Northwestern University

Real Analysis

MATH 321-2

# uniform continuity go brrrr

*Author:*
Elliott Yoon

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# 1 The Riemann Stieljis Integral

**Remark 1.1.** For this section, let there be a standing assumption that $f$ is bounded.

**Definition 1.2.** Let $[a, b]$ be a given interval. A **partition** $P$ of $[a, b]$ is a finite set of points $\{x_0, x_1, \ldots, x_n\}$ where

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b.$$

We will adopt the following notation: $\Delta x_i = x_i - x_{i-1}$. Now, let $P$ be any partition of $[a, b]$. We put

1. $M_i = \sup f(x) \quad (x_{i-1} \leq x \leq x_i)$,
2. $m_i = \inf f(x) \quad (x_{i-1} \leq x \leq x_i)$,
3. $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$,
4. $L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$,

and finally obtain the *upper* and *lower Riemann integrals* of $f$ over $[a, b]$:

1. $\overline{\int_a^b} f\, dx = \inf_{P \in \mathscr{P}} U(P, f)$,
2. $\underline{\int_a^b} f\, dx = \sup_{P \in \mathscr{P}} L(P, f)$

where $\mathscr{P}$ is the set of all partitions $P$ of $[a, b]$.

**Lemma 1.3**

The set $\{U(P, f) \mid P \in \mathscr{P}\}$ is bounded below.

*Proof.* Since $f$ is bounded, $f(x) \geq m$ for all $x \in [a, b]$. Notice that

$$U(P, f) = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x)\, \Delta x_i \geq \sum_{i=1}^{n} m \cdot \Delta x_i = m(b - a).$$

□

**Definition 1.4.** We say that $f$ is **Riemann-integrable** and write $f \in \mathscr{R}([a, b])$ if

$$\overline{\int_a^b} f\, dx = \underline{\int_a^b} f\, dx.$$

**Remark 1.5.** Notice that $L(P, f)$ and $U(P, f)$ are bounded by $m(b - a)$ and $M(b - a)$, where $m \leq f(x) \leq M$ for all $x \in [a, b]$. In other words, the upper and lower integrals exist for *every* bounded real function.

**Definition 1.6.** Let $\alpha$ be nondecreasing (monotonically increasing) function on $[a, b]$. We write $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. (Clearly, $\Delta \alpha_i \geq 0$. We put

1. $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i$,
2. $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i$.

where $M_i, m_i$ have the same meaning as in Definition 1.2 and we define

$$\overline{\int_a^b} f\,d\alpha = \inf_{P\in\mathscr{P}} U(P,f,\alpha) \tag{1}$$

and

$$\underline{\int_a^b} f\,d\alpha = \sup_{P\in\mathscr{P}} L(P,f,\alpha). \tag{2}$$

If (1) and (2) are equal, then we say $f$ is *integrable with respect to $\alpha$ over $[a,b]$*, written $f \in \mathscr{R}(\alpha)$, and notate their common value, known as the **Riemann-Stieltjes integral** as

$$\int_a^b f\,d\alpha.$$

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**Question 1.7.** When is $f \in \mathscr{R}(\alpha)$

It may be helpful to rephrase the question to ask when $f$ is *not* in $\mathscr{R}(\alpha)$.

- Nonexample: The function
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$
is *not* in $\mathscr{R}([a,b])$. Notice that for any partition $P$, $U(P,f) = 1 \neq 0 = L(P,f)$.

So whenever $\inf_{P\in\mathscr{P}} U(P,f,\alpha)$ is *strictly* greater than $\sup_{P\in\mathscr{P}} L(P,f,\alpha)$, we know $f \notin \mathscr{R}(\alpha)$.

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**Definition 1.8.** For partitions $P, Q \in \mathscr{P}$,

1. If $Q \supset P$, we say $Q$ is a **refinement** of $P$.
2. We call $P^* = P \cup Q$ a **common refinement**.

**Lemma 1.9**
If $Q \supset P$, then $U(Q,f,\alpha) \leq U(P,f,\alpha)$ and $L(Q,f,\alpha) \geq L(P,f,\alpha)$.

*Proof.* Let $Q = P \cup \{x_0,\ldots,x_k\}$. If $k = 0$, the conclusion obviously holds. Now, suppose $k \in \mathbb{N}$ and $U(Q,f,\alpha) \leq U(P,f,\alpha)$, and let $P^*$ contain just one more point than $P$, $x^*$, where $x_{i-1} < x^* < x_i$. Write $w_1 = \sup_{x_{i-1}\leq x\leq x^*} f(x)$ and $w_2 = \sup_{x^*\leq x\leq x_i} f(x)$. Notice $w_1, w_2 \leq M_i$ where $M_i = \sup_{x_{i-1}\leq x\leq x_i} f(x)$. Then

$$\begin{aligned} U(P,f,\alpha) - U(P^*,f,\alpha) &= M_i[\alpha(x_i) - \alpha(x_{i-1}) - w_1[\alpha(x^*) - \alpha(x_{i-1})] - w_2[\alpha(x_i) - \alpha(x^*)] \\ &= (M_i - w_1)[\alpha(x^*) - \alpha(x_{i-1})] + (M_i - w_2)[\alpha(x_i) - \alpha(x^*)] \\ &\geq 0. \end{aligned}$$

The proof for the lower integrals is the same. $\square$

**Remark 1.10.** Notice that for any partitions $P_1, P_2$, it follows with the common refinement $P^* = P_1 \cup P_2$ that

$$L(P_1,f,\alpha) \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P_2,f,\alpha).$$

**Corollary 1.11**

$$\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha.$$

We now arrive at a useful lemma relating integrability to being able to find partitions that allow the distance between upper and lower integrals to be arbitrarily small:

**Lemma 1.12**

$f \in \mathscr{R}(\alpha)$ if, and only if, $\forall \epsilon > 0$, there exists a partition $P$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

*Proof.* Let $f \in \mathscr{R}$. Then there exists a partition $P_1$ such that $0 \leq U(P_1, f, \alpha) - \overline{\int_a^b} f\, d\alpha < \epsilon/2$. Similarly, there exists $P_2$ such that $0 \leq \underline{\int_a^b} f\, d\alpha - L(P_2, f, \alpha) < \epsilon/2$. (Notice that $f \in \mathscr{R}(\alpha)$, so $\overline{\int} f\, d\alpha = \underline{\int} f\, d\alpha = \int f\, d\alpha$.) Let $P = P_1 \cup P_2$ be the common refinement of $P_1$ and $P_2$. Then

$$U(P, f, \alpha) \leq U(P_1, f, \alpha) < \int f\, d\alpha + \epsilon/2 < L(P_2, f, \alpha) + \epsilon \leq L(P, f, \alpha) + \epsilon.$$

Now assume the converse. Recall that $\overline{\int} f\, d\alpha \leq U(P, f, \alpha)$ and $\underline{\int} f\, d\alpha \geq L(P, f, \alpha)$ for any partition $P$. Let $\epsilon > 0$. Then there exists a partition $P$ such that

$$0 \leq \overline{\int} f\, d\alpha - \underline{\int} f\, d\alpha \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

□

Now, let's introduce a bit of notation to make our lives easier. We can write that $f$ is continuous on a metric space $X$ as $f \in \mathscr{C}(X)$. Furthermore, we can improve upon our notation of integrability to write $f \in \mathscr{R}(\alpha, S)$ to mean that $f$ is integrable on with respect to $\alpha$ over $S$.

