Solutions
Math 291-3
Homework 5
May 6, 2022

Exercise 1 (Colley 7.1.27, 7.1.29) For each part, compute the surface area of the given surface $S \subset \mathbb{R}^3$.

(a) S is the surface cut from the paraboloid $z = 2x^2 + 2y^2$ by the planes $z = 2$ and $z = 8$.

(b) S is the surface defined by the equation $z = f(x, y)$ for a C^1 function $f : D \to \mathbb{R}$ over an elementary region $D \subset \mathbb{R}^2$, assuming that there is $a \in (0, \infty)$ such that for each $(x, y) \in D$, $(f_x(x, y))^2 + (f_y(x, y))^2 = a$. Your answer should be in terms of the area of D.

(a) First we parametrize this surface using

$$\vec{X}(r, \theta) = (r\cos(\theta), r\sin(\theta), 2r^2),\ 0 \le \theta \le 2\pi,\ 1 \le r \le 2.$$

The choice of $r\cos(\theta)$ and $r\sin(\theta)$ for x and y come from the $x^2 + y^2$ term showing up in the equation for the surface, the equation for z comes from plugging these values of x and y into the surface equation, and the bounds on r come from the values which give $z = 2$ and $z = 8$.

From this, we have

$$\vec{X}_r(r, \theta) = (\cos(\theta), \sin(\theta), 4r) \text{ and } \vec{X}_\theta = (-r\sin(\theta), r\cos(\theta), 0).$$

This gives $N_{\vec{X}}(r, \theta) = \vec{X}_r(r, \theta) \times \vec{X}_\theta(r, \theta) = (-4r^2\cos(\theta), -4r^2\sin(\theta), r)$, so

$$\|N_{\vec{X}}(r, \theta)\| = \sqrt{16r^4 + r^2} = r\sqrt{16r^2 + 1}.$$

Thus the surface area is

$$\begin{aligned}\iint_D \|N_{\vec{X}}(r, \theta)\|\, dr\, d\theta &= \int_0^{2\pi} \int_1^2 r\sqrt{16r^2 + 1}\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{48}(16r^2 + 1)^{3/2}\Big|_1^2\, d\theta \\ &= \frac{\pi}{24}(65^{3/2} - 17^{3/2}).\end{aligned}$$

(b) We take as parametric equations:

$$\vec{X}(x, y) = (x, y, f(x, y)),\ (x, y) \in D.$$

We get:

$$N_{\vec{X}}(x, y) = (1, 0, f_x(x, y)) \times (0, 1, f_y(x, y)) = (-f_x(x, y), -f_y(x, y), 1).$$

Thus the surface area of S is

$$\iint_D \|N_{\vec{X}}(x, y)\|\, dA = \iint_D \sqrt{(f_x(x, y))^2 + (f_y(x, y))^2 + 1}\, dA = \iint_D \sqrt{a + 1}\, dA = \sqrt{a + 1} \iint_D 1\, dA,$$

which is $\sqrt{a + 1}\text{Area}(D)$.

■

Exercise 2 (Colley 7.1.30) Let S be the surface defined by

$$z = \frac{1}{\sqrt{x^2+y^2}} \quad \text{for } z \geq 1.$$

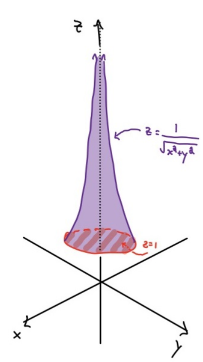
The point of this problem is to show that S has infinite surface area, but nevertheless encloses a region of finite volume.

(a) Sketch the graph of this surface.

(b) Show that the volume of the region bounded by S and the plane $z = 1$ is finite. (You can compute this as an improper integral. To do so, let W_a denote the region bounded above by S, below by $z = 1$, and that lies outside of the cylinder $x^2 + y^2 = a^2$. Then the volume you are after should be given as $\lim_{a \to 0+} \mathrm{Vol}_3(W_a)$. Compute $\mathrm{Vol}_3(W_a)$, and then compute the limit using single-variable calculus techniques.)

(c) Show that the surface area of S is infinite. (You will need to use another improper integral here.)

(a) In cylindrical coordinates the surface is given by $z = \frac{1}{r}$. Thus, it looks like a branch of a hyperbola rotated around the z-axis:



In particular, the surface gets thinner and thinner as z increases.

