Northwestern University

MATH 291-3 Final Examination
Spring Quarter 2022
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Instructions

- This examination consists of 7 questions.
- Read all problems carefully before answering.
- You have 120 minutes to complete this examination.
- Do not use books, notes, calculators, computers, tablets, phones, smart watches, or similar devices.
- Possession of a digital communication device during a bathroom break will be treated as a *prima facie* violation of Northwestern's academic integrity policy.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Determine whether each of the following statements is true or false. If true, then prove it. If false, then give an argument showing that the statement is false.

   (a) (5 points) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous throughout $[0,1] \times [0,1]$ except at one point, then f is integrable on $[0,1] \times [0,1]$.

   Solution: This is false. The function

   $$f(x,y) \stackrel{def}{=} \begin{cases} (x^2+y^2)^{-1} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   is continuous on $\mathbb{R}^2$ except at $(0,0)$, but is not integrable on $[0,1] \times [0,1]$ because it is not bounded on $[0,1] \times [0,1]$.

   (b) (5 points) There is a C^2 1-form ω on $\mathbb{R}^3$ such that $d\omega = (y^2x + z)dx \wedge dy$.

   Solution: This is false. To see why, note that if such a 1-form ω existed, then we would have

   $$0dx \wedge dy \wedge dz = d^2\omega = d((y^2x+z)dx \wedge dy) = 1dz \wedge dx \wedge dy = 1dx \wedge dy \wedge dz,$$

   an impossibility.

   (c) (5 points) If $C \subset \mathbb{R}^n$ is a smooth oriented curve, then $\text{Length}(-C) = \text{Length}(C)$.

   Solution: This is true. Let $\vec{x} : [a,b] \to \mathbb{R}^n$ be a C^1 orientation-preserving parametrization of C. Then $\vec{x}$ is an orientation-reversing orientation of $-C$ (and therefore a parametrization of $-C$), so

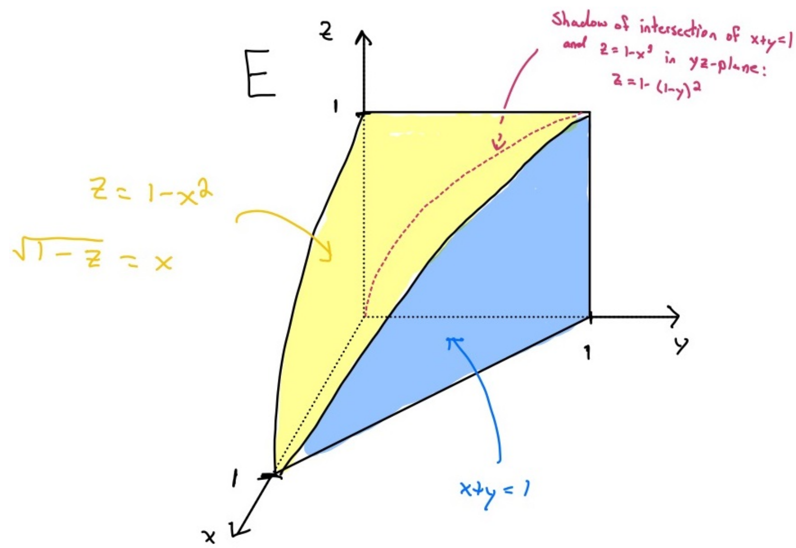
   $$\text{Length}(-C) = \int_a^b \|\vec{x}\,'(t)\|\, dt = \text{Length}(C).$$

2. (10 points) Assume $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous. Rewrite the iterated integral

$$\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x, y, z)\, dzdydx$$

as an iterated integral (or sum of iterated integrals) in the order $dxdzdy$.