**Theorem 1.13**

Let $f \in \mathscr{C}([a, b])$. Then $f \in \mathscr{R}(\alpha, [a, b])$.

*Proof.* Notice $[a, b]$ is compact. Thus $f$ is uniformly continuous on $[a, b]$, so for $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{\alpha(b) - \alpha(a)}$ whenever $|x - y| < \delta$. Now, pick a partition $P$ (with $n$ elements) such that $\Delta x_j < \delta$ for all $j$. Then

$$\begin{aligned}
\overline{\int} f\, d\alpha - \underline{\int} f\, d\alpha \leq U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{j=1}^{n} \sup_{I_j} f\, \Delta x_j - \sum_{j=1}^{n} \inf_{I_j} f\, \Delta x_j \\
&= \sum_{j=1}^{n} \left( \sup_{I_j} f - \inf_{I_j} f \right) \Delta x_j \\
&< \sum_{j=1}^{n} \frac{\epsilon}{\alpha(b) - \beta(a)} \Delta x_j \\
&= \frac{\epsilon}{\alpha(b) - \alpha(a)} (\alpha(b) - \alpha(a)) = \epsilon.
\end{aligned}$$

Since $\epsilon$ is arbitrary, the proof is complete. □

**Remark 1.14.** It turns out, we need not require that $f$ is continuous on the entire interval; it suffices for $f$ to be continuous *except at finitely many points*, with $\alpha$ continuous where $f$ is not! Zoo wee mama!

**Remark 1.15.** In fact, the **Lebesgue Criterion for Riemann Integrability** states that

$$f \in \mathscr{R} \iff f \text{ is discontinuous on a set of } \textit{measure zero}!$$

(As a reminder, a set $E$ has measure zero if for $\epsilon > 0$, there exists a collection of intervals $\{I_n\} \supset E$ such that $\sum_n \operatorname{diam}(I_n) < \epsilon$.)

**Theorem 1.16** (The cooler Daniel)

If $f$ is continuous at except finitely many points and $\alpha$ is continuous at the points of $f$'s discontinuity, then $f \in \mathscr{R}(\alpha)$.

*Proof.* Let $f$ be continuous except at finitely many points, say $\{x_0, \ldots, x_n\}$. Because $f$ is continuous at except *finitely* many points, we can let $M = |f|$.

Since the set of discontinuities $S = \{x_0, \ldots, x_n\}$ is finite, $\alpha$ is uniformly continuous on $S$. It follows from the triangle inequality and the monotone increasing property of $\alpha$ that for all $\epsilon > 0$ there exists a $\delta > 0$ such that $\alpha(x_j + \delta) - \alpha(x_j - \delta) < \epsilon$.

We can always choose $\delta$ to be smaller, so without loss of generality, assume the set of $[x_j - \delta, x_j + \delta]$ is disjoint. Let $F = [a, b] \setminus \bigcup_{i=1}^{n} (x_j - \delta, x_j + \delta)$. $F$ is compact, so for all $\epsilon > 0$, there exists $\delta' > 0$ such that $|f(u) - f(v)| < \epsilon$ for all $u, v \in F$ where $|u - v| < \delta'$.

We can now partition $F$ into intervals $I_j$ with $\Delta x_j < \delta'$. Let $J_i = [x_i - \delta, x_i + \delta]$. We can now partition $[a, b]$ into a partition $P$ consisting of the $I_i$'s and $J_i$'s. Then

$$\begin{aligned}
\overline{\int} f \, d\alpha - \underline{\int} f \, d\alpha &\leq \sum_j \sup_{I_j} f \, \Delta x_j - \sum_j \inf_{I_j} f \, \Delta x_j + \sum_j \sup_{J_j} f \, \Delta x_j - \sum_j \inf_{J_j} f \, d\alpha \\
&= \sum_j \left( \sup_{I_j} f - \inf_{I_j} f \right) \Delta x_j + \sum_j \left( \sup_{J_j} f - \inf_{J_j} f \right), \Delta x_j \\
&\leq \epsilon \sum_j \Delta x_j + \sum_j 2M\epsilon \\
&= K\epsilon,
\end{aligned}$$

where $K \in \mathbb{R}$. ∎

**Remark 1.17.** What if we want to compose functions? Will their composition be integrable? Well it turns out that if the inner function is integrable, then the outer function being continuous on the range of the inner function is sufficient for integrability of the composition.

**Theorem 1.18** (Integrability of composition of functions)

If $f$ takes values in $[m, M]$ on $[a, b]$, $f \in \mathscr{R}(\alpha, [a, b])$, and $\phi$ continuous on $[m, M]$, then $\phi \circ f \in \mathscr{R}(\alpha, [a, b])$.

The proof for this theorem is pretty funny, so hang on.

*Proof.* $\phi$ is uniformly continuous on $[m, M]$ (why?) so for some $\epsilon > 0$ there exists a $\delta < \epsilon$ such that $|\phi(u) - \phi(v)| < \epsilon$ whenever $|u - v| < \delta$. Note that if we find a sufficiently small $\delta$, then any value less than $\delta$

also works so we can restrict ourselves to only working with $\delta < \epsilon$. It turns out, this restriction will become very useful later on!

Since $f \in \mathscr{R}(\alpha)$, it follows from **Lemma 1.12** that there exists a partition $P$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2.$$

For each $j = 1, \dots, n$ (where $n = |P|$), if $\sup_{I_j} f - \inf_{I_j} f < \delta$, place $j \in A$. Otherwise place $j \in B$.

1. If $j \in A$, then $|\phi(f(x)) - \phi(f(y))| < \epsilon$, $x, y \in I_j$.
2. If $j \in B$, then $\sup_{I_j}(\phi \circ f) - \inf_{I_j}(\phi \circ f) \le 2 \sup_{[m,M]} |\phi|$, and let's notate $K = \sup_{[m,M]} |\phi|$.

   But $U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$, so

   $$\sum_{j \in B} \delta \Delta\alpha_j \le \sum \left( \sup_{I_j} f - \inf_{I_j} f \right) \Delta x_j \le \delta^2.$$

   Dividing both sides by $\delta$, we get $\sum_{j \in B} \Delta\alpha_j < \delta$.

