Northwestern University

MATH 291-3 Second Midterm Examination - Practice A
Spring Quarter 2022
May 12, 2022

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Instructions

- This examination consists of 5 questions.
- Read all problems carefully before answering.
- You have 50 minutes to complete this examination.
- Do not use books, notes, calculators, computers, tablets, phones, smart watches, or similar devices.
- Possession of a digital communication device during a bathroom break will be treated as a *prima facie* violation of Northwestern's academic integrity policy.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is true or false. If true, then explain why. If false, then give a counterexample.

   (a) If $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is C^1 and satisfies $\text{div}\vec{F} = x$, then there does not exist a C^2 vector field $\vec{G}$ with $\text{curl}\vec{G} = \vec{F}$.

   (b) If $f : \mathbb{R}^3 \to \mathbb{R}$ is C^2, then the 2-form

$$\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial x}\right) dz \wedge dx + \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

   is closed on $\mathbb{R}^3$.

Solution: (a) is true. If such a $\vec{G}$ existed, then $0 = \text{div}(\text{curl}\vec{G}) = \text{div}\vec{F}$. Since $\text{div}\vec{F} = x$ is not the zero function, we conclude that no such $\vec{G}$ exists.

(b) is true. To see why, note that if ω is the given differential form, then (throwing out the terms that are 0 by antisymmetry)

$$\begin{aligned} d(f\,dx + f\,dy + f\,dz) &= \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial f}{\partial z} dz \wedge dx + \frac{\partial f}{\partial x} dx \wedge dy + \frac{\partial f}{\partial z} dz \wedge dy + \frac{\partial f}{\partial x} dx \wedge dz + \frac{\partial f}{\partial y} dy \wedge dz \\ &= \left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial x}\right) dz \wedge dx + \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy \\ &= \omega. \end{aligned}$$

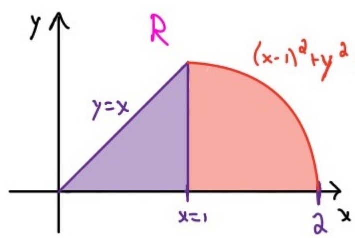
Therefore $d\omega = d^2(f\,dx + f\,dy + f\,dz) = 0$, so ω is closed on $\mathbb{R}^3$.

2. Write the following as a single iterated integral in polar coordinates.

$$\int_0^1 \int_y^1 (x^2 + y^2)\, dxdy + \int_1^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2)\, dydx$$

Note that the order of integration in the first expression is $dxdx$ while in the second it is $dydx$.

Solution: The first iterated integral represents the double integral of x^2+y^2 over the region in $\mathbb{R}^2$ enclosed by the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$. The second iterated integral represents the double integral of x^2+y^2 over the region in $\mathbb{R}^2$ that lies to the right of the line $x = 1$, and between $y = 0$ (the x-axis) and $y = \sqrt{2x - x^2}$ (the upper-half of the circle $y^2 + (x-1)^2 = 1$ centered at $(1,0)$ with radius 1). Therefore the sum of iterated integrals can be written as $\iint_D (x^2 + y^2)\, dA(x,y)$, where D is the region in the first quadrant enclosed by the line $x = y$, the x-axis, and the circle $(x-1)^2 + y^2 = 1$.



The region can be described in polar coordinates by taking $0 \le \theta \le \frac{\pi}{4}$, and then for each θ allowing r to run from 0 (the origin) to the circle $(x-1)^2 + y^2 = 1$. In terms of r and θ, this last equation is $r^2 - 2r\cos(\theta) = 0$, so that $r(r - 2\cos(\theta)) = 0$. Since $r > 0$ on the circle, we must have $r = 2\cos(\theta)$. Because $x^2 + y^2 = r^2$, we can express the original sum of integrals as an iterated integral in polar coordinates as

$$\int_0^{\pi/4} \int_0^{2\cos(\theta)} r^2 \cdot r\, drd\theta.$$

3. Suppose $S \subset \mathbb{R}^3$ is a smooth C^1 surface with parametrization

$$\vec{X}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in E$$

where $E \subseteq \mathbb{R}^2$, and let $\vec{c}(t) = (u(t), v(t))$, $a \leq t \leq b$ be a parametrization of a smooth C^1 curve in E. The composition $\vec{X} \circ \vec{c} \colon [a,b] \to \mathbb{R}^3$ then describes a smooth C^1 curve on S. Show that for every $t \in [a,b]$,

$$(\vec{X} \circ \vec{c})'(t) \cdot N_{\vec{X}}(u(t), v(t)) = 0.$$

Hint: Show that $(\vec{X} \circ \vec{c})'(t)$ is a linear combination of $\vec{X}_u(u(t), v(t))$ and $\vec{X}_v(u(t), v(t))$. (The point is that $(\vec{X} \circ \vec{c})'(t)$ gives a vector tangent to S at the point $\vec{X}(u(t), v(t))$, so this verifies that $N_{\vec{x}}(u(t), v(t)) = (\vec{X}_u \times \vec{X}_v)(u(t), v(t))$ is orthogonal to every vector that is tangent to S at $\vec{X}(u(t), v(t))$, which is why $N_{\vec{X}}$ is indeed normal to S.)

