or

(undirected) simple graph

no self loops

directed graph

weighted graph

(undirected) graph

multigraph

"graph"

Def A graph $\boxed{G=(V,E)}$ is an ordered pair of a vertex set, V, and an edge set, E. For every edge $e \in E$, $\exists$ associated vertices $a, b \in V$, called the endpoints. (For directed graphs, we want to know the start and end.)

Undirected Graphs

$\forall e \in E$, $e \subseteq V$ is an (n- unordered) 2-element subset of V.

$V = \{1, 2, 3\}$

$E = \{\{1,2\}, \{1,3\}\}$

- can't do directed, multigraphs (or simple) w/ this model.

Directed Graphs

$E \subseteq V \times V$, i.e. each edge is of the form (a,b), $a, b \in V$.

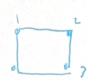
$V = \{1, 2, 3\}$

$E = \{(1,2), (3,1)\}$

- still can't do multigraphs.
- can model undirected graphs: if $a \leftrightarrow b$, $(a,b), (b,a) \in E$

Common Graphs

Cycle graph on $n \geq 3$ vertices, C_n : C_4



C_3

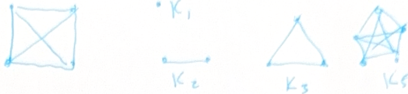
...

$V = \{1, 2, \dots, n\}$, $E = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}, \{n, 1\}\}$

Path graph on n vertices, P_n : P_5

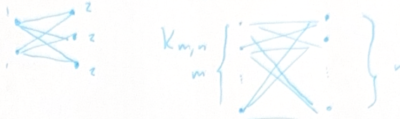
$V = \{1, 2, \dots, n\}$, $E = \{\{1,2\}, \dots, \{n-1, n\}\} = E_{C_n} \setminus \{n, 1\}$

Complete Graph on n vertices, K_n : K_4



$V = \{1, 2, \dots, n\}$, $E = \{ \{i,j\} \subseteq V : i \neq j \}$

Complete Bipartite Graph, $K_{m,n}$: $K_{2,3}$



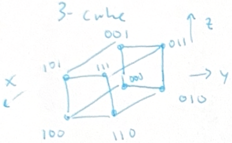
$V = \{ (1,1), \dots, (1,m), (2,1), \dots, (2,n) \}$ $E = \{ \{ (1,i), (2,j) \} : 1 \le i \le m, 1 \le j \le n \}$

$\updownarrow$ $\updownarrow$

$V = \{ a_1, \dots, a_m, b_1, \dots, b_n \}$ $E = \{ \{ a_i, b_j \} \mid 1 \le i \le m, 1 \le j \le n \}$

HyperCube Graph, Q_n

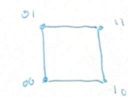
3-cube



V: all n-bit strings

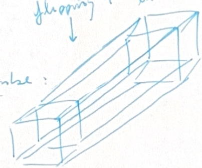
$E = \{ \{a,b\} \mid a \wedge b = k \ll 1, \ 0 \le k < n \}$

2-cube



flipping 4th bit

4-cube:



Graph	# vertices	# edges
C_n	n	n
P_n	n	$n-1$
K_n	n	$\frac{(n-1)(n)}{2} = \binom{n}{2}$
$K_{m,n}$	$m+n$	$m \cdot n$
Q_n	2^n	$\frac{n \cdot 2^n}{2} = n \cdot 2^{n-1}$

every vertex has n-edges

Some useful terms:

degree of a vertex: # of edges connected to vertex

connected vs disconnected

disconnected

connected

connected components

union of two graphs

$G = (V, E)$, $H = (V', E')$ $\to$ $G \cup H = (V \cup V', E \cup E')$

complement of a graph

G

$\bar{G}$

$V = \{1, \dots, n\}$.

$E = E_{K_n} \setminus E_G$

$G = (V, E)$

$\bar{G} = (V, E')$

$e \in E \iff e \notin E'$

Theorem Let $G = (V, E)$ be a simple undirected graph. Then

Recall: no self loops!

$$\sum_{v \in V} \deg(v) = 2|E|$$

Pf

$\deg(v) = $ # of edges incident on v. $= \sum_{e \in E}$ "does e touch v"

$\Rightarrow \sum_{v \in V} \deg(v) = \sum_{v \in V} \sum_{e \in E}$ "does e touch v?"

$= \sum_{e \in E} \sum_{v \in V}$ "does e touch v?" (commutativity of addition)

$= \sum_{e \in E} 2$ (each edge touches exactly 2 vertices)

$= 2|E|$. $\boxtimes$

Q: How many edges does $\bar{C}_n$ have?

