MATH 344-1: Introduction to Topology
Northwestern University, Lecture Notes

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These are notes which provide a basic summary of each lecture for MATH 344-1, the first quarter of "Introduction to Topology", taught by the author at Northwestern University. The book used as a reference is the 2nd edition of *Topology* by Munkres. Watch out for typos! Comments and suggestions are welcome.

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Lecture 1: Topological Spaces

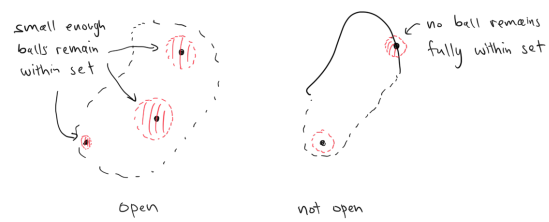
This is a course about *topology*. Topology provides a very general setting in which we can talk about continuity, which is a notion of fundamental interest in pretty much all areas of mathematics. Topology does this by providing a general way in which we can talk about the notion of "near" or "close" *without* the need for a notion of "distance". Why should we strive to avoid using distance? Imagine taking a sphere and stretching it out in one direction to make it thinner and thinner; this procedure does not change the "topology" of the sphere (whatever that means), but it does affect distances. Similarly, we would say that the surface of a donut and a coffee mug are the same topologically since we can (continuously!) "deform" one into the other, but such deformations will certainly affect distance and length. The desire to define "near" without making use of distance is what serves as a guide for much of what we will do.

The focus this quarter will be on the concept of a *topological space* and their fundamental properties. A key underlying idea we will highlight from time to time is that the extent to which we can "separate" structures (such points and subsets) from one another within a given space reflects the extent to which we can control the behavior of continuous functions on that space. (One of the two fundamental results we will study towards the end of the cousre, *Urysohn's lemma*, is explicitly about this exact phenomenon. The other fundamental result we will prove, *Tychnoff's theorem*, is all about guaranteeing that certain spaces will not be too "large".) This is quite a vague idea for now, but we will flesh it out as we go.

Open in $\mathbb{R}^2$. In order to motivate the definition of our fundamental object of study, we first consider the case of $\mathbb{R}^2$. In $\mathbb{R}^2$ we can capture the idea of "close" using the notion of an open set. We say that a subset U of $\mathbb{R}^2$ is *open* if for any $p \in U$ there exists $r > 0$ such that

$$B(p, r) \subseteq U$$

where $B(p, r) = \{q \in \mathbb{R}^2 \mid d(p, q) < r\}$ ($d(p, q)$ here denotes the usual Euclidean distance between points in $\mathbb{R}^2$) is the *open disk* (or *open ball*) of radius r centered at p. So, open sets are ones for which all points within them can be surrounded by small enough open disks which remain fully within those sets. The picture is as follows:



The intuition is that points that are "close enough" to points within an open set are themselves within that set. We cannot "escape" an open set as long as we remain sufficiently "close" to it. Note that an arbitrary open set will be a union of open balls, namely the open balls $B(p, r)$ guaranteed to exist by the definition of open as $p \in U$ varies.

The collection of all open subsets of $\mathbb{R}^2$ has the following two key properties. First, it closed under the taking of arbitrary unions. That is, if U_α is an open set for each α belonging to some indexing set (which could be infinite, even uncountable), then the union $\bigcup_\alpha U_\alpha$ is open as well. To see his, let $p \in \bigcup_\alpha U_\alpha$. Then $p \in U_\beta$ for some index β, which implies there exists $r > 0$ such that

$$B(p, r) \subseteq U_\beta$$

since U_β is open in $\mathbb{R}^2$. This open disk $B(p, r)$ is then contained in the union of all U_α since U_β is, so $\bigcup_\alpha U_\alpha$ is open as claimed. Second, this collection is closed under the taking of finite intersections, by which we mean that if $U_1, \ldots, U_n$ are all open, then so is $U_1 \cap \cdots \cap U_n$. Indeed, if $p \in U_1 \cap \cdots \cap U_n$, then $p \in U_i$ for each $i = 1, \ldots, n$, so for each i there exists $r_i > 0$ such that

$$B(p, r_i) \subseteq U_i$$

since each U_i is open in $\mathbb{R}^2$. The minimum $r := \min\{r_1, \ldots, r_n\}$ of these finitely many radii is then positive and satisfies

$$B(p, r) \subseteq B(p, r_i) \subseteq U_i \text{ for all } i = 1, \ldots, n,$$

so $B(p, r) \subseteq U_1 \cap \cdots \cap U_n$ and hence $U_1 \cap \cdots \cap U_n$ is open as well. (Note we cannot guarantee that the intersection of infinitely many open sets remains open; the argument above breaks down since, although we can replace a minimum radius with an infimum stead, this infimum might be zero. For a concrete example, each open disk $B(0, \frac{1}{n})$ centered at 0 is open for $n \in \mathbb{N}$, but their intersection consists of only the origin, which is not open since no open disk containing it lies fully in $\{(0, 0)\}$.)

It is these two properties of open sets in $\mathbb{R}^2$ that we seek to generalize and which will, eventually, lead to a good notion of "continuity". From the perspective of using open sets to capture the idea of "close", the intuition behind these properties is that, first, if we want to determine whether a given point is "close enough" to a point in a union, we can do so simply by picking out the set in that union containing a point our given one is "close" to, so that if we can measure "closeness" for a bunch of U_α, we can measure "closeness" for their union without trouble. However, if we want to determine whether a given point is "close enough" to a point in an intersection, we must verify that each set in that intersection contains a point to which our give one is "close" to, and since there is only a finite amount of time left in the existence of the universe, this is only guaranteed to be possible to do if we have a *finite* number of sets to check. So, it is only for intersections of finitely many sets for which we can measure "closeness" that it is possible to measure "closeness" for their intersection. (None of these is meant to be rigorously precise—this discussion is only meant to give a sense for why being closed under arbitrary unions and finite intersections are properties we might expect to have when formulating "closeness".)

Topological spaces. With the motivating example of $\mathbb{R}^2$, we now give the core definition for this course. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X such that

- both X and $\emptyset$ are in $\mathcal{T}$,
- if $U_\alpha \in \mathcal{T}$ for each α, then $\bigcup_\alpha U_\alpha \in \mathcal{T}$, and
- if $U_1, \ldots, U_n \in \mathcal{T}$ for each $i = 1, \ldots, n$ (no restriction on how large $n \in \mathbb{N}$ can be), then $U_1 \cap \cdots \cap U_n \in T$.

We refer to the elements of $\mathcal{T}$ as being the *open sets* of this topology, and a *topological space* is a set equipped with a chosen topology.

Thus, a topological space is nothing but a set on which we have specified that certain subsets are the ones we consider to be "open", and the second and third properties above capture the same behavior that open sets in $\mathbb{R}^2$ do. The requirement that X be open in the first requirement guarantees that working with open sets alone does allow us to describe all of X (what really matter is that we can write X as a union of open sets), and the requirement that the empty set be open is just there to guarantee that even the intersection of two disjoint open sets is still considered to be open; the only purpose this serves is to allow us to state the third property above without having carve out any special cases.

Examples. The first example of a topological space is the one we gave as motivation, namely $\mathbb{R}^2$ the notion of open set we gave earlier. More generally, we can do an analogous thing for $\mathbb{R}^n$ by using n-dimensional Euclidean distance when defining open balls. In $\mathbb{R}$, for example, open balls are open intervals, and thus open sets are unions of open intervals. We will refer to this topology on $\mathbb{R}^n$ as being the *standard* (or *Euclidean*) topology.

On any set X we can consider the *discrete* topology, which is the topology defined by declaring *every* subset of X to be open. (In other words, the topology $\mathcal{T}$ consisting of *all* subsets of X.) The requirements in the definition of a "topology" are trivially satisfied just because there is no restriction on what qualifies as "open". The discrete topology on X is the one which in a sense has as many open sets as possible, and at the other extreme is the topology where the only open subsets of X are X itself and the empty set; this is called the *trivial* topology on X, and has the fewest possible number of open sets.

On a two element set $X = \{a, b\}$ we can take the topology where $\emptyset, \{a, b\}$, and $\{a\}$ are declared to be open. This in a sense sits "between" the trivial and discrete topologies on $\{a, b\}$. The collection $\emptyset, \{a, b\}, \{b\}$ gives a different topology on $\{a, b\}$, although one which is "essentially" the same as the first; we will clarify what "essentially" the same means later when discussing the notion of a *homeomorphism*.

Line with two origins. Take two distinct "points" p and q and consider the set

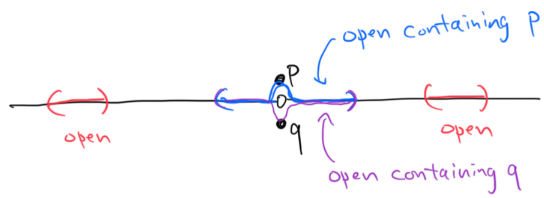
$$(\mathbb{R} - \{0\}) \cup \{p\} \cup \{q\}.$$

(The idea is that we replace the origin 0 in $\mathbb{R}$ with *two* new points.) The *line with two origins* is this set equipped with the following topology. First, any ordinary open set in $\mathbb{R}$ which does not contain 0 remains open in the line with two origins. For open sets U in $\mathbb{R}$ which *do* contain 0, we introduce two copies of U, each containing one of the two new "origins" p and q; to be clear, for U open in $\mathbb{R}$ with $0 \in U$, we take

$$(U - \{0\}) \cup \{p\} \text{ and } (U - \{0\}) \cup \{q\}$$

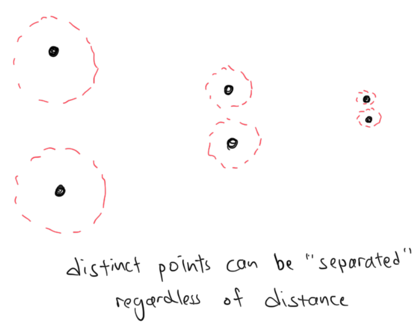
to be open sets in the line with two origins.

Picture this space as an ordinary line, only, as the name suggests, with two origins, usually drawn with one on top of the other:

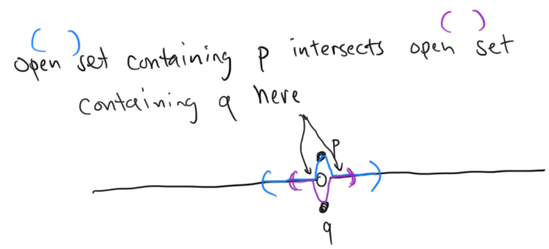


These two "origins" are different, but, in a sense, share the same open sets. To get another visualization, imagine taking two copies of $\mathbb{R}$ (so, two lines) and gluing each point in the first to the corresponding point in the second *except* for the two origins; the space resulting from this gluing procedure is the line with two origins. We'll talk about such gluing constructions later when we discuss *quotient* topologies.

