Solutions
Math 291-3
Homework 7
May 23, 2022

Exercise 1 (Colley 6.3.1, 6.3.2) This problem has two parts.

(a) Consider the line integral $\int_C z^2\,dx + 2y\,dy + xz\,dz$.

(i) Evaluate this integral, where C is the line segment from $(0,0,0)$ to $(1,1,1)$.

(ii) Evaluate this integral, where C is the path from $(0,0,0)$ to $(1,1,1)$ parametrized by $\vec{x}(t) = (t, t^2, t^3),\ 0 \le t \le 1$.

(iii) Is the vector field $\vec{F} = z^2\vec{\imath} + 2y\vec{\jmath} + xz\vec{k}$ conservative? Why or why not?

(b) Let $\vec{F} = 2xy\,\vec{\imath} + (x^2+z^2)\,\vec{\jmath} + 2yz\vec{k}$.

(i) Calculate $\int_C \vec{F}\cdot d\vec{s}$ where C is the path parametrized by $\vec{x}(t) = (t^2, t^3, t^5), 0 \le t \le 1$.

(ii) Calculate $\int_C \vec{F}\cdot d\vec{s}$ where C is the straight-line path from $(0,0,0)$ to $(1,0,0)$, followed by the straight-line path from $(1,0,0)$ to $(1,1,1)$.

(iii) Does $\vec{F}$ have path-independent line integrals? Explain your answer.

(a) (i) Using the parametric equations $\vec{x}(t) = (t,t,t),\ 0 \le t \le 1$ for C we get

$$\int_C z^2\,dx + 2y\,dy + xz\,dz = \int_0^1 (t^2 + 2t + t^2)\,dt = \int_0^1 (2t^2 + 2t)\,dt = \frac{2}{3} + 1.$$

(ii) We have

$$\int_C z^2\,dx + 2y\,dy + xz\,dz = \int_0^1 (t^6 + 2t^2[2t] + t^4[3t^2])\,dt = \int_0^1 (4t^6 + 4t^3)\,dt = \frac{4}{7} + 1.$$

(iii) Since the line integrals over the two curves above are not the same even though they both start at $(0,0,0)$ and end at $(1,1,1)$, line integrals of $\vec{F}$ are not path-independent so $\vec{F}$ cannot be conservative. We can also see this by computing

$$\operatorname{curl}\vec{F} = z\vec{\jmath} + 2\vec{k},$$

which is not zero as it would have to be for a conservative field.

(b) Note that $\vec{F}$ is the gradient of $f(x,y,z) = x^2y + yz^2$, so it is conservative, which makes these problems simple. The book probably expects you to compute the integrals in (a) and (b) using parametric equations and notice you get the same answer, but why go through all that trouble when we have better ways of computing integrals of conservative fields? The answer, of course, is that we shouldn't go through all that trouble.

(i) Since $\vec{x}(0) = (0,0,0)$ and $x(1) = (1,1,1)$, the Fundamental Theorem of Line Integrals gives

$$\int_C \vec{F}\cdot d\vec{s} = f(1,1,1) - f(0,0,0) = 2.$$

(ii) The start point of C is $(0,0,0)$ and the endpoint of C is $(1,1,1)$, so again

$$\int_C \vec{F} \cdot d\vec{s} = f(1,1,1) - f(0,0,0) = 2.$$

(iii) Since $\vec{F}$ is conservative, if has path-independent line integrals.

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Exercise 2 (Colley 6.3.20, 6.3.21) Find all functions $M(x,y)$ and $N(x,y)$ so that the following vector fields are conservative on $\mathbb{R}^2$.

(a) $\vec{F} = M(x,y)\vec{\imath} + (x\sin(y) - y\cos(x))\vec{\jmath}$

(b) $\vec{F} = (ye^{2x} + 3x^2e^y)\vec{\imath} + N(x,y)\vec{\jmath}$

(a) In order for this to be conservative we need a function $f(x,y)$ such that $\vec{F} = \nabla f$, which requires that

$$f_x = M \text{ and } f_y = x\sin(y) - y\cos(x).$$

Based on f_y, f would have to look like

$$f(x,y) = -x\cos(y) - \frac{1}{2}y^2\cos(x) + V(x)$$

where V is some expression only depending on x. This gives

$$f_x = -\cos(y) + \frac{1}{2}y^2\sin(x) + V_x,$$

so this is what M must look like. Thus in order for $\vec{F}$ to be conservative,

$$M = -\cos(y) + \frac{1}{2}y^2\sin(x) + (\text{something only depending on } x).$$

