Northwestern University

MATH 291-3 First Midterm Examination - Practice A
Spring Quarter 2022
April 21, 2022

Last name: SOLUTIONS ______________ Email address: ______________

First name: ______________ NetID: ______________

Instructions

- This examination consists of 5 questions.
- Read all problems carefully before answering.
- You have 50 minutes to complete this examination.
- Do not use books, notes, calculators, computers, tablets, phones, smart watches, or similar devices.
- Possession of a digital communication device during a bathroom break will be treated as a *prima facie* violation of Northwestern's academic integrity policy.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is true or false. If true, then explain why. If false, then give a counterexample.

   (a) Every Riemann sum for $f(x,y) \overset{def}{=} x$ over $[-1,1] \times [-1,1]$ is non-negative.

   (b) If $f : [-1,1] \times [-1,1] \to \mathbb{R}$ is such that all Riemann sums for f have the same value (i.e. there is $c \in \mathbb{R}$ such that for every partition $\mathcal{P}$ of $B = [-1,1] \times [-1,1]$ and every choice of sample points $\mathcal{C}$, $c = R(f, \mathcal{P}, \mathcal{C})$), then f is constant.

Solution: (a) is false. Let $\mathcal{P}$ be the partition of $[-1,1] \times [-1,1]$ consisting of a single box $B_1 = [-1,1] \times [-1,1]$, and choose sample point $\vec{c}_1 = (-1,0) \in B_1$. Then

$$R(f, \mathcal{P}, \mathcal{C}) = f(\vec{c}_1)\mathrm{Vol}_2(B_1) = f(-1,0)\mathrm{Vol}_2([-1,1] \times [-1,1]) = -1(2)(2) = -4 < 0.$$

(b) is true. Suppose such a c exists, and let $(x,y) \in B \overset{def}{=} [-1,1] \times [-1,1]$. Let $\mathcal{P}$ be the partition of B consisting of just the single box B, and choose $\vec{c}_1 = (x,y)$ to be the sample point for B. Then $c = R(f, \mathcal{P}, \mathcal{C}) = f(\vec{c}_1)\mathrm{Vol}_2(B) = 4f(x,y)$, so that $f(x,y) = \frac{c}{4}$. Because (x,y) was arbitrary, $f(x,y) = \frac{c}{4}$ for every $(x,y) \in B$, so f is constant.

2. Fix $K > 0$ and consider all non-negative numbers $x_1, \ldots, x_n$ satisfying $x_1 + \cdots + x_n = K$. Show that among all such numbers there exist ones that maximize the product $x_1 x_2 \ldots x_n$, and find specific the specific values of those that do.

   (You may use without proof that $S = \{(x_1, \ldots, x_n) \ : \ x_j \geq 0 \text{ for } j = 1, \ldots, n \text{ and } x_1 + \cdots + x_n = K\}$ is closed.)

Solution: Let $S = \{(x_1, \ldots, x_n) \ : \ x_j \geq 0 \text{ for } j = 1, \ldots, n \text{ and } x_1 + \cdots + x_n = K\}$. If $\vec{x} = (x_1, \ldots, x_n) \in S$, then for each $j = 1, \ldots, n$ we have $0 \leq x_j \leq x_1 + \cdots + x_n = K$, so that $S \subseteq [0, K] \times \cdots [0, K]$. Therefore S is bounded. Because S is also closed, S is compact and the Extreme Value Theorem implies that the continuous function $f(x_1, \ldots, x_n) \overset{def}{=} x_1 \cdots x_n$ achieves a global maximum value on S.

Suppose that this global maximum value is achieved at $(x_1, \ldots, x_n)$. Since $f(x_1, \ldots, x_n) > 0$ if every $x_1, \ldots, x_n > 0$ and $f(x_1, \ldots, x_n) = 0$ if $x_j = 0$ for some $j = 1, \ldots, n$, we know that the global maximum value of f must occur away from the edge of S, and therefore is a constrained local maximum value of f on the constraint $g(x_1, \ldots, x_n) = K$, where $g(x_1, \ldots, x_n) = x_1 + \cdots + x_n$. Because $\nabla g(x_1, \ldots, x_n) \neq \vec{0}$, we know that the point $(x_1, \ldots, x_n)$ at which f has a global extreme value satisfies, for some $\lambda \in \mathbb{R}$,

$$\begin{cases} \nabla f(x_1, \ldots, x_n) = \lambda \nabla g(x_1, \ldots, x_n) \\ g(x_1, \ldots, x_n) = K \end{cases} \quad \Leftrightarrow \quad \begin{cases} x_2 x_3 \ldots x_n = \lambda \\ x_1 x_3 \cdots x_n = \lambda \\ \vdots \\ x_1 \cdots x_{n-1} = \lambda \\ x_1 + \cdots + x_n = K \end{cases}$$

