Northwestern University

MATH 291-3 Second Midterm Examination
Spring Quarter 2022
May 12, 2022

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Instructions

- This examination consists of 5 questions for a total of 50 points.
- Read all problems carefully before answering.
- You have 50 minutes to complete this examination.
- Do not use books, notes, calculators, computers, tablets, phones, smart watches, or similar devices.
- Possession of a digital communication device during a bathroom break will be treated as a *prima facie* violation of Northwestern's academic integrity policy.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is true or false. If true, then explain why. If false, then give a counterexample.

   (a) (5 points) Suppose $T : [0,1] \times [0,1] \to \mathbb{R}^2$ is a C^1, injective function with $DT(x,y)$ invertible at each point in $[0,1] \times [0,1]$, and let $D = T([0,1] \times [0,1])$. Then

$$\mathrm{Vol}_2(D) = \int_0^1 \int_0^1 \det DT(x,y)\, dx dy.$$

Solution: This is false. To see why, let $T(x,y) = (-x, y)$. Then T is C^1, injective, and $DT(x,y) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible throughout $[0,1] \times [0,1]$. But since $\mathrm{Vol}_2(D) \geq 0$ and

$$\int_0^1 \int_0^1 \det DT(x,y)\, dx dy = \int_0^1 \int_0^1 -1 dx dy = -1 < 0,$$

this double integral cannot give the area of D.

   (b) (5 points) There is a C^2 vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ with $\mathrm{curl}\vec{F}(\vec{x}) = \vec{x}$ for every $\vec{x} \in \mathbb{R}^3$.

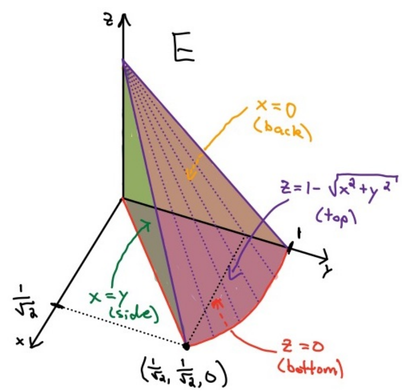
Solution: This is false. If such a vector field existed, then we would have $0 = \mathrm{div}(\mathrm{curl}\vec{F}(\vec{x})) = \mathrm{div}(\vec{x}) = 3$, an impossibility.

2. (10 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be continuous. Rewrite

$$\int_0^{1/\sqrt{2}} \int_0^{y} \int_0^{1-\sqrt{x^2+y^2}} f(x,y,z)\, dz dx dy + \int_{1/\sqrt{2}}^{1} \int_0^{\sqrt{1-y^2}} \int_0^{1-\sqrt{x^2+y^2}} f(x,y,z)\, dz dx dy$$

as a single iterated integral using cylindrical coordinates.

Solution: This sum of integrals represents the triple integral of $f(x,y,z)$ over a region E in $\mathbb{R}^3$. The bounds for z in each iterated integral imply that E is bounded below by the xy-plane $z = 0$ and above by the (downward-opening) cone $z = 1 - \sqrt{x^2+y^2}$ with vertex $(0,0,1)$. The bounds on x and y indicate that the shadow of E in the xy-plane consists of the portion of the unit disc $x^2 + y^2 \leq 1$ that lies to the right of the y-axis and above the line $x = y$. We sketch E below:



To represent E in cylindrical coordinates, note that the shadow of E in the xy-plane can be represented in polar coordinates as $0 \leq r \leq 1$ and $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$. For each fixed point in this shadow, z runs from 0 (the xy-plane) to $1 - \sqrt{x^2+y^2} = 1 - r$ (the cone). Therefore we can express the triple integral of f over E as an iterated integral in cylindrical coordinates as

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r} f(r\cos(\theta), r\sin(\theta), z) r\, dz dr d\theta.$$

3. (10 points) Let $f : [a, b] \to \mathbb{R}$ be differentiable with f' continuous, and assume $f(x) > 0$ for every $x \in [a, b]$. Let

$$S = \{(x, \cos(\theta)f(x), \sin(\theta)f(x)) \ : \ x \in [a, b],\ \theta \in [0, 2\pi]\}$$

be the surface generated by revolving (in $\mathbb{R}^3$) the graph $\{(x, f(x), 0) \ : \ x \in [a, b]\}$ of f in the xy-plane around the x-axis. Prove that

$$\text{Surface Area of } S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

Solution: The surface S is parametrized by

$$\vec{X} : [a, b] \times [0, 2\pi] \to \mathbb{R}^3, \qquad \vec{X}(x, \theta) = (x, \cos(\theta)f(x), \sin(\theta)f(x)).$$