We can also do this using the (scalar curl) curl $\vec{F}$. We have

$$\operatorname{curl}\vec{F} = (\sin(y) + y\sin(x) - M_y),$$

so in order for $\vec{F}$ to be conservative this must be 0. (Assuming that $\vec{F}$ will be C^1 everywhere, which is reasonable here based on the $\vec{\jmath}$-component. Knowing that the curl is zero is enough to imply conservative since $\mathbb{R}^3$ is simply-connected.) This gives

$$M_y = \sin(y) + y\sin(x),$$

which then gives the same expression for M as before.

(b) In order to have $\nabla f = \vec{F}$, we would need

$$f_x = ye^{2x} + 3x^2e^y.$$

This gives

$$f(x,y) = \frac{1}{2}ye^{2x} + x^3e^y + V(y),$$

which gives

$$f_y = \frac{1}{2}e^{2x} + x^3 e^y + V_y,$$

which is what N must thus look like. Thus $\vec{F}$ is conservative when

$$N(x, y) = \frac{1}{2}e^{2x} + x^3 e^y + (\text{something depending only on } y).$$

■

Exercise 3 (Colley 6.3.26, 6.3.27, 6.3.28) Show that the following integrals are path independent (on the domain of the integrand) and evaluate them along the given oriented curve both directly and using the Fundamental Theorem of Line Integrals.

(a) $\int_C (3x - 5y)\,dx + (7y - 5x)\,dy$, C is the line segment from $(1, 3)$ to $(5, 2)$.

(b) $\int_C \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$, C is a semicircular arc of $x^2 + y^2 = 4$ from $(2, 0)$ to $(-2, 0)$.

(c) $\int_C (2y - 3z)\,dx + (2x + z)\,dy + (y - 3x)\,dz$, C is the line segment from the point $(0, 0, 0)$ to $(0, 1, 1)$ followed by the line segment from the point $(0, 1, 1)$ to $(1, 2, 3)$.

(a) The given 1-form is the differential of $f(x, y) = \frac{3}{2}x^2 - 5xy + \frac{7}{2}y^2$, so its integrals are path independent. The Fundamental Theorem gives

$$\int_C (3x - 5y)\,dx + (7y - 5x)\,dy = f(5, 2) - f(1, 3) = \frac{75}{2} - 50 + 14 - \frac{3}{2} + 15 - \frac{63}{2} = \frac{9}{2} - 21.$$

To compute this integral directly, we use the parametrization

$$\vec{x}(t) = (1 + 4t, 3 - t),\ 0 \le t \le 1.$$

We get:

$$\int_C (3x-5y)\,dx+(7y-5x)\,dy = \int_0^1 [(-12+17t)4-(16-27t)]\,dt = \int_0^1 (-64+95t)\,dt = -64+\frac{95}{2},$$

which is the same value as $\frac{9}{2} - 21$.

(b) The given form is the differential of $f(x, y) = \sqrt{x^2 + y^2}$, so its integrals are path independent. The Fundamental Theorem gives

$$\int_C \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = f(-2, 0) - f(2, 0) = 2 - 2 = 0.$$

On the other hand, the parametrization

$$\vec{x}(t) = (2\cos(t), 2\sin(t)),\ 0 \le t \le \pi$$

gives

$$\int_C \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = \int_0^\pi \frac{-4\cos(t)\sin(t) + 4\sin(t)\cos(t)}{4}\,dt = 0.$$

Technically, the orientation of C was not specified, but either choice gives zero.

(c) This is a line integral of the vector field

$$\vec{F} = (2y - 3z)\vec{\imath} + (2x + z)\vec{\jmath} + (y - 3x)\vec{k}.$$

Since

$$\operatorname{curl}\vec{F} = (1 - 1)\vec{\imath} - (-3 + 3)\vec{\jmath} + (2 - 2)\vec{k} = \vec{0}$$

and $\vec{F}$ is continuous on all of $\mathbb{R}^3$, which is simply connected, $\vec{F}$ is conservative and thus its line integrals are path independent. Concretely, $\vec{F}$ is the gradient of

$$f(x, y, z) = 2xy - 3xz + yz.$$

Now, parametrizing the first segment of C using $\vec{x}(t) = (0, t, t),\ 0 \le t \le 1$, we have

$$\int_{\text{1st}} \vec{F} \cdot d\vec{s} = \int_0^1 [(2t - 3t)0 + (2t + t)1 + (t - 3t)1]\, dt = \int_0^1 t\, dt = 1.$$