(b) In cylindrical coordinates, this volume is given by

$$\begin{aligned}
\mathrm{Vol}_3(W) &= \lim_{a \to 0+} \iiint_{W_a} 1 \, dV(x,y,z) \\
&= \lim_{a \to 0+} \int_0^{2\pi} \int_a^1 \int_1^{1/r} r \, dz \, dr \, d\theta. \\
&= \lim_{a \to 0+} \int_0^{2\pi} \int_a^1 (1-r) \, dr \, d\theta \\
&= \lim_{a \to 0+} \int_0^{2\pi} \frac{(1-a)^2}{2} \, d\theta \\
&= \lim_{a \to 0+} \pi(1-a)^2 \\
&= \pi.
\end{aligned}$$

(c) Let S_a denote the portion of S between the planes $z = 1$ and $z = \frac{1}{a}$. (Note that S_a is a portion of the boundary of the region W_a considered in part (b).) Then the surface area of S should be given by $\lim_{a\to 0+}$ Surface Area(S_a). We parametrize S_a using:

$$\vec{X}(r,\theta) = \left(r\cos(\theta), r\sin(\theta), \frac{1}{r}\right), \quad a \leq r \leq 1,\ 0 \leq \theta \leq 2\pi.$$

We have:

$$\vec{X}_r(r,\theta) \times \vec{X}_\theta(r\theta) = \left(\cos(\theta), \sin(\theta), -\frac{1}{r^2}\right) \times (-r\sin(\theta), r\cos(\theta), 0) = \left(\frac{\cos(\theta)}{r}, \frac{\sin(\theta)}{r}, r\right),$$

and hence the surface area of S_a is given by

$$\int_0^{2\pi}\int_a^1 \|N_{\vec{X}}(r,\theta)\|\, dr\, d\theta = \int_0^{2\pi}\int_a^1 \sqrt{\frac{1}{r^2} + r^2}\, dr\, d\theta.$$

The integrand here is never smaller than $\sqrt{\frac{1}{r^2}} = \frac{1}{r}$, so we have:

$$\int_0^{2\pi}\int_a^1 \sqrt{\frac{1}{r^2} + r^2}\, dr\, d\theta \geq \int_0^{2\pi}\int_a^1 \frac{1}{r}\, dr\, d\theta = \int_0^{2\pi} -\ln(a)\, d\theta = -2\pi\ln(a).$$

Therefore we have

$$\text{Surface Area}(S) = \lim_{a\to 0+} \text{Surface Area}(S_a) \geq \lim_{a\to 0+} -2\pi\ln(a) = +\infty,$$

so that S has infinite surface area.

■

Exercise 3 (Colley 7.1.32, 7.1.33) In this problem you determine formulas for the surface area of graphs of functions defined in terms of cylindrical and spherical coordinates.

(a) Suppose that a surface S is given in cylindrical coordinates by the equation $z = f(r,\theta)$, where (r,θ) varies through a region D in the $r\theta$-plane where r is nonnegative. Show that the surface area of S is given by

$$\iint_D \sqrt{1 + (f_r(r,\theta))^2 + \frac{1}{r^2}(f_\theta(r,\theta))^2}\, r\, dA(r,\theta).$$

(b) Suppose that a surface S is given in spherical coordinates by the equation $\rho = f(\phi,\theta)$, where (ϕ,θ) varies through a region D in the $\phi\theta$-plane and $f(\phi,\theta)$ is nonnegative. Show that the surface area of S is given by

$$\iint_D f(\phi,\theta)\sqrt{((f(\phi,\theta))^2 + (f_\phi(\phi,\theta))^2)\sin^2\phi + (f_\theta(\phi,\theta))^2}\, dA(\phi,\theta).$$

(a) In cylindrical coordinates the surface is parametrized by

$$\vec{X}(r,\theta) = (r\cos\theta, r\sin\theta, f(r,\theta)), \quad (r,\theta) \in D.$$

