Lecture 1

Definition Say $\mathcal{U} \subseteq \mathbb{R}^n$ is open if $\forall p \in \mathcal{U}$ $\exists r > 0$ such that $B(p,r) \subseteq \mathcal{U}$.

Definition A topology on X is a collection $\mathcal{U}$ of subsets of X such that ① $X, \phi \in \mathcal{U}$ ② $\{\mathcal{U}_\alpha\}_\alpha \subseteq \mathcal{U} \Rightarrow \bigcup_\alpha \mathcal{U}_\alpha \in \mathcal{U}$, and ③ $\{\mathcal{U}_i\}_{i=1}^n \Rightarrow \bigcap_{i=1}^n \mathcal{U}_i \in \mathcal{U}$. We call any member $\mathcal{U} \in \mathcal{U}$ an open set.

Lecture 2

Definition $A \subseteq X$ is closed in the given topology if $X - A$ is open.

Definition Let X, Y be topologies if $X \subseteq Y$, we say X is coarser than Y, or equivalently, Y is finer than X.

↳ Cofinite topology is the coarsest one in which singletons are closed.

Lecture 3

Definition Say $\mathcal{B} = \{$ collection of subsets of $X\}$ is a basis for a topology on X if ① $X = \bigcup_{\mathcal{U} \in \mathcal{B}} \mathcal{U}$ and ② given $\mathcal{U}, V \in \mathcal{B}$ $\forall p \in \mathcal{U} \cap V$ $\exists W_p \in \mathcal{B}$ such that $p \in W_p$ and $W_p \subseteq \mathcal{U} \cap V$. The topology generated by $\mathcal{B}$ is defined by saying open sets $= \bigcup_{\mathcal{U} \in \mathcal{B}} \mathcal{U}$.

Lecture 4

Definition A metric on X is a function $d: X \times X \to \mathbb{R}$ such that ① $d(p,q) \geq 0$ $\forall p,q \in X$, $d(p,q) = 0 \Leftrightarrow p = q$, ② $d(p,q) = d(q,p)$ $\forall p,q \in X$, and ③ $d(p,q) \leq d(p,r) + d(r,q)$ $\forall p,q,r \in X$.

Definition Given $p \in X$ and $r > 0$, the open ball of radius r centered at p is $B(p,r) = \{q \in X \mid d(p,q) < r\}$.

↳ Given a metric on X, the collection of open balls forms a basis for a topology on X.

Lecture 5

Definition Given topological spaces X, Y, define the product/box topology to be with basis consisting of $\mathcal{U} \times V$, with $\mathcal{U}$ open in X, V open in Y.

Claim Denote by $pr_i : X_1 \times \cdots \times X_n \to X_i$ the function which projects onto X_i. The product topology on $X_1 \times \cdots \times X_n$ is the coarsest one for which $pr_i^{-1}(\mathcal{U})$ is open given $\mathcal{U} \subseteq X_i$ open (i.e. pr_i is continuous) $\forall i = 1, 2, \ldots, n$.

Definition The sequence $(x_n)_{n \in \mathbb{N}}$ converges to $p \in X$ if $\forall$ open $\mathcal{U} \ni p$ $\exists N \in \mathbb{N}$ such that $x_n \in \mathcal{U}$ $\forall n \geq N$.

↳ Note: Convergence in box topology $\not\Leftrightarrow$ componentwise convergence.

Lecture 6

Definition The product topology on $\prod_{\alpha \in \Delta} X_\alpha$ is the one generated by the basis $\prod_{\alpha \in \Delta} Y_\alpha$, where $Y_\alpha = X_\alpha$ for all but a finite number of α's, and Y_α open X_α $\forall \alpha \in \Delta$.

Fact Given a $\prod_\alpha X_\alpha$ equipped with the product topology, convergence in product $\Leftrightarrow$ componentwise convergence.

↳ Special case: $(x_1^n, x_2^n, x_3^n, \ldots)$ in $\mathbb{R}^\omega$ converges to $(y_1, y_2, \ldots)$ $\Leftrightarrow$ each $x_k^n \to y_k$.

Lecture 7

Definition Given $A \subseteq X$, the closure of A is $\bar{A} = \bigcap_{\substack{\text{closed} \\ K \supseteq A}} K$.

↳ Intuition: $\bar{A}$ is "smallest" closed set containing A (so A closed $\iff \bar{A} = A$)

Claim $p \in \bar{A} \iff$ every neighborhood U of p intersects A. or equivalently, $p \notin \bar{A} \iff \exists$ neighborhood U of p with $U \cap A = \phi$.

Claim If $\exists$ sequence in A converging to p, then $p \in \bar{A}$.

↳ Converse true in a metric space.

