Solutions
Math 291-3
Homework 6
May 16, 2022

**Exercise 1** Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$. Denote the coordinates of the domain by $(u_1, \dots, u_n)$ the coordinates of the codomain by $(x_1, \dots, x_n)$, and the components of $T$ by

$$T(u_1, \dots, u_n) = (x_1(u_1, \dots, u_n), \dots, x_n(u_1, \dots, u_n)).$$

Show that

$$T^*(dx_1 \wedge \cdots \wedge dx_n) = (\det DT(u_1, \dots, u_n))\, du_1 \wedge \cdots \wedge du_n.$$

Hint: This is most easily done with the pattern characterization of the determinant.

(Solution using Pattern Characterization of Determinant) For each $i$, we have:

$$dx_i = \frac{\partial x_i}{\partial u_1}\, du_1 + \cdots + \frac{\partial x_i}{\partial u_n}\, du_n,$$

and thus

$$dx_1 \wedge \cdots \wedge dx_n = \left( \sum_j \frac{\partial x_1}{\partial u_j}\, du_j \right) \wedge \cdots \wedge \left( \sum_j \frac{\partial x_n}{\partial u_j}\, du_j \right).$$

When distributing all the resulting wedge products, note that anytime we have an $n$-fold wedge product which involves the same differential $du_j$ appearing more than once, this particular wedge product will be zero. Thus, the only nonzero wedge products obtained after distributing everything out are those of the form

$$du_{i_1} \wedge \cdots \wedge du_{i_n}$$

where $u_{i_1}, \dots, u_{i_n}$ are $u_1, \dots, u_n$ in some order. The coefficient of this particular term is

$$\frac{\partial x_1}{\partial u_{i_1}} \frac{\partial x_2}{\partial u_{i_2}} \cdots \frac{\partial x_n}{\partial u_{i_n}},$$

so overall the resulting wedge product looks like

$$dx_1 \wedge \cdots \wedge dx_n = \sum_{i_1, \dots, i_n} \left( \frac{\partial x_1}{\partial u_{i_1}} \cdots \frac{\partial x_n}{\partial u_{i_n}} \right) du_{i_1} \wedge \cdots \wedge du_{i_n}$$

where the sum is taken over all possible orderings $i_1, \dots, i_n$ of $1, \dots, n$.

Now, consider the matrix $DT(u_1, \dots, u_n)$:

$$DT(u_1, \dots, u_n) = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_n} \end{bmatrix}.$$

Each coefficient

$$\frac{\partial x_1}{\partial u_{i_1}} \cdots \frac{\partial x_n}{\partial u_{i_n}}$$

in the summation above comes from taking a single entry from each row and column of $DT$ and multiplying them together, which is what we referred to last quarter as a *pattern* of this matrix. Thus, the wedge product expression above is a sum over all possible patterns of $DT$:

$$dx_1 \wedge \cdots \wedge dx_n = \sum_{n \times n \text{ patterns}} (\text{products of terms in pattern in } DT(\vec{u}))\, du_{i_1} \wedge \cdots \wedge du_{i_n}.$$

Finally, to put $du_{i_1} \wedge \cdots \wedge du_{i_n}$ into the order $du_1 \wedge \cdots \wedge du_n$ requires swapping terms, and the total number of such swaps required is the same as the number of swaps required to put the corresponding pattern into the standard "diagonal" form:

$$\begin{bmatrix} * & & & \\ & * & & \\ & & \ddots & \\ & & & * \end{bmatrix}.$$

Each such swap changes the sign of the wedge product by a factor of $(-1)$, so we end up with

$$dx_1 \wedge \cdots \wedge dx_n = \left[ \sum_{n \times n \text{ patterns}} (-1)^{\#\text{ swaps}} (\text{products of terms in pattern in } DT(\vec{u})) \right] du_1 \wedge \cdots \wedge du_n.$$

