Math 291-3: Discussion #3 Problems (Solutions)
Northwestern University, Spring 2022

1. By making an appropriate change of variables, compute

$$\iint_D (y^2+y)e^{-2x}\,dA,$$

where D is the region in the first quadrant of $\mathbb{R}^2$ bounded by the curves $y=e^x$, $y=2e^x$, $y=2-x$, and $y=5-x$.

Solution. Note that D is described by $e^x \le y \le 2e^x$ and $2-x \le y \le 5-x$. Rewriting these inequalities yields $1 \le ye^{-x} \le 2$ and $2 \le x+y \le 5$. It therefore seems convenient to make the change of variable $T : [1,2]\times[2,5] \to D$, $T(u,v)=(x,y)$, where $u=ye^{-x}$ and $v=x+y$. The Jacobian of the inverse transformation $T^{-1}(x,y)=(u,v)$ is

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\left(\begin{bmatrix} -ye^{-x} & e^{-x} \\ 1 & 1 \end{bmatrix}\right) = -ye^{-x}-e^{-x} = -(y+1)e^{-x},$$

which, since $y \ge e^x > 0$ throughout D, does not vanish. Because $\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \frac{1}{-(y(u,v)+1)e^{-x(u,v)}}$, we therefore have

$$\begin{aligned}
\iint_D (y^2+y)e^{-2x}\,dA(x,y) &= \iint_{T([1,2]\times[2,5])} ye^{-x}(y+1)e^{-x}\,dA(x,y) \\
&= \iint_{[1,2]\times[2,5]} u(y(u,v)+1)e^{-x(u,v)}\left|\frac{1}{-(y(u,v)+1)e^{-x(u,v)}}\right| dA(u,v) \\
&= \int_1^2\int_2^5 u\,dvdu \\
&= \int_1^2 3u\,du \\
&= \frac{9}{2}.
\end{aligned}$$

2. Suppose $f:\mathbb{R}^n \to \mathbb{R}$ is continuous and odd with respect to some x_i, meaning that

$$f(x_1,\dots,-x_i,\dots,x_n) = -f(x_1,\dots,x_i,\dots,x_n) \text{ for every } \vec{x}\in\mathbb{R}^n.$$

Suppose that $E \subset \mathbb{R}^n$ is an elementary region that is symmetric across the hyperplane $x_i=0$, in the sense that $(x_1,\dots,-x_i,\dots,x_n)\in E$ if and only if $(x_1,\dots,x_i,\dots,x_n)\in E$. Prove that

$$\int_E f(\vec{x})\,dV_n(\vec{x}) = 0.$$

(By "prove", I mean something more rigorous than a hand-wavy argument using Riemann sums. Instead, make a change of variables.)

Solution. Consider the map $T : E \to E$ given by

$$(x_1, \dots, x_i, \dots, x_n) = T(u_1, \dots, u_i, \dots, u_n) \stackrel{def}{=} (u_1, \dots, -u_i, \dots, u_n).$$

Then T is bijective because E is symmetric across the hyperplane $x_i = 0$. Moreover, T is linear with diagonal matrix, where each diagonal entry is 1 except for the entry in the i, i-th spot, which is -1. Therefore T is C^1 and $\det(DT(\vec{u})) = -1$, so that $DT(\vec{u})$ is invertible throughout E. We therefore have

$$\begin{aligned}
\int_E f(\vec{x})\, dV_n(\vec{x}) &= \int_E f(x_1, \dots, x_i, \dots, x_n)\, dV_n(\vec{x}) \\
&= \int_E f(u_1, \dots, -u_i, \dots, u_n) |-1|\, dV_n(\vec{u}) \\
&= \int_E -f(u_1, \dots, u_i, \dots, u_n)\, dV_n(\vec{u}) \\
&= -\int_E f(\vec{u})\, dV_n(\vec{u}) \\
&= -\int_E f(\vec{x})\, dV_n(\vec{x}).
\end{aligned}$$

Therefore $2\int_E f(\vec{x})\, dV_n(\vec{x}) = 0$, so that $\int_E f(\vec{x})\, dV_n(\vec{x}) = 0$.

3. Consider the surface S with C^1 parametrization

$$\vec{\phi}(u, v) = \left(\left(1+v\sin\left(\frac{u}{2}\right)\right)\cos(u),\ \left(1+v\sin\left(\frac{u}{2}\right)\right)\sin(u),\ v\cos\left(\frac{u}{2}\right)\right) \text{ for } (u, v) \in [0, 2\pi]\times\left[-\frac{1}{2}, \frac{1}{2}\right].$$

Show that if $N_{\vec{\phi}}(u, v)$ is the normal vector to S at $\vec{\phi}(u, v)$ arising from $\vec{\phi}$, then $N_{\vec{\phi}}(0, 0) = \vec{e}_1$ and $N_{\vec{\phi}}(2\pi, 0) = -\vec{e}_1$. Does this bother you at all, given the fact that $\vec{\phi}(0, 0) = \vec{\phi}(2\pi, 0)$? (This surface is known as the **Möbius strip**.)

Solution. We compute that

$$\vec{\phi}_u(u, v) = \begin{bmatrix} \frac{v}{2}\cos\left(\frac{u}{2}\right)\cos(u) - \left(1 + v\sin\left(\frac{u}{2}\right)\right)\sin(u) \\ \frac{v}{2}\cos\left(\frac{u}{2}\right)\sin(u) + \left(1 + v\sin\left(\frac{u}{2}\right)\right)\cos(u) \\ -\frac{v}{2}\sin\left(\frac{u}{2}\right) \end{bmatrix}$$

and

$$\vec{\phi}_v(u, v) = \begin{bmatrix} \sin\left(\frac{u}{2}\right)\cos(u) \\ \sin\left(\frac{u}{2}\right)\sin(u) \\ \cos\left(\frac{u}{2}\right) \end{bmatrix},$$

so that

$$N_{\vec{\phi}}(0,0) = \vec{\phi}_u(0,0) \times \vec{\phi}_v(0,0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$N_{\vec{\phi}}(2\pi,0) = \vec{\phi}_u(2\pi,0) \times \vec{\phi}_v(2\pi,0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$