Solution: Note that the given iterated integral is equal to $\iiint\limits_E f\, dV$, where E is the subset of $\mathbb{R}^3$ in the first octant bounded by the yz-plane, the xy-plane, the xz-plane, the plane $x + y = 1$, and the parabolic cylinder $z = 1 - x^2$ (pictured below):



The shadow of E in the yz-plane is $\{(0, y, z) \ : \ 0 \le y \le 1 \text{ and } 0 \le z \le 1\}$. The shadow of the intersection of the surfaces $x + y = 1$ and $z = 1 - x^2$ in the yz-plane is $z = 1 - (1 - y)^2$. When $0 \le z \le 1 - (1 - y)^2$, we have $0 \le x \le 1 - y$. When $1 - (1 - y)^2 \le z \le 1$, we have $0 \le x \le \sqrt{1 - z}$. Therefore we can express this triple integral as a a sum of iterated integrals in the order $dx\, dz\, dy$ as

$$\int_0^1 \int_0^{1-(1-y)^2} \int_0^{1-y} f(x, y, z)\, dx\, dz\, dy + \int_0^1 \int_{1-(1-y)^2}^{1} \int_0^{\sqrt{1-z}} f(x, y, z)\, dx\, dz\, dy.$$

3. (10 points) Let $E \subset \mathbb{R}^2$ denote the region in the first quadrant enclosed by the coordinate axes and the ellipse $4x^2 + y^2 = 1$. Compute

$$\iint_E \cos(4x^2 + y^2)\, dA(x, y).$$

Solution: Let $D = [0, 1] \times [0, \frac{\pi}{2}]$, and write $T : D \to \mathbb{R}^2$ as $T(r, \theta) = (\frac{1}{2} r\cos(\theta), r\sin(\theta))$. Then T is injective (except on ∂D), $T(D) = E$, T is C^1 (and therefore differentiable) throughout D, and

$$\det DT(r, \theta) = \det \begin{bmatrix} \frac{1}{2}\cos(\theta) & -\frac{1}{2} r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} = \frac{1}{2} r > 0$$

except on (part of) the boundary of D. Therefore the Change of Variables Theorem gives

$$\begin{aligned} \int_E \cos(4x^2 + y^2)\, dA(x, y) &= \iint_D \cos\left(4\left(\frac{1}{2} r\cos(\theta)\right)^2 + (r\sin(\theta))^2\right) \frac{1}{2} r\, dA(r, \theta) \\ &= \int_0^1 \int_0^{\pi/2} \frac{1}{2} r\cos(r^2)\, d\theta dr \\ &= \int_0^1 \frac{\pi}{4} r\cos(r^2)\, dr = \frac{\pi}{8}\sin(1). \end{aligned}$$

Solution: Let D be the portion of the unit disc in the first quadrant, and define $T : D \to \mathbb{R}^2$ by $T(u, v) = (\frac{1}{2} u, v)$. Then T is linear with invertible matrix $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ (which is also $DT(u, v)$), and is therefore an injective C^1 function. Moreover, note that for $(x, y) \in E$, $(x, y) = T(2x, y)$, so that T maps D onto E. Therefore the Change of Variables Theorem (followed by a change to polar coordinates) gives

$$\begin{aligned} \int_E \cos(4x^2 + y^2)\, dA(x, y) &= \iint_D \cos\left(4\left(\frac{1}{2} u\right)^2 + (v)^2\right) \left|\det DT(u, v)\right|\, dA(u, v) \\ &= \iint_D \frac{1}{2}\cos(u^2 + v^2)\, dA(u, v) \\ &= \int_0^1 \int_0^{\pi/2} \frac{1}{2}\cos(r^2) r\, d\theta dr \\ &= \int_0^1 \frac{\pi}{4} r\cos(r^2)\, dr = \frac{\pi}{8}\sin(1). \end{aligned}$$

4. (10 points) Let C be a smooth, oriented closed curve in $\mathbb{R}^2 - \{(0,0)\}$. One can show (and you may assume) that there are C^1 functions $r : [0,1] \to (0,\infty)$ and $\theta : [0,1] \to \mathbb{R}$ such that

$$\vec{x}(t) \overset{def}{=} (r(t)\cos(\theta(t)), r(t)\sin(\theta(t))),\ t \in [0,1]$$

is a parametrization of C. Prove that

$$\oint_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \theta(1) - \theta(0).$$

(Remark: Since $(r(0)\cos(\theta(0)), r(0)\sin(\theta(0))) = \vec{x}(0) = \vec{x}(1) = (r(1)\cos(\theta(1)), r(1)\sin(\theta(1)))$, we have that $r(0) = r(1)$ and $\theta(1) - \theta(0) = 2\pi k$ for some integer k. k is the (net) number of times that C wraps around $(0,0)$ in the counterclockwise direction, and is called the **winding number** of C.)

