Solutions
Math 291-3
Homework 8
May 31, 2022

Exercise 1 Compute

$$\int_C (y^2x + x^2 + yx^5)\,dx + (x^2y + x - \sin(y)(y+1)^{y\sin(y)+y^2+3})\,dy$$

where C is the top half of the unit circle oriented clockwise. To be clear, C is NOT closed.

Let C_1 be the line segment from $(1,0)$ to $(-1,0)$. Attaching this to C gives a closed curve $C + C_1$ and we get

$$\int_C \omega = \int_{C+C_1} \omega - \int_{C_1} \omega$$

where $\omega = (y^2x + x^2 + yx^5)\,dx + (x^2y + x - \sin(y)(y+1)^{y\sin(y)+y^2+3})\,dy$. Denoting by D the top-half of the unit disk, Green's Theorem gives

$$\int_{C+C_1} (y^2x + x^2 + yx^5)\,dx + (x^2y + x - \sin(y)(y+1)^{y\sin(y)+y^2+3})\,dy = -\iint_D (1 - x^5)\,dA(x,y)$$

where the negative sign comes from correcting for the orientation. The function x^5 is odd with respect to x and D is symmetric with respect to x (i.e. across the y-axis), so

$$\iint_D (1 - x^5)\,dA(x,y) = \iint_D dA(x,y) - \iint_D x^5\,dA(x,y) = \text{area}(D) + 0 = \frac{\pi}{2}.$$

Thus

$$\int_C \omega = -\frac{\pi}{2} - \int_{C_1} \omega.$$

Parametrizing C_1 with $\vec{x}(t) = (-t, 0),\ -1 \le t \le 1$ gives

$$\int_{C_1} \omega = \int_{-1}^{1} -t^2\,dt = -\frac{2}{3},$$

so we conclude that

$$\int_C (y^2x + x^2 + yx^5)\,dx + (x^2y + x - \sin(y)(y+1)^{y\sin(y)+y^2+3})\,dy = -\frac{\pi}{2} + \frac{2}{3}.$$

■

Exercise 2 Let D be a compact region in $\mathbb{R}^2$ to which Green's Theorem applies. Suppose u is C^2 and **harmonic** on D, meaning that $u_{xx} + u_{yy} = 0$ on D. If $u(x,y) = 0$ for all $(x,y) \in \partial D$, show that $u = 0$ on all of D.

(Thus if a harmonic function is zero on the boundary of a region, then it must be zero throughout the entire region. This implies that the values of a harmonic function throughout a region are fully determined by its values on the boundary alone, which is a key property of harmonic functions.)

Hint: Apply Green's Theorem to the vector field $\vec{F} = -uu_y\vec{i} + uu_x\vec{j}$.

Give ∂D the orientation such that D is "on the left" when traveling along any piece of ∂D. Note that $\vec{F} = \vec{0}$ on ∂D because $u = 0$ on ∂D, and therefore

$$\oint_{\partial D} \vec{F} \cdot d\vec{s} = \int_{\partial D} \vec{F} \cdot \vec{T}\, ds = \int_{\partial D} 0\, ds = 0.$$

On the other hand, Green's Theorem gives

$$\begin{aligned}
0 &= \oint_{\partial D} \vec{F} \cdot d\vec{s} \\
&= \oint_{\partial D} -uu_y\, dx + uu_x\, dy \\
&= \iint_D \Big((uu_x)_x - (-uu_y)_y \Big)\, dA(x,y) \\
&= \iint_D \Big((u_x)^2 + uu_{xx} + (u_y)^2 + uu_{yy} \Big)\, dA(x,y) \\
&= \iint_D \Big(\|\nabla u(x,y)\|^2 + u(x,u) \underbrace{(u_{xx}(x,y) + u_{yy}(x,y))}_{=0} \Big)\, dA(x,y) \\
&= \iint_D \|\nabla u(x,y)\|^2\, dA(x,y).
\end{aligned}$$

Because $\|\nabla u(x,y)\|^2$ is continuous throughout D with $\|\nabla u(x,y)\|^2 \geq 0$, *we must have* $\|\nabla u(x,y)\|^2 = 0$ *throughout* D, and therefore (by a fact from last quarter), $u(x,y)$ is constant throughout D. Because $u(x,y) = 0$ on ∂D, we must have $u(x,y) = 0$ throughout D.

