Northwestern University

MATH 291-3 Final Examination - Practice A Solutions
Spring Quarter 2022
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Last name: SOLUTIONS ______________________ Email address: ______________________

First name: ______________________ NetID: ______________________

Instructions

- This examination consists of 7 questions.
- Read all problems carefully before answering.
- You have 120 minutes to complete this examination.
- Do not use books, notes, calculators, computers, tablets, phones, smart watches, or similar devices.
- Possession of a digital communication device during a bathroom break will be treated as a *prima facie* violation of Northwestern's academic integrity policy.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is true or false. If true, then explain why. If false, then give a counterexample.
   (a) If $f(x, y)$ is continuous everywhere except on the set of points satisfying $x^2 + 2y^2 \leq 1$, then f is integrable over the rectangle $[-3, 3] \times [-3, 3]$.
   (b) If $\vec{F}$ is C^1 on an open set $U \subseteq \mathbb{R}^2$ and if $\text{curl}\vec{F}(x, y) = 0$ on U, then $\vec{F}$ is conservative on U.
   (c) If S_1 and S_2 are smooth oriented surfaces with the same boundary and which induce the same orientation on that boundary, then $\iint_{S_1} \text{curl}\vec{F} \cdot d\vec{S} = \iint_{S_2} \text{curl}\vec{F} \cdot d\vec{S}$ for every C^1 vector field $\vec{F}$.

Solution: (a) is false. The region described by the inequality $x^2 + 2y^2 \leq 1$ is enclosed by an ellipse and has positive area (i.e. 2-volume), and therefore does not have measure zero as a subset of $\mathbb{R}^2$. Because this region is contained in the box $[-3, 3] \times [-3, 3]$, the set of discontinuities of f in the box $[-3, 3] \times [-3, 3]$ does not have measure zero, and therefore f does not satisfy Lebesgue's Criterion for Riemann Integrability.

(As the instructions are written, you are asked to provide a counterexample. Because the counterexample is rather technical to write down, it would have been better if the instructions for 1 were "Prove or disprove each statement." and if 1(a) stated that "There exists a function $f(x, y)$ that is continuous everywhere except on the set of points satisfying $x^2 + 2y^2 \leq 1$, and that is integrable over $[-3, 3] \times [-3, 3]$." You can be assured that the actual exam will not have this type of poor wording. For an explicit example of such a function satisfying the hypotheses of 1(a), consider

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \begin{cases} 0 & \text{if } x \text{ is rational or } x^2 + 2y^2 \leq 1, \\ 1 & \text{if } x \text{ is irrational and } x^2 + 2y^2 > 1.) \end{cases}$$

(b) is false. For a counterexample, take $U = \mathbb{R}^2 - \{(0, 0)\}$ and $\vec{F} = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$. We have shown that $\text{curl}\vec{F}(x, y) = 0$ throughout U, but that $\vec{F}$ is not conservative on U.

(c) is true. Because ∂S_1 has the orientation induced by the orientation of S_1, and ∂S_2 has the orientation induced by the orientation of S_2, and since $\partial S_1 = \partial S_2$ as (possibly unions of) piecewise-smooth oriented closed curves, Stokes' Theorem implies that

$$\iint_{S_1} \text{curl}\vec{F} \cdot d\vec{S} = \oint_{\partial S_1} \vec{F} \cdot d\vec{s} = \oint_{\partial S_2} \vec{F} \cdot d\vec{s} = \iint_{S_2} \text{curl}\vec{F} \cdot d\vec{S}.$$

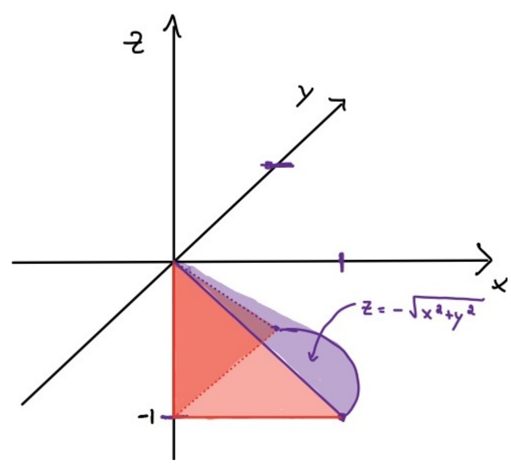
2. Consider the following iterated integral:

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{-1}^{-\sqrt{x^2+y^2}} y^2 \, dzdxdy.$$

(a) Rewrite this as an iterated integral in cylindrical coordinates.

