Northwestern University

MATH 291-3 First Midterm Examination - Practice B
Spring Quarter 2022
April 21, 2022

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Instructions

- This examination consists of 5 questions.
- Read all problems carefully before answering.
- You have 50 minutes to complete this examination.
- Do not use books, notes, calculators, computers, tablets, phones, smart watches, or similar devices.
- Possession of a digital communication device during a bathroom break will be treated as a *prima facie* violation of Northwestern's academic integrity policy.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is true or false. If true, then explain why. If false, then give a counterexample.
   (a) If $f : [-1,1] \times [-2,2] \times [-3,3] \to \mathbb{R}$ is a constant function, then all Riemann sums for f have the same value.
   (b) If $f, g : \mathbb{R}^n \to \mathbb{R}$ are C^1 and f has a local maximum at $\vec{a}$ among points satisfying $g(\vec{x}) = 10$ and $\nabla f(\vec{a}) \neq \vec{0}$, then there exists $\lambda \in \mathbb{R}$ such that $\nabla g(\vec{a}) = \lambda \nabla f(\vec{a})$.

Solution: (a) is true. Let $\mathcal{P}$ be any partition of $[-1,1] \times [-2,2] \times [-3,3]$ and let $\mathcal{C}$ be any choice of sample points. Then if $k \in \mathbb{R}$ is sucht hat $f(x,y,z) = k$ for every (x,y,z), then

$$R(f, \mathcal{P}, \mathcal{C}) = \sum_i f(\vec{c}_i)\mathrm{Vol}_3(B_i) = \sum_i k\mathrm{Vol}_3(B_i) = k\mathrm{Vol}_3([-1,1] \times [-2,2] \times [-3,3]) = k(2)(4)(6) = 48k.$$

(b) is true. First note that if $\nabla g(\vec{a}) = \vec{0}$, then the claim holds with $\lambda = 0$. if $\nabla g(\vec{a}) \neq \vec{0}$, then the Lagrange Multiplier theorem implies that there is $\nu \in \mathbb{R}$ with $\nabla f(\vec{a}) = \nu \nabla g(\vec{a})$. Because $\nabla f(\vec{a}) \neq \vec{0}$, $\nu \neq 0$, and therefore the desired equation holds with $\lambda = \frac{1}{\nu}$.

2. (10 points) Find and classify the critical points of $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^3y + y^3x + xy$.

Solution: Note that f is C^2 on $\mathbb{R}^2$, and therefore every critical point (x, y) of f will satisfy

$$\begin{bmatrix} 0 & 0 \end{bmatrix} = Df(x, y) = \begin{bmatrix} 3x^2y + y^3 + y & x^3 + 3y^2x + x \end{bmatrix},$$

so that $0 = y(3x^2 + y^2 + 1)$ and $0 = x(x^2 + 3y^2 + 1)$. Since $3x^2 + y^2 + 1 \geq 1 > 0$, $y = 0$. Since $x^2 + 3y^2 + 1 \geq 1 > 0$, $x = 0$. Therefore $(0, 0)$ is the only critical point of f.

To classify this critical point, note that

$$D^2f(x, y) = \begin{bmatrix} 6xy & 3x^2 + 3y^2 + 1 \\ 3x^2 + 3y^2 + 1 & 6xy \end{bmatrix}, \quad \text{so that} \quad D^2f(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The characteristic equation of $D^2f(0, 0)$ is $0 = \det(D^2f(0, 0) - \lambda I_2) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$, so the eigenvalues of $D^2f(0, 0)$ are 1 and -1. Therefore $D^2f(0, 0)$ is indefinite, so f has a saddle point at $(0, 0)$.

3. Consider rectangles of a fixed area $A > 0$ whose sides have length at most $100\sqrt{A}$. Show that among all such rectangles there is one of minimal perimeter and determine this minimal perimeter.

Solution: If x, y denote the length and height of such a rectangle, then we are attempting to show that $f(x,y) = 2x + 2y$ has a global minimum value on $S = \{(x,y) \ : \ xy = A,\ 0 \leq x \leq 100\sqrt{A},\ 0 \leq y \leq 100\sqrt{A}\}$ (and compute this global minimum value). Note that f is continuous and, since $S \subset [0, 100\sqrt{A}] \times [0, 100\sqrt{A}]$, S is bounded. S is a segment of the curve $xy = A$ that contains its endpoints $(100\sqrt{A}, \frac{\sqrt{A}}{100})$ and $(\frac{\sqrt{A}}{100}, 100\sqrt{A})$, and is therefore closed. Therefore the Extreme Value Theorem implies that f attains a global minimum value on S. This global minimum value occurs at one of the endpoints of S or is a constrained local extreme value of f on S. In this last case, this would occur at a point (x,y) satisfying, for some $\lambda \in \mathbb{R}$ (and setting $g(x,y) = xy$),

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = A \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} 2 = \lambda y \\ 2 = \lambda x \\ xy = A \end{cases}$$

The first two equations imply that $\lambda x = \lambda y$, so that $\lambda(x - y) = 0$. But if $\lambda = 0$ because $2 \neq 0$, and therefore we must have $x = y$. Since $xy = A$, we have $x = y = \sqrt{A}$ (and $\lambda = \frac{1}{2\sqrt{A}}$). Therefore the only point on S where f may have a constrained local extreme value is $(\sqrt{A}, \sqrt{A})$.