(For a more rigorous justification of the italicized claim above, note that if $\|\nabla u(x_0,y_0)\|^2 > 0$ at some point (x_0,y_0) in D, then by continuity there is $r > 0$ such that $\|\nabla u(x,y)\|^2 \geq \frac{1}{2}\|\nabla u(x_0,y_0)\|^2 > 0$ for all $(x,y) \in B_r(x_0,y_0) \subset D$, and therefore

$$\iint_D \|\nabla u(x,y)\|^2\, dA(x,y) \geq \iint_{B_r(x_0,y_0)} \|\nabla u(x,y)\|^2\, dA(x,y) \geq \frac{1}{2}\|\nabla u(x_0,y_0)\|^2 \pi r^2 > 0,$$

contradicting our previous conclusion that $\iint_D \|\nabla u(x,y)\|^2\, dA(x,y) = 0$.) ∎

Exercise 3 (Colley 7.2.3, 7.2.24) This problem has two unrelated parts.

(a) Find the flux of $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ across the surface S consisting of the triangular portion of the plane $2x - 2y + z = 2$ that is cut out by the coordinate planes. Here assume that S is oriented with upward-pointing normal vectors.

(b) Let $F = 2x\vec{i} + 2y\vec{j} + z^2\vec{k}$. Find $\displaystyle\iint_S \vec{F} \cdot d\vec{S}$, where S is the portion of the cone $x^2 + y^2 = z^2$ between the planes $z = -2$ and $z = 1$, oriented with outward-pointing normal vectors.

(a) To visualize S, note that the given plane intersects the x-axis at $(1,0,0)$, the y-axis at $(0,-1,0)$, and the z-axis at $(0,0,2)$. We parametrize S using

$$\vec{X}(x,y) = (x, y, 2 - 2x + 2y), \ (x,y) \in D$$

where D is the region in the xy-plane lying under S since it is this region which restricts the values of x and y. Setting $z = 0$ in the equation of the plane, we find that D is the region in the fourth quadrant with boundaries the two axes and the line $x - y = 1$.

We have $\vec{X}_x = (1, 0, -2)$ and $\vec{X}_y = (0, 1, 2)$, so

$$\vec{X}_x \times \vec{X}_y = (2, -2, 1).$$

Note that we could have also found this normal vector simply by taking the coefficients of x, y, z in the equation of the plane; seeing this would have avoided a bit of work. This normal vector indeed points upward so $\vec{X}$ is orientation-preserving and there is no need to correct the orientation. Thus the flux is:

$$\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= \iint_D \vec{F}(\vec{X}(x, y)) \cdot (\vec{X}_x \times \vec{X}_y)\, dx\, dy \\
&= \int_{-1}^{0} \int_{0}^{1+y} (x, y, 2 - 2x - 2y) \cdot (2, -2, 1)\, dx\, dy \\
&= \int_{-1}^{0} \int_{0}^{1+y} (2x - 2y + 2 - 2x - 2y)\, dx\, dy \\
&= \int_{-1}^{0} \int_{0}^{1+y} 2\, dx\, dy \\
&= 2\,\text{area}(D) \\
&= 1.
\end{aligned}$$

(b) Denote the piece of S above the xy-plane by S_1 and the piece below by S_2. We do this since the equation for the cone is different in cylindrical coordinates over these two pieces: $z = r$ on top and $z = -r$ on bottom. We parametrize S_1 using

$$\vec{X}^1(r, \theta) = (r\cos(\theta), r\sin(\theta), r),\ 0 \le \theta \le 2\pi,\ 0 \le r \le 1$$

and S_2 using

$$\vec{X}^2(r, \theta) = (r\cos(\theta), r\sin(\theta), -r),\ 0 \le \theta \le 2\pi,\ 0 \le r \le 2.$$

This gives:

$$\vec{X}^1_r \times \vec{X}^1_\theta = (-r\cos(\theta), -r\sin(\theta), r) \text{ and } \vec{X}^2_r \times \vec{X}^2_\theta = (r\cos(\theta), r\sin(\theta), r)$$

as normal vectors to S_1 and S_2 respectively. This normal vector to S_2 is indeed outward-pointing, but this normal vector for S_1 is actually inward-pointing. (On the top part of the cone, outward-pointing normal vectors should actually have a negative $\vec{k}$ component.) Thus we use $\vec{X}^1_\theta \times \vec{X}^1_r = (r\cos(\theta), r\sin(\theta), -r)$ for the normal vector of S_1 instead.

