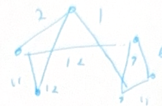
(undirected) simple graph — no self loops

directed graph

weighted graph



(undirected) graph

multigraph

$\underbrace{\hspace{6em}}$ "graph"

Def A **graph** $\boxed{G=(V,E)}$ is an ordered pair of ~~...~~ a vertex set, V, and an edge set, E. For every edge $e \in E$, $\exists$ associated vertices $a, b \in V$, called the endpoints. (For directed graphs, we want to know the start and end.)

Undirected Graphs

$\forall e \in E$, $e \subseteq V$ is an (n- unordered) 2-element subset of V.

$V = \{1, 2, 3\}$

$E = \{\{1,2\}, \{1,3\}\}$

- can't do directed, multigraphs (or simple) w/ this model.

Directed Graphs

$E \subseteq V \times V$, i.e. each edge is of the form (a,b), $a, b \in V$.

$V = \{1, 2, 3\}$

$E = \{(1,2), (3,1)\}$

- still can't do multigraphs.
- can model undirected graphs: if $a \leftrightarrow b$, $(a,b), (b,a) \in E$.

Common Graphs

Cycle graph on $n \geq 3$ vertices, C_n : C_4

C_3

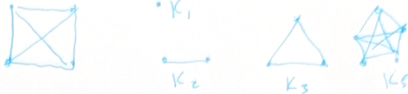
...

$V = \{1, 2, \ldots, n\}$, $E = \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}, \{n, 1\}\}$

Path graph on n vertices, P_n : P_5

$V = \{1, 2, \ldots, n\}$, $E = \{\{1,2\}, \ldots, \{n-1, n\}\} = E_{C_n} \setminus \{n, 1\}$

Complete Graph on n vertices, K_n : K_4



$V = \{1, 2, \dots, n\}$, $E = \{\{i,j\} \subseteq V : i \neq j\}$

Complete Bipartite Graph, $K_{m,n}$: $K_{2,3}$

$K_{m,n}$

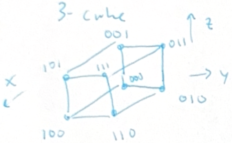
$V = \{(1,1), \dots, (1,m), (2,1), \dots, (2,n)\}$ $\quad E = \{\{(1,i), (2,j)\} : 1 \le i \le m, 1 \le j \le n\}$

$\updownarrow$ $\quad\quad \updownarrow$

$V = \{a_1, \dots, a_m, b_1, \dots, b_n\}$ $\quad E = \{\{a_i, b_j\} \mid 1 \le i \le m, 1 \le j \le n\}$

HyperCube Graph, Q_n

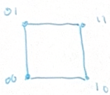
3-cube



V: all n-bit strings

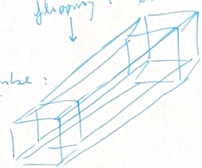
$E = \{\{a,b\} \mid a \wedge b = k \ll 1, \ 0 \le k < n\}$

2-cube



flipping 4th bit

4-cube:



Graph	# vertices	# edges
C_n	n	n
P_n	n	$n-1$
K_n	n	$\frac{(n-1)(n)}{2} = \binom{n}{2}$
$K_{m,n}$	$m+n$	$m \cdot n$
Q_n	2^n	$\frac{n \cdot 2^n}{2} = n \cdot 2^{n-1}$

every vertex has n-edges

Some useful terms:

degree of a vertex: # of edges connected to vertex

connected vs disconnected

disconnected

connected

connected components

union of two graphs

$G = (V, E)$, $H = (V', E')$ $\rightarrow$ $G \cup H = (V \cup V', E \cup E')$

complement of a graph

G

$\bar{G}$

$V = \{1, \dots, n\}$.

$E = E_{K_n} \setminus E_G$

$G = (V, E)$

$\bar{G} = (V, E')$

$e \in E \iff e \notin E'$

Recall: no self loops!

Theorem Let $G = (V, E)$ be a simple undirected graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

Pf

$\deg(v) = $ # of edges incident on v. $= \sum_{e \in E}$ "does e touch v"

$$\Rightarrow \sum_{v \in V} \deg(v) = \sum_{v \in V} \sum_{e \in E} \text{"does e touch v?"}$$

$$= \sum_{e \in E} \sum_{v \in V} \text{"does e touch v?"} \quad \text{(commutativity of addition)}$$

$$= \sum_{e \in E} 2 \quad \text{(each edge touches exactly 2 vertices)}$$

$$= 2|E|. \quad \blacksquare$$

Q: How many edges does $\bar{C}_n$ have?

