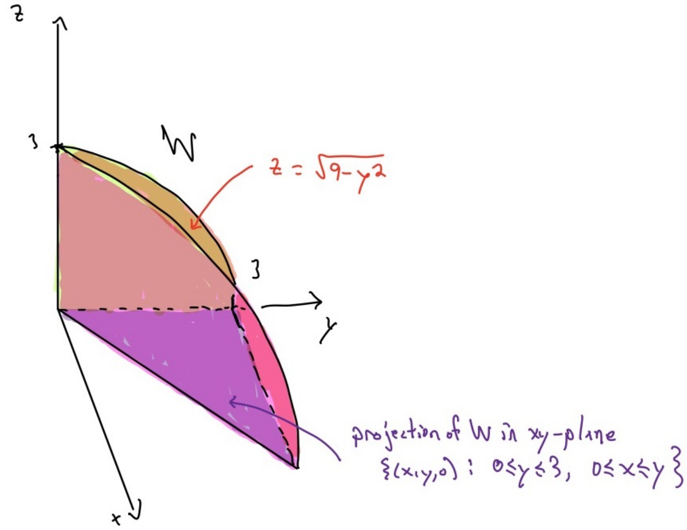
Solutions
Math 291-3
Homework 3
April 25, 2022

Exercise 1 (Colley 5.4.14 and 5.4.19) For each part, integrate the given function over the indicated region W.

(a) $f(x, y, z) = z$; W is the region in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and the planes $y = x$, $x = 0$, and $z = 0$.

(b) $f(x, y, z) = 4x + y$; W is the region bounded by $x = y^2$, $y = z$, $x = y$, and $z = 0$.

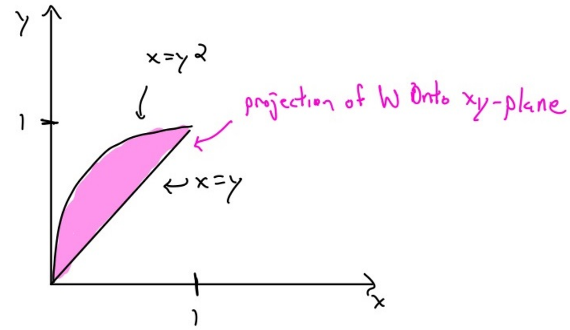
(a) The region W we are integrating over looks like:



We integrate with respect to $dz\,dx\,dy$, although other orders are equally doable. The projection of E on the xy-plane is the following triangular region described by $0 \le y \le 3$ and $0 \le x \le y$. For fixed (x, y) in the xy-plane we have $0 \le z \le \sqrt{9 - y^2}$. Thus

$$\begin{aligned}
\iiint_W z\, dV &= \int_0^3 \int_0^y \int_0^{\sqrt{9-y^2}} z\, dz\, dx\, dy \\
&= \int_0^3 \int_0^y \frac{1}{2}(9 - y^2)\, dx\, dy \\
&= \frac{1}{2} \int_0^3 (9y - y^3)\, dy \\
&= \frac{1}{2} \left(\frac{9}{2} y^2 - \frac{1}{4} y^4 \right) \Big|_0^3 \\
&= \frac{1}{2} \left(\frac{81}{2} - \frac{81}{4} \right) \\
&= \frac{81}{8}.
\end{aligned}$$

(b) The projection of W onto the xy-plane is characterized by the given bounds on x and y (forgetting z) alone, so it looks like:



For a fixed (x, y) in this shadow, the values of $0 \le z \le y$. Thus the required integral is:

$$\iiint_W (4x+y)\,dV = \int_0^1 \int_{y^2}^{y} \int_0^{y} (4x+y)\,dz\,dx\,dy$$
$$= \int_0^1 \int_{y^2}^{y} (4xy+y^2)\,dx\,dy$$
$$= \int_0^1 (3y^3 - 2y^5 - y^4)\,dy$$
$$= \frac{3}{4} - \frac{1}{3} - \frac{1}{5}.$$

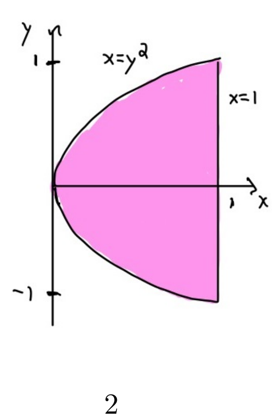
■

Exercise 2 (Colley 5.4.25 and 5.4.27) For each iterated integral, sketch the region of integration and rewrite the integral as an equivalent iterated integral in each of the five other orders of integration.

