

(undirected) simple graph
no self loops

(undirected) graph

"graph"

directed graph

multigraph

weighted graph

If a graph $[G = (V, E)]$ is an ordered pair of a vertex set, V , and an edge set, E . For every edge $e \in E$, 2 associated vertices $a, b \in V$, called the endpoints. (For directed graphs, we want to know the start and end.)

Undirected Graphs

$\forall e \in E, e \subseteq V$ is an 2-element subset of V .

(= undirected)



$$V = \{1, 2, 3\}$$

$$E = \{\{1, 2\}, \{1, 3\}\}$$

- can't do directed, multigraphs w/ this model.
(or simple)

Directed Graphs

$E \subseteq V \times V$, i.e. each edge is of the form (a, b) , $a, b \in V$.

$$V = \{1, 2, 3\}$$

$$E = \{(1, 2), (3, 1)\}$$

- still can't do multigraphs.

- can model undirected graphs: if $a \sim b$, $(a, b), (b, a) \in E$.

Common Graphs

Cycle graph on n^3 vertices, C_n



$$V = \{1, 2, \dots, n\}, E = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{n, 1\}\}$$

Path graph on n vertices, P_n



$$V = \{1, 2, \dots, n\}, E = \{\{1, 2\}, \dots, \{n-1, n\}\} = E_n \setminus \{n, 1\}$$

Complete Graph on n vertices, $K_n = K_{nn}$

$$V = \{1, 2, \dots, n\}, E = \{ \{i, j\} : i \neq j \}$$



K_1

\longleftarrow
 K_2



K_3

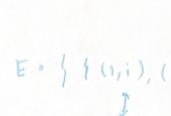


K_5

Complete Bipartite Graph, K_{mn}

vertex partition, $\{\}$

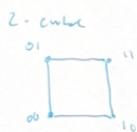
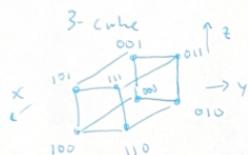
$$V = \{(1,1), \dots, (1,m), (2,1), \dots, (2,n)\}$$



$$E = \{ \{(1,i), (2,j)\} : 1 \leq i \leq m, 1 \leq j \leq n \}$$

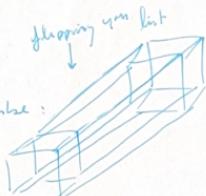
$$V = \{a_1, \dots, a_m, b_1, \dots, b_n\} \quad E = \{ \{a_i, b_j\} \mid 1 \leq i \leq m, 1 \leq j \leq n \}$$

HyperCube Graph, Q_n



$\left. \begin{array}{l} V = \text{all } n\text{-bit binary strings} \\ E = \{ \{a, b\} \mid a \wedge b = k \ll 1, 0 \leq k \leq n \} \end{array} \right\}$

 ↓



4-cube:

flipping your first

Graph	# vertices	# edges
C_n	n	n
P_n	n	$n-1$
K_n	n	$\frac{(n-1)(n)}{2} = \binom{n}{2}$
K_{mn}	$m+n$	$m \cdot n$
Q_n	2^n	$\frac{n \cdot 2^n}{2} = n \cdot 2^{n-1}$ every vertex has n edges

Some useful terms:

degree of a vertex: # of edges connected to vertex

connected vs disconnected



disconnected



connected



connected components

union of two graphs

$$G = (V, E), H = (V', E') \rightarrow G \cup H = (V \cup V', E \cup E')$$



complement of a graph



G



\bar{G}

$$V = \{1, \dots, n\}$$

$$E = E_{kn} \setminus E_G$$

Result: no self loops!

