Northwestern University

Lebesgue Integration and Measure Theory

MATH 321-3

size matters

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1 The Lebesgue Measure

1.1 Desired Properties of the Lebesgue Measure

In our study of measure theory, we wish to find a function (or *measure*) that denotes size of sets, some $\mu(E) \in [0, \infty)$ for all sets $E \in \mathbb{R}$. Let's write down some intuitive axioms:

1. **Normalization of Length.** For an open interval $E = (a, b)$, we want $\mu(E) = b - a$.
2. **Translation Invariance**. First note that for some scalar c and a set A, the set $A + c = \{a + c \mid a \in A\}$. We want $\mu(E) = \mu(E + c)$ for all $c \in \mathbb{R}$.
3. **Countable Additivity** If $E_i \subset \mathbb{R}$, $i \in \mathbb{N}$, then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$. Moreover, if the E_i's are pairwise disjoint (i.e. $E_i \cap E_j = \varnothing$ for all $i \neq j$), then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$.

Unfortunately, no such measure satisfying these properties exists. Rats :/

Fact: It's impossible to define μ satisfying (1)-(3) and defined for all (bounded) $E \subset \mathbb{R}$.

1.2 Null Sets

When working with Riemann integration, there's an often repeated motto that "finite sets don't matter". In the field of measure theory, we want to generalize this statement to be that sets of "generalized length 0", or **measure zero**, don't matter. In fact, we can explore these sets of measure zero without even needing to properly define the Lebesgue measure (though, of course, we will).

In our search for a measure of satisfactory compatibility with the previously proposed "measure axioms" of sorts, we will describe the notion of the **outer measure**, which is defined for all bounded sets of real numbers, satisfies Properties (1) and (2), and satifies the inequality of Property (3), called *subadditivity*. The outer measure fails to be additive (the equality portion of (3)) for certain disjoint sets, so we'll restrict its definition to a large collection of nice (measurable) sets to which additivity holds. What's a measurable set? Let's find out!

Before jumping into some definitions, let's first formalize a notion of length of intervals. We define the length of an open interval $I = (a, b)$ to be $\text{len}(I) = b - a$. Great! We're all set now.

Definition 1.1 (Lebesgue Outer Measure)**.** Suppose $A \subset \mathbb{R}$ is bounded and $\mathscr{U}(A)$ is the set of all *countable* coverings of A by open intervals. We define the **Lebesgue Outer Measure**, $\mu^*(A)$, by

$$\mu^*(A) = \inf_{\{U_n\} \in \mathscr{U}(A)} \left\{ \sum_{i=1}^{n} \text{len}(U_n) \right\},$$

where the infimum is taken over the set of all countable coverings of A by open intervals.

Remark 1.2. It seems silly, but just to be safe, let's note that $\inf\{\infty\} = \infty$.

Example 1.3

- Let $A = (a, b)$. Then $\mu^*(A) = b - a$. (Clearly, $A \subset (a, b)$, so $\mu^*(A) \leq b - a$. Why does $\mu^*(A) \geq b - a$ hold?).
- Let $A = \varnothing$. Then $\varnothing \subset (0, \epsilon)$ for all $\epsilon > 0$, so $\mu^*(A) \leq \inf_{\epsilon} \text{len}((0, \epsilon)) = \inf_{\epsilon} \epsilon = 0$.
- Let $A = \{c\}, where c \in \mathbb{R}$. Then $A \subset (c - \epsilon, c + \epsilon)$, so $\mu^*(A) = 0$.
- Let $A = \mathbb{Q}$. Then $\mu^*(A) = 0$. (Why?)

Proposition 1.4

The outer measure of a closed interval is the same as the outer measure of its correspondent open interval. In other words, if $A = [a, b]$, then $\mu^*(A) = b - a$.

Proof. We can encapsulate A inside an open interval: $A \subset (a - \epsilon, b + \epsilon)$, which has length $b - a + 2\epsilon$ for all ϵ. Thus $\mu^*(A) \leq b - a$. Now, note that if $\{U_n\}$ is a cover of A by open intervals, then compactness gives a finite subcover $A \subset \bigcup_{i=1}^n U_i$. Thus, it suffices to show that for any finite cover $\{U_i\}_{i=1}^n$, $\sum_{i=1}^n \text{len}(U_i) \geq b - a$. We'll do so by induction:

The $n = 1$ case is trivial. Now, suppose that for coverings of $n - 1$ intervals, the $(n - 1)$-sum of lengths of the covering open intervals is greater than or equal to $b - a$. Let $A \subset \bigcup_{i=1}^n U_i$. Since A is connected, then if $A \cap U_i$ for all $1 \leq i \leq n$, there are $i \neq j$ such that $U_i \cap U_j \neq \varnothing$. Reordering without loss of generality, assume $i = 1$ and $j = 2$, and let $V = U_1 \cup U_2$ (which is also an open interval). Then $A \subset V \cup \bigcup_{i=3}^n$, which is a union of $n - 1$ open sets, so we're done by the induction hypothesis. □

Definition 1.5 (Null Sets)**.** A set $A \subset \mathbb{R}$ is said to be a **null set** provided that $\mu^*(A) = 0$.

Remark 1.6. Null sets can also defined without the machinery of the Lebesgue outer measure as follows: If for all $\epsilon > 0$, there exists a collection of open intervals $\{U_i\}_{i=1}^\infty$ such that

$$\sum_{i=1}^\infty \text{len}(U_i) < \epsilon \quad \text{and} \quad A \subset \bigcup_{i=1}^\infty U_i.$$

then we say A is a null set.

