Math 291-3: Discussion #4 Problems (Solutions)
Northwestern University, Spring 2022

1. Suppose that $\vec{X}(s,t)$ is a C^1 parametrization of a surface S, and let $N_{\vec{X}}(s_0,t_0)$ denote the normal vector to S at $\vec{X}(s_0,t_0)$ arising from the parametrization $\vec{X}$. Consider the C^1 parametrization of S given by $\vec{Y}(t,s) \stackrel{def}{=} \vec{X}(s,t)$. Show that the normal vector $N_{\vec{Y}}(t_0,s_0)$ to S at $\vec{Y}(t_0,s_0) = \vec{X}(s_0,t_0)$ arising from the parametrization $\vec{Y}$ satisfies $N_{\vec{Y}}(t_0,s_0) = -N_{\vec{X}}(s_0,t_0)$.

   Solution. Note that
   $$\vec{Y}_t(t,s) = \frac{\partial}{\partial t}[\vec{Y}(t,s)] = \frac{\partial}{\partial t}[\vec{X}(s,t)] = \vec{X}_t(s,t)$$
   and, similarly, $\vec{Y}_s(t,s) = \vec{X}_s(s,t)$. Therefore we have
   $$N_{\vec{Y}}(t_0,s_0) = \vec{Y}_t(t_0,s_0)\times\vec{Y}_s(t_0,s_0) = \vec{X}_t(s_0,t_0)\times\vec{X}_s(s_0,t_0) = -\vec{X}_s(s_0,t_0)\times\vec{X}_t(s_0,t_0) = -N_{\vec{X}}(s_0,t_0).$$

2. Suppose $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field, and that $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^1 function. Show that
   $$\nabla \cdot (f(\vec{x})\vec{F}(\vec{x})) = f(\vec{x})(\nabla \cdot \vec{F})(\vec{x}) + \vec{F}(\vec{x}) \cdot \nabla f(\vec{x})$$
   and
   $$\nabla \times (f(\vec{x})\vec{F}(\vec{x})) = f(\vec{x})(\nabla \times \vec{F})(\vec{x}) + \nabla f(\vec{x}) \times \vec{F}(\vec{x}).$$
   To be clear, in this notation if $\vec{G}$ is a vector field, then $f(\vec{x})\vec{G}(\vec{x})$ is the vector field obtained by multiplying each component of $\vec{G}(\vec{x})$ by $f(\vec{x})$. These two equalities are the analogs of the product rules for divergence and curl.

