Solutions
Math 291-3
Homework 4
April 29, 2022

Exercise 1 (Colley 5.5.32 and 5.5.36) Determine the value of each of the following integrals, where W is as descirbed.

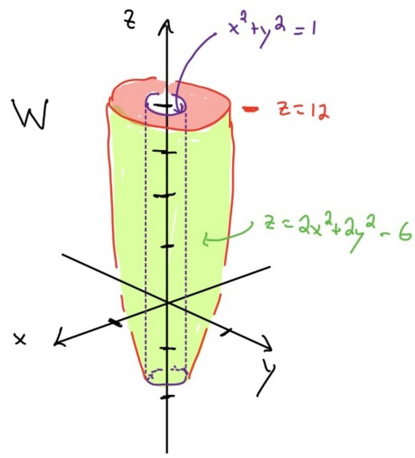
(a) $\iiint_W \frac{z}{\sqrt{x^2+y^2}}\, dV(x,y,z)$, where W is the solid region bounded below by the plane $z = 12$, below by the paraboloid $z = 2x^2 + 2y^2 - 6$, and lies outside the cylinder $x^2 + y^2 = 1$.

(b) $\iiint_W (x+y+z)\, dV(x,y,z)$, where W is the solid region in the first octant between the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, where $0 < a < b$.

(If you set this up directly in terms of spherical coordinates, you'll get an integral which involves a lot of messy computation, and which will require some not-so-obvious trig identity. Perhaps try to find a way to cut down on the amount of computation required.)

(a) Note that this problem is not the same as the version in the book, as the original problem in the book suffers from the same issues that we discussed in Problem 5a on Homework 3.

The region W looks like:



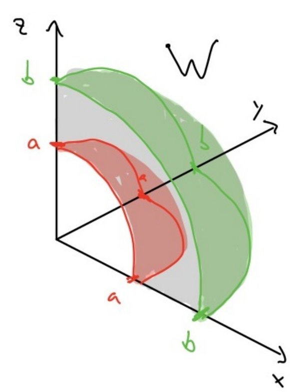
We work this out in cylindrical coordinates. We need θ to make a full revolution, so we take $0 \le \theta \le 2\pi$. To find the bounds on r we determine the projection of W to the xy-plane, which is precisely what we get when projecting the annulus (i.e. the region bounded between two concentric circles) on top at $z = 12$. The "inner radius" of this annulus is 1, since this portion of the boundary of the annulus lies on the cylinder $x^2 + y^2 = 1$. The "outer radius" of the annulus is 3, which we find by the setting the equations for the plane $z = 12$ and the paraboloid $z = 2x^2 + 2y^2 - 6$ equal to each other:

$$12 = 2x^2 + 2y^2 - 6$$

and solving for $\sqrt{x^2+y^2}$. Thus $1 \le r \le 3$. Finally, z goes from the paraboloid on the bottom, which has equation $z = 2r^2 - 6$ in cylindrical coordinates, up to the plane $z = 12$. Hence

$$\begin{aligned}
\iiint_W \frac{z}{\sqrt{x^2+y^2}}\,dV(x,y,z) &= \int_0^{2\pi}\int_1^3\int_{2r^2-6}^{12} \frac{z}{r}\,r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^3 \frac{1}{2}z^2\Big|_{2r^2-6}^{12}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^3 (72 - 2r^4 + 12r^2 - 18)\,dr\,d\theta \\
&= \int_0^{2\pi}\left(54r - \frac{2r^5}{5} + 4r^3\right)\Big|_1^3\,d\theta \\
&= \frac{1152\pi}{5}.
\end{aligned}$$

(b) The region W (shaded in grey) is inside the green sphere and outside the red sphere in the picture:



We will use spherical coordinates, where $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$, and $a \le \rho \le b$ since ρ runs from the smaller sphere to the larger sphere. If you set this up and start computing directly, at some point you end up having to integrate $\sin^2(\phi)$, and to do this you need the half-angle formula

$$\sin^2(\phi) = \frac{1}{2}(1 - \cos(2\phi)).$$

Instead, here is another approach which avoids having to do this which exploits symmetry.

