Northwestern University

MATH 291-3 Second Midterm Examination - Practice B
Spring Quarter 2022
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Instructions

- This examination consists of 5 questions.
- Read all problems carefully before answering.
- You have 50 minutes to complete this examination.
- Do not use books, notes, calculators, computers, tablets, phones, smart watches, or similar devices.
- Possession of a digital communication device during a bathroom break will be treated as a *prima facie* violation of Northwestern's academic integrity policy.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is true or false. If true, then explain why. If false, then give a counterexample.

   (a) If $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 vector field with $\text{div}\vec{F} = 0$ on $\mathbb{R}^2$, then $\vec{F}$ is conservative on $\mathbb{R}^2$.

   (b) There does not exist a C^2 1-form ω on $\mathbb{R}^3$ such that $d\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$.

Solution: (a) is false. For a counterexample, consider $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$. Then $\text{div}\vec{F} = 0$ throughout $\mathbb{R}^2$, but if $\vec{F} = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}$, then $f_x(x, y) = -y$ and $f_y(x, y) = x$ throughout $\mathbb{R}^2$, so that $f_{xx}(x, y) = 0$, $f_{xy}(x, y) = -1$, $f_{yx}(x, y) = 1$, and $f_{yy}(x, y) = 0$. Therefore f is C^2, but $f_{xy}(x, y) \neq f_{yx}(x, y)$, contradicting Clairaut's Theorem. Therefore no such f exists.

(b) is true. If such a 1-form ω existed, then we would have

$$0 = d^2\omega = d(d\omega) = dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy = 3\,dx \wedge dy \wedge dz,$$

an absurdity.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying
$$\int_0^1 (1-x)f(x)\,dx = 5.$$
Find the value of the iterated integral
$$\int_0^1 \int_0^x f(x-y)\,dydx.$$
Hint: Let $u = x - y$ and use this as one of the new variables in a suitable change of variables application.

Solution: (The Intended Solution) Consider the change of variables described by $u = x-y$, $v = x$. Then we have $(x(u,v), y(u,v)) = T(u,v) = (v, v-u)$, so that the coordinate transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear with standard matrix $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. Therefore T is C^1 and, since the matrix of T is invertible, bijective. Moreover, we have
$$\det(DT(u,v)) = \det \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = 1.$$
Note that the region R of integration of the original iterated integral is described by $0 \le x \le 1$, $0 \le y \le x$. In terms of the new coordinates, this becomes $0 \le v \le 1$, $0 \le v-u \le v$, or rather $0 \le v \le 1$, $-v \le -u \le 0$, or $0 \le v \le 1$, $0 \le u \le v$. This region in the uv-plane is enclosed by the triangle with vertices at $(0,0)$, $(1,1)$, $(0,1)$, and can therefore also be expressed at $0 \le u \le 1$, $u \le v \le 1$. Therefore the Change of Variable Theorem gives that
$$\int_0^1 \int_0^x f(x-y)\,dydx = \int_0^1 \int_u^1 f(u) \cdot |1|\,dvdu = \int_0^1 (1-u)f(u)\,du = 5$$
by the assumption about f in the problem statement.

Solution: (The Easier, Unintended Solution) For fixed x, we make the (Calculus I) change of variable $y = x - u$ (so that $dy = (-1)du$) to see that
$$\int_0^x f(x-y)\,dy = \int_x^0 f(u)(-1)\,du = \int_0^x f(u)\,du,$$
so that
$$\int_0^1 \int_0^x f(x-y)\,dydx = \int_0^1 \int_0^x f(u)\,dudx.$$
This last iterated integral represents a double integral over the region enclosed by the triangle (in the xu-plane) with vertices $(0,0)$, $(1,1)$, $(1,0)$. Changing the order of integration then gives
$$\int_0^1 \int_0^x f(x-y)\,dydx = \int_0^1 \int_0^x f(u)\,dudx = \int_0^1 \int_u^1 f(u)\,dxdu = \int_0^1 (1-u)f(u)\,du = 5$$
by the assumption about f in the problem statement.

3. Recall that the surface area of a smooth C^1 surface with parametrization $\vec{X}(u,v)$ where $(u,v) \in D$ is given by
$$\iint_D \|N_{\vec{X}}(u,v)\| \, dA(u,v).$$
Compute the surface area of the paraboloid $z = x^2 + y^2$ lying below the plane $z = 4$.