The term in brackets is precisely the pattern/inversion definition of $\det DT$, so we get

$$dx_1 \wedge \cdots \wedge dx_n = (\det DT(u_1, \ldots, u_n))\, du_1 \wedge \cdots du_n$$

as claimed. ■

**Exercise 2** A $k$-form $\omega$ on $\mathbb{R}^n$ is called **closed** if $d\omega = 0$, and is called **exact** if there exists a $(k-1)$-form $\alpha$ on $\mathbb{R}^n$ such that $d\alpha = \omega$. Show that every exact $C^1$ $k$-form is closed. In other words, this is asking to show that

$$d^2\alpha \overset{def}{=} d(d\alpha) \text{ is zero}$$

for every $C^2$ $(k-1)$-form alpha.

Hint: Since $d$ is linear, it is enough to check that applying $d$ twice to something of the form

$$f(x_1, \ldots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $f$ is $C^2$, results in zero. Clairaut's Theorem is important here.

We will write the proof without following the hint, as the hint only delays the inevitable summation. Suppose that

$$\alpha = \sum_{i_1, \ldots, i_k = 1}^{n} f_{i_1, \ldots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

is a $C^2$ $k$-form on $\mathbb{R}^n$. Then we have

$$\begin{aligned}
d^2\alpha &= d\left( \sum_{i_1,\ldots,i_k=1}^{n} df_{i_1,\ldots,i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) \\
&= d\left( \sum_{i_1,\ldots,i_k=1}^{n} \sum_{p=1}^{n} \frac{\partial f_{i_1,\ldots,i_k}}{\partial x_p}\, dx_p \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} \sum_{p=1}^{n} d\left( \frac{\partial f_{i_1,\ldots,i_k}}{\partial x_p} \right) \wedge dx_p \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= \sum_{i_1,\ldots,i_k=1}^{n} \sum_{p=1}^{n}\sum_{q=1}^{n} \frac{\partial^2 f_{i_1,\ldots,i_k}}{\partial x_q \partial x_p} dx_q \wedge dx_p \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= \sum_{i_1,\ldots,i_k=1}^{n} \sum_{p,q=1}^{n} \frac{\partial^2 f_{i_1,\ldots,i_k}}{\partial x_q \partial x_p} dx_q \wedge dx_p \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= \sum_{i_1,\ldots,i_k=1}^{n} \sum_{p,q=1}^{n} \frac{\partial^2 f_{i_1,\ldots,i_k}}{\partial x_p \partial x_q} dx_q \wedge dx_p \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= \sum_{i_1,\ldots,i_k=1}^{n} \sum_{p,q=1}^{n} -\frac{\partial^2 f_{i_1,\ldots,i_k}}{\partial x_p \partial x_q} dx_p \wedge dx_q \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= -\sum_{i_1,\ldots,i_k=1}^{n} \sum_{p,q=1}^{n} \frac{\partial^2 f_{i_1,\ldots,i_k}}{\partial x_p \partial x_q} dx_p \wedge dx_q \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= -\sum_{i_1,\ldots,i_k=1}^{n} \sum_{\tilde{p},\tilde{q}=1}^{n} \frac{\partial^2 f_{i_1,\ldots,i_k}}{\partial x_{\tilde{q}} \partial x_{\tilde{p}}} dx_{\tilde{q}} \wedge dx_{\tilde{p}} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= -d^2\alpha,
\end{aligned}$$

so that $d^2\alpha = 0$. Here we used the definition of exterior derivative in the first and third steps, and in the second and fourth steps we used the face that the wedge product distributes over sums. In the sixth step we used Clairaut's Theorem and the fact that each $f_{i_1,\ldots,i_k}$ is $C^2$. In the seventh step we use antisymmetry to replace $dx_q \wedge dx_p$ with $-dx_p \wedge dx_q$. In the penultimate step, we relabeled the sum with $\tilde{p} \overset{def}{=} q$ and $\tilde{q} \overset{def}{=} p$. ∎