Thus

$$\begin{aligned}
\overline{\int} \phi \circ f \, d\alpha - \underline{\int} \phi \circ f \, d\alpha &\le U(P, \phi \circ f, \alpha) - L(P, \phi \circ f, \alpha) \\
&\le \sum_{j=1}^{n} \epsilon \Delta\alpha_j + \sum_{j \in B} 2K \Delta\alpha_j \\
&\le \epsilon(\alpha(b) - \alpha(a)) + 2K\delta \\
&< \epsilon(\alpha(b) - \alpha(a) + 2K).
\end{aligned}$$

□

> **Remark 1.19.** Note that we (stupidly, in the words of Jared Wunsch,) overcount in the third-to-last line of the extended equation; summing over *all* $j$ instead of just $j \in A$.

> **Remark 1.20.** You've probably caught on to the style of proving a function is integrable: find a partition such that the difference $U - L$ is bounded above by an arbitrary $\epsilon$.

We will now explore the properties of the integral, which pretty much agree with the intuition of someone who studied linear algebra and multivariate calculus with Aaron Peterson in MATH 291 @ Northwestern University:

1. The integral is *linear* over $\mathbb{R}$;
2. If a function bounds another from above, then the integral of the first will bound the integral of the second from above;
3. We can split integrals by an intermediate bound;
4. If the magnitude of a function is bounded by a finite number $M$, then the magnitude of the integral of that function will by bounded by the product of $M$ and the width of the integral's bounds.
5. The sum of functions integrable with respect to different "clock speeds" is integrable with respect to the sum of their individual clock speeds. (Really pushing the metaphore here..)

**Theorem 1.21** (Rudin 6.12)

1. If $f_1, f_2 \in \mathscr{R}(\alpha)$ then $f_1 + f_2 \in \mathscr{R}(\alpha)$, $cf \in \mathscr{R}(\alpha)$ for every $c \in \mathbb{R}$, and
$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha \quad \text{and} \quad \int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha.$$
2. If $f_1(x) \le f_2(x)$ on $[a, b]$, then
$$\int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha.$$
3. If $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and $a < c < b$, then $f \in \mathscr{R}(\alpha)$ on $[a, c]$ and $[c, b]$, and
$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha.$$
4. If $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and if $|f(x)| \le M$ on $[a, b]$, then
$$\left|\int_a^b f\, d\alpha\right| \le M[\alpha(b) - \alpha(a)];$$
5. If $f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$, then $f \in \mathscr{R}(\alpha_1 + \alpha_2)$ and
$$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2;$$
If $f \in \mathscr{R}(\alpha)$ and $c \in \mathbb{R}^+$, then $f \in \mathscr{R}(c\alpha)$ and
$$\int_a^b f\, d(c\alpha) = c\int_a^b f\, d\alpha.$$

*Proof.* The proofs for each part are very similar, so we will only prove (1). However Wunsch messed up here so we'll skip this for now. A proof is in Rudin if you really want to read it. ☐

The previous theorem (**Rudin 6.12**) gives us a lot of power to determine the integrability of functions; we just need to be adept at manipulating expressions into sums and compositions of continuous functions. Thankfully, $x \to x^2$ is continuous and we have a useful identity to translate multiplication into addition:

**A useful identity:** $xy = \frac{1}{4}\left((x + y)^2 - (x - y)^2\right)$.

**Theorem 1.22**

Let $f, g \in \mathscr{R}(\alpha)$. Then

1. $fg \in \mathscr{R}(\alpha)$,
2. $|f| \in \mathscr{R}(\alpha)$, and
3. $\left|\int_a^b f\, d\alpha\right| \le \int_a^b |f|\, d\alpha$.

*Proof.* Notice that $f \pm g \in \mathscr{R}(\alpha)$, so $(f \pm g)^2 \in \mathscr{R}(\alpha)$. Then

$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right) \in \mathscr{R}(\alpha).$$

Since $u \to u^2$ is continuous, $|f| \in \mathscr{R}(\alpha)$. Finally, there exists a $c = \pm 1$ where

$$\left|\int f\, d\alpha\right| = c\int f\, d\alpha = \int cf\, d\alpha \leq \int |f|\, d\alpha.$$

□

**Example 1.23** (Heaviside Function)
We define the **Heaviside Function** as

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 1 \end{cases}$$

If $a < 0 < b$ and $f$ is continuous at $x = 0$, then $f \in \mathscr{R}([a,b], H)$ and $\int_a^b f\, dH = f(0)$.

*Proof.* Again, we choose a funny partition that will result in some clean shit: Let $P = \{x_0, x_1, x_2, x_3\}$, where $x_0 = a, x_1 = 0, x_3 = b$, and $x_2 \in (0, b)$. Then

$$\begin{aligned} U(P,f,H) &= \sup_{[a,0]} f \cdot (H(0) - H(a)) + \sup_{[0,x_2]} f \cdot (H(x_2) - H(0)) + \sup_{[x_2,b]} f \cdot (H(b) - H(x_2)) \\ &= \sup_{[a,0]} f \cdot (0-0) + \sup_{[0,x_2]} f \cdot (1-0) + \sup_{[x_2,b]} f \cdot (1-1) \\ &= \sup_{[0,x_2]} f. \end{aligned}$$

Similarly, $L(P,f,H) = \inf_{[0,x_2]} f$. Letting $x_2$ approach 0 from the right, notice that $U(P,f,H) \to f(0)^+$ and $L(P,f,H) \to f(0)^-$. So $\int_a^b f\, dH = 0$. □

**Corollary 1.24** (Basically Heaviside, with **linearity!**)
Let $\alpha = \sum_{j=1}^N c_j H(x - s_j)$, $s \in [a,b]$, and $f \in \mathscr{C}([a,b])$. Then

$$\int_a^b f\, d\alpha = \sum_{i=1}^N c_j f(s_j)$$

*Proof.* Immediate by Theorem 1.20. □

**Remark 1.25.** Rudin extends $\alpha$ to be an infinite sum, but we don't need to get that crazy here...

**Theorem 1.26**
Say $\alpha'$ exists for all $x \in [a,b]$, $\alpha'$ is bounded, and $f$ is *Riemann*-integrable (i.e. $f \in \mathscr{R}([a,b], x)$. Then $f \in \mathscr{R}(\alpha)$. If $\alpha' \in \mathscr{R}([a,b], x)$, then $\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx$.

*Proof.* Let $\alpha'$ be bounded for all $x \in [a,b]$ and $f \in \mathscr{R}([a,b],x)$. For $\epsilon > 0$, there exists a partition $P$ of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon/K$, where $K = \sup_{[a,b]} \alpha'$. Now, note that for any $j \in P$, the Mean Value Theorem implies there exists some $x_j^* \in [x_{j-1}, x_j]$ such that $\alpha(x_j) - \alpha(x_{j-1}) = \alpha'(x_j^*)\Delta x_j$. With this, notice

$$\begin{aligned} U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{j\in P} \left( \sup_{I_j} f - \inf_{I_j} f \right) (\alpha(x_j) - \alpha(x_{j-1})) \\ &= \sum_{j\in P} (M_j - m_j)\alpha'(x_j^*)\Delta x_j \\ &\leq K \sum_{j\in P} (M_j - m_j)\Delta x_j \\ &< K \cdot \epsilon/K = \epsilon. \end{aligned}$$

Thus $f \in \mathscr{R}(\alpha)$. Now, let $\alpha' \in \mathscr{R}([a,b],x)$. Then for $\epsilon > 0$, there exists a partition $P$ such that

1. $U(P,f) - L(P,f) < \epsilon$,
2. $U(P,f,\alpha') - L(P,f,\alpha') < \epsilon$,
3. $\sum_{j\in P} \left( \sup_{I_j} \alpha' - \inf_{I_j} \alpha' \right) \Delta x_j < \epsilon$, or $U(P,\alpha') - L(P,\alpha') < \epsilon$, and
4. $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$, shown earlier in the proof.

