Math 291-3: Discussion #6 Problems (Solutions)

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1. Suppose $D \subseteq \mathbb{R}^2$ is a compact region whose boundary is a simple, closed piecewise-smooth curve, oriented to that D is "on the left". Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a C^1 vector field defined on some open set containing D. Show that

$$\oint_{\partial D} \vec{F} \cdot \vec{n}\, ds = \iint_D (P_x + Q_y)\, dA(x, y)$$

where $\vec{n}$ denotes the unit normal vector to ∂D that points "outward" from D. To be clear, the integral on the left is *not* a vector line integral because we are taking $\vec{F} \cdot \vec{n}$ instead of $\vec{F} \cdot \vec{T}$. The integral on the left measures the extent to which $\vec{F}$ flows *through* (not *along*) ∂D. The equality here is meant to be an analog of Green's Theorem for this type of integral. (Indeed, this is sometimes called the **Divergence Theorem in the Plane**.)

Hint: How is $\vec{n}$ related to $\vec{T}$?

Solution. Note that $\vec{n}$ is obtained from $\vec{T}$ by rotating $\vec{T}$ clockwise by $\frac{\pi}{2}$ radians. In other words, we can transform $\vec{n}$ into $\vec{T}$ by rotating $\vec{n}$ counterclockwise by $\frac{\pi}{2}$ radians. Moreover, because rotations are orthogonal transformations, at each point on ∂D we have

$$\vec{F} \cdot \vec{n} = (R_{\frac{\pi}{2}}\vec{F}) \cdot (R_{\frac{\pi}{2}}\vec{n}) = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} \right) \cdot \vec{T} = (-Q\vec{i} + P\vec{j}) \cdot \vec{T}.$$

Therefore we apply Green's Theorem to obtain

$$\oint_{\partial D} \vec{F} \cdot \vec{n}\, ds = \oint_{\partial D} (-Q\vec{i} + P\vec{j}) \cdot d\vec{s} = \iint_D (P_x - (-Q)_y)\, dA(x, y) = \iint_D (P_x + Q_y)\, dA(x, y).$$

2. Recall the fact (stated in Exercise 4 of Homework 4) that if f is continuous on $\mathbb{R}^n$ then

$$\lim_{r \to 0+} \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} f(\vec{x})\, dV_n(\vec{x}) = f(\vec{x}_0).$$

Using this, prove that if $\vec{F} = P\vec{i} + Q\vec{j}$ is a C^1 vector field on $\mathbb{R}^2$, then

$$Q_x(x_0, y_0) - P_y(x_0, y_0) = \mathrm{curl}\vec{F}(x_0, y_0) = \lim_{r \to 0+} \frac{1}{\pi r^2} \oint_{C_r} \vec{F} \cdot d\vec{s},$$

where C_r is the circle $(x - x_0)^2 + (y - y_0)^2 = r^2$, oriented counterclockwise. It is in this sense that $\mathrm{curl}\vec{F}(x_0, y_0)$ measures "infinitesimal rotation of $\vec{F}$ at (x_0, y_0)".

Solution. By Exercise 4 on Homework 4 (and Green's Theorem) at $(x_0, y_0) \in \mathbb{R}^2$

$$\mathrm{curl}\vec{F}(x_0, y_0) = \lim_{r \to 0+} \frac{1}{\pi r^2} \iint_{B_r(x_0, y_0)} \mathrm{curl}\vec{F}(x, y)\, dA(x, y) = \lim_{r \to 0+} \frac{1}{\pi r^2} \oint_{C_r} \vec{F} \cdot d\vec{s},$$

where C_r is the circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ oriented in the counterclockwise direction. Therefore $\text{curl}\vec{F}(x_0, y_0)$ does indeed measure "infinitesimal counterclockwise rotation" of $\vec{F}$ at (x_0, y_0).

3. Repeat the previous problem for a C^1 vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ on $\mathbb{R}^3$. That is, suppose that $\text{curl}\vec{F}(x_0, y_0, z_0) \neq \vec{0}$, and let S_r denote the surface consisting of points (x, y, z) on the plane through (x_0, y_0, z_0) that is normal to $\text{curl}\vec{F}(x_0, y_0, z_0)$, such that $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \leq r^2$. Orient S_r so that the unit normal vectors $\vec{n}$ point in the same direction as $\text{curl}\vec{F}(x_0, y_0, z_0)$. Show that

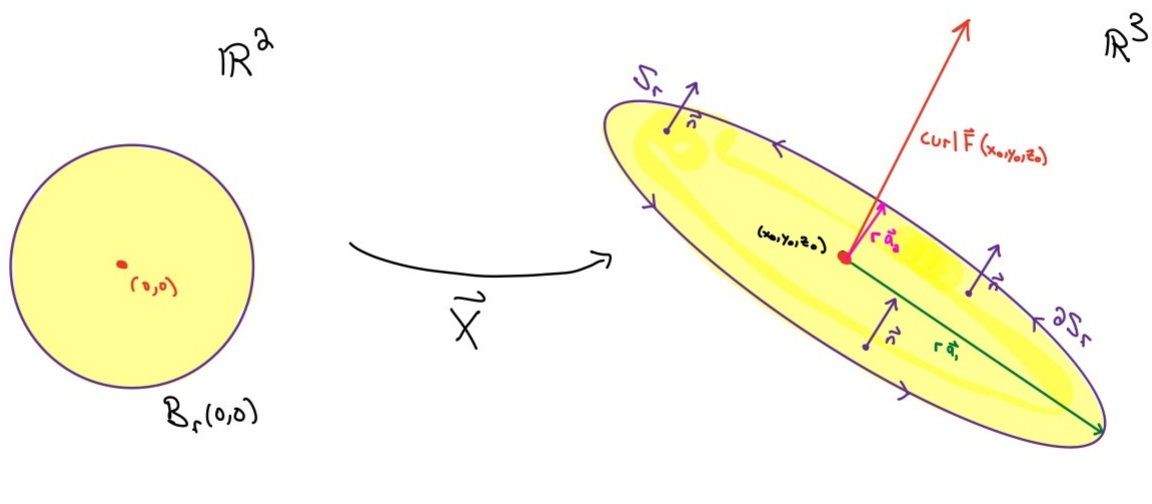
$$\|\text{curl}\vec{F}(x_0, y_0, z_0)\| = \lim_{r \to 0+} \frac{1}{\pi r^2} \oint_{\partial S_r} \vec{F} \cdot d\vec{s},$$