Example 1.7

- $\varnothing$ is a null set.
- Finite sets are null sets.
- The countable collection of null sets $E = \bigcup_{i=1}^\infty E_i \subset \mathbb{R}$ is a null set.
- Countable sets are null sets.
- The Cantor 1/3-set is a null set.

The punchline of the tail end of the previous list of null-set examples is that all null sets are measurable, and for whatever reason, the existence of uncountable null sets implies that describing all measurable sets and functions is, well... complicated.

1.3 σ-algebras

Remark 1.8. Usually, the existence of σ in the nomenclature of an object is to denote that countable operations are allowed.

We're going to now delve into the wonderful mathematical structures called σ-algebras. It turns out that these will be imperative to the study of measurable sets. In fact, as motivation, we shall see that the following holds:

The collection of measurable sets has a structure of a σ-algebra.

First, let's recess quickly for a brief discussion of cardinality: Let X be a set, and write the power set of X as $\mathscr{P}(X) = \{A \subset X\}$. If X is finite and $\operatorname{card}(X) = l$, then $\operatorname{card}(\mathscr{P}(X)) = 2^l$. Instead, if X is countably infinite, then $\operatorname{card}(\mathscr{P}(X))$ is uncountable. (To see why, use a diagonalization argument.)

Definition 1.9 (σ-algebra on X)**.** Suppose X is a set and A is a collection of subsets of X, i.e. $A \subset \mathscr{P}(X)$. A is a *sigma algebra of subsets of* X if

1. $\varnothing, X \in A$,
2. A is closed under complements, and
3. A is closed under countable unions, i.e. if $E_i \subset A$ for $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} E_i \in A$.

Remark 1.10. It's often written as fourth necessary condition that A be closed under countable intersections, but if $E_i \in A$ for $i \in \mathbb{N}$, then

$$\bigcap_{i=1}^{\infty} = \left(\bigcup_{i=1}^{\infty} E_i^C\right)^C \in A,$$

so closure under intersection follows immediately from (2) and (3). Moreover, if $U, V \in A$, then $U \setminus V = U \cap V^C \in A$.

Example 1.11 (Degenerate $\sigma-$algebras)

1. $\mathscr{P}(X)$,
2. $\{\varnothing, X\}$ (called the *trivial σ-algebra*)

Example 1.12 (The Null-Conull $\sigma-$algebra)

A more fun (and illuminating) example of a σ-algebra is defined as follows: the set $A \subset \mathscr{P}(\mathbb{R})$ such that $E \in A$ if either E is a null set or E is a null set.

Definition 1.13. Let $\mathscr{F} \subset \mathscr{P}(X)$. The σ-algebra generated by $\mathscr{F}$, written $\sigma(\mathscr{F})$, is the smallest $\sigma-$algebra containing $\mathscr{F}$.

Remark 1.14. Baked into the definition of generated σ-algebras is the guarantee that a $\sigma-$algebra containing $\mathscr{F}$ exists int he first place! (Proven in homework.)

Example 1.15 (The Borel σ-algebra)

Take $\mathscr{F} \subset \mathscr{P}(\mathbb{R})$ to be all open subsets of the real line. $\mathscr{B} \subset \sigma(\mathscr{F})$, the $\sigma-$algebra generated by open sets, is called the **Borel σ-algebra**.

Remark 1.16. Thinking about basic topology of the real line, closure under complements, unions, and intersections means that there are a lot of interesting structures contained in the Borel $\sigma-$algebra. A few of the more interesting ones are as follows:

- Countable unions of closed sets, and
- Countable intersections of open sets.

Indeed,

$$\mathscr{B} = \sigma(\text{open sets}) = \sigma(\text{closed sets}) = \sigma(\text{open intervals}) = \sigma(\text{open intervals of the form } (a, \infty)).$$

Theorem 1.17 (yo this bih kinda slaps)
The σ−algebra of **Lebesgue-measurable sets** is generated by (1) Borel sets and (2) Null sets.

Zoo wee mama! We don't have sufficient machinery to prove this right now, but it should serve as sufficient motivation for what's to come.

1.4 Properties of the Outer Measure μ^*

So far, we've defined the outer measure μ^* (which isn't a true measure) and checked that $\mu^*([a,b]) = b - a$. At the very beginning, we defined some desired properties of this theoretical notion of a measure, and we'll now explore which of these properties the outer measure has.

Proposition 1.18 (Monotonicity)
If $A \subset B \subset \mathbb{R}$, then $\mu^*(A) \le \mu^*(B)$.

Proof. Since $A \subset B$, every countable cover of B by open intervals $\{U_n\} \in \mathscr{U}(B)$ also covers A. Thus

$$\inf_{\{U_n\}\in\mathscr{U}(A)} \sum_{i=1}^{\infty} \text{len}(U_n) \le \inf_{\{U_n\}\in\mathscr{U}(B)} \sum_{i=1}^{\infty} \text{len}(U_n),$$

so $\mu^*(A) \le \mu^*(B)$. □

We'd previously stated that $\mu^*((a,b)) = b - a$. Let's finish the proof from before:

Proof. Obviously, $\mu^*((a,b)) \le b - a = \text{len}(a,b)$ since $(a,b) \subset (a,b)$. Moreover, note that $[a+\epsilon, b-\epsilon] \subset (a,b)$ for all sufficiently small $\epsilon > 0$. So $\mu^*((a,b)) \ge \mu^*([a+\epsilon, b-\epsilon] = b - a + 2\epsilon$. □

Corollary 1.19
$\mu^*(\mathbb{R}) = +\infty$ and $\mu^*((a,\infty)) = +\infty$.

