Math 291-3: Discussion #2 Problems (Solutions)

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1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) \overset{def}{=} \begin{cases} x+y & \text{if } |x|+|y| \leq 1, \\ x-y & \text{if } |x|+|y| > 1. \end{cases}$$

Show that f is integrable over $[-5,5] \times [-5,5]$ and that

$$\iint_{[-5,5]\times[-5,5]} f(x,y)\, dA = \int_{-5}^{5} \int_{-5}^{5} f(x,y)\, dxdy.$$

Solution. Because the functions $x+y$ and $x-y$ are continuous on $\mathbb{R}^2$, the points at which f is possibly discontinuous satisfy $|x|+|y| = 1$, and therefore lie on the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$. Note that the line segment connecting $(1,0)$ to $(0,1)$ is a subset of the line L described by $x+y=1$. This line is the image of the C^1 function $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ given by $\vec{r}(t) = (t, 1-t)$, and therefore the Measure Zero Theorem implies that L (and therefore the line segment connecting $(1,0)$ to $(0,1)$) has measure zero. A similar argument works for the other line segments as well, and so the Measure Zero Theorem implies that the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$ has measure zero. Therefore f is integrable on $[-5,5] \times [-5,5]$ by the Lebesgue Criterion for Riemann Integrability. Moreover, note that every horizontal or vertical line intersects with the square in either 0, 1, or 2 points. Therefore f satisfies the hypotheses of Fubini's Theorem on $[-5,5] \times [-5,5]$, so that

$$\iint_{[-5,5]\times[-5,5]} f(x,y)\, dA = \int_{-5}^{5} \int_{-5}^{5} f(x,y)\, dxdy.$$

2. Compute the double integral in Problem 1.

Solution. There are many ways to compute this integral, but here is an approach using Riemann sums instead of iterated integrals. Note that f is odd, in the sense that $f(-x,-y) = -f(x,y)$. Moreover, $(x,y) \in [-5,5] \times [-5,5]$ if and only if $(-x,-y) \in [-5,5] \times [-5,5]$. Therefore we might expect that the integral of f over this box is 0. To argue this, note that because f is integrable over $[-5,5] \times [-5,5]$, we can evaluate f as the limit of Riemann sums for *any* sequence of partitions $\mathcal{P}_n$ of $[-5,5] \times [-5,5]$ with $\|\mathcal{P}_n\| \to 0$ as $n \to \infty$. For each n, choose the partition $\mathcal{P}_n$ of $[-5,5] \times [-5,5]$ so that each box in the partition has edges of length $\frac{5}{n}$. Note that $\mathcal{P}_n$ will consist of 4^n smaller boxes. For each box B_i in $\mathcal{P}_n$, the reflection of B_i through the origin $B_i^* = \{(-x,-y) \ : \ (x,y) \in B_i\}$ will be another box in $\mathcal{P}_n$. We can therefore arrange it so that if $\vec{c}_i$ is the sample point we choose in B_i, then $\vec{c}_i^* \overset{def}{=} -\vec{c}_i$ is the sample point for B_i^*. Let $\mathcal{C}_n$ denote this particular choice of sample points. Then the summands in the Riemann sum $R(f, \mathcal{P}_n, \mathcal{C}_n)$ can be split into pairs of the form

$$f(\vec{c}_i)\mathrm{Vol}_2(B_i) + f(\vec{c}_i^*)\mathrm{Vol}_2(B_i^*) = f(\vec{c}_i)\mathrm{Vol}_2(B_i) - f(\vec{c}_i)\mathrm{Vol}_2(B_i) = 0.$$

Therefore $R(f, \mathcal{P}_n, \mathcal{C}_n) = 0$. Because $\|\mathcal{P}_n\| = \frac{5}{n} \to 0$ as $n \to \infty$, we therefore have

$$\iint_{[-5,5]\times[-5,5]} f(x,y)\, dA = \lim_{n\to\infty} R(f, \mathcal{P}_n, \mathcal{C}_n) = \lim_{n\to\infty} 0 = 0.$$