The surface integral over S_1 is:

$$\begin{aligned}
\iint_{S_1} \vec{F} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^1 (2r\cos(\theta), 2r\sin(\theta), r^2) \cdot (r\cos(\theta), r\sin(\theta), -r)\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 (2r^2 - r^3)\, dr\, d\theta \\
&= 2\pi\left(\frac{2}{3} - \frac{1}{4}\right) \\
&= \frac{5\pi}{6}.
\end{aligned}$$

The surface integral over S_2 is:

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^2 (2r\cos(\theta), 2r\sin(\theta), r^2) \cdot (r\cos(\theta), r\sin(\theta), r)\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^2 (2r^2 + r^3)\, dr\, d\theta$$
$$= 2\pi \left(\frac{16}{3} + 4 \right)$$
$$= \frac{56\pi}{3}.$$

The surface integral over all of S is then $\frac{5\pi}{6} + \frac{56\pi}{3} = \frac{117\pi}{6}$.

∎

Exercise 4 (Colley 7.3.11, 7.3.13b) This problem has two unrelated parts.

(a) Let S be the surface defined by $y = 10 - x^2 - z^2$ with $y \geq 1$, oriented with normals pointing in the positive y-direction. Let

$$\vec{F} = (2xyz + 5z)\,\vec{i} + e^x \cos(yz)\,\vec{j} + x^2 y\,\vec{k}.$$

Determine

$$\iint_S \text{curl}\vec{F} \cdot d\vec{S}.$$

(b) Evaluate

$$\oint_C (y^3 + \cos(x))\, dx + (\sin(y) + z^2)\, dy + x\, dz$$

where C is the smooth closed curve parametrized (and oriented by) the path $\vec{x}(t) = (\cos(t), \sin(t), \sin(2t)), 0 \leq t \leq 2\pi$. Note that this path lies on the surface $z = 2xy$.

(a) The surface S is a paraboloid opening up in the negative y-direction, cut off at $y = 1$ and with rightmost point at $(0, 10, 0)$. This has geometric boundary equal to the circle of radius 3 centered on the y-axis in the plane $y = 1$, which we get from setting $y = 1$ in the equation for S. The induced orientation on ∂S is the one which appears counterclockwise when viewed from the rightmost point at $(0, 10, 0)$. Stokes' Theorem gives

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S \text{curl}\vec{F} \cdot d\vec{S}.$$

But now, if S_1 is the disk of radius 3 centered on the y-axis in the plane $y = 1$ with rightward orientation, then $\partial(S_1) = \partial S$ so Stokes' Theorem now gives

$$\int_{\partial S = \partial S_1} \vec{F} \cdot d\vec{s} = \iint_{S_1} \text{curl}\vec{F} \cdot d\vec{S}.$$

This final integral is simpler to compute than the original. Indeed, the unit normal vector on S_1 is given by $\vec{n} = \vec{j}$, so in dot product on the right only the $\vec{j}$-component of curl$\vec{F}$ will matter. Since curl$\vec{F}$ looks like

$$\text{curl}\vec{F} = (\text{something}, 5, \text{something}),$$

we get

$$\iint_{S_1} \text{curl}\vec{F} \cdot d\vec{S} = \iint_{S_1} 5\, dS,$$

which is 5 times the surafce area of S_1. Hence we conclude that

$$\iint_{S} \text{curl}\vec{F} \cdot d\vec{S} = \iint_{S_1} \text{curl}\vec{F} \cdot d\vec{S} = 45\pi.$$

(b) We use Stokes' Theorem. Note that the given integral is the circulation of $\vec{F} = (y^3+\cos(x))\vec{i}+(\sin(y)+z^2)\vec{j}+x\vec{k}$ around the curve C. Let S be the piece of the surface $z = 2xy$ enclosed by C. Since C has counterclockwise orientation when viewed from the positive z-direction (based on the x and y equations), S should have upward orientation. Since the shadow of C in the xy-plane is the unit circle (based on the x and y parametric equations for C), the shadow of S in the xy-plane is the unit disk. Thus S has parametric equations

$$\vec{X}(s,t) = (s, t, 2st), \ (s,t) \text{ in the unit disk } D.$$

Then

$$\vec{X}_s \times \vec{X}_t = (1, 0, 2t) \times (0, 1, 2s) = (-2t, -2s, 1),$$

which gives the correct orientation on S. Also,

$$\text{curl}\vec{F} = (-2z, -1, -3y^2),$$

so

$$\iint_{S} \text{curl}\vec{F} \cdot d\vec{S} = \iint_{D} (-4st, -1, -3t^2) \cdot (-2t, -2s, 1)\, dA(s,t) = \iint_{D} (8st^2 + 2s - 3t^2)\, dA(s,t).$$