Proof. $(a,m) \subset (a,\infty)$ for all $m > a$. Use monotonicity. □

Theorem 1.20 (Translation invariance)
For all subsets $E \subset \mathbb{R}$ and scalars $c \in \mathbb{R}$

$$\mu^*(E) = \mu^*(E + c).$$

Proof. Homework (use intervals). □

Theorem 1.21 (Countable subadditivity)
Given $E_i \subset \mathbb{R}$, $\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mu^*(E_i)$.

Proof. Fix $\epsilon > 0$. For each i, pick a cover $\{U_n^i\}$ of E_i by open intervals with

$$\sum_{n=1}^{\infty} \operatorname{len}\left(U_n^i\right) - \frac{\epsilon}{2^i} \leq \mu^*(E_i) \leq \sum_{n=1}^{\infty} \operatorname{len}(U_n^i).$$

Let $E = \bigcup_{i=1}^{\infty} E_i$. Now, the set $\{U_n^i \mid i, n \in \mathbb{N}\}$ is a cover of E by countably many open intervals, and

$$\mu^*(E) \leq \sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} \operatorname{len}(U_n^i) \right) \leq \sum_i \left(\mu^*(E_i) + \frac{\epsilon}{2^i} \right) = \left(\sum_{i=1}^{\infty} \mu^*(E_i) \right) + \epsilon.$$

□

Remark 1.22. Unlike our desired measure properties, we might not have equality even if all our subsets are pairwise disjoint! (this is really sad)

In fact, there exists $A, B \subset [0,1]$ such that

1. $A \cup B = [0,1]$,
2. $A \cap B = \varnothing$, *but*
3. $\mu^*(A) + \mu^*(B) > 1$.

This defect, of sorts, is why ”outer measure” is not a measure.

1.5 A non-measurable set

To concretely illustrate the shortfall of the Lebesgue outer measure, we'll construct a non-measurable set.

Theorem 1.23

There is no $\lambda : \mathscr{P}(\mathbb{R}) \to [0, \infty)$ satisfying

1. λ is **translation invariant,**
2. **monotonicity** holds,
3. $\lambda([0,1]) = 1$ (this can be any non-zero, noninfinite value), and
4. **countable additivity** holds

Remark 1.24. Note that **countable additivity** in (4) can be split into **countable *sub*-additivity** (i.e. $\lambda\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \lambda(E_i)$), and the equality statement:

$$E_i \cap E_j = \varnothing \ \forall i \neq j \Rightarrow \lambda\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \lambda(E_i). \qquad (\star)$$

Moreover, Lebesgue Outer Measure μ^* satisfies (1)-(3) and countable subadditivity (but not the equality statement of (4)).

As hinted before, the obvious punchline of Theorem 1.23 is that we will need to restrict the real line $\mathbb{R}$ to a class of sets we ”measure”. To prove this theorem, we will ”build” a non-measurable set.

First, let's define the following equivalence relation: Given $x, y \in \mathbb{R}$, say $x \ y$ if $x - y \in \mathbb{Q}$. (Feel free to check this yourself if the omission of the proof will keep you up at night.) Then, we'll define the following equivalence class:

$$E_x = \{y \in \mathbb{R} \mid y \ x\}.$$

Note that $x + \frac{k}{107}$ x for all $x \in \mathbb{Z}$, so $E_x \cap [0,1] \neq \varnothing$. For each equivalence class, we will pick a *unique* representative $Z_\alpha \in [0,1]$, where $\alpha \in \Delta$, an uncountable index set.

Definition 1.25 (The "Bad Set"). We will define the following set, and later show that it is unmeasurable:

$$B = \{Z_n \mid \alpha \in \Delta\}.$$

Remark 1.26. Note that

1. If $y \in \mathbb{R}$, there exists an index α and rational $q \in \mathbb{Q}$ such that $y = z\alpha + q$,

$$\bigcup_{q \in \mathbb{Q}} B + q = \mathbb{R}.$$

2. If $(B+q) \cap (B+p) \neq \varnothing$ for $p, q \in \mathbb{Q}$, then $p = q$. (This is not entirely obvious, so here's a quick proof: Take $y \in (B+q) \cap (B+p)$. Then there are α, β such that $y = Z_\alpha + q, y = Z_\beta + p$. Thus $Z_\alpha = Z_\beta + p - q$, so Z_α Z_β. Since the representatives in B are unique, $Z_\alpha = Z_\beta$, and thus $p = q$.

We can now prove Theorem 1.23:

Proof. Note that $B \subset [0,1]$. So, $\lambda(B) \leq \lambda([0,1]) \leq 1$. The proof of the theorem is immediate from the following two propositions:

1. If λ satisfies (1)-(3) and countable subadditivity, then $\lambda(B) > 0$. *Proof:* Enumerate $\mathbb{Q} = \{q_i\}$ and write $B_i = B + q_i$ for each $i \in \mathbb{N}$. Since $\mathbb{R} = \bigcup_{i=1}^\infty B_i$,

$$1 \leq \lambda(\mathbb{R}) \leq \sum_{i=1}^\infty \lambda(B_i) \leq \sum_{i=1}^\infty \lambda(B),$$

so $\lambda(B) > 0$.