The normal vector arising from this parameterization is

$$N_{\vec{X}}(x, \theta) = \begin{bmatrix} 1 \\ \cos(\theta)f'(x) \\ \sin(\theta)f'(x) \end{bmatrix} \times \begin{bmatrix} 0 \\ -\sin(\theta)f(x) \\ \cos(\theta)f(x) \end{bmatrix} = \begin{bmatrix} \cos^2(\theta)f'(x)f(x) + \sin^2(\theta)f'(x)f(x) \\ -\cos(\theta)f(x) \\ -\sin(\theta)f(x) \end{bmatrix} = f(x) \begin{bmatrix} f'(x) \\ -\cos(\theta) \\ -\sin(\theta) \end{bmatrix},$$

so that the surface area of S is given by

$$\begin{aligned} \int_a^b \int_0^{2\pi} \|N_{\vec{X}}(x, \theta)\| d\theta dx &= \int_a^b \int_0^{2\pi} |f(x)|\sqrt{(f'(x))^2 + \cos^2(\theta) + \sin^2(\theta)}\, d\theta dx \\ &= \int_a^b \int_0^{2\pi} f(x)\sqrt{1 + (f'(x))^2}\, d\theta dx \\ &= 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx. \end{aligned}$$

4. (10 points) Assume that $f, g : \mathbb{R}^3 \to \mathbb{R}$ are C^2. Prove that

$$\operatorname{curl}(f\nabla g) = \nabla f \times \nabla g.$$

Here $(f\nabla g)(\vec{x}) = f(\vec{x})\nabla g(\vec{x})$ and $(\nabla f \times \nabla g)(\vec{x}) = \nabla f(\vec{x}) \times \nabla g(\vec{x})$.

Solution: Note that

$$\begin{aligned}
\operatorname{curl}(f\nabla g) &= \operatorname{curl}(fg_x\vec{i} + fg_y\vec{j} + fg_z\vec{k}) \\
&= ((fg_z)_y - (fg_y)_z)\vec{i} + ((fg_x)_z - (fg_z)_x)\vec{j} + ((fg_y)_x - (fg_x)_y)\vec{k} \\
&= (f_yg_z + fg_{zy} - f_zg_y - fg_{yz})\vec{i} + (f_zg_x + fg_{xz} - f_xg_z - fg_{zx})\vec{j} + (f_xg_y + fg_{yx} - f_yg_x - fg_{xy})\vec{k} \\
&= (f_yg_z - f_zg_y)\vec{i} + (f_zg_x - f_xg_z)\vec{j} + (f_xg_y - f_yg_x)\vec{k} \\
&= \nabla f \times \nabla g,
\end{aligned}$$

where we applied Clairaut's Theorem in the fourth step.

5. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function and let ω be a C^1 1-form on $\mathbb{R}^2$. Prove that

$$d(f\omega) = df \wedge \omega + f d\omega.$$

(Suggestion: Compute and simplify both sides.)

Solution: Write $\omega = adx + bdy$ for C^1 functions $a, b : \mathbb{R}^2 \to \mathbb{R}$. Then

$$\begin{aligned} d(f\omega) &= d(fadx + fbdy) \\ &= d(fa) \wedge dx + d(fb) \wedge dy \\ &= ((f_x a + f a_x)dx + (f_y a + f a_y)dy) \wedge dx + ((f_x b + f b_x)dx + (f_y b + f b_y)dy) \wedge dy \\ &= (f_x b + f b_x - (f_y a + f a_y))dx \wedge dy \end{aligned}$$

so that

$$\begin{aligned} df \wedge \omega + f d\omega &= (f_x dx + f_y dy) \wedge (adx + bdy) + f(da \wedge dx + db \wedge dy) \\ &= (f_x b - f_y a)dx \wedge dy + f((a_x dx + a_y dy) \wedge dx + (b_x dx + b_y dy) \wedge dy) \\ &= (f_x b - f_y a)dx \wedge dy + f(b_x - a_y)dx \wedge dy \\ &= (f_x b + f b_x - f_y a - f a_y)dx \wedge dy \\ &= d(f\omega). \end{aligned}$$