13 Normal subgroups and orbit decomposition (10/23)

13.1 Normal subgroups

Remark 13.1. Recall that last time we defined normal subgroups $N \subseteq G$ to be those subgroups such that for every $n \in N$ and every $g \in G$, the conjugate gng^{-1} is in N. We observed that every kernel is normal and that conversely if N is a normal subgroup of G, then the equality holds $(Ng)(Nh) = N(gh)$ and makes the set G/N of right cosets into a group. Also, in this case, the set of left cosets is equal to the set of right cosets and we could have defined G/N via left cosets as well.

Lemma 13.2. *Let G be a group and let $N \subseteq G$ be a normal subgroup. There is a bijection between the set of normal subgroups of G/N and the set of normal subgroups of G containing N.*

Proof. Let $f\colon G \to G/N$ be the quotient homomorphism defined by $f(g) = Ng$. If $K \subseteq G/N$ is normal, then we can construct a further group homomorphism $g_K\colon G/N \to (G/N)/K$. The kernel of the composition $g_K \circ f$ is a normal subgroup of G and contains N. It is $f^{-1}(K)$. This gives a function from normal subgroups of G/N to normal subgroups of G containing N. Now, if $N \subseteq M \subseteq G$ and N, M are normal in G, then I claim that $f(M) \subseteq G/N$ is normal. Indeed, if $m \in M$ and $g \in G$, we have to show that $(Ng)(Nm)(Ng)^{-1} = Nm_0$ for some $m_0 \in M$. We have $(Ng)^{-1} = N(g^{-1})$ by normality and $(Ng)(Nm)(N(g^{-1})) = N(gmg^{-1})$. But, $gmg^{-1} \in M$. Thus, $M \mapsto f(M)$ and $K \mapsto f^{-1}(K)$ give mutually inverse bijections. $\square$

13.2 Orbit decomposition

Remark 13.3. On the practice midterm, we saw that if G is a finite group acting on a set X, then for every element $x \in X$,

$$|G| = |G_x||G \cdot x|.$$

In other words, the number of elements of G is equal to the size of the stabilizer of x in G times the size of the orbit of G containing x.

Lemma 13.4. *Suppose that a finite group G acts on a finite set X. Then,*

$$|X| = \sum_{\mathcal{O} \in X/G} \frac{|G|}{|G_x|},$$

where $\mathcal{O}$ ranges over the orbits of G acting on X and where x is a choice of a representative of $\mathcal{O}$.

Proof. We know that the action of G on X leads to an equivalence relation on X where $x \sim y$ if there exists $g \in G$ such that $g \cdot x = y$. It follows that X is partitioned into equivalence classes, which we have called the orbits of G acting on X and written as X/G. Thus, we have the equality

$$|X| = \sum_{\mathcal{O} \in X/G} |\mathcal{O}|.$$

It suffices to compute $|\mathcal{O}|$. If $x \in \mathcal{O}$, then Remark 13.3 implies that $|G| = |G_x||G \cdot x| = |G_x||\mathcal{O}|$ or $|\mathcal{O}| = \frac{|G|}{|G_x|}$. Substituting into the displayed equation above, the lemma follows. $\square$

Example 13.5. Recall that a group G acts transitively on X if there is only one orbit $\mathcal{O}$ (which must then be equal to X). In this case, it follows that for any $x \in X$ there is an equality $|X| = |\mathcal{O}| = \frac{|G|}{|G_x|}$. Suppose then that D_{2n} is the dihedral group acting on the set $\{1, \ldots, n\}$. This is a transitive action (as one sees by using rotations). The equality

$$n = |\{1, \ldots, n\}| = \frac{|D_{2n}|}{|(D_{2n})_x|} = \frac{2n}{|(D_{2n})_x|}$$

holds for every $x \in \{1, \ldots, n\}$. In particular, we see that the stabilizer of x is a subgroup of order 2 for each $x \in D_{2n}$. These are precisely the reflections. For example, $\{e, sr^k\}$ is the stabilizer of some vertex (which one?) and every stabilizer is of this form.

13.3 Exercises

Exercise 13.1. If G is a group, and $N \subseteq M \subseteq G$ are subgroups where N is normal in G and M is normal in G, then $(G/N)/(M/N) \cong G/M$. Hint: construct a surjective homomorphism $G/N \to G/M$ and compute its kernel.

Exercise 13.2. Find an example of a group G with subgroups $N \subseteq M \subseteq G$ where N is normal in M and M is normal in G but N is not normal in G.

Exercise 13.3. Let H be the stabilizer of n in S_n acting on $\{1, \ldots, n\}$. What is the order of H? Which group that we've studied is H isomorphic to?