Northwestern University

MATH 291-3 First Midterm Examination
Spring Quarter 2022
April 21, 2022

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Instructions

- This examination consists of 5 questions.
- Read all problems carefully before answering.
- You have 50 minutes to complete this examination.
- Do not use books, notes, calculators, computers, tablets, phones, smart watches, or similar devices.
- Possession of a digital communication device during a bathroom break will be treated as a *prima facie* violation of Northwestern's academic integrity policy.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is true or false. If true, then explain why. If false, then give a counterexample.

   (a) (5 points) If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous with global maximum value 1, and if $B \subseteq \mathbb{R}^n$ is a box, then

$$\int_B f\,dV_n \leq \mathrm{Vol}_n(B).$$

Solution: (a) is true. Because f is continuous on B, f is integrable on B. Because $f(\vec{x}) \leq 1$ for every $\vec{x} \in B$,

$$\int_B f\,dV_n \leq \int_B 1\,dV_n = \mathrm{Vol}_n(B).$$

   (b) (5 points) If $f : \mathbb{R}^n \to \mathbb{R}$ is bounded, if $B \subseteq \mathbb{R}^n$ is a box with $\mathrm{Vol}_n(B) > 0$, and if $R(f, \mathcal{P}, \mathcal{C}) = 0$ for every partition $\mathcal{P}$ of B and every choice of sample points $\mathcal{C}$, then $f(\vec{x}) = 0$ for every $\vec{x} \in B$.

Solution: (b) is true. Let $\vec{x} \in B$, let $\mathcal{P} = \{B_1\}$ be the partition of B into a single box $B_1 = B$, and let $\mathcal{C} = \{\vec{c}_1\}$ where $\vec{c}_1 = \vec{x}$. Then

$$0 = R(f, \mathcal{P}, \mathcal{C}) = f(\vec{c}_1)\mathrm{Vol}_n(B_1) = f(\vec{x})\mathrm{Vol}_n(B).$$

Since $\mathrm{Vol}_n(B) \neq 0$, $f(\vec{x}) = 0$.

2. (10 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be C^1 functions, and let $\Gamma \stackrel{def}{=} \{(x, y, z) \in \mathbb{R}^3 \ : \ z = g(x, y)\}$ be the graph of g. Prove that if $f : \Gamma \to \mathbb{R}$ has a constrained local extreme value at $\vec{p}_0 = (x_0, y_0, z_0)$, then

$$\begin{bmatrix} f_x(\vec{p}_0) \\ f_y(\vec{p}_0) \end{bmatrix} = -f_z(\vec{p}_0)\nabla g(x_0, y_0).$$

(Hint: Γ is a level set of $G : \mathbb{R}^3 \to \mathbb{R}$, $G(x, y, z) \stackrel{def}{=} g(x, y) - z$.)

Solution: Note that G is C^1 on $\mathbb{R}^3$, and that $\nabla G(\vec{p}_0) = \begin{bmatrix} g_x(x_0, y_0) \\ g_y(x_0, y_0) \\ -1 \end{bmatrix} \neq \vec{0}$, so the Lagrange Multiplier Theorem implies that there is $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \nabla f(\vec{p}_0) = \lambda \nabla G(\vec{p}_0), \\ G(\vec{p}_0) = 0, \end{cases}$$

or rather

$$\begin{cases} f_x(\vec{p}_0) = \lambda g_x(x_0, y_0), \\ f_y(\vec{p}_0) = \lambda g_y(x_0, y_0), \\ f_z(\vec{p}_0) = -\lambda, \\ g(x_0, y_0) = z_0. \end{cases}$$

Making the substitution $\lambda = -f_z(\vec{p}_0)$ in the first two equations implies that

$$\begin{cases} f_x(\vec{p}_0) = -f_z(\vec{p}_0)g_x(x_0, y_0), \\ f_y(\vec{p}_0) = -f_z(\vec{p}_0)g_y(x_0, y_0), \end{cases}$$

which gives the result.

3. (10 points) Find and classify the critical points of $f : \{(x, y) \ : \ x + y > 0\} \to \mathbb{R}$, $f(x, y) = \ln(x + y) - x^2 - y$.

(Because it hasn't yet come up, recall that $\ln : (0, +\infty) \to \mathbb{R}$ is differentiable and $(\ln)'(t) = \dfrac{1}{t}$ for $t \in (0, +\infty)$.)

Solution: Note that f is C^2 on its domain $\{(x, y) \ : \ x + y > 0\}$, and therefore every critical point (x, y) of f will satisfy

$$\begin{bmatrix} 0 & 0 \end{bmatrix} = Df(x, y) = \begin{bmatrix} \frac{1}{x+y} - 2x & \frac{1}{x+y} - 1 \end{bmatrix},$$

so that $0 = \frac{1}{x+y} - 2x$ and $0 = \frac{1}{x+y} - 1$. Therefore $x + y = 1$ and so $2x = 1$, so that $x = \frac{1}{2}$ and $y = 1 - \frac{1}{2} = \frac{1}{2}$. One can quickly verify that $(\frac{1}{2}, \frac{1}{2})$ is indeed a (and therefore, by the above arugment, the only) critical point of f.