The region D is symmetric across the t-axis and $8st^2 + 2s$ is odd with respect to s, so the double integral of $8st - 2s$ over D is zero. Hence we're left computing the double integral of $-3t^2$, and for this we convert to polar coordinates:

$$\begin{aligned}
\iint_{D} -3t^2\, dA(s,t) &= \int_0^{2\pi} \int_0^1 (-3r^2 \sin^2(\theta)) r\, dr\, d\theta \\
&= \int_0^{2\pi} -\frac{3}{4} \sin^2(\theta)\, d\theta \\
&= \int_0^{2\pi} -\frac{3}{8}(1 - \cos(2\theta))\, d\theta \\
&= -\frac{3\pi}{4}.
\end{aligned}$$

Thus the line integral in question has value $-\frac{3\pi}{4}$ as well.

■

Exercise 5 (Colley 7.3.12) Let S be the surface defined as $z = 4-4x^2-y^2$ with $z \geq 0$ and oriented with normal vectors that have a nonnegative $\vec{k}$-component. Let $\vec{F}(x,y,z) = x^3\,\vec{i} + e^{y^2}\,\vec{j} + ze^{xy}\,\vec{k}$. Find $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$.

Since the region S_1 enclosed by the ellipse $4x^2 + y^2 = 4$ in the xy-plane has this same boundary as S (where the ellipse is oriented in the counterclockwise direction when viewed from the positive z-axis), by Stokes' Theorem the line integral of $\vec{F}$ over this ellipse equals the surface integral of $\nabla \times \vec{F}$ over S_1 as long as we give S_1 the correct orientation. Thus

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_{S_1} \nabla \times \vec{F} \cdot d\vec{S}$$

where S_1 has the upward orientation.

Now we compute $\iint_{S_1} \nabla \times \vec{F} \cdot d\vec{S}$. The normal vector to S_1 is simply $\vec{k}$, so since all we need is $(\nabla \times F) \cdot \vec{k}$ the only thing that matters is the $\vec{k}$-component of $\nabla \times \vec{F}$. We have

$$\nabla \times \vec{F} = \text{curl}\vec{F} = (\text{something}, \text{something}, 0),$$

so $(\nabla \times \vec{F}) \cdot \vec{k} = 0$. Thus $\iint_{S_1} \nabla \times \vec{F} \cdot d\vec{S} = 0$, so our original surface integral is zero as well, a fact which is pretty much impossible to determine any other way. ■

Exercise 6 The goal of this problem is to prove a special case of Stokes' Theorem. Suppose S is the portion of the graph of $z = f(x, y)$, where f is C^2, for (x, y) in a compact region D in the xy-plane with boundary consisting of a single smooth curve. Thus S is parametrized by

$$\vec{X}(x, y) = (x, y, f(x, y)), \ (x, y) \in D.$$

Give S the upward orientation and ∂S the induced orientation. Let $\vec{x}(t) = (x(t), y(t)), \ a \le t \le b$ be parametric equations for ∂D and suppose that ∂S is parametrized by

$$(x(t), y(t), f(x(t), y(t))), \ a \le t \le b.$$

Let $\vec{F}$ be a C^1 vector field of the form $\vec{F} = (P, Q, R)$.

(a) Show that

$$\int_{\partial S} (P, Q, R) \cdot d\vec{s} = \int_{\partial D} (P(\vec{X}(x, y)) + R(\vec{X}(x, y)) f_x(x, y) \, , \, Q(\vec{X}(x, y)) + R(\vec{X}(x, y)) f_y(x, y)) \cdot d\vec{s}$$

(b) Use Green's Theorem to replace the right-hand-side of part (a) with an equivalent double integral over D. This will involve the use of the chain rule.