Parametrizing the second segment using $\vec{x}(t) = (t, t + 1, 2t + 1),\ 0 \le t \le 1$, we have

$$\int_{\text{2nd}} \vec{F} \cdot d\vec{s} = \int_0^1 [(-4t - 1)1 + (4t + 1)1 + (-2t + 1)2]\, dt = \int_0^1 (-4t + 2)\, dt = 0.$$

Thus the line integral over all of C is $1 + 0 = 1$. On the other hand, using the Fundamental Theorem we have

$$\int_C \nabla f \cdot d\vec{s} = f(\text{end point of } C) - f(\text{start point of } C) = f(1, 2, 3) - f(0, 0, 0) = 1,$$

which agrees with our direct computation.

■

Exercise 4 Let $\vec{F}$ be the vector field

$$(-y + ye^y)\vec{\imath} + [x + xe^y(1 + y) + z]\vec{\jmath} + (y + 2)\vec{k}.$$

Compute the line integral of $\vec{F}$ over the left half of the unit circle in the xy-plane oriented clockwise as viewed from the positive z-direction.
Hint: $\vec{F}$ is not conservative, but find a way to use the Fundamental Theorem of Line Integrals anyway.

Say we don't know beforehand that $\vec{F}$ is conservative and we try to find a potential f. We first need

$$f_x = -y + ye^y,$$

and thus $f(x, y, z) = -yx + xye^y + V(y, z)$. Differentiating this with respect to y gives

$$f_y = -x + xe^y(1 + y) + V_y,$$

which would have to equal $x + xe^y(y + 1) + z$ in order for $\vec{F}$ to equal ∇f. However now we see that this is not possible: our f_y has a $-x$ term whereas we need an x term to give the field $\vec{F}$, and since $V(y, z)$ does not depend on x this can't be resolved. Thus $\vec{F}$ is not conservative.

But the point is that if the terms causing trouble—$-y$ in the $\vec{i}$-component and x in the $\vec{j}$-component—weren't there, the field would be conservative since

$$\nabla(xye^y + yz + 2z) = ye^y\vec{i} + (xe^y(1+y) + z)\vec{j} + (y+2)\vec{k}.$$

Thus the given field $\vec{F}$ can be written as the sum

$$\vec{F} = \nabla f + (-y\vec{i} + x\vec{j})$$

where $f(x, y, z) = xye^y + yz + 2z$. Thus

$$\int_C \vec{F} \cdot d\vec{s} = \int_C \nabla f \cdot d\vec{s} + \int_C (-y\vec{i} + x\vec{j}) \cdot d\vec{s}.$$

For the first integral we can apply the Fundamental Theorem of Line Integrals:

$$\begin{aligned}\int_C \nabla f \cdot d\vec{s} &= f(\text{end point of } C) - f(\text{start point of } C)\\ &= f(0,1,0) - f(0,-1,0)\\ &= 0 - 0\\ &= 0.\end{aligned}$$

For the second integral we compute directly using the parametric equations $\vec{x}(t) = (\cos(t), -\sin(t), 0)$, $\pi/2 \leq t \leq 3\pi/2$ for the left half of the unit circle oriented clockwise. We get:

$$\int_C (-y\vec{i} + x\vec{j}) \cdot d\vec{s} = \int_{\pi/2}^{3\pi/2} (\sin(t), \cos(t)) \cdot (-\sin(t), -\cos(t))\, dt = \int_{\pi/2}^{3\pi/2} -\, dt = -\pi.$$

Thus

$$\int_C \vec{F} \cdot d\vec{s} = \int_C \nabla f \cdot d\vec{s} + \int_C (-y\vec{i} + x\vec{j}) \cdot d\vec{s} = 0 - \pi = -\pi.$$