C_4

$\bar{C}_4$

edges in $C_4 + \bar{C}_4$ = # edges in K_4 $\longrightarrow$ # edges in $\bar{C}_n$ = # edges in K_n − # edges in C_n

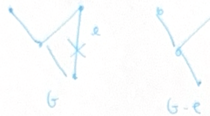
$$= \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

Remove an edge $e \in E$

$G' = G - e$

$V' = V$

$E' = E - e$

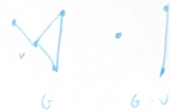


Remove a vertex $v \in V$

$G' = G - v$

$V' = V - v$

$E' = E -$ all edges incident on v



Def $H = (V', E')$ is a subgraph of G if it is a graph w/ $V' \subseteq V$, $E' \subseteq E$.

↳ ex $G = (V, E)$

$G - v$ is a subgraph of G $\forall v \in V$

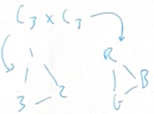
$G - e$ is a subgraph of G $\forall e \in E$ → any subgraph can be obtained by removing edges & vertices.

Def The product graph $\underset{(V,E)}{G} \times \underset{(V',E')}{H} = (V \times V', E \times E')$

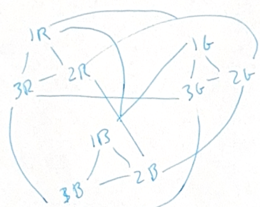
$V \times V' = \{(a,b) \mid a \in V, b \in V'\}$

$E \times E' = \{(a,x),(a,y) \mid (x,y) \in E', a \in V\} \cup \{(x,b),(y,b) \mid (x,y) \in E, b \in V'\}$

ex $C_3 \times C_3$

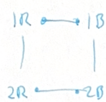


=

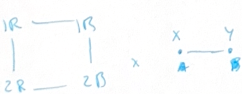


ex $K_2 \times K_2$

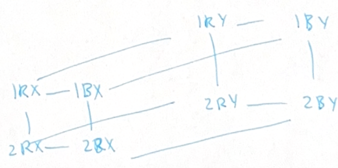
=



ex $K_2 \times K_2 \times K_2$



=



How many vertices does $G \times E$ have? edges?

$|V \times V'| = |V| \cdot |V'|$

$\#$ edges caused by G + $\#$ edges caused by H

$= |E| \cdot |V'| + |E'| \cdot |V|$

Exercise! Using $\prod_{i=1}^{n} K_2 = Q_n$, prove $\#$ vertices in $Q_n = 2^n$, $\#$ edges in $Q_n = n \cdot 2^{n-1}$.

Let $G = (V, E)$ be a graph, $u, w \in V$ be vertices.

Def: ① a walk from u to w is a sequence of vertices

$$u = v_0 - v_1 - v_2 - \cdots - v_n = w,$$

such that each v_i is adjacent to v_{i+1}.

② the length of the walk is n.

ex

$1 - 2 - 3 - 4 - 2 - 3$ is a walk from 1 to 3 of length 5.

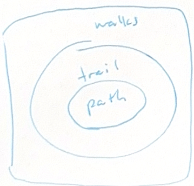
4 is a trivial walk (of length 0).

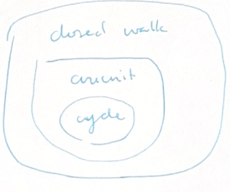
Def a closed walk starts and ends at the same vertex.

e.g. $1 - 2 - 3 - 4 - 2 - 3 - 2 - 1$.

Def a trail is a walk where no edge is repeated (closed trail of length ≥ 3 = circuit)

a path is a walk where no vertex is repeated (closed path of length ≥ 3 = cycle)





Def a graph $G = (V, E)$ is connected if $\forall u, v \in V$ $\exists$ a walk from u to v.

Theorem Let $G=(V,E)$ with $u,v \in V$. Then if $\exists$ a walk from u to v, then $\exists$ a path from u-v.

Pf (sketch)

$u-1-2-3-4-5-6-7-8-9$ is a walk from u-v.

↳ $u-1-2-8-9-v$ is a path from u-v, which we obtained by removing closed loops from our walk.

Def Let $G=(V,E)$. The distance, $d(u,v)$, from $u \in V$ to $v \in V$ is the length of the shortest path from u to v. If no paths exist, write $d(u,v)=+\infty$.

- We define the geodesic from u to v to be the shortest path from u to v.

Theorem Let $G=(V,E)$ be a graph.

1) For any $a,b \in V$, $d(a,b) \geq 0$ w/ $d(a,b)=0 \iff a=b$.
2) For any $a,b,c \in V$, $d(a,c) \leq d(a,b)+d(b,c)$
3) For any $a,b \in V$, $d(a,b)=d(b,a)$.

