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## Question 1

Prove a linear function $f: V_1 \to V_2$ is continuous $\iff$ it is continuous at $x_0 = 0 \in V_1$.

($\Rightarrow$) Let $f: V_1 \to V_2$ be continuous at every point $x_0 \in V_1$. Since $V_1$ is a Hilbert space, $0 \in V_1$, so $f$ is continuous at $0$.

($\Leftarrow$) We have that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\| f(y) - f(0) \| = \| f(y-0) \| = \| f(y) \| < \varepsilon$ whenever $\|y\| < \delta$, $y \in V_1$.

Fix $x_0 \in V_1$, $\varepsilon_0 > 0$. Then $\exists \delta_0 > 0$, $y = x - x_0$ such that $\varepsilon > \| f(y) \| = \| f(x - x_0) \| = \| f(x) - f(x_0) \|$ whenever $\| x - x_0 \| = \| y \| < \delta$. $\blacksquare$

## Question 2

Suppose $\mathcal{H}$ is a Hilbert Space, $\{u_i\}_{i=0}^{\infty}$ is an orthonormal family. Define $P: \mathcal{H} \to \mathcal{H}$ by

$$P(w) = \sum_{i=0}^{\infty} \langle w, u_i \rangle u_i$$

which converges by Proposition 5.4.6.

1) Prove $P$ and $(1-P)$ are continuous and linear, where $1: \mathcal{H} \to \mathcal{H}$ is the identity function.

Let $x, y \in \mathcal{H}$, $a, b \in \mathbb{R}$. Then bilinearity gives

$$P(ax+by) = \sum_{i=0}^{\infty} \langle ax+by, u_i \rangle u_i = \sum_{i=0}^{\infty} \big( a\langle x, u_i \rangle + b \langle y, u_i \rangle \big) u_i = a\Big( \sum_{i=0}^{\infty} \langle x, u_i \rangle u_i \Big) + b \Big( \sum_{i=0}^{\infty} \langle y, u_i \rangle u_i \Big) = aP(x) + bP(y)$$

Now, let $\{x_n\} \in \mathcal{H}$ be a sequence converging to $x$. By Question 1, $P$ is continuous if $\| P(x) - P(0) \| = \| P(x) - \sum_{i=0}^{\infty} \langle 0, u_i \rangle u_i \| = \| P(x) \| < \varepsilon$ $\forall x$ with $\| x - 0 \| < \delta$. Choose $\delta = \varepsilon$. Then for all $x$ with $\|x\| < \delta$, we have that

$$\begin{aligned}
\| P(x) \|^2 &= \lim_{n \to \infty} \Big\| \sum_{i=0}^{n} \langle x, u_i \rangle u_i \Big\|^2 \\
&= \lim_{n \to \infty} \sum_{i=0}^{n} |\langle x, u_i \rangle|^2 && \text{(orthogonality + Pythagorean theorem)} \\
&= \sum_{i=0}^{\infty} |\langle x, u_i \rangle|^2 \\
&\leq \| x \|^2 && \text{(Bessel)} \\
&\leq \delta^2 \\
&= \varepsilon^2
\end{aligned}$$

So $P$ is continuous. Note that $I$ is clearly continuous and linear, and since sums of continuous/linear functions are continuous/linear, $I - P$ is continuous and linear.

2) The kernel of $P$ is defined as $K := \{ u \in \mathcal{H} \mid P(u) = 0 \}$. Prove $K, P(\mathcal{H}) \subset \mathcal{H}$ are closed subspaces.