Lecture 8

Note: $\overline{\prod_\alpha A_\alpha} = \prod_\alpha \bar{A}_\alpha$, where $A_\alpha \subseteq X_\alpha$ holds in product and box topologies.

Definition: X is Hausdorff if $\forall p \neq q$ in X, $\exists$ open $U \ni p$, $V \ni q$ such that $U \cap V = \phi$.

Claim If X is Hausdorff, then limits of convergent sequences are unique

Definition X is T_1 if $\forall p \neq q$ in X $\exists$ open $U \ni p$ with $q \notin U$, and $\exists$ open $V \ni q$ with $p \notin V$.

↳ Note: Hausdorff $\Rightarrow T_1$.

Claim: A space X is $T_1 \iff \{p \mid p \in X\}$ are closed.

Lecture 9

Definition: We say $f: X \to Y$ is continuous if $f^{-1}(U)$ is open in X whenever U is open in Y.

↳ Note: As we saw earlier, the product topology on $\prod_\alpha X_\alpha$ is the coarsest topology relative to which each projection is continuous

$pr_\beta: \prod_\alpha X_\alpha \to X_\beta$

↳ Note: Since $f^{-1}(Y-B) = X - f^{-1}(B)$, we get that f is continuous $\iff$ pre-image of closed are closed.

Claim $f: X \to Y$ is continuous $\iff \forall A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$.

Lecture 10

Claim Given maps $X \xrightarrow{f_\alpha} Y_\alpha$ for each α, show that $X \xrightarrow{g} \prod_\alpha Y_\alpha$, $p \mapsto (f_\alpha(p))_\alpha = g(p)$ is continuous $\iff$ each f_α is continuous, where $\prod_\alpha Y_\alpha$ has the product topology.

↳ Note Recall that product is the finest topology relative to which continuity $\iff$ componentwise continuity.

Definition A homeomorphism from X to Y is a continuous bijection $X \to Y$ with a continuous inverse. If such a thing exists, say $X \cong Y$ are homeomorphic.

Lecture 11

Definition Given a space X and a surjection $X \xrightarrow{p} Y$, the quotient topology induced by p is defined by requiring that $p^{-1}(U)$ open in $X \iff U$ open in Y.

↳ Fact: this is the finest one on Y relative to which p is continuous.

↳ Intuition: All elements of X in a fiber $p^{-1}(\{y\})$ all glued/collapsed into one another.

Lecture 12

Lecture 13

Definition X is disconnected if $\exists\, U, V$ open in X, disjoint and non-empty such that $X = U \cup V$.

X is connected if it is not disconnected, i.e. whenever $X = U \cup V$ with U, V open distinct, one of $U, V = \emptyset$.

Theorem If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

Claim Suppose A_α are connected and $\exists\, p \in \bigcap_\alpha A_\alpha$. Then $\bigcup_\alpha A_\alpha$ is connected.

Lecture 14

Definition We can alternatively define X to be connected if $\nexists$ nonempty clopen proper subset.

↳ Note: Closures of connected sets are connected.

Claim If X, Y are connected, then $X \times Y$ is connected.

Definition X is path-connected if $\forall p, q \in X$ $\exists$ continuous $\gamma : [a,b] \to X$ such that $\gamma(a) = p$, $\gamma(b) = q$.

Claim Path-connected $X \implies$ connected X.

Lecture 15

Definition Given X, a connected component C of X is a maximally connected subset of X, i.e. C is connected and if $C \subseteq S$ where S is connected, then $C = S$.

Definition Given X and $p \in X$, a local base of open sets at p is a collection $\mathcal{B}_p$ of neighborhoods of p such that if U is a neighborhood of p, $\exists B \in \mathcal{B}_p$ with $B \subseteq U$.

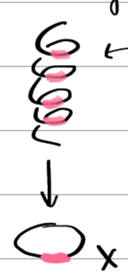
Definition X is locally connected if every $p \in X$ has a local base of connected sets.

↳ Intuition: X is "connected" near p.

Claim If X is locally connected, then every component is open in X.

Lecture 15

Definition Given X, a covering space of X is a continuous surjection $Y \xrightarrow{p} X$ such that $\forall x \in X$ $\exists$ neighborhood U of x such that $p^{-1}(U) =$ union of spaces $\cong U$.



↳ Fact: To guarantee a universal cover exists, need to assume X is connected, locally path connected, and semi-locally simply-connected.

Definition Given X, an open cover of X is a collection $\{U_\alpha\}_{\alpha \in \Delta}$ of open sets such that $X \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$.

X is compact if any open cover can be reduced to a finite one, i.e. if $X \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$ then $\exists\, U_{\alpha_1}, \dots, U_{\alpha_n}$ such that $X \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.