Now, for each $j \in P$ pick any $u_j \in I_j$. By the Mean Value Theorem, there exists some $x_j^* \in I_j$ such that

$$\sum_{j\in P} f(u_j)\Delta\alpha_j = \sum_{j\in P} f(u_j)\alpha'(x_j^*)\Delta x_j = \left( \sum_{j\in P} f(u_j)\alpha'(u_j)\Delta x_j \right) + \left( \sum_{j\in P} f(u_j)(\alpha'(x_j^*) - \alpha'(u_j))\Delta x_j \right).$$

Define, for sake of brevity, $A = \sum_{j\in P} f(u_j)\alpha'(u_j)\Delta x_j$ and $B = \sum_{j\in P} f(u_j)(\alpha'(x_j^*) - \alpha'(u_j))\Delta x_j$. (These are the last two sums in the previous equation.) Letting $M = \sup|f|$, we can bound $B$ as follows:

$$|B| \leq \sum_{j\in P} \sup|f| \cdot \left( \sup_{I_j} \alpha' - \inf_{I_j} \alpha' \right) \Delta x_j \leq M\epsilon.$$

Since $L(P,f,\alpha) \leq A + B \leq U(P,f,\alpha)$, we have $L(P,f,\alpha) - M\epsilon \leq A \leq U(P,f,\alpha) + M\epsilon$, and thus

$$A - \int_a^b f\,d\alpha < \epsilon + M\epsilon.$$

Since $A$ is a Riemann Sum, we also have

$$\left| A - \int_a^b f\alpha'\,dx \right| < \epsilon.$$

Combining all our Pokémon card inequality cards collected throughout the proof, we finally get

$$\left| \int_a^b f\,d\alpha - \int_a^b f\alpha'\,dx \right| < \epsilon + M\epsilon + \epsilon.$$

□

# 2 Integration and Differentiation

We will explore the dynamics between integration and differentiation, and as expected, the two act as quasi-inverse functions.

**Theorem 2.1** (Fundamental Theorem of Calculus 1)
Let $f \in \mathscr{R}([a,b])$ and $f$ be continuous at a point $x_0 \in [a,b]$. Then

$$f(x_0) = \frac{d}{dx}\int_a^x f(s)\,ds\bigg|_{x=x_0}.$$

*Proof.* Differentiating our funny integral, we have that

$$\frac{d}{dx}\int_a^x f(s)\,ds = \lim_{h\to 0}\frac{\int_0^{x_0+h} f(s)\,ds - \int_a^{x_0} f(s)\,ds}{h}.$$

We now have to inspect both right and left hand limits, but as the proofs for each case are analogous, we'll just look at the right hand limit: $h \to 0^+$. Since $f$ is continuous at $x_0$, for $\epsilon > 0$, there exists $\delta > 0$ such that if $0 \le |y - x_0| < \delta$, then $|f(y) - f(x_0)| < \epsilon$. Since we're taking the limit as $h$ approaches 0, we can limit our choice of $h$ to only those with $h < \delta$. Pick any of them. Then $|f(s) - f(x_0)| < \epsilon|$ for all $s \in (x_0, x_0 + h)$.

We'll now employ a slick trick: since $f(x_0)$ is constant, we can write $f(x_0) = \frac{1}{h}\int_{x_0}^{x_0+h} ds$.

Then

$$\begin{aligned}\left|\frac{1}{h}\int_{x_0}^{x_0+h} f(s)\,ds - f(x_0)\right| &= \left|\frac{1}{h}\int_{x_0}^{x_0+h} f(s)\,ds - \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)\,ds\right| \\ &= \left|\frac{1}{h}\int_{x_0}^{x_0+h} f(s) - f(x_0)\,ds\right| \\ &\le \frac{1}{h}\int_{x_0}^{x_0+h} |f(s) - f(x_0)|\,ds \\ &< \frac{1}{h}\int_{x_0}^{x_0+h} \epsilon\,ds \\ &= \epsilon.\end{aligned}$$

□

**Remark 2.2.** Notice that $F(x) = \int_a^x f(s)\,ds$ is continuous on $[a,b]$.

*Proof.* By continuity, if $x < y$ then

$$|F(x) - F(y)| = \left|\int_x^y f(s)\,ds\right| \le \int_x^y |f(s)|\,ds \le \sup|f|(y-x).$$

So for $\epsilon > 0$, take $\delta = \epsilon/\sup|f|$. □

**Continuous things have antiderivatives!**

**Theorem 2.3** (Fundamental Theorem of Calculus 2 (le célèbre))
Let $f \in \mathscr{R}([a,b])$, and there exist a differentiable $F$ such that $F' = f$ on $[a,b]$. Then

$$\int_a^b f(s)\,ds = F(b) - F(a).$$

> **Remark 2.4.** Recall that integrable functions need not be continuous. (What's an example of a finitely discontinuous function that is is Riemann-integrable?) However, the large majority of commonly used integrable functions are continuous, so we'll prove this theorem for continuous functions first, and then weaken our hypothesis for the *real* proof.

*Proof.* *(naive)* Let $f$ be continuous, and set $G(x) = \int_c^x f(s)\,ds$. By FTC1,

$$\frac{d}{dx}G(x) = f(x) = F'(x),$$

so $G(x) = F(x) + C$, where $C$ is constant. Thus

$$\int_a^b f(s)\,ds = \int_c^b f(s)\,ds - \int_c^a f(s)\,ds = G(b) - G(a) = F(b) - F(a).$$

$\square$

*Proof.* *(The real one..)* You know the drill: For $\epsilon > 0$, there exists a partition $P = \{x_0, x_1, \dots, x_n\}$ such that $U(P,f) - L(P,f) < \epsilon$. By the Mean Value Theorem, there exists a $x_j^* \in [x_{j-1}, x_j]$ such that $F(x_j) - F(x_{j-1}) = f(x_j^*)\Delta x_j$. Then

$$F(b) - F(a) = \sum_{i=1}^n F(x_j) - F(x_{j-1}) = \sum_{i=1}^n f(x_j^*)\Delta x_j.$$

Thus,

$$\left| F(b) - F(a) - \int_a^b f(s)\,ds \right| < \epsilon.$$

$\square$

We now approach the topic of **integration by parts**, which is often thought of as a computational integration tool by calculus students. As it turns out, it also carries much importance in analysis by showing that one can move derivatives around inside the integrand at the cost of a negative sign:

**Theorem 2.5** (Integration by Parts)
Say $F, G$ are differentiable functions, $F' = f$, $G' = g$, and $f, g \in \mathscr{R}$. Then

$$\int_a^b Fg\,dx = FG\Big|_a^b - \int_a^b fG\,dx.$$

*Proof.* By the chain rule, $(FG)' = Fg + fG$. Rearrange to isolate $Fg$ and apply FTC2. $\square$

**Corollary 2.6**

If $G = 0$ and $a, b$, then $\int_a^b FG'\,dx = -\int_a^b F'G\,dx$.