where the circle ∂S_r is oriented so that, when viewed from "above" (i.e. from the direction of $\text{curl}\vec{F}(x_0, y_0, z_0)$), S_r is "on the left".

Hint: You can use the fact that S_r is a disc of radius r, and therefore there is a matrix $A \in M_{3\times 2}(\mathbb{R})$ with orthonormal columns such that S_r is parametrized by

$$\vec{X} : B_r(0,0) \to \mathbb{R}^3, \ \vec{X}(s,t) = (x_0, y_0, z_0) + A \begin{bmatrix} s \\ t \end{bmatrix}.$$

(Such a parametrization can be produced by taking the columns of A to be an orthonormal basis $\vec{a}_1, \vec{a}_2$ for $(\text{span}(\text{curl}\vec{F}(x_0, y_0, z_0)))^\perp$. To ensure that $\vec{X}$ is orientation-preserving, we should order $\vec{a}_1, \vec{a}_2$ so that $\det(\text{curl}\vec{F}(x_0, y_0, z_0), \vec{a}_1, \vec{a}_2) > 0$.)



Solution. Let $\vec{X}$ be as in the hint. Note that since $\vec{a}_1, \vec{a}_2$ form an orthonormal basis for $\text{curl}\vec{F}(x_0, y_0, z_0)$, $\vec{a}_1 \times \vec{a}_2 = \lambda \text{curl}\vec{F}(x_0, y_0, z_0)$ for some $\lambda \in \mathbb{R}$. But since

$$\text{curl}\vec{F}(x_0, y_0, z_0) \cdot (\vec{a}_1 \times \vec{a}_2) = \det(\text{curl}\vec{F}(x_0, y_0, z_0), \vec{a}_1, \vec{a}_2) > 0,$$

we must have that $\lambda > 0$. Because $\vec{a}_1$ and $\vec{a}_2$ are orthonormal, $\|\vec{a}_1 \times \vec{a}_2\| = \mathrm{Vol}_2(\vec{a}_1, \vec{a}_2) = \|\vec{a}_1\|\|\vec{a}_2\| = 1$. Therefore we have

$$\vec{a}_1 \times \vec{a}_2 = \frac{1}{\|\mathrm{curl}\vec{F}(x_0, y_0, z_0)\|}\mathrm{curl}\vec{F}(x_0, y_0, z_0) = \vec{n}.$$

We therefore have, by Stokes' Theorem and Exercise 4 from Homework 4,

$$\begin{aligned}
\lim_{r\to 0+} \frac{1}{\pi r^2} \oint_{\partial S_r} \vec{F} \cdot d\vec{s} &= \lim_{r\to 0+} \frac{1}{\pi r^2} \iint\limits_{S_r} \mathrm{curl}\vec{F} \cdot d\vec{s} \\
&= \lim_{r\to 0+} \frac{1}{\pi r^2} \iint\limits_{B_r(0,0)} \mathrm{curl}\vec{F}(\vec{X}(s,t)) \cdot N_{\vec{X}}(s,t)\, dA(s,t) \\
&= \lim_{r\to 0+} \frac{1}{\pi r^2} \iint\limits_{B_r(0,0)} \mathrm{curl}\vec{F}(\vec{X}(s,t)) \cdot (\vec{a}_1 \times \vec{a}_2)\, dA(s,t) \\
&= \lim_{r\to 0+} \frac{1}{\pi r^2} \iint\limits_{B_r(0,0)} \mathrm{curl}\vec{F}(\vec{X}(s,t)) \cdot \left(\frac{1}{\|\mathrm{curl}\vec{F}(x_0, y_0, z_0)\|}\mathrm{curl}\vec{F}(x_0, y_0, z_0) \right) dA(s,t) \\
&= \mathrm{curl}\vec{F}(\vec{X}(0,0)) \cdot \left(\frac{1}{\|\mathrm{curl}\vec{F}(x_0, y_0, z_0)\|}\mathrm{curl}\vec{F}(x_0, y_0, z_0) \right) \\
&= \mathrm{curl}\vec{F}(x_0, y_0, z_0) \cdot \left(\frac{1}{\|\mathrm{curl}\vec{F}(x_0, y_0, z_0)\|}\mathrm{curl}\vec{F}(x_0, y_0, z_0) \right) \\
&= \|\mathrm{curl}\vec{F}(x_0, y_0, z_0)\|.
\end{aligned}$$