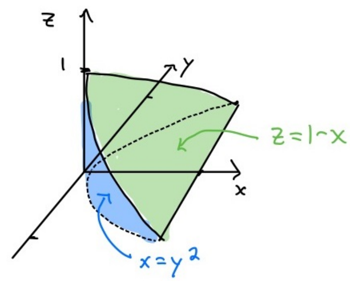
(a) $\displaystyle\int_{-1}^{1} \int_{y^2}^{1} \int_0^{1-x} f(x,y,z)\,dz\,dx\,dy$

(b) $\displaystyle\int_0^2 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx$

(a) First we sketch the region described by the bounds on x and y. The smallest and largest values of y are -1 and 1 respectively and x moves from the parabola $x = y^2$ to the right until $x = 1$, so we get:



This is the region in the xy-plane lying directly below our region of integration. Now, at a fixed (x, y) in the xy-plane z starts on the xy-plane at $z = 0$ and moves up to $z = 1 - x$, which is the plane obtained by taking the line $z = 1 - x$ on the xz-plane and sliding it out in the y-direction. On the xz-plane the line $z = 1 - x$ and surface $x = y^2$ intersection at $(0, 0, 1)$, and as we move $z = 1 - x$ along the y-axis this intersection point slides down until at $y = 1$ the intersection occurs at $(1, 1, 0)$ and at $y = -1$ it occurs at $(1, -1, 0)$. Thus the region of integration looks like:



Now, with respect to $dz\,dy\,dx$ the only thing which changes are the bounds on x and y, where now x goes from 0 to 1 and y goes from the one part of the parabola in the xy-plane $y = -\sqrt{x}$ to the other part at $y = -\sqrt{x}$. Thus we get

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{1-x} f(x, y, z)\,dz\,dy\,dx.$$

Next we consider the order $dy\,dx\,dz$. The projection of our region to the xz-plane is the region below the line $z = 1 - x$ in the first quadrant. Thus z goes from 0 to 1 and then x from 0 to $1 - z$. At a fixed (x, z) the value of y goes from $-\sqrt{x}$ to $\sqrt{x}$, so we get

$$\int_0^1 \int_0^{1-z} \int_{-\sqrt{x}}^{\sqrt{x}} f(x, y, z)\,dy\,dx\,dz.$$

With respect to $dy\,dz\,dx$ the only changes are the bounds on x and z, where now z goes from 0 to 1 and z from 0 to $1 - x$. Thus we get

$$\int_0^1 \int_0^{1-x} \int_{-\sqrt{x}}^{\sqrt{x}} f(x, y, z)\,dy\,dz\,dx.$$

Finally we integrate with respect to dx on the inside. The projection of our region to the yz-plane is the region in the upper half of the plane below $z = 1 - y^2$; this curve is found by eliminating z from the equations of the surfaces $x = y^2$ and $z = 1 - x$. With respect to $dx\,dy\,dz$, z goes from 0 to 1 and y from $-\sqrt{1-z}$ to $\sqrt{1-z}$, and then x from $x = y^2$ on the back to the plane $x = 1 - z$ on front, giving:

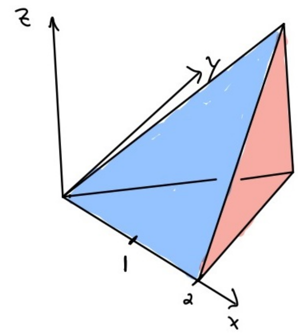
$$\int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{y^2}^{1-z} f(x, y, z)\,dx\,dy\,dz.$$

With respect to $dx\,dz\,dy$, y goes from -1 to 1, z from 0 to $1 - y^2$, and the bounds on x are the same as before. This gives

$$\int_{-1}^1 \int_0^{1-y^2} \int_{y^2}^{1-z} f(x, y, z)\,dx\,dz\,dy$$

as the final iterated integral.

(b) The shadow of the region of integration in the xy-plane is given by the bounds $0 \le x \le 2$ and $0 \le x \le y$, so this looks like a triangular region. For a fixed (x, y) in this shadow, the z values move from $z = 0$ to $z = y$, so overall the region of integration looks like a tetrahedron:



First, with respect to $dz\,dx\,dy$, the z bounds remain the same and only the x, y-bounds change to describe the shadow in the xy-plane; we get:

$$\int_0^2 \int_y^2 \int_0^y f(x, y, z)\,dz\,dx\,dy.$$

The shadow of the given region in the yz-plane is the triangle with boundaries $z = 0, z = y$, and $y = 2$. At a fixed (y, z) in this shadow, the values of x range from $x = y$, which is the back side of this region to $x = 2$, which is the front. Thus we get

$$\int_0^2 \int_0^y \int_y^2 f(x, y, z)\,dx\,dz\,dy \quad \text{and} \quad \int_0^2 \int_z^2 \int_y^2 f(x, y, z)\,dx\,dy\,dz$$

for the two orders with dz on the inside.