We can break up the given integral as:

$$\iiint_W x\,dV(x,y,z) + \iiint_W y\,dV(x,y,z) + \iiint_W z\,dV(x,y,z).$$

The key observation is that the first two integrals are precisely equal to the third one! Intuitively, this should be true since the region W is "symmetric" (meaning looks the same) in the x, y, and z directions: if you rotate the original picture so that x is now the vertical axis, the rotation region will look exactly the same so integrating x (the new vertical coordinate) is the same as integrating z in the original picture. To be more precise, if you make the change of variables

$$(x,y,z) \mapsto (z,y,x)$$

which exchanges x and z, then the integrand x in the first triple integral becomes z, the region W stays the same, and the Jacobian factor turns out to be 1, so

$$\iiint\limits_W x\,dV(x,y,z) = \iiint\limits_W z\,dV(x,y,z)$$

under this change of variables. Similar reasoning shows that $\iiint\limits_W y\,dV(x,y,z) = \iiint\limits_W z\,d(x,y,z)V$ as well.

Thus

$$\iiint\limits_W (x+y+z)\,dV(x,y,z) = 3\iiint\limits_W z\,dV(x,y,z),$$

and the point is that integrating z alone in spherical coordinates is more straightforward since it avoids the use of any trig identities. We get:

$$\begin{aligned}
\iiint\limits_W z\,dV(x,y,z) &= \int_0^{\pi/2}\int_0^{\pi/2}\int_a^b \rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2} \frac{b^4-a^4}{4}\sin\phi\cos\phi\,d\phi\,d\theta \\
&= \frac{b^4-a^4}{4}\int_0^{\pi/2} \frac{1}{2}\sin^2\phi\Big|_0^{\pi/2}\,d\theta \\
&= \frac{(b^4-a^4)\pi}{16},
\end{aligned}$$

so $\iiint\limits_W (x+y+z)\,dV(x,y,z) = \dfrac{3\pi(b^4-a^4)}{16}$.

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Exercise 2 (Colley 5.5.29 and 5.5.37, altered)

(a) Evaluate

$$\int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}}\int_0^{4-x^2-4y^2} e^{x^2+4y^2+z}\,dz\,dx\,dy$$

by using cylindrical-like coordinates.

(b) Determine the value of

$$\iiint\limits_W z^2\,dV(x,y,z)$$

where W is the solid region lying above the cone $z=\sqrt{3x^2+3y^2}$ and inside the sphere $x^2+y^2+z^2=6z$. In addition (not in the book), determine the value of this integral if W is the region lying above the elliptic cone $z=\sqrt{3x^2+12y^2}$ and inside the ellipsoid $x^2+4y^2+z^2=6z$ instead.

(a) We make the change of variables

$$x = 2r\cos\theta,\ \ y = r\sin\theta,\ \ z = z.$$

Then

$$x^2 + 4y^2 = 4r^2\cos^2\theta + 4r^2\sin^2\theta = 4r^2,$$

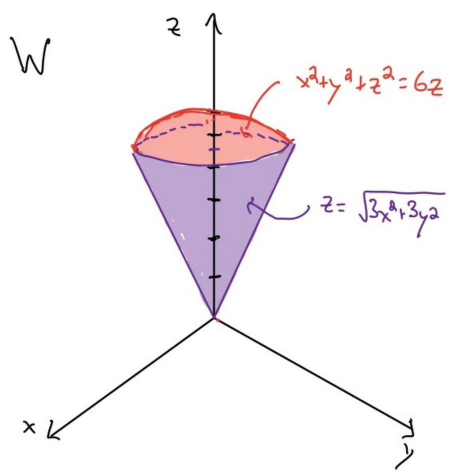
so the ellipse $x^2 + 4y^2 = 4$ is described in these new coordinates by $r^2 = 1$. The region enclosed by this ellipse gives the shadow of the region of integration in the xy-plane since this shadow has bounds $x = \pm\sqrt{4-4y^2}$. Then, at a fixed (x, y) in this shadow, z goes from $z = 0$ to the paraboloid $z = 4 - x^2 - 4y^2 = 4 - 4r^2$. The Jacobian for this change of variables is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\left(\begin{bmatrix} 2\cos(\theta) & -2r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 2r \geq 0,$$