Relation between "open' and "near". We can use the line with two origins to further clarify the notion between "open" and "near". Consider two distinct points in $\mathbb{R}^2$. Certainly if these points are drawn far enough apart we can easily surround each by open disks which do not intersect each other. The point is that no matter how visually close these points appear to be to one another (say the distance between them is the size of an electron), this is still true: the open disks we need might be incredibly small, but they still exist:



Thus such points can still be "separated" in a topological sense, and so really are not that "near" each after all. However, in the line with two origins something new happens: the two origins themselves cannot be separated in this way. To be precise, the claim is that there do not exist open sets containing the two origins which are disjoint, which is true since any open set containing one origin has to intersect an open set containing the other by the way in which defined open sets in that topology:



Intuitively this says that the two origins are distinct and yet "infinitely close" to one another, even though there is no notion of "distance" defined a priori in this space.

Lecture 2: More on Topologies

Warm-Up. For a set X, we define the *cofinite* topology (also called the *finite complement* topology) on X to be the one where we take as open sets $\emptyset$ and complements of finite sets. We verify that this is indeed a topology on X. First, $\emptyset$ is open by definition and the complement $X - X = \emptyset$ of X is finite, which means that X is open as well. Now, suppose $\{U_\alpha\}_\alpha$ is a collection of open sets. If any U_α is X, then $\bigcup_\alpha U_\alpha = X$ is open. Thus we need only consider the case where each U_α has finite complement, say

$$X - U_\alpha = \{p_1^\alpha, \ldots, p_{n_\alpha}^\alpha\} \text{ for some } p_j^\alpha \in X.$$

Then the complement of $\bigcup_\alpha U_\alpha$ is

$$X - \bigcup_\alpha U_\alpha = \bigcap_\alpha (X - U_\alpha) = \bigcap_\alpha \{p_1^\alpha, \ldots, p_{n_\alpha}^\alpha\}.$$

This is an intersection of finite sets and so is finite itself, which means that $\bigcup_\alpha U_\alpha$ is open.

If now $U_1, \ldots, U_n$ is a collection of finitely many open sets, then if any of these is empty their intersection is empty and thus open. Otherwise each U_i has finite complement, say

$$X - U_i = \{p_1^i, \ldots, p_{n_i}^i\}.$$

Then

$$X - (U_1 \cap \cdots \cap U_n) = (X - U_1) \cup \cdots \cup (X - U_n) = \bigcup_{i=1}^n \{p_1^i, \ldots, p_{n_i}^i\}$$

is a union of finitely many finite sets, so is finite itself. Thus $U_1 \cap \cdots \cap U_n$ is open, so the cofinite topology is indeed a topology.

Closed sets. Having to argue via complements as above can get a bit cumbersome, so it can be useful to rephrase the definition of a topology in terms of such complements directly. We say that $A \subseteq X$ is a *closed* set in the given topology on X if its complement $X - A$ is open. Since unions become intersections and intersections become unions when taking complements, the definition of a topology phrased in terms of closed sets then becomes:

- X and $\emptyset$ are both closed,
- the intersection of arbitrarily many closed sets is closed, and
- the union of finitely many closed sets is closed.

In terms of closed sets, the cofinite topology is then defined by taking X and finite sets to be closed. Arguing that this is a topology then comes down to the fact that intersections of (any number) of finite sets is finite and the union of finitely many finite sets is finite, as we saw above.

Zariski topology. The *Zariski topology* on $\mathbb{R}^2$ is defined by taking closed sets to be common zero sets of polynomials in two variables. To be clear, if S is a collection of polynomials in two variables on $\mathbb{R}^2$, the set

$$V(S) = \{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 0 \text{ for all } p \in S\}$$

is closed in the Zarisky topology. For example, $V(y - x^2)$ is the set of zeroes of the single two-variable polynomial $y-x^2$, so is thus the set of points in $\mathbb{R}^2$ satisfying $y = x^2$, which is a parabola. (The set $V(S)$ is an example of what is called a *variety* in the subject of algebraic geometry, which is where the Zariski topology finds its natural home. We will touch on this a bit later.)

A problem on the first homework asks to show that this indeed gives a topology on $\mathbb{R}^2$, and here we verify just two special cases. If f and g are each polynomials in two variables, then we claim that $V(f) \cap V(g)$ and $V(f) \cup V(g)$ are both closed. Indeed, a point in $V(f) \cap V(g)$ is one which is a zero of f and g simultaneously, meaning that it is a common zero of the polynomials in the set $\{f, g\}$. Hence

$$V(f) \cap V(g) = V(\{f, g\}),$$

so $V(f) \cap V(g)$ is closed. (Alternatively, a common zero of f and g is a zero of $f^2 + g^2$ since we are working over $\mathbb{R}$, so $V(f) \cap V(g) = V(f^2 + g^2)$. This is not true if we work over $\mathbb{C}$ instead since squares can be negative.) Now, a point in $V(f) \cup V(g)$ is one which is a zero either of f or g. But to say that $f(x, y) = 0$ or $g(x, y) = 0$ is the same as saying that $f(x, y)g(x, y) = 0$ since a product is zero when one factor is zero. Hence

$$V(f) \cup V(g) = V(fg),$$

so $V(f) \cup V(g)$ is closed in the Zariski topology.

Comparing topologies. The standard parabola $y = x^2$ defines a closed subset of $\mathbb{R}^2$ in the Zariski topology since it is the zero set of the polynomial $y-x^2$, and for example the cubic $y = x^3$ is another example of a closed set since it is the zero set of $y - x^3$. Now, both of these sets are also closed in the standard Euclidean topology on $\mathbb{R}^2$, which we can see either by convincing ourselves that its complement is open (in the "drawing small open disks" sense) or by recalling some facts from analysis, namely that the set of zeroes of any continuous function always defines a closed set. It is true more generally that the set of common zeroes of *any* collection of polynomials is closed in $\mathbb{R}^2$ in the standard topology for a similar reason. This means that any set which is open (respectively closed) in the Zarisksi topology on $\mathbb{R}^2$ is also open (respectively closed) in the standard topology, so we say that the Zariski topology is *coarser* than the standard topology. (A coarser topology is one which has fewer open sets. On the flip side, a *finer* topology is one which has more open sets, so the standard topology on $\mathbb{R}^2$ is finer than the Zariski topology. So, given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on a set, when $\mathcal{T}_1 \subseteq \mathcal{T}_2$ we say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ or that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$. Note that not all topologies are comparable in this way: a given topology might be neither courses for finer than another given one. For sure, however, the trivial topology is coarser than any other and the discrete is finer than any other.)

However, it is not true that any set which is closed in the standard topology is also closed in the Zariski topology. For instance, the graph of $y = e^x$ is closed in the standard topology and yet we claim that it is not closed in the Zariski topology. Now, $y - e^x$ is certainly not a polynomial in two variables (infinite polynomials do not count!), but this alone does not guarantee that its zero set is not open in the Zariski topology since there *could* be a polynomial in two variables which had the same zero set as $y - e^x$; there is not, but this is not so straightforward to prove without more tools from analysis. A simpler example is the interval $[0,1]$ on the x-axis, which is closed in the standard topology but not in the Zariski topology; this is more straightforward to prove and we encourage you do so for practice! Thus, the Zariski topology is actually *strictly coarser* than the standard topology on $\mathbb{R}^2$.

Coarsest/finest topologies. We will often resort to defining topologies by specifying that they should be the *coarsest* ones in which some stated property should be true. To be precise, to say that $\mathcal{T}$ is the coarsest topology satisfying some property means that if $\mathcal{T}'$ is any other topology satisfying that same property, we should have $\mathcal{T} \subseteq \mathcal{T}'$. (We can make a similar definition of what it means for a topology to be the *finest* one satisfying some property.) In practice this means that we allow as open sets whatever we need in order to guarantee that the stated property holds, and then we also take as open sets anything else we need to include to ensure we get a topology, but no more. (So, the coarsest topology in which a property holds is the one which has the fewest open sets needed to ensure that property holds.) This should become clearer as we actually start using this terminology. For now, notice that in the cofinite topology on a set, single points are always closed, and indeed we can characterize the cofinite topology on a set as the being the coarsest one in which this is true.

Back to cofinite and Zariski. Here is one last observation. The Zariski topology on $\mathbb{R}$ is defined in an analogous way where we take zero sets of polynomials in one variable. In this case, since a nonzero polynomial in one variable can only have finitely many roots (it has no more than the degree of the polynomial), we see that any closed set in the Zariski topology on $\mathbb{R}$ is either $\mathbb{R}$ itself (this is the zero set of the zero polynomial) or consists of finitely many points. But this is precisely the characterization of the cofinite topology on $\mathbb{R}$, so we conclude that the cofinite and Zariski topologies on $\mathbb{R}$ are one and the same. This is not true in $\mathbb{R}^n$ for $n > 1$, where these two topologies are differen: the cofinite topology will be strictly coarser than the Zariski topology. Proving this comes down to showing that singleton sets $\{(a,b)\}$ are Zariski-closed, which is true since this is the common zero set of $x - a$ and $y - b$.

Lecture 3: Bases for Topologies

Warm-Up. We find coarsest topology on $\{a,b,c,d,e\}$ in which $\{a,b\}$ and $\{b,d\}$ are closed. In practice this means that we determine which other sets *must* be closed if these two are to be closed, and then take no more closed sets beyond these. Of course, we need $\emptyset$ and $\{a,b,c,d,e\}$ to both be closed as required in the definition of a topology. Since unions of finitely many closed sets should be closed, we also need

$$\{a,b\} \cup \{b,d\} = \{a,b,d\}$$

to be closed. Intersections are closed sets are closed, so

$$\{a,b\} \cap \{b,d\} = \{b\}$$

must be closed as well. And that is it: taking unions or intersections of the closed sets

$$\emptyset,\ \{a,b,c,d,e\},\ \{a,b\},\ \{b,d\},\ \{a,b,d\},\ \{b\}$$

we have thus far results in one of these same sets, so no more closed sets are needed in order to ensure we satisfy the definition of a topology. The open sets of this topology, obtained by taking complements of those above, are then

$$\{a, b, c, d, e\},\ \emptyset,\ \{c, d, e\},\ \{a, c, e\},\ \{c, e\},\ \{a, c, d, e\}.$$

Back to open balls. Describing all open sets of a topology directly is not always necessary as long as we can describe a potentially smaller collection of open sets from which all others can be obtained, which is useful in practice since it leads to simpler/cleaner arguments. Our goal now is to characterize such "smaller" collections. To do so, we return to the original motivation we gave for the definition of a topology on a set, namely the case of open sets (in the standard sense) in $\mathbb{R}^2$. The definition of being open here implies that we can write an arbitrary open set as a union of open balls via

$$U = \bigcup_{p \in U} B(p, r_p)$$

where $B(p, r_p) \subseteq U$ guaranteed to exist for each $p \in U$ by the definition of open in $\mathbb{R}^2$. The point is that all open sets can be built from open balls alone, so that perhaps studying open balls alone is enough to make general topological arguments about $\mathbb{R}^2$.