Finally, the shadow in the xz-plane is the triangle with boundaries $x = 2, z = 0$, and $z = x$, where $z = x$ is found by eliminating y from the planes $z = y$ and $y = x$. At a fixed (x, z) in this shadow, y ranges from $y = z$ to $y = x$, so we get

$$\int_0^2 \int_0^x \int_z^x f(x, y, z)\,dy\,dz\,dx \quad \text{and} \quad \int_0^2 \int_z^2 \int_z^x f(x, y, z)\,dy\,dx\,dz$$

for the iterated integrals with dy on the inside.

■

Exercise 3 (Colley 5.4.29) Consider the iterated integral

$$\int_{-2}^{2}\int_{0}^{\frac{1}{2}\sqrt{4-x^2}}\int_{x^2+3y^2}^{4-y^2}(x^3+y^3)\,dzdydx.$$

(a) This integral is equal to the triple integral over a solid region W in $\mathbb{R}^3$. Describe W.

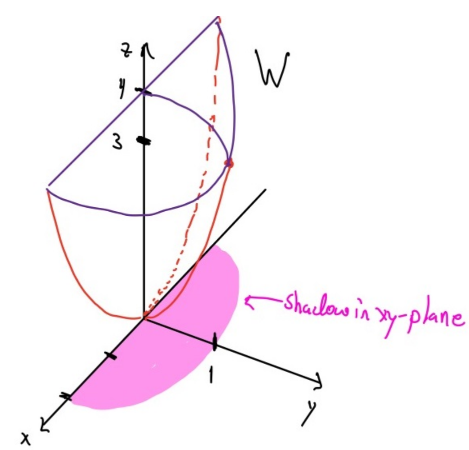
(b) Set up an equivalent iterated integral by integrating first with respect to z, then with respect to x, then with respect to y. Do not evaluate your answer.

(c) Set up an equivalent iterated integral by integrating first with respect to x, then with respect to z, then with respect to y. Do not evaluate your answer.

(d) Now consider integrating first with respect to x, then y, then z. Set up a sum of iterated integrals that, when evaluated, give the same result. Do not evaluate your answer.

(e) Repeat part (d) for integration first with respect to y, then z, then x.

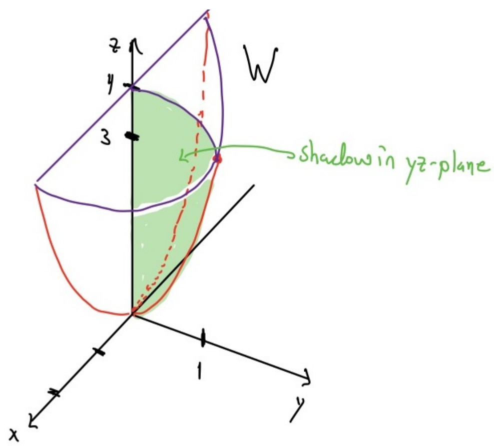
(a) The region W is the region to the right of the xz-plane which is above the paraboloid $z = x^2+3y^2$ and below the curved surface $z = 4-y^2$. The shadow in the xy-plane is the top half of the region enclosed by the ellipse $4 = x^2+4y^2$, where this ellipse lies directly below the intersection of $z = x^2+3y^2$ and $z = 4-y^2$. (The equation of the ellipse is found by eliminating z in the equation of the bottom and top surfaces.) The region thus looks like:



(b) The only difference here is that we describe the shadow in the opposite order. The min/max values of y are $0, 1$ and x moves from one side of the ellipse at $x = -\sqrt{4-4y^2}$ to the other side at $x = \sqrt{4-4y^2}$. Hence we get

$$\int_{0}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}}\int_{x^2+3y^2}^{4-y^2}(x^3+y^3)\,dz\,dx\,dy.$$

(c) The shadow in the yz-plane looks like:



The curve surface $z = 4 - y^2$ contributes nothing to the back and front boundaries of W since it is independent of x, so only the paraboloid $z = x^2 + 3y^2$ determines the back and front boundaries on x. Thus in this order we have:

$$\int_0^1 \int_{3y^2}^{4-y^2} \int_{-\sqrt{z-3y^2}}^{\sqrt{z-3y^2}} (x^3 + y^3)\, dx\, dz\, dy.$$

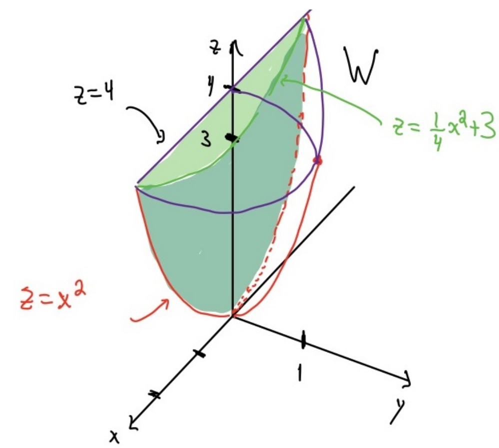
(d) With $dy\, dz$ as the middle and outer bounds, the right boundary on y in the yz-shadow consists of two different curves, which is why we must split this up region along the horizontal line $y = 1$ where this switch occurs. The inner bounds on x will be the same for both pieces since regardless of which piece we're behind or in front of, x always goes from the back side of the paraboloid to the front side. Thus we get:

