8 The adjoint homomorphism (10/06)

Our next theorem says that group actions of G on X are "the same" as group homomorphisms $G \to S_X$.

Theorem 8.1. *Let G be a group and X as set. There is a bijection*

$$\{\textit{actions } k \textit{ of } G \textit{ on } X\} \xrightarrow{k \mapsto f_k} \mathrm{Hom}(G, S_X),$$

where $\mathrm{Hom}(G, S_X)$ *denotes the set of group homomorphisms from G to S_X.*

Proof. Let $k \colon G \times X \to X$ be a group action; we will write $g \cdot_k x$ for $k(g, x)$ in this proof. For $g \in G$, let $f_k(g)$ be the function $X \to X$ defined by $f_k(g)(x) = k(g, x) = g \cdot_k x$. This is a bijection as one sees by observing that $f_k(g^{-1})$ is an inverse using (a) and (b) from the definition of a group action. Therefore, f_k is a function $G \to S_X$. In fact, this is a group homomorphism. Indeed, $f_k(gh)(x) = gh \cdot_k x = g \cdot_k (h \cdot_k x) = f_k(g)(f_k(h)(x))$ for all $g, h \in G$ and $x \in X$. Therefore, $f_k(gh) = f_k(g) \circ f_k(h)$, as desired.

To show that the assignment $k \mapsto f_k$ is bijective, assume first that k and n are distinct group actions. Then, there exists a pair $(g, x) \in G \times X$ such that $g \cdot_k x \neq g \cdot_n x$. It follows that $f_k(g) \neq f_n(g)$. This shows injectivity.

Given a group homomorphism $f \colon G \to S_X$, we define a new group action k_f of G on X by letting $g \cdot_{k_f} x = f(g)(x)$. By definition, $f_{k_f}(g)(x) = g \cdot_{k_f} x = f(g)(x)$, so $f_{k_f}(g) = f(g)$ for all $g \in G$ and hence $f_{k_f} = f$, which proves surjectivity. □

Definition 8.2. If k is an action of G on X, then $f_k \colon G \to S_X$ is called the **adjoint homomorphism**. If $f \colon G \to S_X$ is a homomorphism, then k_f is called the **action associated to** f.

Example 8.3. Let G be a group and consider its left regular action on itself $m \colon G \times G \to G$. The adjoint homomorphism $\ell = f_m \colon G \to S_G$ is the homomorphism used in the proof of Cayley's Theorem 6.10.

Example 8.4. Recall the group $K = \mathbf{Z}/2 \times \mathbf{Z}/2$, sometimes known as the **Klein four-group**. It has four elements, which we label as follows:

$$\begin{pmatrix} (0,0) & (1,0) & (0,1) & (1,1) \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

The adjoint homomorphism $\ell \colon K \to S_K$ we view, using the labeling above, as a homomorphism $\ell \colon K \to S_4$. A cycle decomposition of $\ell(1,0)$ is $((0,0)\,(1,0))((0,1)\,(1,1)) = (1\,2)(3\,4)$.

8.1 Exercises

Exercise 8.1. Say that an action of a group G on a set X is **trivial** if $g \cdot x = x$ for all $g \in G$ and x on X. Suppose that p is a prime and that X is a set with fewer than p elements. Show that all actions of $\mathbf{Z}/p$ on X are trivial.

Exercise 8.2. Compute the set $\mathrm{Hom}(\mathbf{Z}/2, S_4)$ of group homomorphisms into S_4. Use your computation to describe all group actions of $\mathbf{Z}/2$ on $\{1, 2, 3, 4\}$.