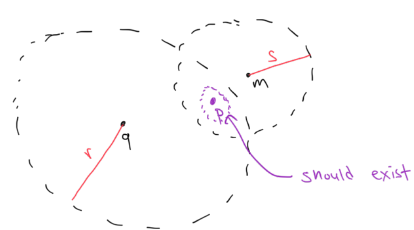
But to truly see what it is about open balls that matters here, consider the following. Suppose we define $U \subseteq \mathbb{R}^2$ to be open if for any $p \in U$ there exists a disk $B(q, r)$ such that $p \in B(q, r) \subseteq U$. The subtlety is that now we are no longer requiring that the disk be centered at p itself, which is important because the notion of "centered at" has no meaning in a general topological setting since there is no such thing as "distance" in general. The question is: if we use this definition of open, how do we show that the intersection of two open sets is still open, as would be required in the definition of a topology? If you work through the details, this boils down to showing that if p is in the intersection of two open disks

$$p \in B(q, r) \cap B(m, s),$$

there exists exists a third open disk $B(n, t)$ containing p and contained in this intersection:

$$p \in B(n, t) \subseteq B(q, r) \cap B(m, s).$$

Visually this looks like the following:



It is this property which, as $p \in B(q, r) \cap B(m, s)$ varies, guarantees that $B(q, r) \cap B(m, s)$ can itself be expressed as the union of open balls, so that open balls alone are enough to describe everything.

Topological bases. With this in mind, we make the following definition. A collection $\mathcal{B}$ of subsets of X is a *basis* for a topology on X if

- the union of all sets in $\mathcal{B}$ is all of X, and
- for any $B_1, B_2 \in \mathcal{B}$ and $p \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $p \in B_3 \subseteq B_1 \cap B_2$.

We then define the topology on X *generated* by $\mathcal{B}$ to be the one whose open sets are unions of elements of $\mathcal{B}$. We refer to elements of $\mathcal{B}$ as *basic open sets* for this topology. The first requirement in the definition of basis says that X is open in the topology generated by $\mathcal{B}$, and the second guarantees (as $p \in B_1 \cap B_2$ and hence B_3 vary) that the intersection of two basic open sets is open.

Indeed, we should verify that taking unions of basic open sets to be open does actually give a valid topology on X. For sure X is open and, if we take an "empty" union—i.e., a union of no basic open sets at all—we get that $\emptyset$ is open. Unions of unions of basic open sets are still unions of basic open sets, so such things are open too. Now, for the intersection of finitely many opens property, it is enough by induction to verify this in the case of only two open sets: if U, V are open in X then $U \cap V$ is open in X. If U, V are open in X, we have

$$U = \bigcup_{\alpha} B_\alpha \text{ and } V = \bigcup_{\gamma} C_\gamma \text{ where } B_\alpha, C_\gamma \in \mathcal{B}$$

for α, γ. Then

$$U \cap V = \bigcup_{\alpha,\gamma} B_\alpha \cap C_\gamma.$$

For each p in some $B_\alpha \cap C_\gamma$, pick $W_p^{\alpha,\gamma} \in \mathcal{B}$ such that

$$p \in W_p^{\alpha,\gamma} \subseteq B_\alpha \cap C_\gamma,$$

which is possible by the basis property. This gives

$$U \cap V = \bigcup_{\alpha,\gamma} B_\alpha \cap C_\gamma = \bigcup_{\alpha,\gamma} \bigcup_{p \in B_\alpha \cap C_\gamma} W_p^{\alpha,\gamma},$$

which is a union of basic open sets and thus $U \cap V$ is open as desired.

Back to $\mathbb{R}^2$. Open balls/disks form a basis for the standard topology on $\mathbb{R}^2$. Note that there are two things being said here: first that the collection of open balls form a basis for *a* topology on $\mathbb{R}^2$, and second that the topology which they generate is in fact the the standard one. That open balls form a basis for a topology is what we used to motivate the definition of basis in the first place (we will come back to this fact in more generality next time), and that the topology they generate is the standard one comes from how we originally defined "open" in $\mathbb{R}^2$.

But there are other bases for this same topology. For example, we claim that the collection of *open squares* $(a-r, a+r) \times (b-r, b+r)$ also forms a basis for the standard topology on $\mathbb{R}^2$. Here are two relevant pictures:

The first shows (informally) why open squares form a basis for a topology, and the second shows (informally) that they generate the standard topology. (The first fact will be made precise next time, and the second in a discussion problem.)

Lower limit and K-topologies. Using the language of bases we can now describe two more key topologies on $\mathbb{R}$. The *lower limit* topology on $\mathbb{R}$ is the one generated by the basis of half-closed/half-open intervals $[a, b)$. So, something like $[1, 3)$ is open in this topology. We can see that intervals of this type do form a basis for a topology by noting that the intersection of two $[a,b),[c,d)$ is either empty or of the same form:

$$[a,b)\cap[c,d)=[\max\{a,c\},\min\{b,d\}),$$

so we can take this intersection itself as the "third basic open set" required in the second part of the basis definition. We use the notation $\mathbb{R}_\ell$ to denote $\mathbb{R}$ equipped with this specific topology.

In fact, the lower limit topology is strictly finer than the standard topology. To see that it is finer it is enough to argue a basic open set (a,b) for the standard topology is still open in $\mathbb{R}_\ell$ since unions of open intervals will then be unions of unions of half-closed/half-open intervals. We have

$$(a,b)=\bigcup_{a<c}[c,d)$$

where the union ranges over all real numbers larger than a, which shows that (a,b) is open in $\mathbb{R}_\ell$. The comparison between these topologies is strict since $[1,2)$, for example, is open in $\mathbb{R}_\ell$ but not in the standard topology since there is no open interval containing $1\in[1,2)$ which is fully contained in $[1,2)$. The lower limit topology will give us an example of a topological space which has some nice properties but not others, and it is also a natural setting in which to discuss the notion of *right-sided limits*—we will mention this concept later, but it will not play an important role for us.

Another interested non-standard topology on $\mathbb{R}$ is the *K-topology*, which denoted by $\mathbb{R}_K$. Here K denotes the set of reciprocals of positive integers:

$$K=\{\tfrac{1}{n}\mid n\in\mathbb{N}\}.$$

The K-topology is generated by the basis consisting of open intervals *and* complements of K inside open intervals. For example, both $(-1,1)$ and $(-1,1)-K$ are open in $\mathbb{R}_K$. The fact that these give a basis for a topology also comes from the fact that intersections of sets of these forms are either empty or of the same form. We will compare this topology with the standard one next time.

Lecture 4: Metric Spaces

Warm-Up. We show that the K-topology on $\mathbb{R}$ is strictly finer than the standard topology. First, note that by definition open intervals are basic open sets for the K-topology on $\mathbb{R}$, which immediately implies that the K-topology is finer than the standard topology since anything open in the standard topology—i.e. unions of open intervals—will still be open in $\mathbb{R}_K$.

But the containment is strict since, for example, $(-1,1)-K$, where $K=\{\frac{1}{n}\mid n\in\mathbb{N}\}$ is open in $\mathbb{R}_K$ but not in standard $\mathbb{R}$. Indeed, in order for this to be open in $\mathbb{R}$ would require in particular that for $0\in(-1,1)-K$ there exist an open interval (a,b) such that

$$0\in(a,b)\subseteq(-1,1)-K,$$

which is not possible since for $b>0$ there always exists $\frac{1}{n}<b$ for some $n\in\mathbb{N}$, so no (a,b) containing 0 can exclude all elements of K. (Using the language of closed sets, this argument shows that K itself is not closed in $\mathbb{R}$, but it is closed in $\mathbb{R}_K$.) Note that for something like $(0,1)-K$ there is no problem and that this *is* open in $\mathbb{R}$ since it is the union

$$(0,1)-K=\bigcup_{n\in\mathbb{N}}(\tfrac{1}{n+1},\tfrac{1}{n}).$$

Metric spaces. As we seek to build up more examples of topological spaces, we begin with the types of spaces which are most comparable to $\mathbb{R}^2$ with its standard topology, namely ones where we *do* we have an appropriate notion of distance available. These are what are called *metric spaces*. A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying

- $d(p, q) \geq 0$ for all $p, q \in X$, and $d(p, q) = 0$ if and only if $p = q$,
- $d(p, q) = d(q, p)$ for all $p, q \in X$, and
- $d(p, q) \leq d(p, r) + d(r, q)$ for all $p, q, r \in X$.

A metric space is then a set X equipped with a specific metric. (Technically the metric should be part of the notation, but it is common to exclude it if all is clear from context.)

The three properties above required of a metric are meant to capture the way in which "distance" should behave. The first (often called *nondegeneracy*) says that distances are never negative, and are zero only when the points between which the distance being measured are one and the same. The second property (called *symmetry* says that the order in which we measure distance between points does not matter. The third property (called the *triangle inequality* is the most important and says that $d(p, q)$ gives the "shortest" distance from p to q in the sense that going through some intermediate point r can only increase distance overall. In $\mathbb{R}^2$ with the usual Euclidean distance, this says that the sum of lengths of two sides of a triangle is always at least as large as the length the remaining side, which is where the name "triangle inequality" comes from.

Given a metric d on X, we define the *open ball* $B(p, r)$ of radius $r > 0$ around $p \in X$ to be the set of all points whose distance to p is less than r as measured by d:

$$B(p, r) = \{q \in X \mid d(p, q) < r\}.$$

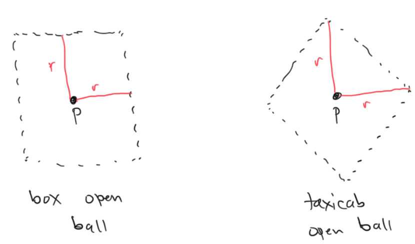
These form, as we will see, a basis for a topology on X called the *metric topology* generated by d.