- We know $P(0) = \sum_{i=0}^{\infty} \langle 0, u_i \rangle u_i = 0$, so $0 \in P(0)$. If $u, v \in \ker(P)$, $P(\lambda v + u) = \lambda P(v) + P(u) = \lambda 0 + 0 = 0$, so $u + v \in \ker(P)$. By the subspace criterion, $K$ is a subspace of $\mathcal{H}$. Now, let $v_n \in K$, $\lim v_n = w$. Thus $P(v_n) = 0$ $\forall n$. Since $P$ is continuous and linear, $P(w) = P(\lim v_n) = \lim P(v_n) = 0$. So $w \in K$ and $K$ closed.
- Now, since $P(0) = 0$, $0 \in P(\mathcal{H})$. Letting $u, v \in P(\mathcal{H})$, $\exists x, y \in \mathcal{H}$ such that $P(x) = u$, $P(y) = v$. Then

$P(x+y) = P(x) + P(y) = u+v$, so $u+v \in P(\mathcal{H})$. Moreover, if $c \in \mathbb{R}$, $P(cx) = cP(x) = cu$, so $cu \in P(\mathcal{H})$.
Then the subspace criterion shows $P(\mathcal{H})$ is a subspace of $\mathcal{H}$. If $\{u_n\}$ is a sequence in $P(\mathcal{H})$ converging to $w$, then for $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that Bessel + convergence of $x_n \to x$ give

$$\left\| \sum_{i=0}^{\infty} \langle x_n, u_i \rangle u_i - \sum_{i=0}^{\infty} \langle x, u_i \rangle u_i \right\| \leq \left\| \sum_{i=0}^{\infty} \langle x_n - x, u_i \rangle u_i \right\|^2 \leq \sum_{i=0}^{\infty} |\langle x_n - x, u_i \rangle|^2 \leq \| x_n - x \|^2 < \varepsilon^2$$

Thus $x = \sum_{i=0}^{\infty} \langle x, u_i \rangle u_i$, so $x \in P(\mathcal{H})$, which we can now conclude to be closed.

3) Prove every $w \in \mathcal{H}$ can be written uniquely as $w = u + v$ with $u \in K$, $v \in P(\mathcal{H})$.

We can find $u \in K$ such that $w - u \in P(\mathcal{H})$, as $P(w-u) = P(w) - 0 \in P(\mathcal{H})$.
Now, let $v = w - u$. Then $v \in P(\mathcal{H})$ and $w = u + v$. Now, suppose $w_1, w_2 \in K$, $v_1, v_2 \in P(\mathcal{H})$ with $w = u_1 + v_1 = u_2 + v_2$. Then $u_1 - u_2 = v_2 - v_1$. Since $K$ is a subspace, $u_1 - u_2 \in K$, and similarly, $v_2 - v_1 \in P(\mathcal{H})$. Then $u_1 - u_2 = v_2 - v_1 \in K \cap P(\mathcal{H}) = \{0\}$, so $u_1 - u_2 = 0 = v_2 - v_1$. ∎

## Question 3

Show a Hilbert space $\mathcal{H}$ is seperable $\Leftrightarrow$ $\mathcal{H}$ has a complete orthonormal family.

a) If $\{u_n\}$ is a complete countable orthonormal family, show $\mathcal{H}$ has a countable dense subset

Let $\{u_n\}$ be a complete countable orthonormal family, and consider the sets

$$S_k := \left\{ \sum_{i=1}^{k} q_i u_i : q_i \in \mathbb{Q},\ 1 \leq i \leq k \right\}.$$

Notice each $S_k$ is countable, as we have a countable set of finite sums.
Now, let $S = \bigcup_{k=1}^{\infty} S_k$ be the countable union of countable sets $S_k$.
Since every $x \in \mathcal{H}$ is the limit of some sequence of elements of $S$, we have from Question 6 that $\overline{S} = \mathcal{H}$ and $S$ is dense in $\mathcal{H}$.

b) If $\mathcal{H}$ is seperable and $S \subset \mathcal{H}$ is any orthonormal set, $S$ is countable.

