Name: Solutions ______________________________

Do not use books, notes, calculators, phones, tablets, or software applications. You have 15 minutes to complete this quiz.

1. (5 points) Determine whether the following statement is true or false. If true, then explain why. If false, then give a counterexample with brief justification.

   The function

   $$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) \stackrel{def}{=} \begin{cases} \dfrac{1}{xy} & \text{if } x \neq 0 \text{ and } y \neq 0, \\ 0 & \text{if } x = 0 \text{ or } y = 0 \end{cases}$$

   is integrable over every box in $\mathbb{R}^2$.

   **Solution.**

   This is false. To see why, note that $f$ is not bounded on $[0, 1] \times [0, 1]$, since for each $M > 0$ we have $(\frac{1}{\sqrt{M+1}}, \frac{1}{\sqrt{M+1}}) \in [0, 1] \times [0, 1]$ and $f(\frac{1}{\sqrt{M+1}}, \frac{1}{\sqrt{M+1}}) = M + 1 > M$.

2. (10 points) Let $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ be $C^1$ functions, let $c_1, \ldots, c_m \in \mathbb{R}$, let

   $$S = \{\vec{x} \in \mathbb{R}^n \ : \ g_i(\vec{x}) = c_i, \ 1 \leq i \leq m\},$$

   let $\vec{a} \in S$, and assume that $f : S \to \mathbb{R}$ has a constrained local extreme value at $\vec{a}$. Prove that $\operatorname{rank}(D\vec{H}(\vec{a})) \leq m$, where $\vec{H} : \mathbb{R}^n \to \mathbb{R}^{m+1}$, $\vec{H}(\vec{x}) = (f(\vec{x}), g_1(\vec{x}), g_2(\vec{x}), \ldots, g_m(\vec{x}))$.

   (Suggestion: Consider the case where $\nabla g_1(\vec{a}), \ldots, \nabla g_m(\vec{a})$ is linearly dependent separately from the case where it is linear independent.)

   **Solution.**

   Note first that since $f, g_1, \ldots, g_m$ are $C^1$, they are differentiable at $\vec{a}$ and therefore the Component Characterization of Differentiability implies that $H$ is differentiable at $\vec{a}$ and

   $$DH(\vec{a}) = \begin{bmatrix} Df(\vec{a}) \\ Dg_1(\vec{a}) \\ \vdots \\ Dg_m(\vec{a}) \end{bmatrix}.$$

   If $\nabla g_1(\vec{a}), \ldots, \nabla g_m(\vec{a})$ is a linearly dependent set, then one of $Dg_1(\vec{a}), \ldots, Dg_m(\vec{a})$ is a linear combination of the others and therefore $\operatorname{rref}(DH(\vec{a}))$ has a row of all zeros, so that $\operatorname{rank}(DH(\vec{a})) \leq m$.

   If $\nabla g_1(\vec{a}), \ldots, \nabla g_m(\vec{a})$ is a linearly independent set, then the Lagrange Multiplier Theorem implies that $\nabla f(\vec{a})$ is a linear combination of $\nabla g_1(\vec{a}), \ldots, \nabla g_m(\vec{a})$ (so that $Df(\vec{a})$ is a linear combination of $Dg_1(\vec{a}), \ldots, Dg_m(\vec{a})$). Therefore $\operatorname{rref}(DH(\vec{a}))$ has a row of all zeros, so that $\operatorname{rank}(DH(\vec{a})) \leq m$.