C_4

$\bar{C}_4$

edges in C_4 + $\bar{C}_4$ =

edges in K_4

$\to$ # edges in $\bar{C}_n$ =

edges in K_n − # edges in C_n

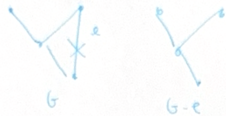
$= \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$

Remove an edge $e \in E$

$G' = G - e$

$V' = V$

$E' = E - e$

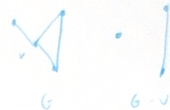


Remove a vertex $v \in V$

$G' = G - v$

$V' = V - v$

$E' = E -$ all edges incident on v



~~Defn~~ : ~~the~~ ~~E~~

Def $H = (V', E')$ is a subgraph of G if it is a graph w/ $V' \subseteq V$, $E' \subseteq E$.

↳ ex $G = (V, E)$

$G - v$ is a subgraph of G $\forall v \in V$

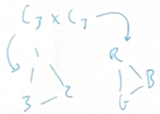
$G - e$ is a subgraph of G $\forall e \in E$ ⟶ any subgraph can be obtained by removing edges & vertices.

Def The product graph $\underset{(V,E)}{G} \times \underset{(V',E')}{H} = (V \times V', E \times E')$

$V \times V' = \{(a,b) \mid a \in V, b \in V'\}$

$E \times E' = \{(a,x),(a,y) \mid (x,y) \in E', a \in V\} \cup \{(x,b),(y,b) \mid (x,y) \in E, b \in V'\}$

ex $C_3 \times C_3$



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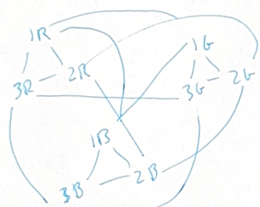
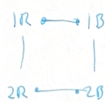
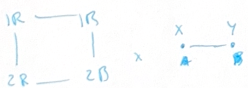


ex $K_2 \times K_2$

=



ex $K_2 \times K_2 \times K_2$



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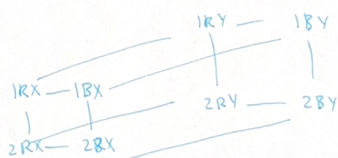


How many vertices does $G \times E$ have? edges?

$\downarrow$

$|V \times V'| = |V| \cdot |V'|$

$\searrow$

edges caused by G + # edges caused by H

$= |E| \cdot |V'| + |E'| \cdot |V|$

Exercise! Using $\prod_{i=1}^{n} K_2 = Q_n$, prove # vertices in $Q_n = 2^n$, # edges in $Q_n = n \cdot 2^{n-1}$.

Let $G = (V, E)$ be a graph, $u, w \in V$ be vertices.

Def: ① a walk from u to w is a sequence of vertices

$$u = v_0 - v_1 - v_2 - \cdots - v_n = w,$$

such that each v_i is adjacent to v_{i+1}.

② the length of the walk is n.

ex

$1 - 2 - 3 - 4 - 2 - 3$ is a walk from 1 to 3 of length 5.

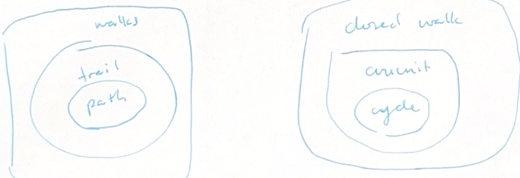
4 is a trivial walk (of length 0).

Def a closed walk starts and ends at the same vertex.

e.g. $1 - 2 - 3 - 4 - 2 - 3 - 2 - 1$.

Def a trail is a walk where no edge is repeated (closed trail of length ≥ 3 = circuit)

a path is a walk where no vertex is repeated (closed path of length ≥ 3 = cycle)



Def a graph $G = (V, E)$ is connected if $\forall u, v \in V$ $\exists$ a walk from u to v.

Theorem Let $G=(V,E)$ with $u,v\in V$. Then if $\exists$ a walk from u to v, then $\exists$ a path from u-v.

Pf (Sketch)

$u-1-2-3-4-5-6-7-8-9$ is a walk from u-v.

↳ $u-1-2-8-9-v$ is a path from u-v, which we obtained by removing closed loops from our walk.

Def Let $G=(V,E)$. The distance, $d(u,v)$, from $u\in V$ to $v\in V$ is the length of the shortest path from u to v. If no paths exist, write $d(u,v)=+\infty$.