Let $\mathcal{H}$ be seperable and $S \subset \mathcal{H}$ be orthonormal. Since $\mathcal{H}$ separable, so is $S$.
Thus $S$ has a countable dense subset, say $\{x_n\}$. Suppose for contradiction that $S$ is uncountable. If $u_1, u_2 \in S$, $u_1 \neq u_2$, then

$$\|u_1 - u_2\|^2 = \langle u_1 - u_2, u_1 - u_2 \rangle = \langle u_1, u_1 \rangle + \langle u_2, u_2 \rangle - 2\langle u_1, u_2 \rangle = \|u_1\| + \|u_2\| = 2$$

For all $u_i \in S$, consider the set of balls of radius $\frac{1}{2}$: $\{B_{\frac{1}{2}}(u_i)\}$. Since the radii of each ball is $\frac{1}{2} < 1$, each only contains one element of $S$, and the balls are disjoint.
But $\{x_n\}$ is dense in $S$, so in each ball there exists at least one element of $\{x_n\}$ in each ball. Thus the mapping $\{x_n\} \to \{B_{\frac{1}{2}}(u_i)\}$ is surjective, a contradiction since there must exist an uncountable number of $x_n$'s. ∎

# Question 4

In infinite dimensions, closed + bounded $\not\Rightarrow$ compact.

a) Show the closed unit ball in $C^0[0,1]$ is not compact.

Note that the closed unit ball $B(0,1) := \{ f \in C^0[0,1] \mid \sup \|f\| \leq 1 \}$.

Define $f_n : [0,1] \to \mathbb{R}$ as $f_n(x) = x^n$, so $\sup \|f\| = 1$. Then $\{f_n\}$ is a sequence on the ball $B[0,1]$, but

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases},$$

is discontinuous. Thus $f_n$ doesn't have a convergent subsequence, so $B(0,1)$ is not compact.

b) Show the closed unit ball in $\mathcal{L}^2[0,1]$ is not compact.

Note that the closed unit ball $B(0,1) := \{ f \in \mathcal{L}^2[0,1] \mid \|f\|_{\mathcal{L}^2} \leq 1 \}$

Define $f_n : [0,1] \to \mathbb{R}$ as $f_n(x) = \sin(2\pi n x)$. Then $\left( \int_0^1 \sin^2(2\pi n x) \right)^{1/2} = 1/\sqrt{2}$ $\forall n \in \mathbb{N}$.

By trig, we have

$$\begin{aligned} \sin(2\pi n x)\sin(2\pi m x) &= \tfrac{1}{2}\left[\cos(2\pi n x - 2\pi m x) - \cos(2\pi n x + 2\pi m x)\right] \\ &= \tfrac{1}{2}\left[\cos(2\pi x(n-m)) - \cos(2\pi x(n+m))\right] \\ &= \tfrac{1}{2}[1 - 1] \\ &= 0 \end{aligned}$$

since $m, n \in \mathbb{Z}$.

Thus when $m \neq n$,

$$\begin{aligned} \|f_n - f_m\|_{\mathcal{L}^2}^2 &= \|\sin(2\pi n x) - \sin(2\pi m x)\|^2 \\ &= \int_0^1 (\sin(2\pi n x) - \sin(2\pi m x))^2 \, dx \\ &= \int_0^1 \sin^2(2\pi n x)\, dx - \int_0^1 2\sin(2\pi n x)\sin(2\pi m x)\, dx + \int_0^1 \sin^2(2\pi m x)\, dx \\ &= \tfrac{1}{2} + 0 + \tfrac{1}{2} \end{aligned}$$

so $\|f_n - f_m\|_{\mathcal{L}^2} = 1$ and thus no subsequence of $\{f_n\}$ converges. So $\mathcal{L}^2[0,1]$ is not compact.

$\boxtimes$

# Question 5

Let $\mathbb{R}^{\mathbb{Z}} = \{\{x_n\}_{n\in\mathbb{Z}}\}$ be the space of bi-infinite sequences of real numbers. Let $\ell^2 \subset \mathbb{R}^{\mathbb{Z}}$ be the subset of square-integrable sequences $x = \{x_n\}_{n\in\mathbb{Z}} \in \ell^2 \iff \sum_{n\in\mathbb{Z}} (x_n)^2 < \infty$. Define the inner product on $\ell^2$ as follows: given $x = \{x_n\}_{n\in\mathbb{Z}} \in \ell^2$, $y = \{y_n\}_{n\in\mathbb{Z}} \in \ell^2$, $\langle x, y\rangle = \sum_{n\in\mathbb{Z}} x_n y_n$.

a) Show the formula defining $\langle x,y\rangle$ converges (absolutely) and that $\langle \cdot,\cdot\rangle$ defines an inner product on $\ell^2$.