■

Exercise 5 Suppose the continuous vector field $\vec{F} = P\,\vec{i} + Q\,\vec{j}$ on $\mathbb{R}^2$ has the property that its line integrals are path-independent, meaning that whenever C_1 and C_2 are piecewise-smooth oriented curves with the same starting point and the same ending point, we have

$$\int_{C_1} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy.$$

Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) \stackrel{def}{=} \int_C P\,dx + Q\,dy,$$

where C is any oriented piecewise-smooth curve that starts at $(0,0)$ and ends at (x, y). (The fact that line integrals of $\vec{F}$ are path-independent guarantees that any such path gives the same value for the integral, so that $f(x, y)$ is indeed well-defined.) Show that $\nabla f = \vec{F}$, thus showing that $\vec{F}$ is conservative. (A similar reasoning shows that any vector field field on $\vec{R}^n$ with path-independent line integrals must be conservative.)

Hint: To determine the partial derivative $\frac{\partial f}{\partial x}$, consider the path from $(0,0)$ to (x, y) consisting of the vertical line segment from $(0,0)$ to $(0, y)$ followed by the horizontal segment from $(0, y)$ to (x, y). You'll need to consider a different path when computing $\frac{\partial f}{\partial y}$. The single-variable Fundamental Theorem of Calculus will be crucial.

We computed $\frac{\partial f}{\partial x}$ in class using the following argument reproduced from the lecture notes. Consider the curve consisting of the vertical line segment C_1 from $(0,0)$ to $(0,y)$ followed by the horizontal line segment C_2 from $(0,y)$ to (x,y). With these we get

$$f(x,y) = \int_{C_1} (P\,dx + Q\,dy) + \int_{C_2} (P\,dx + Q\,dy).$$

Now, we can parametrize C_1 and C_2 using

$$\vec{x}_1(t) = (0,t),\ 0 \le t \le y \quad \text{and} \quad \vec{x}_2(t) = (t,y),\ 0 \le t \le x$$

respectively. With these, we get

$$\int_{C_1} P\,dx + Q\,dy = \int_0^y Q(0,t)\,dy \quad \text{and} \quad \int_{C_2} P\,dx + Q\,dy = \int_0^x P(t,y)\,dt,$$

so

$$f(x,y) = \int_0^y Q(0,t)\,dy + \int_0^x P(t,y)\,dt.$$

Now we can differentiate: the first term does not depend on x, so its derivative with respect to x is zero, while the derivative of the second term with respect to x is $P(x,y)$ by the Fundamental Theorem of Calculus. Thus

$$\frac{\partial f}{\partial x}(x,y) = P(x,y)$$

as desired.

To compute $\frac{\partial f}{\partial y}$, we instead use the curve consisting of the horizontal line segment C_3 from $(0,0)$ to $(x,0)$ followed by the vertical line segment C_4 from $(x,0)$ to (x,y). Parametrizations for these are respectively given by

$$\vec{x}_3(t) = (t,0),\ 0 \le t \le x \quad \text{and} \quad \vec{x}_4(t) = (x,t),\ 0 \le t \le y.$$

With these we get

$$\int_{C_3} P\,dx + Q\,dy = \int_0^x P(t,0)\,dt \quad \text{and} \quad \int_{C_4} P\,dx + Q\,dy = \int_0^y Q(x,t)\,dt,$$

so

$$f(x,y) = \int_0^x P(t,0)\,dt + \int_0^y Q(x,t)\,dt.$$

The derivative of the first term on the right with respect to y is zero since this term doesn't depend on y, and the derivative of the second term on the right with respect to y is $Q(x,y)$ by the Fundamental Theorem of Calculus. Thus

$$\frac{\partial f}{\partial y}(x,y) = Q(x,y),$$

so we conclude that $\nabla f(x,y) = (P(x,y), Q(x,y)) = \vec{F}(x,y)$ and hence that $\vec{F}$ is conservative as claimed. ■

Exercise 6 (Colley 6.2.8) Let $\vec{F} = 3xy\,\vec{i} + 2x^2\,\vec{j}$ and suppose C is the oriented curve consisting of the top half of the circle $(x-1)^2 + y^2 = 1$ oriented counterclockwise, followed by the line segment from $(0,0)$ to $(0,-2)$, followed by the line segment from $(0,-2)$ to $(2,-2)$, followed by the line segment from $(2,-2)$ to $(2,0)$. Evaluate $\oint_C \vec{F} \cdot d\vec{s}$ both directly and also by means of Green's Theorem.

