Northwestern University

Lebesgue Integration and Measure Theory

MATH 321-3

size matters

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29 March 2023

1 The Lebesgue Measure

1.1 Desired Properties of the Lebesgue Measure

In our study of measure theory, we wish to find a function (or *measure*) that denotes size of sets, some $\mu(E) \in [0, \infty)$ for all sets $E \in \mathbb{R}$. Let's write down some intuitive axioms:

1. **Normalization of Length.** For an open interval $E = (a, b)$, we want $\mu(E) = b - a$.
2. **Translation Invariance**. First note that for some scalar c and a set A, the set $A + c = \{a + c \mid a \in A\}$. We want $\mu(E) = \mu(E + c)$ for all $c \in \mathbb{R}$.
3. **Countable Additivity** If $E_i \subset \mathbb{R}$, $i \in \mathbb{N}$, then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$. Moreover, if the E_i's are pairwise disjoint (i.e. $E_i \cap E_j = \varnothing$ for all $i \neq j$), then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$.

Unfortunately, no such measure satisfying these properties exists...

Fact: It's impossible to define μ satisfying (1)-(3) and defined for all (bounded) $E \subset \mathbb{R}$.

1.2 Null Sets

When working with Riemann integration, there's an often repeated motto that "finite sets don't matter". In the field of measure theory, we want to generalize this statement to be that sets of "generalized length 0", or **measure zero**, don't matter. In fact, we can explore these sets of measure zero without even needing to properly define the Lebesgue measure (though, of course, we will).

In our search for a measure of satisfactory compatibility with the previously proposed "measure axioms" of sorts, we will describe the notion of the **outer measure**, which is defined for all bounded sets of real numbers, satisfies Properties (1) and (2), and satifies the inequality of Property (3), called *subadditivity*. The outer measure fails to be additive (the equality portion of (3)) for certain disjoint sets, so we'll restrict its definition to a large collection of nice (measurable) sets to which additivity holds. What's a measurable set? Let's find out!

Before jumping into some definitions, let's first formalize a notion of length of intervals. We define the length of an open interval $I = (a, b)$ to be $\text{len}(I) = b - a$. Great! We're all set now.

Definition 1.1 (Lebesgue Outer Measure)**.** Suppose $A \subset \mathbb{R}$ is bounded and $\mathscr{U}(A)$ is the set of all *countable* coverings of A by open intervals. We define the **Lebesgue Outer Measure**, $\mu^*(A)$, by

$$\mu^*(A) = \inf_{\{U_n\} \in \mathscr{U}(A)} \left\{ \sum_{i=1}^{n} \text{len}(U_n) \right\},$$

where the infimum is taken over the set of all countable coverings of A by open intervals.

Remark 1.2. It seems silly, but just to be safe, let's note that $\inf\{\infty\} = \infty$.

Example 1.3

- Let $A = (a, b)$. Then $\mu^*(A) = b - a$. (Clearly, $A \subset (a, b)$, so $\mu^*(A) \leq b - a$. Why does $\mu^*(A) \geq b - a$ hold?).
- Let $A = \varnothing$. Then $\varnothing \subset (0, \epsilon)$ for all $\epsilon > 0$, so $\mu^*(A) \leq \inf_\epsilon \text{len}((0, \epsilon)) = \inf_\epsilon \epsilon = 0$.
- Let $A = \{c\}, where c \in \mathbb{R}$. Then $A \subset (c - \epsilon, c + \epsilon)$, so $\mu^*(A) = 0$.
- Let $A = \mathbb{Q}$. Then $\mu^*(A) = 0$. (Why?)

Proposition 1.4

The outer measure of a closed interval is the same as the outer measure of its correspondent open interval. In other words, if $A = [a, b]$, then $\mu^*(A) = b - a$.

Proof. We can encapsulate A inside an open interval: $A \subset (a - \epsilon, b + \epsilon)$, which has length $b - a + 2\epsilon$ for all ϵ. Thus $\mu^*(A) \leq b - a$. Now, note that if $\{U_n\}$ is a cover of A by open intervals, then compactness gives a finite subcover $A \subset \bigcup_{i=1}^n U_i$. Thus, it suffices to show that for any finite cover $\{U_i\}_{i=1}^n$, $\sum_{i=1}^n \operatorname{len}(U_i) \geq b - a$. We'll do so by induction:

The $n = 1$ case is trivial. Now, suppose that for coverings of $n - 1$ intervals, the $(n - 1)$-sum of lengths of the covering open intervals is greater than or equal to $b - a$. Let $A \subset \bigcup_{i=1}^n U_i$. Since A is connected, then if $A \cap U_i$ for all $1 \leq i \leq n$, there are $i \neq j$ such that $U_i \cap U_j \neq \varnothing$. Reordering without loss of generality, assume $i = 1$ and $j = 2$, and let $V = U_1 \cup U_2$ (which is also an open interval). Then $A \subset V \cup \bigcup_{i=3}^n$, which is a union of $n - 1$ open sets, so we're done by the induction hypothesis. □

Definition 1.5 (Null Sets)**.** A set $A \subset \mathbb{R}$ is said to be a **null set** provided that $\mu^*(A) = 0$.