Notice that $\sum |x_n y_n| \le \sum x_n^2 + \sum y_n^2 < \infty$ so the series defined by $\langle x,y\rangle$ converges.

· (Conjugate) Symmetry: Since $x_n, y_n \in \mathbb{R}$, $\langle x,y\rangle = \sum x_n y_n = \sum y_n x_n = \langle y, x\rangle$.

· Bilinearity: Let $a,b \in \mathbb{R}$, $z = \{z_n\}_{n\in\mathbb{Z}} \in \ell^2$. Then

$$\langle ax+by, z\rangle = \sum (ax_n + by_n) z_n = \sum a x_n z_n + b y_n z_n = a\left(\sum x_n z_n\right) + b\left(\sum y_n z_n\right) = a\langle x,z\rangle + b\langle y,z\rangle.$$

· Positive definite: If $x \neq 0$, then $\exists N \in \mathbb{Z}$ such that $x_N \neq 0$. So $\langle x,x\rangle = \sum_{n\in\mathbb{Z}} x_n^2 \ge x_N^2 > 0$. ✓

b) Show $\ell^2$ is a vector space.

Let $x,y \in \ell^2$, $c,d \in \mathbb{R}$.

1: Closure

Notice $\langle x,y\rangle < \infty$. Then $\sum (x_n+y_n)^2 = \sum x_n^2 + 2x_n y_n + y_n^2 = \underbrace{\left(\sum x_n^2\right)}_{<\infty} + 2\underbrace{\langle x,y\rangle}_{<\infty} + \underbrace{\left(\sum y_n^2\right)}_{<\infty} < \infty$.

Also, $\sum (c x_n)^2 = \sum c^2 x_n^2 = c^2 \sum x_n^2 < \infty$.

2: Assoc. Vector Add

By associativity of $\mathbb{R}$, $\sum ((x_n+y_n)+z_n)^2 = \sum (x_n + (y_n+z_n))^2$.

3: Comm. Vector Add

Commutativity in $\mathbb{R}$ gives $\sum (x_n+y_n)^2 = \sum (y_n + x_n)^2$

4: Add. Identity

$\sum (x_n+0)^2 = \sum x_n^2$

5: Add. Inverse

$\{x_n\} + \{-x_n\} = \{x_n - x_n\} = \{0\}$.

6: Assoc. Scalar Mult

$(cd)\{x_n\} = c\{dx_n\}$

7,8: Distributivity

$\{(c+d)x_n\} = \{cx_n + dx_n\} = c\{x_n\} + d\{x_n\}$ ✓ $c(\{x_n\}+\{y_n\}) = c(\{x_n+y_n\}) = \{c x_n + c y_n\} = c\{x_n\} + c\{y_n\}$

9: Mult. Identity

$1\cdot\{x_n\} = \{1\cdot x_n\} = \{x_n\}$.

c) Show $\ell^2$ is complete.

Let $\{u_n\}_{n\in\mathbb{Z}} \in \ell^2$ be Cauchy. For ease of notation, if $\{(u_1)_n\}_{n\in\mathbb{N}} = ((u_1)_1, (u_1)_2, \dots)$, we write $u_1(k) := (u_1)_k$.

Fix $\varepsilon > 0$. Since $\{u_n\}$ Cauchy, $\exists N \in \mathbb{N}$ such that $\|u_n - u_m\| < \varepsilon$ $\forall |n|,|m| \ge N$. Thus for all $k$,

$|u_n(k) - u_m(k)| \le |u_n(k) u_m(k)| < \varepsilon$, and so $\{u_n(k)\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$ for each $k$.