$$\int_0^3 \int_0^{\sqrt{z/3}} \int_{-\sqrt{x-3y^2}}^{\sqrt{z-3y^2}} (x^3 + y^3)\, dx\, dy\, dz + \int_3^4 \int_0^{\sqrt{4-z}} \int_{-\sqrt{x-3y^2}}^{\sqrt{z-3y^2}} (x^3 + y^3)\, dx\, dy\, dz.$$

(e) With y on the inside, the right boundary of W consists of $z = 4 - y^2$ over part of W but $z = x^2 + 3y^2$ over other parts, so we must split up our region. To see where this split should occur, consider the intersection of the surfaces $z = 4 - y^2$ and $z = x^2 + 3y^2$. Pushing this intersection to the left onto the xy-plane gives the curve

$$z = x^2 + 3(4 - z), \text{ or } z = \frac{1}{4}x^2 + 3$$

whose equation is found by eliminating y in the equations of these surfaces. Hence the shadow of W in the xz-plane looks like:



Over the top part of this shadow y should come out as far to the right as the curved surface $z = 4 - y^2$ while over the rest of the shadow y should come out as far as the paraboloid $z = x^2 + 3y^2$. Thus, using the picture of this shadow to split up our region, we get:

$$\int_{-2}^{2}\int_{x^2}^{\frac{1}{4}x^2+3}\int_{0}^{\sqrt{(z-x^2)/3}}(x^3+y^3)\,dy\,dz\,dx + \int_{-2}^{2}\int_{\frac{1}{4}x^2+3}^{4}\int_{0}^{\sqrt{4-z}}(x^3+y^3)\,dy\,dz\,dx.$$

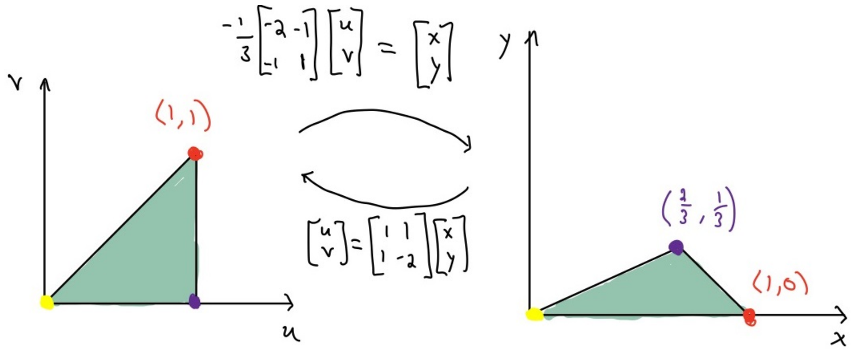
■

Exercise 4 (Colley 5.5.10 (Altered) and 5.5.12)

(a) Evaluate
$$\iint_D \sqrt{(x+y)(x-2y)}\,dA(x,y)$$
where D is the region in $\mathbb{R}^2$ enclosed by the lines $y = x/2, y = 0$, and $x + y = 1$.

(b) Evaluate
$$\iint_D \frac{(2x+y-3)^2}{(2y-x+6)^2}\,dx\,dy$$
where D is the (filled-in) square with vertices $(0,0),(2,1),(3,-1)$, and $(1,-2)$.

(a) We use the change of variables $u = x + y$ and $v = x - 2y$. Note that this is C^1 and injective since it is determined by the linear transformation with standard matrix
$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix},$$
which is invertible. Now, the boundaries $y = x/2, y = 0$, and $x + y = 1$ of D become $v = 0, u = v$, and $u = 1$ respectively in the uv-plane:



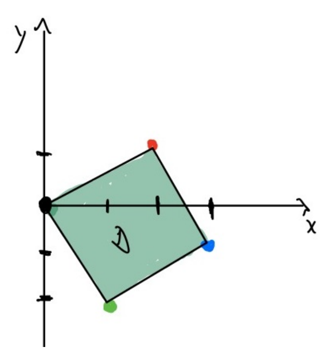
The function $\sqrt{(x+y)(x-2y)}$ becomes $\sqrt{uv}$, and since

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\det\left(\begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix}\right)\right| = 3,$$

the Jacobian of the transformation is $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \frac{1}{3}$. Thus

$$\begin{aligned}\iint\limits_D \sqrt{(x+y)(x-2y)}\,dA(x,y) &= \iint\limits_{D^*} \frac{1}{3}\sqrt{uv}\,dv\,du\\ &= \frac{1}{3}\int_0^1\int_0^u \sqrt{uv}\,dv\,du\\ &= \frac{1}{3}\int_0^1 \frac{2}{3}\sqrt{u}v^{3/2}\Big|_0^u\,du\\ &= \frac{1}{3}\int_0^1 \frac{2}{3}u^2\,du\\ &= \frac{2}{27}.\end{aligned}$$

