Name: Solutions ______________________________

Do not use books, notes, calculators, phones, tablets, or software applications. You have 15 minutes to complete this quiz.

1. (5 points) Determine whether the following statement is true or false. If true, then explain why. If false, then give a counterexample with brief justification.

   Let $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be $\vec{F}(x,y) = (1+xy)e^{xy}\vec{\imath} + x^2e^{xy}\vec{\jmath}$. Let C_1 be the smooth curve from $(0,0)$ to $(1,1)$ parametrized by $\vec{x}_1(t) = (t, t^2)$, $0 \le t \le 1$, and let C_2 be the smooth curve from $(0,0)$ to $(1,1)$ parametrized by $\vec{x}_2(t) = (t,t)$, $0 \le t \le 1$. Then

   $$\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}.$$

   Solution.

   This is true. To see why, note that $\vec{F}$ is C^1 on (the simply connected set) $\mathbb{R}^2$, and that

   $$\text{curl}\vec{F}(x,y) = (x^2e^{xy})_x - ((1+xy)e^{xy})_y = (2x + yx^2)e^{xy} - (x + x + x^2y)e^{xy} = 0$$

   throughout $\mathbb{R}^2$. By Poincaré's Lemma, $\mathbb{F}$ is conservative on $\mathbb{R}^2$ (and therefore has path-independent line integrals). Since C_1 and C_2 have the same starting points and ending points, the claim follows.

2. (10 points) Let $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ be C^1, suppose $\vec{x}(t)$, $a \le t \le b$ is a flow line of $\vec{F}$, and let C be the curve parametrized by $\vec{x}$. Show that

   $$\int_C \vec{F} \cdot d\vec{s} \ge 0.$$

   Solution.

   Note that because $\vec{x}(t)$ is a flow line of $\vec{F}$, $\vec{x}'(t) = \vec{F}(\vec{x}(t))$ for each $a \le t \le b$. Therefore

   $$\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\, dt = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{F}(\vec{x}(t))\, dt = \int_a^b \|\vec{F}(\vec{x}(t))\|^2\, dt \ge \int_a^b 0\, dt = 0.$$