**Exercise 3** Which of the following differential forms on $\mathbb{R}^3$ are closed?

(a) $e^{xyz}\, dy \wedge dz \wedge dx$

(b) $xy^2\, dx - zx\, dy + y\, dz$

(c) $(x^2y + e^z)\, dx \wedge dy + (e^{xy} - xe^z)\, dy \wedge dz + ((x^2+1)^{\sin(xz)} + 2)\, dz \wedge dx$

(a) We can see that this is closed without doing any computation: the exterior derivative of a 3-form is a 4-form, and there are no nonzero 4-forms on $\mathbb{R}^3$ since the expression for any 4-form will necessarily involve a wedge product with at least one of $dx, dy$, or $dz$ repeated.

(b) We have:

$$\begin{aligned} d(xy^2\,dx - zx\,dy + y\,dz) &= d(xy^2)\wedge dx - d(zx)\wedge dy + dy\wedge dz \\ &= (y^2\,dx + 2xy\,dy)\wedge dx - (z\,dx + x\,dz)\wedge dy + dy\wedge dz \\ &= 2xy\,dy\wedge dx - z\,dx\wedge dy - x\,dz\wedge dy + dy\wedge dz \\ &= (2xy+z)\,dy\wedge dz + (y+x)\,dy\wedge dz. \end{aligned}$$

Since this is not zero, the given 1-form is not closed.

(c) The exterior derivative of this 2-form is:

$$d(x^2y+e^z)\wedge dx\wedge dy + d(e^{xy} - xe^z)\wedge dy\wedge dz + d((x^2+1)^{\sin(xz)}+2)\wedge dz\wedge dx.$$

Now, only the $dz$ term from $d(x^2y+e^z)$ gives a nonzero contribution after wedging with $dx\wedge dy$, and similarly only the $dx$ term from $d(e^{xy}-xe^z)$ and only the $dy$ term form $d((x^2+1)^{\sin(xz)}+2)$ matter. Thus the above expression is equal to:

$$e^z\,dz\wedge dx\wedge dy + (ye^{xy} - e^z)\,dx\wedge dy\wedge dz = ye^{xy}\,dx\wedge dy\wedge dz.$$

Since this is not zero, the given 2-form is not closed.

■

**Exercise 4** This problem has two parts.

(a) Show that the differential form
$$\omega = \frac{-y\,dx + x\,dy}{x^2+y^2}$$
is closed on the region $U = \mathbb{R}^2 - \{(0,0)\}$ obtained by removing the origin from $\mathbb{R}^2$.

(b) The form in (a) is not exact on $U$, but show that it is exact on the region obtained by removing the positive $x$-axis from $U$. In other words, find a $C^1$ function on this region whose differential is $\omega$. (This is essentially the same as Problem 10 on Homework 5, only in that case the question was phrased in terms of vector fields and here it is phrased in terms of differential forms. Nonetheless, the steps outlined there and in the solution to that problem are the ones you need here as well.)

(a) We have:

$$\begin{aligned} d\omega &= d\left(-\frac{y}{x^2+y^2}\right)\wedge dx + d\left(\frac{x}{x^2+y^2}\right)\wedge dy \\ &= \frac{(x^2+y^2)(-1)+y(2y)}{(x^2+y^2)^2}\,dy\wedge dx + \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2}\,dx\wedge dy \\ &= \frac{-y^2+x^2}{(x^2+y^2)^2}\,dx\wedge dy + \frac{y^2-x^2}{(x^2+y^2)^2}\,dx\wedge dy \\ &= \frac{-y^2+x^2+y^2-x^2}{(x^2+y^2)^2}\,dx\wedge dy \\ &= 0, \end{aligned}$$

so $\omega$ is closed.