Solution. Here is a way to compute the double-integral using iterated integrals. Note that $f(x,y) = x+y$ if $|y|-1 \le x \le 1-|y|$, and $f(x,y) = x-y$ if $x > 1-|y|$ or $x < |y|-1$. This second case automatically holds when $|y| > 1$. Therefore we can write

$$\begin{aligned}
\iint_{[-5,5]\times[-5,5]} f(x,y)\, dA &= \int_{-5}^{5}\int_{-5}^{5} f(x,y)\, dxdy \\
&= \int_{-5}^{-1}\int_{-5}^{5}(x-y)\, dxdy + \int_{-1}^{1}\int_{-5}^{5} f(x,y)\, dxdy + \int_{1}^{5}\int_{-5}^{5}(x-y)\, dxdy \\
&= \int_{-5}^{-1}(-10y)\, dy + \int_{-1}^{1}\int_{-5}^{5} f(x,y)\, dxdy + \int_{1}^{5}(-10y)\, dy.
\end{aligned}$$

Because

$$\underbrace{\int_{-5}^{-1}(-10y)\, dy}_{y=-u,\ dy=-du} = \int_{5}^{1} 10u(-1)du = \int_{1}^{5} 10udu = -\int_{1}^{5}(-10y)\, dy,$$

we have

$$\begin{aligned}
\iint_{[-5,5]\times[-5,5]} f(x,y)\, dA &= \int_{-1}^{1}\int_{-5}^{5} f(x,y)\, dxdy \\
&= \int_{-1}^{1}\int_{-5}^{|y|-1}(x-y)\, dxdy + \int_{-1}^{1}\int_{|y|-1}^{1-|y|}(x+y)\, dxdy + \int_{-1}^{1}\int_{1-|y|}^{5}(x-y)\, dxdy \\
&= \int_{-1}^{1}\Big(\frac{(|y|-1)^2-25}{2} - y(4+|y|)\Big)\, dy + \int_{-1}^{1} y(2-2|y|)dy \\
&\qquad + \int_{-1}^{1}\Big(\frac{25-(1-|y|)^2}{2} - y(4+|y|)\Big)dy \\
&= \int_{-1}^{1}\Big(-2y(4+|y|) + y(2-2|y|)\Big)\, dy \\
&= 0,
\end{aligned}$$

where the last step follows because we are computing the integral of an odd function $-6y - 4y|y|$ over the symmetric interval $[-1,1]$.

3. Set up, but do not evaluate, the iterated integrals (both of them) which give the integral of the function f in Problem 1 over the region D in $\mathbb{R}^2$ consisting of the left half of the closed unit disc $x^2+y^2 \leq 1$, the portion of the unit disc in the first quadrant above the line $y=x$, and the portion of the unit disc in the fourth quadrant below the line $y=-x$. Find a way to do this where the integrand of each iterated integral used is either $x+y$ or $x-y$; in other words, "$f(x,y)$" should not appear as the integrand of any iterated integral.

Solution. This is accomplished as follows:

$$\begin{aligned}\iint\limits_D f(x,y)\,dA &= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{|y|-1}(x-y)\,dxdy + \int_{-1}^{-1/\sqrt{2}}\int_{y+1}^{\sqrt{1-y^2}}(x-y)\,dxdy + \int_{-1/\sqrt{2}}^{-1/2}\int_{y+1}^{-y}(x-y)\,dxdy \\ &+ \int_{1/2}^{1/\sqrt{2}}\int_{1-y}^{y}(x-y)\,dxdy + \int_{1/\sqrt{2}}^{1}\int_{1-y}^{\sqrt{1-y^2}}(x-y)\,dxdy \\ &+ \int_{-1}^{-1/2}\int_{-1-y}^{1+y}(x+y)\,dxdy + \int_{-1/2}^{1/2}\int_{|y|-1}^{|y|}(x+y)\,dxdy + \int_{1/2}^{1}\int_{y-1}^{1-y}(x+y)\,dxdy.\end{aligned}$$

We also have

$$\begin{aligned}\iint\limits_D f(x,y)\,dA &= \int_{-1}^{0}\int_{-x-1}^{x+1}(x+y)\,dydx + \int_{0}^{1/2}\int_{x}^{1-x}(x+y)\,dydx + \int_{0}^{1/2}\int_{x-1}^{-x}(x+y)\,dydx \\ &+ \int_{1/2}^{-1}\int_{1-|x|}^{\sqrt{1-x^2}}(x-y)\,dydx + \int_{1/2}^{1/\sqrt{2}}\int_{x}^{\sqrt{1-x^2}}(x-y)\,dydx \\ &+ \int_{1/2}^{-1}\int_{-\sqrt{1-x^2}}^{|x|-1}(x-y)\,dydx + \int_{1/2}^{1/\sqrt{2}}\int_{-\sqrt{1-x^2}}^{-x}(x-y)\,dydx.\end{aligned}$$