Finally, we introduce machinery that will facilitate changing the bounds of integration. In doing so, we must account for the "stretch" factor when stretching or shrinking the region of integration.

**Theorem 2.7** (Change of Variables)

Let $\phi : [a, b] \to [A, B]$ be strictly increasing, where $\phi(a) = A$ and $\phi(b) = B$. Let $\phi$ be differentiable, with $\phi' \in \mathscr{R}$, and $f : [A, B] \to \mathbb{R}$ be continuous. Then

$$\int_a^b f(\phi(x))\phi'(x)\,dx = \int_A^B f(y)\,dy.$$

*Proof.* Set $F(x) = \int_A^x f(s)\,ds$. By FTC1, $F' = f$. By the chain rule, $\frac{d}{dx}F(\phi(x)) = f(\phi(x))\phi'(x)$. By FTC2, we have

$$\int_a^b f(\phi(x))\phi'(x)\,dx = \frac{d}{dx}\int_a^b F(\phi(x)) = F(\phi(x))\Big|_a^b = \int_A^B f(s)\,ds.$$

□

## 2.1 Appendix

There are some arguments utilized throughout the section worth having in writing for posterity:

**Theorem 2.8**

Recall that $F \in \mathscr{R}(\alpha)$ on $[a, b]$ if and only if for every $\epsilon > 0$ there exists a partition $P$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon. \tag{3}$$

It is now the case that

1. If (3) holds for some $P$ and $\epsilon$, then (3) holds (with the same $\epsilon$) for *every* refinement of $P$.
2. If (3) holds for $P = \{x_0, \dots, x_n\}$ and if $s_i, t_i$ are arbitrary points in $[x_{i-1}, x_i]$, then
$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i < \epsilon.$$
3. If $f \in \mathscr{R}(\alpha)$ and the hypothesis of (2) hold, then
$$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha\right| < \epsilon.$$

*Proof.* Immediate after noting $\sum_{i=1}^n |f(s_i) - f(t_i)|\Delta x_i \le U(P, f, \alpha) - L(P, f, \alpha)$,

$$L(P, f, \alpha) \le \sum f(t_i)\Delta\alpha_i \le U(P, f, \alpha) \quad \text{and} L(P, f, \alpha) \le \int f\,d\alpha \le U(P, f, \alpha).$$

□

**Theorem 2.9**

If $f$ is monotonic on $[a, b]$ and $\alpha$ is continuous on $[a, b]$, then $f \in \mathscr{R}(\alpha)$.

*Proof.* Monotonic functions are discontinuous at most countably many times. Countable subsets have measure zero, so we're done. Thanks, Lebesgue! □

# 3 Sequences and Series of Functions

Say $f_n : E \to \mathbb{C}$ are functions.

**Definition 3.1.** The sequence $\{f_n\}$ **converges pointwise** on $E$ (to $f(x)$) if for all $x \in E$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

**Question 3.2.** What good properties of $f_n$ might $f$ inherit?

LMAO none. Anyway, an example:

**Example 3.3**

Let $f_n = \arctan(nx) \subset \mathbb{R}$. $f_n$ converges pointwise to

$$f(x) = \begin{cases} -\pi/2 & x < 0 \\ 0 & x = 0 \\ \pi/2 & x > 0. \end{cases}$$

So $f_n$ is infinitely differentiable, but $\lim_{n\to\infty} f(n)$ is not even continuous!

**Definition 3.4.** $f_n$ **converges uniformly** on $E$ if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$ and $x \in E$, then

$$|f_n - f(x)| < \epsilon.$$

**Remark 3.5.** $N$ is *independent* of $x$! (What else does this independence remind you of?)

**Example 3.6** (3.3, revisited.)

Pick $\epsilon = \frac{\pi}{4}$. Given $N$, there exists $x > 0$ such that $\arctan(Nx) < \frac{\pi}{4}$ (since $\lim_{n\to\infty} \arctan(Nx) = 0$). Then

$$\left| f_N(x) - f(x)| < \frac{\pi}{4} \right|.$$

**Theorem 3.7** (Cauchy Criterion for sequences of functions, kinda.)

$f_n \to f$ uniformly on $E$ if, and only if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $m, n \geq N$, for all $x \in E$,

$$|f_m(x) - f_n(x)| < \epsilon.$$

*Proof.* Say $f_n \to f$ uniformly. Then for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $x$. Then for $m \geq N$, we obtain the same inequality and the proof follows directly from the triangle inequality. Conversely, suppose for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n \geq N$, for all $x \in E$, $|f_m(x) - f_n(x)| < \frac{\epsilon}{2}$. Then for all $x$, if we fix $m$, $\{f_n(x)\}$ is Cauchy in $\mathbb{C}$. By completeness of $\mathbb{C}$, there exists a $f(x)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$. By uniform convergence, for all $m \geq N$

$$\lim_{n\to\infty} |f_m(x) - f_n(x)| = |f_m(x) - f(x)| \leq \frac{\epsilon}{2}.$$

□

**Theorem 3.8**

If $f_n$ are continuous functions on $X$, a metric space, and $f_n \to f$ uniformly on $X$, then $f$ is continuous.

*Proof.* Fix $y \in X$. Then for $\epsilon > 0$:

1. There exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $n \geq N$ and $x \in X$.
2. If $f_N$ continuous, there exists $\delta > 0$ such that $|f_N(x) - f_N(y)| < \frac{\epsilon}{3}$ whenever $d(x, y) < \delta$.

Now for all $x$ such that $d(x, y) < \delta$, (1) and (2) give

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}.$$

□

We will now introduce an important notion of distance between functions, which will nicely lead to the notion of a metric space of continuous functions!

**Definition 3.9.** If $f : X \to \mathbb{C}$ is bounded, set $\|f\| = \sup |f|$, where $X$ is a nonempty metric space. Let

$$\mathscr{C}(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous and bounded}\}.$$

For $f, g \in \mathscr{C}(X)$, we define $d_{\mathscr{C}}(f, g) = \|f - g\|$.

Of course, we wouldn't be defining a distance function if we didn't think we could use it as a metric...

**Lemma 3.10**

$\|f + g\| \leq \|f\| + \|g\|$.

*Proof.* As one would intuitively expect,

$$\|f + g\| = \sup |f(x) + g(x)| \leq \sup(|f(x)| + |g(x)|) \leq \sup |f| + \sup |g| = \|f\| + \|g\|.$$

□

**Proposition 3.11**

$d_{\mathscr{C}}$ is a metric on $\mathscr{C}(X)$.

