Math 291-3 Quiz 3 Spring 2022

Name: Solutions ________________________________

Do not use books, notes, calculators, phones, tablets, or software applications. You have 15 minutes to complete this quiz.

1. (5 points) Determine whether the following statement is true or false. If true, then explain why. If false, then give a counterexample with brief justification.

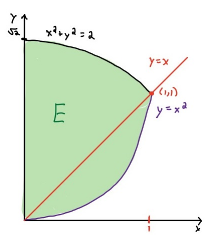
   Let $a > 0$. If $C \subset \mathbb{R}^n$ is a smooth curve with C^1 parametrization $\vec{r} : [0,1] \to \mathbb{R}^n$ satisfying $\|\vec{r}\,'(t)\| \geq a$ for each $t \in [0,1]$, then Length$(C) \geq a$ as well.

   Solution.
   This is true. Since $\|\vec{r}\,'(t)\| \geq a$ for each $t \in [0,1]$, Length$(C) = \int_0^1 \|\vec{r}\,'(t)\|\,dt \geq \int_0^1 a\,dt = a$.

2. (10 points) Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous. Write the following as an iterated integral (or sum of iterated integrals) in polar coordinates:

$$\int_0^1 \int_{x^2}^{\sqrt{2-x^2}} f(x^2+y^2)\,dydx$$

   Solution.
   The given iterated integral is equivalent to the double integral of $f(x^2+y^2)$ over the region E enclosed on the left by the y-axis, above by the line $y = 1$, and below the circle $x^2 + y^2 = 2$.



   To describe E in polar coordinates, note that we should have $0 \leq \theta \leq \frac{\pi}{2}$, and that $0 \leq r$ regardless of θ. The upper bound for r, though, changes. For $0 \leq \theta \leq \frac{\pi}{4}$ we have that r runs from the origin $(r = 0)$ to the curve $y = x^2$, which in polar coordinates is $r\sin(\theta) = r^2\cos^2(\theta)$, so that $r = \frac{\sin(\theta)}{\cos^2(\theta)}$. For $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$, r will run from the origin $(r = 0)$ to the circle $x^2 + y^2 = 2$ (i.e. $r = \sqrt{2}$). Because $x^2 + y^2 = r^2$, we have Therefore we can express the integral in polar coordinates as

$$\int_0^1 \int_{x^2}^1 f(x^2+y^2)\,dydx = \int_0^{\frac{\pi}{4}} \int_0^{\sin(\theta)/\cos^2(\theta)} f(r^2)r\,drd\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} f(r^2)r\,drd\theta.$$