13.1 Let G be a group and $N \trianglelefteq M \trianglelefteq G$. Show $(G/N)/(M/N) \cong G/M$

Define $\phi : G/N \to G/M$ by $\phi(Ng) = Mg$.

- Suppose $Ng_0 = Ng_1$, $g_0, g_1 \in G$. Then $g_0 g_1^{-1} \in N$, so $\phi(Ng_0) = Mg_0 = Mg_0g_1^{-1}g_1 = \phi(N\overbrace{g_0g_1^{-1}}^{\in N}g_1) = \phi(Ng_1)$.
  Thus ϕ is well defined.
- Notice ϕ is obviously surjective by its definition. Now, from Lemma 12.9 we have
  $$\phi(Ng_0)\phi(Ng_1) = Mg_0Mg_1 = Mg_0g_1 = \phi(Ng_0g_1) = \phi(Ng_0Ng_1),$$
  so ϕ is a surjective homomorphism.
- Furthermore, if $Mg = \phi(Ng) = M$, then $g \in M$, so
  $$\ker(\phi) = \{Ng \in G/N \mid \phi(Ng) = M\} = \{Ng \mid g \in M\} = M/N$$

Thus $M/N = \ker(\phi) \trianglelefteq G$, and by Exercise 12.3, $G/M \cong (G/N)/(M/N)$ ■

13.2 Find an example of a group G with subgroups $N \leq M \leq G$ where $N \trianglelefteq M$, $M \trianglelefteq G$, but $N \ntrianglelefteq G$.

Take $G = D_8$, $M = \langle s, r^2 \rangle$ $(= \{s, sr^2, r^2, e\})$, $N = \langle s \rangle$ $(= \{e, s\})$

$N \trianglelefteq M$

Note that $mem^{-1} = e \in N$ $\forall m \in M$, and

$sss^{-1} = s \in N$, $sr^2(s)(sr^2)^{-1} = sr^2sr^2s = s \in N$, $r^2s(r^2)^{-1} = r^2r^2s = s \in N$, $ese^{-1} = s \in N$ ✓

so $N \trianglelefteq M$.

$M \trianglelefteq G$

Similarly, $geg^{-1} = e \in M$ $\forall g \in G$. Also, $M \trianglelefteq M$ so if $m_0 \in M$, $g(m)g^{-1} \in M$ $\forall g \in M = \{e, s, sr^2, r^2\}$.

Still need to check $gmg^{-1} \in M$ $\forall g \in \{r, r^3, sr^3\}$:

- $rsr^{-1} = sr^{-2} = sr^2 \in M$, $r^3s(r^3)^{-1} = s(r^3)^{-1}(r^3)^{-1} = sr^2 \in M$, $(sr^3)(s)(sr^3)^{-1} = (r^3)^{-1}(r^3)^{-1}s^{-1} = sr^2 \in M$
- $rsr^2r^{-1} = rsr = s \in M$. $r^3(sr^2)(r^3)^{-1} = srr^2r = s \in M$, $(sr^3)(sr^2)(sr^3)^{-1} = ssrr^2rs = s \in M$.
- $rr^2r^{-1} = r^2 \in M$, $r^3r^2(r^3)^{-1} = r^2 \in M$, $sr^3r^2(sr^3)^{-1} = srrs = r^2 \in M$

Thus $M \trianglelefteq G$.

$N \ntrianglelefteq G$

We have $s \in N$. Note that $(sr)(s)(sr)^{-1} = ssr^{-1}r^{-1}s = r^2s = sr^2 \notin N$.

So N is not normal in G.

13.3 Let H be the stabalizer of n in S_n acting on $\{1, \dots, n\}$. What is the order of H? What is $H \cong$ to?

- $H = (S_n)_n = \{\sigma \in S_n \mid \sigma(n) = n\} \cong S_{n-1}$
- Note that $|S_{n-1}| = (n-1)!$. We can also observe that $|(S_n)_n| = \dfrac{|S_n|}{|S_n \cdot n|} = \dfrac{n!}{n} = (n-1)!$

14.1

Let G be a p-group of order p^2. By Theorem 14.8, $|Z(G)| \in \{p, p^2\}$. If $|Z(G)| = p^2$, $Z(G) = G$ and we're done. (G is abelian). If $|Z(G)| = p$, $\left|\frac{G}{Z(G)}\right| = \frac{|G|}{|Z(G)|} = \frac{p^2}{p} = p$, so $G/Z(G) \cong \mathbb{Z}/p$

Write $G/Z(G) = \langle Z(G)g \rangle$ for some $g \in G$, abusing the notation of $\tilde{g} := Z(G)g$. Then, writing $x, y \in G$ as

$x = z_1 g^n$, $y = z_2 g^m$, where $z_1, z_2 \in Z(G)$ and $0 \leq n, m < p$.