- We define the geodesic from u to v to be the shortest path from u to v.

Theorem Let $G=(V,E)$ be a graph.

1) For any $a,b\in V$, $d(a,b)\geq 0$ w/ $d(a,b)=0 \iff a=b$.
2) For any $a,b,c\in V$, $d(a,c)\leq d(a,b)+d(b,c)$
3) For any $a,b\in V$, $d(a,b)=d(b,a)$.

$\Rightarrow$ d is a metric on V.

Pf ~~(sketch)~~ - Trivial. ☒

~~① Length of path~~

~~② Suppose $d(a,c)$ ...~~

Theorem Let $G=(V,E)$. For $a,b\in V$, write $a\sim b$ if $\exists$ a path from a to b.

1) $u\sim u$ $\forall u\in V$. (reflexive)
2) $u\sim v \iff v\sim u$ $\forall u,v\in V$ (symmetric)
3) $u\sim v$, $v\sim w \Rightarrow u\sim w$ $\forall u,v,w\in V$ (transitive)

Pf (Trivial ☒)

↳ Every graph is the union of its connected components

Pf Partition V by $\sim$ from above; equivalence classes are connected components. ✓

Bipartitions

Def an undirected simple graph $G=(V,E)$ is bipartite if $V=V_1\cup V_2$ where $e=(a,b)$, $a\in V_1$, $b\in V_2$ $\forall e\in E$.

(can "colour" vertices w/ two different colors where adjacent vertices are different colours)

Theorem (Book): G bipartite $\iff$ G doesn't contain any odd cycles.

11

Def The <u>diameter</u> $d(G)$ of a graph G is $\max_{a,b\in V} \text{dist}(a,b)$

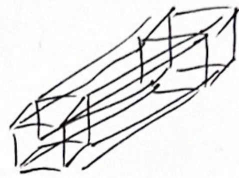
ex

1) $\text{diam}(K_n) = 1$ for $n \geq 2$

2) $\text{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$

3) $\text{diam}(P_n) = n-1$

4) $\text{diam}(Q_n) = n$ (worst case, need to flip n bits)



<u>Claim</u> G is bipartite $\iff$ G has no odd cycles.

pf ($\Rightarrow$) Suppose G has an odd length cycle: $v_0 - \cdots - v_{2k+1} = v_0$. If G bipartite, $V = V_1 \cup V_2$ w/ $E = \{(a,b) \mid a \in V_1, b \in V_2\}$. WLOG, $v_0 \in V_1$. Then $v_i \in \begin{cases} V_1 & i = 2j \\ V_2 & i = 2j+1 \end{cases}$ (induction), $j \in \mathbb{Z}$. Then $v_0 = v_{2k+1} \in V_2$ so $v_0 \in V_1 \cap V_2 = \emptyset$.

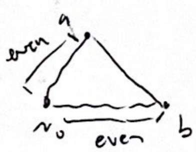
($\Leftarrow$) Suppose G has no odd length cycles. Suffices to show claim holds for each connected component. Thus WLOG, assume G connected. Let $v_0 \in V$ be any vertex. Take

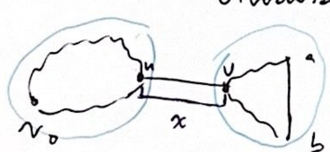
$$V = V_1 \cup V_2$$

s.t.

$$V_1 = \{v \in V \mid \text{dist}(v_0, v) \text{ even}\}, \quad V_2 = \{v \in V \mid \text{dist}(v_0, v) \text{ odd}\}.$$

Just need to check $e = (a,b)$ s.t. $a \in V_1, b \in V_2$ $\forall e \in E$. Consider an edge ~~of~~ $e = (a,b)$ where ~~...~~ $a, b \in V_1$. If $v_0 - a$, $v_0 - b$ don't overlap, then $v_0 - a - b - v_0$ is a cycle of odd length. Otherwise, if $v_0 - a$, $v_0 - b$ overlap from $u - v$, ~~...~~, a path of length x. Note





# of edges from $v_0 - b$ (even) ~~(odd)~~

$+$ # of edges from $a - v_0$ (even)

$+$ $\text{dist}(a,b)$ ~~...~~ odd

$= 2x + \ell(v_0 - u - v_0) + \ell(v - a - b - v)$ is ~~even~~ odd. Thus $\ell(u - v_0 - u) + \ell(u,v) + \ell(v-b-a-v) + \ell(u,v)$ is odd so one of either cycle is odd. ∎