   Solution. Let F_1, F_2, F_3 denote the component functions of F. Then
   $$\begin{aligned}\nabla \cdot (f(\vec{x})\vec{F}(\vec{x})) &= \sum_{k=1}^{3} \frac{\partial}{\partial x_k}(f(\vec{x})F_k(\vec{x}))\\ &= \sum_{k=1}^{3} f_{x_k}(\vec{x})F_k(\vec{x}) + \sum_{k=1}^{3} f(\vec{x})(F_k)_{x_k}(\vec{x})\\ &= \nabla f(\vec{x}) \cdot \vec{F}(\vec{x}) + f(\vec{x})\sum_{k=1}^{3}(F_k)_{x_k}(\vec{x})\\ &= \vec{F}(\vec{x}) \cdot \nabla f(\vec{x}) + f(\vec{x})(\nabla \cdot \vec{F})(\vec{x})\end{aligned}$$
   and
   $$\nabla \times (f(\vec{x})\vec{F}(\vec{x})) = \begin{bmatrix} \frac{\partial}{\partial x_2}(f(\vec{x})F_3(\vec{x})) - \frac{\partial}{\partial x_3}(f(\vec{x})F_2(\vec{x})) \\ \frac{\partial}{\partial x_3}(f(\vec{x})F_1(\vec{x})) - \frac{\partial}{\partial x_1}(f(\vec{x})F_3(\vec{x})) \\ \frac{\partial}{\partial x_1}(f(\vec{x})F_2(\vec{x})) - \frac{\partial}{\partial x_2}(f(\vec{x})F_1(\vec{x})) \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} f_{x_2}(\vec{x})F_3(\vec{x}) + f(\vec{x})(F_3)_{x_2}(\vec{x}) - f_{x_3}(\vec{x})F_2(\vec{x}) - f(\vec{x})(F_2)_{x_3}(\vec{x}) \\ f_{x_3}(\vec{x})F_1(\vec{x}) + f(\vec{x})(F_1)_{x_3}(\vec{x}) - f_{x_1}(\vec{x})F_3(\vec{x}) - f(\vec{x})(F_3)_{x_1}(\vec{x}) \\ f_{x_1}(\vec{x})F_2(\vec{x}) + f(\vec{x})(F_2)_{x_1}(\vec{x}) - f_{x_2}(\vec{x})F_1(\vec{x}) - f(\vec{x})(F_1)_{x_2}(\vec{x}) \end{bmatrix} \\
&= \begin{bmatrix} f_{x_2}(\vec{x})F_3(\vec{x}) - f_{x_3}(\vec{x})F_2(\vec{x}) \\ f_{x_3}(\vec{x})F_1(\vec{x}) - f_{x_1}(\vec{x})F_3(\vec{x}) \\ f_{x_1}(\vec{x})F_2(\vec{x}) - f_{x_2}(\vec{x})F_1(\vec{x}) \end{bmatrix} + f(\vec{x}) \begin{bmatrix} (F_3)_{x_2}(\vec{x}) - (F_2)_{x_3}(\vec{x}) \\ (F_1)_{x_3}(\vec{x}) - (F_3)_{x_1}(\vec{x}) \\ (F_2)_{x_1}(\vec{x}) - (F_1)_{x_2}(\vec{x}) \end{bmatrix} \\
&= (\nabla f(\vec{x})) \times \vec{F}(\vec{x}) + f(\vec{x})(\nabla \times \vec{F})(\vec{x}).
\end{aligned}
$$

3. Suppose C is a (portion of a) flow line of a C^1 nowhere-vanishing vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$. Let $\vec{T}(\vec{x})$ denotes the unit tangent vector to C (in the direction of flow) at the point $\vec{x} \in C$. Then the expression $\vec{F}(\vec{x}) \cdot \vec{T}(\vec{x})$ describes a function from C to $\mathbb{R}$. Why is the value of the integral
$$\int_C (\vec{F}(\vec{x}) \cdot \vec{T}(\vec{x}))\, ds$$
positive? (Recall that this is a *scalar line integral*, which for a function $f : C \to \mathbb{R}$ is defined via the formula
$$\int_C f(\vec{x})\, ds \stackrel{def}{=} \int_a^b f(\vec{r}(t))\|\vec{r}'(t)\|\, dt$$
where $\vec{r} : [a, b] \to \mathbb{R}^n$ is a parametrization of C.)

Solution. Let $\vec{x} : [a, b] \to \mathbb{R}^n$ be a parametrization of C satisfying $\vec{x}'(t) = \vec{F}(\vec{x}(t))$. Then since $\vec{F}(\vec{x}) \neq \vec{0}$ at every $\vec{x} \in \mathbb{R}^n$, we have
$$\vec{T}(\vec{x}(t)) = \frac{1}{\|\vec{F}(\vec{x}(t))\|}\vec{F}(\vec{x}(t)) = \frac{1}{\|\vec{x}'(t)\|}\vec{F}(\vec{x}(t)),$$
so that
$$\int_C (\vec{F}(\vec{x}) \cdot \vec{T}(\vec{x}))\, ds = \int_a^b (\vec{F}(\vec{x}(t)) \cdot \vec{T}(\vec{x}(t)))\|\vec{x}'(t)\|\, dt = \int_a^b \|\vec{F}(\vec{x}(t))\|^2\, dt > 0$$
since $\|\vec{F}(\vec{x}(t))\|^2$ is continuous and positive throughout the interval $[a, b]$.