so the integral in question becomes

$$\int_0^{2\pi}\int_0^1\int_0^{4-4r^2} e^{4r^2+z}2r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r(e^4 - e^{4r^2})\,dr\,d\theta = \int_0^{2\pi}\frac{1}{4}(3e^4+1)\,d\theta = \frac{\pi}{2}(3e^4+1).$$

(b) The given sphere is the same as $x^2 + y^2 + (z-3)^2 = 9$, so is centered at $(0, 0, 3)$ and has radius 3. The region W looks like an ice-cream cone:



We compute this using spherical coordinates. First, we need $0 \leq \theta \leq 2\pi$. To find the bounds on ϕ we look at the piece of the cone on the yz-plane, which looks like a triangle with hypotenuse on the line $z = \sqrt{3}y$. The value ϕ pointing along this line satisfies

$$\tan(\phi) = \frac{y}{z} = \frac{y}{\sqrt{3}y} = \frac{1}{\sqrt{3}}, \text{ so } \phi = \frac{\pi}{6}.$$

Thus $0 \leq \phi \leq \frac{\pi}{6}$, starting along the positive z-axis and ending along the cone. Finally, ρ goes from 0 at the origin out to the sphere, which is $\rho = 6\cos(\phi)$ in spherical coordinates. Hence

$$\begin{aligned}
\iiint_W z^2 \, dV(x,y,z) &= \int_0^{2\pi}\int_0^{\pi/6}\int_0^{6\cos(\phi)} (\rho^2\cos^2(\phi))\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/6} \frac{1}{5}\rho^5\cos^2(\phi)\sin(\phi)\Big|_0^{6\cos(\phi)} \,d\phi\,d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/6} 6^5\cos^7(\phi)\sin(\phi)\,d\phi\,d\theta \\
&= \frac{6^5}{5}\int_0^{2\pi} -\frac{1}{8}\cos^8(\phi)\Big|_0^{\pi/6} \,d\theta \\
&= \frac{6^5}{5}\int_0^{2\pi} -\frac{1}{8}\left(\frac{(\sqrt{3})^8}{2^8} - 1\right) d\theta \\
&= 2\pi\frac{6^5}{40}\left(1 - \frac{3^4}{2^8}\right) \\
&= \frac{8505\pi}{32}.
\end{aligned}$$

Now, suppose instead that W lies above the cone $z = \sqrt{3x^2 + 12y^2}$ and inside the ellipsoid $x^2 + 4y^2 + z^2 = 6z$. To compute the required integral we could instead use the change of variables

$$x = \rho\sin(\phi)\cos(\theta),\ \ y = \frac{1}{2}\rho\sin(\phi)\sin(\theta),\ \ z = \rho\cos(\phi),$$

which is just like spherical coordinates only with the y term modified to account for the $4y^2$ in the ellipsoid equation. The expansion factor works out to be $\frac{1}{2}\rho^2\sin(\phi)$, the cone is $z = \sqrt{3}r$, the ellipsoid is $\rho = 6\cos(\phi)$, and the bounds on ϕ and θ are the same as they were previously. Computing the resulting integral will just change the previous value by a factor of $\frac{1}{2}$, so that we get

$$\frac{8505\pi}{64}$$

instead. Alternatively, note that the change of variables which leaves x and z alone and replaces y by $\frac{1}{2}y$ transforms the new region in the previous region, leaves the integrand z^2 unchanged, and has expansion factor $\frac{1}{2}$, so that

$$\text{new integral } = \frac{1}{2}(\text{old integral}),$$

again giving the value $8505\pi/64$.