Examples. Here are three metrics on $\mathbb{R}^n$, the so-called *Euclidean* metric d_E, the *box* metric d_{box}, and the *taxicab* metric d_{taxi}:

$$\begin{aligned}
d_E((x_1, \ldots, x_n), (y_1, \ldots, y_n)) &= \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \\
d_{box}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) &= \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\} \\
d_{taxi}(x_1, \ldots, x_n), (y_1, \ldots, y_n)) &= |x_1 - y_1| + \cdots + |x_n - y_n|.
\end{aligned}$$

(We will leave the verification that these are indeed metrics to the reader.) The Euclidean metric generates the standard topology where open balls are open intervals $(p - r, p + r)$ in the $n = 1$ case, open disks in the $n = 2$ case, and usual open balls (i.e. regions within spheres) in the $n = 3$ case.

Open balls with respect to the box metric are open squares in the $n = 2$ case and analogs thereof in higher dimensions, and open balls with respect to the taxicab metric are open diamonds in the $n = 2$ case and analogs thereof in higher dimensions:



In fact, these three metrics on $\mathbb{R}^n$ all generate the same topology! You will verify this formally in a discussion section problem, but visually it comes from the fact that we can fill up open balls, open squares, and open diamonds with one of the other two types of shapes. (We hinted at this last time when suggesting that open squares in $\mathbb{R}^2$ also formed a basis for the standard topology.)

Open balls form a basis. To see that open balls in a metric space form a basis for a topology, we first note that X (equipped with a metric) is the union of all possible open balls of radius, say, 1 as the center varies:

$$X = \bigcup_{p \in X} B(p, 1).$$

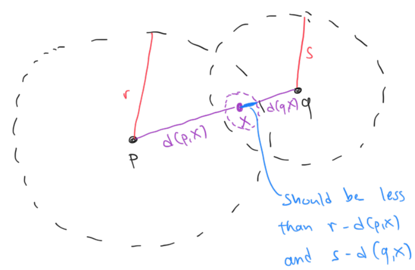
Now, given two open balls and a point in their intersection, say

$$x \in B(p, r) \cap B(q, s),$$

we need to know there exists an open ball within this intersection and containing x. We claim that the ball $B(x, t)$ of any radius

$$0 < t < \min\{r - d(p, x), s - d(q, x)\}$$

works. (Note that both quantities of which we are taking the minimum are positive since $x \in B(p, r)$ and $x \in B(q, s)$ so that $d(p, x) < r$ and $d(q, x) < s$.) Here is the picture to use as a guide:



Indeed, let $y \in B(x, t)$. The triangle inequality then gives

$$d(p, y) \leq d(p, x) + d(x, y) < d(p, x) + t \leq d(p, x) + (r - d(p, x)) = r$$

and

$$d(q, y) \leq d(q, x) + d(x, y) < d(q, x) + t \leq d(q, x) + (s - d(q, x)) = s.$$

Hence $y \in B(p, r)$ and $y \in B(q, s)$, so $B(x, t) \subseteq B(p, r) \cap B(q, s)$ as desired.

Space of real sequences. We let $\mathbb{R}^\omega$ (the superscript is *omega*) denote the set of sequences of real numbers:

$$\mathbb{R}^\omega = \{(x_1, x_2, x_3, \ldots) \mid \text{each } x_i \in \mathbb{R}\}.$$

(Note that in some other subjects this set of sequences is often denoted by $\mathbb{R}^\infty$, but in topology $\mathbb{R}^\infty$ is typically reserved for the set of only those sequences which are *eventually zero*:

$$\mathbb{R}^\infty = \{(x_1, x_2, x_3, \ldots) \in \mathbb{R}^\omega \mid x_i = 0 \text{ for large enough } i\}.$$

This distinction comes from the idea that X^∞ in topology typically denotes a union $\bigcup_n X^n$—so in some sense the "limit of X^n as n goes to ∞"—where we view the X^n's as belonging to a common larger space inside of which we take the union. So, if we view $\mathbb{R}^n$ as the subset of $\mathbb{R}^\omega$ consisting of those sequences for which only the first n-components can be nonzero, then the union of these inside $\mathbb{R}^\omega$ is indeed $\mathbb{R}^\infty$. Similarly, S^∞ denotes the "infinite-dimensional sphere" obtianed by taking the unions of spheres S^n in varying dimensions, where S^n is the standard unit sphere in $\mathbb{R}^{n+1} \subseteq \mathbb{R}^\omega$.)

We define a metric on $\mathbb{R}^\omega$ as follows. We first take the distances $|x_n - y_n|$ between corresponding terms in two sequences

$$\mathbf{x} = (x_1, x_2, \ldots),\ \mathbf{y} = (y_1, y_2, \ldots) \in \mathbb{R}^\omega.$$

We would like to take the supremum of these termwise distances (in other words we want an infinite-dimensional version of the box metric), but the issue is that this supremum might be infinite. So we first cut off a maximal termwise distance value by using

$$\min\{|x_n - y_n|, 1\} \text{ instead of } |x_n - y_n|$$

and then define a metric $\overline{\rho}$ on $\mathbb{R}^\omega$ by taking the supremum of these minimums:

$$\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\min\{|x_n - y_n|, 1\} \mid n \in \mathbb{N}\}.$$

(In the end all that really matters in terms of topology is what happens for *small* distances, so cutting off the minimums to never be larger than 1 does not affect anything topologically.) We call this the *uniform metric* on $\mathbb{R}^\omega$ and the topology it generates the *uniform topology*.

We will work with this topology a bit as we go. To start with, note that, for example, any open balls of radius, say, 2 is all of $\mathbb{R}^\omega$ since $\overline{\rho}(\mathbf{x}, \mathbf{y})$ is always bounded by 1 by construction:

$$B(\mathbf{x}, 2) = \{\mathbf{y} \in \mathbb{R}^\omega \mid \overline{\rho}(\mathbf{x}, \mathbf{y}) < 2\} = \mathbb{R}^\omega.$$

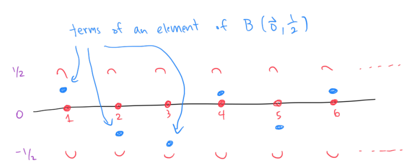
The same is true for any open balls of any radius $r \geq 1$. To start to get a sense of what open balls of smaller radii look like, consider $B(\mathbf{0}, \frac{1}{2})$ where $\mathbf{0} = (0, 0, 0, \ldots)$ is the constant zero sequence. To have $\mathbf{x} \in B(\mathbf{0}, \frac{1}{2})$ requires that

$$\overline{\rho}(\mathbf{x}, \mathbf{0}) = \sup\{\min\{|x_n|, 1\} \mid n \in \mathbb{N}\} < \tfrac{1}{2}.$$

For such $\mathbf{x}$, the minimum of $|x_n|$ and 1 must always be $|x_n|$ since otherwise the supremum above could not be smaller than $\frac{1}{2}$, so $\mathbf{x} \in B(\mathbf{0}, \frac{1}{2})$ satisfies, if nothing else,

$$|x_n| < \tfrac{1}{2}, \text{ or equivalently } x_n \in (-\tfrac{1}{2}, \tfrac{1}{2}) \text{ for all } n.$$

Thus if we take the (vertical) interval $(-\frac{1}{2}, \frac{1}{2})$ around each positive integer in $\mathbb{R}$, elements of $B(\mathbf{0}, \frac{1}{2})$ will have terms that fall in these intervals for varying n:



This might suggest that $B(\mathbf{0}, \frac{1}{2})$ is the product of such intervals, but we will see that this not quite the case as a Warm-Up next time when we come up with a clear visualization of $B(\mathbf{0}, \frac{1}{2})$.

Lecture 5: Product Topology

Warm-Up 1. We show that the open ball $B(\mathbf{0}, \frac{1}{2})$ in $\mathbb{R}^\omega$ with respect to the uniform metric is not the product

$$(-\tfrac{1}{2}, \tfrac{1}{2}) \times (-\tfrac{1}{2}, \tfrac{1}{2}) \times (-\tfrac{1}{2}, \tfrac{1}{2}) \times \cdots$$

and moreover determine what this and other open balls look like concretely. We argued last time that this open ball is contained in this product, so the claim is that this containment is strict.

Take a sequence of terms in $(-\frac{1}{2}, \frac{1}{2})$ which converge to $\frac{1}{2}$ in the usual analytic sense, so for example

$$\mathbf{x} = (\tfrac{1}{2} - \tfrac{1}{2}, \tfrac{1}{2} - \tfrac{1}{3}, \tfrac{1}{2} - \tfrac{1}{4}, \tfrac{1}{2} - \tfrac{1}{5}, \ldots)$$

with n-th term equal to $x_n = \frac{1}{2} - \frac{1}{n+1}$. For this sequence we have

$$\min\{|x_n|, 1\} = \tfrac{1}{2} - \tfrac{1}{n+1}, \text{ so } \overline{\rho}(\mathbf{x}, \mathbf{0}) = \sup\{\tfrac{1}{2} - \tfrac{1}{n+1} \mid n \in \mathbb{N}\} = \tfrac{1}{2}.$$

Thus $\overline{\rho}(\mathbf{x}, \mathbf{0}) \not< \frac{1}{2}$, so this $\mathbf{x}$ is an element of

$$(-\tfrac{1}{2}, \tfrac{1}{2}) \times (-\tfrac{1}{2}, \tfrac{1}{2}) \times (-\tfrac{1}{2}, \tfrac{1}{2}) \times \cdots$$

that is not in $B(\mathbf{0}, \frac{1}{2})$, and hence this product is strictly larger than this open ball.