2. If λ satisfies (1)-(4), then $\lambda([0,2]) = +\infty$. *Proof:* Enumerate $\mathbb{Q} \cap [0,1] = \{q_j\}$, and set $B_j = B + q_j$. Since $B \subset [0,1]$ and $0 \leq q_j \leq 1$, translation is limited and thus $\bigcup_{j=1}^\infty B_j \subset [0,2]$ so

$$\lambda([0,2]) \geq \lambda\left(\bigcup_{j=1}^\infty B_j\right) = \sum_{j=1}^\infty \lambda(B_j) = \sum_{j=1}^\infty \lambda(B) = +\infty.$$

□

Remark 1.27. Observe the following:

1. Our bad set B is non-measurable. If μ is our Lebesgue measure, then $\mu(B)$ is undefined.
2. μ^* satisfies (1)-(3) and countable subadditivity, so $0 \leq \mu^*(B) < 1$.
3. *Claim:* The set $N = [0,1] \smallsetminus B$ is also non-measurable and $\mu^*(N) = 1$. (Think about we're building measurable sets up to have structure similar to σ-algebras.) So $[0,1] = B \cup N$, $B \cap N = \varnothing$, and $\mu^*(B) + \mu^*(N) > 1 = \mu^*(B \cup N)$.

Proposition 1.28 (Outer Regularity)

If $A \subset \mathbb{R}$ is a set with finite outer measure, then for any $\epsilon > 0$, there exists an open set v with

1. $A \subset V$, and
2. $\mu^*(A) \leq \mu^*(V) \leq \mu^*(A + \epsilon)$.

In particular, $\mu^*(A) = \inf\{\mu^*(V) \mid A \subset V, V \text{ open }\}$.

Proof. If $U = \{U_n\}$ is a cover by countably many open intervals with $\sum_{n=1}^{\infty} \text{len } U_n \leq \mu^*(A) + \epsilon$. Take $V = \bigcup_{n=1}^{\infty} U_n$. Then $A \subset V$ and $\mu^*(V) \leq \sum_{n=1}^{\infty} \mu^*(U_n) = \sum_{n=1}^{\infty} \text{ len } U_n \leq \mu^*(A) + \epsilon$. □

Zooming out a bit to gain some perspective, we can see that we've found sets A, B such that

$$\mu^*(A) + \mu^*(B) > \mu^*(A \cup B).$$

In particular, we found A, B, where $\mu^*(A \cap [0,1]) + \mu^*(A^C \cap [0,1]) > 1$. We will soon say that $A \subset \mathbb{R}$ is **measurable** if for any $E \subset \mathbb{R}$,

$$\mu^*(A \cap E) + \mu^*(A^C \cap E) = \mu^*(E).$$

1.6 Measurable Sets

Definition 1.29. Let $M_0 \subset \mathscr{P}(\mathbb{R})$. Denote all sets $\mathscr{A}$ with the following property: for any $X \subset \mathbb{R}$,

$$\mu^*(A \cap X) + \mu^*(A^C \cap X) = \mu^*(A). \qquad (\star)$$

For a set $\mathscr{A} \in M_0$, define the **Lebesgue measure** of $\mathscr{A}$ to be $\mu(A) = \mu^*(A)$.

Proposition 1.30

Let $A \subset \mathbb{R}$.

1. $A \in M_0$ (is measurable) if, and only if, A^C is measurable.
2. $A \in M_0$ if, and only if, for all $X \subset \mathbb{R}$, $\mu^*(A \cap X) + \mu^*(A^C \cap X) \leq \mu^*(X)$.

Proof. (obvious from definitions) □

1.7 M, the $\sigma-$algebra generated by Borel sets and Null sets

Surprise! M_0 is a $\sigma-$algebra, $M = M_0$, and μ defined on $M = M_0$ has the desired properties of a measure outlined in the beginning of the chapter.

Definition 1.31. M_0 (which we'll later show to be exactly M) is the $\sigma-$algebra of **Lebesgue measurable sets**.

Recall that $\mu^*(A \cap X) + \mu^*(A^C \cap X) \geq \mu^*(X)$. To show M_0 is Lebesgue measurable, it therefore suffices to check $(\star)$ for sets with bounded outer measure. By countable subadditivity, it further suffices to check for only bounded sets; in fact, it's enough to check $(\star)$ when X is an open set or interval. (Shown in week 2 problem set).

Proposition 1.32

IF $A \subset M_0$ is bounded (or even $\mu^*(A) < \infty$), then there exists a Borel set B and a null set $N = A^C \cap B$ such that $A = B \setminus N$.

Proof. For $\epsilon > 0$, there exists an open set V_ϵ with

- $A \subset V_\epsilon$, and
- $\mu^*(A) \leq \mu^*(V_\epsilon) \leq \mu^*(A) + \epsilon$.

Set $B = \bigcap_{k=1}^{\infty} V_{1/k}$. Then

- B is Borel,
- $A \subset B \subset V_{1/k}$ for all k, and
- $\mu^*(A) \le \mu^*(B) \le \mu^*(A) + \frac{1}{k}$ for all k.

So $\mu^*(A) = \mu^*(B)$. Let $N = B \setminus A$, so $\mu^*(A \cap B) + \mu^*(A^C \cap B) = \mu^*(B)$. Since $\mu^*(A) = \mu^*(B) = \mu^*(A \cap B)$, we have $\mu^*(N) = \mu^*(A^C \cap B) = 0$. □

Proposition 1.33

$A \subset \mathbb{R}$ is null if, and only if, $A \in M_0$ and $\mu(A) = 0$.