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Exercise 3 (Colley 5.5.42) Find the volume of the intersection of the three solid cylinders

$$x^2 + y^2 \le a^2, \quad x^2 + z^2 \le a^2, \quad y^2 + z^2 \le a^2.$$

(Hint: First draw a careful sketch, then note that, by symmetry, it suffices to calculate the volume of a portion of the intersection.)

Note that the shadow of the given region in the xy-plane is determined by the cylinder $x^2+y^2 \le a^2$, so this shadow is a disk of radius a. Now, at a fixed (x, y), the bounds on z come from the remaining cylinders $x^2 + z^2 \le a^2$ and $y^2 + z^2 \le a^2$; in particular, the lower bound on z comes from whichever of

$$z = -\sqrt{a^2 - x^2} \quad \text{and} \quad z = -\sqrt{a^2 - y^2}$$

is *larger* (i.e. closest to the xy-plane), and the upper bound on z comes from whichever of

$$z = \sqrt{a^2 - x^2} \quad \text{and} \quad z = \sqrt{a^2 - y^2}$$

is smaller (i.e. closest to the xy-plane). Which one is smaller or larger depends on where exactly (x, y) is in the xy-plane. For instance, in the region of the xy-plane where $|y| \le |x|$ (so that $y^2 \le x^2$), the bounds on z are given by

$$z = -\sqrt{a^2 - x^2} \quad \text{to} \quad z = \sqrt{a^2 - x^2}$$

since $a^2 - x^2 \le a^2 - y^2$ for such values of (x, y).

By symmetry, the volume of the entire region in question should be 16 times the portion of which lies in the first octant above the region where $y \le x$. The piece of the shadow in this region is the eighth of the disk of radius a below $y = x$ in the first quadrant, so this is given in polar coordinates by

$$0 \le r \le a \quad \text{and} \quad 0 \le \theta \le \frac{\pi}{4}.$$

Thus in cylindrical coordinates, the required volume is:

$$16 \int_0^{\pi/4} \int_0^a \int_0^{\sqrt{a^2 - r^2 \cos^2(\theta)}} r \, dz \, dr \, d\theta,$$

where the upper bound on z comes from $z = \sqrt{a^2 - x^2}$. Computing this gives:

$$\begin{aligned} 16 \int_0^{\pi/4} \int_0^a r\sqrt{a^2 - r^2 \cos^2(\theta)} \, dr \, d\theta &= 16 \int_0^{\pi/4} -\frac{1}{3} \frac{(a^2 - r^2 \cos^2(\theta))^{3/2}}{\cos^2(\theta)} \Bigg|_0^a \, d\theta \\ &= \frac{16a^3}{3} \int_0^{\pi/4} \frac{1 - \sin^3(\theta)}{\cos^2(\theta)} \, d\theta. \end{aligned}$$

Note that

$$\frac{1 - \sin^3(\theta)}{\cos^2(\theta)} = \sec^2(\theta) - \frac{(1 - \cos^2(\theta)) \sin(\theta)}{\cos^2(\theta)} = \sec^2(\theta) - \sec(\theta) \tan(\theta) + \sin(\theta),$$

this final integral is

$$\frac{16a^3}{3} (\tan(\theta) - \sec(\theta) - \cos(\theta)) \Big|_0^{\pi/4} = a^3 (16 - 8\sqrt{2}),$$

