mathscr{R}([a, b], dx)$. Then $f \in \mathscr{R}(\alpha)$ and $\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx$.

Proof. TBD. Wunsch messed this one up too. □

2 Integration and Differentiation

We will explore the dynamics between integration and differentiation, and as expected, the two act as quasi-inverse functions.

Theorem 2.1

Let $f \in \mathscr{R}([a,b])$ and f be continuous at a point $x_0 \in [a,b]$. Then

$$f(x_0) = \frac{d}{dx} \int_a^x f(s)\,ds \bigg|_{x=x_0}.$$

Proof. Differentiating our funny integral, we have that

$$\frac{d}{dx} \int_a^x f(s)\,ds = \lim_{h \to 0} \frac{\int_0^{x_0+h} f(s)\,ds - \int_a^{x_0} f(s)\,ds}{h}.$$

We now have to inspect both right and left hand limits, but as the proofs for each case are analogous, we'll just look at the right hand limit: $h \to 0^+$. Since f is continuous at x_0, for $\epsilon > 0$, there exists $\delta > 0$ such that if $0 \le |y - x_0| < \delta$, then $|f(y) - f(x_0)| < \epsilon$. Since we're taking the limit as h approaches 0, we can limit our choice of h to only those with $h < \delta$. Pick any of them. Then $|f(s) - f(x_0)| < \epsilon|$ for all $s \in (x_0, x_0 + h)$.

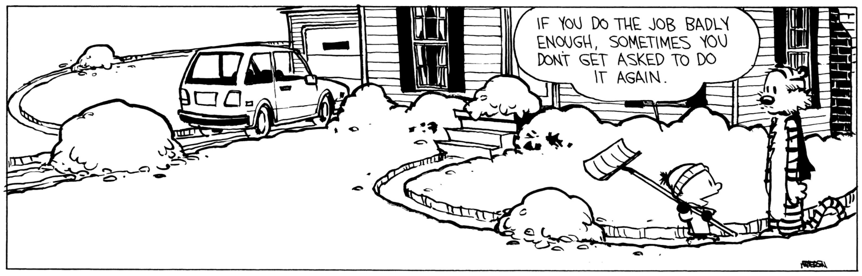
We'll now employ a slick trick: since $f(x_0)$ is constant, we can write $f(x_0) = \frac{1}{h}\int_{x_0}^{x_0+h} ds$.

Then

$$\begin{aligned}
\left| \frac{1}{h} \int_{x_0}^{x_0+h} f(s)\,ds - f(x_0) \right| &= \left| \frac{1}{h} \int_{x_0}^{x_0+h} f(s)\,ds - \frac{1}{h} \int_{x_0}^{x_0+h} f(x_0)\,ds \right| \\
&= \left| \frac{1}{h} \int_{x_0}^{x_0+h} f(s) - f(x_0)\,ds \right| \\
&\le \frac{1}{h} \int_{x_0}^{x_0+h} |f(s) - f(x_0)|\,ds \\
&< \frac{1}{h} \int_{x_0}^{x_0+h} \epsilon\,ds \\
&= \epsilon.
\end{aligned}$$

□

Remark 2.2. Notice that $F(x) = \int_a^x f(s)\,ds$ is continuous on $[a,b]$.


IF YOU DO THE JOB BADLY ENOUGH, SOMETIMES YOU DON'T GET ASKED TO DO IT AGAIN.