Proof. If $A \in M_0$ and $\mu(A) = 0$, then $\mu^*(A) = 0$, so A is null. On the other hand, suppose A is null, so $\mu^*(A) = 0$. Fix $X \subset \mathbb{R}$. Then monotonicity gives

$$\mu^*(A \cap X) + \mu^*(A^C \cap X) \le \mu^*(A) + \mu^*(X) = \mu^*(X).$$

□

Proposition 1.34

If $A, B \in M_0$, then $A \cup B$ and $A \cap B \in M_0$.

Remark 1.35. Since $A \cap B = \left(A^C \cup B^C\right)^C$, it suffices to show $A \cup B \in M_0$.

Proof. Fix $A, B \in M_0$ and pick any $X \subset \mathbb{R}$. Note that

1. $(A \cup B) \cap X = (B \cap X) \cup (A \cap B^C \cap X)$, ad
2. $(A \cup B)^C \cap X = A^C \cap B^C \cap X$.

So

$$\begin{aligned}
\mu^*((A \cup B) \cap X) + \mu^*((A \cup B)^C \cap X) &\le \mu^*(B \cap X) + \mu^*(A \cap B^C \cap X) + \mu^*(A^C \cap B^C \cap X) \\
&= \mu^*(B \cap X) + \mu^*(B^C \cap X) \\
&= \mu^*(X).
\end{aligned}$$

□

Remark 1.36. If A_i are in M_0, so are $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$. (To prove, induct on n)

Proposition 1.37

All intervals are in M_0.

Remark 1.38. Using complements and finite intersections/unions, we can build any interval from intervals of the form $(-\infty, a], [b, \infty)$.

Example 1.39

$(1,7] = (-\infty,7] \cap ((-\infty,1])^C$.

This finally leads to the following claim: If U is an interval, set $U^- = (-\infty, b) \cap U$ and $U^+ = [b, \infty) \cap U$. Then $\mu^*(U) = \text{len } U$, $\mu^*(U^-) = \text{len } U^-$, and $\mu^*(U^+) = \text{len } U^+$. So by additivity of length,

$$\mu^*(U) = \text{len } U = \mu^*(U^-) + \mu^*(U^+).$$

1.8 TA recitation 1

Definition of outer measure can take open intervals disjoint (uses lemma: any open $U \subset \mathbb{R}$ is countable union of disjoint intervals)

1.9 An equivalence of σ-algebras

An equivalence that we'll end up using naively, going forward is that the $\sigma-$algebra generated by Borel sets and Null sets is exactly the same set as the $\sigma-$algebra of measurable sets. In other words, $M = M_0$.

Proposition 1.40

Intervals are in M_0.

Proof. By the magic of complements, countable unions, and the like, it suffices to show that $[b, \infty) \in M_0$ (and $(-\infty, a] \in M_0$). Since the argument to prove either is the same, we'll proceed by showing $[b, \infty) \in M_0$. Let $A = [a, \infty), X \subset \mathbb{R}$. Fix $\epsilon > 0$ and countably many open intervals N_n such that

$$X \subset \bigcup_{n=1}^{\infty} U_n \quad \text{and} \quad \mu^*(X) \leq \sum_{n=1}^{\infty} \text{len } U_n \leq \mu^*(X) + \epsilon.$$

Set $X^+ = A \cap X$, $X^- = A^C \cap X$, $U_n^+ = U_n \cap X$, $U_n^- = U_n^C \cap X$. Note that U_n^- is an open interval and U_n^+ is a half-open interval, so $\text{len } U_n^+ + \text{len } U_n^- = \text{len } U_n$. So

$$\begin{aligned} \mu^*(X^+) &\leq \sum_{n=1}^{\infty} \mu^*(U_n^+) = \sum_{n=1}^{\infty} \text{len } U_n^+ \\ \mu^*(X^-) &\leq \sum_{n=1}^{\infty} \mu^*(U_n^-) = \sum_{n=1}^{\infty} \text{len } U_n^-, \end{aligned}$$

thus

$$\begin{aligned} \mu^+(A \cap X) + \mu^*(A^C \cap X) &= \mu^*(X^+) + \mu^*(X^-) \\ &\leq \sum_{n=1}^{\infty} \text{len } U_n^+ + \text{len } U_n^- \\ &= \mu^*(X) + \epsilon. \end{aligned}$$

□

Proposition 1.41 (Finite additivity, of sorts)
Let $A \in M_0$. Fix $B, X \subset \mathbb{R}$. Suppose that $A \cap B = \varnothing$ Then $\mu^*(A \cap X) + \mu^*(B \cap X) = \mu^*((A \cup B) \cap X)$. In particular, if $A, B \in M_0$, then $\mu(A) + \mu(B) = \mu(A \cup B)$.

Proof. Notice that
$$A \cap ((A \cup B) \cap X) = A \cap X,$$
and because $A \cap B = \varnothing$,
$$A^C \cap ((A \cup B) \cap X) = B \cap X.$$
Thus,
$$\begin{aligned}\mu^*((A \cup B) \cap X) &= \mu^*(A \cap ((A \cup B) \cap X)) + \mu^*(A^C \cap ((A \cup B) \cap X) \\ &= \mu^*(A \cap X) + \mu^*(B \cap X).\end{aligned}$$

□

Corollary 1.42
Given countably many $E_i \in M_0$ such that $E_i \cap E_j = \varnothing$ for all $i \neq j$ and $X \subset \mathbb{R}$,
$$\mu^*\left(\bigcup_{i=1}^n E_i \cap X\right) = \sum_{i=1}^n \mu^*(E_i \cap X).$$

Great, so we now have everything we need to proceed with the main proof of the section!