$$G = (V, E)$$

$$\bar{G} = (\bar{V}, \bar{E})$$

$$e \in E \Leftrightarrow e \notin \bar{E}$$

Theorem Let $G = (V, E)$ be a simple undirected graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

$$\text{deg}(v) = \# \text{ of edges incident on } v. = \sum_{e \in E} \text{"does } e \text{ touch } v?"$$

$$\Rightarrow \sum_{v \in V} \deg(v) = \sum_{v \in V} \sum_{e \in E} \text{"does } e \text{ touch } v?"$$

$$= \sum_{e \in E} \sum_{v \in V} \text{"does } e \text{ touch } v?"$$

(commutativity of addition)

$$= \sum_{e \in E} 2$$

(each edge touches exactly 2 vertices)

$$= 2|E|. \quad \blacksquare$$

Q. How many edges does \bar{C}_n have?



$$\# \text{ edges in } C_n + \bar{C}_n =$$

$$\# \text{ edges in } K_4$$

$$\rightarrow$$

$$\# \text{ edges in } \bar{C}_n =$$

$$\# \text{ edges in } K_n - \# \text{ edges in } C_n$$

$$= \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

Remove an edge $e \in E$:

$$G' = G - e$$

$$V' = V$$

$$E' = E - e$$



Remove a vertex $v \in V$:

$$G' = G - v$$

$$V' = V - v$$

$$E' = E - \text{all edges incident to } v$$



Def: $H = (V, E)$

Def: $H = (V, E)$ is a subgraph of G if it is a graph w/ $V' \subseteq V$, $E' \subseteq E$.

Ex: $G = (V, E)$

$v \in V$ is a subgraph of $G \forall v \in V$

$e \in E$ is a subgraph of $G \forall e \in E$

$$(V, E) \subset (V', E')$$

Any subgraph can be obtained by removing edges & vertices.

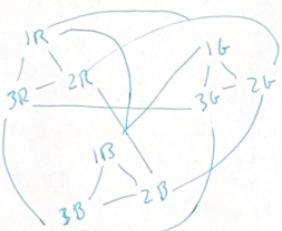
Def: The product graph $G \times H = (V \times V', E \times E')$

$$\{(a, b) \mid a \in V, b \in V'\}$$

$$\{(a, x), (a, y) \mid \forall y (x, y) \in E' \}$$

$$V \cup \{(x, b), (y, b) \mid (x, y) \in E \}$$

Ex: $C_3 \times C_3$



Ex: $K_2 \times K_2$

$$K_2 \times K_2 = \begin{array}{c} \bullet R \\ \times \\ \bullet B \end{array}$$

$$\begin{array}{c} IR \rightarrow 1B \\ | \\ 2R \leftarrow 2B \end{array}$$

Ex: $K_2 \times K_2 \times K_2$

$$\begin{array}{c} IR \rightarrow 1B \\ | \\ 2R \leftarrow 2B \end{array} \times \begin{array}{c} x \rightarrow y \\ | \\ z \end{array} =$$

$$\begin{array}{c} IRX \rightarrow 1BX \\ | \\ 2RX \leftarrow 2BX \end{array} \quad \begin{array}{c} IRY \rightarrow 1BY \\ | \\ 2RY \leftarrow 2BY \end{array}$$

How many vertices does $G \times E$ have? edges?

$$|V \times V'| = |V| \cdot |V'|$$

$$\begin{aligned} &\downarrow \\ &\# \text{ edges caused by } G + \# \text{ edges caused by } H \\ &= |E| \cdot |V'| + |E'| \cdot |V| \end{aligned}$$

Exercise! Using $\prod_{i=1}^n K_2 = Q_n$, prove #vertices in $Q_n = 2^n$, #edges in $Q_n = n \cdot 2^{n-1}$.

Def. $G = (V, E)$ be a graph, $v, w \in V$ be vertices.

Def: $\textcircled{1}$ a walk from v to w is a sequence of vertices

$$v = v_0 - v_1 - v_2 - \dots - v_n = w,$$

such that each v_i is adjacent to v_{i+1} .

$\textcircled{2}$ the length of the walk is n .

ex



$1 - 2 - 3 - 4 - 2 - 3$ is a walk from 1 to 3 of length 5.

4 is a trivial walk (of length 0).