Testing f at these points yields

$$f(\sqrt{A}, \sqrt{A}) = 4\sqrt{A}, \quad f\left(100\sqrt{A}, \frac{\sqrt{A}}{100}\right) = f\left(\frac{\sqrt{A}}{100}, 100\sqrt{A}\right) = \left(200 + \frac{1}{50}\right)\sqrt{A} > 4\sqrt{A}.$$

Therefore the global minimum value of f on S is $4\sqrt{A}$ (and is achieved when $x = y = \sqrt{A}$).

4. Show that for any compact region $D \subseteq \mathbb{R}^2$ with $\text{Area}(D) = 10$, the following inequality holds:

$$\iint_D (3 - x^2 + 2x - y^2 + 2y)\, dA \leq 50.$$

You may assume that any local maximum value of $f(x,y) = 3 - x^2 + 2x - y^2 + 2y$ is actually a global maximum value.

Solution: Let D be such a region. Because f is C^2 (and therefore differentiable) throughout $\mathbb{R}^2$, every critical point (x,y) of f satisfies

$$\begin{bmatrix} 0 & 0 \end{bmatrix} = Df(x,y) = \begin{bmatrix} -2x+2 & -2y+2 \end{bmatrix},$$

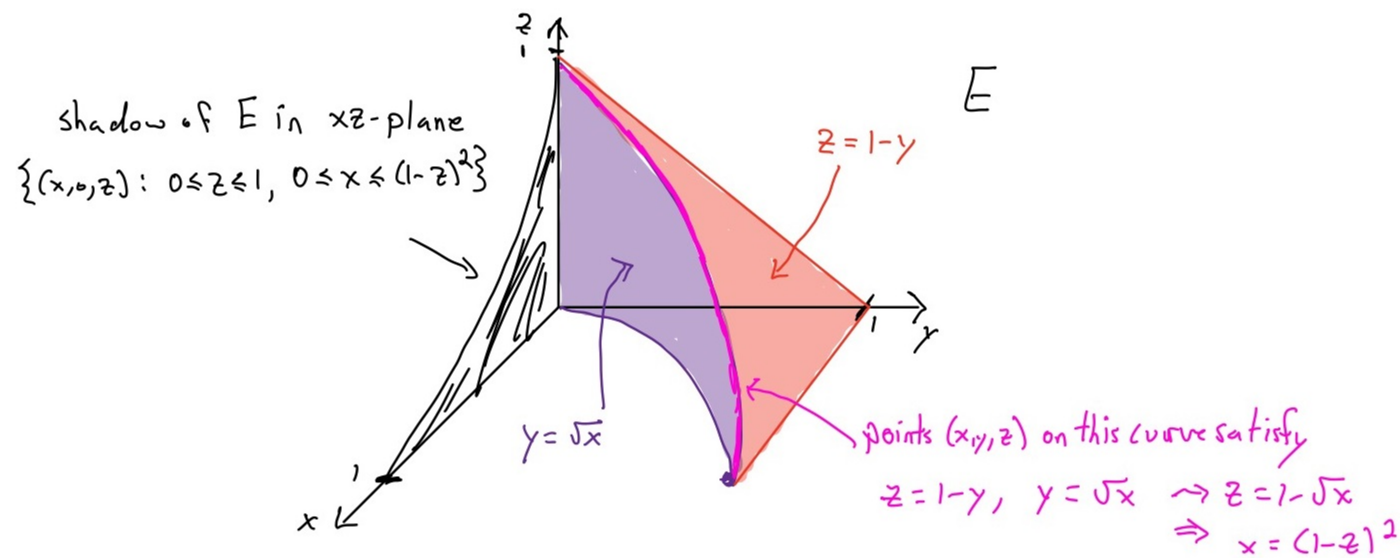
so that $(1,1)$ is the only critical point of f. Indeed, note that $f(x,y) = 5 - x^2 + 2x - 1 - y^2 + 2y - 1 = 5 - (x-1)^2 - (y-1)^2$, so that f has a global maximum value of 5 at $(1,1)$. In other words, $f(x,y) \leq 5$ for every $(x,y) \in \mathbb{R}^2$, and therefore for every $(x,y) \in D$. Therefore

$$\iint_D f(x,y)\, dA \leq \iint_D 5\, dA = 5\text{Area}(D) = 50.$$

5. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous. Rewrite the following as an iterated integral with respect to the order $dy\,dx\,dz$:

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$$

Solution: Note that the given iterated integral is equal to $\iiint_E f(x,y,z)\,dV$, where E is the subset of $\mathbb{R}^3$ in the first octant bounded by the yz- and xy-coordinate planes, the plane $z = 1-y$, and the parabolic cylinder $y = \sqrt{x}$ (pictured below):



The shadow of E in the xz-plane is $\{(x,0,z) \ : \ 0 \le z \le 1 \text{ and } 0 \le x \le (1-z)^2\}$, and for each choice of x and z satisfying these inequalities, y will run from its smallest value $\sqrt{x}$ to its largest value $1-z$. Therefore we can express this triple integral as an iterated integral in the order $dy\,dx\,dz$ as

$$\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz.$$