(b) Define $f$ by

$$f(x,y) = \begin{cases} \arctan\left(\frac{y}{x}\right) & x > 0, y > 0 \\ -\arctan\left(\frac{x}{y}\right) + \frac{\pi}{2} & x \le 0, y > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & x < 0, y \le 0 \\ -\arctan\left(\frac{x}{y}\right) + \frac{3\pi}{2} & x \ge 0, y < 0. \end{cases}$$

It follows from the solution of Problem 10 of Homework 5 that $f$ is $C^1$ on the given region and that the gradient of $f$ is $\frac{-y\vec{i}+x\vec{j}}{x^2+y^2}$, or equivalently that $df = \omega$. Thus $f$ is the desired function.

Note that, as explained in Problem 10 of Homework 5, $f$ would not be $C^1$, let alone continuous, if we including the positive $x$-axis in the domain by taking the first piece in the definition of $f$ to be valid for $x > 0, y \ge 0$. This is because the limit of $f$ as we approach the positive $x$-axis from the fourth quadrant is actually $\arctan\left(\frac{y}{x}\right) + 2\pi$ instead of $\arctan\left(\frac{y}{x}\right)$. This actually reflects an important property of angle measurement: note that, in polar coordinates $\theta = \arctan\left(\frac{y}{x}\right)$, so that $\omega$ is $d\theta$, and the point is that the function $\theta$ measuring angles cannot be made continuous everywhere. Indeed, if we restrict the allowed angle values to be between 0 and $2\pi$, the $\theta$ function is not continuous on the positive $x$-axis since points on there have $\theta$ value 0 but the $\theta$ values for points in the fourth quadrant approaching the $x$-axis approach $2\pi$ instead. This observation accounts for many subtleties we'll see in vector calculus, and accounts for many interesting properties you would see in a complex analysis course. The moral is that $d\theta$ is an interesting thing to work with!

■

**Exercise 5** Show that the differential 1-form

$$\frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z^2}$$

defined on $\mathbb{R}^3 - \{(0,0,0)\}$ is closed and exact.

By Problem 2, any exact $C^2$ form is automatically closed, so it is enough to show that this 1-form is exact. Let

$$f(x,y,z) = \frac{1}{2}\ln(x^2 + y^2 + z^2),$$

which is $C^2$ on $\mathbb{R}^3$ with the origin removed. Then:

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = \frac{2x}{2(x^2+y^2+z^2)}\,dx + \frac{2y}{2(x^2+y^2+z^2)}\,dy + \frac{2z}{2(x^2+y^2+z^2)}\,dz,$$

which simplifies to

$$\frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z^2}$$

as desired. ■

**Exercise 6** (Colley 6.1.15, 6.1.19) This problem has two parts.

(a) Find $\displaystyle\int_{\vec{x}} \vec{F} \cdot d\vec{s}$ where

$$\vec{F}(x, y, z) = 3z\,\vec{i} + y^2\,\vec{j} + 6z\,\vec{k} \quad \text{and} \quad \vec{x}(t) = (\cos(t), \sin(t), t/3),\ 0 \le t \le 4\pi.$$

(b) If $\vec{x}(t) = (e^{2t}\cos 3t, e^{2t}\sin 3t), 0 \le t \le 2\pi$, find

$$\int_{\vec{x}} \frac{x\,dx + y\,dy}{(x^2+y^2)^{3/2}}$$

(a) We compute:

$$\begin{aligned}
\int_{\vec{x}} \vec{F} \cdot d\vec{s} &= \int_0^{4\pi} \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\,dt \\
&= \int_0^{4\pi} (t, \sin^2(t), 2t) \cdot (-\sin(t), \cos(t), 1/3)\,dt \\
&= \int_0^{4\pi} \left( -t\sin(t) + \sin^2(t)\cos(t) + \frac{2}{3}t \right) dt \\
&= 4\pi + \frac{16\pi^2}{3}.
\end{aligned}$$

