# 14 The class equation (10/23)

**Definition 14.1.** Recall the conjugation action of $G$ on itself defined by $g \cdot h = ghg^{-1}$. We write $G//G$ for the of orbits for the conjugation action. The set $G//G$ is also called the set of **conjugacy classes** of $G$ as two elements $h$ and $k$ satisfy $h \sim k$ if and only if they are conjugate: there exists a $g \in G$ such that $ghg^{-1} = k$.

**Definition 14.2.** Let $G$ be a group and let $x \in G$. The **normalizer** of $x$ in $G$, written $N_G(x)$, is the subgroup of elements $g \in G$ such that $gxg^{-1} = x$. Note that the normalizer $N_g(x)$ is just the stabilizer of $x$ with respect to the conjugation action.

**Theorem 14.3** (The class equation)**.** *If $G$ is a finite group, then*

$$|G| = \sum_{\mathcal{O} \in G//G} \frac{|G|}{|N_G(x_{\mathcal{O}})|},$$

*where $\mathcal{O}$ ranges over the conjugacy classes in $G$ and $x_{\mathcal{O}}$ is the choice of an element in $\mathcal{O}$.*

*Proof.* This is an example of the class equation for group actions, Lemma 13.4. □

**Remark 14.4.** Here is another way the class equation is often stated. The **center** of a group $G$ is the subgroup $Z(G)$ consisting of elements $h \in G$ such that $ghg^{-1} = h$ for all $g \in G$. In other words, it is the set up elements that commute with all elements in $G$. Note that $h \in Z(G)$ if and only if $N_g(h) = G$. In particular, the orbit of the conjugation action containing $h \in Z(G)$ is just $\{h\}$. It follows that we can write the class equation as

$$|H| = \sum_{h \in Z(G)} 1 + \sum_{\mathcal{O} \in G//G \text{ non-central}} \frac{|G|}{|N_G(x_{\mathcal{O}})|} = |Z(G)| + \sum_{\mathcal{O} \in G//G \text{ non-central}} \frac{|G|}{|N_G(x_{\mathcal{O}})|},$$

where the sum on the right ranges over the *non-central* conjugacy classes $\mathcal{O}$ and $x_{\mathcal{O}}$ is a representative of $\mathcal{O}$.

**Notation 14.5.** If $G$ is a finite group and $H \subseteq G$ is a subgroup, then the **index** of $H$ in $G$, written $|G : H|$ is the number of right cosets $G/H$. In other words, $|G : H| = \frac{|G|}{|H|}$, which is an integer by Lagrange's theorem.

**Remark 14.6** (Class equation, final form)**.** Using the notation above and the simplification of Remark 14.4, we have

$$|G| = |Z(G)| + \sum_{\mathcal{O} \in G//G \text{ non-central}} |G : N_G(x_{\mathcal{O}})|,$$

where $x_{\mathcal{O}} \in \mathcal{O}$.

**Definition 14.7.** A $p$-group is a finite group $G$ whose order is a prime power $p^n$ for some prime $p$ and natural number $n \geqslant 0$.

**Theorem 14.8.** *If $G$ is a $p$-group of order $p^n$ for some $n \geqslant 1$, then $Z(G)$ is non-trivial.*

*Proof.* Use the class equation. If $\mathcal{O} \in G//G$ is non-central, then $N_G(x_{\mathcal{O}})$ is a proper subgroup of $G$ (so it has order $p^{m_{\mathcal{O}}}$ for some $m_{\mathcal{O}} < n$ by Lagrange's theorem). Thus, using the class equation, we have

$$p^n = |G| = |Z(G)| + \sum_{\mathcal{O} \in G//G \text{ non-central}} p^{n - m_{\mathcal{O}}}.$$

Working modulo $p$ and using that $n - m_{\mathcal{O}} \geqslant 1$ for all non-central conjugacy classes $\mathcal{O}$, we find that $|Z(G)| \equiv 0 \mod p$. So, either $|Z(G)| = 0$ or it is non-trivial. But, $e \in Z(G)$, so $|Z(G)| > 0$ so $Z(G)$ has at least $p$ elements, so it is non-trivial. □

## 14.1 Exercises

**Exercise 14.1.** Use Theorem 14.8 to show that if $p$ is a group of order $p^2$, then either $G \cong \mathbf{Z}/p^2$ or $G \cong \mathbf{Z}/p \times \mathbf{Z}/p$. In particular, $G$ is abelian.

**Exercise 14.2.** Let $G$ be a finite *abelian* group such that $p \mid |G|$ where $p$ is a prime number. Prove that $G$ has an element of order $p$.

**Exercise 14.3.** Suppose that $n \geqslant 1$ and let $1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_r \leqslant$ be integers such that $\sum_{i=1}^{r} n_i = n$. (Such a sequence is called a **partition** of $n$.) Say that an element $g \in S_n$ has cycle type $(n_1, \ldots, n_r)$ if it can be written as a product of *disjoint* cycles of lengths $n_1, \ldots, n_r$. Prove the following statements.

(a) If $f, g \in S_n$ are conjugate, then they have the same cycle types.

(b) If $f, g \in S_n$ have the same cycle types, then they are conjugate.

This proves that the set of conjugacy classes $S_n//S_n$ is in bijection to the set of partitions of $n$.