so the required volume is $a^3(16 - 8\sqrt{2})$. ∎

Exercise 4 For now, take the following fact for granted: if $f : \mathbb{R}^n \to \mathbb{R}$ continuous and $\vec{x}_0 \in \mathbb{R}^n$, then
$$\lim_{r\to 0^+} \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} f(\vec{x})\, dV_n(\vec{x}) = f(\vec{x}_0).$$
In the solutions I'll prove this result.
Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is C^1, injective, and that $DT(\vec{x})$ is invertible at every $\vec{x} \in \mathbb{R}^n$. Show that for each $\vec{x}_0 \in \mathbb{R}^n$,
$$\lim_{r\to 0^+} \frac{\mathrm{Vol}_n(T(B_r(\vec{x}_0)))}{\mathrm{Vol}_n(B_r(\vec{x}_0))} = |\det(DT(\vec{x}_0))|.$$
(This emphasizes the "infinitesimal expansion factor" interpretation of $|\det(DT(\vec{x}_0)|$, since it says that the ratio between the volume of the image of a ball under T and the volume of that ball itself for very small radii is essentially $|\det(DT(\vec{x}_0)|$.)

Hint: Express the numerator of the limit as an integral and use a change of variables.

Let $\vec{x}_0 \in \mathbb{R}^n$. Since T is C^1, injective, and has invertible derivative everywhere, the change of variables formula applies to give:
$$\mathrm{Vol}_n(T(B_r(\vec{x}_0))) = \int_{T(B_r(\vec{x}_0))} 1\, dV_n(\vec{x}) = \int_{B_r(\vec{x}_0)} |\det DT(\vec{x})|\, dV_n(\vec{x}).$$
Thus:
$$\lim_{r\to 0^+} \frac{\mathrm{Vol}_n(T(B_r(\vec{x}_0)))}{\mathrm{Vol}_n(B_r(\vec{x}_0))} = \lim_{r\to 0^+} \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} |\det DT(\vec{x})|\, dV_n(\vec{x}).$$
Since T is C^1, the function sending $\vec{x}$ to $DT(\vec{x})$ is continuous since $\det DT(\vec{x})$ is made up by adding products of the continuous entries of $DT(\vec{x})$, and hence the function sending $\vec{x}$ to $|\det DT(\vec{x})|$ is continuous as well. Hence by the quoted fact we are taking for granted with f with the function $f(\vec{x}) = |\det DT(\vec{x})|$, we have that
$$\lim_{r\to 0^+} \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} |\det DT(\vec{x})|\, dV_n(\vec{x}) = |\det DT(\vec{x}_0)|,$$
so
$$\lim_{r\to 0^+} \frac{\mathrm{Vol}_n(T(B_r(\vec{x}_0)))}{\mathrm{Vol}_n(B_r(\vec{x}_0))} = |\det DT(\vec{x}_0)|$$
as claimed.

As a bonus, we prove the "taken for granted" fact. We must prove that for every $\epsilon > 0$ there is $\delta > 0$ such that if $0 < r < \epsilon$, then
$$\left| \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} f(\vec{x})\, dV_n(\vec{x}) - f(\vec{x}_0) \right| < \epsilon.$$

To this end, let $\epsilon > 0$, and (using continuity of f at $\vec{x}_0$) choose $\delta > 0$ such that if $\|\vec{x} - \vec{x}_0\| < \delta$ then $|f(\vec{x}) - f(\vec{x}_0)| < \frac{\epsilon}{2}$. Then when $0 < r < \delta$, $\vec{x} \in B_r(\vec{x}_0)$ satisfies $\|\vec{x} - \vec{x}_0\| < r < \delta$, so that