Thus

$$\begin{aligned}\vec{X}_r(r,\theta)\times\vec{X}_\theta(r,\theta) &= (\cos(\theta),\sin(\theta),f_r(r,\theta))\times(-r\sin(\theta),r\cos(\theta),f_\theta(r,\theta))\\ &= (f_\theta(r,\theta)\sin(\theta)-rf_r(r,\theta)\cos(\theta),-rf_r(r,\theta)\sin(\theta)-f_\theta(r,\theta)\cos(\theta),r),\end{aligned}$$

so

$$\begin{aligned}&\|N_{\vec{X}}(r,\theta)\|\\ &=\sqrt{(f_\theta(r,\theta)\sin(\theta)-rf_r(r,\theta)\cos(\theta))^2+(-rf_r(r,\theta)\sin(\theta)-f_\theta(r,\theta)\cos(\theta))^2+r^2}\\ &=\sqrt{(f_\theta(r,\theta))^2+r^2(f_r(r,\theta))^2+r^2}\\ &=r\sqrt{\frac{1}{r^2}(f_\theta(r,\theta))^2+(f_r(r,\theta))^2+1}.\end{aligned}$$

Hence the surface area is

$$\iint\limits_D r\sqrt{\frac{1}{r^2}(f_\theta(r,\theta))^2+(f_r(r,\theta))^2+1}\,dA(r,\theta)$$

as claimed.

(b) In spherical coordinates the surface is parametrized by

$$\vec{X}(\phi,\theta)=(f(\phi,\theta)\sin(\phi)\cos(\theta),f(\phi,\theta)\sin(\phi)\sin(\theta),f(\phi,\theta)\cos(\phi)),\quad(\phi,\theta)\in D.$$

We have

$$\vec{X}_\phi(\phi,\theta)=\begin{bmatrix}f_\phi(\phi,\theta)\sin(\phi)\cos(\theta)+f(\phi,\theta)\cos(\phi)\cos(\theta)\\ f_\phi(\phi,\theta)\sin(\phi)\sin(\theta)+f(\phi,\theta)\cos(\phi)\sin(\theta)\\ f_\phi(\phi,\theta)\cos(\phi)-f(\phi,\theta)\sin(\phi)\end{bmatrix}$$

and

$$\vec{X}_\theta(\phi,\theta)=\begin{bmatrix}f_\theta(\phi,\theta)\sin(\phi)\cos(\theta)-f(\phi,\theta)\sin(\phi)\sin(\theta)\\ f_\theta(\phi,\theta)\sin(\phi)\sin(\theta)+f(\phi,\theta)\sin(\phi)\cos(\theta)\\ f_\theta(\phi,\theta)\cos(\phi)\end{bmatrix},$$

so

$$\vec{X}_\phi(\phi,\theta)\times\vec{X}_\theta(\phi,\theta)=\begin{bmatrix}f(\phi,\theta)f_\theta(\phi,\theta)\sin(\theta)\\ f(\phi,\theta)(f(\phi,\theta)\sin^2(\phi)\sin(\theta)-f_\phi(\phi,\theta)\sin(\phi)\cos(\phi)\sin(\theta)-f_\theta(\phi,\theta)\cos^2(\phi)\cos(\theta))\\ f(\phi,\theta)f_\phi(\phi,\theta)\sin^2(\phi)+(f(\phi,\theta))^2\sin(\phi)\cos(\phi)\end{bmatrix}.$$

Thus (after a very lengthy computation using $\sin^2(\alpha)+\cos^2(\alpha)=1$ a bunch of times) we get

$$\|N_{\vec{X}}(\phi,\theta)\|=\sqrt{(f(\phi,\theta))^2((f(\phi,\theta))^2+(f_\phi(\phi,\theta))^2)\sin^2(\phi)+(f(\phi,\theta))^2(f_\theta(\phi,\theta))^2},$$

so the surface area is

$$\iint\limits_D f(\phi,\theta)\sqrt{((f(\phi,\theta))^2+(f_\phi(\phi,\theta))^2)\sin^2(\phi)+(f_\theta(\phi,\theta))^2}\,dA(\phi,\theta)$$

as claimed.