*Proof.* Using the lemma,

- $d_{\mathscr{C}}$ is symmetric since $|f-g| = |g-f|$.
- $d_{\mathscr{C}}(f,g) = 0 \iff \sup|f-g| = 0 \iff |f(x)-g(x)| = 0 \quad \forall x \iff f = g$.
- $d(f,h) = \|f-h\| = \|f-g+g-h\| \leq \|f-g\| + \|g-h\| = d(f,g) + d(g,h)$.

□

Sick, so $(\mathscr{C}(X), d_{\mathscr{C}})$ is a metric space.

But what's the point of going through all this work to verify this fact?

**Proposition 3.12** (Convergence in $(\mathscr{C}(X), d_{\mathscr{C}})$ is analogous to uniform convergence of functions.)
$f_n \to f$ in $\mathscr{C}(X)$ if, and only if, $f_n \to f$ uniformly.

*Proof.* Let $f_n \to f$ in $\mathscr{C}(X)$. Then for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup|f(x) - f_n(x)| = \|f_n - f\| < \epsilon$$

whenever $n \geq N$. Thus, $|f(x) - f_n(x)| < \epsilon$ for all $x$. Conversely, let $f_n \to f$ uniformly. Then for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, $x \in X$, then

$$|f_n(x) - f(x)| < \frac{\epsilon}{2},$$

so

$$\|f_n - f\| = \sup|f_n(x) - f(x)| \leq \frac{\epsilon}{2}.$$

□

**Theorem 3.13** (this seems important)
$\mathscr{C}(X)$ is complete.

*Proof.* Say $\{f_n\}$ is Cauchy in $\mathscr{C}(X)$. Then for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n \geq N$, then

$$\|f_n - f_m\| = \sup|f_n - f_m| < \epsilon.$$

Thus for all $x \in X$, $|f_n(x) - f_m(x)| < \epsilon$ and thus $\{f_n\}$ is uniformly convergent to some $f(x)$. By the previous proposition, $f$ is continuous. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$, $|f_n - f(x)| < 1$ for all $x$, and thus $|f(x)| < 1 + |f_n(x)|$, so $f$ is bounded. □

**Some notation:** If $E \subset \mathbb{R}$, we write $\mathscr{C}^k(E) = \{f : E \to \mathbb{C} \mid f, f', \ldots, f^{(k)} \in \mathscr{C}(E)\}$. (Notice $\mathscr{C}^0(E) = \mathscr{C}(E)$.)

**Theorem 3.14**
Let $\alpha$ be nondecreasing on $[a,b] \subset \mathbb{R}$, $f_n \in \mathscr{R}(\alpha)$ for all $n$, and assume $f_n \to f$ uniformly. Then

1. $f \in \mathscr{R}(\alpha)$, and
2. $\lim_{n\to\infty} \int_a^b f_n \, d\alpha = \int_a^b f \, d\alpha$.

*Proof.* TBD. □

## 3.1 I missed class

wip: need to catch up on the lecture I missed (1/13/2023)

**Definition 3.15.** Two notions of **boundedness**:

1. A sequence of functions $\{f_n\} \in \mathbb{C}(X)$ is said to be **pointwise bounded** if for all $x \in X$, there exists $C(x)$ such that $|f_n(x)| \leq C(x)$ for all $x$.
2. A sequence of functions $\{f_n\} \in \mathbb{C}(X)$ is said to be **uniformly bounded** if for all $x \in X$, there exists a constant $M$ such that $|f_n(x)| \leq M$ for all $x$.

---

**Definition 3.16.** A family $\mathscr{F}$ of complex functions $f$ defined on a set $E$ of a metric space $X$ is said to be **equicontinuous** on $E$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

whenever $d(x, y) < \delta$, where $x, y \in E$, $f \in \mathscr{F}$.

Indeed, it is the case that uniform convergence of sequences of functions and this notion of equicontinuity are related to one another.

**Theorem 3.17**

If $K$ is compact set, $f_n \in \mathscr{C}(K)$ and $f_n \to f$ uniformly on $K$ for $n \in \mathbb{N}$, then $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous.

*Proof.* For $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $x \in K$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

whenever $n \geq N$. Furthermore, since $f_n \to f$ uniformly, $f$ is continuous; because $K$ is compact, we even have $f$ uniformly continuous. So there exists a $\delta' > 0$ such that if $d(x, y) < \delta'$, then

$$|f(x) - f(y)| < \frac{\epsilon}{3}.$$

So, if $d(x, y) < \delta'$ and $m, n \geq N$, we have

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)| < \epsilon.$$

Finally, $f_1, \ldots, f_N$ are continuous on a compact metric space, so they are uniformly continuous. Thus for each $1 \leq j \leq N$, there exists a $\delta_j > 0$ such that

$$|f_j(x) - f_j(y)| < \epsilon \quad (j = 1, \ldots, N).$$

Set $\delta = \min(\delta', \delta_1, \ldots, \delta_N)$, and we're done. □

**Lemma 3.18**

If $K$ is compact, then there exists countable dense subset of $K$.

*Proof.* We can cover $K$ with $\{B(x, \frac{1}{n}) \mid x \in K\}_{n\in\mathbb{N}}$. (Here, we shall abuse the notation $B(a, b)$ to represent the neighborhood of radius $b$ centered at $a$, and $x_j^n$ to represent the element $x_j$ with corresponding neighborhood $B(x_j, \frac{1}{n})$.) For any $n \in \mathbb{N}$, there exists a finite subcover $\{B(x_j^n, \frac{1}{n}\}$ of balls of fixed radius $\frac{1}{n}$. Then for all $n \in \mathbb{N}$ and $y \in K$, there exists $x_j^n$ such that $d(y, x_j^n) < \frac{1}{n}$. Take

$$S = \{x_j^n \mid n \in \mathbb{N}, j = 1, \ldots, N\}.$$

□

**Lemma 3.19**

Given countable $S$ and uniformly bounded sequence of functions $f_n$, there exits subsequence converging to every element of $S$

*Proof.* Let $S = \{x_1, \dots\}$. Since the sequence $f_n(x_1)$ is bounded in $\mathbb{C}$ (i.e $\sup |f_n| \leq M$ for all $n \in \mathbb{N}$), there exists a subsequence $f_{n_j^1}$ such that $f_{n_j^1}(x_1)$ converges. (Abusing more notation, let the sequence of subindices $n_j^n$ only consist of the sequence of indices $n_j^m$ if $m < n$.) Similary, because $f_{n_j^1}(x_2) \leq M$ for all $j$, there exists a subsequence $n_j^2$ of $n_j^1$ such that $f_{n_j^2}(x_2)$ converges. Since $f_{n_j^2}$ is a subsequence of $f_{n_j^1}$, it also converges at $x_1$. Then $f_{n_j^3}$ converges at $x_3$ (and thus at $x_1$ and $x_2$); by induction, it can be seen that $f_{n_j^k}$ converges at $x_1, \dots, x_k$. Now, to obtain an explicit subsequence, we shall **diagonalize** (recall Cantor's diagonalization argument from proving the countable union of countable sets is countable!), by setting

$$g_j = f_{n_j^j}.$$

For all $k$, if $j \geq k$, then $f_{n_j^j}$ is a subsequence of $f_{n_j^k}$, so $g_j(x_1), \dots, g_j(x_j)$ converge as $j \to \infty$. □

**Theorem 3.20** (Arzelà–Ascoli)

$K$ compact, $f_n \in \mathscr{C}(K)$ for $n = 1, 2, \dots$. Assume $\{f_n\}_{n \in \mathbb{N}}$ bounded in $\mathscr{C}(K)$ (i.e. uniformly bounded) and equicontinuous. Then there exists a convergent subsequence in $\mathscr{C}(K)$ (i.e. uniformly convergent).