Multiplying each of the first n equations (respectively) by x_1, x_2, and so on gives $\lambda x_1 = x_1 x_2 \cdots x_n = \lambda x_2 = \lambda x_3 = \cdots = \lambda x_n$. Because $x_j \neq 0$ for all $j = 1, \ldots, n$ at the point we are seeking, we know that $x_1 x_2 \cdots x_n \neq 0$ and therefore $\lambda \neq 0$. We can therefore divide by λ to see that $x_1 = x_2 = \cdots = x_n$. Therefore, for each $j = 1, \ldots, n$, $K = x_1 + \cdots + x_n = n x_j$, so that $x_j = \frac{K}{n}$. Because $(\frac{K}{n}, \ldots, \frac{K}{n})$ is the only point on S where $x_1, \ldots, x_n > 0$ and where f may have a constrained local extreme value, this must be the location of the global maximum value. Therefore, the global maximum values of f on S occurs at $(\frac{K}{n}, \ldots, \frac{K}{n})$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, and define $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x, y) \stackrel{def}{=} \begin{cases} 1 & \text{if } y = f(x), \\ 0 & \text{if } y \neq f(x). \end{cases}$$

Prove that g is integrable over any box $B \subset \mathbb{R}^2$.

(You may use without proof the fact that the graph $\{(x, f(x)) \ : \ x \in \mathbb{R}\}$ is a closed subset of $\mathbb{R}^2$.)

Solution: Let B be a box. Since $|g(x, y)| \leq 1$ for every $(x, y) \in B$, g is bounded on B. Let $D \stackrel{def}{=} \{(x, f(x)) \ : \ x \in \mathbb{R}\}$. Because g is C^1 (as it is identically 0) on the open set $\mathbb{R}^2 \backslash D$, g is differentiable and therefore continuous at every point in $\mathbb{R}^2 \backslash D$. Therefore g is only possibly discontinuous at points in D. But $D = \vec{h}(\mathbb{R})$, where $\vec{h}(t) = (t, f(t))$. Since $\vec{h}$ is a C^1 function from $\mathbb{R}^1$ to $\mathbb{R}^2$ and $1 < 2$, the Measure Zero Theorem implies that $\text{Vol}_2(D) = 0$. By the Measure Zero Theorem, since $D \cap B \subseteq D$, $D \cap B$ has measure zero. Therefore g is continuous on B except on a set of measure zero, so the Lebesgue Criterion implies that g is integrable on B.

4. Let D be the region in $\mathbb{R}^2$ enclosed by the circle with equation $(x-4)^2 + (y-\frac{1}{2})^2 = \frac{1}{16}$. Show that

$$\frac{3\pi}{8} \leq \iint_D \left(xy + \frac{8}{x} + \frac{1}{y} \right) dA.$$

You may assume that the global minimum value of the integrand on D does not occur on the boundary of D.

Solution: Note that the function $f(x,y) = xy + \frac{8}{x} + \frac{1}{y}$ is C^1 when $x \neq 0$ and $y \neq 0$, and is therefore differentiable throughout D (which is the disc of radius $\frac{1}{4}$ centered at $(4, \frac{1}{2})$, and therefore contains no points where $x = 0$ or $y = 0$). The Extreme Value Theorem implies that f achieves a global minimum value on D that (according to the simplifying assumption in the problem statement) does not occur on the boundary of D. Therefore the global minimum value of f on D is a local minimum value, so we will identify the points in D where f may have a local minimum value.