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Exercise 4 Suppose $S \subset \mathbb{R}^3$ is a smooth C^1 surface with two parametrizations:

$$\vec{X} : D \to \mathbb{R}^3 \quad \text{and} \quad \vec{Y} : E \to \mathbb{R}^3$$

where D is a region in the uv-plane and E a region in the st-plane. To be clear, u and v denote the parameters in the $\vec{X}$-parametric equations and $\vec{X}(u,v)$ is defined for $(u,v) \in D$, and s and t denote the parameters in the $\vec{Y}$-parametric equations and $\vec{Y}(s,t)$ is defined for $(s,t) \in E$. Suppose further that these two parametrizations are related by $\vec{Y}(s,t) = \vec{X} \circ T(s,t)$ where $T : E \to D$ is some C^1 bijective map with $\det DT(s,t) \neq 0$ for every $(s,t) \in E$.

(a) (This part is only to establish the standard algebraic properties of the cross product, which will be useful in part (b) and going forward.) Recall from Problem 3 on Homework 3 from MATH 291-2, for $\vec{u}, \vec{v} \in \mathbb{R}^3$ the cross product $\vec{u} \times \vec{v} \in \mathbb{R}^3$ is the unique vector that satisfies $\vec{x} \cdot (\vec{u} \times \vec{v}) = \det(\vec{x}, \vec{u}, \vec{v})$ for every $\vec{x} \in \mathbb{R}^3$. Prove that for every $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$,

  (i) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$,

  (ii) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ and $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$, and

  (iii) $(\lambda \vec{u}) \times \vec{v} = \lambda(\vec{u} \times \vec{v}) = \vec{u} \times (\lambda \vec{v})$.

(b) If $N_{\vec{X}}(u,v)$ and $N_{\vec{Y}}(s,t)$ denote the normal vectors arising from $\vec{X}$ and $\vec{Y}$ respectively at a point $(u,v) = T(s,t)$, show that

$$N_{\vec{Y}}(s,t) = (\det DT(s,t)) N_{\vec{X}}(T(s,t)).$$

This says that the Jacobian $\det DT(s,t)$ describes how to express normal vectors with respect to one parametrization in terms of normal vectors with respect to another parametrization. (Suggestion: First apply the chain rule to $\vec{Y}(s,t) = \vec{X}(T(s,t))$, and then use the result to write $\vec{Y}_s(s,t)$ and $\vec{Y}_t(s,t)$ as linear combinations of $\vec{X}_u(T(s,t))$ and $\vec{X}_v(T(s,t))$.)

(c) Show that

$$\iint_E \|N_{\vec{Y}}(s,t)\| \, dA(s,t) = \iint_D \|N_{\vec{X}}(u,v)\| \, dA(u,v),$$

and therefore the surface area of S is well-defined (i.e. it does not depend on which parametrization for S we use).

(a) Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Then for every $\vec{x} \in \mathbb{R}^3$ we have

$$\vec{x} \cdot (\vec{v} \times \vec{u}) = \det(\vec{x}, \vec{v}, \vec{u}) = -\det(\vec{x}, \vec{u}, \vec{v}) = -(\vec{x} \cdot (\vec{u} \times \vec{v})) = \vec{x} \cdot (-(\vec{u} \times \vec{v})),$$

so by uniqueness of the cross product we have $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$. Note by a similar argument we have, for each $\vec{x} \in \mathbb{R}^3$,

$$\vec{x} \cdot (\vec{u} \times (\vec{v} + \vec{w})) = \det(\vec{x}, \vec{u}, \vec{v} + \vec{w}) = \det(\vec{x}, \vec{u}, \vec{v}) + \det(\vec{x}, \vec{u}, \vec{w}) = \vec{x} \cdot (\vec{u} \times \vec{v}) + \vec{x} \cdot (\vec{u} \times \vec{w}) = \vec{x} \cdot (\vec{u} \times \vec{v} + \vec{u} \times \vec{w})$$

and

$$\vec{x} \cdot ((\lambda \vec{u}) \times \vec{v}) = \det(\vec{x}, \lambda \vec{u}, \vec{v}) = \lambda \det(\vec{x}, \vec{u}, \vec{v}) = \lambda(\vec{x} \cdot (\vec{u} \times \vec{v}) = \vec{x} \cdot (\lambda(\vec{u} \times \vec{v})),$$

whence it follows that $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ and $(\lambda \vec{u}) \times \vec{v} = \lambda(\vec{u} \times \vec{v})$. For the other parts of (ii) and (iii), we simply apply the results we just proved to see that

$$(\vec{u} + \vec{v}) \times \vec{w} = -(\vec{w} \times (\vec{u} + \vec{v})) = -(\vec{w} \times \vec{u} + \vec{w} \times \vec{v}) = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$$

and

$$\vec{u} \times (\lambda \vec{v}) = -((\lambda \vec{v}) \times \vec{u}) = -\lambda(\vec{v} \times \vec{u}) = \lambda(\vec{u} \times \vec{v}).$$