If we unwind the condition needed to have some $\mathbf{x}$ belong to $B(0, \frac{1}{2})$, we see that we need

$$\sup\{|x_n| \mid n \in \mathbb{N}\} < \tfrac{1}{2}.$$

The point is that this not only requires $|x_n| < \frac{1}{2}$, but in fact it requires that the $|x_n|$ be *uniformly* bounded away from $\frac{1}{2}$ by some fixed amount: if C denotes the supremum above, we need

$$|x_n| \leq C < \tfrac{1}{2} \text{ for all } n.$$

The situation in the first part of the Warm-Up arises when we have $|x_n|$'s that can get arbitrarily close to $\frac{1}{2}$ rather than arbitrarily close to something *smaller* than $\frac{1}{2}$. As long as we remain away from $\frac{1}{2}$ (or $-\frac{1}{2}$) by some positive amount, we will get elements of $B(\mathbf{0}, \frac{1}{2})$. Thus, while

$$(-\tfrac{1}{2}, \tfrac{1}{2}) \times (-\tfrac{1}{2}, \tfrac{1}{2}) \times (-\tfrac{1}{2}, \tfrac{1}{2}) \times \cdots$$

is not contained in $B(\mathbf{0}, \frac{1}{2})$, the product

$$(-\tfrac{1}{2} + \epsilon, \tfrac{1}{2} - \epsilon) \times (-\tfrac{1}{2} + \epsilon, \tfrac{1}{2} - \epsilon) \times (-\tfrac{1}{2} + \epsilon, \tfrac{1}{2} - \epsilon) \times \cdots$$

is contained in $B(\mathbf{0}, \frac{1}{2})$ for every $\epsilon > 0$, and indeed $B(\mathbf{0}, \frac{1}{2})$ is precisely the union of such things:

$$B(\mathbf{0}, \tfrac{1}{2}) = \bigcup_{\epsilon > 0} [(-\tfrac{1}{2} + \epsilon, \tfrac{1}{2} - \epsilon) \times (-\tfrac{1}{2} + \epsilon, \tfrac{1}{2} - \epsilon) \times (-\tfrac{1}{2} + \epsilon, \tfrac{1}{2} - \epsilon) \times \cdots].$$

We can express this more succinctly using product notation

$$B(\mathbf{0}, \tfrac{1}{2}) = \bigcup_{\epsilon > 0} \prod_{n \in \mathbb{N}} (-\tfrac{1}{2} + \epsilon, \tfrac{1}{2} - \epsilon)$$

where $\prod_\alpha X_\alpha$ in general denotes the product of sets X_α indexed by some α's. (In the example above, all of the X_α's are the same interval $(-\frac{1}{2} + \epsilon, \frac{1}{2} - \epsilon)$ and they are indexed by positive integers.) In general, the uniform ball of radius r around $\mathbf{x} \in \mathbb{R}^\omega$ is

$$B(\mathbf{x}, r) = \bigcup_{\epsilon > 0} \prod_{n \in \mathbb{N}} (x_n - r + \epsilon, x_n + r - \epsilon)$$

where we restrict points from getting arbitrarily close to the endpoints of any $(x_n - r, x_n + r)$.

Warm-UP 2. Given distinct p, q in a metric space X, we show there exist open sets U, V with

$$p \in U, \ q \in V, \text{ and } U \cap V = \emptyset.$$

This condition is what it means for a space in general to be *Hausdorff*, so the claim here is that metric spaces are always Hausdorff. (The Hausdorff condition is one that we will revisit quite a bit as we go and is an example of what is called a *separation axiom*. We will clarify later.)

Since $p \neq q$, we have $d(p,q) > 0$. We claim that $B_{d(p,q)/2}(p)$ and $B_{d(p,q)/2}(q)$ are then disjoint open sets containing p and q respectively. Indeed, if there exists $x \in B_{d(p,q)/2}(p) \cap B_{d(p,q)/2}(q)$, then

$$d(x,p) < \tfrac{d(p,q)}{2} \text{ and } d(x,q) < \tfrac{d(p,q)}{2},$$

so the triangle inequality gives

$$d(p,q) \leq d(p,x) + d(x,q) < \tfrac{d(p,q)}{2} + \tfrac{d(p,q)}{2} = d(p,q).$$

This is not possible, so there is no such x and hence $B_{d(p,q)/2}(p)$ and $B_{d(p,q)/2}(q)$ are disjoint as claimed. These are then the required open sets in the Hausdorff condition.

Metrizability. If the topology on a topological space arises from a metric, we say that that space is *metrizable*. The second Warm-Up says that any metrizable space must be Hausdorff, so we can now give examples of topologies which do not arise from metrics. For instance, the cofinite topology on an infinite set is not Hausdorff (any nonempty set open set in such a topology only excludes finitely many points, so any two such open sets will always have infinitely many points in common and so are not disjoint) and so cannot be given by a metric. The Zariski topology on $\mathbb{R}^n$ is also non-Hausdorff (we will come back to this), and so is also not given by a metric. The line with two origins is another example as open sets around the two origins will never be disjoint.

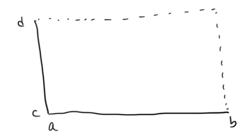
However, note that we can also have Hausdroff spaces which are not metrizable. For instance, $\mathbb{R}_\ell$ ($\mathbb{R}$ with the lower limit topology) is actually Hausdroff, but it turns out not metrizable. Showing that there is no metric on $\mathbb{R}$ which gives the lower limit topology is not something we can do just yet, but will follow from some other properties of metric spaces we will look at later. (If you want to hear the buzzwords now, the key fact is that a metric space is "separable" if and only if it is "second countable", and $\mathbb{R}_\ell$ is separable but not second countable, so it is not metrizable.)

Finite products. Given two spaces X, Y, we seek to define a topology on the product $X \times Y$ that incorporates the given topologies on X and Y. The natural thing to do is declare $U \times V$ to be open in $X \times Y$ when U is open in X and V is open in Y. Unions of such products are not necessarily products again, but such products do form a basis for a topology, and this is what we call the *product/box* topology on $X \times Y$. Similarly, given a finite number of spaces $X_1, \ldots, X_n$, the product/box topology on $X_1 \times \cdots \times X_n$ is the one generated by the basic open sets $U_1 \times \cdots \times U_n$ where U_i is open in X_i. (The distinction between "product" and "box" will only manifest itself when considering, as we will do next time, products of infinitly many spaces; for finitely many spaces as above the notions are the same.)

For example, if $\mathbb{R}$ is equipped with the standard topology,the product/box topology on $\mathbb{R}^2$ (or $\mathbb{R}^n$ in general) is the same as the standard topology on $\mathbb{R}^2$ (or $\mathbb{R}^n$). This comes from the fact that a basic open set in the product topology on $\mathbb{R}^2$ is a product of open intervals

$$(a,b) \times (c,d)$$

which are thus open rectangles and hence generate the same topology as open disks. (Open rectangles are not quite open balls with respect to the box metric—open squares are—but they are unions of such things.) A basic open set in $\mathbb{R}_\ell \times \mathbb{R}_\ell$, by contrast, is a rectangle which is only open at the top and right edges in the usual drawing of $\mathbb{R}^2$:



$[a,b) \times [c,d)$ open in $\mathbb{R}_\ell \times \mathbb{R}_\ell$

Characterization via projections. Here we single out one important aspect of the product topology in the finite case, which is essentially in the book if you read between the lines but is not made explicit. The claim is that the product topology on $X_1 \times \cdots \times X_n$ is the coarsest one relative to which the preimage of any open set under any projection is itself open: for any $i = 1, \ldots, n$

$$pr_i^{-1}(U) \text{ is open in } X_1 \times \cdots \times X_n \text{ whenever } U \text{ is open in } X_i.$$

Here, the i-th projection $pr_i : X_1 \times \cdots X_n \to X_i$ is the function which picks out i-th components:

$$pr_i(x_1, \ldots, x_n) = x_i.$$

The condition given above in terms of preimages is (as we will see soon enough) precisely what it means to say that each projection is continuous, so the claim is that the product topology is the coarsest one relative to which all projections are continuous.

To prove this, suppose $\mathcal{T}$ is any topology on $X_1 \times \cdots \times X_n$ having the property that the preimage of any open set under any projection is open in $X_1 \times \cdots \times X_n$. We want to show that $\mathcal{T}$ is finer than the product topology. To this end, suppose $U \subseteq X_1 \times \cdots \times X_n$ is open in the product topology. Then U can be written as the union of open sets of the form $U_{1\alpha} \times \cdots \times U_{n\alpha}$:

$$U = \bigcup_{\alpha \in U} (U_{1\alpha} \times \cdots \times U_{n\alpha})$$

for α in some indexing set IU and where $U_{I\alpha}$ is open in X_i for each α. The preimage of such a $U_{I]alpha}$ under the projection pr_i is

$$pr_i^{-1}(U_{i\alpha}) = X_1 \times \cdots \times \underbrace{U_{i\alpha}}_{i\text{-th location}} \times \cdots X_n,$$

which we can write using product notation more succinctly as

$$pr_i^{-1}(U_{i\alpha}) = \prod_{j=1}^{n} U_j, \text{ where } U_i = U_{i\alpha} \text{ and } U_j = X_j \text{ for } i \neq j.$$

By the assumption on $\mathcal{T}$ this preimage is open in $\mathcal{T}$. But then the intersection of finitely many such preimages is also open in $\mathcal{T}$, and such an intersection is precisely of the form

$$U_{1\alpha} \times \cdots \times U_{n\alpha} = pr_1^{-1}(U_{1\alpha}) \cap \cdots \cap pr_n^{-1}(U_{n\alpha}).$$

Thus

$$U = \bigcup_{\alpha \in I} (U_{1\alpha} \times \cdots \times U_{n\alpha})$$

is open in $\mathcal{T}$ as well, and hence $\mathcal{T}$ is finer than the product topology as claimed.

Convergence in $\mathbb{R}^\omega$ with box. We finish by hinting at why we have to careful when trying to define the "product topology" in the case of infinite products. We can attempt to generalize the case we had for finite products as is and declare that the topology we want is the one generated by products of open sets. In the case of $\mathbb{R}^\omega$ (the space of infinite sequences of real numbers), this would say that the topology we want is the one generated by the basis consisting of things of the form

$$\prod_{n \times \mathbb{N}} U_n = U_1 \times U_2 \times U_3 \times \cdots$$

where each U_i is open in $\mathbb{R}$. The topology arising in this way is the *box* topology on $\mathbb{R}^\omega$, which is now distinguished from the *product* topology we will define next time. For instance, the infinite product

$$\prod_n (-\tfrac{1}{n}, \tfrac{1}{n}) = (-1,1) \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \times \left(-\tfrac{1}{3}, \tfrac{1}{3}\right) \times \cdots ,$$

where the n-th term is $(-\frac{1}{n}, \frac{1}{n})$, is open in the box topology on $\mathbb{R}^\omega$.