Hence $u_n(k) \to u(k)$ pointwise $\forall k$. Now, choose $n_0 \in \mathbb{N}$ such that $\|u_n - u_m\|_2 < \varepsilon$ $\forall |n|,|m| \ge n_0$.

Then for all $N \in \mathbb{N}$, $|n| > n_0$,

$$\sum_{k=0}^{N,-N} |u(k) u_n(k)| = \lim_{m\to\pm\infty} \sum_{k=1}^{N,-N} |u_m(k) u_n(k)|^2 \le \varepsilon^2.$$

Thus $\|u - u_n\|_2 < \varepsilon$ $\forall |n| \ge n_0$, so $u_n \to u$ in $\ell_2$.

d) Show $\ell^2$ has a complete orthonormal family.

Define $\mathcal{J} = \{e_i\} = \{(1,0,0\dots), (0,1,0,\dots), (0,0,1,0,\dots)\dots\}$. Notice if $e_i, e_j \in \mathcal{J}$, $\langle e_i, e_j\rangle = \left(\sum_{n\neq i,j} 0\right) + 0\cdot 1 + 1\cdot 0 = 0$

and $\langle e_i, e_i\rangle = \left(\sum_{n\neq i} 0\right) + 1\cdot 1 = 1$. If $x = \{x_n\}_{n\in\mathbb{Z}} \in \ell^2$, then we can choose $c_n = x_n$ $\forall n$, so $x = \sum c_n e_n$. ▨

## Question 6

Let $\mathcal{H}$ be a Hilbert space, $\{u_n\}_{n=0}^{\infty}$ be an orthonormal family in $\mathcal{H}$.

Show the following are equivalent:

a) $\{u_n\}$ is a complete orthonormal family

b) $\overline{\text{span}\{u_n\}} = \mathcal{H}$

c) If $x \in \mathcal{H}$, $\langle x, u_n \rangle = 0 \ \forall n$, then $x = 0$.

We'll show (c) $\Leftrightarrow$ (a) $\Leftrightarrow$ (b)

- (a) $\Rightarrow$ (b). Suppose $\{u_n\}$ is a complete orthonormal family and let $x \in \overline{\text{span}\{u_n\}}$.
  - If $x \in \text{span}\{u_n\}$, then $x \in \mathcal{H}$, so let's assume that $x \notin \text{span}\{u_n\}$. Then $\forall \varepsilon > 0$, $\exists m = \sum_{i=0}^{N} c_i u_i$ such that $\| x - \sum_{i=0}^{N} c_i u_i \| < \varepsilon$.

    Notice that $\forall n \geq N$, $0 \leq i \leq n$,
    $$\left( x - \sum_{i=0}^{n} \langle x, u_i \rangle u_i \right) \perp u_i$$
    so
    $$\left( x - \sum_{i=0}^{n} \langle x, u_i \rangle u_i \right) \perp \left( \sum_{i=0}^{n} \langle x, u_i \rangle u_i - \underbrace{\sum_{i=0}^{N} c_i u_i}_{m} \right)$$
    Thus, by the Pythagorean Theorem,
    $$\varepsilon^2 > \| x - m \|^2 = \| x - \sum_{i=0}^{n} \langle x, u_i \rangle u_i \|^2 + \| \sum_{i=0}^{n} \langle x, u_i \rangle u_i - m \|^2$$
    $$\geq \| x - \sum_{i=0}^{n} \langle x, u_i \rangle u_i \|^2$$
    $\forall n \geq N$. Thus $x = \sum_{i=0}^{\infty} \langle x, u_i \rangle u_i \in \mathcal{H}$ since $\{u_i\}$ is a complete family.
  - Now, suppose $x \in \mathcal{H}$. Since $\{u_i\}$ is complete, $\exists c_0, c_1, \ldots \in \mathbb{R}$ such that $x = \sum_{i=0}^{\infty} c_i u_i$. Thus, for $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\| x - \sum_{i=0}^{n} c_i u_i \| < \varepsilon \quad \forall n \geq N$. Thus $x \in \overline{\text{span}\{u_i\}}$.
- (b) $\Rightarrow$ (a) Suppose $\overline{\text{span}\{u_n\}} = \mathcal{H}$, and let $x \in \mathcal{H}$.

