Problem set 6: Due Monday, May 15

Definition 1 (Dual spaces)**.** Let V be a vector space and let $\|\cdot\|$ be any norm on V. We define the *dual space* of V to be the vector space of bounded linear functionals,

$$V^* = \{f\colon V \to \mathbb{R} : f \text{ is a bounded linear functional}\}.$$

Problems

1. 5.3.7 (1)

2. 5.4.8 (1)–(3). What happens if $\{u_i\}$ is a complete orthonormal family

3. Show that a Hilbert space $\mathcal{H}$ is separable[1] if and only if $\mathcal{H}$ has a complete orthonormal family.

   Remark 1. It seems implicit in Franks' text that an orthonormal family is always assumed to be countable. Thus in Problem 3. you should do the following:

   (a) If $\{u_n\}$ is a complete *countable* orthonormal family, show that $\mathcal{H}$ has a countable dense subset. (Use Problem 6.)

   (b) If $\mathcal{H}$ is separable and if $S \subset \mathcal{H}$ is any orthonormal set, then S is countable. You can then argue that there exists a complete (countable) orthonormal family.

4. In infinite dimensions, being closed and bounded does not imply being compact.

   (a) Show the closed unit ball in $C^0[0,1]$ is not compact

   (b) Show the closed unit ball in $L^2[0,1]$ is not compact

5. Let $\mathbb{R}^{\mathbb{Z}} = \{\{x_n\}_{n\in\mathbb{Z}}\}$ be the space of bi-infinite sequences of real numbers. Let $\ell^2 \subset \mathbb{R}^{\mathbb{Z}}$ be the subset of square-integrable sequences:

   $$x = \{x_n\}_{n\in\mathbb{Z}} \in \ell^2$$

   if and only if

   $$\sum_{n\in\mathbb{Z}} (x_n)^2 < \infty.$$

   Define the inner product on ℓ^2 as follows: given $x = \{x_n\}_{n\in\mathbb{Z}} \in \ell^2$, and $y = \{y_n\}_{n\in\mathbb{Z}} \in \ell^2$,

   $$\langle x, y\rangle = \sum_{n\in\mathbb{Z}} x_n y_n.$$

   (a) Show that the formula defining $\langle x, y\rangle$ converges (absolutely) and that $\langle\cdot,\cdot\rangle$ defines an inner product on ℓ^2. (In particular, $\langle\cdot,\cdot\rangle$ satisfies Cauchy-Schwarz.)

   (b) Show ℓ^2 is a vector space (with the obvious operations)

   (c) Show ℓ^2 is complete (and so is a Hilbert space)

   (d) Show that ℓ^2 has a complete orthonormal family (and so is a separable Hilbert space)

6. Let $\mathcal{H}$ be a Hilbert space and let $\{u_n\}_{n=0}^{\infty}$ be an orthonormal family in $\mathcal{H}$. Show that the following are equivalent:

   (a) $\{u_n\}$ is a complete orthonormal family

   (b) $\overline{\operatorname{span}\{u_n\}} = \mathcal{H}$ [2]

   (c) If $x \in \mathcal{H}$ and $\langle x, u_n\rangle = 0$ for every n, then $x = 0$.

[1] recall that a subspace of separable metric space is always separable

[2] Recall, that span denotes the set of *finite* linear combinations. The overline indicates taking the closure.

7. Let V be a vector space equipped with a norm $\|\cdot\|$.
   (a) Show that the operator norm
   $$\|f\| = \sup\{|f(v)| : \|v\| \leq 1\}$$
   defines a norm on the dual space V^*.
   (b) Suppose $\{f_n\}$ is a Cauchy sequence in V^* (with the above norm).
      (i) For every $x \in V$ show that $f(x) = \lim_{n\to\infty} f_n(x)$ exists.
      (ii) Show that $f\colon V \to \mathbb{R}$ is a linear function
      (iii) Show that $f_n \to f$ in V^*. (In particular, conclude that V^* is complete with respect to this norm, regardless of whether or not V is complete)

8. Let $\mathcal{H}$ be a Hilbert space. Given $x \in \mathcal{H}$, let $f_x\colon \mathcal{H} \to \mathbb{R}$ be
   $$f_x(v) = \langle x, v\rangle.$$
   Show that the map $x \mapsto f_x$ is an isometry (with respect to the norm from problem 7.) from $\mathcal{H} \to \mathcal{H}^*$.