Theorem Let $G = (V,E)$ be a graph w/ n vertices. Suppose $\forall$ nonadjacent $u, v \in V$,

$$\deg(u) + \deg(v) \geq n-1$$

Then G is connected, and $\text{diam}(G) \leq 2$.

pf Let $x, y \in V$. If $\{x,y\} \in E$ or $x = y$, ~~...~~ $\exists$ a path. Suppose instead $x \neq y$ and x, y nonadjacent. Then ~~...~~ $\exists$ path between $x - y$ by pigeonhole. ∎

11

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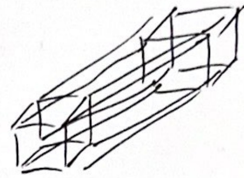

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Then $v_0 = v_{2k+1} \in V_2$ so $v_0 \in V_1 \cap V_2 = \emptyset$.

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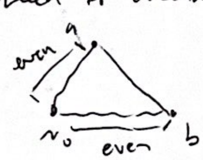

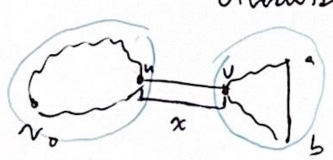

$\#$ of edges from $v_0 - b$ (~~odd~~ even)

$+$ $\#$ of edges from $a - v_0$ (even)

$+$ $\mathrm{dist}(a,b)$ ~~even~~ odd

$= 2x + \ell(v_0 - u - v_0) + \ell(v - a - b - v)$ is ~~even~~ odd. Thus $\ell(u - v_0 - u) + \ell(u,v) + \ell(v\text{-}b\text{-}a\text{-}v) + \ell(u,v)$ is odd so one of either cycle is odd. ∎

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Then G is connected, and $\mathrm{diam}(G) \leq 2$.

pf Let $x, y \in V$. If $\{x,y\} \in E$ or $x = y$, $\exists$ a path. Suppose instead $x \neq y$ $\wedge$ x, y nonadjacent. Then $\exists$ path between $x - y$ by pigeonhole. ∎

Theorem Let $G=(V,E)$ have n vertices. Suppose $\min_{v \in V} \deg(v) \geq \frac{n-1}{2}$. Then G is connected.

Pf

Follows immediately from previous example.

Def We say $G=(V,E)$ is k-regular if $\forall v \in V$, $\deg(v) = k$.

ex

K_n is $(n-1)$ regular

C_n is 2-regular

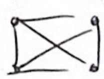
Q_n is n-regular

P_n is not k-regular.

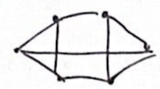
ex Any 3-regular graph on at most 7 vertices must be connected

Pf If $n \leq 7$, $\frac{n-1}{2} \leq 3 \Rightarrow \min_{v \in V} \deg(v) \geq \frac{n-1}{2}$

Q are there 3-regular graphs on:

4 vertices!  K_4

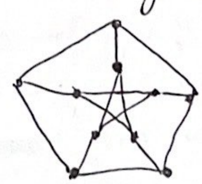
5 vertices? $\sum_{v \in V} \deg(v) = 15 = 2|E|$ requires 7.5 edges.

6 vertices! 

(!!) 7 vertices? $\sum_{v} \deg(v) = 21 \Rightarrow$ impossible!

Another famous graph

The Petersen Graph is a 3-regular graph on 10 vertices:

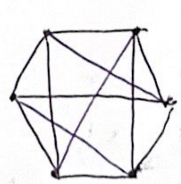


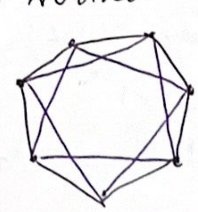
* Famously serves as a counterexample to many different hypotheses *

Q are there 4-regular graphs on:

5 vertices! 6 vertices? 7 vertices! Connect each vertex i to $i \pm 1$, $i \pm 2$.

K_5





Theorem Let $0 \le k \le n-1$. Then $\exists$ a k-regular graph on n vertices $\iff$ $k \cdot n$ ~~kleo~~ is even.

Pf.

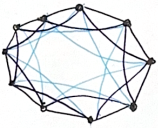
($\Rightarrow$) Assume $\exists$ k-regular graph on n-vertices $G = (V, E)$. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$
$$\|$$
$$\sum_{v \in V} k = k|V| = kn$$

So kn is even. ~~Note~~ ~~suppose~~ Thus k or n (or k, n) is even.

