THE LOCAL ATIYAH-SINGER INDEX THEOREM OVER FOUR-MANIFOLDS

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JUNE 2020

Abstract. After introducing the theory of Clifford algebras and Dirac operators over four-manifolds, we discuss Getzler's proof of the local Atiyah-Singer index theorem [3] in the case of four-manifolds, analysing heat kernels of certain heat operators. We also discuss various formulations of the Atiyah-Singer index theorem and some of it's consequences in understanding the signature of compact oriented four-manifolds.

Contents

1. Introduction

The Atiyah-Singer index theorem [11] is an important and widely-celebrated theorem in geometry as it provides a deep link between analysis and topology. More precisely, it asserts that the analytical index of certain elliptic differential operators on compact manifolds is equal to the topological index which is defined solely in terms of certain topological invariants assigned to the manifold and the vector bundle on the manifold whose sections are acted on by the differential operator. There are many famous consequences of the Atiyah-Singer index theorem, including the Gauss-Bonnet theorem and the Riemann-Roch theorem. The differential operators we'll be discussing in this report are Dirac operators, which are used widely in mathematics and physics, and can be thought of as a generalisation of the square-root of the Euclidean Laplacian. Furthermore, we'll be interested in Dirac operators on compact oriented four-manifolds.

Theorem 1.1 (The Atiyah-Singer Index Theorem over Four-Manifolds)**.** *Let $D\colon C^\infty(W) \to C^\infty(W)$ be the Dirac operator on spinor fields over a compact oriented four-manifold M, and let $D^+\colon C^\infty(W_+) \to C^\infty(W_+)$ be the corresponding positive chiral Dirac operator with W_+ the positive chiral spinor bundle. Then, the analytical index of D^+ is equal to the topological index. That is,*

$$Ind(D^+) = \int_M \hat{A}(\Omega).$$

The statement of the Atiyah-Singer index theorem as stated above generalises to considerations of Dirac operators with coefficients in a complex vector bundle with Hermitian metric and unitary connection, often called twisted Dirac operators. Corollaries of the index theorem as stated in Theorem 1.1 include Rochlin's theorem, the Lichnerowicz theorem and the Hirzebruch signature theorem all of which we discuss in Section 3 and concern the signature of our four-manifold. In Section 4, we discuss the proof of Theorem 1.1 and whilst there are multiple proofs of the Atiyah-Singer index theorem, we will discuss the proof of the local index theorem due to Getzler [3] involving the concept of supersymmetry.

The idea of Getzler's short and clever proof is as follows. We can express the index by the McKean-Singer formula in terms of the supertrace of the heat kernel corresponding to the heat operator e^{-tD^2}, namely

$$\mathrm{Ind}(D^+) = \int_M \mathrm{Tr}_s(K(t,x,x))\, dV.$$

Leveraging the time-independence of the left-hand side, we can apply small-time asymptotics to the heat kernel $K(t,x,x)$ to localise the problem. The key idea of Getzler is a clever rescaling of the heat kernel which turns out to converge to the heat kernel of a generalised harmonic oscillator, and Mehler's formula gives us an explicit form of this heat kernel and we are marvellously left with the right topological classes used to define the topological index.

Whilst much of what we discuss can easily be generalised to higher-dimensional manifolds, we will focus on the case of four-manifolds as some of the theory is simplified. For example, a property of compact oriented four-manifolds is that they always possess a spin^c-structure, and some of the consequences of the index theorem that we discuss in Section 3 don't hold for higher-dimensional manifolds. Furthermore, the theory and results discussed in this report over four-manifolds is useful in Seiberg-Witten theory (see [2] and [7]), which is concerned with studying the differential topology of four-manifolds.

2. Clifford Algebras and Dirac Operators on 4-Manifolds

Intuitively, one can think of a Dirac operator as a 'square root' of a Laplacian. Indeed, if we first consider our manifold M to be $\mathbb{R}^4$ with global Euclidean coordinates (x_1, x_2, x_3, x_4), then the Dirac operator $D_A\colon C^\infty(M \times \mathbb{C}^4) \to C^\infty(M \times \mathbb{C}^4)$ is given by

$$D_A\psi = \sum_{i=1}^{4} e_i \frac{\partial \psi}{\partial x_i},$$

where $e_1, e_2, e_3, e_4 \in \mathrm{End}(\mathbb{C}^4)$ are the matrices

$$e_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ \text{and } e_4 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

Notice that these four matrices satisfy the *Clifford relation*, i.e. for all $1 \leq i, j \leq 4$ we have that

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

where δ_{ij} is the Kronecker delta. Then, a direct computation shows that D_A is a square root of the Euclidean Laplacian $\Delta = -\sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2}$. Indeed, we see that

$$\begin{aligned}
(D_A \circ D_A)\psi &= \sum_{i=1}^{4} e_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{4} e_j \frac{\partial \psi}{\partial x_j}\right) \qquad \text{(by definition)} \\
&= \sum_{i=1}^{4}\sum_{j=1}^{4} e_i e_j \frac{\partial \psi}{\partial x_i \partial x_j} \\
&= \sum_{i=1}^{4} e_i^2 \frac{\partial^2 \psi}{\partial x_i^2} + \sum_{1 \leq i \neq j \leq 4} e_i e_j \frac{\partial \psi}{\partial x_i \partial x_j} \\
&= -\sum_{i=1}^{4} \frac{\partial^2 \psi}{\partial x_i^2} \qquad (\text{as } e_i^2 = -1 \text{ and } e_i e_j = -e_j e_i \text{ for } i \neq j) \\
&= \Delta\psi.
\end{aligned}$$

Note that we wouldn't be able to write down a square root of the Euclidean Laplacian Δ without taking our coefficients e_i to be matrix-valued. In order to give the general definition of a Dirac operator on a

4-manifold M (which isn't necessarily $\mathbb{R}^4$) where we aren't just looking at the trivial bundle $M \times \mathbb{C}^4$, we need to introduce the language of Clifford algebras.