(b) Rewrite this as an iterated integral in spherical coordinates.

Solution: Note that this iterated integral represents the triple integral of y^2 over the region E in $\mathbb{R}^3$ whose shadow in the xy-plane in the portion of the unit disc in the first quadrant, and that is bounded below by the plane $z = -1$ and above by the cone $z = -\sqrt{x^2 + y^2}$:



In cylindrical coordinates, the shadow of this region in the xy-plane can be represented by $0 \le r \le 1$ and $0 \le \theta \le \frac{\pi}{2}$, while we have $-1 \le z \le -\sqrt{x^2 + y^2} = -r$. Therefore

$$\iiint_E y^2 \, dV_3(x, y, z) = \int_0^{\frac{\pi}{2}} \int_0^1 \int_{-1}^{-r} (r\sin(\theta))^2 r \, dzdrd\theta = \int_0^{\frac{\pi}{2}} \int_0^1 \int_{-1}^{-r} r^3 \sin^2(\theta) \, dzdrd\theta.$$

In spherical coordinates, we still have $0 \le \theta \le \frac{\pi}{2}$. We also see that ϕ runs from $\frac{3\pi}{4}$ (the cone) to π (the negative z-axis). Then ρ runs from the origin ($\rho = 0$) to the plane $-1 = z = \rho\cos(\phi)$, so $\rho = -\sec(\phi)$. Therefore we write the triple integral as an iterated integral in spherical coordinates as

$$\iiint_E y^2 \, dV_3(x, y, z) = \int_0^{\frac{\pi}{2}} \int_{\frac{3\pi}{4}}^{\pi} \int_0^{-\sec(\phi)} (\rho\sin(\theta)\sin(\phi))^2 \rho^2 \sin(\phi) \, d\rho d\phi d\theta.$$

3. Suppose $A \in M_{n\times n}(\mathbb{R})$ is invertible. Let $D \subset \mathbb{R}^n$ be compact, and let $A(D)$ denote the image of D under the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ with standard matrix A. Assume that ∂D and $\partial A(D)$ each have measure zero. Show that
$$\mathrm{Vol}_n(A(D)) = |\det(A)|\mathrm{Vol}_n(D).$$
(That is, prove the geometric interpretation of the determinant as an expansion factor.)

Solution: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ denote the linear transformation $T(\vec{x}) = A\vec{x}$. Then T is C^1 throughout $\mathbb{R}^n$ (since its component functions are polynomials), and $DT(\vec{x}) = A$ is invertible throughout $\mathbb{R}^n$. Then we have (using the fact that $\partial A(D)$ and ∂D have measure zero, so that the constant function 1 is integrable on both D and $A(D)$) by the Change of Variables Theorem

$$\mathrm{Vol}_n(A(D)) = \int_{T(D)} 1\, dV_n = \int_D 1\, |\det(A)|\, dV_n = |\det(A)| \int_D 1\, dV_n = |\det(A)|\mathrm{Vol}_n(D).$$

4. Let C be the curve where the cylinder $y^2 + z^2 = 1$ and the plane $x = y$ intersect. Show that C is smooth.

Solution: The curve C can be parametrized by

$$\vec{x}(t) = (\cos(t), \cos(t), \sin(t)), \quad 0 \leq t \leq 2\pi.$$

Because $\vec{x}$ is differentiable with

$$\|\vec{x}'(t)\| = \|(-\sin(t), -\sin(t), \cos(t))\| = \sqrt{\sin^2(t) + \sin^2(t) + \cos^2(t)} = \sqrt{\sin^2(t) + 1} \geq \sqrt{1} = 1 > 0,$$

$\vec{x}'(t) \neq \vec{0}$ for every $t \in [0, 2\pi]$. Therefore C is smooth.