(b) We have:

$$\begin{aligned}
&\int_{\vec{x}} \frac{x\,dx + y\,dy}{(x^2+y^2)^{3/2}} \\
&= \int_0^{2\pi} \frac{(e^{2t}\cos(3t))(2e^{2t}\cos(3t) - 3e^{2t}\sin(3t)) + (e^{2t}\sin(3t))(2e^{2t}\sin(3t) + 3e^{2t}\cos(3t))}{e^{6t}}\,dt \\
&= \int_0^{2\pi} 2e^{-2t}\,dt \\
&= -e^{-2\pi} + 1.
\end{aligned}$$

■

**Exercise 7** (Colley 6.1.23, 6.1.32) This problem has two parts.

(a) Let $\vec{F}(x, y, z) = (2z^5 - 3yx)\,\vec{i} - x^2\,\vec{j} + x^2 z\,\vec{k}$. Calculate the line integral of $\vec{F}$ around the perimeter of the square with vertices $(1, 1, 3), (-1, 1, 3), (-1, -1, 3), (1, -1, 3)$, oriented counterclockwise about the $z$-axis as viewed from the positive $z$-axis.

(b) Calculate $\displaystyle\int_C z\,dx + x\,dy + y\,dz$ where $C$ is the curve obtained by intersecting the surfaces $z = x^2$ and $x^2 + y^2 = 4$ and is oriented counterclockwise around the $z$-axis as seen from the positive $z$-axis.

(a) We parametrize the four sides of the square using:

$$\vec{x}_1(t) = (-t, 1, 3), -1 \le t \le 1 \quad \vec{x}_2(t) = (-1, -t, 3), -1 \le t \le 1$$

$$\vec{x}_3(t) = (t, -1, 3), -1 \le t \le 1 \quad \vec{x}_4(t) = (1, t, 3), -1 \le t \le 1.$$

We have:

$$\begin{aligned}
\int_{\vec{x}_1} \vec{F} \cdot d\vec{s} &= \int_{-1}^{1} (486 + 3t, -t^2, 3t^2) \cdot (-1, 0, 0)\, dt = \int_{-1}^{1} (-486 - 3t)\, dt = -972 \\
\int_{\vec{x}_2} \vec{F} \cdot d\vec{s} &= \int_{-1}^{1} (486 - 3t, -1, 3) \cdot (0, -1, 0)\, dt = \int_{-1}^{1} dt = 2 \\
\int_{\vec{x}_3} \vec{F} \cdot d\vec{s} &= \int_{-1}^{1} (486 + 3t, -t^2, 3t^2) \cdot (1, 0, 0)\, dt = \int_{-1}^{1} (486 + 3t)\, dt = 972 \\
\int_{\vec{x}_4} \vec{F} \cdot d\vec{s} &= \int_{-1}^{1} (486 - 3t, -1, 3) \cdot (0, 1, 0)\, dt = \int_{-1}^{1} -\, dt = -2.
\end{aligned}$$

Adding these up gives that the value of the line integral over the entire perimeter is 0. (We'll see later that Stokes' Theorem gives a much quicker way of determining this value.)

(b) The equation for the second surface suggests $x = 2\cos(t)$ and $y = 2\sin(t)$ as the parametric equations for $x$ and $y$, while the equation for the first surface then gives $z = 4\cos^2(t)$. Hence we use

$$\vec{x}(t) = (2\cos(t), 2\sin(t), 4\cos^2(t)),\ 0 \le t \le 2\pi$$

as parametric equations for $C$. Then

$$\begin{aligned}
\int_C z\, dx &+ x\, dy + y\, dz \\
&= \int_C (z\vec{i} + x\vec{j} + y\vec{k}) \cdot d\vec{s} \\
&= \int_0^{2\pi} \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\, dt \\
&= \int_0^{2\pi} (4\cos^2(t)\vec{i} + 2\cos(t)\vec{j} + 2\sin(t)\vec{k}) \cdot (-2\sin(t)\vec{i} + 2\cos(t)\vec{j} - 8\cos(t)\sin(t)\vec{k})\, dt \\
&= \int_0^{2\pi} (-8\cos^2(t)\sin(t) + 4\cos^2(t) - 16\cos(t)\sin^2(t))\, dt \\
&= \frac{8}{3}\cos^3(t)\Big|_0^{2\pi} + 2\int_0^{2\pi} (1 + \cos(2t))\, dt - \frac{16}{3}\sin^3(t)\Big|_0^{2\pi} \\
&= 0 + 2\left(t + \frac{1}{2}\sin(2t)\right)\Big|_0^{2\pi} + 0 \\
&= 4\pi.
\end{aligned}$$