$\Rightarrow$ d is a metric on V.

Pf (~~sketch~~) - Trivial. ☒

~~① length of path ...~~
~~② I.neq $d(a,c) \leq d(a,b)+d(b,c)$ ...~~

Theorem Let $G=(V,E)$. For $a,b \in V$, write $a \sim b$ if $\exists$ a path from a to b.

1) $u \sim u$ $\forall u \in V$. (reflexive)
2) $u \sim v \iff v \sim u$ $\forall u,v \in V$ (symmetric)
3) $u \sim v,\ v \sim w \Rightarrow u \sim w$ $\forall u,v,w \in V$ (transitive)

Pf (Trivial ☒)

↳ Every graph is the union of its connected components

Pf Partition V by $\sim$ from above; equivalence classes are connected components. ✓

Bipartitions

Def an undirected simple graph $G=(V,E)$ is bipartite if $V=V_1 \cup V_2$ where $e=(a,b)$, $a \in V_1$, $b \in V_2$ $\forall e \in E$.

(can "colour" vertices w/ two different colors where adjacent vertices are different colors)

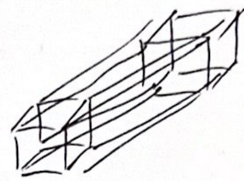
Theorem (Book): G bipartite $\iff$ G doesn't contain any odd cycles.

11

Def The ~~diameter~~ $d(G)$ of a graph G is $\max_{a,b\in V} \operatorname{dist}(a,b)$

ex

1) $\operatorname{diam}(K_n) = 1$ for $n \geq 2$ 
2) $\operatorname{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$ 
3) $\operatorname{diam}(P_n) = n-1$ 
4) $\operatorname{diam}(Q_n) = n$ (worst case, need to flip n bits)



Claim G is bipartite $\iff$ G has no odd cycles.

pf ($\Rightarrow$) Suppose G has an odd length cycle: $v_0 - \cdots - v_{2k+1} = v_0$. If G bipartite, $V = V_1 \cup V_2$ w/ $E = \{(a,b) \mid a \in V_1, b \in V_2\}$. WLOG, $v_0 \in V_1$. Then $v_i \in \begin{cases} V_1 & i = 2j \\ V_2 & i = 2j+1 \end{cases}$ (induction). Then $v_0 = v_{2k+1} \in V_2$ so $v_0 \in V_1 \cap V_2 = \emptyset$.

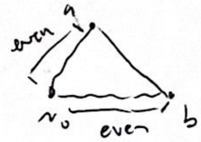
($\Leftarrow$) Suppose G has no odd length cycles. Suffices to show claim holds for each connected component. Thus WLOG, assume G connected. Let $v_0 \in V$ be any vertex. Take

$$V = V_1 \cup V_2$$

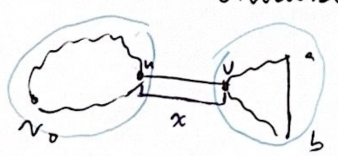
s.t.

$$V_1 = \{v \in V \mid \operatorname{dist}(v_0, v) \text{ even}\}, \quad V_2 = \{v \in V \mid \operatorname{dist}(v_0, v) \text{ odd}\}.$$

Just need to check $e = (a,b)$ ∍ $a \in V_1, b \in V_2$ $\forall e \in E$. Consider an edge $e = (a,b)$ where $a, b \in V_1$. If $v_0 - a$, $v_0 - b$ don't overlap, then $v_0 - a - b - v_0$ is a cycle of odd length.



Otherwise, if $v_0 - a$, $v_0 - b$ overlap from $u - v$, a path of length x. Note



$$\# \text{ of edges from } v_0 - b \ (\text{even})$$
$$+ \# \text{ of edges from } a - v_0 \ (\text{even})$$
$$+ \operatorname{dist}(a,b) \quad \text{odd}$$

$= 2x + l(v_0 - u - v_0) + l(v - a - b - v)$ is odd. Thus $l(u - v_0 - u) + l(u,v) + l(v-b-a-v) + l(u,v)$ is odd so one of either cycle is odd. ☒

Theorem Let $G = (V,E)$ be a graph w/ n vertices. Suppose $\forall$ nonadjacent $u, v \in V$,

$$\deg(u) + \deg(v) \geq n-1$$

Then G is connected, and $\operatorname{diam}(G) \leq 2$.

pf Let $x, y \in V$. If $\{x,y\} \in E$ or $x = y$, $\exists$ a path. Suppose instead $x \neq y$ and x, y nonadjacent. Then $\exists$ path between $x - y$ by pigeonhole. ☒