Def a closed walk starts and ends at the same vertex.

e.g. $1 - 2 - 3 - 4 - 2 - 3 - 2 - 1$.

Def a trail is a walk where no edge is repeated (closed trail of length 3 = circuit)
a path is a walk where no vertex is repeated (closed path of length 3 = cycle)



Def a graph $G = (V, E)$ is connected if $\forall u, v \in V$ Ja walk from u to v .

Theorem Let $G = (V, E)$ with $u, v \in V$. Then if \exists a walk from u to v , - then \exists a path from u to v .

Pf (sketch)



$u - 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - v$ is a walk from u to v .

$u - 1 - 2 - 3 - 4 - v$ is a path from u to v , which we obtained by removing closed loops from our walk.

Def Let $G = (V, E)$. The distance, $d(u, v)$, from $u \in V$ to $v \in V$ is the length of the shortest path from u to v . If no paths exist, write $d(u, v) = +\infty$.

- We define the geodesic from u to v to be the shortest path from u to v .

Theorem Let $G = (V, E)$ be a graph.

- 1) For any $a, b \in V$, $d(a, b) \geq 0$ w/ $d(a, b) = 0 \Leftrightarrow a = b$.
- 2) For any $a, b, c \in V$, $d(a, c) \leq d(a, b) + d(b, c)$
- 3) For any $a, b \in V$, $d(a, b) = d(b, a)$.

$\Rightarrow d$ is a metric on V .

Pf (sketch) - Trivial. \square

~~(length of path \geq distance between endpoints)~~
~~If $d(a, b) < d(a, c) + d(c, b)$, then $d(a, b) \leq d(a, c) + d(c, b)$ is established~~

Theorem Let $G = (V, E)$. For $a, b \in V$, write $a \sim b$ if \exists a path from a to b .

- 1) $u \sim u$ $\forall u \in V$. (reflexive)
- 2) $u \sim v \Leftrightarrow v \sim u \quad \forall u, v \in V$. (symmetric)
- 3) $u \sim v, v \sim w \Rightarrow u \sim w \quad \forall u, v, w \in V$ (transitive)

Pf (Trivial \square)

\hookrightarrow Every graph is the union of its connected components

Pf Partition V by \sim from above; equivalence classes are connected components. \checkmark

Bipartition

Def an undirected simple graph $G = (V, E)$ is bipartite if $V = V_1 \cup V_2$ where $e = (a, b)$, $a \in V_1, b \in V_2 \quad \forall e \in E$.

\checkmark can "color" vertices w/ two different colors where adjacent vertices are different colors

Theorem (Book): G bipartite $\Leftrightarrow G$ doesn't contain any odd cycles.

2) The diameter $d(G)$ of a graph G is $\max_{a,b \in V} \text{dist}(a,b)$

ex

$$1) \text{diam}(K_n) = 1 \text{ for } n \geq 2$$



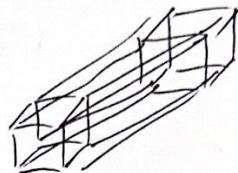
$$2) \text{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$$



$$3) \text{diam}(P_n) = n-1$$

(with cone, need to flip
n bits)

$$4) \text{diam}(Q_n) = n$$



Claim G is bipartite $\Leftrightarrow G$ has no odd cycles.

Pl (\Rightarrow) Suppose G has an odd length cycle: $v_0 - \dots - v_{2k+1} = v_0$. If G bipartite,

$V = V_1 \cup V_2$ w/ $E = \{(a,b) \mid a \in V_1, b \in V_2\}$. WLOG, $v_0 \in V_1$. Then $v_i \in V_2$, $v_{i+1} \in V_1$, \dots , $v_{2k+1} \in V_2$. (induction)

(\Leftarrow) Suppose G has no odd length cycles. Sufficient to show claim holds for each connected component. Thus WLOG, assume G connected. Let $v \in V$ be any vertex. Take

$$V = V_1 \cup V_2$$

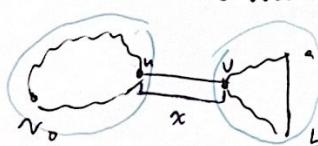
st.