■

**Exercise 8** (Colley 6.1.33) Show that $\int_{\vec{x}} \vec{T} \cdot d\vec{s}$ equals the length of the path $\vec{x}$, where $\vec{T}$ denotes the unit tangent vector of the path.

Supposing that the given parametrization is $\vec{x}(t), a \le t \le b$, we have

$$\vec{T}(\vec{x}(t)) = \frac{\vec{x}\,'(t)}{\|\vec{x}\,'(t)\|}.$$

Thus
$$\int_{\vec{x}} \vec{T} \cdot d\vec{s} = \int_a^b \vec{T}(\vec{x}(t)) \cdot \vec{x}\,'(t)\, dt = \int_a^b \frac{\vec{x}\,'(t) \cdot \vec{x}\,'(t)}{\|\vec{x}\,'(t)\|}\, dt = \int_a^b \|\vec{x}\,'(t)\|\, dt,$$
which is the length of the path as claimed. ■

**Exercise 9** (Colley 6.1.37) Suppose $C$ is the curve $y = f(x)$, oriented from $(a, f(a))$ to $(b, f(b))$ where $a < b$ and where $f$ is positive and $C^1$ on $[a, b]$. If $\vec{F} = y\,\vec{i}$, show that the value of $\int_C \vec{F} \cdot d\vec{s}$ is the area under the graph of $f$ between $x = a$ and $x = b$.

Take as a parametrization of $C$:
$$\vec{x}(t) = (t, f(t)),\ \ a \le t \le b.$$
Then
$$\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\, dt = \int_a^b (f(t), 0) \cdot (1, f'(t))\, dt = \int_a^b f(t)\, dt,$$
which gives the area under the graph of $f$ between $x = a$ and $x = b$ as claimed. ■

**Exercise 10** (Colley 6.1.38, 6.1.39) This problem has two parts.

(a) Let $\vec{F}$ be the radial vector field $\vec{F}(x, y, z) = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$. Show that if $\vec{x}(t), a \le t \le b$ is any path that lies on the sphere $x^2 + y^2 + z^2 = c^2$, then $\int_{\vec{x}} \vec{F} \cdot d\vec{s} = 0$.

(b) Suppose that $C$ is a smooth oriented curve that lies in the level set of a $C^1$ function $f : \mathbb{R}^2 \to \mathbb{R}$. Show that $\int_C \nabla f \cdot d\vec{s} = 0$ .

(a) We have
$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\, dt = \int_a^b (x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t))\, dt.$$
As was shown in a previous homework assignment, since $\vec{x}(t)$ lies on a sphere we have
$$0 = \vec{x}(t) \cdot \vec{x}\,'(t) = (x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t)) \text{ for all } t.$$
Thus the line integral expression above becomes
$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{x}(t) \cdot \vec{x}\,'(t)\, dt = \int_a^b 0\, dt = 0$$
as claimed.

(b) Let $\vec{x}(t), a \le t \le b$ be a parametrization of $C$. Then
$$\int_C \nabla f \cdot d\vec{s} = \int_a^b \nabla f(\vec{x}(t)) \cdot \vec{x}\,'(t)\, dt.$$
As we saw last quarter, at any point on the level set $C$, $\nabla f$ is perpendicular to the level set itself, which means that $\nabla f$ is perpendicular to any vector tangent to $C$. Thus $\nabla f(\vec{x}(t)) \cdot \vec{x}\,'(t) = 0$ for all $t$, so the line integral in question is zero.

■