*Proof.* Using Lemma 3.18, pick a countable and dense $S \subset K$. Using Lemma 3.19, pick a subsequence $g_j$ of $f_n$ converging on $x$ for all $x \in S$. If we show $g_j$ is uniformly convergent on $K$, then we're done:

We can do so by showing $g_j$ is uniformly Cauchy. It follows from equicontinuity that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) < \delta$, then for all $j$,

$$|g_j(x) - g_j(y)| < \frac{\epsilon}{3}. \tag{4}$$

By Lemma 3.18, we can obtain, from the open cover $\{B(x_j, \delta) \mid x_j \in S\}$, a finite subcover $\{B(x, \delta) \mid x \in S_\delta\}$, where $S_\delta$ is finite. Now, since $g_j$ converges to every element of $S$, there exists $N \in \mathbb{N}$ such that for all $x \in S_\delta$

$$|g_n(x) - g_m(x)| < \frac{\epsilon}{3} \tag{5}$$

whenever $m, n \geq N$. (Take $N = \max\{N_x \mid x \in S_\delta\}$.) Now, for all $i, j \geq N$, $y \in K$, there exists $x \in S_\delta$ such that $d(y, x) < \delta$, so (4) and (5) give

$$\begin{aligned} |g_i(y) - g_j(y)| &\leq |g_i(y) - g_i(x)| + |g_i(x) - g_j(x)| + |g_j(x) - g_j(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

□

**Remark 3.21.** Rudin displays the Arzelà–Ascoli theorem with slightly weaker conditions: he requires $\{f_n\}$ to be a pointwise bounded sequence of complex functions on a countable set. However, pointwise boundedness is almost never used, so we'll choose not to think about it.

**Remark 3.22.** TODO: bounds on derivatives (or difference quotients) give equicontinuity. (MVT probably comes into play.....)

# 4 A Special Function

**Definition 4.1.** For $z \in \mathbb{C}$, we define a **power series** to be the infinite series

$$\sum_{n=0}^{\infty} c_n z^n = \lim_{n\to\infty} \sum_{n}^{N} c_n z^n. \tag{6}$$

**Lemma 4.2** (Weierstrauss M-Test)
Consider the series of functions $\sum_{j=0}^{\infty} f_j(x)$. If there exists $M_j$ such that $\sup |f(x)| \le M_j$ and $\sum M_j < \infty$, then $\sum_{j=0}^{\infty} f_j(x)$ converges uniformly.

*Proof.* Let $s_n = \sum_{j=0}^{n} f_j(x)$. For $m < n$,

$$|(s_n - s_m)(x)| \le \sum_{j=m+1}^{n} |f_j(x)| \le \sum_{j=m+1}^{n} M_j,$$

and if $\sum M_j$ converges, then for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\sum_{j=m+1}^{n} M_j < \epsilon$ if $m, n \ge N$. $\square$

**Theorem 4.3**
There exists an $R \in [0, +\infty)$ such that

1. (6) converges absolutely for all $z \in \mathbb{C}$ with $|z| < R$, and converges uniformly on $\{z \mid |z| \le R'\}$ for all $0 \le R' < R$.
2. (6) diverges for $|z| > R$, with no information on $R$.

*Proof.* Recall that $\sum a_n$ converges if $\limsup |a_n|^{\frac{1}{n}} =: \alpha < 1$, and diverges if $\alpha > 1$. Now, notice that

$$\limsup |c_n z^n|^{\frac{1}{n}} = |z| \limsup |c_n|^{\frac{1}{n}} = \frac{|z|}{R}.$$

If $\frac{|z|}{R} < 1$, we get absolute convergence (and divergence if $\frac{|z|}{R} > 1$). We'll now check uniform convergence on the closure of $B(0, R')$: If $R' < R$, then $\frac{1}{R} < \frac{1}{R'}$. Pick $s$ to be sandwiched such that $\frac{1}{R} < s < \frac{1}{R'}$. Now, $\limsup |c_n|^{\frac{1}{n}} = \frac{1}{R}$, so there exists $N \in \mathbb{N}$ such that if $n \ge N$, then $|c_n|^{\frac{1}{n}} \le s$. So, for $n \ge N$, if $|z| < R'$, observe

$$|c_n z^n| \le s_n (R')^n = (sR')^n < 1.$$

Defining $\beta := sR'$, we have $|c_n z^n| < \beta^n$, so $\beta < 1$ and we get uniform convergence for $|z| \le R'$ by the M-test! $\square$

**Theorem 4.4**

Fix a series $\sum_n^\infty c_n z^n$ with radius of convergence $R$. For $|z| < R$, let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Then the derivative evaluated at real $z \in \mathbb{R}$

$$f'(z) = \sum_{n=1}^{\infty} c_n n z^{n-1} = \sum_{n=0}^{\infty} c_{k+1}(k+1)z^k$$

has the same radius of convergence $R$.

*Proof.* It suffices to verify that both $f(z)$ and $f'(z)$ have the same radius of convergence. Notice that $\limsup_{n\to\infty} |n \cdot c_n|^{\frac{1}{n}} = \limsup n^{\frac{1}{n}} |c_n|^{\frac{1}{n}} = \frac{1}{R_{f'}}$, so

$$\frac{1}{R_{f'}} = \lim_{n\to\infty} n^{\frac{1}{n}} \cdot \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = 1 \cdot \frac{1}{R}.$$

So for any $R' < R$, $\sum n c_n z^{n-1} = f'(z)$ converges uniformly if $|z| \le R'$. $\square$

**Corollary 4.5**

For all $k$, $f^{(k)}(x) = \sum_{n=k}^{\infty} c_n n(n+1) \cdots (n-k+1) z^{n-k}$ for $|x| < R$.

**Corollary 4.6**

$f^{(k)}(0) = k! c_k$.

**Remark 4.7.** By the last corollary, we get that the infinite series we've been working with were Taylor series for $f$.

What about the converse? Can we represent every function by some Taylor series?