To see why the box topology is in some sense the "wrong" one to consider, take the sequence of elements in $\mathbb{R}^\omega$ given by

$$\mathbf{x}_n = \left(\tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \ldots\right).$$

To be clear, the first term in this sequence is $(1,1,1,\ldots)$, the second term is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$, and so on. (So we are taking a "sequence of sequences".) The question is: does this sequence in $\mathbb{R}^\omega$ converge? We will define what *convergence* means in an arbitrary topological space next time, but for now we are just thinking about it in an intuitive sense. You would hope that since the sequence $\frac{1}{n}$ in $\mathbb{R}$ converges to 0, the sequence we are looking at in $\mathbb{R}^\omega$ should converge to

$$(0,0,0,\ldots) \in \mathbb{R}^\omega.$$

However, this is *not* true in the box topology! In fact, the sequence

$$\left(\tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \ldots\right)$$

in $\mathbb{R}^\omega$ does not converge at all with respect to the box topology, the problem being that in a sense the box topology has "too many" open sets. However, this sequence *will* converge as we expect it to with respect to the product topology. We will elaborate on all this next time, but is essentially the key distinguishing feature of the product topology vs the box topology.

Lecture 6: More on Products

Warm-Up. Denote $\mathbb{R}^n$ with the Zariski topology by $\mathbb{R}^n_{Zar}$. We will determine the relation between $\mathbb{R}^2_{Zar}$ and the product topology on $\mathbb{R}_{Zar} \times \mathbb{R}_{Zar}$. (Of course, as sets both of these spaces are just $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.) First, recalling that the Zariski topology on $\mathbb{R}$ is the same as the cofinite topology, we note that closed sets in $\mathbb{R}_{Zar} \times \mathbb{R}_{Zar}$ (apart from $\mathbb{R} \times \mathbb{R}$ itself) are of the form

$$\{\text{finite set}\} \times \mathbb{R},\ \mathbb{R} \times \{\text{finite set}\},\ \{\text{finite set}\} \times \{\text{finite set}\},$$

or finite unions of such things. (In general, if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$ under the product topology since its complement is the union of open sets $(X - A) \times Y$ and $X \times (Y - B)$.) Furthermore, these three types of closed subsets are finite unions of closed sets of the form

$$\{\text{point}\} \times \mathbb{R},\ \mathbb{R} \times \{\text{point}\},\ \{\text{point}\} \times \{\text{point}\},$$

so if each of these is closed in $\mathbb{R}^2_{Zar}$ we will be able to conclude that anything closed in $\mathbb{R}_{Zar} \times \mathbb{R}_{Zar}$ is closed in $\mathbb{R}^2_{Zar}$, meaning that $\mathbb{R}^2_{Zar}$ is finer than $\mathbb{R}_{Zar} \times \mathbb{R}_{Zar}$.

The set

$$\{a\} \times \mathbb{R}$$

is the vertical line $x = a$, which is the zero set of the polynomial $x - a$ and hence is closed in $\mathbb{R}^2_{Zar}$, and the set

$$\mathbb{R} \times \{b\}$$

is the horizontal line $y = b$ and hence is closed in $\mathbb{R}^2_{Zar}$ since it is the zero set of $y - b$. A single point $\{(a, b)\}$ is thus the common zero set of the collection of polynomials given by $\{x - a, y - b\}$, so is also closed in $\mathbb{R}^2_{Zar}$. Thus $\mathbb{R}^2_{Zar}$ is finer than $\mathbb{R}_{Zar} \times \mathbb{R}_{Zar}$.

But we claim that the opposite inclusion does not hold: $\mathbb{R}_{Zar} \times \mathbb{R}_{Zar}$ is not finer than $\mathbb{R}^2_{Zar}$. Indeed, the parabola $y = x^2$ is closed in $\mathbb{R}^2_{Zar}$ since it is the zero set of $y - x^2$, but this is not closed in $\mathbb{R}_{Zar} \times \mathbb{R}_{Zar}$. Indeed, note that the types of closed sets in $\mathbb{R}_{Zar} \times \mathbb{R}_{Zar}$ mentioned above are all finite or collections of lines, and the parabola $y = x^2$ is none of these forms.

Box and convergence. We now come back to understanding why the box topology on, say, $\mathbb{R}^\omega$ was not really the right one to consider. First, let us be clear now about what convergence means in a general topological space. A sequence (p_n) in a space X *converges* to $p \in X$ if for any open set U containing p, there exists $N \in \mathbb{N}$ such that $p_n \in U$ for $n \geq N$. This is precisely the same notion of convergence you would have seen for sequences in $\mathbb{R}$ in an analysis course if you replace the arbitrary open set U with one of the form $(p - \epsilon, p + \epsilon)$. One key difference, as we will see later, is that in general topological spaces limits of sequences are *not* necessarily unique in the sense that a sequence can converge to possibly more than one point.

With this we can now justify the claim we finished with last time, namely that the sequence

$$\left(\tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \ldots\right)$$

in $\mathbb{R}^\omega$ does not converge to $(0, 0, 0, \ldots)$ with respect to the box topology. The set

$$(-1, 1) \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \times \left(-\tfrac{1}{3}, \tfrac{1}{3}\right) \times \cdots$$

is open in the box topology and contains $(0, 0, 0, \ldots)$. Thus if the given sequence did converge to $(0, 0, 0, \ldots)$, there would have to exist $N \in \mathbb{N}$ such that

$$\left(\tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \ldots\right) \in (-1, 1) \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \times \left(-\tfrac{1}{3}, \tfrac{1}{3}\right) \times \cdots$$

for $n \geq N$. But since all terms in this sequence are the same, this would require that

$$\tfrac{1}{n} \in \left(-\tfrac{1}{i}, \tfrac{1}{i}\right) \text{ for } n \geq N$$

for all $i \in \mathbb{N}$. In particular, all of these intervals would have $\frac{1}{N}$ in their intersection, which is nonsense because the intersection only consists of 0:

$$\bigcap_{i=1}^{\infty} \left(-\tfrac{1}{i}, \tfrac{1}{i}\right) = \{0\}.$$

Thus $(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots)$ does not converge to $(0, 0, 0, \ldots)$ with respect to the box topology as claimed.

The product topology on $\mathbb{R}^\omega$. To see what the "right" thing to consider is instead, let us return to characterization of the product topology we gave in the finite case as being the coarsest one satisfying the property that preimages of open sets under projects are always open. In the case of $\mathbb{R}^\omega$, say, we thus seek the coarsest topology on $\mathbb{R}^\omega$ with the property that

$$pr_i^{-1}(U) \text{ is open in } \mathbb{R}^\omega \text{ whenever } U \text{ is open in } \mathbb{R}$$

for every projection $pr_i : \mathbb{R}^\omega \to \mathbb{R}$? Note that such a preimage concretely looks like

$$pr_i^{-1}(U) = \mathbb{R} \times \cdots \times \mathbb{R} \times \underbrace{U}_{i\text{-th location}} \times \mathbb{R} \times \cdots .$$

Such a set would have to be open in the coarsest topology for which we are looking. But then the intersection of finitely many such sets would also have to be open, and such intersections look like

$$pr_{i_1}^{-1}(U_{i_1}) \cap \cdots \cap pr_{i_k}^{-1}(U_{i_k}) = \text{product with } U_{i_t} \text{ in the } i_t\text{-th location and } \mathbb{R}\text{'s elsewhere.}$$

Such products form a basis for a topology, and the topology they generate is the coarsest one we seek—*this* is what we call the *product topology* on $\mathbb{R}^\omega$.

The key difference between this and the box topology is that, while in the box topology any product of the form

$$U_1 \times U_2 \times U_3 \times \cdots ,$$

where each U_i is open in $\mathbb{R}$, is open, in the product topology such products are open only when all but *finitely many* factors are actually $\mathbb{R}$ itself (or, only finitely many factors are not all of $\mathbb{R}$). In the case of $\mathbb{R}^\omega$, this can also be phrased as saying that

$$U_1 \times U_2 \times U_3 \times \cdots ,$$

is open if there exists N such that $U_n = \mathbb{R}$ for $n \geq N$. Thus,

$$(-1, 1) \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \times \left(-\tfrac{1}{3}, \tfrac{1}{3}\right) \times \cdots$$

is not open in the product topology on $\mathbb{R}^\omega$, so the argument we gave for why $(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots)$ does not converge to $(0, 0, 0, \ldots)$ does not apply here.

Indeed, we now show that this sequence *does* converge to $\mathbf{0} = (0, 0, 0, \ldots)$ with respect to the product topology on $\mathbb{R}^\omega$. When checking the definition of convergence it is enough to check it for basic open sets, so we take a basic open set

$$U_1 \times U_2 \times U_3 \times \cdots$$

containing $\mathbf{0}$ in the product topology. Then there exists $K \geq 1$ such that $U_i = \mathbb{R}$ for $i > K$. For $i = 1, \ldots, K$, U_i is an open set containing 0 in $\mathbb{R}$, so since $\frac{1}{n} \to 0$ in $\mathbb{R}$ there exists N_i such that $\frac{1}{n} \in U_i$ for $n \geq N_i$. For $n \geq N := \max\{N_1, \ldots, N_K\}$ we then have

$$\tfrac{1}{n} \in U_i \text{ for } i = 1, \ldots, K, \text{ so that } (\tfrac{1}{n}\tfrac{1}{n}, \ldots) \in U_1 \times U_2 \times \cdots \text{ for } n \geq N.$$

This shows that $(\frac{1}{n}, \frac{1}{n}, \ldots)$ converges to $(0, 0, \ldots)$ with respect to the product topology as claimed.

Arbitrary products. For an arbitrary collection of spaces X_α indexed by α's coming from some indexing set, the product $\prod_\alpha X_\alpha$ consists of "tuples" of elements $(x_\alpha)_\alpha$ indexed by α's, where $x_\alpha \in X_\alpha$ for each α. (We think of x_α as the term at the "α-th location" of the tuple.) The product topology on $\prod_\alpha X_\alpha$ is then the one with basic open sets of the form

$$\prod_\alpha U_\alpha \text{ where } U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha = X_\alpha \text{ for all but a finite number of } \alpha\text{'s.}$$

If we drop the restriction that $U_\alpha = X_\alpha$ for all but a finite number of indices, we get a basic open set of the box topology on $\prod_\alpha X_\alpha$. As in the $\mathbb{R}^\omega$ case, the product topology on $\prod_\alpha X_\alpha$ in general is the coarsest one relative to which preimages of open sets under projections are always open, where, as we saw in $\mathbb{R}^\omega$, the finiteness restriction in the product topology definition boils down to the fact that intersections of only finitely many open sets are required to be open.

One main benefit of the product topology is that it is one relative to which convergence is equivalent to "componentwise convergence". This is the general version of the fact that in, say, $\mathbb{R}^3$ a sequence (a_n, b_n, c_n) converges if and only if each of the component sequences $(a_n), (b_n), (c_n)$ converge in $\mathbb{R}$. You will justify this in full generality as a homework problem.