$|f(\vec{x}) - f(\vec{x}_0)| < \epsilon$. Therefore for $0 < r < \delta$, we have

$$\begin{aligned}
&\left| \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} f(\vec{x})\, dV_n(\vec{x}) - f(\vec{x}_0) \right| \\
&= \left| \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} f(\vec{x})\, dV_n(\vec{x}) - f(\vec{x}_0) \frac{\mathrm{Vol}_n(B_r(\vec{x}_0))}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \right| \\
&= \left| \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} f(\vec{x})\, dV_n(\vec{x}) - f(\vec{x}_0) \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} 1\, dV_n(\vec{x}) \right| \\
&= \left| \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} f(\vec{x})\, dV_n(\vec{x}) - \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} f(\vec{x}_0)\, dV_n(\vec{x}) \right| \\
&= \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \left| \int_{B_r(\vec{x}_0)} (f(\vec{x}) - f(\vec{x}_0))\, dV_n(\vec{x}) \right| \\
&\leq \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} |f(\vec{x}) - f(\vec{x}_0)|\, dV_n(\vec{x}) \\
&\leq \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} \frac{\epsilon}{2}\, dV_n(\vec{x}) \\
&= \frac{\epsilon}{2\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} 1\, dV_n(\vec{x}) \\
&= \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Therefore we have $\lim\limits_{r \to 0^+} \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} f(\vec{x})\, dV_n(\vec{x}) = f(\vec{x}_0)$, as claimed! ∎

Exercise 5 Let C be a smooth curve in $\mathbb{R}^n$ with parametrization $\vec{x} : [a, b] \to \mathbb{R}^n$. Show that C lies on a hypersphere (i.e. the set of points at a fixed distance away from the origin) if, and only if, $\vec{x}(t)$ and $\vec{x}\,'(t)$ are orthogonal for every $t \in [a, b]$.

Hint: To say that C lies on a hypersphere is to say that $\|\vec{x}(t)\|$ equals the same value for all t, which is the same as saying that $\|\vec{x}(t)\|^2$ is constant in t.

The curve C lies on a hypersphere if, and only if, the single-variable function $\|\vec{x}(t)\|$ is constant in t, which is true if, and only if, $\|\vec{x}(t)\|^2$ is constant in t, which is true if, and only if,

$$\frac{d}{dt}\|\vec{x}(t)\|^2 = 0 \text{ for all } t.$$

Since $\|\vec{x}(t)\|^2 = \vec{x}(t) \cdot \vec{x}(t)$, we have (see the solution to Exercise 6 for a proof of the derivative formula)

$$\frac{d}{dt}\|\vec{x}(t)\|^2 = \frac{d}{dt}(\vec{x}(t) \cdot \vec{x}(t)) = \vec{x}\,'(t) \cdot \vec{x}(t) + \vec{x}(t) \cdot \vec{x}\,'(t) = 2(\vec{x}(t) \cdot \vec{x}\,'(t)).$$

Thus C lies on a hypersphere if and only if $2(\vec{x}(t) \cdot \vec{x}\,'(t)) = 0$ for all t, which is true if, and only if, $\vec{x}(t) \cdot \vec{x}'(t) = 0$ for all t, meaning that $\vec{x}(t)$ and $\vec{x}'(t)$ are orthogonal for all t as claimed. ∎

Exercise 6 (Colley 3.1.35) Let $\vec{x}(t)$ be a path of class C^1 that does not pass through the origin in $\mathbb{R}^3$. Suppose $\vec{x}(t_0)$ is a point on the image of $\vec{x}$ closest to the origin and $\vec{x}\,'(t_0) \neq \vec{0}$. Show that $\vec{x}(t_0)$ is orthogonal to $\vec{x}\,'(t_0)$.