(b) Note first that since $\vec{Y}(s,t) = \vec{X}(T(s,t))$, the Chain Rule gives

$$D\vec{Y}(s,t) = D\vec{X}(T(s,t))DT(s,t).$$

Inspecting the columns of the matrix on the left-hand-side, we have

$$\vec{Y}_s(s,t) = D\vec{X}(T(s,t))T_s(s,t) = u_s(s,t)\vec{X}_u(T(s,t)) + v_s(s,t)\vec{X}_v(T(s,t))$$

and

$$\vec{Y}_t(s,t) = D\vec{X}(T(s,t))T_t(s,t) = u_t(s,t)\vec{X}_u(T(s,t)) + v_t(s,t)\vec{X}_v(T(s,t)).$$

Therefore

$$\begin{aligned}
N_{\vec{Y}}(s,t) &= \vec{Y}_s(s,t) \times \vec{Y}_t(s,t) \\
&= (u_s(s,t)\vec{X}_u(T(s,t)) + v_s(s,t)\vec{X}_v(T(s,t))) \times (u_t(s,t)\vec{X}_u(T(s,t)) + v_t(s,t)\vec{X}_v(T(s,t))) \\
&= u_s(s,t)u_t(s,t)\Big(\underbrace{\vec{X}_u(T(s,t)) \times \vec{X}_u(T(s,t))}_{=\vec{0}}\Big) + u_s(s,t)v_t(s,t)\Big(\vec{X}_u(T(s,t)) \times \vec{X}_v(T(s,t))\Big) \\
&\quad + v_s(s,t)u_t(s,t)\Big(\underbrace{\vec{X}_v(T(s,t)) \times \vec{X}_u(T(s,t))}_{=-(\vec{X}_u(T(s,t))\times\vec{X}_v(T(s,t)))}\Big) + v_s(s,t)v_t(s,t)\Big(\underbrace{\vec{X}_v(T(s,t)) \times \vec{X}_v(T(s,t))}_{=\vec{0}}\Big) \\
&= (u_s(s,t)v_t(s,t) - v_s(s,t)u_t(s,t))\Big(\vec{X}_u(T(s,t)) \times \vec{X}_v(T(s,t))\Big) \\
&= (\det DT(s,t))N_{\vec{X}}(T(s,t)).
\end{aligned}$$

(c) By the change of variables formula for $(u,v) = T(s,t)$, we have:

$$\begin{aligned}
\iint\limits_{D=T(E)} \|N_{\vec{X}}(u,v)\|\, dA(u,v) &= \iint\limits_{E} \|N_{\vec{X}}(T(s,t))\| |\det DT(s,t)|\, dA(s,t) \\
&= \iint\limits_{E} \|(\det DT(s,t))N_{\vec{X}}(T(s,t))\|\, dA(s,t) \\
&= \iint\limits_{E} \|N_{\vec{Y}}(s,t)\|\, dA(s,t),
\end{aligned}$$

where we applied (b) in the final step.

■

Exercise 5 (Colley 3.3.20) Calculate the flow line $\vec{x}(t)$ of the vector field $\vec{F}(x,y) = -x\vec{i} + y\vec{j}$ that passes through the point $\vec{x}(0) = (2,1)$. In addition (not in the book), sketch the field and the flow line you found.

Flow lines $\vec{x}(t) = (x(t), y(t))$ of $\vec{F}$ satisfy

$$(x'(t), y'(t)) = \vec{F}(x(t), y(t)) = (-x(t), y(t)),$$

or equivalently

$$x'(t) = -x(t) \quad \text{and} \quad y'(t) = y(t).$$

Thus, $x(t)$ must be of the form $x(t) = ae^{-t}$ for some $a \in \mathbb{R}$ and $y(t)$ is of the form $y(t) = be^t$ for some $b \in \mathbb{R}$. The specific flow line passing through $\vec{x}(0) = (2, 1)$ thus satisfies

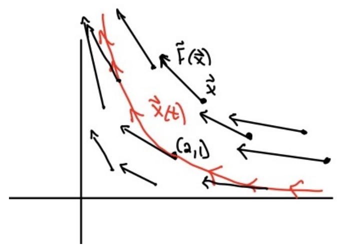
$$(2, 1) = (x(0), y(0)) = (a, b),$$

so

$$\vec{x}(t) = (2e^{-t}, e^t)$$

is the flow line passing through $\vec{x}(0) = (2, 1)$.