(b) The region D looks like



Set $u = y+2x$ and $v = 2y-x$, so that the equations of the boundaries become $u = 5, u = 0, v = 0$, and $v = -5$. This transformation is given by the linear transformation $T^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix

$$\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix},$$

so it is injective and C^1. The Jacobian is

$$|\det DT(u,v)| = |\det DT^{-1}(x,y)|^{-1} = \left|\det \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\right|^{-1} = \frac{1}{5}.$$

Hence after changing variables we get

$$\int_0^5 \int_{-5}^0 \frac{1}{5}\frac{(u-3)^2}{(v+6)^2}\,dv\,du = \int_0^5 \frac{1}{6}(u-3)^2\,du = \frac{35}{18}$$

$\blacksquare$

Exercise 5 (Colley 5.5.17 (Altered) and 5.5.27)

(a) Exercise 5.5.17 in our book directs students to transform the following double integral into polar coordinates:

$$\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dA(x,y)$$

where D is the triangular region with vertices at $(0,0)$, $(3,0)$, and $(3,3)$. This is a bad problem because the Change of Variables Theorem does not apply. Why not?

(b) Use polar coordinates to evaluate $\iint_D \frac{x}{\sqrt{x^2+y^2}}\,dA(x,y)$ where D is the unit square $[0,1]\times[0,1]$. Why does this integral not suffer from the same issue as the one in part (a)?

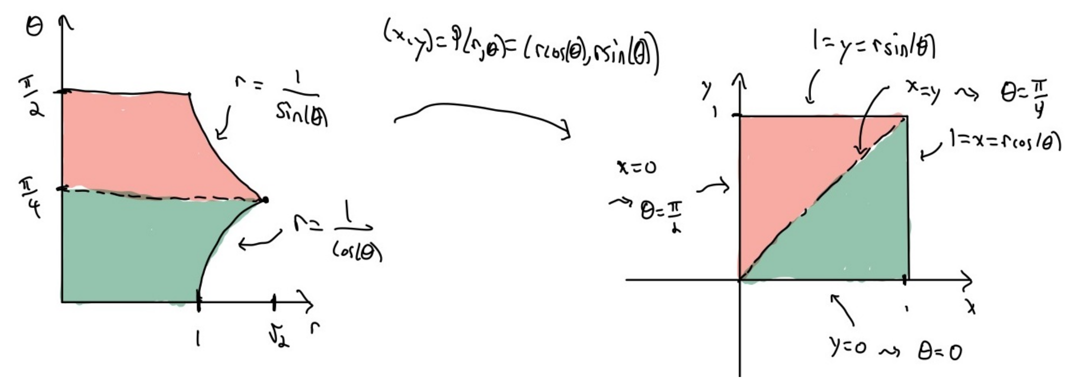
(a) This problem is defective because the integrand, $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$, is not bounded on D. Indeed, as $(x,y) \to (0,0)$ along the curve $x = y$, we have $\lim_{x\to 0+} f(x,x) = \lim_{x\to 0+} \frac{1}{\sqrt{2}x} = \infty$. Therefore f is not integrable on D, so that the Change of Variables Theorem does not directly apply.

(b) Here we note that the integrand $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$ is discontinuous only at $(0,0)$, and that

$$|f(x,y)| = \frac{|x|}{\sqrt{x^2+y^2}} \le \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = 1$$

for every $(x,y) \in D$ (except at $(0,0)$, of course). Therefore f is bounded on D and its set of discontinuities consists of a single point, which therefore has measure zero. Therefore f is integrable on D.

We proceed with our change of variables. In polar coordinates, the right side of the square $x = 1$ is $r\cos\theta = 1$ and the top side $y = 1$ is $r\sin\theta = 1$, so we must split up the square along the diagonal and change variables in each piece separately:



The diagonal is at $\theta = \pi/4$, so we have:

$$\begin{aligned}
\iint_D \frac{x}{\sqrt{x^2+y^2}}\,dA(x,y) &= \int_0^{\pi/4}\int_0^{1/\cos\theta} \frac{r\cos\theta}{r}\,r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{1/\sin\theta} \frac{r\cos\theta}{r}\,r\,dr\,d\theta \\
&= \int_0^{\pi/4}\int_0^{1/\cos\theta} r\cos\theta\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{1/\sin\theta} r\cos\theta\,dr\,d\theta \\
&= \int_0^{\pi/4} \frac{1}{2\cos\theta}\,d\theta + \int_{\pi/4}^{\pi/2} \frac{\cos\theta}{2\sin^2\theta}\,d\theta \\
&= \frac{1}{2}\ln(\sec\theta+\tan\theta)\Big|_0^{\pi/4} - \frac{1}{2\sin\theta}\Big|_{\pi/4}^{\pi/2} \\
&= \frac{1}{2}\ln(\sqrt{2}+1) - \frac{1}{2}\ln 1 - \frac{1}{2} + \frac{1}{\sqrt{2}} \\
&= \frac{1}{2}\ln(\sqrt{}+1) - \frac{1}{2} + \frac{\sqrt{2}}{2}.
\end{aligned}$$

(No, you do not have to know what the integral of $1/\cos\theta = \sec\theta$ is on the midterm.)

