Name: Solutions ______________________

Do not use books, notes, calculators, phones, tablets, or software applications. You have 15 minutes to complete this quiz.

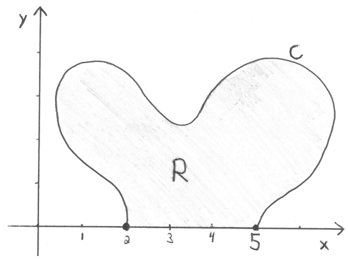
1. (5 points) Determine whether the following statement is true or false. If true, then explain why. If false, then disprove.

   Let $S = \{(x, y, z) \ : \ z = x^2 + y^2, \ 0 \leq z \leq 1\}$, with the upward orientation, and let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be $\vec{F}(x, y, z) = y\vec{i} - x\vec{j} + 0\vec{k}$. Then
$$\iint_S \vec{F} \cdot d\vec{S} = 0.$$

   Solution.

   This is **true**. S is parametrized by $\vec{X}(x, y) = (x, y, x^2 + y^2)$ for $x^2 + y^2 \leq 1$, and
$$N_{\vec{X}}(x, y) = \begin{bmatrix} 1 \\ 0 \\ 2x \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 2y \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \\ 1 \end{bmatrix},$$
   so that
$$\iint_S \vec{F} \cdot d\vec{S} = \iint\limits_{x^2+y^2 \leq 1} \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2x \\ -2y \\ 1 \end{bmatrix} \, dA(x, y) = \iint\limits_{x^2+y^2 \leq 1} 0 \, dA(x, y) = 0.$$

2. (10 points) A region $R \subset \mathbb{R}^2$ is enclosed by the x-axis and a smooth curve C as shown below.



Suppose we are given that

$$\iint_R 1\, dA(x,y) = 17, \quad \iint_R x\, dA(x,y) = 12, \quad \text{and} \quad \iint_R y\, dA(x,y) = 8.$$

If

$$\vec{F}(x,y) = (x^2 + xy + 3y)\vec{\imath} + (\arctan(y^3) + 3y^2 + 2xy + x)\vec{\jmath},$$

calculate $\int_C \vec{F} \cdot d\vec{s}$, where C is oriented so that it starts at $(2,0)$ and ends at $(5,0)$.

(You do **not** need to algebraically simplify your final answer.)

Solution.

Let $\tilde{C}$ denote the line segment from $(2,0)$ to $(5,0)$, which has parametrization $\vec{x}(t) = (t,0)$, $2 \le t \le 5$ (with $\vec{x}'(t) = \vec{\imath}$). Then note that $(-C) \cup \tilde{C}$ is the counterclockwise-oriented boundary of R, so that we can apply Green's Theorem (in the third step below) to see that

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{s} &= -\int_{-C} \vec{F} \cdot d\vec{s} - \int_{\tilde{C}} \vec{F} \cdot d\vec{s} + \int_{\tilde{C}} \vec{F} \cdot d\vec{s} \\
&= -\int_{(-C)\cup\tilde{C}} \vec{F} \cdot d\vec{s} + \int_2^5 \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\, dt \\
&= -\iint_R \operatorname{curl}\vec{F}(x,y)\, dA(x,y) + \int_2^5 \vec{F}(t,0) \cdot \vec{\imath}\, dt \\
&= -\iint_R (2y + 1 - (x+3))\, dA(x,y) + \int_2^5 t^2\, dt \\
&= 2\iint_R 1\, dA(x,y) + \iint_R x\, dA(x,y) - 2\iint_R y\, dA(x,y) + \left[\frac{t^3}{3}\right]_2^t \\
&= 2(17) + 12 - 2(8) + \frac{125-8}{3} \qquad \longleftarrow \text{(You can stop here.)} \\
&= 69.
\end{aligned}$$