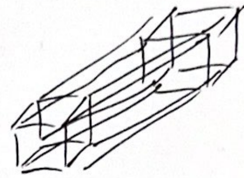
11

Def The <u>diameter</u> $d(G)$ of a graph G is $\max_{a,b\in V} dist(a,b)$

ex

1) $diam(K_n) = 1$ for $n \geq 2$
2) $diam(C_n) = \lfloor \frac{n}{2} \rfloor$
3) $diam(P_n) = n-1$
4) $diam(Q_n) = n$

(worst case, need to flip n bits)



Claim G is bipartite $\iff$ G has no odd cycles.

pf ($\Rightarrow$) Suppose G has an odd length cycle: $v_0 - \cdots - v_{2k+1} = v_0$. If G bipartite, $V = V_1 \cup V_2$ w/ $E = \{(a,b) \mid a \in V_1, b \in V_2\}$. WLOG, $v_0 \in V_1$. Then $v_i \in \begin{cases} V_1 & i = 2j \\ V_2 & i = 2j+1 \end{cases}$ (induction), $j \in \mathbb{Z}$. Then $v_0 = v_{2k+1} \in V_2$ so $v_0 \in V_1 \cap V_2 = \emptyset$.

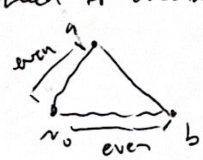
($\Leftarrow$) Suppose G has no odd length cycles. Suffices to show claim holds for each connected component. Thus WLOG, assume G connected. Let $v_0 \in V$ be any vertex. Take

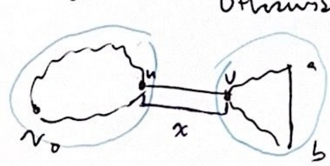
$$V = V_1 \cup V_2$$

s.t.

$$V_1 = \{v \in V \mid dist(v_0, v) \text{ even}\}, \quad V_2 = \{v \in V \mid dist(v_0, v) \text{ odd}\}.$$

Just need to check $e = (a,b)$ s.t. $a \in V_1, b \in V_2$ $\forall e \in E$. Consider an edge $e = (a,b)$ where $a, b \in V_1$. If $v_0 - a$, $v_0 - b$ don't overlap, then $v_0 - a - b - v_0$ is a cycle of odd length. Otherwise, if $v_0 - a$, $v_0 - b$ overlap from $u - v$, a path of length x. Note





\# of edges from $v_0 - b$ (even)
\+ \# of edges from $a - v_0$ (even)
\+ $dist(a,b)$ odd

$= 2x + l(v_0 - u - v_0) + l(v - a - b - v)$ is ~~even~~ odd. Thus $l(u - v_0 - u) + l(u,v) + l(v-b-a-v) + l(u,v)$ is odd so one of either cycle is odd. ∎

Theorem Let $G = (V,E)$ be a graph w/ n vertices. Suppose $\forall$ nonadjacent $u, v \in V$,

$$\deg(u) + \deg(v) \geq n-1$$

Then G is connected, and $diam(G) \leq 2$.

pf Let $x, y \in V$. If $\{x,y\} \in E$ or $x = y$, $\exists$ a path. Suppose instead $x \neq y$ $\wedge$ x, y nonadjacent. Then $\exists$ path between $x - y$ by pigeonhole. ∎

Theorem Let $G = (V,E)$ have n vertices. Suppose $\min_{v \in V} \deg(v) \geq \frac{n-1}{2}$. Then G is connected.

Pf

Follows immediately from previous example.

Def We say $G = (V,E)$ is k-regular if $\forall v \in V$, $\deg(v) = k$.

ex

K_n is $(n-1)$ regular

C_n is 2-regular

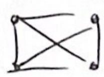
Q_n is n-regular

P_n is not k-regular.

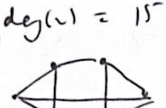
ex Any 3-regular graph on at most 7 vertices must be connected

Pf If $n \leq 7$, $\frac{n-1}{2} \leq 3 \Rightarrow \min_{v \in V} \deg(v) \geq \frac{n-1}{2}$

Q are there 3-regular graphs on:

4 vertices?  K_4

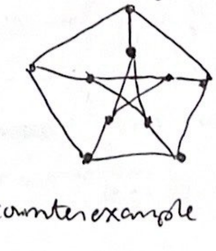
5 vertices? $\sum_{v \in V} \deg(v) = 15 = 2|E|$ requires 7.5 edges.

6 vertices? 

(!!) 7 vertices? $\sum_{v} \deg(v) = 21 \Rightarrow$ impossible!