  Then $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $\exists c_1, \ldots, c_n \in \mathbb{R}$ such that $\| x - \sum_{i=0}^{n} c_i u_i \| < \varepsilon$.

  Thus $\sum_{i=0}^{\infty} c_i u_i = x$.
- (a) $\Rightarrow$ (c). Let $\{u_n\}$ be a complete orthonormal family. Now, suppose $\langle x, u_n \rangle = 0 \ \forall n$.

  Then by Proposition 5.4.6, $x = \sum_{n=0}^{\infty} \langle x, u_n \rangle u_n = 0$.
- (c) $\Rightarrow$ (a) Let $x \in \mathcal{H}$ and (c) hold. Suppose now, for contradiction, that $\{u_n\}$ is not a complete orthonormal family, so $\exists w$ such that $w \neq \sum_{n=0}^{\infty} c_n u_n$ (*) for any $\{c_n\} \in \mathbb{R}$. By question 2(3), we can write $w$ uniquely as $w = u + v$, where $v \in P(\mathcal{H})$, $u \in \ker\{P\}$. Note that $u \neq 0$ because of (*).

  Then $P(u) = \sum_{n=0}^{\infty} \langle u, u_n \rangle u_n = 0$, and thus $\langle u, u_n \rangle = 0 \ \forall n$. But $u \neq 0$! Which contradicts our assumption (c). So $\{u_n\}$ must be a complete orthonormal family. ∎

Question 7

a) Show the operator norm $\|f\| = \sup\{|f(v)| : \|v\| \le 1\}$ defines a norm on the dual space $V^*$.

- Let $f \in V^*$. Since $|f(v)| \ge 0 \ \forall v$, $\|f\| = \sup\{|f(v)|\} \ge 0$. If $f = 0$, then $\|f\| = \sup\{0(v) : \|v\| \le 1\} = 0$, and if $f(x) \ne 0$ for $x \in V$, then $\exists$ basis vector $u_i \in V$ where $x = c_1 u_1 + \cdots + c_i u_i + \cdots$ such that $|f(u_i)| > 0$. Thus linearity of $f$ gives
$$\|f\| = \sup\{|f(v)| : \|v\| \le 1\} \ge \left|f\left(\frac{u_i}{\|u_i\|}\right)\right| = \frac{1}{\|u_i\|}|f(u_i)| > 0.$$

- Let $f, g \in V^*$. Then the triangle inequality on $\mathbb{R}$ gives
$$\begin{aligned}\|f+g\| &= \sup\{|f(v)+g(v)| : \|v\| \le 1\} \\ &\le \sup\{|f(v)| + |g(v)| : \|v\| \le 1\} \\ &\le \sup\{|f(v)| : \|v\| \le 1\} + \sup\{|g(v)| : \|v\| \le 1\} \\ &= \|f\| + \|g\|.\end{aligned}$$

- Let $f \in V^*$, $a \in \mathbb{R}$. Then
$$\|af\| = \sup\{|af(v)| : \|v\| \le 1\} = |a| \sup\{|f(v)| : \|v\| \le 1\} = |a|\,\|f\|$$ ■

b) Suppose $\{f_n\}$ is Cauchy in $V^*$.

(i) For every $x \in V$, show $f(x) = \lim_{n\to\infty} f_n(x)$ exists.

Fix $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, $\exists N \in \mathbb{N}$ such that $\forall n, m \ge N$,
$$\sup\{|f_n(v) - f_m(v)| : \|v\| \le 1\} = \|f_n - f_m\| < \varepsilon.$$

Thus for all $v$ where $\|v\| \le 1$, $f_n(v)$ is Cauchy in $\mathbb{R}$, so $\exists f(v)$ such that $f_n(v) \to f(v)$.

Now, let $x \in V$.