($\Leftarrow$) Suppose kn even. If k even, connect i to $i \pm 1, i \pm 2, \dots, i \pm \frac{k}{2}$ (mod n) $\forall i \in V$. Otherwise, connect i to $i \pm 1, i \pm 2, \dots, i \pm \frac{k-1}{2}$, $i + \frac{n}{2}$ (mod n), $(\forall i \in V)$. ☒

ex: $k = 6$, $n = 10$.



$i \pm 1, 2, 3$

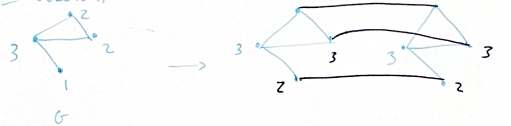
$k = 5$, $n = 8$

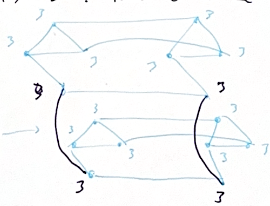
$i \pm 1, \pm 2$

~~i~~ $+ 4$

Theorem Let $G = (V, E)$ be a graph w/ max degree of any vertex $\le r$. Then G is a subgraph of some r-regular graph.

Pf (Sketch)





"copy the graph and connect deficient vertices"

$$G = (V, E) \quad \begin{array}{l} \max_{v \in V} \deg(v) = r \\ \min_{v \in V} \deg(v) = r - k \end{array}$$

$$H := G \cup G', \text{ where } (v, v') \in H \text{ if } \deg(v) < r$$

$(\{v' := v \mid v \in V(G)\}, \{$ ~~edges~~ $e' := e \mid e \in E(G)\})$ $=$ G'

$$\rightarrow H: \max_{v \in V(H)} \deg(v) = r, \quad \min_{v \in V(H)} \deg(v) = r - k + 1.$$

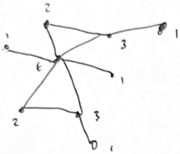
Repeat this process for H', H'', ... ☒

Notation: $\delta(G) = \min_{v \in V} \deg(v)$, $\Delta(G) = \max_{v \in V} \deg(v)$.

G is r-regular $\Leftrightarrow$ $\delta(G) = \Delta(G) = r$.

Def Let $G=(V,E)$. The degree sequence of G is a list of degrees $\deg(v)$ for $v \in V$ (with repeats as needed). Often, the list is written in non-increasing order.

ex



The degree sequence is: 6, 3, 3, 2, 2, 1, 1, 1, 1

ex

Does any graph have degree sequence 4, 4, 2, 2, 1

Note: $\sum_v \deg(v) = 13$, odd but sum of degrees must be even. (So no.)

"this sequence is not graphical"

ex

Does any graph have degree sequence 7, 6, 5, 4, 3, 2, 1 ?

Even though $\sum \deg(v)$ even, still impossible because such a graph would have 7 vertices so each vertex has degree ≤ 6. ∎

ex

Does any graph have degree sequence 3, 3, 3, 1.
 A B C D

Such a graph would have A, B, C, D each connected to all 3 vertices other than themselves. In particular, D would be connected to each of them, contradicting $\deg(D) = 1$.

Theorem Let $n \geq 2$ and consider the non-increasing sequence of non-negative integers

$$s: \underbrace{d_1}_{= k}, d_2, \ldots, d_n$$

Then s is a degree sequence for some graph $\Leftrightarrow$

$$s' = d_2 - 1, d_3 - 1, \ldots, d_{k+1} - 1, d_{k+2}, \ldots, d_n$$

is a degree sequence for some graph. In particular, $k \leq n-1$.

<u>ex</u>

same vertices

$\cancel{3}$, 3, 2, 2, 1, 1 $\quad$ <u>k = 3</u>

-1 for k = 3 terms

$\rightsquigarrow$ $\cancel{2}$, 1, 1, 1, 1 $\quad$ k = 2

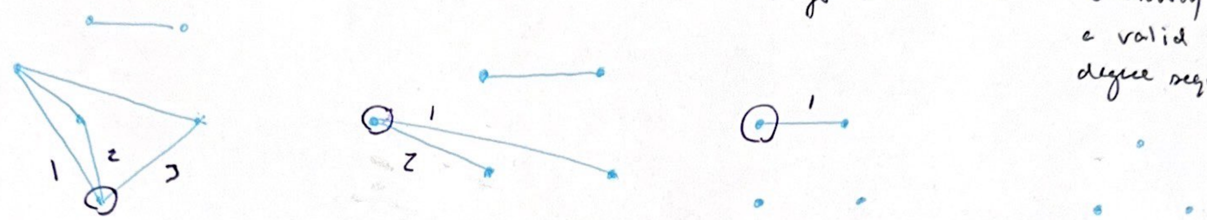
-1 for k = 2 terms

$\rightsquigarrow$ 0, 0, 1, 1 $\downarrow$ sort $\cancel{1}$, 1, 0, 0 $\quad$ <u>k = 1</u>

-1 for k = 1 terms

$\rightsquigarrow$ 0, 0, 0

obviously a valid degree sequence.