■

> **Exercise 6** Suppose D is a elementary region in $\mathbb{R}^n$, and that $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible affine transformation of the form $T(\vec{x}) = A\vec{x} + \vec{b}$. Show that $\mathrm{Vol}_n(T(D)) = |\det(A)|\mathrm{Vol}_n(D)$.

Since T is invertible it is injective, and $DT(\vec{u}) = A$ at any $\vec{u} \in \mathbb{R}^n$. Thus T satisfies the hypotheses of the change of coordinates theorem. The change of variables formula gives

$$\mathrm{Vol}_n(T(D)) = \int_{T(D)} 1 dV_n(\vec{x}) = \int_D |\det DT(\vec{u})|\,dV_n(\vec{u}) = |\det A| \int_D dV_n(\vec{u}) = |\det A|\mathrm{Vol}_n(D).$$

This justifies a fact we mentioned last quarter, that determinants not only give expansion factors for parallelopipeds, but for more general regions as well. ■

> **Exercise 7** Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1 and injective and that $\det(DT(\vec{x})) > 0$ for all $\vec{x}$ not on the line $y = x$. Suppose that $D \subset \mathbb{R}^2$ is an elementary region, and that $f : T(D) \to \mathbb{R}$ is integrable. Show that
>
> $$\iint_{T(D)} f(x,y)\,dA(x,y) = \iint_D f(x(u,v),y(u,v))\det(DT(u,v))\,dA(u,v)$$
>
> where $(x(u,v),y(u,v)) = T(u,v)$ denote the component functions of T and give the expressions for x, y in terms of u, v.

The change of variables formula gives

$$\iint_D f(x(u,v),y(u,v))|\det DT(u,v)|\,dA(u,v) = \iint_{T(D)} f(x,y)\,dA(x,y).$$

Now, when $u \neq v$, $DT(u,v)$ has positive determinant, and thus

$$|\det DT(u,v)| = \det DT(u,v) \text{ for } u \neq v.$$

On the other hand, because the component functions of T are C^1, $\det DT(u,v) = x_u(u,v)y_v(u,v) - y_u(u,v)x_v(u,v)$ is continuous on D, so that (in particular) since $\det DT(u,v) > 0$ when $u \neq v$, $\det DT(u,v) \geq 0$ when $u = v$. Therefore $|\det DT(u,v)| = \det DT(u,v)$ in this case as well.

$$f(x(u,v),y(u,v))|\det DT(u,v)|$$

being integrated on the left side above agrees with the function

$$f(x(u,v),y(u,v))\det DT(u,v)$$

throughout the square. This implies that the integral of both of these functions over D is the same, so

$$\iint_{T(D)} f(x,y)\,dA(x,y) = \iint_D f(x(u,v),y(u,v))\det(DT(u,v))\,dA(u,v)$$

as claimed. ∎

Exercise 8 Evaluate

$$\iint_D \frac{2xy^3}{x^2y^2+1}\,dA(x,y)$$

where D is the region in the first quadrant of $\mathbb{R}^2$ bounded by $xy = 1, xy = 4, y = 1$ and $y = 2$.

Based on the boundaries of D we use the change of variables

$$u = xy \text{ and } v = y.$$

Then the bounds in terms of u and v are all constant, and the Jacobian factor is

$$|\partial(x,y)(u,v)| = |\partial(u,v)(x,y)|^{-1} = \left|\det\begin{bmatrix} y & x \\ 0 & 1\end{bmatrix}\right|^{-1} = \frac{1}{|y|} = \frac{1}{y}$$

where in the last step we use the fact that y is positive in our region. Thus we get:

$$\begin{aligned}
\iint_D \frac{2xy^3}{x^2y^2+1}\,dA(x,y) &= \int_1^2\int_1^4 \frac{2xy^3}{u^2+1}\frac{1}{y}\,du\,dv \\
&= \int_1^2\int_1^4 \frac{2uv}{u^2+1}\,du\,dv \\
&= \int_1^2 v\ln(u^2+1)\Big|_1^4\,dv \\
&= \int_1^2 (\ln 17 - \ln 2)v\,dv \\
&= \frac{3}{2}\ln\frac{17}{2}.
\end{aligned}$$

∎

Exercise 9 (Colley 5.5.34) Determine the value of

$$\iiint_W \frac{1}{\sqrt{x^2+y^2+z^2}}\, dV(x,y,z)$$

where $W \subset \mathbb{R}^3$ is the region bounded by the two spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$ for $0<a<b$.