Comparing topologies on $\mathbb{R}^\omega$. We now have nice examples of three topologies on $\mathbb{R}^\omega$, namely the product, uniform, and box topologies. The fact is that these are related to one another in the following way

$$\text{product} \subseteq \text{uniform} \subseteq \text{box}$$

but where each containment is strict. (We already know that the box topology is strictly finer than the product topology.) The fact that the box topology is finer than the uniform topology comes from the description we gave in the first Warm-Up from last time of uniform open balls in general:

$$B(\mathbf{x}, r) = \bigcup_{\epsilon>0} \prod_{n\in\mathbb{N}} (x_n - r + \epsilon, x_n + r - \epsilon).$$

The products on the right are open in the box topology, so their union is as well, and hence any uniform open ball is open in the box topology. This containment is strict since, as we saw before, allowing ourselves to get arbitrarily close to the endpoints of intervals as in

$$\prod_{n\in\mathbb{N}} (-x_n, x_n)$$

gives something which is not open in the uniform topology.

The fact that the uniform topology is strictly finer than the product topology will be left to a discussion problem.

Lecture 7: Closed Sets

Warm-Up. Recall that $\mathbb{R}^\infty$ denotes the set of sequences of real numbers which are eventually zero, so elements are $(x_1, x_2, x_3, \ldots)$ such that there exists $N \in \mathbb{N}$ for which $x_n = 0$ for $n \geq N$. We show that $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ with respect to the box topology but not the product topology. First, let $\mathbf{x} \in \mathbb{R}^\omega - \mathbb{R}^\infty$. Then $\mathbf{x}$ is not eventually zero, so $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ contains infinitely many terms which are nonzero, say $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$. Since $x_{n_k} \neq 0$, for each k there exists an open interval U_{n_k} in $\mathbb{R}$ containing x_{n_k} and not containing 0; for example we can take

$$U_{n_k} = (2x_{n_k}, 0) \text{ or } U_{n_k} = (0, 2x_{n_k})$$

depending on whether x_{n_k} is negative or positive. The product

$$\prod_n U_n \text{ where } U_{n_k} \text{ are as above and } U_n = \mathbb{R} \text{ for } n \neq n_{k_1}, n_{k_2}, \ldots$$

is open in the box topology on $\mathbb{R}^\omega$ and contains $\mathbf{x}$. Since U_{n_k} does not contain 0 for all k, any element of this product is not eventually zero, so

$$\mathbf{x} \in \prod_n U_n \subseteq \mathbb{R}^\omega - \mathbb{R}^\infty,$$

which shows that $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ with respect to the box topology.

Now, the argument given above does not apply when we have the product topology, since the product

$$U_1 \times U_2 \times \cdots$$

defined above is not open in the product topology since infinitely many factors are strictly smaller than $\mathbb{R}$ itself. Indeed, for $\mathbf{x} \in \mathbb{R}^\omega - \mathbb{R}^\infty$ let

$$V_1 \times V_2 \times \cdots$$

be a basic open set under the product topology containing it. Then only finitely many V_i are not $\mathbb{R}$, so

$$V_n = \mathbb{R} \text{ for } n \text{ past some index } N.$$

Define the element $\mathbf{y} \in \mathbb{R}^\omega$ by taking any terms from $V_1, \ldots, V_N$ as the first N components, but then taking 0 as the component in V_n for $n > N$. (Here we use the fact that $V_n = \mathbb{R}$ for $n > N$ to guarantee that V_n contains zero.) Then

$$\mathbf{y} \in V_1 \times V_2 \times \cdots$$

and $\mathbf{y}$ is eventually zero, so the basic open set $V_1 \times V_2 \times \cdots$ is not contained in the complement $\mathbb{R}^\omega - \mathbb{R}^\infty$. Since any open set must contain one of these basic ones, we conclude that no open set around $\mathbf{x}$ under the product topology is contained fully within $\mathbb{R}^\omega - \mathbb{R}^\infty$. Hence $\mathbb{R}^\omega - \mathbb{R}^\infty$ is not open, so $\mathbb{R}^\infty$ is not closed in the product topology on $\mathbb{R}^\omega$.

Function spaces. When all X_α in a product $\prod_{\alpha \in I} X_\alpha$ are the same, say $X_\alpha = X$ for all indices $\alpha \in I$, it is common to write the product as

$$X^I := \prod_{\alpha \in I} X$$

and to think of it as a product "I-many" copies of X. The elements of this product are then elements of X indexed by elements of I, so that for each $\alpha \in I$ we get some $x_\alpha \in X$. But this is precisely the data of a *function* $I \to X$, so that we can view the *function space*

$$X^I = \{\text{functions } I \to X\}$$

as a special case of a product of topological spaces. (Note that when $I = \mathbb{N}$ we get a function $\mathbb{N} \to X$, which is precisely the formal definition of a sequence of elements of X and in this special case it is common to denote $X^{\mathbb{N}}$ as X^ω as we have been doing in the case $X = \mathbb{R}$. When $I = \{1, \ldots, n\}$ is finite, a function $I \to X$ is the same as an n-tuple in $X^n = X \times \cdots \times X$ (n times).)

Viewing function spaces as product spaces allows us to thus consider various topologies on them. The box topology on this product is then generated by the basis consisting of sets of the form

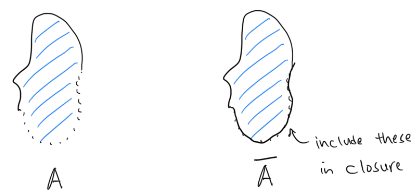
$$\{f : I \to X \mid f(\alpha) \in U_\alpha \text{ for all } \alpha\},$$

where the U_α are some prescribed open subsets of X, and the product topology is similar only that there are no restrictions on $f(\alpha)$ for all but a finite number of α. In the case of $\mathbb{R}^{\mathbb{R}}$, so functions $\mathbb{R} \to \mathbb{R}$, convergence with respect to the product topology is what is usually called *pointwise convergence* in analysis, which is just "componentwise convergence" for the product $\mathbb{R}^{\mathbb{R}} = \prod_{x \in \mathbb{R}} \mathbb{R}$. It is a (difficult) fact of analysis that there is no metric on this space of functions relative to which convergence corresponds to pointwise convergence, but there is at least a (non-metrizable) topology for which is this is true. The definition we gave before for the uniform metric on $\mathbb{R}^\omega$ works just the same on $\mathbb{R}^\mathbb{R}$, and convergence with respect to this uniform topology is what is usually called *uniform convergence* in analysis. (Note that although the product topology on $\mathbb{R}^\mathbb{R}$ is not metrizable, the product topology on $\mathbb{R}^\omega$ *is* metrizable, as shown in a discussion section problem. The product topology on $\mathbb{R}^I$ only becomes non-metrizable when I is uncountable!)

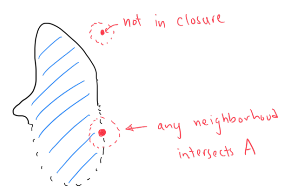
Closures. A key tool in building up more properties of topological spaces is the notion of *closure*. Given $A \subseteq X$, the closure $\overline{A}$ of A in X is the intersection of all closed subsets of X containing A:

$$\overline{A} := \bigcap_{\text{closed } C \supseteq A} C.$$

This makes the closure the "smallest" closed subset of X that contains A since it is contained in all other closed sets containing A. Note that $\overline{A}$ is always closed itself, and A equals its own closure if and only if A is already closed. Visually (in $\mathbb{R}^n$ at least), closures look like



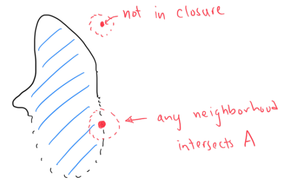
Describing the closure as an intersection of closed sets is a fine definition but not always of practical use since it does not say much about what points in the closure actually look like. A more practical characterization is given by the following. First, a quick piece of terminology: a *neighborhood* of $p \in X$ is simply an open subset of X that contains p. The result is that $p \in \overline{A}$ if and only if every neighborhood of p intersects A. This is intuitively clear in the picture above:



To justify this formally, we instead consider the contrapositive statements: $p \notin \overline{A}$ if and only if there exists a neighborhood of p in X that does not intersect A. So, suppose $p \notin \overline{A}$. Then p is not in the intersection of all closed sets containing A, so there exists a closed set C containing A and not p. Then $X - C$ is open and contains p but no element of A (since all elements of A are contained in C), so $X - C$ is the desired neighborhood.

Conversely, suppose there exists a neighborhood U of p that does not intersect A. Then $X - U$ is closed and contains A, so it is one of the sets being intersected when forming $\overline{A}$. This particular closed set does not contain A (since $p \in U$), so p cannot be in the closure of A as claimed.

Limits of sequences. One way in which we can construct elements of a closure is by looking at limits of convergent sequences. If (p_n) is a sequence of points in A that converges to p, then it is true that $p \in \overline{A}$. Indeed, let U be a neighborhood of p. Since (p_n) converges to p, there exists some p_N in U, so this p_N is an element of A in U, meaning that every neighborhood of p intersects A. Thus $p \in \overline{A}$. Here is the picture:



This picture suggests that the converse (elements in the closure are limits of sequences) might be true as well, but here we must be careful since the picture implicitly assumes a notion of distance. Indeed, the converge of the result above is true in any metric space but not necessarily in non-metrizable spaces. To see that the converse holds in a metric space, suppose X is a metric space, $A \subseteq X$, and that $p \in \overline{A}$. Then for each $n \in \mathbb{N}$ we have that the open ball $B(p, \frac{1}{n})$ is a neighborhood of p, so since $p \in \overline{A}$ there must exist $p_n \in B(p, \frac{1}{n}) \cap A$. The fact that $d(p, p_n) < \frac{1}{n}$ implies that (p_n) converges to p (any neighborhood around p will contain some open ball of the form $B(p, \frac{1}{n})$, and once it does it contains all open balls of smaller radii as well), so this (p_n) is a sequence of elements of A that converges to p.

$\mathbb{R}^\omega$ box ix not metrizable. We this we can justify the fact that the box topology on $\mathbb{R}^\omega$ is not metrizable. This is a consequence of the fact that we can find an element of the closure of some subset that is not the limit of any sequence in that subset, which by the result above is not possible in a metric space. Indeed, let $\mathbb{R}^\omega_+$ be the set of sequences of *positive* real numbers. Any neighborhood about the zero sequence $\mathbf{0}$ with respect to the box topology contains a basic open box set of the form

$$(a_1, b_1) \times (a_2, b_2) \times \cdots$$

with $a_i < 0 < b_i$ for all i. But then any such product contains a sequence of positive numbers, say

$$\tfrac{1}{2}b_1,\ \tfrac{1}{2}b_2,\ \tfrac{1}{2}b_3,\ \ldots,$$

so any neighborhood of $\mathbf{0}$ intersects $\mathbb{R}^\omega_+$. Thus $\mathbf{0}$ is in the closure of $\mathbb{R}^\omega_+$.