- Since $V$ is a vector space, there exist orthonormal basis vectors $\{u_1, \ldots, u_k\}$ and $c_1, \ldots, c_k \in \mathbb{R}$ such that $x = c_1 u_1 + \cdots + c_k u_k$. For each $u_i$, $\exists N_i \in \mathbb{N}$ such that $\forall n, m \ge N_i$, $|f_m(u_i) - f_n(u_i)| < \frac{\varepsilon}{(\max\{|c_1|, \ldots, |c_k|\}) \cdot k}$

Then $\forall n, m \ge \max\{N_i\}_1^k$, linearity of $f_n$ gives
$$\begin{aligned}|f_n(x) - f_m(x)| &= |c_1 f_n(u_1) + \cdots + c_k f_n(u_k) - c_1 f_m(u_1) - \cdots - c_k f_m(u_k)| \\ &\le |c_1||f_n(u_1) - f_m(u_1)| + \cdots + |c_k||f_n(u_k) - f_m(u_k)| \\ &< \frac{|c_1|}{k} \frac{\varepsilon}{\max\{|c_1|, \ldots, |c_k|\}} + \cdots + \frac{|c_k|}{k} \frac{\varepsilon}{\max\{|c_1|, \ldots, |c_k|\}} \\ &< \varepsilon\end{aligned}$$

Thus $f_n(x)$ cauchy in $\mathbb{R}$, so $f(x) = \lim_{n\to\infty} f_n(x)$ exists.

(ii) Show $f : V \to \mathbb{R}$ is linear.

Let $x, y \in V$, $a \in \mathbb{R}$. Then since each $f_n$ linear and $\lim_{n\to\infty} f_n(x)$ exists $\forall x \in V$,

$$f(ax+y) = \lim_{n\to\infty} f_n(ax+y) = \lim_{n\to\infty} af_n(x) + f_n(y) = a\lim_{n\to\infty} f_n(x) + \lim_{n\to\infty} f_n(y) = af(x) + f(y).$$

(iii) Show that $f_n \to f$ in $V^*$.

Because each $f_n$ bounded and the sequence is Cauchy, $\exists M \in \mathbb{N}$ such that $\|f_n\| < M$ $\forall n \in \mathbb{N}$.

Thus $\|f\| = \lim_{n\to\infty} \|f_n\| \leq M$, so $f$ bounded.

Let $\varepsilon > 0$. Since $\{f_n\}$ Cauchy in $V^*$, $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$,

$$\sup\{ |f_n(x) - f_m(x)| < \varepsilon : \|x\| \leq 1\} = \|f_n - f_m\| < \varepsilon.$$

Then for $n \geq N$,

$$\|f - f_n\| = \sup\{|f(x) - f_n(x)| : \|x\| \leq 1\} = \lim_{m\to\infty} \sup\{|f_m(x) - f_n(x)| : \|x\| \leq 1\} \leq \varepsilon$$

Finally, $f$ is linear by (ii). So $f_n \to f$ in $V^*$. ▨

# Question 8

Let $\mathcal{H}$ be a Hilbert space. Given $x \in \mathcal{H}$, let $f_x : \mathcal{H} \to \mathbb{R}$ be

$$f_x(v) = \langle v, x \rangle$$

Show that the map $x \mapsto f_x$ is an isometry (with norm from (7)) from $\mathcal{H} \to \mathcal{H}^*$.

First note that by Cauchy-Schwarz, we can see that if $x, y, v \in \mathcal{H}$, then $|\langle v, x-y \rangle| \leq \|x-y\|$.

Then bilinearity of $\langle \cdot, \cdot \rangle$ gives

$$\begin{aligned}
\|f_x - f_y\| &= \sup\{|f_x(v) - f_y(v)| : \|v\| \leq 1\} \\
&= \sup\{|\langle v, x \rangle - \langle v, y \rangle| : \|v\| \leq 1\} \\
&= \sup\{|\langle v, x-y \rangle| : \|v\| \leq 1\} \\
&= |2||\langle x-y \rangle| \\
&= \|x-y\|
\end{aligned}$$

▨