Then
$$xy = (z_1 g^n)(z_2 g^m) = z_2 z_1 g^{n+m} = (z_2 g^m)(z_1 g^n) = yx,$$

so G is abelian. (Thus $G = Z(G)$).

If $\exists g \in G$ with $|g| = p^2$, $G \cong \mathbb{Z}/p^2$. Otherwise $G \cong \mathbb{Z}/p \times \mathbb{Z}/p$ by Lagrange. ∎

14.2

Let G be a finite abelian group with $p \mid |G|$, p prime. Since G abelian, note that
$$ghg^{-1} = gg^{-1}h = h \qquad \forall g, h \in G$$
so $Z(G) = G$. We'll proceed by strong induction on the order of G: write $|G| = pk$, $k \in \mathbb{N}$.

If $k = 1$, $|G| = p$ and thus $|G| \cong \mathbb{Z}/p$.

Now suppose for all $1 \leq n \leq k$, if $|G| = pn$, then $\exists g \in G$ with $|g| = p$. Let $|G| = p(k+1)$ and choose $g \in G$ with $t := |g| + 1$. If $p \mid t$, then $|g^{t/p}| = p$, so assume $p \nmid t$. Since G abelian, $xgx^{-1} = xx^{-1}g = g \in \langle g \rangle$ $\forall x \in G$.
Thus $\langle g \rangle \trianglelefteq G$, so $G/\langle g \rangle$ is an abelian group of order $p(k+1)/t \Rightarrow t \mid k+1 \Rightarrow p \mid \left|\frac{G}{\langle g \rangle}\right|$.
Since $\left|\frac{G}{\langle g \rangle}\right| < p(k+1)$, we have from the induction hypothesis that $\exists g' \in G/\langle g \rangle$ such that $|g'| = p$. Since the quotient map $\varphi: G \to G/\langle g \rangle$ is surjective, then $\varphi^{-1}(g') \in G$ with $|\varphi^{-1}(g')| = np$, $n \in \mathbb{N}$. Thus $|\varphi^{-1}(g')^n| = p$ ∎

14.3

a) Note that we can write $k \in S_n$ as a product of transpositions. Let $\sigma \in S_n$ be a transposition and write $\sigma = (a_1\ a_2)$.

1) If a_1, a_2 are in the same disjoint cycle of $f \in S_n$, then writing $f = (a_1\ a_{11}\ a_{12} \cdots a_2\ a_{21} \cdots)\underbrace{\cdots}_{\substack{\text{disjoint cycles} \\ \text{not containing } a_1, a_2}}$, we have
$$(a_1 a_2) f (a_2\ a_1) = (a_2\ a_{11}\ a_{12} \cdots a_1\ a_{21}\ a_{22} \cdots) \cdots$$
so cycle type of f = cycle type of $\sigma f \sigma^{-1}$.

2) Otherwise, a_1, a_2 are in different disjoint cycles of $f \in S_n$, write $f = (a_1\ a_{11} \cdots)(a_2\ a_{21} \cdots)\underbrace{\cdots}_{\substack{\text{other} \\ \text{disjoint} \\ \text{cycles.}}}$
Then
$$(a_1 a_2) f (a_2\ a_1) = (a_2\ a_{11} \cdots)(a_1\ a_{21} \cdots) \cdots$$
So cycle type of f = cycle type of $\sigma f \sigma^{-1}$.

