or (undirected) simple graph | no self loops

directed graph

weighted graph

(undirected) graph

multigraph

"graph"

Def A graph $G = (V, E)$ is an ordered pair of a vertex set, V, and an edge set, E. For every edge $e \in E$, $\exists$ associated vertices $a, b \in V$, called the endpoints. (For directed graphs, we want to know the start and end.)

Undirected Graphs

$\forall e \in E$, $e \subseteq V$ is an (n- unordered) 2-element subset of V.

$V = \{1, 2, 3\}$

$E = \{\{1,2\}, \{1,3\}\}$

- can't do directed, multigraphs w/ this model. (or simple)

Directed Graphs

$E \subseteq V \times V$, i.e. each edge is of the form (a, b), $a, b \in V$.

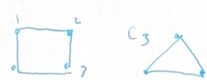
$V = \{1, 2, 3\}$

$E = \{(1,2), (3,1)\}$

- still can't do multigraphs.
- can model undirected graphs: if $a \leftrightarrow b$, $(a,b), (b,a) \in E$

Common Graphs

Cycle graph on $n \geq 3$ vertices, C_n : C_4



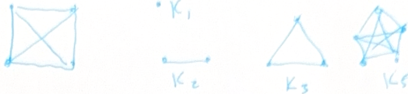
...

$V = \{1, 2, \dots, n\}$, $E = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}, \{n, 1\}\}$

Path graph on n vertices, P_n : P_5

$V = \{1, 2, \dots, n\}$, $E = \{\{1,2\}, \dots, \{n-1, n\}\} = E_{C_n} \setminus \{n, 1\}$

Complete Graph on n vertices, K_n :



$V = \{1, 2, \ldots, n\}$, $E = \{ \{i,j\} \subseteq V : i \neq j \}$

Complete Bipartite Graph, $K_{m,n}$: $K_{2,3}$

$K_{m,n}$

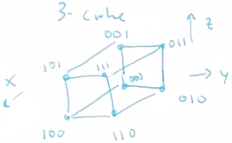
$V = \{ (1,1), \ldots, (1,m), (2,1), \ldots, (2,n) \}$ $\quad E = \{ \{ (1,i), (2,j) \} : 1 \leq i \leq m,\ 1 \leq j \leq n \}$

$\updownarrow$ $\qquad$ $\updownarrow$

$V = \{ a_1, \ldots, a_m, b_1, \ldots, b_n \}$ $\quad E = \{ \{a_i, b_j\} \mid 1 \leq i \leq m,\ 1 \leq j \leq n \}$

HyperCube Graph, Q_n

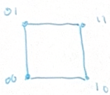
3-cube



V: all n-bit strings

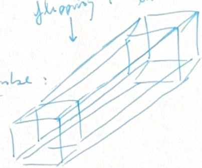
$E = \{ \{a,b\} \mid a \wedge b = k \ll 1,\ 0 \leq k < n \}$

2-cube



flipping 4th bit

4-cube:



Graph	# vertices	# edges
C_n	n	n
P_n	n	$n-1$
K_n	n	$\frac{(n-1)(n)}{2} = \binom{n}{2}$
$K_{m,n}$	$m+n$	$m \cdot n$
Q_n	2^n	$\frac{n \cdot 2^n}{2} = n \cdot 2^{n-1}$

every vertex has n-edges

Some useful terms:

degree of a vertex: # of edges connected to vertex

connected vs disconnected

disconnected

connected

connected components

union of two graphs

$G = (V, E)$, $H = (V', E')$ $\rightarrow$ $G \cup H = (V \cup V', E \cup E')$

complement of a graph

G

$\bar{G}$

$V = \{1, \dots, n\}$.

$E = E_{K_n} \setminus E_G$

$G = (V, E)$

$\bar{G} = (V, E')$

$e \in E \iff e \notin E'$

Recall: no self loops!

Theorem Let $G = (V, E)$ be a simple undirected graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

Pf

$\deg(v) = $ # of edges incident on v. $= \sum_{e \in E}$ "does e touch v"

$\Rightarrow \sum_{v \in V} \deg(v) = \sum_{v \in V} \sum_{e \in E}$ "does e touch v?"

$= \sum_{e \in E} \sum_{v \in V}$ "does e touch v?" (commutativity of addition)

$= \sum_{e \in E} 2$ (each edge touches exactly 2 vertices)

$= 2|E|$. $\boxtimes$

Q: How many edges does $\bar{C}_n$ have?

C_4

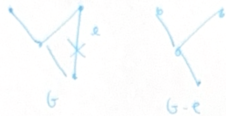
$\bar{C}_4$

edges in $C_4 + \bar{C}_4$ = # edges in K_4 $\rightarrow$ # edges in $\bar{C}_n$ = # edges in K_n − # edges in C_n

$= \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$

Remove an edge $e \in E$

$G' = G - e$
$V' = V$
$E' = E - e$



Remove a vertex $v \in V$

$G' = G - v$
$V' = V - v$
$E' = E -$ all edges incident on v

Def $H = (V', E')$ is a subgraph of G if it is a graph w/ $V' \subseteq V$, $E' \subseteq E$.