5. Show that a C^1 vector field $\vec{F}$ on $\mathbb{R}^n$ has path-independent line integrals if, and only if, its line integral over every closed, oriented, piecewise-smooth curve is 0.

   (Recall that $\vec{F}$ has path-independent line integrals if whenever C_1 and C_2 are piecewise-smooth oriented curves which begin at the same point and end at the same point, it must be that $\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}$. You may not use the fact that these properties are equivalent to $\vec{F}$ being conservative.)

Solution: Suppose that $\vec{F}$ has path-independent line integrals. Let C be a closed, oriented, piecewise-smooth curve, and denote the starting point of C by $\vec{a}$ (note that $\vec{a}$ is also the ending point of C). Choose a point $\vec{b}$ on C with $\vec{b} \neq \vec{a}$, and let C_1 be the portion of C that starts at $\vec{a}$ and ends at $\vec{b}$. Let C_2 be the portion of C that starts at $\vec{b}$ and ends at $\vec{a}$, so that $C = C_1 \cup C_2$. Then C_1 and $-C_2$ both start at $\vec{a}$ and end at $\vec{b}$, so that

$$\oint_C \vec{F} \cdot d\vec{s} = \int_{C_1} \vec{F} \cdot d\vec{s} + \int_{C_2} \vec{F} \cdot d\vec{s} = \int_{C_1} \vec{F} \cdot d\vec{s} - \int_{-C_2} \vec{F} \cdot d\vec{s} = 0$$

because $\vec{F}$ has path-independent line integrals. This proves the "only if" direction.

Now suppose that the line integral of $\vec{F}$ over every closed, oriented, piecewise-smooth curve is 0. Let C_1, C_2 be piecewise-smooth oriented curves which start at the same point and end at the same point. Then $C \stackrel{def}{=} C_1 \cup (-C_2)$ is a piecewise-smooth closed oriented curve, so that

$$0 = \oint_C \vec{F} \cdot d\vec{s} = \int_{C_1} \vec{F} \cdot d\vec{s} + \int_{-C_2} \vec{F} \cdot d\vec{s} = \int_{C_1} \vec{F} \cdot d\vec{s} - \int_{C_2} \vec{F} \cdot d\vec{s},$$

whence it follows that $\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}$. Therefore $\vec{F}$ has path-independent line integrals.

6. Compute the line integral
$$\int_C (x\sin(e^x) - xz)\,dx - 2xy\,dy + (z^2+y)\,dz$$
where C is the piecewise-smooth curve consisting of the line segment from $(2,0,0)$ to $(0,2,0)$, followed by the line segment from $(0,2,0)$ to $(0,0,2)$, followed by the line segment from $(0,0,2)$ to $(2,0,0)$.

(Hint: C lies on the plane $x+y+z=2$.)

Solution: Let $\vec{F} = (x\sin(e^x) - xz)\vec{i} - 2xy\vec{j} + (z^2+y)\vec{k}$. Let D be the region in the plane $x+y+z=2$ that is enclosed by the triangle C, so that $C = \partial D$. Then because C is orient counterclockwise when viewed from above, we oriented D with upward-pointing normal vectors. Then the orientation of C is the one induced by the orientation of D, so that Stokes' Theorem applies. Using the parametrization