Consider the single-variable function $f(t) = \|\vec{x}(t)\|^2$, which gives the square of the distance from a point $\vec{x}(t)$ on the curve to the origin and is always positive since the curve does not pass through the origin. If $\vec{x}(t_0)$ is the point on the curve closest to the origin, then t_0 minimizes $f(t)$. Thus the derivative of $f(t) = \vec{x}(t) \cdot \vec{x}(t)$ at t_0 is zero, so

$$f'(t_0) = \vec{x}\,'(t_0) \cdot \vec{x}(t_0) + \vec{x}(t_0) \cdot \vec{x}\,'(t_0) = 0.$$

(Here we note that the product rule from Calculus I implies that

$$\frac{d}{dt}(\vec{x}(t) \cdot \vec{y}(t)) = \frac{d}{dt}\sum_{k=1}^{n} x_k(t)y_k(t) = \sum_{k=1}^{n} x_k'(t)y_k(t) + \sum_{k=1}^{n} x_k(t)y_k'(t) = \vec{x}\,'(t) \cdot \vec{y}(t) + \vec{x}(t) \cdot \vec{y}\,'(t)$$

for vector-valued functions $\vec{x}(t)$ and $\vec{y}(t)$ of a single variable.) This gives $2(\vec{x}(t_0) \cdot \vec{x}\,'(t_0)) = 0$, so $\vec{x}(t_0) \cdot \vec{x}\,'(t_0) = 0$ and $\vec{x}(t_0), \vec{x}\,'(t_0)$ are orthogonal as claimed. ∎

Exercise 7 (Colley 3.2.14) Consider the path $\vec{x}(t) = (e^{-t}\cos(t), e^{-t}\sin(t))$.

(a) Argue that the path spirals toward the origin as $t \to +\infty$.

(b) Show that, for any a, the improper integral

$$\int_a^\infty \|\vec{x}\,'(t)\|\, dt$$

converges.

(c) Interpret what the result in part (b) says about the path $\vec{x}$.

(a) Note that

$$\|\vec{x}(t)\| = e^{-2t}cos^2(t) + e^{-2t}\sin^2(t) = e^{-2t}.$$

As $t \to +\infty$, $\|\vec{x}(t)\| = e^{-2t} \to 0$ so the path approaches the origin. The $\cos(t), \sin(t)$ terms in the parametric equations for x and y indicate circular motion only with decreasing radius due to the e^{-t} factor, so the curve indeed spirals towards the origin as $t \to +\infty$.

(b) We have

$$\vec{x}'(t) = (-e^{-t}\cos(t) - e^{-t}\sin(t), -e^{-t}\sin(t) + e^{-t}\cos(t)),$$

so

$$\begin{aligned}\|\vec{x}'(t)\| &= \sqrt{(-e^{-t}\cos(t) - e^{-t}\sin(t))^2 + (-e^{-t}\sin(t) + e^{-t}\cos(t))^2} \\ &= \sqrt{e^{-2t} + e^{-2t}} \\ &= \sqrt{2}e^{-t}.\end{aligned}$$

Thus for any a, we have:

$$\lim_{b\to\infty}\int_a^b \|\vec{x}'(t)\|\, dt = \lim_{b\to\infty}\int_a^b \sqrt{2}e^{-t}\, dt = \lim_{b\to\infty}\sqrt{2}(e^{-a} - e^{-b}) = \sqrt{2}e^{-a},$$

so the given improper integral converges.

(c) The integral in (b) gives the length of the curve starting at $t = a$ and letting t increase without bound, so the result is that this length is always finite and has value $\sqrt{2}e^{-a}$. Note that the origin is never actually reached along this curve since e^{-t} will never be zero, but the length of the curve is finite nonetheless.

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Exercise 8 (Colley 3.2.15) Suppose that a curve is given in polar coordinates by an equation of the form $r = f(\theta)$, where f is C^1. Derive the formula

$$L = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$$

for the length L of the curve between the points $(f(\alpha), \alpha)$ and $(f(\beta), \beta)$ given in polar coordinates.