Solution: Writing $x(t) = r(t)\cos(\theta(t))$ and $y(t) = r(t)\sin(\theta(t))$ for short, we note that $x(t)^2 + y(t)^2 = r(t)^2$ and therefore

$$\begin{aligned}
&\oint_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \\
&\qquad = \int_0^1 \begin{bmatrix} -y(t)/r(t)^2 \\ x(t)/r(t)^2 \end{bmatrix} \cdot \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} dt \\
&\qquad = \int_0^1 \frac{1}{r(t)} \begin{bmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{bmatrix} \cdot \begin{bmatrix} r'(t)\cos(\theta(t)) - r(t)\sin(\theta(t))\theta'(t) \\ r'(t)\sin(\theta(t)) + r(t)\cos(\theta(t))\theta'(t) \end{bmatrix} dt \\
&\qquad = \int_0^1 \left(\frac{r'(t)}{r(t)} \begin{bmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{bmatrix} + \theta'(t) \begin{bmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{bmatrix} \cdot \begin{bmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{bmatrix} \right) dt \\
&\qquad = \int_0^1 \left(\frac{r'(t)}{r(t)} \cdot 0 + \theta'(t)1 \right) dt \\
&\qquad = \int_0^1 \theta'(t)\,dt \\
&\qquad = \theta(1) - \theta(0)
\end{aligned}$$

by the Fundamental Theorem of Calculus.

5. (10 points) Produce $\lambda : \mathbb{R}^2 \to \mathbb{R}$ so that for every simple, closed, piecewise-smooth oriented curve $C \subset \mathbb{R}^2$,

$$\oint_C (x - y^3)\,dx + x^3\,dy = \oint_C \lambda(x,y)\,dy.$$

(This appeared on your homework; you must produce a proof here. If needed, you may assume that every simple, closed, piecewise-smooth curve $C \subset \mathbb{R}^2$ is the boundary of a bounded region $D \subset \mathbb{R}^2$.)

Solution: Take $\lambda(x,y) = x^3 + 3y^2x$. Then $\operatorname{curl}\begin{bmatrix} x - y^3 \\ x^3 - \lambda(x,y) \end{bmatrix} = 3x^2 - (3x^2 + 3y^2) + 3x^2 = 0$ on $\mathbb{R}^2$, and therefore (because $\mathbb{R}^2$ is simply connected) Poincarè's Lemma implies that there is a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ with $\nabla f(x,y) = \begin{bmatrix} x - y^3 \\ x^3 - \lambda(x,y) \end{bmatrix}$ on $\mathbb{R}^2$. Let C be a closed, piecewise-smooth oriented curve in $\mathbb{R}^2$. By the Conservative Vector Field Theorem,

$$0 = \oint_C \nabla f \cdot d\vec{s} = \oint_C (x - y^3)\,dx + x^3\,dy - \oint_C \lambda(x,y)dy,$$

and the result follows.

Solution: Let $\lambda : \mathbb{R}^2 \to \mathbb{R}$, $\lambda(x,y) \overset{def}{=} x^3 + 3y^2x$. Let C be a simple, closed, piecewise-smooth oriented curve in $\mathbb{R}^2$, and let D be the region enclosed by C. Then Green's Theorem gives

$$\begin{aligned}
\oint_C (x - y^3)dx + x^3dy - \oint_C \lambda(x,y)dy &= \oint_C (x - y^3)dx + (x^3 - \lambda(x,y))dy \\
&= \pm \iint_D (3x^2 - \lambda_x(x,y) - (0 - 3y^2))dA(x,y) \\
&= \pm \iint_D (3x^2 + -(3x^2 + 3y^2) + 3y^2)dA(x,y) \\
&= \pm \iint_D 0\,dA(x,y) \\
&= 0,
\end{aligned}$$

where $\pm$ depends on the orientation of C as the boundary of D.