First we evaluate directly. The curve on top has parametrization

$$x = 1 + \cos(t),\ \ y = \sin(t),\ \ 0 \le t \le \pi,$$

so the integral over this top half is

$$\int_0^{\pi} [-3(1+\cos(t))\sin^2(t) + 2(1+\cos(t))^2\cos(t)]\, dt = \frac{\pi}{2}.$$

The line segment from $(0,0)$ to $(0,-2)$ has parametrization

$$\vec{x}(t) = (0, -t),\ \ 0 \le t \le 2,$$

so the integral over this segment is

$$\int_0^2 0\, dt = 0.$$

The line segment from $(0,-2)$ to $(2,-2)$ has parametrization

$$\vec{x}(t) = (t, -2),\ \ 0 \le t \le 2,$$

so the integral over this segment is

$$\int_0^2 -6t\, dt = -12.$$

The line segment from $(2,-2)$ to $(2,0)$ has parametrization

$$\vec{x}(t) = (2, t-2),\ \ 0 \le t \le 2,$$

so the integral over this segment is

$$\int_0^2 8\, dt = 16.$$

Thus the integral over C is

$$\frac{\pi}{2} + 0 - 12 + 16 = \frac{\pi}{2} + 4.$$

Now, by Green's Theorem we have

$$\int_C 3xy\, dx + 2x^2\, dy = \iint\limits_D (4x - 3x)\, dA = \iint\limits_D x\, dA$$

where D is the region enclosed by C. This double integral can be split up into the integral over the portion of D above the x-axis, which is a half-disk, and the integral over the bottom portion, which is a square. The circle forming the top boundary of the half-disk is $r = 2\cos(\theta)$ is polar coordinates, so the integral over the top portion is:

$$\int_0^{\pi/2}\int_0^{2\cos(\theta)} r^2\cos(\theta)\, dr\, d\theta = \int_0^{\pi/2} \frac{8}{3}\cos^4(\theta)\, d\theta = \frac{\pi}{2}.$$

The integral over the square is

$$\int_0^2\int_{-2}^0 x\, dy\, dx = \int_0^2 2x\, dx = 4.$$

Hence the integral over D is

$$\frac{\pi}{2} + 4,$$

which agrees with the value we found via direct computation. ∎

Exercise 7 (Colley 6.2.16, 6.2.27) This problem has two parts.

(a) Use Green's Theorem to find the area between the ellipse $x^2/9 + y^2/4 = 1$ and the circle $x^2 + y^2 = 25$.

(b) Show that if C is the boundary of any rectangular region in $\mathbb{R}^2$ (oriented counterclockwise), then

$$\int_C (x^2y^3 - 3y)\, dx + x^3y^2\, dy$$

depends only on the area of the rectangle, not on its placement in $\mathbb{R}^2$.

(a) Orient the circle counterclockwise and the ellipse clockwise, and let D be the region between the two. Then ∂D consists of both the circle and the ellipse, and Green's Theorem gives:

$$\frac{1}{2}\oint_{\partial D} -y\, dx + x\, dy = \iint_D 1\, dA = \text{area of } D.$$

The line integral on the left breaks up into the integral over the circle plus the integral over the ellipse. The circle can be parametrized by

$$\vec{x}(t) = (5\cos(t), 5\sin(t)),\ \ 0 \le t \le 2\pi,$$

so the integral over the circle is

$$\frac{1}{2}\int_0^{2\pi} (25\sin^2(t) + 25\cos^2(t))\, dt = 25\pi.$$

The (clockwise) ellipse can be parametrized by

$$\vec{x}(t) = (3\cos(t), -2\sin(t)),\ \ 0 \le t \le 2\pi,$$

so the integral over the ellipse is

$$\frac{1}{2}\int_0^{2\pi} (-6\sin^2(t) - 6\cos^2(t))\, dt = -6\pi.$$

Thus the area of D is

$$\frac{1}{2}\oint_{\partial D} -y\, dx + x\, dy = 25\pi - 6\pi = 19\pi.$$

(b) The field $\vec{F} = (x^2y^3 - 3y)\vec{i} + x^3y^2\vec{j}$ is C^1 on all of $\mathbb{R}^2$, so Green's Theorem is applicable. Giving C the counterclockwise orientation, we get

$$\int_C (x^2y^3 - 3y)\, dx + x^3y^2\, dy = \iint_D (3x^2y^2 - 3x^2y^2 + 3)\, dA = \iint_D 3\, dA = 3(\text{area of } D)$$

where D is the rectangular region enclosed by C. So the line integral depends only on the area of D and not where in the xy-plane it is actually located.