(c) Use the given parametrization for S to show that the double integral over D produced in part (b) is equal to

$$\iint_S (R_y - Q_z, P_z - R_x, Q_x - P_y) \cdot d\vec{S}.$$

The vector field $(R_y - Q_z, P_z - R_x, Q_x - P_y)$ is the curl of (P, Q, R), so we have shown that Stokes' Theorem holds in this special case.

(a) By the chain rule, the derivative of $z = f(x(t), y(t))$ with respect to t is

$$z'(t) = \frac{\partial f}{\partial x}(x(t), y(t)) x'(t) + \frac{\partial f}{\partial y}(x(t), y(t)) y'(t).$$

Thus we have

$$
\begin{aligned}
&\int_{\partial S} (P, Q, R) \cdot d\vec{s} \\
&\quad = \int_a^b \begin{bmatrix} P(x(t), y(t), f(x(t), y(t))) \\ Q(x(t), y(t), f(x(t), y(t))) \\ R(x(t), y(t), f(x(t), y(t))) \end{bmatrix} \cdot \begin{bmatrix} x'(t) \\ y'(t) \\ f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) \end{bmatrix} dt \\
&\quad = \int_a^b \Bigg(P(x(t), y(t), f(x(t), y(t)))x'(t) + Q(x(t), y(t), f(x(t), y(t)))y'(t) \\
&\qquad\qquad + R(x(t), y(t), f(x(t), y(t)))f_x(x(t), y(t))x'(t) \\
&\qquad\qquad + R(x(t), y(t), f(x(t), y(t)))f_y(x(t), y(t))y'(t) \Bigg) dt \\
&\quad = \int_a^b \begin{bmatrix} P(x(t), y(t), f(x(t), y(t))) + R(x(t), y(t), f(x(t), y(t)))f_x(x(t), y(t)) \\ Q(x(t), y(t), f(x(t), y(t))) + R(x(t), y(t), f(x(t), y(t)))f_y(x(t), y(t)) \end{bmatrix} \cdot \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} dt \\
&\quad = \int_{\partial D} (P(\vec{X}(x, y)) + R(\vec{X}(x, y))f_x(x, y), Q(\vec{X}(x, y)) + R(\vec{X}(x, y))f_y(x, y)) \cdot d\vec{s}
\end{aligned}
$$

as claimed.

(b) Green's Theorem gives

$$
\begin{aligned}
&\int_{\partial D} (P(\vec{X}(x, y)) + R(\vec{X}(x, y))f_x(x, y), Q(\vec{X}(x, y)) + R(\vec{X}(x, y))f_y(x, y)) \cdot d\vec{s} \\
&= \iint_D \left[\Big(Q(\vec{X}(x, y)) + R(\vec{X}(x, y))f_y(x, y) \Big)_x - \Big(P(\vec{X}(x, y)) + R(\vec{X}(x, y))f_x(x, y) \Big)_y \right] dA(x, y).
\end{aligned}
$$

The first term in brackets on the right is the derivative of

$$Q(x, y, f(x, y)) + R(x, y, f(x, y))f_y(x, y)$$

with respect to x. Differentiating $Q(x, y, f(x, y))$ requires the chain rule:

$$\Big(Q(x, y, f(x, y))) \Big)_x = Q_x(x, y, f(x, y)) + Q_z(x, y, f(x, y))f_x(x, y).$$

Differentiating $R(x, y, f(x, y))f_y(x, y)$ requires the product rule and chain rule:

$$
\begin{aligned}
&\Big(R(x, y, f(x, y))f_y(x, y)) \Big)_x \\
&\qquad = \Big(R(x, y, f(x, y)) \Big)_x f_y(x, y) + R(x, y, f(x, y))(f_y)_x(x, y) \\
&\qquad = (R_x(x, y, f(x, y)) + R_z(x, y, f(x, y))f_x(x, y))\, f_y(x, y) + R(x, y, f(x, y))f_{yx}(x, y).
\end{aligned}
$$

