7 Group actions (10/04)

Definition 7.1. Let G be a group and X a set. An **action** of G on X is a function $k\colon G \times X \to X$, written $a \cdot x = k(a, x)$ for $a \in G$ and $x \in X$, satisfying the following axioms:

(a) $e \cdot x = x$ for all $x \in X$ where e is the identity element of G;

(b) $a \cdot (b \cdot x) = (ab) \cdot x$ for all $a, b \in G$ and $x \in X$.

Example 7.2. The group $\mathbf{Z}$ acts on $\mathbf{R}$ by $n \cdot x = n + x$ for $n \in \mathbf{Z}$ and $x \in \mathbf{R}$.

Example 7.3. The group S_X acts on X by $f \cdot x = f(x)$ for $f \in S_X$ and $x \in X$. In particular, S_n acts on the set $\{1, \ldots, n\}$.

Example 7.4. If V is a real vector space, then the group $\mathbf{R}^\times$ of non-zero real numbers acts on V by scalar multiplication: if $v \in V$ and $\alpha \in \mathbf{R}^\times$, then $\alpha \cdot v = \alpha v$.

Example 7.5. If G is a group, it acts on itself by left multiplication: for $g, h \in G$, we let $g \cdot h = gh$. Here, we view the G which acts as the *left* G in $m\colon G \times G \to G$. This is called the *left regular action* of G on itself. The formula $g \cdot h = hg$ would not generally be a group action of G on itself. Why not?

Example 7.6 (Return to Exercise 4.2)**.** We can learn about a group G via its actions. For example, consider a symmetric group S_n. The symmetric group acts on the set F of functions $\mathbf{R}^n \to \mathbf{R}$ as follows. Given $a \ni S_n$ and $f\colon \mathbf{R}^n \to \mathbf{R}$, we let $(a \cdot f)(x_1, \ldots, x_n) = f(x_{a(1)}, x_{a(2)}, \ldots, x_{a(n)})$, i.e., by reordering the inputs. Let $g(x) = \prod_{1 \leqslant i < j \leqslant n}(x_i - x_j)$. This polynomial is called the Vandermonde polynomial. Note that for any $a \in S_n$, either $a \cdot g = g$ or $a \cdot g = -g$. For example, if $n = 4$, this polynomial is

$$g(x_1, x_2, x_3, x_4) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4).$$

The element $a = (1\,2\,3\,4)$ of S_4 then acts as

$$(a \cdot g)(x_1, x_2, x_3, x_4) = (x_2 - x_3)(x_2 - x_4)(x_2 - x_1)(x_3 - x_4)(x_3 - x_1)(x_4 - x_1) = -g(x_1, x_2, x_3, x_4).$$

Let S_n act on $\{1, -1\}$ by letting $a \cdot \epsilon = \gamma$ if $a \cdot (\epsilon g) = \gamma g$. In the example above, the 4-cycle a has $a \cdot 1 = -1$ and $a \cdot (-1) = 1$. If $a \in S_n$ is a transposition, then $a \cdot 1 = -1$. To see this, suppose that $a = (c\,d)$ where $1 \leqslant c < d \leqslant n$. If $i < c$, then $a \cdot (x_i - c) = (x_i - d)$ and if $d < j$, then $a \cdot (c - x_j) = (d - x_j)$. We also have $a \cdot (x_i - x_j) = (x_j - x_i) = -(x_i - x_j)$. Finally, if $c < i < d$,

$$a \cdot (x_c - x_i)(x_i - x_d) = (x_d - x_i)(x_i - x_c) = (-(x_i - x_d))(-(x_c - x_i)) = (x_c - x_i)(x_i - x_d).$$

Collating these calculations, it follows that $a \cdot v = -v$ for $a = (c\,d)$. Thus, by axiom (b) of a group action, if a is a product of k transpositions, then $a \cdot 1 = (-1)^k$. This proves the claim from Exercise 4.2 as if $(-1)^k = (-1)^m$, then $k \equiv m \mod 2$.

The next theorem says that group actions of G on X are "the same" as group homomorphisms $G \to S_X$.

Theorem 7.7. *Let G be a group and X as set. There is a bijection*

$$\{\textit{actions } k \textit{ of } G \textit{ on } X\} \xrightarrow{k \mapsto f_k} \mathrm{Hom}(G, S_X).$$

Proof. Next time. □

Example 7.8. The action of S_n on the Vandermonde polynomial induces, via the theorem, a surjective group homomorphism $S_n \to S_{\{1,-1\}}$, which we view as a group homomorphism $\epsilon\colon S_n \to S_2 \cong \mathbf{Z}/2 \cong \{1, -1\}$, where $\{1, -1\}$ is a group under multiplication. The **sign** of an element $a \in S_n$ is $\epsilon(a) \in \{1, -1\}$.

7.1 Exercises

Exercise 7.1. Suppose that G is a finite group of even order. Show that there exists $x \neq e$ in G with $x^2 = e$.

Exercise 7.2. Show that every finite group G of order 4 is isomorphic to either $\mathbf{Z}/4$ or to $K = \mathbf{Z}/2 \times \mathbf{Z}/2$.

Exercise 7.3. Show that a finite group G of order 5 is isomorphic to $\mathbf{Z}/5$.