6. (10 points) Let $\vec{☺} = a\vec{i}+b\vec{j}+c\vec{k}$ be a constant vector field, and $\vec{F} = x\vec{i}+y\vec{j}+z\vec{k}$. Assume that S is a smooth, oriented surface with geometric boundary ∂S consisting of a single closed piecewise-smooth curve. Give ∂S the orientation induced by the orientation of S. Prove that

$$\iint_S \vec{☺} \cdot d\vec{S} = \frac{1}{2}\oint_{\partial S} (\vec{☺} \times \vec{F}) \cdot d\vec{s}.$$

Solution: Note that

$$\vec{☺} \times \vec{F} = (bz - cy)\vec{i} + (cx - az)\vec{j} + (ay - bx)\vec{k}$$

is a C^1 vector field on $\mathbb{R}^3$, and that S and ∂S satisfy the hypotheses of Stokes' Theorem. Therefore we can apply Stokes' Theorem to see that

$$\frac{1}{2}\oint_{\partial S} (\vec{☺} \times \vec{F}) \cdot d\vec{s} = \frac{1}{2}\iint_S \operatorname{curl}(\vec{☺} \times \vec{F}) \cdot d\vec{S}.$$

But

$$\operatorname{curl}(\vec{☺} \times \vec{F}) = \operatorname{curl}((bz - cy)\vec{i} + (cx - az)\vec{j} + (ay - bx)\vec{k}) = 2a\vec{i} + 2b\vec{j} + 2c\vec{k} = 2\vec{☺},$$

so that

$$\frac{1}{2}\oint_{\partial S} (\vec{☺} \times \vec{F}) \cdot d\vec{s} = \frac{1}{2}\iint_S \operatorname{curl}(\vec{☺} \times \vec{F}) \cdot d\vec{S} = \frac{1}{2}\iint_S 2\vec{☺} \cdot d\vec{S} = \iint_S \vec{☺} \cdot d\vec{S},$$

as desired.

7. (10 points) Let $S \overset{def}{=} \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2+y^2+z^2=1, \ z \geq 0\}$ denote the top half of the unit sphere in $\mathbb{R}^3$, oriented with upward-pointing normal vectors. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be C^2, and assume that $f_z(x,y,0)=0$ for every $x, y \in \mathbb{R}$. Prove that if f is **harmonic**, in the sense that $f_{xx}+f_{yy}+f_{zz}=0$ at each point in $\mathbb{R}^3$, then

$$\iint_S \nabla f \cdot d\vec{S} = 0.$$

(Hint: Note that S is not a closed surface!)

Solution: Let $S' = \{(x,y,0) \in \mathbb{R}^3 \ : \ x^2+y^2 \leq 1\}$ denote the closed unit disc in the xy-plane, oriented with downward-pointing normal vectors. (Note that the normal vector for S' is $-\vec{k}$ at each point!) Then $S \cup S'$ is the "outward" oriented boundary of $E = \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2+y^2+z^2 \leq 1, \ z \geq 0\}$, the closed top half of the unit ball in $\mathbb{R}^3$. Then Gauss's Theorem gives

$$\begin{aligned}
\iint_S \nabla f \cdot d\vec{S} &= \iint_{S\cup S'} \nabla f \cdot d\vec{S} - \iint_{S'} \nabla f \cdot d\vec{S} \\
&= \iiint_E \operatorname{div}(\nabla f)\, dV - \iint_{S'} \nabla f(x,y,z) \cdot (-\vec{k})\, dS \\
&= \iiint_E (f_{xx}+f_{yy}+f_{zz})\, dV + \iint_{S'} \underbrace{f_z(x,y,z)}_{=0 \text{ on } S' \text{ since } z=0} dS \\
&= \iiint_E 0\, dV + \iint_{S'} 0\, dS \\
&= 0.
\end{aligned}$$