Such a point (x,y) must satisfy

$$\begin{bmatrix} 0 & 0 \end{bmatrix} = Df(x,y) = \begin{bmatrix} y - \frac{8}{x^2} & x - \frac{1}{y^2} \end{bmatrix},$$

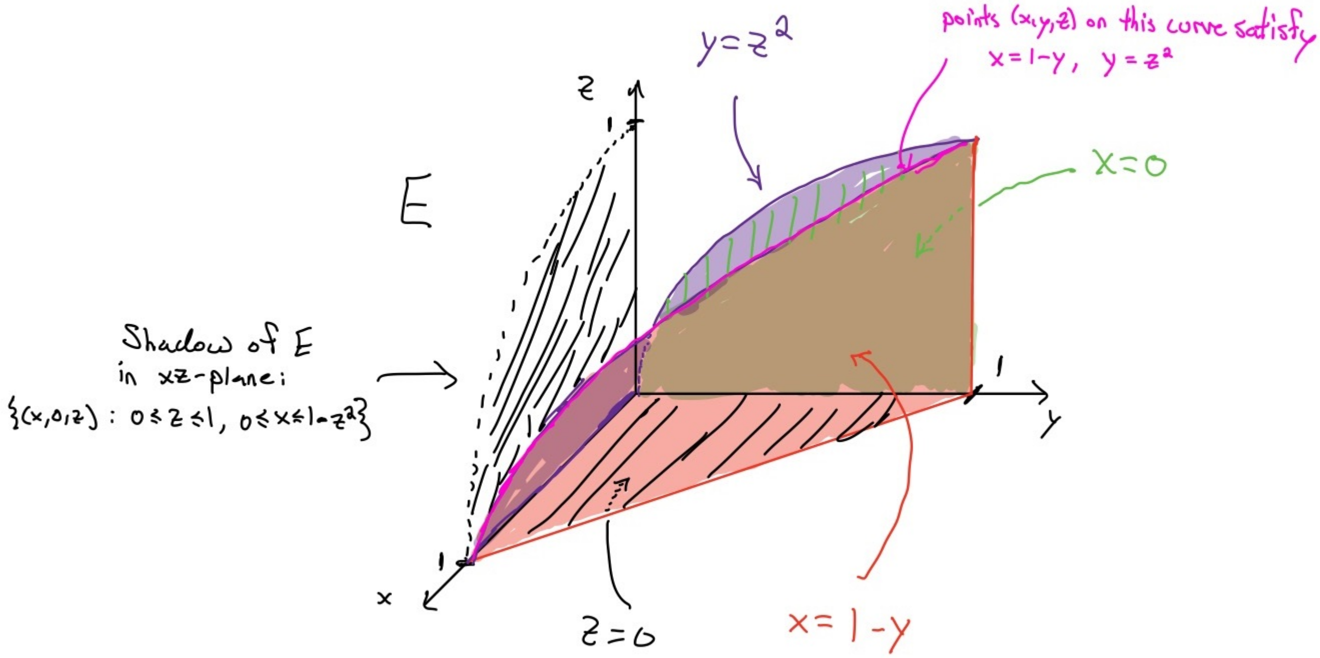
so that $8 = yx^2$ and $1 = xy^2$. Therefore $\frac{1}{y} = yx = \frac{8}{x}$, so that $8y = x$. Therefore $64 = 8yx^2 = x^3$, so that $x = 4$ and $y = \frac{1}{2}$. It follows that the only possible location of a local minimum value of f on D is $(4, \frac{1}{2})$, so that $f(x,y) \geq f(4, \frac{1}{2}) = 2 + 2 + 2 = 6$ on D. Therefore

$$\iint_D f(x,y)\, dA \geq \iint_D 6\, dA = 6\text{Area}(D) = 6\frac{\pi}{16} = \frac{3\pi}{8}.$$

5. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous. Rewrite the following as an iterated integral with respect to the order $dy\,dx\,dz$:

$$\int_0^1 \int_{z^2}^1 \int_0^{1-y} f(x,y,z)\,dx\,dy\,dz$$

Solution: Note that the given iterated integral is equal to $\iiint_E f(x,y,z)\,dV$, where E is the subset of $\mathbb{R}^3$ in the first octant bounded by the zy- and xy-coordinate planes, the plane $x = 1-y$, and the parabolic cylinder $y = z^2$ (pictured below):



The shadow of E in the xz-plane is $\{(x,0,z) \ : \ 0 \le z \le 1 \text{ and } 0 \le x \le 1-z^2\}$, and for each choice of x and z satisfying these inequalities, y will run from its smallest value z^2 to its largest value $1-x$. Therefore we can express this triple integral as an iterated integral in the order $dy\,dx\,dz$ as

$$\int_0^1 \int_0^{1-z^2} \int_{z^2}^{1-x} f(x,y,z)\,dy\,dx\,dz.$$