$$\vec{X}(x,y) = (x, y, 2-x-y), \qquad 0 \le x \le 2,\ 0 \le y \le 2-x$$

of D therefore gives

$$\begin{aligned}
\oint_C \vec{F}\cdot d\vec{s} &= \iint_D \operatorname{curl}\vec{F}\cdot d\vec{S} \\
&= \iint_D \begin{bmatrix} 1 \\ -x \\ -2y \end{bmatrix} \cdot d\vec{S} \\
&= \int_0^2 \int_0^{2-x} \begin{bmatrix} 1 \\ -x \\ -2y \end{bmatrix} \cdot N_{\vec{X}}(x,y)\,dydx \\
&= \int_0^2 \int_0^{2-x} \begin{bmatrix} 1 \\ -x \\ -2y \end{bmatrix} \cdot \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right) dydx \\
&= \int_0^2 \int_0^{2-x} \begin{bmatrix} 1 \\ -x \\ -2y \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} dydx \\
&= \int_0^2 \int_0^{2-x} (1-x-2y)\,dydx \\
&= \int_0^2 \Big((2-x)(1-x) - (2-x)^2 \Big)\,dx \\
&= \int_0^2 (x-2)\,dx \\
&= -2.
\end{aligned}$$

7. Compute the surface integral $\iint_S \vec{F}\cdot d\vec{S}$, where

$$\vec{F} = (3x - ye^{\cos(z)})\vec{i} + (e^{x^{10}z^8} - 2yz)\vec{j} + (z^2 + ye^x)\vec{k},$$

and where S is the portion of the cylinder $x^2 + y^2 = 1$ that lies between $z = 0$ and $z = 1$, oriented with inward-pointing normal vectors.

Solution: Let S_0 denote the disc $x^2 + y^2 \le 1$ in the plane $z = 0$, and orient S_0 with upward-pointing normal vector. Let S_1 denote the disc $x^2 + y^2 \le 1$ in the plane $z = 1$, and orient S_1 with upward-pointing normal vector. Then $S \cup S_0 \cup (-S_1)$ is a closed surface, and $-(S \cup S_0 \cup (-S_1)) = (-S) \cup (-S_0) \cup S_1$ is the (outward-oriented) boundary of the solid cylinder E described by $x^2 + y^2 \le 1$ and $0 \le z \le 1$. Therefore we can apply Gauss's Theorem (in the third step below) to see that

$$\begin{aligned}
\iint_S \vec{F}\cdot d\vec{S} &= \iint_S \vec{F}\cdot d\vec{S} + \iint_{S_0} \vec{F}\cdot d\vec{S} - \iint_{S_0} \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} - \iint_{S_1} \vec{F}\cdot d\vec{S} \\
&= -\iint_{(-S)\cup(-S_0)\cup(S_1)} \vec{F}\cdot d\vec{S} - \iint_{S_0} \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} \\
&= -\iiint_E \operatorname{div}\vec{F}\, dV - \iint_{S_0} \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} \\
&= -\iiint_E 3\, dV - \iint_{S_0} \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} \\
&= -3\mathrm{Vol}(E) - \iint_{S_0} \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} \\
&= -3\pi - \iint_{S_0} \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S}.
\end{aligned}$$

Parametrizing $\vec{S}_0$ with $\vec{X}_0 = (x, y, 0)$ for $x^2 + y^2 \le 1$ gives $N_{\vec{X}_0}(x, y) = \vec{k}$, and therefore we have

$$\iint_{S_0} \vec{F}\cdot d\vec{S} = \iint_{x^2+y^2\le 1} (0^2 + ye^x)\, dA(x, y) = \iint_{x^2+y^2\le 1} ye^x\, dA(x, y) = 0$$

because the integrand ye^x is odd in y and the disc $x^2 + y^2 \le 1$ is symmetric across the y-axis.

Parametrizing $\vec{S}_1$ with $\vec{X}_0 = (x, y, 1)$ for $x^2 + y^2 \le 1$ gives $N_{\vec{X}_1}(x, y) = \vec{k}$, and therefore we have

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{x^2+y^2\le 1} (1 + ye^x)\, dA(x, y) = \iint_{x^2+y^2\le 1} (1 + ye^x)\, dA(x, y) = \iint_{x^2+y^2\le 1} 1\, dA(x, y) = \pi,$$

where we used our previous computation and the fact that the area of the unit disc is π.

Therefore we have $\iint_S \vec{F}\cdot d\vec{S} = -3\pi - 0 + \pi = -2\pi$.