Solutions
Math 291-3
Homework 2
April 15, 2022

Exercise 1 Let $A \subseteq B \subseteq \mathbb{R}^n$, and assume that B has measure zero. Prove that A has measure zero.

Let $\epsilon > 0$. Choose boxes $B_1, B_2, B_3, \ldots$ such that $B \subset \bigcup_i B_i$ and $\sum_i \mathrm{Vol}_n(B_i) < \epsilon$. Because $A \subset B$, $A \subset \bigcup_i B_i$ and $\sum_i \mathrm{Vol}_n(B_i) < \epsilon$. Therefore A has measure zero. ∎

Exercise 2 Suppose that B is a box in $\mathbb{R}^3$ and let B_1, B_2 be the boxes obtained by cutting B with a plane parallel to the xy-plane. If $f : B \to \mathbb{R}$ is integrable, show that $f : B_1 \to \mathbb{R}$ and $f : B_2 \to \mathbb{R}$ are each integrable and that

$$\iiint\limits_B f\,dV = \iiint\limits_{B_1} f\,dV + \iiint\limits_{B_2} f\,dV.$$

You may use the result of Exercise 1 without proof.

(Note: It is not necessary for the plane that cuts B to be horizontal; any plane will do. This is the analog of the fact in single-variable calculus that $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any c between a and b, provided that f is integrable on $[a, b]$. This exercise generalizes to $\mathbb{R}^n$ and for more general regions of integration.)

Let S denote the set of points in B at which f is discontinuous, which (by Lebesgue's Criterion) has measure zero since f is integrable over B. Then the set of points A_1 in B_1 at which f is discontinuous is a subset of S, so A_1 has measure zero by the first problem. Similarly, the set of points in B_2 at which f is discontinuous has measure zero since it is a subset of S as well. Since f is integrable over B, it is bounded over B and hence over B_1 and B_2 as well. Thus f is integrable over both B_1 and B_2.

Let $\mathcal{P}_1$ be a partition of B_1, and let $\mathcal{P}_2$ be a partition of B_2. We can combine $\mathcal{P}_1$ and $\mathcal{P}_2$ to form a partition $\mathcal{P}$ of B. Let $\mathcal{C}_1$ be a choice a sample points for $\mathcal{P}_1$, and let $\mathcal{C}_2$ be a choice of sample points for $\mathcal{P}_2$. Then we can combine $\mathcal{C}_1$ and $\mathcal{C}_2$ to get a choice of sample points $\mathcal{C}$ for $\mathcal{P}$. We then have

$$R(f, \mathcal{P}, \mathcal{C}) = \sum_{B_i \in \mathcal{P}_1} f(\vec{c}_i)\mathrm{Vol}_3(B_i) + \sum_{B_i \in \mathcal{P}_2} f(\vec{c}_i)\mathrm{Vol}_3(B_i) = R(f, \mathcal{P}_1, \mathcal{C}_1) + R(f, \mathcal{P}_2, \mathcal{C}_2).$$

Taking $\|\mathcal{P}_1\| \to 0$ and $\|\mathcal{P}_2\| \to 0$ (so that $\|\mathcal{P}\| = \max(\|\mathcal{P}_1\|, \|\mathcal{P}_2\|) \to 0$ as well) then gives the desired equation. ∎

Exercise 3 Let $B = [1, 3] \times [2, 5] \times [5, 8]$, let $g : B \to \mathbb{R}$ be any continuous function, and define $f : B \to \mathbb{R}$ by

$$f(x, y, z) \overset{def}{=} \begin{cases} g(x, y, z) & \text{if } x^2 + y^2 - z^2 \neq 1, \\ -10 & \text{if } x^2 + y^2 - z^2 = 1. \end{cases}$$

Show that f is integrable over B.

Because $g(x,y,z)$ and the constant function $(x,y,z) \mapsto -10$ are continuous on B, the set of discontinuities of f are a subset of the one-sheeted hyperboloid described by $x^2+y^2-z^2=1$. Because this one-sheeted hyperboloid is a level-set of a C^1 function with gradient $\begin{bmatrix} 2x \\ 2y \\ -2z \end{bmatrix} \neq \vec{0}$ everywhere (except for at $(0,0,0)$, which does not lie on the hyperboloid), the Measure Zero Theorem implies that this one-sheeted hyperboloid has measure zero, and therefore the set of discontinuities of f (which is a subset of this hyperboloid) has measure zero as well. Therefore f is integrable over B.