↳ ex $G = (V, E)$

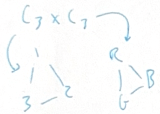
$G - v$ is a subgraph of G $\forall v \in V$
$G - e$ is a subgraph of G $\forall e \in E$ → any subgraph can be obtained by removing edges & vertices.

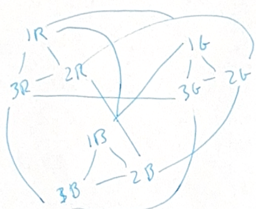
Def The product graph $\underset{(V,E)}{G} \times \underset{(V',E')}{H} = (V \times V', E \times E')$

$V \times V' = \{(a,b) \mid a \in V, b \in V'\}$

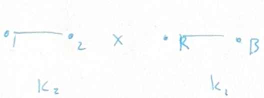
$E \times E' = \{(a,x),(a,y) \mid (x,y) \in E', a \in V\} \cup \{(x,b),(y,b) \mid (x,y) \in E, b \in V'\}$

ex $C_3 \times C_3$ =





ex $K_2 \times K_2$



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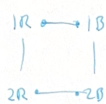
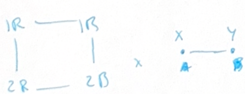


ex $K_2 \times K_2 \times K_2$



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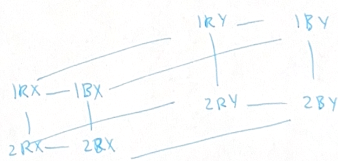


How many vertices does $G \times E$ have? edges?

$|V \times V'| = |V| \cdot |V'|$

edges caused by G + # edges caused by H

$= |E| \cdot |V'| + |E'| \cdot |V|$

Exercise! Using $\prod_{i=1}^{n} K_2 = Q_n$, prove # vertices in $Q_n = 2^n$, # edges in $Q_n = n \cdot 2^{n-1}$.

Let $G = (V, E)$ be a graph, $u, w \in V$ be vertices.

Def: ① a walk from u to w is a sequence of vertices

$$u = v_0 - v_1 - v_2 - \cdots - v_n = w,$$

such that each v_i is adjacent to v_{i+1}.

② the length of the walk is n.

ex

$1 - 2 - 3 - 4 - 2 - 3$ is a walk from 1 to 3 of length 5.

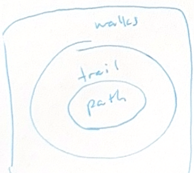
4 is a trivial walk (of length 0).

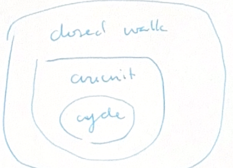
Def a closed walk starts and ends at the same vertex.

e.g. $1 - 2 - 3 - 4 - 2 - 3 - 2 - 1$.

Def a trail is a walk where no edge is repeated (closed trail of length ≥ 3 = circuit)

a path is a walk where no vertex is repeated (closed path of length ≥ 3 = cycle)





Def a graph $G = (V, E)$ is connected if $\forall u, v \in V$ $\exists$ a walk from u to v.

Theorem Let $G=(V,E)$ with $u,v \in V$. Then if $\exists$ a walk from u to v, then $\exists$ a path from u-v.

Pf (sketch)

$u-1-2-3-4-5-6-7-8-9$ is a walk from u-v.

↳ $u-1-2-8-9-v$ is a path from u-v, which we obtained by removing closed loops from our walk.

Def Let $G=(V,E)$. The distance, $d(u,v)$, from $u \in V$ to $v \in V$ is the length of the shortest path from u to v. If no paths exist, write $d(u,v) = +\infty$.

- We define the geodesic from u to v to be the shortest path from u to v.

Theorem Let $G=(V,E)$ be a graph.

1) For any $a,b \in V$, $d(a,b) \geq 0$ w/ $d(a,b)=0 \iff a=b$.
2) For any $a,b,c \in V$, $d(a,c) \leq d(a,b) + d(b,c)$
3) For any $a,b \in V$, $d(a,b) = d(b,a)$.

$\Rightarrow$ d is a metric on V.

Pf - Trivial. ☒

Theorem Let $G=(V,E)$. For $a,b \in V$, write $a \sim b$ if $\exists$ a path from a to b.

1) $u \sim u$ $\forall u \in V$. (reflexive)
2) $u \sim v \iff v \sim u$ $\forall u,v \in V$. (symmetric)
3) $u \sim v, v \sim w \Rightarrow u \sim w$ $\forall u,v,w \in V$ (transitive)

Pf (Trivial ☒)

↳ Every graph is the union of its connected components

Pf Partition V by $\sim$ from above; equivalence classes are connected components. ✓

Bipartitions

Def an undirected simple graph $G=(V,E)$ is bipartite if $V = V_1 \cup V_2$ where $e=(a,b)$, $a \in V_1$, $b \in V_2$ $\forall e \in E$.

(can "colour" vertices w/ two different colors where adjacent vertices are different colors)

Theorem (Book): G bipartite $\iff$ G doesn't contain any odd cycles.