Backtracking to construct graph.

<u>Pf of theorem</u>

too long. look it up lmao.

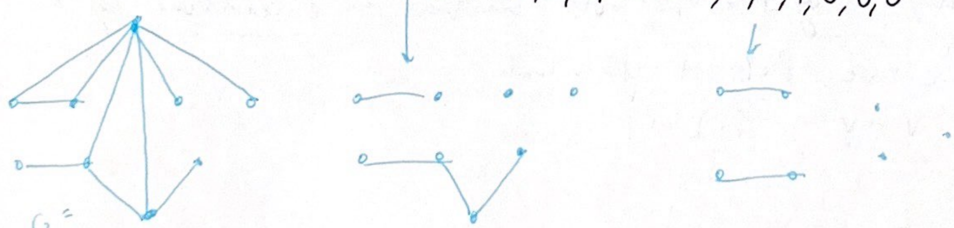
<u>ex</u>

6, 3, 3, 2, 2, 1, 1, 1, 1

2, 2, 1, 1, 0, 0, 1, 1 → 2, 2, 1, 1, 1, 1, 0, 0

1, 0, 0, 1, 1, 0, 0 → 1, 1, 1, 1, 0, 0, 0

G =



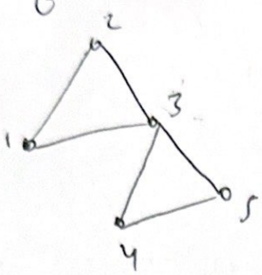
Def Let $G = (V, E)$, $H = (V', E')$ be graphs. A <u>morphism</u> from G to H is a <u>relabeling</u> of G's vertices in such a way that all the edges are still valid. That is, a morphism from G to H denotes

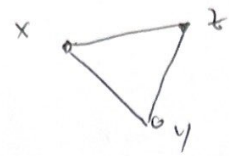
$$f : V \to V'$$

which also induces a function on the edges

$$f : E \to E'$$
$$\{a, b\} \mapsto \{f(a), f(b)\}.$$

ex





$f : \{1,2,3,4,5\} \to \{x,y,z\}$

$f(1) = x,\ f(2) = y,\ f(3) = z$

$f(4) = x,\ f(5) = y$

is a morphism from G to H.

Why? Check all the edges:

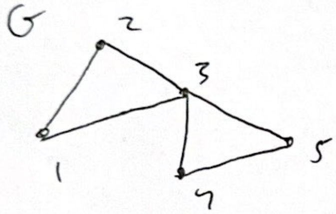
$\{1,2\} \to \{x,y\}$ $\quad$ $\{3,4\} \to \{x,z\}$

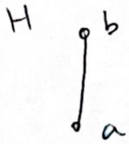
$\{1,3\} \to \{x,z\}$ $\quad$ $\{3,5\} \to \{y,z\}$

$\{2,3\} \to \{y,z\}$ $\quad$ $\{4,5\} \to \{x,y\}$

all resultant edges are valid edges.

ex





Is there a morphism $G \to H$?

No. 1 has to go to either a or b. WLOG, $f(1) = a$. Then since $(1,2) \in E(G)$, $f(2) = b$ to preserve the edge. However,

$f(3) = b \Rightarrow \{2,3\} \to \{b,b\}$ X

but $f(3) = a \Rightarrow \{1,3\} \to \{a,a\}$ X

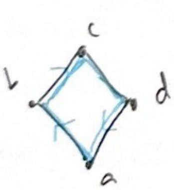
ex

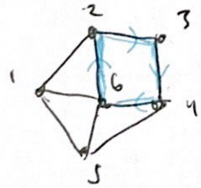
If $G = (V,E)$ is a subgraph of $H = (V', E')$ then we have an <u>embedding</u> of G into H. That is, we have $f : G \to H$ for which

$$f : V \to V', \quad f : E \to E'$$

are both injective.

ex





The morphism $f : G \to H$ via

$a \mapsto 6$

$b \mapsto 2$

$c \mapsto 3$

$d \mapsto 4$

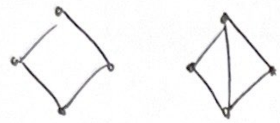
is an embedding.

