Northwestern University

MATH 291-3 Final Examination - Practice B Solutions
Spring Quarter 2022
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Last name: SOLUTIONS ______________________ Email address: ______________________

First name: ______________________ NetID: ______________________

Instructions

- This examination consists of 7 questions.
- Read all problems carefully before answering.
- You have 120 minutes to complete this examination.
- Do not use books, notes, calculators, computers, tablets, phones, smart watches, or similar devices.
- Possession of a digital communication device during a bathroom break will be treated as a *prima facie* violation of Northwestern's academic integrity policy.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is true or false. If true, then explain why. If false, then give a counterexample.

   (a) If a bounded function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous everywhere except on the ellipse $2x^2 + 3y^2 = 4$, then

$$\int_{-5}^{5} \int_{-6}^{6} f(x, y)\, dxdy = \int_{-6}^{6} \int_{-5}^{5} f(x, y)\, dydx.$$

   (b) The sum

$$\int_{-1}^{0} \int_{-x}^{\sqrt{2-x^2}} x^2 y\, dydx + \int_{0}^{1} \int_{x}^{\sqrt{2-x^2}} x^2 y\, dydx$$

   can be written as a single iterated integral in polar coordinates.

   (c) Suppose that $D \subset \mathbb{R}^2$ is a region whose boundary consists of a simple piecewise-smooth closed curve C. Then the value of the line integral $\oint_C -y\, dx + (y^2 + x)\, dy$ depends only on the area of D.

Solution: (a) is true. To see why, note that f is integrable on the box $[-6, 6] \times [-5, 5]$ by Lebesgue's Criterion for Riemann Integrability because the ellipse $2x^2 + 3y^2 = 4$ has measure zero. Also, for each fixed x the function $y \mapsto f(x, y)$ has at most 2 points of discontinuity, and therefore is integrable on $[-5, 5]$. Similarly, for each fixed y the function $x \mapsto f(x, y)$ has at most 2 points of discontinuity, and therefore is integrable on $[-6, 6]$. Therefore f satisfies the hypotheses of Fubini's Theorem, so that each of the integrals in the problem statement are equal to $\iint_{[-6,6]\times[-5,5]} f\, dA$.

(b) is true. In polar coordinates, this is the integral of the function $x^2 y = r^3 \cos^2(\theta) \sin(\theta)$ (times the Jacobian factor r) over the region described by $0 \le r \le \sqrt{2}$ and $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$. Therefore the integral can be written as

$$\int_{0}^{\sqrt{2}} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} r^3 \cos^2(\theta) \sin(\theta) r\, d\theta dr.$$

(c) is false. Suppose that D is the unit disc $x^2 + y^2 \le 1$, so that C is the unit circle $x^2 + y^2 = 1$. Then Green's Theorem implies that

$$\oint_C -y\, dx + (y^2 + x)\, dy = \pm \iint_D 2\, dA = \pm 2\pi,$$

where we have $+2\pi$ if C is oriented counterclockwise, and -2π if C is oriented clockwise. Therefore the value of the line integral might also depend on the orientation of C.

2. Consider the following iterated integral (in spherical coordinates):

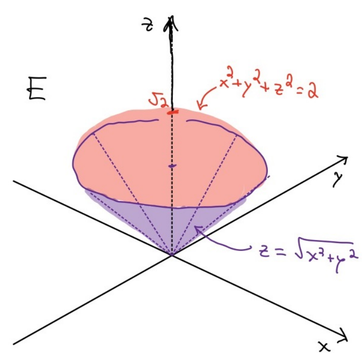
$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} \rho^3 \sin^2(\phi)\cos(\theta)\, d\rho d\phi d\theta.$$

(a) Rewrite this as a **single** iterated integral in rectangular coordinates.

(b) Rewrite this as a **sum** of iterated integrals in cylindrical coordinates.

The point is that you have to determine for yourself which orders of integration give a single integral in (a) and a sum of integrals in (b).

Solution: Since the Jacobian of the spherical coordinate change of variables is $\rho^2\sin(\phi)$, the given iterated integral represents the triple integral of $x = \rho\sin(\phi)\cos(\theta)$ over a region E in $\mathbb{R}^3$. Since $0 \le \theta \le 2\pi$, the shape of E is completely determined by the bounds in ρ and ϕ. Since $0 \le \rho \le \sqrt{2}$, E is enclosed by the sphere $x^2+y^2+z^2 = 2$ of radius $\sqrt{2}$ centered at the origin. The restriction $0 \le \phi \le \frac{\pi}{4}$ says that E also lies above the cone $z = \sqrt{x^2+y^2}$. We sketch E below.