■

Exercise 4 Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) \stackrel{def}{=} \begin{cases} e^{x^2+y^4} & \text{if } x^2+4y^2 \neq 4, \\ x+y & \text{if } x^2+4y^2 = 4, \end{cases}$$

which is integrable over $[1,2]\times[0,1]$. Show that f satisfies the hypotheses of Fubini's Theorem on $[1,2]\times[0,1]$. What does the conclusion of Fubini's Theorem tell us in this case?

First, f is bounded over $[1,2]\times[0,1]$ because $e^{x^2+y^4}$ and $x+y$ are both continuous over the compact set $[1,2]\times[0,1]$, and are therefore bounded by the Extreme Value Theorem. Second, f is discontinuous only at points on the ellipse

$$x^2+4y^2=4.$$

Because the ellipse is the level set of $g(x,y)=x^2+4y^2$ and $\nabla g(x,y) \neq \vec{0}$ at each point on the ellipse, the Measure Zero Theorem implies that the ellipse has measure zero, and therefore (again by the Measure Zero Theorem) the set of discontinuities of f (which is a subset of the ellipse) has measure zero. Lebesgue's Criterion for Integrability implies that f is integrable on $[1,2]\times[0,1]$.

Note that each vertical or horizontal line in $[1,2]\times[0,1]$ intersects the ellipse (and therefore the set of discontinuities of f) in either zero, one, or two points, and therefore the hypotheses of Fubini's Theorem Apply. In this case, the conclusion of Fubini's Theorem would tell us that the double integral of f over $[1,2]\times[0,1]$ is equal to either iterated integral of f over the same box. ■

Exercise 5 (Colley 5.2.41) The point of this problem is to show that the notion of a multiple integral (i.e. the integral of a function over a box in $\mathbb{R}^n$, defined as a limit of Riemann sums) is different than the notion of an iterated integral (i.e. repeatedly computing single-variable integrals of a function, one variable at a time). In particular, this problem shows that the conclusion of Fubini's Theorem—that a multiple integral can be computed as an iterated integral under some circumstances—is nontrivial. To do this, we will investigate a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that f is not integrable over a box $B = [0,1] \times [0,2]$ (in the sense that the double integral $\iint_B f\, dA$, which is defined as a limit of Riemann sums, fails to exist), but that nevertheless the iterated integral $\int_0^1 \int_0^2 f(x,y)\, dy\, dx$ can indeed be computed.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) \stackrel{def}{=} \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational and } y \le 1, \\ 2 & \text{if } x \text{ is irrational and } y > 1. \end{cases}$$

(a) Show that $\displaystyle\int_0^2 f(x,y)\, dy$ does not depend on whether x is rational or irrational.

(b) Using (a), show that the iterated integral $\displaystyle\int_0^1 \int_0^2 f(x,y)\, dy\, dx$ exists and find its value.

(c) Let $B = [0,1] \times [0,2]$. For a partition $\mathcal{P}$ of B, make the choice of sample points $\mathcal{C}$ such that each sample point $\vec{c}_i$ has a rational x-coordinate. If we always choose the sample points in this way, what should be $\displaystyle\lim_{\|\mathcal{P}\| \to 0} R(f, \mathcal{P}, \mathcal{C})$?

(d) For a partition $\mathcal{P}$ of B, make the choice of sample points $\mathcal{C}$ such that each sample point $\vec{c}_i = (x_i^*, y_i^*)$ satisfies that x_i^* is rational if $y_i^* \le 1$ and x_i^* is irrational if $y_i^* > 1$. If we always choose the sample points in this way, what should be $\displaystyle\lim_{\|\mathcal{P}\| \to 0} R(f, \mathcal{P}, \mathcal{C})$?

(Suggestion: To make the reasoning easier, you can assume that each small box B_i in the partition $\mathcal{P}$ of B always satisfies $B_i \subseteq [0,1] \times [0,1]$ or $B_i \subseteq [0,1] \times [1,2]$.)

(e) Using parts (c) and (d), conclude that $\displaystyle\lim_{\|\mathcal{P}\| \to 0} R(f, \mathcal{P}, \mathcal{C})$ does not exist, and therefore that the double integral $\iint_B f(x,y)\, dA(x,y)$ does not exist. Thus, we see that multiple integrals and iterated integrals are actually different notions.