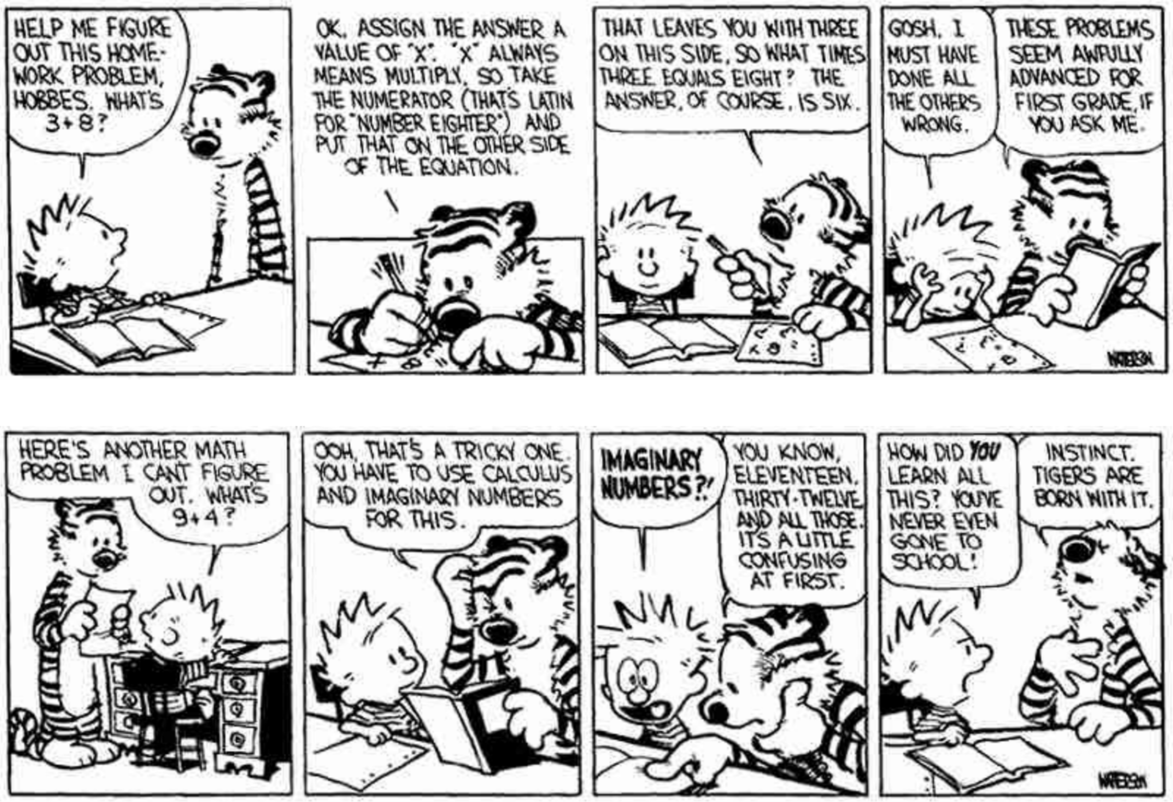
Remark 1.6. Null sets are also defined without the machinery of the Lebesgue outer measure as follows: If for all $\epsilon > 0$, there exists a collection of open intervals $\{U_i\}_{i=1}^\infty$ such that

$$\bigcup_{i=1}^\infty \operatorname{len}(U_i) < \epsilon \quad \text{and} \quad A \subset \bigcup_{i=1}^\infty U_i.$$

Example 1.7

- $\varnothing$ is a null set.
- Finite sets are null sets.
- The countable collection of null sets $E = \bigcup_{i=1}^\infty E_i \subset \mathbb{R}$ is a null set.
- Countable sets are null sets.
- The Cantor 1/3-set is a null set.

The punchline of the tail end of the previous list of null-set examples is that all null sets are measurable, and for whatever reason, the existence of uncountable null sets implies that describing all measurable sets and functions is, well... complicated.


HELP ME FIGURE OUT THIS HOME-WORK PROBLEM, HOBBES. WHAT'S 3+8?
OK, ASSIGN THE ANSWER A VALUE OF "X". "X" ALWAYS MEANS MULTIPLY, SO TAKE THE NUMERATOR (THAT'S LATIN FOR "NUMBER EIGHTER") AND PUT THAT ON THE OTHER SIDE OF THE EQUATION.
THAT LEAVES YOU WITH THREE ON THIS SIDE, SO WHAT TIMES THREE EQUALS EIGHT? THE ANSWER, OF COURSE, IS SIX.
GOSH, I MUST HAVE DONE ALL THE OTHERS WRONG.
THESE PROBLEMS SEEM AWFULLY ADVANCED FOR FIRST GRADE, IF YOU ASK ME.
HERE'S ANOTHER MATH PROBLEM I CAN'T FIGURE OUT. WHAT'S 9+4?
OOH, THAT'S A TRICKY ONE. YOU HAVE TO USE CALCULUS AND IMAGINARY NUMBERS FOR THIS.
IMAGINARY NUMBERS?!
YOU KNOW, ELEVENTEEN, THIRTY-TWELVE AND ALL THOSE. IT'S A LITTLE CONFUSING AT FIRST.
HOW DID YOU LEARN ALL THIS? YOU'VE NEVER EVEN GONE TO SCHOOL!
INSTINCT. TIGERS ARE BORN WITH IT.