In rectangular coordinates, we can express this triple integral as a single iterated integral in the order $dzdydx$. To see why, note that the shadow of E in the xy-plane is exactly the unit disc $x^2+y^2 \le 1$ (and therefore we have $-1 \le x \le 1$ and $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$). For each of x and y in this disc, z runs from the cone $z = \sqrt{x^2+y^2}$ to the sphere $z = \sqrt{2-x^2-y^2}$. Therefore we can write this as a single iterated integral as

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} x\, dzdydx.$$

In cylindrical coordinates, we see that if r is the 'innermost' integration variable, then the upper bound of r changes from $r = z$ (when $0 \le z \le 1$) to $r = \sqrt{2-z^2}$ (when $1 \le z \le \sqrt{2}$). Therefore we can express the triple integral as a sum of iterated integrals in cylindrical coordinates as

$$\int_0^{2\pi}\int_0^{1}\int_0^{z} r^2\cos(\theta)\, drdzd\theta + \int_0^{2\pi}\int_1^{\sqrt{2}}\int_0^{\sqrt{2-z^2}} r^2\cos(\theta)\, drdzd\theta.$$

3. Let D be the region in $\mathbb{R}^2$ enclosed by the circle with equation $x^2 + y^2 = 4x$. Show that

$$\iint_D (y^{101} + \sqrt{x^2 + y^2})\, dA(x, y) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{64}{3} \cos^3(\theta)\, d\theta.$$

You do **NOT** need to compute the integral on the right.

Solution: D is enclosed by the circle $(x-2)^2 + y^2 = 4$ of radius 2 centered at $(2, 0)$, and therefore is symmetric about the x-axis. Because y^{101} is an odd function of y, we immediately have

$$\iint_D y^{101}\, dA(x, y) = 0,$$

and therefore

$$\iint_D (y^{101} + \sqrt{x^2 + y^2})\, dA(x, y) = \iint_D \sqrt{x^2 + y^2}\, dA(x, y).$$

We convert to polar coordinates. Note that the disc D contains at least one point on the ray from $(0, 0)$ through (x, y) as long as $x > 0$, and therefore we can take $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. For each θ in this interval r runs from 0 (the origin) until it hits the circle, where $r^2 = x^2 + y^2 = 4x = 4r\cos(\theta)$. This equation is satisfied when $r = 0$ (since the circle passes through the origin) and when $r = 4\cos(\theta)$. Therefore $0 \le r \le 4\cos(\theta)$, and we have

$$\begin{aligned}
\iint_D (y^{101} + \sqrt{x^2 + y^2})\, dA(x, y) &= \iint_D \sqrt{x^2 + y^2}\, dA(x, y) \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{4\cos(\theta)} r \cdot r\, dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right|_0^{4\cos(\theta)} d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{64}{3} \cos^3(\theta)\, d\theta
\end{aligned}$$

as desired.

4. Prove the **Fundamental Theorem of Line Integrals** for smooth curves: If C is a smooth, oriented curve in $\mathbb{R}^n$ that starts at $\vec{a} \in \mathbb{R}^n$ and ends at $\vec{b} \in \mathbb{R}^n$, and if $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 on C, then

$$\int_C \nabla f \cdot d\vec{s} = f(\vec{b}) - f(\vec{a}).$$

Solution: Let $\vec{x} : [a, b] \to \mathbb{R}^n$ be a C^1 parametrization of C with $\vec{x}'(t) \neq \vec{0}$ for each t, and such that $\vec{x}(b) = \vec{b}$ and $\vec{x}(a) = \vec{a}$. Then

$$\begin{aligned}
\int_C \nabla f \cdot d\vec{s} &= \int_a^b \nabla f(\vec{x}(t)) \cdot \vec{x}'(t)\, dt \\
&= \int_a^b Df(\vec{x}(t)) D\vec{x}(t)\, dt \\
&= \int_a^b \frac{d}{dt}[f(\vec{x}(t))]\, dt \\
&= f(\vec{x}(b)) - f(\vec{x}(a)) \\
&= f(\vec{b}) - f(\vec{a}).
\end{aligned}$$

5. Suppose that $S \subset \mathbb{R}^3$ is a smooth oriented surface and that $\vec{X} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ and $\vec{Y} : E \subset \mathbb{R}^2 \to \mathbb{R}^3$ are two orientation preserving parametrizations of S. Suppose further that $\vec{Y} = \vec{X} \circ T$ for some C^1 bijective function $T : E \to D$ such that $DT(s,t)$ is invertible throughout E. Prove that

$$\iint_E \vec{Y}(s,t) \cdot N_{\vec{Y}}(s,t)\, dA(s,t) = \iint_D \vec{X}(u,v) \cdot N_{\vec{X}}(u,v)\, dA(u,v).$$

You may take it for granted (as proved earlier in the course) that $N_{\vec{Y}}(s,t) = \det DT(s,t) N_{\vec{X}}(T(s,t))$.

Solution: First note that since $\vec{X}$ and $\vec{Y}$ are orientation-preserving parametrizations, we have

$$\frac{1}{\|N_{\vec{X}}(T(s,t))\|} N_{\vec{X}}(T(s,t)) = \vec{n}(\vec{X}(T(s,t))) = \vec{n}(\vec{Y}(s,t)) = \frac{1}{\|N_{\vec{Y}}(s,t)\|} N_{\vec{Y}}(s,t)$$

at each $(s,t) \in E$, where $\vec{n}(\vec{p})$ is the unit normal vector to S at $\vec{p} \in S$ given by the orientation of S.