Claim If X is compact and $A \subseteq X$ is closed, then A is compact.

Claim If X is Hausdorff and $K \subseteq X$ is compact, then K is closed in X.

↳ Note: In Hausdorff space, get that if K compact, $p \notin K$, then $\exists$ disjoint open $U \supseteq K$, $V \ni p$, i.e. K can be "separated" from p.

Claim If $f: X \to Y$ continuous and X is compact, then $f(X)$ is compact.

Subtyping & Variance

Ignore → Rust stuff...

Subtyping is the idea that one type can be used in place of another. Define Sub <: Super. This defines the set of requirements that 'Super' defines are completely satisfied by Sub. (Sub may have more requirements)

Definition: 'a defines a region of code

↳ 'long <: 'short $\iff$ 'long defines a region of code that completely contains 'short.

('long may define a region larger than 'short)

Example: 'static <: 'world.

Variance

We cannot assume &mut &'static str <: &mut &'b str, even if 'static is a subtype of 'b.

Variance is the concept that Rust borrows to define relationships about subtypes through their generic parameters.

(Define for convenience a generic type F<T>)

Definition: The type F's variance is how the subtyping of its inputs affects the subtyping of its outputs. There are 3 types of variance in Rust:

1) F is covariant if F<Sub> <: F<Super> (i.e. the subtype property is passed through)
2) F is contravariant if F<Super> <: F<Sub> (i.e. the subtype property is inverted)
3) F is invariant otherwise (i.e. no subtyping relationship exists)

Example:

- Recall it was ok to treat &'a T as a subtype of &'b T if 'a <: 'b. Thus &'a T is covariant over 'a
- It was not ok to treat &mut &'a U as a subtype of &mut &'b U ⇒ &mut T invariant over T

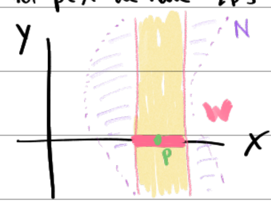
Lecture 17

Claim $[a,b]$ is compact.

Claim If X, Y are compact, then $X \times Y$ is compact. (This is called the finite Tychonoff theorem)

Lemma (Existence of Tubes) If for $p \in X$ we have $\{p\} \times Y \subseteq$ open N in $X \times Y$, then $\exists$ neighborhood W of p in X such that $\underline{W \times Y} \subseteq N$

the tube.



Theorem (Heine-Borel) In $\mathbb{R}^n$, $K \subseteq \mathbb{R}^n$ is compact $\iff$ K is closed and bounded.

Definition X is Lindelöf if every open cover has a countable subcover.

Lecture 18

Definition X is locally compact if $\forall p \in X$ $\exists$ compact $K \subseteq X$ containing a neighborhood U of p.

Definition If X is locally compact Hausdorff, define the one-point compactification of X to be $Y = X \cup \{\infty\}$ with open sets ① U open in X, neighborhoods $\ni \infty$, ② $Y - K$, where $K \subseteq X$ is compact.

Claim Y, the one-point compactification of X, is a compact Hausdorff space and $X \subseteq Y$ is a subspace

Lecture 19

Claim The one-point compactification is unique: if $Y = X \cup \{\infty\}$, $Y' = X \cup \{\infty'\}$ are compact Hausdorff spaces containing X as a subspace, then $Y \cong Y'$.

Definition If Y compact with $X \subseteq Y$ dense, Y is called a compactification of X

↳ 1-pt is the "smallest" one; Stone-Čech is the "universal" one

Claim Suppose X is Hausdorff. Then X is locally compact $\iff$ $\forall p \in X$ $\exists$ local base of neighborhoods with compact closure, i.e. $\forall$ open $V \ni p$ $\exists p \in U \subseteq \underset{\text{compact}}{\overline{U}} \subseteq V$.

↳ Point: If X locally compact Hausdorff, then given $p \in X$, closed $A \overset{\not\ni p}{\subseteq} X$, $\exists$ open $U \ni p$, $V \supseteq A$ with $U \cap V = \emptyset$.

Lecture 20

Definition X is first-countable if every $p \in X$ has a countable local base $\{U_i\}_{i \in \mathbb{N}}$, i.e. every neighborhood of p contains some U_i.

Claim If X is first-countable and $A \subseteq X$, then $p \in \overline{A}$ $\iff$ $\exists$ sequence (a_n) in A converging to p.

Claim If X is first-countable, $f: X \to Y$ continuous $\iff$ $f(p_n) \to f(p)$ in Y whenever $p_n \to p$ in X.

Definition X is second-countable if X has a countable basis.

↳ Note: 2^{nd} countable $\Rightarrow$ 1^{st} countable

Claim If X is compact metric space, then it is second-countable.