Note that the components of this flow line satisfy the equation $xy = 2$, which means that the flow line we want lies on this hyperbola. As $t \to \infty$, $2e^{-t} \to 0$ so the specific flow line we found approaches the y-axis. Thus, the field and the flow line looks like:



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Exercise 6 (Colley 3.3.25) If $\vec{x}(t)$ is a flow line of a gradient vector field $\vec{F} = \nabla f$, show that the function $G(t) = f(\vec{x}(t))$ is an increasing function of t.

By the chain rule, we have:

$$G'(t) = \nabla f(\vec{x}(t)) \cdot \vec{x}'(t) \text{ for all } t.$$

Since $\vec{x}$ is a flow line of ∇f, we have

$$\vec{x}'(t) = \nabla f(\vec{x}(t)) \text{ for all } t.$$

Thus

$$G'(t) = \nabla f(\vec{x}(t)) \cdot \vec{x}'(t) = \vec{x}'(t) \cdot \vec{x}'(t) = \|\vec{x}'(t)\|^2 \geq 0,$$

so G is increasing as claimed. (This all makes sense using what we know about gradients. We saw last quarter that the gradient of f at a point always points in the direction in which f increases most rapidly, so that if we are always *moving* in the direction *of* the gradient, f should never decrease. Saying that $\vec{x}(t)$ is a flow line of ∇f indeed says that at each point we are always moving in the direction of the gradient.) ■

Exercise 7 Suppose $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a *linear* vector field, meaning that there is $A \in M_{n\times n}(\mathbb{R})$ such that

$$\vec{F}(\vec{x}) = A\vec{x} \text{ for every } \vec{x} \in \mathbb{R}^n.$$

A flow line $\vec{x}(t)$ of $\vec{F}$ then satisfies

$$\vec{x}'(t) = \vec{F}(\vec{x}(t)) = A\vec{x}(t).$$

This problem shows that finding flow lines of linear vector fields reduces to computing eigenvalues and eigenvectors, giving an example of the key role linear algebra plays in the study of systems of differential equations, of which $\vec{x}\,'(t) = A\vec{x}(t)$ is an example.

(a) Suppose $\vec{x}(t)$ is of the form $\vec{x}(t) = e^{\lambda t}\vec{v}$ where $\vec{v}$ is some nonzero constant vector in $\mathbb{R}^n$, so that $\vec{x}(t)$ looks like

$$\vec{x}(t) = (v_1 e^{\lambda t}, \dots, v_n e^{\lambda t}).$$

Show that such an $\vec{x}(t)$ is a flow line of $\vec{F}$ if, and only if, $\vec{v}$ is an eigenvector of A with eigenvalue λ.

(b) We will take for granted the following fact from differential equations: the collection of all $\vec{x}(t)$ satisfying the equation $\vec{x}\,'(t) = A\vec{x}(t)$ forms an n-dimensional real vector space, and thus any n linearly independent solutions will span the entire space of solutions. Using the result of (a), find a basis for the space of flow lines of the two-dimensional linear vector field given by

$$\vec{F}(x, y) = (-3x + 3y, 3x + 5y).$$

(c) Find the flow line of $\vec{F}(x, y) = (-3x + 3y, 3x + 5y)$ passing through $(1, 1)$ at time $t = 0$. What happens along this flow line as $t \to \infty$?