(a) If x is rational then

$$\int_0^2 f(x,y)\, dy = \int_0^2 1\, dy = 2.$$

If x is irrational then

$$\int_0^2 f(x,y)\, dy = \int_0^1 0\, dy + \int_1^2 2\, dy = 0 + 2 = 2.$$

Therefore $\displaystyle\int_0^2 f(x,y)\, dy = 2$ regardless of whether x is rational or irrational.

(b) Because $x \mapsto \displaystyle\int_0^2 f(x,y)\,dy = 2$ is constant, it is integrable on $[0,1]$. Moreover,

$$\int_0^1 \int_0^2 f(x,y)\,dydx = \int_0^1 2\,dx = 2.$$

(c) For such a partition $\mathcal{P}$, we have

$$R(f,\mathcal{P},\mathcal{C}) = \sum_i f(\vec{c}_i)\mathrm{Vol}_2(B_i) = \sum_i \mathrm{Vol}_2(B_i) = \mathrm{Vol}_2(B) = 2,$$

so that the only possible value for $\lim\limits_{\|\mathcal{P}\|\to 0} R(f,\mathcal{P},\mathcal{C})$ is 2.

(d) Make the assumption in the suggestion and choose $y_i^* > 1$ if $B_i \subset [0,1]\times[1,2]$. For such a partition $\mathcal{P}$, if

$$f(\vec{c}_i)\mathrm{Vol}_2(B_i) = \mathrm{Vol}_2(B_i) \quad \text{if } B_i \subset [0,1]\times[0,1]$$

and

$$f(\vec{c}_i)\mathrm{Vol}_2(B_i) = 2\mathrm{Vol}_2(B_i) \quad \text{if } B_i \subset [0,1]\times[1,2].$$

Therefore we have

$$\begin{aligned} R(f,\mathcal{P},\mathcal{C}) &= \sum_{i:\ B_i\subset[0,1]\times[0,1]} \mathrm{Vol}_2(B_i) + \sum_{i:\ B_i\subset[0,1]\times[1,2]} 2\mathrm{Vol}_2(B_i) \\ &= \mathrm{Vol}_2([0,1]\times[0,1]) + 2\mathrm{Vol}_2([0,1]\times[1,2]) \\ &= 3, \end{aligned}$$

so that the only possible value for $\lim\limits_{\|\mathcal{P}\|\to 0} R(f,\mathcal{P},\mathcal{C})$ is 3.

(e) Because we can construct partitions $\mathcal{P}$ of the type in (c) and (d) for $\|\mathcal{P}\|$ as close to 0 as we'd like, we conclude that $\lim\limits_{\|\mathcal{P}\|\to 0} R(f,\mathcal{P},\mathcal{C})$ does not exist, and therefore f is not integrable on B.

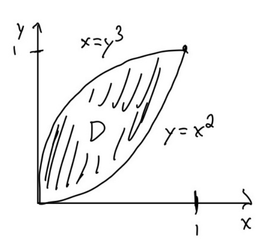
■

Exercise 6 (Colley 5.2.18 and 5.2.22) This problem has two parts.

(a) Evaluate $\displaystyle\iint_D xy\,dA(x,y)$, where D is the region bounded by $x = y^3$ and $y = x^2$.

(b) Evaluate $\displaystyle\iint_D (x^2+y^2)\,dA(x,y)$, where D is the region in the first quadrant bounded by $y = x$, $y = 3x$ and $xy = 3$.

(a) First note that the curves $x = y^3$ and $y = x^2$ intersect at $(0,0)$ and $(1,1)$, and since $\sqrt[3]{x} \geq x^2$ for $x \in [0,1]$, D is the region consisting of points (x,y) where $x \in [0,1]$ and $x^2 \leq y \leq \sqrt[3]{x}$.

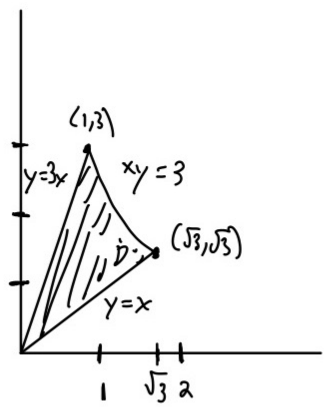


Because $(x, y) \mapsto xy$ is continuous throughout $\mathbb{R}^2$ (and therefore bounded and integrable on D), we can apply Fubini's Theorem to see that

$$\iint_D xy\, dA(x, y) = \int_0^1 \int_{x^2}^{\sqrt[3]{x}} xy\, dydx = \int_0^1 (\frac{1}{2}x^{5/3} - \frac{1}{2}x^5)\, dx = \frac{3}{16} - \frac{1}{12} = \frac{5}{48}.$$