To classify this critical point, note that

$$D^2 f(x, y) = \begin{bmatrix} -\dfrac{1}{(x+y)^2} - 2 & -\dfrac{1}{(x+y)^2} \\ -\dfrac{1}{(x+y)^2} & -\dfrac{1}{(x+y)^2} \end{bmatrix}, \qquad \text{so that} \qquad D^2 f\left(\frac{1}{2}, \frac{1}{2}\right) = \begin{bmatrix} -3 & -1 \\ -1 & -1 \end{bmatrix}.$$

Because $\begin{bmatrix} -3 & -1 \\ -1 & -1 \end{bmatrix}$ has determinant $2 > 0$, $\begin{bmatrix} -3 & -1 \\ -1 & -1 \end{bmatrix}$ is invertible and the two eigenvalues of $\begin{bmatrix} -3 & -1 \\ -1 & -1 \end{bmatrix}$ have the same sign. Because $\operatorname{tr} \begin{bmatrix} -3 & -1 \\ -1 & -1 \end{bmatrix} = -4 < 0$, we see that both eigenvalues of $\begin{bmatrix} -3 & -1 \\ -1 & -1 \end{bmatrix}$ are negative and therefore the Second Derivative Test implies that f has a local maximum value at $(\frac{1}{2}, \frac{1}{2})$.

4. (10 points) Let $E = \{(x, y) \ : \ -1 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 - x^2\}$ be the region in $\mathbb{R}^2$ bounded below by the x-axis and above by the parabola $y = 1 - x^2$. Show that

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \begin{cases} -3 & \text{if } (x, y) \in E, \\ 2 & \text{if } (x, y) \notin E \end{cases}$$

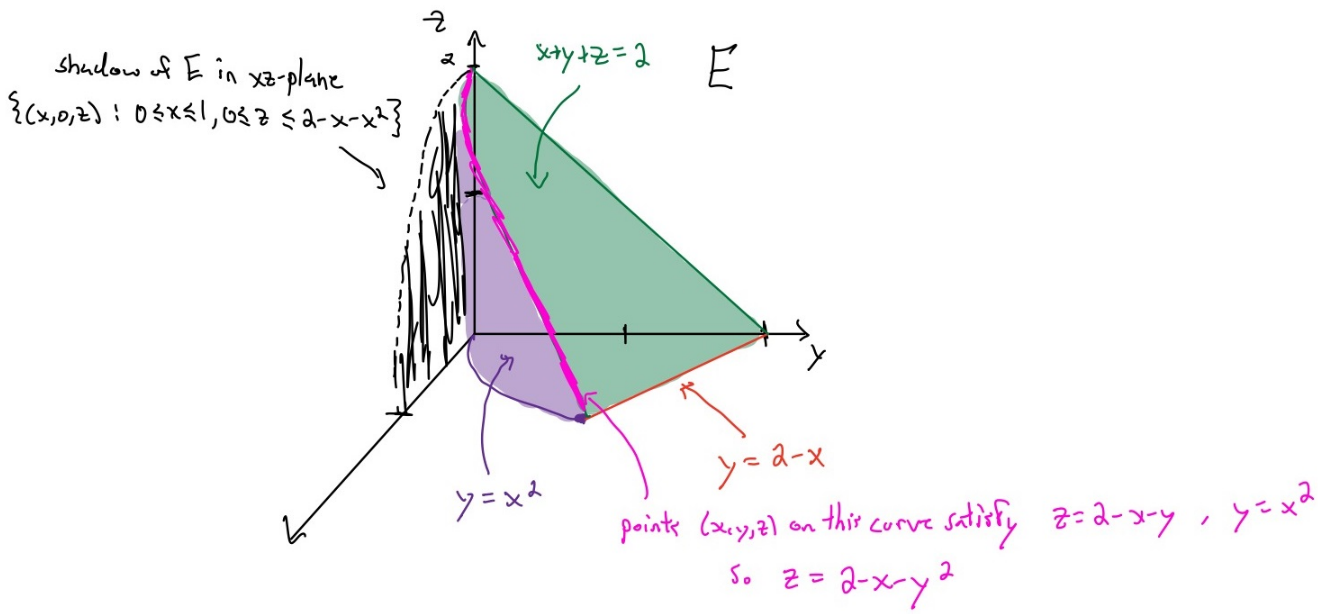
is integrable over the box $B = [-2, 2] \times [-2, 2]$.

Solution: Note that f is continuous on B except at points that lie on ∂E. Since ∂E is the union of the two curves $\{(x, 0) \ : \ x \in [-1, 1]\}$ and $\{(x, 1 - x^2) \ : \ x \in [-1, 1]\}$ (parametrized respectively by the C^1 functions $\vec{r}_1, \vec{r}_2 : [-1, 1] \to \mathbb{R}^2$, $\vec{r}_1(t) = (t, 0)$ and $\vec{r}_2(t) = (t, 1 - t^2)$), the Measure Zero Theorem implies that these two curves have measure zero and therefore ∂E has measure zero. Since f is bounded on B ($|f(x, y)| \leq 3$ for every $(x, y) \in B$), Lebesgue's Criterion for Riemann Integrability implies that f is integrable on B.

5. (10 points) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous. Rewrite the following as an iterated integral with respect to the order $dy\,dz\,dx$:

$$\int_0^1 \int_{x^2}^{2-x} \int_0^{2-x-y} f(x,y,z)\,dz\,dy\,dx$$

Solution: Note that the given iterated integral is equal to $\iiint_E f\,dV$, where E is the subset of $\mathbb{R}^3$ in the first octant bounded by the xz- and xy-coordinate planes, the plane $z = 2 - x - y$, and the parabolic cylinder $y = x^2$ (pictured below):



The shadow of E in the xz-plane is $\{(x,0,z) \;:\; 0 \le x \le 1 \text{ and } 0 \le z \le 2 - x - x^2\}$, and for each choice of x and z satisfying these inequalities, y will run from its smallest value x^2 to its largest value $2 - x - z$. Therefore we can express this triple integral as an iterated integral in the order $dy\,dz\,dx$ as

$$\int_0^1 \int_0^{2-x-x^2} \int_{x^2}^{2-x-z} f(x,y,z)\,dy\,dz\,dx.$$