Thus, by an inductive argument on the transpositions of k, if $kfk^{-1} = g$, $f, g \in S_n$, then f and g have the same cycle type. ✓

b) Let f, g have the same cycle types $(n_1 \cdots n_r)$, and write $\begin{cases} f = (a_{11} \cdots a_{1n_1}) \cdots (a_{r1} \cdots a_{rn_r}) \\ g = (b_{11} \cdots b_{1n_1}) \cdots (b_{r1} \cdots b_{rn_r}) \end{cases}$

If we choose $\sigma \in S_n$ with $\sigma(a_{ij}) = b_{ji}$ $\forall i, j$, then
$$\sigma f \sigma^{-1} = g$$
by a similar transpositions argument to (a). ∎

15.1

Let p be prime, $p \leq n \leq 2p-1$. We can write $|S_n| = n! = p \cdot k$, $p \nmid k$. Since $n < 2p$, p-Sylow groups will not be 2-cycles (or more) of order p, and must be cycles of length p. There are $\frac{n!}{(n-p)!\,p}$ cycles of length p in S_n, and since those cycles are exactly the p-Sylow subgroups, there are $\frac{n!}{(n-p)!\,p}$ p-Sylow groups in S_n.

15.2

Note $|D_{12}| = 12 = 2^2 \times 3$. Thus D_{12} has 2-Sylow subgroups of order 4 and at least one 3-Sylow subgroup of order 3. Since D_{12} has one subgroup of order 3 and three 2-Sylow subgroups of order 4, we have:

3-sylow: $\{e, r^2, r^4\}$

2-sylow: $\{e, s, r^3, sr^3\}$, $\{e, r^3, sr, sr^4\}$, $\{e, r^3, sr^2, sr^5\}$.

15.3

Prove if $|G| = 108$, then $\exists N \trianglelefteq G$ with either $|N| = 9$ or $|N| = 27$.

Note that $108 = 2^2 \times 3^3$. By Sylow $\exists N \leq G$ with $|N| = 27$. We can define the following group action of G on G/N:
$$g \cdot Ng' := N(gg), \quad g \in G,\ Ng' \in G/N.$$

- If $Ng_1 = Ng_2$, $g_1 = ng_2$, $n \in N$. So $Ng_1g = Nng_2g = Ng_2g$, so group action is well-defined.

Letting $\varphi : G \to S_{G/N} (\cong S_4)$ be the adjoint homomorphism of the defined group action, note that
$$|\ker\varphi|\,\underbrace{|\operatorname{Im}\varphi|}_{\leq 27} = 108 \quad \Rightarrow |\ker\varphi| \geq \frac{108}{27} = 4 > 3$$

Since $g \cdot Ne = Ng = Ne \iff g \in N$, $\ker\varphi \leq N$ and thus $|\ker\varphi| \mid 27$ by Lagrange.
Since $|\ker\varphi| > 3$, $|\ker\varphi| \in \{9, 27\}$. $\blacksquare$

15.4

Let G be a finite abelian group $p_1^{r_1} \cdots p_k^{r_k}$. Let $P_1, \dots, P_k$ be p_i-Sylow subgroups of G for $1 \leq i \leq k$. Show $G \cong P_1 \times \cdots \times P_k$ consisting k-tuples $(a_1, \dots, a_k)$, $a_i \in P_i$, $1 \leq i \leq k$.

First, notice that if $(|H|, |K|) = 1$ for $H, K \leq G$, then $H \cap K = \{e\}$.
(Pf: Let $g \in H \cap K$. Then $g \mid |H|$ and $g \mid |K|$. Thus $g \mid (|H|,|K|) = 1 \Rightarrow g = e$.)

By Midterm Q5, $HK = KH \Rightarrow HK$ is a group.

Consider $f : H \times K \to HK$ defined as $f(h,k) = hk$, $h \in H$, $k \in K$. This is obviously well-defined, surjective. If $f(h,k) = e$, then $hk = e$ so $h = k^{-1}$ but $h \in H \cap K = \{e\}$ so $h = k = e$. Thus f injective, and so is a bijection.

Thus $|H| \cdot |K| = |HK|$. By an inductive argument, $|P_1P_2 \cdots P_k| = p_1^{r_1} \cdots p_k^{r_k} = |G|$. Thus
$$P_1P_2\cdots P_k = \{p_1p_2\cdots p_k : p_1 \in P_1, \dots, p_k \in P_k\} = G$$

so $\underbrace{\forall g \in G\ \exists!\,(p_{i_1}, \dots, p_{i_k}) \in P_1 \times \cdots \times P_k}_{(*)}$ so $\gamma : G \to P_1 \times \cdots \times P_k$ defined by $(*)$ is a bijection.
$(\gamma(g) = (p_{i_1}, \dots, p_{i_k}))$

Since G abelian, γ obviously a homomorphism. Thus $G \cong P_1 \times \cdots \times P_k$. $\blacksquare$