(b)

We first sketch the region D:



Note that $(x, y) \mapsto x^2 + y^2$ is continuous throughout $\mathbb{R}^2$ (and therefore bounded and integrable on D), so we can apply Fubini's Theorem. To do this, we split this region in two along the line $x = 1$, and then set up iterated integrals to compute each piece. We get

$$\begin{aligned}\iint_D (x^2 + y^2)\, dA(x, y) &= \int_0^1 \int_x^{3x} (x^2 + y^2)\, dydx + \int_1^{\sqrt{3}} \int_x^{3/x} (x^2 + y^2)\, dydx \\ &= \int_0^1 \frac{32}{3}x^3\, dx + \int_1^{\sqrt{3}} 3x + 9x^{-3} - \frac{4}{3}x^3\, dx \\ &= 6.\end{aligned}$$

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Exercise 7 (Colley 5.2.34) Let D be the region in $\mathbb{R}^2$ with $y \geq 0$ that is bounded by $x^2 + y^2 = 9$ and the line $y = 0$. Without resorting to any explicit calculation of an iterated integral, determine, with explanation, the value of $\iint_D (2x^3 - y^4 \sin(x) - 2)\, dA(x, y)$.

Note that because D is a bounded set, ∂D has measure zero, and $f(x, y) = 2x^3 - y^3 \sin(x) - 2$ is continuous on $\mathbb{R}^2$, we can write

$$\iint_D (2x^3 - y^3 \sin(x) - 2)\, dA = \int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (2x^3 - y^3 \sin(x) - 2)\, dxdy.$$

For each fixed y, the function $x \mapsto 2x^3 - y^3 \sin(x)$ is odd, and therefore we have

$$\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (2x^3 - y^3 \sin(x))\, dx = 0.$$

Therefore we have

$$\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (2x^3 - y^3\sin(x) - 2)\,dx = \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} -2\,dx,$$

so that

$$\iint_D (2x^3 - y^3\sin(x) - 2)\,dA(x,y) = \int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} -2\,dxdy = \iint_D (-2)\,dA = (-2)\mathrm{Vol}_2(D) = -2\frac{9\pi}{2} = -9\pi.$$

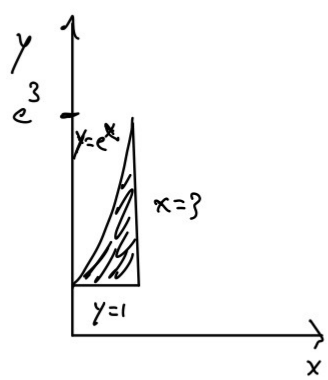
■

Exercise 8 (Colley 5.3.6 and 5.3.8) For each of the following integrals: sketch the region of integration, reverse the order or integration, and evaluate both iterated integrals.

(a) $\displaystyle\int_0^3 \int_1^{e^x} 2\,dy\,dx$

(b) $\displaystyle\int_0^{\pi/2} \int_0^{\cos(x)} \sin(x)\,dy\,dx.$

(a) The region of integration is as pictured:



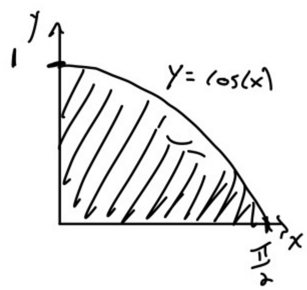
As written, the value of the iterated integral is

$$\int_0^3 \int_1^{e^x} 2\,dydx = \int_0^3 (2e^x - 2)\,dx = 2e^3 - 8.$$

If we change the order or integration, we get

$$\int_1^{e^3} \int_{\ln(y)}^3 2\,dxdy = \int_1^{e^3} (6 - 2\ln(y))\,dy = 6e^3 - 6 - 2(3e^3 - e^3) + 2(0-1) = 2e^3 - 8.$$

(b) The region of integration is a pictured:



As written, the value of the iterated integral is

$$\int_0^{\pi/2}\int_0^{\cos(x)} \sin(x)\,dydx = \int_0^{\pi/2} \sin(x)\cos(x)\,dx = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}.$$

If we change the order of integration, we get

$$\int_0^1\int_0^{\arccos(y)} \sin(x)\,dxdy = \int_0^1 (-y+1)\,dy = 1 - \frac{1}{2} = \frac{1}{2}.$$

■

Exercise 9 (Colley 5.3.10 and 5.3.11 and 5.3.12 and 5.3.13) When you reverse the order of integration in parts (a) and (b), you should obtains a sum of iterated integrals. Make the reversals and evaluate. In (c) and (d), rewrite the given sum of iterated integrals as a single iterated integral by reversing the order or integration, and then evaluate.