In spherical coordinates, the region W is given by $(\rho,\phi,\theta) \in B \stackrel{def}{=} [a,b]\times[0,\pi]\times[0,2\pi]$. The integral then becomes

$$\iiint_W \frac{1}{\sqrt{x^2+y^2+z^2}}\, dV(x,y,z) = \iiint_B \frac{1}{\rho}\rho^2\sin(\phi)\, dV(\rho,\phi,\theta) = \int_0^{2\pi}\int_0^{\pi}\int_a^b r\sin(\phi)\, dr d\phi d\theta = 2\pi(b^2-a^2).$$

■

Exercise 10 For $r > 0$, D_r^n denotes the closed ball in $\mathbb{R}^n$ of radius R centered at the origin, which is the set of all points in $\mathbb{R}^n$ whose distance to the origin is less than or equal to r:

$$D_r^n \overset{def}{=} \{\vec{x} \in \mathbb{R}^n \ : \ \|\vec{x}\| \leq r\}.$$

In other words, this is the region enclosed by the (hyper)sphere of radius r in $\mathbb{R}^n$ centered at the origin. For instance, D_1^2 is the unit disk in $\mathbb{R}^2$ and D_1^3 is the 3-dimensional unit ball in $\mathbb{R}^3$. The point of this problem is to compute $\mathrm{Vol}_n(D_r^n)$ in general. In class we showed that

$$\mathrm{Vol}_1(D_r^1) = 2r \quad \text{and} \quad \mathrm{Vol}_2(D_r^2) = \pi r^2,$$

which give the 1-volume (i.e. length) of $D_r^1 = [-r, r]$ and 2-volume (i.e. area) of D_r^2, which is the disk of radius r centered at the origin in $\mathbb{R}^2$.

(a) Show that $\mathrm{Vol}_n(D_r^n) = r^n \mathrm{Vol}_n(D_1^n)$. (Suggestion: First find a linear transformation that sends D_1^n onto D_r^n.)

(b) The n-volume of D_r^n is given by the n-dimensional integral of the constant 1 over D_r^n:

$$\mathrm{Vol}_n(D_r^n) = \int_{D_r^n} 1 \, dV_n,$$

which we can compute using Fubini's Theorem. Denote the coordinates of $\mathbb{R}^n$ by $(x_1, \ldots, x_n)$. For $n \geq 3$, express the n-dimensional integral above as an iterated integral of the form:

$$\int_?^? \int_?^? (\text{something}) \, dx_1 \, dx_2.$$

(To do this, think about what the "slice" of D_r^n occurring at a fixed (x_1, x_2) looks like. The "something" expression you come up with should involve volumes of lower dimensional balls.)

(c) Use polar coordinates in the x_1x_2-plane to compute the iterated integral above, and as a result derive the recursive relation:

$$\mathrm{Vol}(D_r^n) = \frac{2\pi r^2}{n} \mathrm{Vol}_{n-2}(D_r^{n-2}) \text{ for } n \geq 3.$$

(d) Use induction to show that for n even:

$$\mathrm{Vol}_n(D_r^n) = \frac{2^{\frac{n}{2}} \pi^{\frac{n}{2}} r^n}{2 \cdot 4 \cdot 6 \cdots n}$$

and for n odd:

$$\mathrm{Vol}_n(D_r^n) = \frac{2^{\frac{n+1}{2}} \pi^{\frac{n-1}{2}} r^n}{1 \cdot 3 \cdot 5 \cdots n}.$$

Note that part (d) is not required to be turned in, but you can certainly do it if you like.

(a) The ball of radius r can be obtained from the ball of radius 1 by scaling each direction (i.e. each axis) by a factor of r, meaning that D_r^n is the image of D_1^n under the linear transformation with matrix rI_n. This matrix has determinant r^n, so Problem 6 gives

$$\mathrm{Vol}_n(D_r^n) = r^n \mathrm{Vol}_n(D_1^n)$$

as claimed.