Another famous graph

The Petersen Graph is a 3-regular graph on 10 vertices:

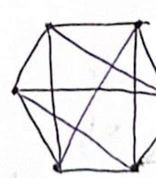


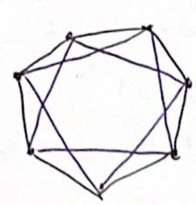
* Famously serves as a counterexample to many different hypotheses *

Q are there 4-regular graphs on:

5 vertices? 6 vertices? 7 vertices? Connect each vertex i to $i \pm 1$, $i \pm 2$.

K_5





Theorem Let $0 \le k \le n-1$. Then $\exists$ a k-regular graph on n vertices $\iff$ $k \cdot n$ is even.

Pf

($\Rightarrow$) Assume $\exists$ k-regular graph on n-vertices $G=(V,E)$. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

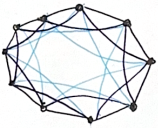
$$\|$$

$$\sum_{v \in V} k = k|V| = kn$$

($\Leftarrow$) Suppose kn even.

So kn is even. Thus k or n (or both) is even. If k even, connect i to $i \pm 1, i \pm 2, \dots, i \pm \frac{k}{2} \pmod n$ $\forall i \in V$. Otherwise, connect i to $i \pm 1, i \pm 2, \dots, i \pm \frac{k-1}{2}, i + \frac{n}{2} \pmod n$ $(\forall i \in V)$. ☒

ex: $k=6$, $n=10$.



$i \pm 1, 2, 3$

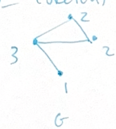
$k=5$, $n=8$

$i \pm 1, \pm 2$

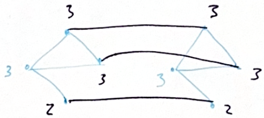
~~skip~~ $+4$

Theorem Let $G=(V,E)$ be a graph w/ max degree of any vertex $\le r$. Then G is a subgraph of some r-regular graph.

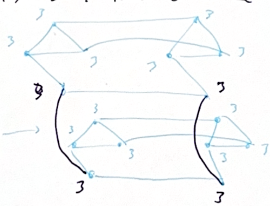
Pf (Sketch)



$\longrightarrow$



$\longrightarrow$



"copy the graph and connect deficient vertices"

$$G = (V,E), \quad \max_{v \in V} \deg(v) = r, \quad \min_{v \in V} \deg(v) = r-k$$

$$\longrightarrow H := G \cup G', \text{ where } (v,v') \in H \text{ if } \deg(v) < r$$

$G' = (\{v' := v \mid v \in V(G)\}, \{e' := e \mid e \in E(G)\})$

$$\to H: \max_{v \in V(H)} \deg(v) = r, \quad \min_{v \in V(H)} \deg(v) = r-k+1.$$

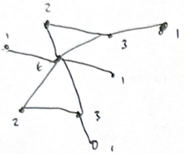
Repeat this process for $H', H'', \dots$ ☒

Notation: $\delta(G) = \min_{v \in V} \deg(v)$, $\Delta(G) = \max_{v \in V} \deg(v)$.

G is r-regular $\iff$ $\delta(G) = \Delta(G) = r$.

Def Let $G=(V,E)$. The degree sequence of G is a list of degrees $\deg(v)$ for $v \in V$ (with repeats as needed). Often, the list is written in non-increasing order.

ex



The degree sequence is: $6,3,3,2,2,1,1,1,1$

ex

Does any graph have degree sequence $4,4,2,2,1$

Note: $\sum_v \deg(v) = 13$, odd but sum of degrees must be even. (So no.)

"this sequence is not graphical"

ex

Does any graph have degree sequence $7,6,5,4,3,2,1$?

Even though $\sum \deg(v)$ even, still impossible because such a graph would have 7 vertices so each vertex has degree ≤ 6. ∎

ex

Does any graph have degree sequence $\underset{A}{3}, \underset{B}{3}, \underset{C}{3}, \underset{D}{1}$.

Such a graph would have

A,B,C,D each connected to all 3 vertices other than themselves. In particular, D would be connected to each of them, contradicting $\deg(D)=1$.

Theorem Let $n \geq 2$ and consider the non-increasing sequence of non-negative integers

$$s: \underset{k}{\underset{\parallel}{d_1}}, d_2, \ldots, d_n$$

Then s is a degree sequence for some graph $\iff$

$$s' : d_2 - 1, d_3 - 1, \ldots, d_{k+1} - 1, d_{k+2}, \ldots, d_n$$

is a degree sequence for some graph. In particular, $k \leq n-1$.