Thus overall (and omitting the evaluation of P, Q, R and their partial derivatives at $\vec{X}(x, y)$ and the evaluation of the partial derivatives of f at (x, y) to simplify the notation), we get

$$\Big(Q + Rf_y \Big)_x = Q_x + Q_z f_x + R_x f_y + R_z f_x f_y + Rf_{yx}.$$

A similar computation using the chain and product rules gives:

$$\Big(P + Rf_x \Big)_y = P_y + P_z f_y + R_y f_x + R_z f_y f_x + Rf_{xy}.$$

Putting everything together gives:

$$(Q + Rf_y)_x - (P + Rf_x)_y = Q_x - P_y + Q_z f_x - P_z f_y + R_x f_y - R_y f_x,$$

where we use the fact that f is C^2 to say that $Rf_{yx} - Rf_{xy} = 0$. Thus we have so far

$$\int_{\partial S} (P, Q, R) \cdot d\vec{s} = \iint_D [Q_x - P_y + Q_z f_x - P_z f_y + R_x f_y - R_y f_x] \, dA(x, y).$$

(c) Using the given parametrization for S, we see that normal vectors to S are given by

$$N_{\vec{X}}(x, y) = (1, 0, f_x(x, y)) \times (0, 1, f_y(x, y)) = (-f_x(x, y), -f_y(x, y), 1).$$

Rewriting the integrand in the double integral above as:

$$Q_x - P_y + Q_z f_x - P_z f_y + R_x f_y - R_y f_x = (R_y - Q_z, P_z - R_x, Q_x - P_y) \cdot (-f_x, -f_y, 1)$$

shows that

$$\begin{aligned} \int_{\partial S} (P, Q, R) \cdot d\vec{s} &= \iint_D (R_y - Q_z, P_z - R_x, Q_x - P_y) \cdot (-f_x, -f_y, 1) \, dA(x, y) \\ &= \iint_S (R_y - Q_z, P_z - R_x, Q_x - P_y) \cdot d\vec{S} \end{aligned}$$

which is exactly the conclusion of Stokes' Theorem!

■

Exercise 7 (Colley 7.3.26) Let $\vec{n}(x, y, z)$ be a unit normal vector to a smooth surface S. The directional derivative of a differentiable function $f(x, y, z)$ in the direction of $\vec{n}$ is called a **normal derivative** of f, denoted $\frac{\partial f}{\partial n}$. In particular, from our results on directional derivatives we have

$$\frac{\partial f}{\partial n} = \nabla f \cdot \vec{n}.$$

Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^2 function such that for any closed, oriented smooth surface S,

$$\iint_S \frac{\partial f}{\partial n} \, dS = 0.$$

Prove that f is **harmonic**, in the sense that $f_{xx} + f_{yy} + f_{zz} = 0$ throughout $\mathbb{R}^3$.

Fix $\vec{p}_0 \stackrel{def}{=} (x_0, y_0, z_0) \in \mathbb{R}^3$. Then since f is C^2, $f_{xx} + f_{yy} + f_{zz}$ is continuous at $\vec{p}_0$. Therefore we have (using the result from Exercise 4 on Homework 4 and Gauss' Theorem)

$$\begin{aligned}
f_{xx}(\vec{p}_0) + f_{yy}(\vec{p}_0) + f_{zz}(\vec{p}_0) &= \lim_{r\to 0+} \frac{1}{\mathrm{Vol}_3(B_r(\vec{p}_0))} \iiint\limits_{B_r(\vec{p}_0)} \Big(f_{xx}(\vec{x}) + f_{yy}(\vec{x}) + f_{zz}(\vec{x})\Big)\, dV(\vec{x}) \\
&= \lim_{r\to 0+} \frac{1}{\mathrm{Vol}_3(B_r(\vec{p}_0))} \iiint\limits_{B_r(\vec{p}_0)} \mathrm{div}(\nabla f)(\vec{x})\, dV(\vec{x}) \\
&= \lim_{r\to 0+} \frac{1}{\mathrm{Vol}_3(B_r(\vec{p}_0))} \iint\limits_{\partial B_r(\vec{p}_0)} \nabla f \cdot d\vec{S} \\
&= \lim_{r\to 0+} \frac{1}{\mathrm{Vol}_3(B_r(\vec{p}_0))} \iint\limits_{\partial B_r(\vec{p}_0)} \nabla f \cdot \vec{n} dS \\
&= \lim_{r\to 0+} \frac{1}{\mathrm{Vol}_3(B_r(\vec{p}_0))} \iint\limits_{\partial B_r(\vec{p}_0)} \frac{\partial f}{\partial n} dS \\
&= \lim_{r\to 0+} \frac{1}{\mathrm{Vol}_3(B_r(\vec{p}_0))} 0 \\
&= \lim_{r\to 0+} 0 \\
&= 0,
\end{aligned}$$

where in the antepenultimate step we used the face that the sphere $\partial B_r(\vec{p}_0)$ (oriented with outward-pointing normals) is a smooth, closed surface. ∎