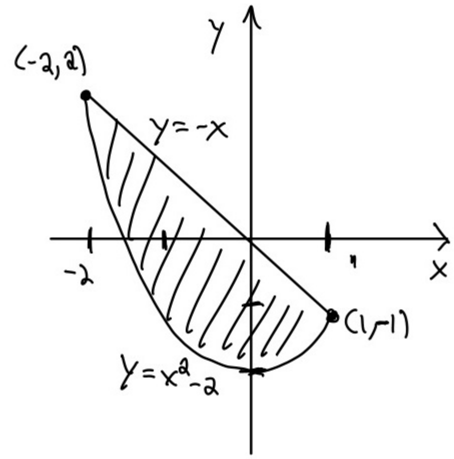
(a) $\displaystyle\int_{-2}^{1}\int_{x^2-2}^{-x} (x-y)\,dy\,dx$

(b) $\displaystyle\int_{-1}^{4}\int_{y-4}^{4y-y^2} (y+1)\,dx\,dy$

(c) $\displaystyle\int_0^1\int_0^x \sin(x)\,dy\,dx + \int_1^2\int_0^{2-x} \sin(x)\,dy\,dx$

(d) $\displaystyle\int_0^8\int_0^{\sqrt{y/3}} y\,dx\,dy + \int_8^{12}\int_{\sqrt{y-8}}^{\sqrt{y/3}} y\,dx\,dy$

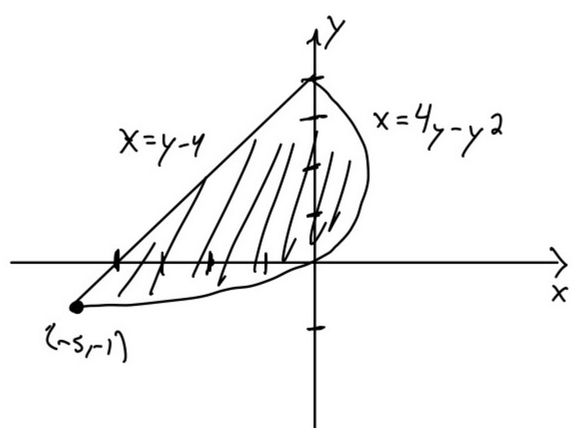
(a) The region of integration is as pictured:



We split the region into the portion above $y = -1$ and the portion below $y = -1$. After reversing the order we get

$$\int_{-1}^{2}\int_{-\sqrt{y+2}}^{-y}(x-y)\,dx\,dy + \int_{-2}^{-1}\int_{-\sqrt{y+2}}^{\sqrt{y+2}}(x-y)\,dx\,dy$$
$$= \int_{-1}^{2}\left(\frac{3}{2}y^2 - \frac{1}{2}y - 1 - y\sqrt{y+2}\right)dy + \int_{-2}^{-1} -2y\sqrt{y+2}\,dy$$
$$= -\frac{139}{60} + \frac{4}{3}.$$

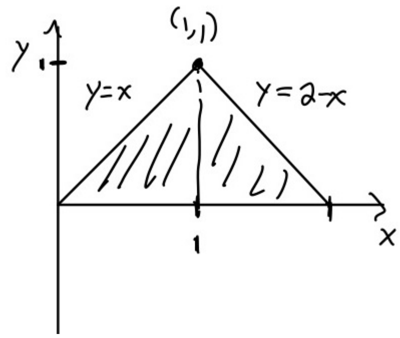
(b) The region of integration is as pictured:



We split the region into the portion to the left of the y-axis and the portion to the right of the y-axis. We get

$$\begin{aligned}
&\int_{-4}^{0}\int_{2-\sqrt{4-x}}^{x+4}(y+1)\,dy\,dx+\int_{0}^{4}\int_{2-\sqrt{4-x}}^{2+\sqrt{4-x}}(y+1)\,dy\,dx\\
&=\int_{-4}^{0}\left[\frac{1}{2}(x+5)^2+x+4-\frac{1}{2}(1-\sqrt{4-x})^2-2+\sqrt{4-x}\right]dx\\
&+\int_{0}^{4}\left[\frac{1}{2}(3+\sqrt{4-x})^2+2+\sqrt{4-x}-\frac{1}{2}(1-\sqrt{4-x})^2-2+\sqrt{4-x}\right]dx\\
&=\frac{625}{12}
\end{aligned}$$

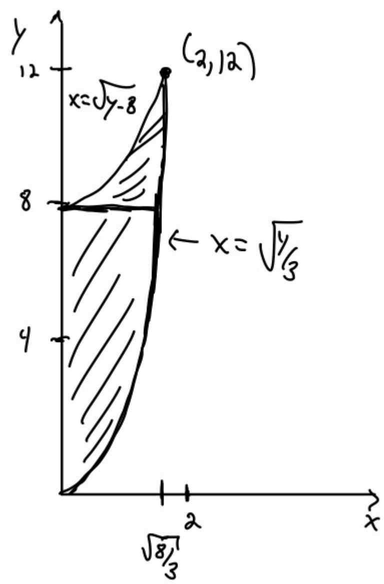
(c) The region of integration is as pictured:



To be clear, the bounds on the first iterated integral give the left part of this region and the bounds on the second give the right part. Thus, if we integrate with respect to $dx\,dy$ instead, the smallest and largest values of y in the region are 0 and 1 respectively and at a fixed y the values of x start at $x=y$ on the left boundary and move to $x=2-y$ on the right boundary. We get

$$\begin{aligned}
\int_{0}^{1}\int_{0}^{x}\sin(x)\,dy\,dx+\int_{1}^{2}\int_{0}^{2-x}\sin(x)\,dy\,dx &= \int_{0}^{1}\int_{y}^{2-y}\sin(x)\,dx\,dy\\
&=\int_{0}^{1}-\cos(x)\Big|_{y}^{2-y}\,dy\\
&=\int_{0}^{1}[\cos(y)-\cos(2-y)]\,dy\\
&=[\sin(y)+\sin(2-y)]\Big|_{0}^{1}\\
&=2\sin(1)-\sin(2).
\end{aligned}$$

(d) The region of integration is as pictured:



After reversing the order we get

$$\int_0^2 \int_{3x^2}^{x^2+8} y\, dy\, dx = \int_0^2 \left[\frac{1}{2}(x^2+8)^2 - \frac{1}{2}9x^4\right] dx = \frac{896}{15}.$$

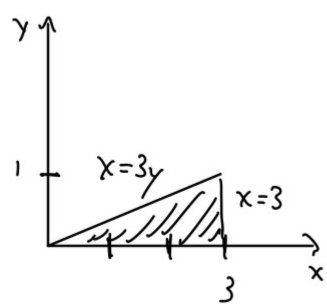
■

Exercise 10 (Colley 5.3.14 and 5.3.18) Evaluate each of the following iterated integrals.

(a) $\displaystyle\int_0^1 \int_{3y}^3 \cos(x^2)\, dx\, dy$

(b) $\displaystyle\int_0^2 \int_{y/2}^1 e^{-x^2}\, dx\, dy$

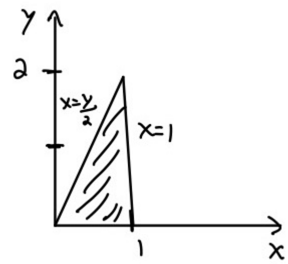
(a) The region of integration is as pictured:



After reversing the order or integration we get

$$\int_0^1 \int_{3y}^3 \cos(x^2)\, dxdy = \int_0^3 \int_0^{x/3} \cos(x^2)\, dydx = \int_0^3 \frac{x}{3}\cos(x^2)\, dx = \frac{1}{6}\sin(x^2)\Big]_0^3 = \frac{\sin(9)}{6}.$$

(b) The region of integration is as pictured:



After reversing the order or integration we get

$$\int_0^2 \int_{y/2}^1 e^{-x^2}\, dxdy = \int_0^1 \int_0^{2x} e^{-x^2}\, dydx = \int_0^1 2xe^{-x^2}\, dx = -e^{-x^2}\Big]_0^1 = 1 - e^{-1}.$$

■