Def $f: G \to H$ is an <u>isomorphism</u> of graphs if the resulting maps

$$f: V \to V'$$
$$f: E \to E'$$

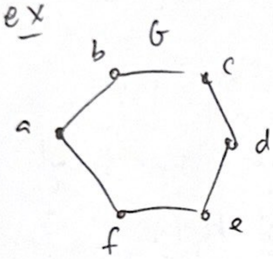
are both bijective.

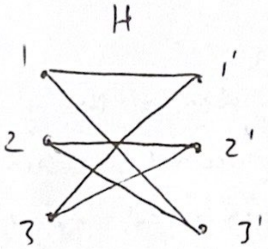
<u>ex</u>



can <u>not</u> be isomorphic. (different number of edges)

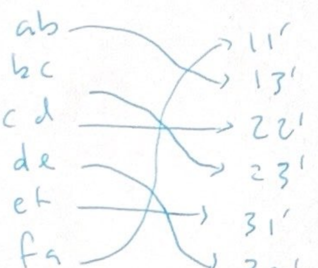
<u>ex</u>





Are G & H isomorphic?
Hint: they're both C_6 :)

$f: G \to H$, ~~[illegible]~~,
$f(a) = 1$, $f(b) = 3'$, $f(c) = 2$, $f(d) = 2'$
$f(e) = 3$, $f(f) = 1'$
is an isomorphism. (Bijection on vertices & edges)

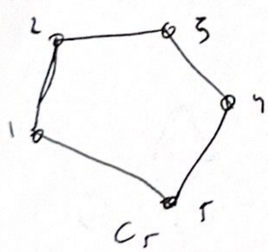


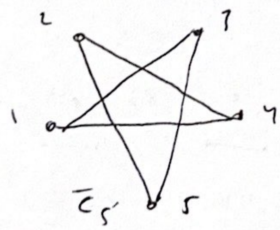
So $G \cong H$

"is isomorphic"

<u>ex</u>

Are C_5 & $\overline{C_5}$ isomorphic?





yes
$1 \mapsto 1$
$2 \mapsto 3$
$3 \mapsto 5$
$4 \mapsto 2$
$5 \mapsto 4$

will give a bijection on vertices & edges.

<u>Note</u>:

When G, H isomorphic, we write $G \cong H$.

Equivalence Relation Properties.

Let G, H be graphs.

- Clearly, $G \cong G$ (id map)
- $G \cong H \Rightarrow H \cong G$ (bijections are invertible)
- Let K be a graph. $G \cong H$, $H \cong K \Rightarrow G \cong K$

} $\cong$ is an equivalence relation.

Basic Properties

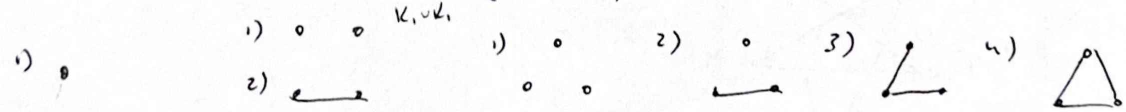
1) If $G = (V, E)$, $H = (V', E')$, $G \cong H$, then
   - $|V| = |V'|$
   - $|E| = |E'|$
   - the degree sequence of G is the same as that of H.
2) If $G \cong H$, then G connected $\Leftrightarrow$ H connected. (In fact, the # of connected components of G is the # of connected components of H).
3) If $G \cong H$, then G bipartite $\Leftrightarrow$ H bipartite.
4) $G \cong H \Leftrightarrow \overline{G} \cong \overline{H}$

these are all pretty intuitive....

Counting Isomorphism Classes

Up to isomorphism, how many (simple, undirected) graphs are there on n vertices?

$n = 1$ (1 total) | $n = 2$ (2 total) | $n = 3$ (4 total)

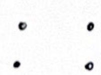


$n=1$: 1) K_1

$n=2$: 1) $K_1 \cup K_1$ 2) $K_2 \cong P_2$

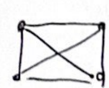
$n=3$: 1) $K_1 \cup K_1 \cup K_1$ 2) $K_1 \cup K_2$ 3) P_3 4) $K_3 \cong C_3$

$n = 4$ (11 total)

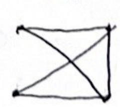
0 edges: 

1 edge: 

$K_2 \cup K_1 \cup K_1$

6 edges: 

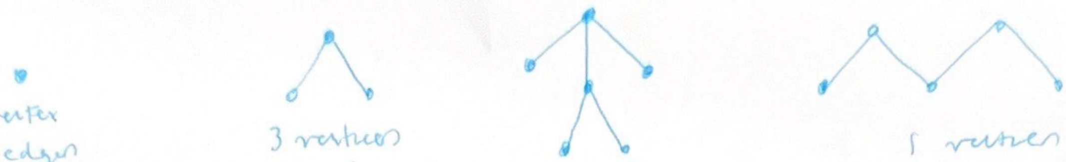
K_4

5 edges: 

complements.

