# 10 Some properties of group actions (10/11)

Recall the following definition from Section 8.

**Definition 10.1** (Trivial actions)**.** Say that an action of $G$ on $X$ is **trivial** if $g \cdot x = x$ for all $x \in X$ and all $g \in G$. This is the case if and only if the adjoint homomorphism $f\colon G \to S_X$ satisfies $f(g) = e$ for all $g \in G$.

At the opposite extreme, we have the faithful actions.

**Definition 10.2.** The action of a group $G$ on a set $X$ is **faithful** if the adjoint homomorphism $G \to S_X$ is injective.

**Remark 10.3.** In other words, an action of $G$ on $X$ is faithful if different elements of $G$ produce different permutations on $X$. Unwinding, this means that for each pair of distinct elements $f, g \in G$ there exists $x \in X$ such that $f \cdot x \neq g \cdot x$.

**Remark 10.4.** If $X$ is a set and $S_X$ is the permutation group of $X$, then any subgroup $G \subseteq S_X$ comes with an action on $X$ which is faithful.

**Example 10.5.** As $D_{2n}$ is a subgroup of $S_n$, its action on $\{1, \ldots, n\}$ is faithful.

**Definition 10.6** (Orbits and stabilizers)**.** Let $G$ be a group acting on a set $X$.

(i) If $x \in X$, the **orbit** of $G$ containing $x$ is the set $G \cdot x = \{g \cdot x | g \in G\}$. Alternatively, if $k\colon G \times X \to X$ denotes the action map, it is the image of $G \times \{x\}$ under $k$.

(ii) If $x \in X$, the **stabilizer** of $x$ in $G$ is the set $G_x = \{g \in G | g \cdot x = x\}$.

**Lemma 10.7.** *If $G$ acts on a set $X$ and if $x \in X$, then the stabilizer $G_x \subseteq G$ is a subgroup.*

*Proof.* Of course, $e \in G_x$. We also have that if $g \in G_x$, then $g^{-1} \in G_x$ as $g^{-1} \cdot x = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = e \cdot x = x$. Similarly, if $g, h \in G_x$, then $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$, so $gh \in G_x$. □

**Example 10.8.** Consider $D_{2n}$ acting on the $n$-gon $X_n$ with vertex set $\{1, \ldots, n\}$ as in Definition 9.1. The orbit of any vertex is $\{1, \ldots, n\}$. What about the orbit of a point on $X$ that is not a vertex? The stabilizer of 1 in $G$ is $G_1 = \{e, s\}$. Indeed, any rotation must "move" 1. Any element $f$ which fixes 1 must either send 2 to itself, in which case $f = 1$ or it sends 2 to $n$ and $n$ to 2, in which case $sf = e$, or $f = s$. The stabilizer of a point which is not a vertex is trivial if $n$ is even and usually trivial if $n$ is odd, the exception being the points opposite to vertices which are fixed by appropriate reflections.

**Philosophy 10.9.** The approach to defining the dihedral group is very helpful in finding new groups. For example, let $T$ in $\mathbf{R}^3$ be a regular tetrahedron with vertex set $\{1, 2, 3, 4\}$. Among all rigid motions of $\mathbf{R}^3$, there are those which act bijectively on $T$, and must send vertices to vertices, edges to edges, and faces to faces. How many are there? I can send 1 to any vertex $i \in \{1, 2, 3, 4\}$, which amounts to four choices of where 1 goes. Once that is fixed, 2 must go to one an element of $\{1, 2, 3, 4\} - \{i\}$, so there are three more choices. But, then it is fixed. For example, if 1 maps to 3 and 2 maps to 1, then one sees by rigidity that 3 maps to 2 and 4 maps to 3.

**Example 10.10** (The conjugation action)**.** Let $G$ be a group. We define a new action of $G$ on itself, given by conjugation. Namely, let $c\colon G \times G \to G$ be defined by $c(g, h) = ghg^{-1}$. This is the result of *conjugating $h$ by $g$*. We have $c(e, h) = ehe^{-1} = h$ for all $h \in G$ and we have $c(f, c(g, h)) = f(ghg^{-1})f^{-1} = (fg)h(fg)^{-1} = c(fg, h)$. So, conjugation defines a group action of $G$ on itself. The conjugation action is always different from the left regular action if $G$ is not the trivial group $\{e\}$.

**Question 10.11.** When is the conjugation action trivial?

**Definition 10.12** (Orbit set)**.** Let a group $G$ act on a set $X$. For $x, y \in X$, write $x \sim y$ if there exists $g \in G$ such that $g \cdot x = y$. This defines an equivalence relation on $X$. Indeed, $e \cdot x = x$ so $x \sim x$ (reflexivity), if $g \cdot x = y$, then $g^{-1} \cdot y = x$ (reflexivity), and if $g \cdot x = y$ and $h \cdot y = z$, then $(hg) \cdot x = z$ (transitivity). The equivalence classes are precisely the orbits. We write $X/G$ for the set of orbits. The quotient function $f \colon X \to X/G$ sends $x \in X$ to $G \cdot x \in X/G$.

**Question 10.13.** What does the orbit set of $D_{2n}$ acting on $X_n$ look like? It is bijective to the half-open line segment $L$ from vertex 1 (inclusive) to vertex 2 (not inclusive). Indeed, for each $x \in X_n$ there is a unique $y$ on $L$ such that $g \cdot x = y$ for some $g \in D_{2n}$. Note that this is the same orbit set as that corresponding to the action of $\mathbf{Z}/n$ on $X_n$ by rotations by multiples of $\frac{2\pi}{n}$.

**Definition 10.14** (Transitive actions)**.** The action of a group $G$ on a set $X$ is **transitive** if $X/G$ is a point or, equivalently, if there is only one orbit or, equivalently, if for all pairs $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$.

## 10.1 Exercises

**Exercise 10.1.** Let $G = S_n$ act on $X = \{1, \ldots, n\}$ via permutations.

(a) What is the orbit $G \cdot 1$?

(b) What is the stabilizer $G_1$ of 1 in $G$? (It is isomorphic to a group we have a name for.)

(c) What is the set of orbits $X/G$?

(d) Is the action faithful?

(e) Is the action transitive?

**Exercise 10.2.** Repeat Exercise 8.2(a)-(e) for the left regular action of a group $G$ on itself (where 1 is replaced by $e$ in parts (a) and (b)).

**Exercise 10.3.** Repeat Exercise 8.2(a)-(e) for the conjugation action of $G = D_8$ on itself (where 1 is replaced by $e$ in parts (a) and (b)).

**Exercise 10.4.** Arguing as in Philosophy 10.9, compute the order of the group of rigid motions of an icosahedron in $\mathbf{R}^3$.