(b) The closed ball D_r^n is described by the inequality

$$x_1^2 + \cdots + x_n^2 \leq r^2.$$

The shadow of this region in the x_1x_2-plane is given by the intersection of this region with this plane, so this shadow is described by the inequality

$$x_1^2 + x_2^2 \leq r^2.$$

Thus, x_2 ranges from $-r$ to r, and at a fixed x_2 the value of x_1 ranges from $-\sqrt{r^2 - x_2^2}$ to $\sqrt{r^2 - x_2^2}$. The remaining integrals over $x_3, \ldots, x_n$ take place over the slice of D_r^n at a fixed (x_1, x_2), which is described by

$$x_3^2 + \cdots + x_n^2 \leq r^2 - x_1^2 - x_2^2$$

and is hence the closed ball of radius $\sqrt{r^2 - x_1^2 - x_2^2}$ in $\mathbb{R}^{n-2}$. Since these remaining integrals involve integrating the constant function 1, these integrals compute the volume of this lower-dimensional closed ball. Hence we get:

$$\begin{aligned}
\mathrm{Vol}_n(D_r^n) = \int_{D_r^n} 1 \, dV_n(\vec{x}) &= \int_{-r}^{r} \int_{-\sqrt{r^2-x_2^2}}^{\sqrt{r^2-x_2^2}} \underbrace{\int \cdots \int 1 \, dx_n \ldots dx_3}_{\text{takes place over lower-dimensional closed ball}} \, dx_1 \, dx_2 \\
&= \int_{-r}^{r} \int_{-\sqrt{r^2-x_2^2}}^{\sqrt{r^2-x_2^2}} \mathrm{Vol}_n\left(D^{n-2}_{\sqrt{r^2-x_1^2-x_2^2}}\right) dx_1 \, dx_2 \\
&= \int_{-r}^{r} \int_{-\sqrt{r^2-x_2^2}}^{\sqrt{r^2-x_2^2}} \left(r^2 - x_1^2 - x_2^2\right)^{\frac{n-2}{2}} \mathrm{Vol}_n\left(D_1^{n-2}\right) dx_1 \, dx_2,
\end{aligned}$$

where in the final equality we use the result of part (a) to express the volume of the ball of radius $\sqrt{r^2 - x_1^2 - x_2^2}$ in terms of the ball of radius 1.

(c) Converting the expression derived above into polar coordinates $x_1 = t\cos\theta, x_2 = t\sin\theta$ gives

$$\mathrm{Vol}_n(D_r^n) = \int_0^{2\pi} \int_0^r (r^2 - t^2)^{\frac{n}{2}-1} \mathrm{Vol}_{n-2}(D_1^{n-2}) \, t \, dt \, d\theta.$$

Computing this gives:

$$\int_0^{2\pi} -\frac{1}{n}(r^2 - t^2)^{\frac{n}{2}} \Big|_0^r \mathrm{Vol}_{n-2}(D_1^{n-2}) \, d\theta = \frac{2\pi t^n}{n} \mathrm{Vol}_{n-2}(D_1^{n-2}).$$

Using part (a) again this can be written as

$$\mathrm{Vol}_n(D_r^n) = \frac{2\pi r^2}{n} \mathrm{Vol}_{n-2}(D_r^{n-2}),$$

which is the desired recursive relation.

(d) Recall that $\mathrm{Vol}_2(D_r^2) = \pi r^2 = \frac{2\pi r^2}{2}$. Suppose

$$\mathrm{Vol}_n(D_r^n) = \frac{2^{\frac{n}{2}}\pi^{\frac{n}{2}}r^n}{2\cdot 4\cdot 6\cdots n}$$

for some even $n \geq 1$. Then

$$\mathrm{Vol}_{n+2}(D_r^{n+2}) = \frac{2\pi r^2}{n+2}\mathrm{Vol}_n(D_r^n) = \left(\frac{2\pi r^2}{n+2}\right)\frac{2^{\frac{n}{2}}\pi^{\frac{n}{2}}r^n}{2\cdot 4\cdot 6\cdots n} = \frac{2^{\frac{n+2}{2}}\pi^{\frac{n+2}{2}}r^{n+2}}{2\cdot 4\cdot 6\cdots (n+2)},$$

so the claimed equality holds for the next even integer $n+2$. Hence by induction we conclude that the claimed equality holds for all even $n \geq 1$.

Now recall that $\mathrm{Vol}_1(D_r^1) = 2r$. Suppose

$$\mathrm{Vol}_n(D_r^n) = \frac{2^{\frac{n+1}{2}}\pi^{\frac{n-1}{2}}r^n}{1\cdot 3\cdot 5\cdots n}$$

for some odd $n \geq 1$. Then

$$\mathrm{Vol}_{n+2}(D_r^{n+2}) = \frac{2\pi r^2}{n+2}\mathrm{Vol}_n(D_r^n) = \left(\frac{2\pi r^2}{n+2}\right)\frac{2^{\frac{n+1}{2}}\pi^{\frac{n-1}{2}}r^n}{1\cdot 3\cdot 5\cdots n} = \frac{2^{\frac{(n+2)+1}{2}}\pi^{\frac{(n+2)-1}{2}}r^{n+2}}{1\cdot 3\cdot 5\cdots (n+2)},$$

so the claimed equality holds for the next odd integer $n+2$. Hence by induction we conclude that the claimed equality holds for all odd $n \geq 1$.

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