(a) For $\vec{x}(t) = e^{\lambda t}(v_1, \dots, v_n) = (e^{\lambda t}v_1, \dots, e^{\lambda t}v_n)$, we have

$$\vec{x}'(t) = (\lambda e^{\lambda t}v_1, \dots, \lambda e^{\lambda t}v_n) = \lambda e^{\lambda t}(v_1, \dots, v_n).$$

Thus $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a flow line of $\vec{F}(\vec{x}) = A\vec{x}$ if and only if

$$\lambda e^{\lambda t}\vec{v} = \vec{x}'(t) = A\vec{x}(t) = A(e^{\lambda t}\vec{v}) = e^{\lambda t}A\vec{v}.$$

Since $e^{\lambda t}$ is never zero, this condition is satisfied if and only if

$$\lambda\vec{v} = A\vec{v},$$

so if and only if $\vec{v}$ is an eigenvector of A with eigenvalue λ.

(b) By part (a) we know that $\vec{x}(t) = e^{\lambda t}\vec{v}$ will be a flow line of this $\vec{F}$ when λ is an eigenvalue of $A = \begin{bmatrix} -3 & 3 \\ 3 & 5 \end{bmatrix}$ and $\vec{v}$ a corresponding eigenvector. The eigenvalues of A are 6 and -4, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

respectively. Thus,

$$\vec{x}_1(t) = e^{6t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{x}_2(t) = e^{-4t}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

are flow lines of $\vec{F}$. Since these are linearly independent, then form a basis for the space of all flow lines of $\vec{F}$.

(c) As a result of part (b), any flow line of $\vec{F}$ can be written as

$$\vec{x}(t) = c_1 e^{6t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

for some $c_1, c_2 \in \mathbb{R}$. In order to satisfy $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we need:

$$c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solving for c_1, c_2 gives

$$c_1 = \frac{2}{5} \quad \text{and} \quad c_2 = -\frac{1}{5},$$

so

$$\vec{x}(t) = \frac{2}{5} e^{6t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \frac{1}{5} e^{-4t} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

gives the flow line of $\vec{F}$ passing through $(1,1)$. As $t \to \infty$, $e^{-4t} \to 0$, so this flow line approaches the one given by

$$\vec{x}_1(t) = e^{6t} \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

which is on the line spanned by $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, as $t \to \infty$.

■

Exercise 8 (Colley 3.4.4, 3.4.11) Find the curl and divergence of each of the following vector fields:

$$\vec{F}(x,y,z) = z\cos(e^{y^2})\,\vec{\imath} + x\sqrt{z^2+1}\,\vec{\jmath} + e^{2y}\sin(3x)\,\vec{k} \qquad \vec{F}(x,y,z) = y^2 z\,\vec{\imath} + e^{xyz}\,\vec{\jmath} + x^2 y\,\vec{k}$$

(3.4.4) We have

$$\begin{aligned} \operatorname{curl}(\vec{F}(x,y,z)) &= \Big((F_3)_y - (F_2)_z\Big)\vec{\imath} + \Big((F_1)_z - (F_3)_x\Big)\vec{\jmath} + \Big((F_2)_x - (F_1)_y\Big)\vec{k} \\ &= \left(2e^{2y}\sin 3x - \frac{xz}{\sqrt{x^2+1}}\right)\vec{\imath} + (\cos(e^{y^2}) - 3e^{2y}\cos 2x)\vec{\jmath} + (\sqrt{z^2+1} + z\sin(e^{y^2})2ye^{y^2})\vec{k}. \end{aligned}$$

Also,

$$\operatorname{div}(\vec{F}(x,y,z)) = (z\cos(e^{y^2}))_x + (x\sqrt{z^2+1})_y + (e^{2y}\sin 2x)_z = 0.$$

(3.4.11) We have

$$\operatorname{curl}(\vec{F}(x,y,z)) = (x^2 - xye^{xyz})\vec{\imath} + (y^2 - 2xy)\vec{\jmath} + (yze^{xyz} - 2yz)\vec{k}.$$

Also,

$$\operatorname{div}(\vec{F}(x,y,z)) = (y^2 z)_x + (e^{xyz})_y + (x^2 y)_z = 0 + xze^{xyz} + 0 = xze^{xyz}.$$

■

Exercise 9 (Colley 3.4.28bc) The Laplacian operator, denoted Δ or ∇^2, is the second-order partial differential operator defined by

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Part (a) asks you to explain why it makes sense to think of ∇^2 as $\nabla \cdot \nabla$, which is just because $\nabla^2 f$ is obtained by taking the divergence of the gradient of f, so that $\nabla^2 f = \nabla \cdot (\nabla f)$. In what follows, you may use any of the results of Exercises 21 through 25 in Colley 3.4 without justification.