**Example 4.8**

Let

$$f(x) = \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x^2}} & x > 0 \end{cases}.$$

Then $f \in \mathscr{C}^\infty$, but $f^{(j)}(0) = 0$ for all $j$, so $f$ is obviously not equal to its Taylor series.

Then what functions work?

**Definition 4.9.** A function that can be represented by a series $\sum c_n z^n$ is said to be an **analytic function**.

**Example 4.10**

The function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ converges if $|z| < 1$ and diverges if $|z| > 1$.

**Example 4.11**

Define

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

By the ratio test, $\lim_{n\to\infty} \left|\frac{z^{n+1}}{(n+1)!}\right| \left|\frac{z^n}{n!}\right| = \lim_{n\to\infty} \frac{|z|}{n+1} = 0$ for all $z \in \mathbb{C}$. Thus $E(z)$ converges uniformly on any disc. Furthermore, notice that

$$E'(x) = \sum_{n=0}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^n}{n!} = E(x),$$

and $E(0) = 1$. Taking $z, w \in \mathbb{C}$, we get from some sad algebraic manipulation that

$$\begin{aligned}
E(z)E(w) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{\infty} \frac{w^k}{k!} \\
&= \sum_{m=0}^{\infty} \sum_{n+k=m} \frac{z^n w^k}{n!k!} \\
&= \sum_{m=0}^{\infty} \left( \sum_{k=0}^{m} \frac{z^{m-k} w^k}{(m-k)!k!} \right) \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} z^{m-k} w^k \\
&= \sum_{m=0}^{k} \frac{1}{m!} (z+w)^m \\
&= E(z+w).
\end{aligned}$$

**Corollary 4.12**

$E(-x) = \frac{1}{E(x)}$, $x \in \mathbb{C}$. Thus $E(z) \neq 0$ for all $z \in \mathbb{C}$.

**Definition 4.13.**

$$e = E(1) = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Notice

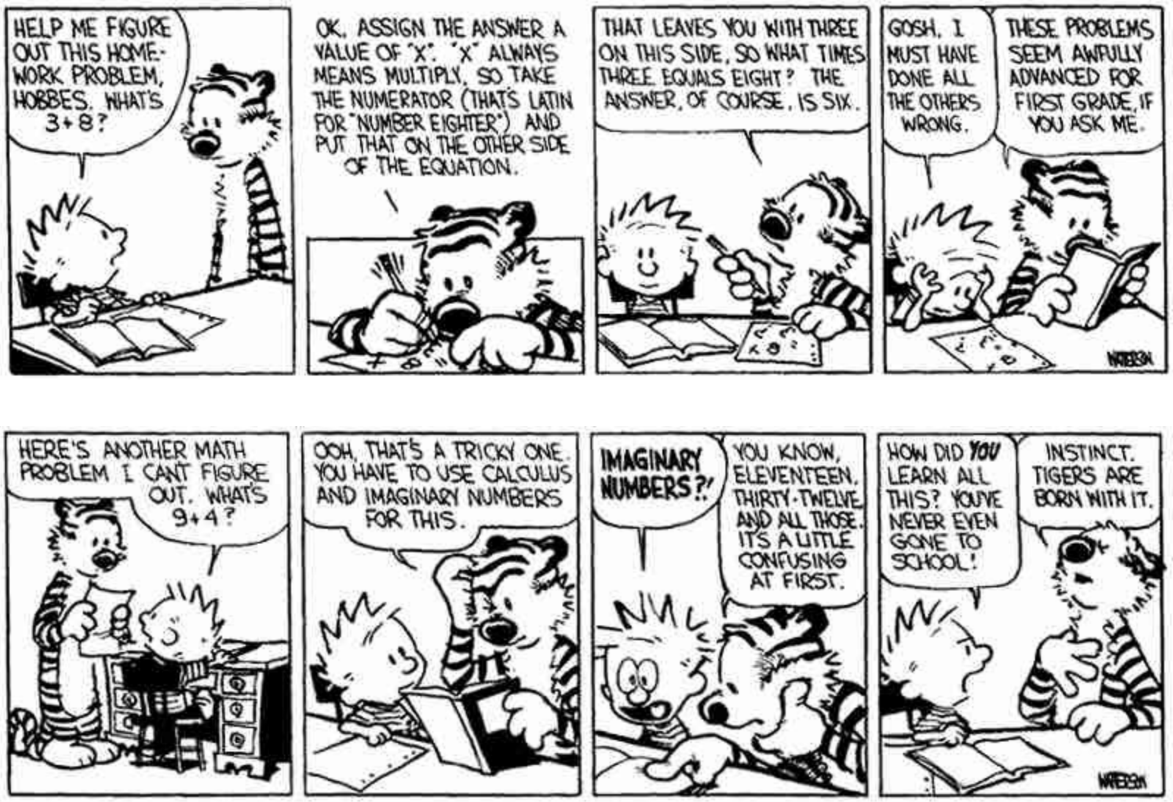
1. $E(w) = E(1 + \cdots + 1) = E(1)^n = e^n$ for all $n \in \mathbb{N}$,
2. $E\left(\frac{p}{q}\right)^q = E\left(\frac{p}{q} + \cdots + \frac{p}{q}\right) = E(p) = e^p$, so $E\left(\frac{p}{q}\right) = e^{p/q}$.

**Proposition 4.14**

$E(x)$ is strictly increasing on $\mathbb{R}$.

*Proof.* $E'(x) = E(x) > 0$ for all $x$. □

**Remark 4.15.** $e^x$ was also defined (at some point) as $\sup_{p/q<x} E\left(\frac{p}{q}\right) = E(x)$.


HELP ME FIGURE OUT THIS HOME-WORK PROBLEM, HOBBES. WHAT'S 3+8?
OK, ASSIGN THE ANSWER A VALUE OF "X". "X" ALWAYS MEANS MULTIPLY, SO TAKE THE NUMERATOR (THAT'S LATIN FOR "NUMBER EIGHTER") AND PUT THAT ON THE OTHER SIDE OF THE EQUATION.
THAT LEAVES YOU WITH THREE ON THIS SIDE, SO WHAT TIMES THREE EQUALS EIGHT? THE ANSWER, OF COURSE, IS SIX.
GOSH, I MUST HAVE DONE ALL THE OTHERS WRONG.
THESE PROBLEMS SEEM AWFULLY ADVANCED FOR FIRST GRADE, IF YOU ASK ME.
WATTERSON
HERE'S ANOTHER MATH PROBLEM I CAN'T FIGURE OUT. WHAT'S 9+4?
OOH, THAT'S A TRICKY ONE. YOU HAVE TO USE CALCULUS AND IMAGINARY NUMBERS FOR THIS.
IMAGINARY NUMBERS?!
YOU KNOW, ELEVENTEEN, THIRTY-TWELVE AND ALL THOSE. IT'S A LITTLE CONFUSING AT FIRST.
HOW DID YOU LEARN ALL THIS? YOU'VE NEVER EVEN GONE TO SCHOOL!
INSTINCT. TIGERS ARE BORN WITH IT.
WATTERSON