Math 291-3: Discussion #1 Problems (Solutions)
Northwestern University, Spring 2022

1. Find the global extreme values of $f(x, y, z) = e^{1-x^2-y^2+2y-z^2-4z}$ over the region in $\mathbb{R}^3$ described by the inequality $x^2 + y^2 - 2y + z^2 + 4z \leq 0$.

   Solution. By manipulating the inequality describing the region, we see that the region is described by $x^2 + (y-1)^2 + (z+2)^2 \leq 5$, or rather $\|(x, y, z) - (0, 1, -2)\|^2 \leq 5$. Therefore we are finding the extreme values of f on the closed ball of radius $\sqrt{5}$ centered at $(0, 1, -2)$. Because this ball is compact and f is continuous, f does indeed have global extreme values on this ball. Note that f is C^1 (and therefore differentiable) on $\mathbb{R}^3$. Note that $f_x(x, y, z) = (-2x)f(x, y, z)$ and $f_y(x, y, z) = (-2y + 2)f(x, y, z)$ and $f_z(x, y, z) = (-2z - 4)f(x, y, z)$. Because $f(x, y, z) \neq 0$ for every $(x, y, z) \in \mathbb{R}^3$, the only critical point of f is $(0, 1, -2)$ (which does indeed lie in the ball). We also note that $f(0, 1, -2) = e^6$. It remains to check the boundary of the ball for extreme values of f, but since the boundary consists of points satisfying $\|(x, y, z) - (0, 1, -2)\| = \sqrt{5}$ and

   $$f(x,y,z) = \exp(1-(x^2+y^2-2y+z^2+4z)) = \exp(6-(x^2+(y-1)^2+(z+2)^2)) = \exp(6-5) = e^1 = e$$

   for all such points. Therefore the global maximum value of f is e^6 (which occurs at $(0, 1, -2)$), and the global minimum value of f is e (which occurs at every point (x, y, z) with $\|(x, y, z) - (0, 1, -2)\| = \sqrt{5}$).

2. For $r > 0$, find the maximum value of $f(x, y, z) = xyz$ on the portion S of the sphere $x^2 + y^2 + z^2 = r^2$ where $x \geq 0$ and $y \geq 0$ and $z \geq 0$. Use the result to prove that for $a > 0$ and $b > 0$ and $c > 0$,

   $$\sqrt[3]{abc} \leq \frac{a+b+c}{3}.$$

   In particular this shows that the **geometric mean** of a, b, c (the left-hand-side) is no greater than the **arithmetic mean** of a, b, c (the right-hand side).

   Solution. Note that f is continuous on $\mathbb{R}^3$ and S is closed and bounded, and therefore the Extreme Value Theorem implies that f attains a global maximum value on S. Because $f(x, y, z) > 0$ if $x, y, z \neq 0$ and $f(x, y, z) = 0$ if $x = 0$ or $y = 0$ or $z = 0$, the global maximum must occur at some points where $x, y, z > 0$, and will therefore be a constrained local maximum of f on $g(x, y, z) = x^2 + y^2 + z^2 = r^2$. We will detect this constrained local maximum using the method of Lagrange multipliers. The point (x, y, z) where this occurs will satisfy, for some $\lambda > 0$,

   $$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = r^2 \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} yz = \lambda 2x \\ xz = \lambda 2y \\ xy = \lambda 2z \\ x^2 + y^2 + z^2 = r^2 \end{cases}$$

Multiplying the first three equations (respectively) by x and y and z and then adding yields $3xyz = 2\lambda(x^2 + y^2 + z^2)$. Because $x^2 + y^2 + z^2 = r^2 > 0$, this implies that $2\lambda = \frac{3xyz}{r^2}$, so that

$$yz = \frac{3xyz}{r^2}x \quad \text{and} \quad xz = \frac{3xyz}{r^2}y \quad \text{and} \quad xy = \frac{3xyz}{r^2}z.$$

Because $x, y, z > 0$ at the point(s) we are seeking, these equations simplify to

$$x^2 = y^2 = z^2 = \frac{r^2}{3},$$

so that $x = y = z = \sqrt{\frac{r^2}{3}} = \frac{r}{\sqrt{3}}$. Therefore the unique point on S at which f has a global maximum value is $(x, y, z) = (\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}})$, and the global maximum value of f at this point is $\left(\frac{r}{\sqrt{3}}\right)^3$.