Theorem 1.43
M_0 is a σ-algebra.

Proof. By definition, if $A \subset M_0$, then $A^C \subset M_0$. Moreover, $\mathbb{R}, \varnothing \in M_0$ since $\mu^*(\mathbb{R} \cap X) + \mu^*(\varnothing \cap X) = \mu^*(X)$. With the first two conditions out of the way, we just need to show that M_0 closed under countable unions. Take countably many $A_i \in M_0$, and let $E_n = \bigcup_{i=1}^n A_i$. Set $B_1 = A_1$, and $B_{n+1} = A_{n+1} \setminus E_n$. (This constructs pairwise disjoint sets from $\{E_i\}$.) Then $E_{n+1} = E_n \cup B_{n+1}$ and $E_n \cap B_{n+1} = \varnothing$. Both $E_n, B_n \in M_0$. If $i < j$ and $B_i \subset E_i \subset E_{j-1}$, then $B_i \cap B_j = \varnothing$.

Now, let $E = \bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty B_i = \bigcup_{i=1}^\infty E_i$. Note that $E^C = \bigcap_{i=1}^\infty B_i^C \subset \bigcap_{i=1}^n B_i^C = \left(\bigcup_{i=1}^n B_i\right)^C = E_n^C$. Fix any $X \subset \mathbb{R}$. Then for *any* n, **Proposition 1.41** and $E_n \in M_0$ gives
$$\begin{aligned}\left(\sum_{i=1}^n \mu^*(B_i \cap X)\right) + \mu^*(E^C \cap X) &= \mu^*(E_n \cap X) + \mu^*(E^C \cap X) \\ &\leq \mu^*(E_n \cap X) + \mu^*(E_n^C \cap X) \\ &= \mu^*(X).\end{aligned}$$
So
$$\left(\sum_{i=1}^\infty \mu^*(B_i \cap X)\right) + \mu^*(E^C \cap X) \leq \mu^*(X)$$

and thus

$$\begin{aligned}
\mu^*(E \cap X) + \mu^*(E^C \cap X) &= \mu^* \left(\bigcup_{i=1}^{\infty} B_i \cap X \right) + \mu^*(E^C \cap X) \\
&\le \left(\sum_{i=1}^{\infty} \mu^*(B_i \cap X) \right) + \mu^*(E_C \cap X) \\
&\le \mu^*(X).
\end{aligned}$$

So $E = \bigcup_i A_i = \bigcup_i B_i \in M_0$. □

Corollary 1.44

$M_0 = M$.

Proof. First note that both are $\sigma-$algebras.

1. Since null sets are in M_0 and open intervals—and *all open sets*—are in M_0, so $M \subset M_0$.
2. Fix $A \in M_0$. Given $n \in \mathbb{Z}$, set $A_[n] = [n, n+1] \cap A$. Since $[n, n+1], A_n \in M_0$, $A_n \in M$. (We've showed before that bounded measurable sets are in M.) In particular, there is a Borel set B_n and Null set N_n such that $A_n = B_n \setminus N_n$. So $A = \bigcup_{n \in \mathbb{Z}} A_n \in M$. So $M_0 \subset M$.

□

It might be helpful to take a step back and list some notes that might be helpful going forward:

- M ($= M_0$) is the $\sigma-$algebra of Lebesgue measurable sets.
- $A \subset \mathbb{R}$ is **measurable** if $A \in M$.
- M is generated by Borel sets and null sets.
- In particular, the following are measurable
  - Intervals,
  - Open sets,
  - Closed sets (like the **Cantor middle 1/3 set**!),
  - Null sets (and thus countable sets).
- Define the Lebesgue measure $\mu : M \to [0, \infty)$, $\mu(A) = \mu^*(A)$. If $A, B \in M$, then

$$\mu(A \cap B) + \mu(A^C \cap B) = \mu(B).$$

Example 1.45

Suppose $A_i, B_i \in M$, $A_1 \subset A_2 \subset A_3 \subset \cdots$, and $B_1 \supset B_2 \supset B_3 \supset \cdots$. Set $A = \bigcup_{i=1}^{\infty} A_i$ and $B = \bigcap_{i=1}^{\infty} B_i$.

1. Show $\mu(A_i) \to \mu(A)$.

   - Set $E_j = A_j \setminus A_{j-1}$. Then E_j are pairwise disjoint and $E_j \in M$, $A = \bigcup_{j=1}^{\infty} E_j$. So

   $$\mu(A_n) \leq \mu(A) = \mu\left(\bigcup_{i=1}^{\infty} E_i\right) < \sum_{i=1}^{\infty} \mu(E_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i) = \lim_{n\to\infty} \mu(A_n).$$

   Note that the last equality holds because the E_i's are measurable.

2. If $\mu(B_1) < \infty$, show $\mu(B_i) \to \mu(B)$.

   - Set $F_j = B_1 \setminus B_j$, so $F_1 \subset F_2 \subset \cdots$. Then

   $$\bigcup_{j=1}^{\infty} F_j = B_1 \setminus B = B^C \text{ (inside } B_1).$$

   By bullet (1) above,
   $$\mu(F_i) \to \mu(B_1 \setminus B) = \mu(B_1) - \mu(B).$$

   Also,
   $$\mu(F_i) = \mu(B_1 \setminus B_i) = \mu(B_1) - \mu(B_i),$$

   so $(\mu(B_1) - \mu(B_i)) \to \mu(B_1) - \mu(B)$.