Parametric equations for the curve in cartesian coordinates are given by

$$\vec{x}(\theta) = (f(\theta)\cos(\theta), f(\theta)\sin(\theta)) \text{ for } \alpha \le \theta \le \beta$$

where we use the fact that $r = f(\theta)$ along the curve in question. We compute:

$$\vec{x}\,'(\theta) = (f'(\theta)\cos(\theta) - f(\theta)\sin(\theta), f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)),$$

so

$$\begin{aligned} \|\vec{x}\,'(\theta)\| &= \sqrt{(f'(\theta)\cos(\theta) - f(\theta)\sin(\theta))^2 + (f'(\theta)\sin(\theta) + f(\theta)\cos(\theta))^2} \\ &= \sqrt{f'(\theta)^2 + f(\theta)^2}. \end{aligned}$$

Thus the length of L is given by

$$\int_\alpha^\beta \|\vec{x}\,'(\theta)\|\, d\theta = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$$

as claimed. ■

Exercise 9 Suppose C is a smooth curve in $\mathbb{R}^n$ with two parametrizations

$$\vec{x} : [a, b] \to \mathbb{R}^n \quad \text{and} \quad \vec{y} : [c, d] \to \mathbb{R}^n$$

related by $\vec{y} = \vec{x} \circ \tau$ for some C^1, bijective map $\tau : [c, d] \to [a, b]$ with $\tau'(u) \neq 0$ for every $u \in [c, d]$. Show that the tangent vector at a point along C determined by $\vec{x}$ points in the same direction as the one determined by $\vec{y}$ if, and only if, $\tau'(u)$ is positive for all $u \in [c, d]$.

Since $\vec{y}(u) = \vec{x}(\tau(u))$ for any $u \in [c, d]$, the chain rule gives

$$\vec{y}\,'(u) = D\vec{x}(\tau(u))D\tau(u) = \tau'(u)\vec{x}\,'(\tau(u)).$$

Thus the vectors $\vec{y}\,'(u)$ and $\vec{x}\,'(\tau(u))$ point in the same direction if, and only if, the scalar $\tau'(u)$ is positive for all $u \in [c, d]$ as claimed. ■

Exercise 10 Suppose C is a smooth curve in $\mathbb{R}^n$ with parametrization $\vec{x} : [a,b] \to \mathbb{R}^n$, and that $f : C \to \mathbb{R}$ is a continuous function on C. The **scalar line integral** of f over C is defined to be:

$$\int_C f\,ds \stackrel{def}{=} \int_a^b f(\vec{x}(t))\|\vec{x}\,'(t)\|\,dt.$$

Show that this definition is independent of parametrization. To be clear, this means that if $\vec{y} : [c,d] \to \mathbb{R}^n$ is another parametrization of C which is related to $\vec{x}$ via

$$\vec{y} = \vec{x} \circ \tau$$

for some C^1, bijective map $\tau : [c,d] \to [a,b]$ with $\tau'(u) \neq 0$ for every $u \in [c,d]$, you want to show that the integral above is the same as the one obtained by using $\vec{y}$ instead of $\vec{x}$.

This is almost the same as the proof we gave in the lecture notes to show that arclength was independent of parametrization, only that now we throw in the function f as well. The change of variables formula gives:

$$\int_{\tau([c,d])} f(\vec{x}(t))\|\vec{x}\,'(t)\|\,dV_1(t) = \int_{[c,d]} f(\vec{x}(\tau(u)))\|\vec{x}\,'(\tau(u))\||\tau'(u)|\,dV_1(u).$$

As in the previous problem, the chain rule gives that

$$\vec{x}\,'(\tau(u))\tau'(u) = \vec{y}\,'(u),$$

so the integral on the right above becomes

$$\int_c^d f(\vec{y}(u))\|\vec{y}\,'(u)\|\,du.$$

Hence since $\tau([c,d]) = [a,b]$, the equality of integrals above becomes

$$\int_c^d f(\vec{y}(u))\|\vec{y}\,'(u)\|\,du = \int_a^b f(\vec{x}(t))\|\vec{x}\,'(t)\|\,dt,$$

which says that the value of the scalar line integral as computed using the $\vec{x}$-parametrization on the right is the same as the value obtained using the $\vec{y}$-parametrization on the left. ∎