Name: Solutions ______________________________

Do not use books, notes, calculators, phones, tablets, or software applications. You have 15 minutes to complete this quiz.

1. (5 points) Determine whether the following statement is true or false. Prove your assertions.

   The function
   $$f : \mathbb{R}^3 \to \mathbb{R}, \quad f(\vec{x}) = \begin{cases} 1 & \text{if } \|\vec{x}\| \leq 3, \\ 0 & \text{if } \|\vec{x}\| > 3 \end{cases}$$
   is integrable over the box $B = [0,3] \times [0,3] \times [0,3]$.

   Solution.

   This is true. Note that $|f(\vec{x})| \leq 1$ for every $\vec{x} \in B$, so f is bounded on B. f is continuous on B except at points where $\|\vec{x}\| = 3$, or rather on the surface described by $x^2 + y^2 + z^2 = 9$. Because $F : \mathbb{R}^3 \to \mathbb{R}$, $F(x,y,z) \overset{def}{=} x^2 + y^2 + z^2$ is a C^1 function and $\nabla F(\vec{x}) = 2\vec{x} \neq \vec{0}$ for every $\vec{x} \in \{\vec{x} \ : \ F(\vec{x}) = 9\}$, the Measure Zero Theorem implies that $\{\vec{x} \ : \ F(\vec{x}) = 9\}$ has measure zero. By the Measure Zero Theorem, $B \cap \{\vec{x} \ : \ F(\vec{x}) = 9\}$ also has measure zero (since it is contained in a set of measure zero). Therefore f is continuous on B except for on a set of measure zero, so f is integrable on B by Lebesgue's Criterion for Riemann Integrability.

2. (10 points) Compute the following iterated integral. Explain the steps of your reasoning.
   $$\int_0^1 \int_y^1 \frac{y}{x^3+1} \, dxdy$$

   Solution.

   Because the integrand $f(x,y) = \frac{y}{x^3+1}$ is continuous on $\{((x,y) \ : \ x > -1\}$ and therefore on an open set containing the triangle $T = \{(x,y) \ : \ 0 \leq y \leq 1, \ y \leq x \leq 1\}$, Fubini's Theorem implies that
   $$\begin{aligned} \int_0^1 \int_y^1 \frac{y}{x^3+1} \, dxdy = \iint_T \frac{y}{x^3+1} \, dA(x,y) &= \int_0^1 \int_0^x \frac{y}{x^3+1} \, dydx \\ &= \int_0^1 \left[\frac{y^2}{2(x^3+1)} \right]_{y=0}^{y=x} dx \\ &= \int_0^1 \frac{x^2}{2(x^3+1)} \, dx \\ &= \left[\frac{1}{6} \ln(x^3+1) \right]_{x=0}^{x=1} \\ &= \frac{1}{6} \ln(2), \end{aligned}$$
   where use used the Fundamental Theorem of Calculus in steps 3 and 5.