16 Statement of Sylow's theorem parts 2 and 3 (10/30)

Definition 16.1 (Normalizers). If G is a group and $S \subseteq G$ is a subset, let $N_G(S) = \{g \in G : gSg^{-1} = S\}$. This is called the **normalizer** of S in G. If $x \in G$, then $N_G(x) = N_G(\{x\})$, where it is often also called the *centralizer* of x in G. We will be interested below in normalizers of subgroups of G. Note that if P is a subgroup of G, then P is a subgroup of $N_G(P)$. In fact, P is a normal subgroup of $N_G(P)$.

Remark 16.2. Given a group G, a subgroup $H \subseteq G$, and an element $g \in G$, the conjugate gHg^{-1} is another subgroup of G. (In fact, it is isomorphic abstractly as a group to H.) If P is a p-Sylow subgroup, then gPg^{-1} is another p-Sylow subgroup. Thus, G acts by conjugation on the set $\mathrm{Syl}_p(G)$ of p-Sylow subgroups of G.

Theorem 16.3 (Sylow parts 2 and 3). *Let G be a finite group and fix a prime p. Fix a p-Sylow subgroup P of G.*

(2) *If Q is any p-subgroup of G, then $Q \subseteq gPg^{-1}$ for some $g \in G$. Thus, any two p-Sylow subgroups of G are conjugate.*

(3) *Let n_p be the number of p-Sylow subgroups of G. Then,*

$$n_p = [G : N_G(P)] \equiv 1 \mod p.$$

Of crucial import in studying a group G is the question of whether it has a normal p-Sylow subgroup P. If $|G| = p^r n$ where $(p, n) = 1$ and if $P \subseteq G$ is a *normal* p-Sylow subgroup, then G/P is a group of order n and we have excised the "p-part" from G and simplified our lives.

Example 16.4. Suppose that G is a group of order $56 = 2^3 \cdot 7$. Then, $n_7 \equiv 1 \mod 7$, while $[G : N_G(P_7)]$ is $1, 2, 4, 8$, where P_7 is a 7-Sylow. Since $n_7 \equiv 1 \mod 7$, it follows that n_7 is either 1 or 8. Note that any 7-Sylow subgroup is isomorphic to $\mathbf{Z}/7$. If there are 8 distinct 7-Sylow subgroups, then this gives $8 \cdot 6 = 48$ elements of order 7 in G. Now, let P_2 be a 2-Sylow subgroup. There are 8 elements in P_2 and as $48 + 8 = 56$, it follows that every element of G is either in a 7-Sylow or in P_2. In particular, there is only one 2-Sylow subgroup, which must be normal. In summary, a group of order 48 either has a normal 7-Sylow subgroup or it has a normal 2-Sylow subgroup. (It could have both, as in the case of $\mathbf{Z}/7 \times \mathbf{Z}/8$.)

The following lemma will be used in the proofs of the remaining parts of the Sylow theorems.

Lemma 16.5. *Let G be a finite group, p a prime number, $P \subseteq G$ a p-Sylow subgroup, and $Q \subseteq G$ a sub-p-group. Then, $P \cap Q = N_G(P) \cap Q$.*

Proof. Set $H = N_G(P) \cap Q$. I claim that $PH = HP$, which follows from the fact that every element of H normalizes P. It follows that PH is a subgroup of G. But,

$$|PH| = \frac{|P||H|}{P \cap H}.$$

As H and P are p-groups, it follows that PH is a p-group containing P. But, it must then be isomorphic to P since P has the largest possible p-power order of subgroups of G by Lagrange's theorem. So, $PH = P$, which implies that $H \subseteq P$. Since $H \subseteq Q$ as well, it follows that $N_G(P) \cap Q \subseteq P \cap Q$. The other inclusion follows from the fact that $P \subseteq N_G(P)$. $\square$

16.1 Exercises

Exercise 16.1. Let p be a prime and let n be any integer satisfying $p \leqslant n \leqslant p^2 - 1$. Compute the isomorphism type of the Galois group of S_n.

Exercise 16.2. Using Exercises 16.1 and Exercise 14.3, find the number of p-Sylow subgroups of S_n when n is a prime and $n = p(p-1)$.

Exercise 16.3 (Herstein)**.** Prove, using all the Sylow theorems, that if G has order 42, then its 7-Sylow subgroup is normal.

Exercise 16.4. Show that if H and K are subgroups of G such that HK is a subgroup, then

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$