3. If $\mu(A) < \infty$, show $\mu(A \setminus A_n) = 0$.

   - Set $G_n = A \setminus A_n$. Since $G_1 \supset G_2 \supset \cdots$, notice $\bigcap_{n=1}^{\infty} G_n = \varnothing$. By bullet (2),

   $$\mu(A \setminus A_n) = \mu(G_n) \to \mu(\varnothing) = 0.$$

Remark 1.46. In (2) of the previous example, the assumption that $\mu(B_1 < \infty$ is imperative. To see why, set $B_n = [n, \infty)$. Notice that

1. $\bigcap_{n=1}^{\infty} = \varnothing$, and
2. $\mu(B_n) = \infty$, but $\mu(B) = 0$.

Clearly, this is an issue.

Remark 1.47. Similarly, in (3) of the previous example, the assumption that $\mu(A) < \infty$ is also necessary. Set $A_n = [-n, n]$. Then the union over all n is $\bigcup_{n=1}^{\infty} A_n = \mathbb{R} = A$, but

$$\mu(A \setminus A_n) = \mu((-\infty, -n) \cup (n, \infty)) = \infty \nrightarrow 0.$$

With the results from Example 1.45 under our belt, we can now prove that the Lebesgue Measure actually exists, and is not instead some wacky figment of our horribly rotten mathematical brains:

Theorem 1.48 (Existence of the Lebesgue Measure)

Specifically, there exists a measure (function) $\mu : M \to [0, \infty)$ satisfying:

1. $\mu((a,b)) = b - a$,
2. **translation invariance**, and
3. **countable additivity**.

Moveover, from (1)-(3), we get the following for free:

- **Monotonicity**,
- the null sets are measurable, and
- **outer regularity**: If $A \in M$, then $\mu(A) = \inf\{\mu(U) \mid U \text{ open}, U \supset A\}$.

Proof. We've pretty much proved everything here except countable additivity, so we'll say that it suffices to show (3) holds.

If $A = \bigcup_{i=1}^{\infty} A_i$, then $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$ by subadditivity of outer measure. Moreover, if A_j pairwise disjoint, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$. So $\sum_{i=1}^{\infty} \mu(A_i) \leq \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$. And thus $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ if A_j are pairwise disjoint. $\square$

Remark 1.49. Note that $\mathbb{R}$ conventionally has measure $\mu(\mathbb{R}) = +\infty$. But sometimes we want to restrict to a total space of finite measure. Arbitrarily, fix $I = [0,1]$ (any other bounded closed interval will also do!). Then we write

$$M(I) = \{A \cap I \mid A \in M\} \subset M.$$

If we do this, then we're restricting μ to $M(I)$, so $0 \leq \mu(A) \leq 1$ for all $A \in M(I)$. It's important to be aware of context, as one might be working in either $\mathbb{R}$ or I, and it's often up to the reader to figure out the total space when taking complements of sets, and the like.

Example 1.50

Say $A \in M(I)$. Then $A^C = I \setminus A$, and we can do things like

$$\mu(A^C) = \mu(I) - \mu(A) = 1 - \mu(A).$$

Proposition 1.51 (Inner Regularity)

Suppose that $A \in M(I)$. Then for any $\epsilon > 0$, there exists a closed set $C \subset A$ where

$$\mu(A) - \epsilon \leq \mu(C) \leq \mu(A).$$

In particular,

$$\mu(A) = \sup\{\mu(C) \mid C \subset A \text{ is closed}\}.$$

Proof. Pick an open U (in either $\mathbb{R}$ or I) with $A^c \subset U$. Then $A^C = I \setminus A$ and $\mu(U) \leq \mu(A^c) + \epsilon$. Set $C = U^c$ $(= I \setminus U)$. Then C is closed, $C = U^c \subset (A^c)^c = A$, and

$$\mu(C) = \mu(U^c) = 1 - \mu(U) \geq 1 - (\mu(A^c) + \epsilon) \geq 1 - \mu(A^c) - \epsilon = \mu(A) - \epsilon.$$

$\square$

1.10 A slightly upsetting freak of nature

Let's restrict our perspective to that of $I = [0,1]$ and say we have a set $A \subset I$, with $\mu(A) = 2/3$. Now, if we pick a point in A and examine a small neighborhood of this point, what should we expect to be the density (in the non-mathematical sense) of points of A in this neighborhood? If you have more than two brain cells, you'd probably guess 2/3, 66%, whatever. Unfortunately, Lebesgue definitely only had one.

Theorem 1.52

Define

$$f_{A,X}(\delta) = \frac{\mu((x-\delta, x+\delta) \cap A)}{\mu(x-\delta, x+\delta)} = \frac{((x-\delta, x+\delta) \cap A)}{2\delta}.$$

For any *typical* (whatever that means...) $x \in A$, $f_{X,A}(\delta) \to 1$ as $\delta \to 0$.

Proof. Our professor didn't prove this for us, so I won't for you. Because fuck you, that's why. (Did I mention how mad this theorem makes me?) □

Theorem 1.53

Let A be measurable with $0 < \mu(A) < \infty$. Given $0 < p < 1$, there is an open interval U such that $\mu(U \cap A) \geq p \cdot \mu(U)$.