Lecture 21

Claim If X is second-countable, then X is separable and Lindelöf.

$\exists$ countable dense subcover

every open cover has a countable subcover.

↳ If X metrizable, separable, or Lindelöf, then X is second-countable. Ex $\mathbb{R}_\ell$ is separable, but not 2^{nd} countable.

Definition X is regular (T_3) if it is T_1 and if whenever $A \subseteq X$ is closed and $p \notin A$ $\exists$ disjoint open $V \ni p$ and $W \supseteq A$.

Claim X is regular $\iff$ $\forall p \in X$ and open $\mathcal{U} \ni p$ $\exists$ open V such that $p \in V \subseteq \overline{V} \subseteq \mathcal{U}$.

↳ Note: locally compact Hausdorff $\Rightarrow$ regular.

Lecture 22

Example Some regular spaces include: locally compact Hausdorff, subspaces, products, $\mathbb{R}_\ell$, metric spaces

Definition X is <u>normal</u> if it is T_1 and $\forall$ disjoint closed $A, B \subseteq X$ $\exists$ open $\mathcal{U} \supseteq A$, $V \supseteq B$ such that $\mathcal{U} \cap V = \emptyset$. Equivalently, X is normal if given $\underset{\text{closed}}{A} \subseteq \underset{\text{open}}{\mathcal{U}}$ $\exists$ open V with $A \subseteq V \subseteq \overline{V} \subseteq \mathcal{U}$.

↳ Note: Metric spaces, $\mathbb{R}_\ell$, compact Hausdorff spaces are normal.

Claim If X is regular and second-countable, then X is normal.

Claim If X is regular and $S \subseteq X$, then S is regular.

Lecture 23

Claim Every subspace of X is normal $\iff$ any $A, B \subseteq X$ with $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$ can be separated by open sets.

Lemma (Urysohn) Suppose X is normal. Let $A, B \subseteq X$ be closed and disjoint. Then $\exists$ continuous $f: X \to [0,1]$ such that $f(a) = 0$ $\forall a \in A$, $f(b) = 1$ $\forall b \in B$.

Lecture 24

Claim The converse of Urysohn holds: if $\forall$ closed and disjoint $A, B \subseteq X$ $\exists$ continuous $f: X \to [0,1]$ such that $f|_A = 0$, $f|_B = 1$, then X is normal.

Definition X is <u>completely regular</u> if it is T_1 and given closed $A \subseteq X$, $p \notin A$, $\exists$ continuous $f: X \to [0,1]$ such that $f(p) = 1$ and $f|_A = 0$.

Implications

Every subspace is normal

- normal $\Rightarrow$ completely regular $\Rightarrow$ regular
- $T_0 \Leftarrow T_1 \Leftarrow T_2$ / Hausdorff $\Leftarrow T_3$ / regular $\Leftarrow T_4$ / normal $\Leftarrow T_5$ / completely normal $\Leftarrow T_6$ / perfectly normal

can separate A, B closed by a continuous function precisely $\iff$ closed sets are G_δ sets.

Theorem (Urysohn Metrization) If X is regular and 2^{nd} countable, then X is metrizable.

normal

Lecture 25

Theorem (Tietze Extension) Suppose X is normal and $A \subseteq X$ is closed. Given $f: A \to [-1,1]$ continuous, $\exists$ extension $g: X \to [-1,1]$ such that $g|_A = f$. In fact, can extend $A \to \mathbb{R}$ to $X \to \mathbb{R}$.

Lecture 26

Theorem (Tychonoff) If $\{X_\alpha\}_\alpha$ is a collection of compact spaces, then $\prod_\alpha X_\alpha$ is compact.

Definition A subbasis for a topology on X is a collection of sets whose union is X. The topology generated by the subbasis is the collection of all unions of finite intersections of elements of the subasis.

Theorem (Alexander Subbase) If every open cover of X by subbasis open sets has a finite subcover, then X is compact.

Axiom (of Choice) Suppose $X_\alpha \neq \phi$ $\forall \alpha$. Then $\prod_\alpha X_\alpha \neq \phi$.

↳ Note: Tychonoff $\Rightarrow$ Axiom of Choice

Lemma (Zorn's Lemma) Given a partially-ordered set in which every chain has an upper bound, $\exists$ a maximal element.

Definition A relation $\leq$ on P is a partial order if ① $x \leq x$ $\forall x \in P$ ② $x \leq y, y \leq z \Rightarrow x \leq z$, and ③ $x \leq y, y \leq x \Rightarrow x = y$. A chain in P is a totally-ordered subset where $\forall x, y$, either $x \leq y$ or $y \leq x$. Finally, $x \in P$ is maximal if $\nexists y \in P$ such that $x \leq y$ and $x \neq y$.