$$V_1 = \{v \in V \mid \text{dist}(v, v) \text{ even}\}, \quad V_2 = \{v \in V \mid \text{dist}(v, v) \text{ odd}\}.$$

Just need to check

$e = (a, b) \text{ s.t. } a \in V_1, b \in V_2, e \in E$. Consider an edge ~~of~~ $e = (a, b)$ where ~~a, b~~ $\in V_1$. If $v_0 - a, v_0 - b$ don't

overlap, then $v_0 - a - b - v_0$ is a cycle of odd length. Otherwise, if $v_0 - a, v_0 - b$ overlap from $v - v$, ~~exists~~, a path of length x . Note



of edges from $v_0 - b$ (even)

+ # of edges from $a - v_0$ (even)

+ ~~dist(a, b)~~ ~~odd~~ odd

$$= 2x + l(v_0 - u - v_1) + l(v_1 - b - v_0) \text{ is even. Thus } l(v_0 - u - v_1) + l(u, v_1) + l(v_1 - b - v_0) + l(b, v_0) \text{ is even odd. So one of either cycle is odd. } \blacksquare$$

Theorem Let $G = (V, E)$ be a graph w/ n vertices. Suppose A nonadjacent $u, v \in V$,

$$\deg(u) + \deg(v) \geq n-1$$

Thus G is connected, and $\text{diam}(G) \leq 2$.

Pl let $x, y \in V$. If $\{x, y\} \cap \{u, v\} = \emptyset$, ~~exists~~ \nexists a path. Suppose instead $x, y \wedge u, v$ adjacent. Then ~~exists~~ \exists path between $x - y$ by pigeonhole. \blacksquare

2) The diameter $d(G)$ of a graph G is $\max_{a,b \in V} \text{dist}(a,b)$

ex

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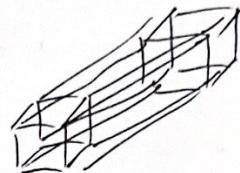
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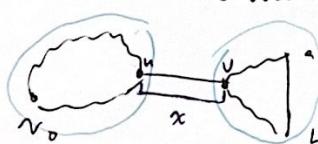
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+ # of edges from $a - v_0$ (even)

+ ~~dist(a, b)~~ ~~odd~~ odd

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Theorem - For $G = (V, E)$ have n vertices. Suppose $\min_{v \in V} \deg(v) \geq \frac{n-1}{2}$. Then G is connected.

Pf

Follow immediately from previous example.

B If we say $G = (V, E)$ is k -regular if $\forall v \in V, \deg(v) = k$.

ex

K_n is $(n-1)$ -regular

C_n is 2 -regular

Q_n is n -regular

P_n is not k -regular.

ex Any 3 -regular graph on at most 7 vertices must be connected

Pf If $n \leq 2$, $\frac{n-1}{2} \leq 3 \Rightarrow \min_{v \in V} \deg(v) \geq \frac{n-1}{2}$

Q Are there 3 -regular graphs on:

4 vertices?



K_4

5 vertices?

$\sum_{v \in V} \deg(v) = 15 = 2|E|$ requires 7.5 edges.

6 vertices?



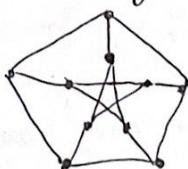
(!!)

7 vertices?

$\sum_v \deg(v) = 21 \Rightarrow$ impossible!

Another famous graph

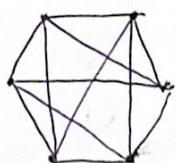
The Petersen Graph is a 3 -regular graph on 10 vertices:



* Famously serves as a counterexample to many different hypotheses *

Q Are there 4 -regular graphs on:

5 vertices?



6 vertices?

7 vertices?



Connect each vertex i to $i \pm 1, i \pm 2$.