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Exercise 8 (Colley 6.2.29, 6.2.30) This problem has two parts.

(a) Let D be a region to which Green's theorem applies and suppose that $u(x,y)$ and $v(x,y)$ are two C^2 functions whose domains include D. Show that

$$\iint_D \frac{\partial(u,v)}{\partial(x,y)}\,dA(x,y) = \oint_C (u\nabla v)\cdot d\vec{s}$$

where $C = \partial D$ is oriented as in Green's Theorem.

(b) Let $f(x,y)$ be a C^2 function such that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

(i.e. f is *harmonic*). Show that if C is any piecewise-smooth simple closed curve (i.e. a curve to which Green's Theorem applies), then

$$\oint_C \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = 0.$$

(a) We have

$$u\nabla v = u(v_x, v_y) = (uv_x, uv_y).$$

Green's Theorem then gives

$$\oint_C (u\nabla v)\cdot d\vec{s} = \oint_C uv_x\,dx + uv_y\,dy = \iint_D \left(\frac{\partial(uv_y)}{\partial x} - \frac{\partial(uv_x)}{\partial y}\right)dA.$$

We compute:

$$\frac{\partial(uv_y)}{\partial x} = u_x v_y + uv_{yx} \quad \text{and} \quad \frac{\partial(uv_x)}{\partial y} = u_y v_x - uv_{xy}.$$

Since v is C^2, $v_{yx} = v_{xy}$, so we get

$$\frac{\partial(uv_y)}{\partial x} - \frac{\partial(uv_x)}{\partial y} = u_x v_y - u_y v_x = \frac{\partial(u,v)}{\partial(x,y)}.$$

Hence the previous integral expression above becomes

$$\oint_C (u\nabla v)\cdot d\vec{s} = \iint_D \frac{\partial(u,v)}{\partial(x,y)}\,dA$$

as desired.

(b) Let D denote the region enclosed by C. By Green's Theorem we have (with a $+$ sign if C is oriented counterclockwise, and a $-$ sign if C is oriented clockwise)

$$\oint_C \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = \pm\iint_D \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right)dA = 0,$$

since f is harmonic and thus $f_{xx} + f_{yy} = 0$.

∎

Exercise 9 Consider the 1-form

$$\omega = \frac{(-y+x)\,dx + (x+y)\,dy}{x^2+y^2}.$$

Determine the value of the line integral of ω over *every* simple, closed piecewise-smooth curve in $\mathbb{R}^2$ which does not pass through the origin, which means that the origin does not lie on the curve itself. The value you obtain will depend on whether or not C encloses the origin, and on the orientation of C.

First, we compute:

$$\begin{aligned}
d\omega &= d\left(\frac{-y+x}{x^2+y^2}\right)\wedge dx + d\left(\frac{x+y}{x^2+y^2}\right)\wedge dy \\
&= \frac{(x^2+y^2)(-1)-(-y+x)2y}{(x^2+y^2)^2}\,dy\wedge dx + \frac{(x^2+y^2)(1)-(x+y)2x}{(x^2+y^2)^2}\,dx\wedge dy \\
&= \frac{y^2-2xy-x^2}{(x^2+y^2)^2}\,dy\wedge dx + \frac{y^2-2xy-x^2}{(x^2+y^2)^2}\,dx\wedge dy \\
&= 0
\end{aligned}$$

since $dy\wedge dx = -dx\wedge dy$. If C is a simple closed curve which does not enclose the origin, Green's theorem implies that

$$\int_{\partial D}\omega = \int_D d\omega = 0$$

where D is the region enclosed by C. (Green's Theorem applies since ω is C^1 on D.)