Solution: By the Chain Rule, $\vec{X} \circ \vec{c}$ is differentiable on $[a,b]$ and

$$\begin{aligned}(\vec{X} \circ \vec{c})'(t) &= D\vec{X}(\vec{c}(t))\vec{c}\,'(t) \\ &= \begin{bmatrix} \vec{X}_u(u(t), v(t)) & \vec{X}_v(u(t), v(t)) \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} \\ &= u'(t)\vec{X}_u(u(t), v(t)) + v'(t)\vec{X}_v(u(t), v(t)).\end{aligned}$$

Therefore, for each $t \in [a,b]$,

$$(\vec{X} \circ \vec{c})'(t) \cdot (\vec{X}_u \times \vec{X}_v)(u(t), v(t)) = \det(u'\vec{X}_u + v'\vec{X}_v, \vec{X}_u, \vec{X}_v) = 0$$

because the vectors $u'(t)\vec{X}_u(u(t), v(t)) + v'(t)\vec{X}_v(u(t), v(t)), \vec{X}_u(u(t), v(t)), \vec{X}_v(u(t), v(t))$ form a linearly dependent set.

4. Suppose $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a C^2 vector field on $\mathbb{R}^3$. Show that

$$\text{curl}(\text{curl}\vec{F}) = \nabla(\text{div}\vec{F}) - \Big(\text{div}(\nabla P)\vec{i} + \text{div}(\nabla Q)\vec{j} + \text{div}(\nabla R)\vec{k}\Big).$$

Start by computing the left-hand side.

Solution: Note first that for each C^2 function $f : \mathbb{R}^3 \to \mathbb{R}$, $\text{div}(\nabla f) = \text{div}(f_x\vec{i}+f_y\vec{j}+f_z\vec{k}) = f_{xx}+f_{yy}+f_{zz}$. Using this observation, we compute that

$$\begin{aligned}
\text{curl}(\text{curl}\vec{F}) &= \text{curl}((R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}) \\
&= (Q_{xy} - P_{yy} - P_{zz} + R_{xz})\vec{i} + (R_{yz} - Q_{zz} - Q_{xx} + P_{yx})\vec{j} + (P_{zx} - R_{xx} - R_{yy} + Q_{zy})\vec{k} \\
&= (Q_{yx} + R_{zx})\vec{i} + (R_{zy} + P_{xy})\vec{j} + (P_{xz} + Q_{yz})\vec{k} - (P_{yy} + P_{zz})\vec{i} - (Q_{xx} + Q_{zz})\vec{j} - (R_{xx} + R_{yy})\vec{k} \\
&= (P_{xx} + Q_{yx} + R_{zx})\vec{i} + (P_{xy} + Q_{yy} + R_{zy})\vec{j} + (P_{xz} + Q_{yz} + R_{zz})\vec{k} \\
&\qquad - (P_{xx} + P_{yy} + P_{zz})\vec{i} - (Q_{xx} + Q_{yy} + Q_{zz})\vec{j} - (R_{xx} + R_{yy} + R_{zz})\vec{k} \\
&= \nabla(P_x + Q_y + R_z) - \text{div}(\nabla P)\vec{i} - \text{div}(\nabla Q)\vec{j} - \text{div}(\nabla R)\vec{k} \\
&= \nabla(\text{div}\vec{F}) - -\text{div}(\nabla P)\vec{i} - \text{div}(\nabla Q)\vec{j} - \text{div}(\nabla R)\vec{k}.
\end{aligned}$$

5. Let $g : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be C^1. Prove that $d(f \circ g) = (f' \circ g)dg$.

Solution: Note that the chain rule implies that, at each $\vec{x} \in \mathbb{R}^n$

$$\begin{aligned} d(f \circ g) &= (f \circ g)_{x_1}(\vec{x})dx_1 + \cdots + (f \circ g)_{x_n}(\vec{x})dx_n \\ &= f'(g(\vec{x}))g_{x_1}(\vec{x})dx_1 + \cdots + f'(g(\vec{x}))g_{x_n}(\vec{x})dx_n \\ &= f'(g(\vec{x}))(g_{x_1}(\vec{x})dx_1 + \cdots + g_{x_n}(\vec{x})dx_n), \end{aligned}$$

so that $d(f \circ g) = (f' \circ g)\, dg$.