Math 291-3 Quiz 4 Spring 2022

Name: Solutions ____________________

Do not use books, notes, calculators, phones, tablets, or software applications. You have 15 minutes to complete this quiz.

1. (5 points) Determine whether the following statement is true or false. If true, then explain why. If false, then give a counterexample with brief justification.

   There is a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla f(x, y) = \begin{bmatrix} \sin(y) \\ \cos(x) \end{bmatrix}$ for every $(x, y) \in \mathbb{R}^2$.

   Solution.

   This is false. If such a function existed, then we would have $0 = \operatorname{curl}(\nabla f(x, y)) = (-\sin(x)) - (\cos(y))$ for every $(x, y) \in \mathbb{R}^2$. But setting $(x, y) = (\frac{\pi}{2}, \frac{\pi}{2})$ would then yield $0 = -2$, an impossibility.

2. (10 points) Compute the surface area of the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies above the box $[1, 2] \times [3, 4]$ in the xy-plane.

   Solution.

   One parametrization of this surface is

$$\vec{X}(x, y) = (x, y, \sqrt{x^2 + y^2}), \quad 1 \le x \le 2,\ 3 \le y \le 4.$$

   We have

$$\|N_{\vec{X}}(x, y)\| = \left\| \begin{bmatrix} 1 \\ 0 \\ \frac{x}{\sqrt{x^2+y^2}} \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix} \right\| = \left\| \begin{bmatrix} -\frac{x}{\sqrt{x^2+y^2}} \\ -\frac{y}{\sqrt{x^2+y^2}} \\ 1 \end{bmatrix} \right\| = \sqrt{\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} + 1} = \sqrt{2},$$

   so that the surface area of the portion of the cone is given by

$$\int_1^2 \int_3^4 \sqrt{2}\, dy dx = \sqrt{2}.$$