Theorem A tree on n vertices has $n-1$ edges.

Ex



1 vertex, 0 edges — 3 vertices, 2 edges — 6 vertices, 5 edges — 5 vertices, 4 edges

Pf

Proceed by induction on n.

- Base case trivial. Any tree on $n=1$ vertices must have $n-1=0$ edges.
- Fix $k \geq 2$. Suppose any tree on $n=k-1$ vertices has $n-1=k-2$ edges.

Let T be a tree on $n=k$ vertices. Since $k \geq 2$, T is a tree on ≥ 2 vertices, so it has ≥ 2 leaves. Let v be a leaf node of T (i.e. $v \in V(T)$, $\deg(v)=1$).

Pick a leaf node.

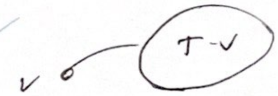
$T-v$ is a tree

- Clearly, $T-v$ doesn't have cycles (T acyclic, we deleted a node).
- Let $u, w \in V(T-v)$ be two arbitrary vertices. Since T connected, $\exists$ a path $u-w$ in T, say $u=v_0 - v_1 - \dots - v_m = w$. Note $v_0, v_m \neq v$ since $u, w \in V(T-v)$, and if $v_i = v$ for some $0 < v_i < m$ then v_{i-1} and v_{i+1} would both be neighbors of v, so $\deg_T(v) \geq 2 \Rightarrow \Leftarrow$

Apply induction hypothesis

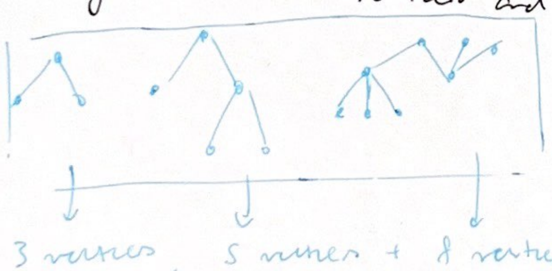
Thus $T-v$ connected tree on $n-1=k-1$ vertices, thus has $n-2$ edges. Thus T has $n-1$ edges ✓

By Induction, the proof is complete. ☒



Theorem a forest with n vertices and m edges has $n-m$ connected components.

ex



3 vertices, 2 edges + 5 vertices, 4 edges + 8 vertices, 7 edges = 16 vertices, 13 edges

"Each tree gets a -1 penalty"

Pf

Let F be a forest w/ n vertices, m edges. Write $F = T_1 \cup \dots \cup T_k$, $\{T_i\}_{1 \le i \le k}$ are trees. Let n_i = # vertices of T_i. Then $\sum_i n_i = n$. Now,

$$m = \# \text{ edges in } F = (\# \text{ edges of } F_1) + \dots + (\# \text{ edges of } F_k)$$
$$= (n_1 - 1) + \dots + (n_k - 1)$$
$$= n - k$$

Thus $k = n - m$. ☒

Corollary If G acyclic w/ n vertices, then G has at most $n-1$ edges.

Pf

G acyclic $\Rightarrow$ forest. So # vertices of G − # edges of G = # connected components of $G \geq 1$.

So # edges in $G \leq$ # vertices in $G - 1 = n - 1$. ☒

Theorem Every connected graph G on n vertices has a subgraph $T \subseteq G$ on n vertices, which is a tree.

Def T is called a spanning tree for G.

Pf

Induct on the # of edges $= m$.

- Base cases: $m = 0, 1, 2$. automatically a tree ✓
- Suppose every connected graph with $m = k-1$ edges contains a spanning tree (fix $k \geq 3$). Let G be a connected graph w/ k edges. Either G has no cycles (we're done ✓) or it has at least one cycle. Then remove an edge from its cycle. The resulting graph is connected on $k-1$ edges, so we're done by induction hypothesis. ☒

Corollary Let G be any connected graph on n vertices. Then $|V(G)| \geq n-1$.

Pf

G connected $\Rightarrow$ G contains a spanning tree, which has $n-1$ edges. ☒