Exercise 8 (Colley 7.3.20) Use Gauss's theorem to evaluate

$$\iint\limits_S \vec{F} \cdot d\vec{S}$$

where $\vec{F} = ze^{x^2}\,\vec{i} + 3y\,\vec{j} + (2 - yz^7)\,\vec{k}$ and S is the union of the five "upper" faces of the unit cube $[0,1] \times [0,1] \times [0,1]$, each oriented with normal vectors that point "away" from center of the cube $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Note that the $z = 0$ face is *not* part of S.

In order to be able to apply Gauss's theorem we need to "close off" S. Let S_1 be the bottom face of the cube with downward orientation, so that the closed surface $S \cup S_1$ has the outward orientation. We have:

$$\iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_{S \cup S_1} \vec{F} \cdot d\vec{S} - \iint\limits_{S_1} \vec{F} \cdot d\vec{S}.$$

Letting E denote the region enclosed by the combined surface $S \cup S_1$, we can compute the first integral on the right using Gauss's theorem:

$$\begin{aligned}
\iint_{S\cup S_1} \vec{F}\cdot d\vec{S} &= \iiint_E \operatorname{div}\vec{F}(x,y,z)\,dV(x,y,z) \\
&= \int_0^1\int_0^1\int_0^1 (2xze^{x^2}+3-7yz^6)\,dx\,dy\,dz \\
&= \int_0^1\int_0^1 (ze-z+3-7yz^6)\,dy\,dz \\
&= \int_0^1 \left(ze-z+3-\frac{7}{2}z^6\right)\,dz \\
&= \frac{e}{2}-\frac{1}{2}+3-\frac{1}{2} \\
&= 2+\frac{e}{2}.
\end{aligned}$$

For the integral over S_1, note that the normal vector to S_1 is $-\vec{k}$ and

$$\vec{F}(x,y,z)\cdot(-\vec{k}) = yz^7-2 = -2 \text{ since } z=0 \text{ on } S_1.$$

Thus

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_1} -2\,dS = -2$$

since S_1 has surface area 1. Putting it all together we get

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S\cup S_1} \vec{F}\cdot d\vec{S} - \iint_{S_1} \vec{F}\cdot d\vec{S} = 2+\frac{e}{2}-(-2) = 4+\frac{e}{2}.$$

■

Exercise 9 Let $\vec{F}$ be the vector field

$$\vec{F} = \frac{x\,\vec{i}+y\,\vec{j}+z\,\vec{k}}{(x^2+y^2+z^2)^{3/2}}.$$

Show that the surface integral of $\vec{F}$ over any closed, outward-oriented, smooth C^1 surface in $\mathbb{R}^3$ which encloses the origin is 4π.

(Hint: Show that the integral of $\vec{F}$ over any such surface is the same as the integral of $\vec{F}$ over a small-enough outward-oriented sphere centered at the origin.)

Let S be a closed, outward-oriented, smooth C^1 surface in $\mathbb{R}^3$ which encloses the origin, and let S_1 be a sphere centered at the origin of small enough radius R such that S_1 lies within the region enclosed by S. (So, S_1 is closer to the origin than S is.) Let E be the solid region lying between S and S_1. If we give S_1 the inward-pointing orientation, then $\partial E = S + S_1$ has the outward orientation. By Gauss's Theorem we get

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} = \iint_E \operatorname{div}\vec{F}\,dV(x,y,z).$$

Note that Gauss's Theorem is applicable since $\vec{F}$ is C^1 on E given that E excludes the origin. (This is why it is not possible to apply Gauss's Theorem directly to the region enclosed by S alone.)

