Math 291-3: Discussion #5 Problems (Solutions)

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1. Let $D \subseteq \mathbb{R}^3$, and suppose that

$$\vec{X} : D \to \mathbb{R}^3, \quad \vec{X}(s,t) = (x(s,t), y(s,t), z(s,t))$$

parametrizes a smooth surface $S \subset \mathbb{R}^3$, and consider a C^1 vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ defined on $\mathbb{R}^3$. Prove that

$$\vec{X}^*(P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy) = \left(\vec{F}(\vec{X}(s,t)) \cdot N_{\vec{X}}(s,t)\right) ds \wedge dt,$$

where $N_{\vec{X}}(s,t)$ is the normal vector to S arising from the parametrization $\vec{X}$ at the point $\vec{X}(s,t)$. (Notational tip: Perhaps suppress the (s,t) when writing to save time. That is, write x_t and $\vec{X}$ instead of $x_t(s,t)$ and $\vec{X}(s,t)$.)

Solution. Following the notational tip, we compute that

$$\begin{aligned}
&\vec{X}^*(P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy) \\
&\quad = P(\vec{X})d(y) \wedge d(z) + Q(\vec{X})d(z) \wedge d(x) \\
&\qquad\qquad + R(\vec{X})d(x) \wedge d(y) \\
&\quad = P(\vec{X})(y_s ds + y_t dt) \wedge (z_s ds + z_t dt) + Q(\vec{X})(z_s ds + z_t dt) \wedge (x_s ds + x_t dt) \\
&\qquad\qquad + R(\vec{X})(x_s ds + x_t dt) \wedge (y_s ds + y_t dt) \\
&\quad = P(\vec{X})(y_s z_t - y_t z_s) ds \wedge dt + Q(\vec{X})(z_s x_t - z_t x_s) ds \wedge dt + R(\vec{X})(x_s y_t - x_t y_s) ds \wedge dt \\
&\quad = \Big(P(\vec{X})(y_s z_t - y_t z_s) + Q(\vec{X})(z_s x_t - z_t x_s) + R(\vec{X})(x_s y_t - x_t y_s)\Big) ds \wedge dt \\
&\quad = \left(\vec{F}(\vec{X}) \cdot \begin{bmatrix} y_s z_t - y_t z_s \\ z_s x_t - z_t x_s \\ x_s y_t - x_t y_s \end{bmatrix}\right) ds \wedge dt \\
&\quad = \left(\vec{F}(\vec{X}(s,t)) \cdot N_{\vec{X}}(s,t)\right) ds \wedge dt.
\end{aligned}$$

2. Determine the value of

$$\int_C (2x^2 - 3y^2)\,dx + (2x + 3y^2)\,dy$$

where C is the piecewise-smooth oriented curve in $\mathbb{R}^2$ consisting of the line segment from $(-2,0)$ to $(2,0)$, followed by the line segment from $(2,0)$ to $(2,2)$, followed by the line segment from $(2,2)$ to $(-2,0)$. (So C is the outline of a triangle.)

Solution. Note that $C = \partial D$, where D is the region enclosed by the triangle with vertices $(-2,0)$, $(2,0)$, and $(2,2)$. Moreover, C is oriented to that D is "on the left" as we trace along C. Green's Theorem then applies to give

$$\begin{aligned}
\int_C (2x^2 - 3y^2)\,dx + (2x+3y^2)\,dy &= \iint\limits_D \Big((2x+3y^2)_x - (2x^2-3y^2)_y\Big)\,dA(x,y) \\
&= \iint\limits_D (2+6y)\,dA(x,y) \\
&= \int_0^2 \int_{2y-2}^2 (2+6y)\,dxdy \\
&= \int_0^2 (2+6y)(2-(2y-2))\,dy \\
&= \int_0^2 (2+6y)(4-2y)\,dy \\
&= 4\int_0^2 (-3y^2+5y+2)\,dy \\
&= 32.
\end{aligned}$$

3. A pair of C^1 functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ is said to satisfy the **Cauchy-Riemann Equations** if

$$u_x(x,y) = v_y(x,y) \quad \text{and} \quad u_y(x,y) = -v_x(x,y) \quad \text{for every } (x,y) \in \mathbb{R}^2.$$

These play a big role in complex analysis, in the sense that a C^1 function $f : \mathbb{C} \to \mathbb{C}$, $f(x+iy) = u(x,y) + iv(x,y)$ is "complex differentiable" if and only if $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy the Cauchy-Riemann Equations. (For more on this, take MATH 325!)

Let C be a simple, closed, piecewise-smooth curve that is the boundary of some set $D \subseteq \mathbb{R}^2$. Show that if u, v satisfy the Cauchy-Riemann equations, then

$$\int_C u dx - v dy = 0 \quad \text{and} \quad \int_C v\,dx + u\,dy = 0.$$

Solution. We apply Green's Theorem and the Cauchy-Riemann equations to see that

$$\int_C u\,dx - v\,dy = \iint\limits_D (-v_y - u_x)\,dA(x,y) = \iint\limits_D 0\,dA(x,y) = 0$$

and

$$\int_C v\,dx + u\,dy = \iint\limits_D (u_y - v_x)\,dA(x,y) = \iint\limits_D 0\,dA(x,y) = 0.$$

This problem allows one to prove that if $f(x + iy) = u(x, y) + iv(x, y)$ is C^1, then every "complex line integral" of f around C (treated as a curve in $\mathbb{C}$ is 0:

$$\int_C f(z)\, dz = 0.$$