(b) Show that if f and g are C^2 functions, then

$$\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g).$$

(c) Show that

$$\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f.$$

(b) By the product rule, we have

$$\nabla(fg) = g\nabla f + f\nabla g.$$

Taking the divergence of both sides gives:

$$\begin{aligned} \nabla \cdot \nabla(fg) &= \nabla \cdot (g\nabla f) + \nabla \cdot (f\nabla g) \\ &= \nabla g \cdot \nabla f + g\nabla^2 f + \nabla f \cdot \nabla g + f\nabla^2 g \\ &= f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g) \end{aligned}$$

as required.

(c) We have:

$$\begin{aligned} \nabla \cdot (f\nabla g - g\nabla f) &= \nabla \cdot (f\nabla g) - \nabla \cdot (g\nabla f) \\ &= \nabla f \cdot \nabla g + f\nabla^2 g - \nabla g \cdot \nabla f + g\nabla^2 f \\ &= f\nabla^2 g - g\nabla^2 f \end{aligned}$$

as claimed.

■

Exercise 10 Consider the following vector field $\vec{F} : D \to \mathbb{R}^2$ defined on the region $D = \{(x, y) \in \mathbb{R}^2 \ : \ (x, y) \neq (0, 0)\}$:

$$\vec{F}(x,y) = \left(-\frac{y}{x^2+y^2} \ , \ \frac{x}{x^2+y^2} \right).$$

This field has curl zero at every point of D, and yet is not conservative on D. However, it is conservative over different portions of D. For instance, on the portion of D in the first quadrant excluding the positive y-axis, we have

$$\vec{F}(x,y) = \nabla \left(\arctan \left(\frac{y}{x} \right) \right).$$

(a) Find a potential function for $\vec{F}$ on the portion of D in the second quadrant excluding the negative x-axis which has the value $\frac{\pi}{2}$ along the positive y-axis.

(b) Find a potential function for $\vec{F}$ on the portion of D in the third quadrant excluding the negative y-axis which has the value π along the negative x-axis.

(c) Find a potential function for $\vec{F}$ on the portion of D in the fourth quadrant excluding the positive x-axis which has the value $\frac{3\pi}{2}$ along the negative y-axis.

(d) Take the potential function you found in part (c), and consider its limit as (x, y) approaches the x-axis (keeping x fixed, and allowing $y \to 0-$) from the fourth quadrant. Does this limiting value agree with the value of the potential function $\arctan\left(\frac{y}{x}\right)$ for $\vec{F}$ in the portion of D in the first quadrant excluding the positive y-axis?

(a) The function

$$-\arctan\left(\frac{x}{y}\right) + \frac{\pi}{2}$$

is C^2 on this given region and has gradient equal to $\vec{F}$. Moreover, along the positive y-axis where $x = 0$ this has the value $\frac{\pi}{2}$ as desired.

(b) The function

$$\arctan\left(\frac{y}{x}\right) + \pi$$

is C^2 on this region and has gradient equal to $\vec{F}$. In addition, when $y = 0$ on the x-axis this has the value π as desired.

(c) The function

$$-\arctan\left(\frac{x}{y}\right) + \frac{3\pi}{2}$$

is C^2 on this region, has gradient $\vec{F}$, and has the value $\frac{3\pi}{2}$ on the y-axis when $x = 0$.

(d) For (x, y) in the fourth quadrant, $x > 0$ and $y < 0$. Thus $\frac{x}{y} < 0$. As (x, y) approaches the x-axis, $y \to 0$ so $\frac{x}{y}$ approaches $-\infty$. Hence, $\arctan\left(\frac{x}{y}\right)$ approaches $-\frac{\pi}{2}$, so

$$-\arctan\left(\frac{x}{y}\right) + \frac{3\pi}{2} \text{ approaches } \frac{\pi}{2} + \frac{3\pi}{2} = 2\pi.$$

The potential function $\arctan\left(\frac{y}{x}\right)$ for the portion of D i the first quadrant has value 0 on the x-axis, which does not agree with the value of 2π for the limit of the potential in the fourth quadrant. (This seems like a random thing to ask right now, but the point of this entire problem will hopefully be made clearer in a later homework.)

■