Now, suppose C is a simple closed curve oriented counterclockwise which does enclose the origin. Pick a small enough circle C_R of radius R centered at the origin which lies fully within the region enclosed by C, and orient it clockwise. Let D denote the region lying between C and C_R, or in other words the region having $C+C_R$ (the union of the two curves) as its boundary. This boundary has the correct orientation required in Green's Theorem and ω is C^1 on D (since D does not include the origin), so Green's Theorem gives

$$\int_{\partial D}\omega = \int_D d\omega = 0.$$

Since $\partial D = C + C_R$, we have

$$\int_{\partial D}\omega = \int_C \omega + \int_{C_R}\omega,$$

so since this is zero we must have

$$\int_C \omega = -\int_{C_R}\omega = \int_{-C_R}\omega,$$

where $-C_R$ denotes the same circle only oriented counterclockwise. Using the parametric equations

$$x(t) = R\cos(t),\ y(t) = R\sin(t),\ 0\le t\le 2\pi$$

for $-C_R$ gives:

$$\int_{-C_R}\omega = \int_0^{2\pi}\left[\frac{R(\cos(t)-\sin(t))}{R^2}(-R\sin(t)) + \frac{R(\cos(t)+\sin(t))}{R^2}(R\cos(t))\right]dt = \int_0^{2\pi} dt = 2\pi.$$

Thus the integral of ω over any counterclockwise-oriented simple closed curve which does enclose the origin is 2π, and it follows by reversing orientation that the integral of ω over any clockwise-oriented simple closed curve which does enclose the origin is -2π. ■

Exercise 10 This problem has two parts.

(a) Evaluate

$$\int_{S^1} (x - y^3)dx + x^3 dy \ ,$$

where S^1 is the unit circle in $\mathbb{R}^2$ with counterclockwise orientation.

(b) Find a function $\lambda(x, y)$ such that for every closed, piecewise-smooth oriented curve C in $\mathbb{R}^2$,

$$\int_C (x - y^3)\, dx + x^3\, dy = \int_C \lambda(x, y)\, dy.$$

Fun Fact: This problem appeared on one of Prof. Peterson's Ph.D. qualifying examinations.

(a) Let D denote the unit disc in $\mathbb{R}^2$. By Green's Theorem (and switching to polar coordinates) we have

$$\int_{S^1} (x - y^3)\, dx + x^3\, dy = \iint_D (3x^2 + 3y^2)\, dA(x, y) = \int_0^{2\pi} \int_0^1 3r^3\, dr d\theta = \frac{3\pi}{2}.$$

(b) Note that the function $\lambda(x, y)$ we produce should satisfy

$$\int_C (x - y^3)\, dx + (x^3 - \lambda(x, y))\, dy = 0$$

for every closed, piecewise-smooth oriented curve C in $\mathbb{R}^2$. By the Fundamental Theorem of Line Integrals, this would be guaranteed if the differential form $(x - y^3)\, dx + (x^3 - \lambda(x, y))\, dy$ were exact on $\mathbb{R}^2$. To this end, if $f : \mathbb{R}^2 \to \mathbb{R}$ is C^1 and satisfies

$$f_x(x, y)\, dx + f_y(x, y)dy = df = (x - y^3)\, dx + (x^3 - \lambda(x, y))\, dy,$$

then we want $f_x(x, y) = x - y^3$ and $f_y(x, y) = x^3 - \lambda(x, y)$. In particular, if we find a C^1 function f such that $f_x(x, y) = x - y^3$, then we can simply take $\lambda(x, y) = x^3 - f_y(x, y)$. But every C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f_x(x, y) = x - y^3$ has the form $f(x, y) = \frac{1}{2}x^2 - xy^3 + C(y)$, where $C : \mathbb{R} \to \mathbb{R}$ is a C^1 function. But then we could take

$$\lambda(x, y) = x^3 - f_y(x, y) = x^3 + 3xy^2 - C'(y),$$

where $C'(y)$ is continuous on $\mathbb{R}$.

In particular (taking $C(y) \equiv 0$ for simplicity), if $\lambda : \mathbb{R}^2 \to \mathbb{R}$, $\lambda(x, y) = x^3 + 3xy^2$, then

$$(x - y^3)\, dx + (x^3 - \lambda(x, y))\, dy = (x - y^3)\, dx + (-3xy^2)\, dy = d(\frac{1}{2}x^2 - xy^3)$$

is conservative on $\mathbb{R}^2$, so that

$$\int_C (x - y^3)\, dx + (x^3 - \lambda(x, y))\, dy = 0$$

for every closed piecewise-smooth oriented curve C in $\mathbb{R}^2$. But then (rearranging)

$$\int_C (x - y^3)\, dx + x^3\, dy = \int_C \lambda(x, y)\, dy$$

for every closed, piecewise-smooth oriented curve C in $\mathbb{R}^2$.

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