20 Inner and outer automorphisms (11/08)

20.1 More on automorphisms

Lemma 20.1. *There is a homomorphism $c\colon G \to \mathrm{Aut}(G)$ given by sending $g \in G$ to conjugation by g, denoted by c_g. The kernel is $Z(G)$, the center of G.*

Proof. Given $g \in G$, the function $c_g\colon G \to G$ is a bijective group homomorphism. It follows that $g \mapsto c_g$ defines a *function* $G \to \mathrm{Aut}(G)$. To see that this is a group homomorphism, it is enough to check that

$$(c_g \circ c_h)(x) = c_g(c_h(x)) = g(hxh^{-1})g^{-1} = (gh)x(gh)^{-1} = c_{gh}(x),$$

for all $g, h, x \in G$. If $g \in Z(G)$, then $c_g(x) = gxg^{-1} = xgg^{-1} = x$, so c_g is the identity automorphism. Conversely, if $c_g = \mathrm{id}_G$, then $gxg^{-1} = x$ for every $x \in G$; in other words, g is in the center. $\square$

Definition 20.2. The image of $c\colon G \to \mathrm{Aut}(G)$ is called $\mathrm{Inn}(G)$. It is a subgroup of $\mathrm{Aut}(G)$. Automorphisms in $\mathrm{Inn}(G)$ are called **inner automorphisms**.

Example 20.3. If G is abelian, then $Z(G) = G$ and $\mathrm{Inn}(G) = G/Z(G)$ is trivial.

Lemma 20.4. *The subgroup $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ is normal. Specifically, if $g \in G$ and $\sigma \in \mathrm{Aut}(G)$, then $\sigma \circ c_g \circ \sigma^{-1} = c_{\sigma(g)}$.*

Proof. Given $x \in G$ we have

$$(\sigma \circ c_g \circ \sigma^{-1})(c) = \sigma(c_g(\sigma^{-1}(x))) = \sigma(g\sigma^{-1}(x)g^{-1}) = \sigma(g)\sigma(\sigma^{-1}(x))\sigma(g^{-1}) = \sigma(g)x\sigma(g)^{-1} = c_{\sigma(g)}(x).$$

Thus, $\mathrm{Inn}(G)$ is normal in $\mathrm{Aut}(G)$ as claimed. $\square$

Definition 20.5 (Outer automorphisms)**.** The quotient of $\mathrm{Aut}(G)$ by $\mathrm{Inn}(G)$ is the group of **outer automorphisms**. Note that, despite the name, the elements of $\mathrm{Out}(G)$ are not, in fact, automorphisms of G but are $\mathrm{Inn}(G)$-cosets of automorphisms.

Definition 20.6 (Characteristic subgroups)**.** A subgroup $H \subseteq G$ is **characteristic** if for every automorphism $\sigma \in \mathrm{Aut}(G)$ one has $\sigma(H) = H$.

Remark 20.7. A characteristic subgroup $H \subseteq G$ is necessarily normal. A normal subgroup is characteristic if every outer automorphism preserves H.

Example 20.8. The center $Z(G)$ of a group G is characteristic.

Example 20.9. A *normal* p-Sylow subgroup of a finite group is characteristic.

Example 20.10. Let $G = \mathbf{Z}/p \times \mathbf{Z}/p$. Every subgroup of G is normal since G is abelian. However, the only characteristic subgroups of G are trivial. To see this, note that $\mathrm{Aut}(G) = \mathbf{GL}_2(\mathbf{F}_p)$. This group acts transitively on the group of non-zero vectors in G. So, no subgroup of order p is fixed by $\mathrm{Aut}(G)$. For concreteness, assume that $H \subseteq G$ is the group of order p generated by the element $(1, 0)$. The element

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

implements the "switch" automorphism $\sigma(a, b) = (b, a)$, And, $H \subseteq G$ is not fixed by σ.

Remark 20.11. In the situation of Example 20.10, $\mathrm{Inn}(G)$ is trivial, so $\mathrm{Aut}(G) \cong \mathrm{Out}(G)$.

20.2 Extensions

Definition 20.12 (Extensions)**.** An extension of a group H by a group N is a sequence

$$1 \to N \to G \to H \to 1$$

of group homomorphisms called an exact sequence (or a short exact sequence). Each arrow is a homomorphism of groups and at each point in the diagram we have ker = im. This is a short-hand way of expressing the following:

(a) $N \to G$ is injective,

(b) $G \to H$ is surjective,

(c) the kernel of $G \to H$ is N, and hence

(d) $G/N \cong H$.

Remark 20.13. In the notation for an exact sequence "1" denotes the trivial group $\{e\}$.

Example 20.14. If N is a normal subgroup of G there is an exact sequence

$$1 \to N \to G \to G/N \to 1,$$

and G is an extension of G/N by N.

Example 20.15. We can write S_3 as an extension of $\mathbf{Z}/2$ by $\mathbf{Z}/3$, resulting in $1 \to \mathbf{Z}/3 \to S_3 \to \mathbf{Z}/2 \to 1$. We *cannot* express S_3 as an extension of $\mathbf{Z}/3$ by $\mathbf{Z}/2$ because S_3 has no normal 2-Sylow subgroups.

Example 20.16. There is an exact sequence

$$1 \to \mathbf{Z}/3 \to \mathbf{Z}/6 \to \mathbf{Z}/2 \to 1.$$

Unlike for S_3, there is *also* an exact sequence

$$1 \to \mathbf{Z}/2 \to \mathbf{Z}/6 \to \mathbf{Z}/3 \to 1.$$

Remark 20.17. In the ideal case when attempting to understand a group G one expresses G as the middle term of an exact sequence, and hence as an extension of G/N by a normal subgroup N. Understanding N and G/N and how they are "glued together" in the exact sequence leads to an understanding of G. The puzzle of finite group theory is that there *are* simple groups, even simple non-abelian groups as we will see, which by definition foil this approach.

20.3 Exercises

Exercise 20.1. Compute $\mathrm{Inn}(\mathbf{Z}/n)$, $\mathrm{Aut}(\mathbf{Z}/n)$, and $\mathrm{Out}(\mathbf{Z}/n)$ for any $n \geqslant 2$.

Exercise 20.2. Compute $\mathrm{Inn}(D_{2n})$, $\mathrm{Aut}(D_{2n})$, and $\mathrm{Out}(D_{2n})$, where D_{2n} is the dihedral group of order $2n$ for $n \geqslant 3$.