However, take an arbitrary sequence $(\mathbf{x}_n)$ in $\mathbb{R}^\omega_+$, where

$$\mathbf{x}_n = (x_{n1}, x_{n2}, x_{n3}, \ldots) \text{ with all } x_{ni} > 0.$$

The product

$$(-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times (-x_{33}, x_{33}) \times \cdots$$

is then open in the box topology and contains $\mathbf{0}$, but does not contain any term from the sequence $(\mathbf{x}_n)$ since the n-th component of x_{nn} specifically is not in the n-th component $(-x_{nn}, x_{nn})$ of the product above. Thus $(\mathbf{x}_n)$ cannot converge to $\mathbf{0}$, so no sequence in $\mathbb{R}^\omega_+$ can converge to $\mathbf{0}$.

Lecture 8: Hausdorff Spaces

Warm-Up. Suppose $A \subseteq X$ and $B \subseteq Y$. We show that closures behave well with respect to products in the sense that

$$\overline{A \times B} = \overline{A} \times \overline{B}$$

where we give $X \times Y$ the product topology. Since $\overline{A}$ is closed in X and $\overline{B}$ is closed in Y, $\overline{A} \times \overline{B}$ is closed in $X \times Y$. This closed set contains $A \times B$ and is thus one of the sets being intersected in forming $\overline{A \times B}$, so

$$\overline{A \times B} \subseteq \overline{A} \times \overline{B}.$$

Let $(p,q) \in \overline{A} \times \overline{B}$ and let $U \times V$ be a basic neighborhood of (p,q) in $X \times Y$. (Any neighborhood contains a basic one, so it is enough to check the closure condition on basic neighborhoods alone.) Then U is a neighborhood of p in X and V is a neighborhood of q in Y, so there exists $a \in A \cap U$ and $b \in B \cap V$ since $p \in \overline{A}$ and $q \in \overline{B}$ respectively. This gives (a,b) as an element of $A \times B$ inside $U \times V$, which shows that (p,q) is in the closure of $A \times B$. Thus $\overline{A \times B} \supseteq \overline{A} \times \overline{B}$, so equality holds.

Note that the same type of argument works for more general products, and in fact with respect to either the product and box topologies. For the forward containment one subtle point is that, although products of arbitrary open sets are not guaranteed to be open in the product topology (unless only finitely many of them are actually proper), there is no such restriction when it comes to products of closed sets: if C_α is closed in X_α for all α, the complement of $\prod_\alpha C_\alpha$ is

$$\prod_\alpha X_\alpha - \prod_\alpha C_\alpha = \bigcup_\beta \left(\prod_\alpha Y_\alpha \right)$$

where $Y_\beta = X_\beta - C_\beta$ and $Y_\alpha = X_\alpha$ for $\alpha \neq \beta$; this is open in the product (and box) topology as it is the union of basic open sets, so $\prod_\alpha C_\alpha$ is closed.

Hausdorff spaces. Recall that a topological space X is *Hausdorff* if for any distinct $p, q \in X$, there exist disjoint open sets U, V in X with $p \in U$ and $q \in V$. We previously showed as a Warm-Up that metrizable spaces are always Hausdorff. The spaces $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are also Hausdorff (but not metrizable, as is still to be shown) simply because they are finer than the standard topology on $\mathbb{R}$, so that open sets U, V separating p, q in the sense of above for the standard topology are still valid separating sets in either $\mathbb{R}_\ell$ or $\mathbb{R}_K$.

The line with two origins is not Hausdorff since no open sets around the two origins are ever disjoint. Similarly, the cofinite topology on $\mathbb{R}$ (or on any infinite set) is not Hausdorff since nonempty open sets always intersect.

Uniqueness of limits. To get a first sense of the benefit of being Hausdorff, note that the sequence

$$1,\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \ldots$$

in the line with origins converges to *both* of the two origins. Indeed, given any open interval $(a,b) \ni p$ around the first origin p (technically this interval is really $[(a,b) - \{0\}] \cup \{p\}$ where we replace 0 by p), we have that $\frac{1}{n} \in (a,b)$ for large enough n, and similarly for such an interval around the second origin q. The upshot is that limits of convergent sequences in general topological spaces are not necessarily unique.

Here is an even worse example. We ask whether the sequence

$$1,\ 2,\ 3,\ 4,\ \ldots$$

converges in $\mathbb{R}$ with the cofinite topology? The answer is that it does, and in fact that it converges to *all* real numbers! Indeed, pick $x \in \mathbb{R}$. Any neighborhood U of x in the cofinite topology excludes only finitely many terms from $1, 2, 3, 4, \ldots$, so for n being this finite number we have $n \in U$, which shows that $1, 2, 3, 4, \ldots$ converges to x. More generally, any sequence with infinitely many distinct terms will converge to everything in the cofinite topology.

But the claim is that if X is Hausdorff we do not run into this issue and limits of convergent sequences *are* unique. Suppose $p_1, p_2, p_3, \ldots$ converges to p and q in X. Let U be a neighborhood of p in X and V a neighborhood of q. Since the given sequence converges to p, there exists $N \in \mathbb{N}$ such that $p_n \in U$ for $n \geq N$, and since it converges to q there exists $M \in \mathbb{N}$ such that $p_n \in V$ for

$n \geq M$. Thus for $n \geq \max\{M, N\}$ we have $p_n \in U \cap V$, so that in particular U and V are not disjoint. Since X is Hausdorff, this requires that $p = q$ as desired.

T_1 **spaces.** The line with two origins and the cofinite topology on $\mathbb{R}$ are not Hausdorff, but they do have a sort of "half-Hausdorff" type of property; namely, they are T_1 *spaces*, which means that for any distinct p and q there exists a neighborhood of p that omits q *and* a neighborhood of q that omits p. We have that Hausdorff implies T_1 since Hausdorff is essentially the "simultaneous" version of T_1: T_1 says that we can separate p from q using an open set and that we can separate q from p, and Hausdorff says that we can do so at the "same time". (We will put this notion of "simultaneous separation" into the proper context in a bit.)

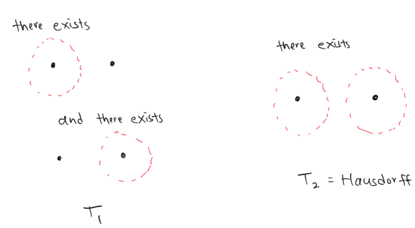
The T_1 condition is equivalent to the fact that finite sets are always closed. (Thus, the cofinite topology on any X is the coarsest T_1-topology one can put on X.) Note that since finite sets are unions of finitely many singletons, it is enough to know that singleton sets are closed. To prove this equivalence, suppose X is T_1 and let $p \in X$. For all $q \neq p$ there exists a neighborhood $U_q \ni q$ that does not containing p, meaning that $U_q \subseteq X - \{p\}$. As $q \neq p$ varies this expresses $X - \{p\}$ as a union

$$X - \{p\} = \bigcup_{q \in X - \{p\}} U_q$$

of open sets, so $X - \{p\}$ is open and hence $\{p\}$ is closed. Conversely, suppose singleton sets are closed in X and suppose $p \neq q$. Then $X - \{q\}$ is a neighborhood of p that does not contain q, and $X - \{p\}$ is a neighborhood of q that does not contain p, so X is T_1.

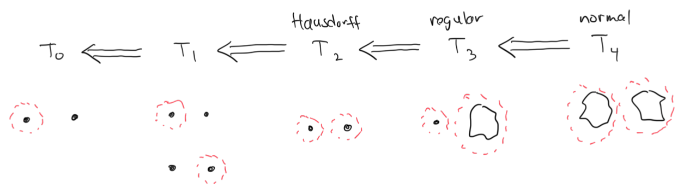
Separation axioms. The T_1 and Hausdorff conditions give two of what are known as the *separation axioms*, which are properties a space can have that specify the extent to which structures within it can be "separated" from one another. We will see that as we introduce more degrees of separation in this sense, we can better control the behavior of our spaces, and perhaps most importantly we can better control the behavior of *continuous* functions on that space.

The T_1 property says that two distinct points can be separated from another one at a time: find U containing p but not q, and then find V containing q but not p. This says that points in a T_1 space are "topologically distinguishable" from another in the sense that we can use open sets to tell them apart. Being Hausdorff (the Hausdorff condition is usually called the "T_2 axiom") says that we can distinguish points topologically simultaneously as we alluded to before:



But we can go further back to what is called the T_0 axiom, which is like T_1 but we can only guarantee that at least one point can be separated from the other, but not necessarily that both can be: given distinct p, q, for at least one of p, q we can find a neighborhood that does not contain the other. So, we can tell that p, say, is topologically distinct from q, but not that q is topologically distinct from p. A standard example of T_0 space which is not T_1 is $\{a, b\}$ with $\emptyset, \{a, b\}$, and $\{b\}$ being open: we can find a neighborhood $\{b\}$ of b that omits a, but there is no neighborhood of a that omits b. (Many spaces that appear in the subject of algebraic geometry are also T_0 but not T_1, which actually plays a big role in applications.)

Next up we have the T_3 axiom, which is what it means for a space to be *regular* and is like Hausdorff only that we replace one point by an entire closed set: closed sets and points not within them can be separated by open sets. (We will give formal definitions of these remaining types of spaces later.) The T_4 axiom—what it means for a space to *normal*—says now that all disjoint closed sets can be separated from another:



The numbering above was developed before more interesting properties of spaces that somehow fit "between" those above were identified. These properties shift the focus from separating using open sets to separating using *functions* instead. The $T_{3\frac{1}{2}}$-axioms (yes, that is the actual name), or what it means for a space to be *completely regular*, sys that points and closed sets can be "separated" using continuous functions in a sense we will define later. The analogous attempt at a "$T_{4\frac{1}{2}}$-axiom" where we try to separate closed sets using continuous functions actually ends up being equivalent to being T_4 (i.e., normal), which is the statement of what is called *Urysohn's lemma*, a cornerstone result in point set topology that we will prove towards the end of the quarter. In the end, Urysohn's lemma is, at its core, a result about certain continuous functions on that space behaving in a controlled way. The T_5 axiom (complete normality) and T_6 (perfect normality) are ones we will also come across, although they are not as impactful as just T_4 (normal) will be.