Solution: We portion S of surface we are computing the area of is parametrized by
$$\vec{X}(x,y) = (x, y, x^2+y^2), \quad x^2+y^2 \le 4.$$
Then we have
$$\|N_{\vec{X}}(x,y)\| = \left\| \begin{bmatrix} 1 \\ 0 \\ 2x \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 2y \end{bmatrix} \right\| = \left\| \begin{bmatrix} -2x \\ -2y \\ 1 \end{bmatrix} \right\| = \sqrt{1+4x^2+4y^2}.$$
Let D be the disc of radius 2 centered at $(0,0)$. Then we have (using polar coordinates in the second step)
$$\begin{aligned} \text{Surface Area of } S &= \iint_D \sqrt{1+4x^2+4y^2} \, dA(x,y) \\ &= \int_0^{2\pi} \int_0^2 \sqrt{1+4r^2} \, r dr d\theta \\ &= \int_0^{2\pi} \left[\frac{1}{12}(1+4r^2)^{3/2} \right]_0^2 d\theta \\ &= \int_0^{2\pi} \frac{(17)^{3/2}-1}{12} \, d\theta \\ &= \frac{\pi((17)^{3/2}-1)}{6}. \end{aligned}$$

4. Suppose $\vec{F}, \vec{G}$ are C^1 vector fields on $\mathbb{R}^3$. Show that

$$\operatorname{div}(\vec{F} \times \vec{G}) = \vec{G} \cdot \operatorname{curl}(\vec{F}) - \vec{F} \cdot \operatorname{curl}(\vec{G})$$

where $\vec{F} \times \vec{G}$ denotes the vector field defined by $(\vec{F} \times \vec{G})(\vec{p}) = \vec{F}(\vec{p}) \times \vec{G}(\vec{p})$.

Solution: We compute that Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ and $\vec{G} = A\vec{i} + B\vec{j} + C\vec{k}$. Then

$$\begin{aligned}
\operatorname{div}(\vec{F} \times \vec{G}) &= \operatorname{div}\Big((QC - RB)\vec{i} + (RA - PC)\vec{j} + (PB - QA)\vec{k}\Big) \\
&= Q_x C + QC_x - R_x B - RB_x + R_y A + RA_y - P_y C - PC_y + P_z B + PB_z - Q_z A - QA_z \\
&= (A\vec{i} + B\vec{j} + C\vec{k}) \cdot \Big((R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}\Big) \\
&\qquad - (P\vec{i} + Q\vec{j} + R\vec{k}) \cdot \Big((C_y - B_z)\vec{i} + (A_z - C_x)\vec{j} + (B_x - A_y)\vec{k}\Big) \\
&= \vec{G} \cdot \operatorname{curl}(\vec{F}) - \vec{F} \cdot \operatorname{curl}(\vec{G}).
\end{aligned}$$

5. Prove that if α is a k-form on $\mathbb{R}^n$ and if β is an ℓ-form on $\mathbb{R}^n$, then

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha.$$

Note: First reduce the proof to the case $\alpha = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and $\beta = dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}$.

Solution: Note that if we knew the result for the basic k-forms and ℓ-forms mentioned in the note, then if

$$\alpha = \sum_{i_1 \ldots, i_k = 1}^{n} f_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \quad \text{and} \quad \beta = \sum_{j_1 \ldots, j_\ell = 1}^{n} g_{j_1,\ldots,j_\ell} dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \ ,$$

distributivity and homogeneity of the wedge product would give

$$\begin{aligned}
\alpha \wedge \beta &= \left(\sum_{i_1 \ldots, i_k = 1}^{n} f_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) \wedge \left(\sum_{j_1 \ldots, j_\ell = 1}^{n} g_{j_1,\ldots,j_\ell} dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \right) \\
&= \sum_{i_1 \ldots, i_k = 1}^{n} \sum_{j_1 \ldots, j_\ell = 1}^{n} f_{i_1,\ldots,i_k} g_{j_1,\ldots,j_\ell} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\
&= \sum_{i_1 \ldots, i_k = 1}^{n} \sum_{j_1 \ldots, j_\ell = 1}^{n} f_{i_1,\ldots,i_k} g_{j_1,\ldots,j_\ell} (-1)^{k\ell} dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= (-1)^{k\ell} \left(\sum_{j_1 \ldots, j_\ell = 1}^{n} g_{j_1,\ldots,j_\ell} dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \right) \wedge \left(\sum_{i_1 \ldots, i_k = 1}^{n} f_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) \\
&= (-1)^{k\ell} \beta \wedge \alpha.
\end{aligned}$$

Therefore let α and β be as mentioned in the note. Then by anticommutativity of the wedge product we can move each term in β in front of α by making k adjacent swaps, which gives

$$\begin{aligned}
\alpha \wedge \beta &= (dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \wedge (dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}) \\
&= (-1)^{k} dx_{j_1} \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \wedge (dx_{j_2} \wedge \cdots \wedge dx_{j_\ell}) \\
&= (-1)^{k2} dx_{j_1} \wedge dx_{j_1} \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \wedge (dx_{j_3} \wedge \cdots \wedge dx_{j_\ell}) \\
&= \cdots = (-1)^{k\ell} dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.
\end{aligned}$$