Note that since $D = T(E)$, and since T satisfies the hypotheses of the Change of Variables Theorem, and since $\vec{X}(u,v) \cdot (\vec{X}_u(u,v) \times \vec{X}_v(u,v))$ is continuous (and therefore integrable) on D, the Change of Variables Theorem implies that

$$\begin{aligned}
&\iint_D \vec{X}(u,v) \cdot N_{\vec{X}}(u,v)\, dA(u,v) \\
&\qquad = \iint_E \vec{X}(T(s,t)) \cdot N_{\vec{X}}(T(s,t)) |\det DT(s,t)|\, dA(s,t) \\
&\qquad = \iint_E (\vec{X} \circ T)(s,t) \cdot N_{\vec{X}}(T(s,t)) |\det DT(s,t)|\, dA(s,t) \\
&\qquad = \iint_E ((\vec{X} \circ T)(s,t) \cdot \vec{n}(\vec{X}(T(s,t)))) \|(\det DT(s,t)) N_{\vec{X}}(T(s,t))\|\, dA(s,t) \\
&\qquad = \iint_E (\vec{Y}(s,t) \cdot \vec{n}(\vec{Y}(s,t))) \|N_{\vec{Y}}(s,t)\|\, dA(s,t) \\
&\qquad = \iint_E \vec{Y}(s,t) \cdot N_{\vec{Y}}(s,t)\, dA(s,t).
\end{aligned}$$

6. Suppose $S \subset \mathbb{R}^3$ is a smooth closed surface and that $\vec{F}$ is C^1 on an open set containing S. Show that $\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = 0$. If needed, you may use without proof the fact that such a surface S can be written as the union of two smooth non-closed surfaces with the same boundary, and that this boundary consists of a single closed piecewise-smooth curve.

Solution: Write $S = S_1 \cup S_2$, where S_1 and S_2 are smooth oriented surfaces such that $\partial S_1 = C = \partial S_2$ for a piecewise-smooth closed curve C. Orient C so that it has the orientation induced by the orientation of S_1. Then S_1 is "on the left" when viewed from "above" (where "above" is the direction of the orientation of S) as one traverses C. Therefore S_2 is "on the right" as one traverses C, so that $-C$ has the orientation induced by the orientation of S_2. We can therefore apply Stokes' Theorem to see that

$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = \iint_{S_1} \operatorname{curl}\vec{F} \cdot d\vec{S} + \iint_{S_2} \operatorname{curl}\vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{s} + \oint_{-C} \vec{F} \cdot d\vec{s} = \oint_C \vec{F} \cdot d\vec{s} - \oint_C \vec{F} \cdot d\vec{s} = 0.$$

7. Suppose that $E \subset \mathbb{R}^3$ is a region whose boundary is a smooth closed surface S, that $u : \mathbb{R}^3 \to \mathbb{R}$ is C^2 with

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

on all of E, and that $u(x, y, z) = 0$ for every $(x, y, z) \in S$. Prove that $u(x, y, z) = 0$ for every $(x, y, z) \in E$.

(Hint: Apply Gauss's Theorem to a well-chosen vector field.)

Solution: Give S the "outward" orientation relative to E. Consider the vector field $\vec{F} \overset{def}{=} uu_x\vec{\imath} + uu_y\vec{\jmath} + uu_z\vec{k}$. Then $\vec{F}$ is C^1 on $\mathbb{R}^3$, and $\vec{F} = \vec{0}$ on S because $u = 0$ on S. Therefore Gauss's Theorem implies that

$$\begin{aligned}
0 &= \iint_S \vec{0} \cdot d\vec{S} \\
&= \iint_S \vec{F} \cdot d\vec{S} \\
&= \iiint_E \operatorname{div}\vec{F}\, dV \\
&= \iiint_E (uu_{xx} + (u_x)^2 + uu_{yy} + (u_y)^2 + uu_{zz} + (u_z)^2)\, dV \\
&= \iiint_E (u \cdot (u_{xx} + u_{yy} + u_{zz}) + \|\nabla u\|^2)\, dV \\
&= \iiint_E \|\nabla u\|^2\, dV.
\end{aligned}$$

Because $\|\nabla u(x, y, z)\|^2$ is continuous and non-negative, if $\|\nabla u(x, y, z)\|^2 > 0$ at any point then we would have $\iiint_E \|\nabla u\|^2\, dV > 0$. But this is not the case, so that $\|\nabla u(x, y, z)\|^2 = 0$ for every $(x, y, z) \in E$. It follows that $\nabla u(x, y, z) = \vec{0}$ throughout E, so that $Du(x, y, z) = [0\,0\,0]$ throughout E. Therefore u is constant on E. By continuity and the fact that $u = 0$ on $S = \partial E$, we have $u = 0$ throughout E.