Therefore, for every $x, y, z > 0$, if we set $r > 0$ such that $x^2 + y^2 + z^2 = r^2$, then we have

$$xyz \leq \left(\frac{r}{\sqrt{3}}\right)^3 = \left(\frac{\sqrt{x^2 + y^2 + z^2}}{\sqrt{3}}\right)^3 = \left(\frac{x^2 + y^2 + z^2}{3}\right)^{3/2}.$$

Raising each side of this equation to the 2/3 power yields

$$\sqrt[3]{x^2y^2z^2} = \frac{x^2 + y^2 + z^2}{3}.$$

Let $a, b, c > 0$. By applying the above identity to $x = \sqrt{a}$, $y = \sqrt{b}$, and $z = \sqrt{c}$, we obtain

$$\sqrt[3]{abc} = \frac{a + b + c}{3},$$

as desired.

3. Suppose that $a, b, c > 0$ and that $B \stackrel{def}{=} [-a, a] \times [-b, b] \times [-c, c]$ is a rectangular box in $\mathbb{R}^3$ centered at the origin. Suppose $f : B \to \mathbb{R}$ is an integrable function such that

$$f(-x, y, z) = -f(x, y, z) \quad \text{for all } (x, y, z) \in B.$$

Show that $\int_B f(\vec{x})\, d\vec{x} = 0$. The point is to give a justification of this fact using the "limit of Riemann sums" definition of $\int_B f(\vec{x})\, d\vec{x}$.

(Note: the fact that we are assuming that this integral exists is important.)

Solution. For $n > 0$ even, partition B into n^3 equally-sized smaller boxes of the form

$$R_{ijk} = \left[-a+(i-1)\frac{2a}{n}, -a+i\frac{2a}{n}\right] \times \left[-b+(j-1)\frac{2b}{n}, -b+j\frac{2b}{n}\right] \times \left[-c+(k-1)\frac{2c}{n}, -c+k\frac{2c}{n}\right], \quad i, j, k = 1, \ldots, n.$$

For each $j, k = 1, \ldots, n$ and $i = 1, \ldots, \frac{n}{2}$, note that the smaller boxes R_{ijk} and $R_{(n+1-i)jk}$ are reflections of each other across the yz-plane. Choose the sample points $\vec{c}_{ijk}$ and $\vec{c}_{(n+1-i)jk}$

in these boxes with the same y- and z-coordinates, but where the x-coordinate of $\vec{c}_{ijk}$ is the negative of the x-coordinate of $\vec{c}_{(n-i)jk}$. Then $f(\vec{c}_{ijk}) = -f(\vec{c}_{(n+1-i)jk})$, so that for each fixed j, k we have

$$\sum_{i=1}^{n} f(\vec{c}_{ijk})\text{Vol}(R_{ijk}) = \sum_{i=1}^{n/2}(f(\vec{c}_{ijk}) + f(\vec{c}_{(n+1-i)jk})\frac{8abc}{n^3} = \sum_{i=1}^{n/2} 0 \cdot \frac{8abc}{n^3} = 0,$$

and therefore the corresponding Riemann sum for f satisfies

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(\vec{c}_{ijk})\text{Vol}(R_{ijk}) = \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=1}^{n} f(\vec{c}_{ijk})\text{Vol}(R_{ijk}) = \sum_{j=1}^{n}\sum_{k=1}^{n} 0 = 0.$$

Because $\displaystyle\int_B f(\vec{x})\, d\vec{x}$ is the limit of these Riemann sums as $n \to \infty$, and because each of these Riemann sums is 0, we conclude that $\displaystyle\int_B f(\vec{x})\, d\vec{x} = 0$.