We compute:

$$\begin{aligned}
\operatorname{div}\vec{F} &= \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)_x + \left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)_y + \left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)_z \\
&= \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} + \frac{(x^2+y^2+z^2)^{3/2} - 3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} \\
&\quad + \frac{(x^2+y^2+z^2)^{3/2} - 3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} \\
&= \frac{3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} \\
&= 0,
\end{aligned}$$

and thus

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} = 0.$$

Hence

$$\iint_S \vec{F}\cdot d\vec{S} + \iint_{S_1} \vec{F}\cdot d\vec{S} = 0, \text{ so } \iint_S \vec{F}\cdot d\vec{S} = -\iint_{S_1} \vec{F}\cdot d\vec{S}.$$

On S_1, the inward-pointing unit normal vectors are given by

$$\vec{n} = -\frac{x}{R}\vec{\imath} - \frac{y}{R}\vec{\jmath} - \frac{z}{R}\vec{k},$$

since at a point (x,y,z) on any sphere the vector (x,y,z) itself is normal to the sphere, and the extra factor of $-\frac{1}{R}$ corrects for the orientation and the length. Thus

$$-\iint_{S_1} \vec{F}\cdot d\vec{S} = -\iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \frac{(x,y,z)}{R^3}\cdot\frac{(x,y,z)}{R}\,dS = \frac{1}{R^4}\iint_{S_1}(x^2+y^2+z^2)\,dS = \frac{1}{R^2}\iint_{S_1} dS$$

where we use at multiple points the fact that $x^2+y^2+z^2 = R^2$ for points on S_1. This final integral gives the surface area of S_1, so it has value $4\pi R^2$ and hence

$$\iint_S \vec{F}\cdot d\vec{S} = -\iint_{S_1} \vec{F}\cdot d\vec{S} = 4\pi$$

as claimed. Note that the integral of $\vec{F}$ over S_1 can also be computed using spherical parametric equations. ∎

Exercise 10 Prove Gauss's Theorem in the special case where E is bounded by the surfaces $x = h_2(y,z)$ on the front and $x = h_1(y,z)$ on the back where $(y,z) \in D$ is the shadow of E in the yz-plane, and $\vec{F}$ has the form $\vec{F} = P(x,y,z)\,\vec{\imath}$.

Parametrize the front half of ∂E by

$$\vec{X}(h_2(y,z),y,z),\ (y,z) \in D$$

and the back half by

$$\vec{X}(h_1(y,z), y, z), \ \ (y,z) \in D.$$

From these we get that normal vectors to the front half of ∂E are given by

$$\left(\frac{\partial h_2}{\partial y}, 1, 0\right) \times \left(\frac{\partial h_2}{\partial z}, 0, 1\right) = \left(1, -\frac{\partial h_2}{\partial y}, -\frac{\partial h_2}{\partial z}\right)$$

and normal vectors to the back half are given by

$$\left(\frac{\partial h_1}{\partial y}, 1, 0\right) \times \left(\frac{\partial h_1}{\partial z}, 0, 1\right) = \left(1, -\frac{\partial h_1}{\partial y}, -\frac{\partial h_1}{\partial z}\right).$$

However, this choice of parametric equations gives the wrong orientation on the back half of ∂E, since the computed normals have positive x-component, meaning that this normals point "into" E rather than "out" of E. After correcting for the orientation we get that

$$\begin{aligned}
\iint\limits_{\partial E} (P,0,0) \cdot d\vec{S} &= \iint\limits_{\text{front}} (P,0,0) \cdot d\vec{S} + \iint\limits_{\text{back}} (P,0,0) \cdot d\vec{S} \\
&= \iint\limits_{D} (P,0,0) \cdot \left(1, -\frac{\partial h_2}{\partial y}, -\frac{\partial h_2}{\partial z}\right) dy\, dz - \iint\limits_{D} (P,0,0) \cdot \left(1, -\frac{\partial h_1}{\partial y}, -\frac{\partial h_1}{\partial z}\right) dy\, dz \\
&= \iint\limits_{D} [P(h_2(y,z), y, z) - P(h_1(y,z), y, z)]\, dy\, dz.
\end{aligned}$$

By the Fundamental Theorem of Calclulus,

$$P(h_2(y,z), y, z) - P(h_1(y,z), y, z) = \int_{h_1(y,z)}^{h_2(y,z)} P_x(x,y,z)\, dx,$$

so

$$\iint\limits_{\partial E} (P,0,0) \cdot d\vec{S} = \iint\limits_{D} \left(\int_{h_1(y,z)}^{h_2(y,z)} P_x(x,y,z)\, dx\right) dy\, dz = \iiint\limits_{E} P_x(x,y,z)\, dV(x,y,z)$$

which is the conclusion of Gauss' Theorem. ∎