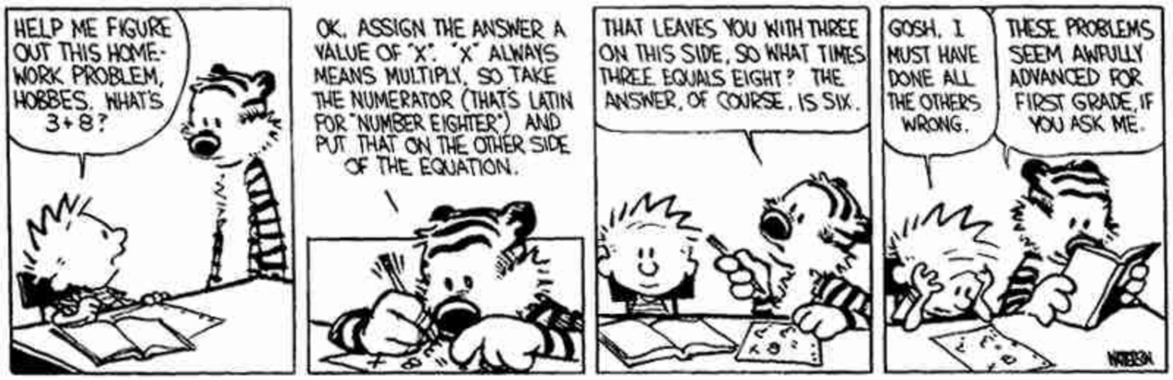
Proof. Yea, this is a proof by contradiction. Enjoy reading through this joke of a proof that's less illuminating than a snuffed out candle. That being said, fix $0 < p < 1$, set $0 < \epsilon = (1-p)\mu(A) < \infty$, and pick disjoint open intervals $\{U_n\}$ such that

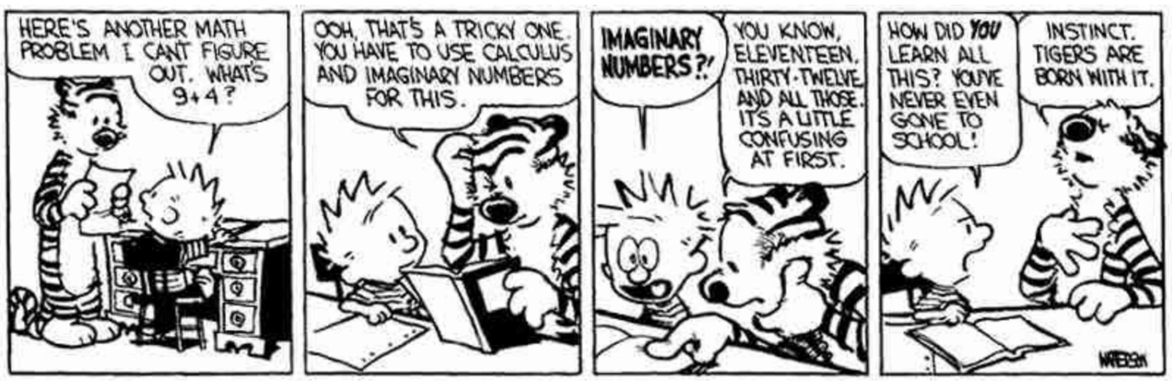
$$A \subset \bigcup_{i=1}^{\infty} U_i \quad \text{and} \quad \mu\left(\bigcup_{i=1}^{\infty} Ui\right) \leq \mu(A) + \epsilon.$$

Suppose $\mu(U_n \cap A) < p \cdot \mu(U_n)$ for all n. Then

$$\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} A \cap U_n\right) = \sum_{n=1}^{\infty} \mu(A \cap U_n) < \sum_{n=1}^{\infty} p \cdot \mu(U_n) \leq p(\mu(A) + \epsilon).$$

Then $(1-p)\mu(A) < p \cdot \epsilon$. ϵ is stupidly defined, so $1 < p$. Oh wait—that's bad, isn't it? Whatever. □


HELP ME FIGURE OUT THIS HOME-WORK PROBLEM, HOBBES. WHAT'S 3+8?
OK, ASSIGN THE ANSWER A VALUE OF "X". "X" ALWAYS MEANS MULTIPLY, SO TAKE THE NUMERATOR (THAT'S LATIN FOR "NUMBER EIGHTER") AND PUT THAT ON THE OTHER SIDE OF THE EQUATION.
THAT LEAVES YOU WITH THREE ON THIS SIDE, SO WHAT TIMES THREE EQUALS EIGHT? THE ANSWER, OF COURSE, IS SIX.
GOSH, I MUST HAVE DONE ALL THE OTHERS WRONG.
THESE PROBLEMS SEEM AWFULLY ADVANCED FOR FIRST GRADE, IF YOU ASK ME.


HERE'S ANOTHER MATH PROBLEM I CAN'T FIGURE OUT. WHAT'S 9+4?
OOH, THAT'S A TRICKY ONE. YOU HAVE TO USE CALCULUS AND IMAGINARY NUMBERS FOR THIS.
IMAGINARY NUMBERS?!
YOU KNOW, ELEVENTEEN, THIRTY-TWELVE AND ALL THOSE. IT'S A LITTLE CONFUSING AT FIRST.
HOW DID YOU LEARN ALL THIS? YOU'VE NEVER EVEN GONE TO SCHOOL!
INSTINCT. TIGERS ARE BORN WITH IT.