2.1. **Clifford Algebras.** Throughout we let W_+ and W_- be two copies of $\mathbb{C}^2$ (both with the standard Hermitian metric $\langle \cdot, \cdot \rangle$) and we write $W = W_+ \oplus W_-$. Furthermore, we denote by V the space of quaternions, which is the real vector space with orthonormal basis

$$\mathbf{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{e_3} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } \mathbf{e_4} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Definition 2.1. The *(complex) Clifford algebra* of the complexification $V \otimes \mathbb{C}$ is the 16-dimensional algebra $Cl(V) = \mathrm{End}(W)$ generated by four matrices e_1, e_2, e_3 and e_4 satisfying the *Clifford relation* that

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}.$$

Typically, we take e_1, e_2, e_3 and e_4 to be the four matrices from before, i.e.

$$e_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \text{ and } e_4 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

So, forgetting the algebra structure we have that the underlying 16-dimensional complex vector space has basis given by 1, e_i, e_ie_j for $i < j$, $e_ie_je_k$ for $i < j < k$, and $e_1e_2e_3e_4$. Comparing the Clifford relation with the exterior algebra relations, we have that the Clifford algebra is isomorphic to the direct sum

$$\sum_{k=0}^{4} \bigwedge^k V \otimes \mathbb{C}.$$

Next, using the 16-dimensional complex Clifford algebra, we can build a 16-dimensional complex vector bundle on a spin 4-manifold M which we call the *Clifford bundle.* First, note that the groups $\mathrm{Spin}(4)$ and $\mathrm{Spin}^c(4)$ act on the complex vector space $\mathrm{End}(W)$ via the adjoint representations Ad and Ad^c as follows, recalling that

$$\mathrm{Spin}(4) = \left\{ \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} : (A_+, A_-) \in SU(2) \times SU(2) \right\}$$

and

$$\mathrm{Spin}^c(4) = \left\{ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} : (A_+, A_-) \in SU(2) \times SU(2) \text{ and } \lambda \in U(1) \right\}.$$

We typically denote by $SU_+(2)$ and $SU_-(2)$ the first and second copies of $SU(2)$ in the product $SU(2) \times SU(2)$ respectively. If $T \in \mathrm{End}(W)$, $A_\pm \in SU_\pm(2)$ and $\lambda \in U(1)$ are arbitrary, then we have that

$$\mathrm{Ad} \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} (T) = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} T \begin{pmatrix} A_+^{-1} & 0 \\ 0 & A_-^{-1} \end{pmatrix},$$

and

$$\mathrm{Ad}^c \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} (T) = \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} T \begin{pmatrix} (\lambda A_+)^{-1} & 0 \\ 0 & (\lambda A_-)^{-1} \end{pmatrix}.$$

Now, if M is a spin manifold, i.e. there is an open cover $\{U_\alpha\}$ of M and a collection of transition functions $\tilde{g}_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{Spin}(4)$ satisfying the cocycle condition, then the *Clifford bundle* is the 16-dimensional complex vector bundle built from the transition functions

$$\mathrm{Ad} \circ \tilde{g}_{\alpha\beta} \colon U_\alpha \cap U_\beta \to GL(\mathrm{End}(W)).$$

We can similarly construct the Clifford bundle with transition functions

$$\mathrm{Ad}^c \circ \tilde{g}_{\alpha\beta} \colon U_\alpha \cap U_\beta \to GL(\mathrm{End}(W))$$

if our manifold M has spin^c structure given by transition functions $\tilde{g}_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{Spin}^c(4)$. Since the fibres of the Clifford bundle are just copies of the Clifford algebra $Cl(V) = \mathrm{End}(W)$, we also call the Clifford bundle $\mathrm{End}(W)$. Looking back at our direct sum decomposition of the Clifford algebra in terms of exterior powers of the complexification of the Euclidean 4-space $V \otimes \mathbb{C}$, we similarly have a direct-sum decomposition of the Clifford bundle in terms of exterior powers of the complexification of the tangent bundle $TM \otimes \mathbb{C}$, i.e. we have that

$$\mathrm{End}(W) = \sum_{k=0}^{4} \bigwedge^k TM \otimes \mathbb{C},$$

which follows from the fact that the adjoint representations preserve the direct sum decomposition of the Clifford algebra. An important result about 4-manifolds (proved using the language of Čech cohomology) is the following.

Theorem 2.2. *If M is a compact oriented 4-manifold, then M has a spin^c structure.*

2.2. **Dirac Operators.** Before we define the Dirac operator, we need to introduce the *spinor bundle* $W \otimes L$ as well as the concept of a *spin connection*. Again, we let W_+ and W_- denote two copies of $\mathbb{C}^2$, where W_+ will correspond to $SU_+(2)$ and W_- will correspond to $SU_-(2)$. Now, observe that the group Spin(4) acts on W_+ and W_- via

$$\rho_+ \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} (w_+) = A_+ w_+$$

and

$$\rho_- \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} (w_-) = A_- w_-.$$

We also have a similar action of $\mathrm{Spin}^c(4)$ on W_+ and W_- given by

$$\rho_+^c \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} (w_+) = \lambda A_+ w_+$$

and

$$\rho_-^c \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} (w_-) = \lambda A_- w_-.$$

Now, if our four-manifold M has a spin structure given by

$$\tilde{g}_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{Spin}(4),$$

then we get two complex vector bundles of dimensional two which we also call W_+ and W_- that have transition functions

$$\rho_+ \circ \tilde{g}_{\alpha\beta} \colon U_\alpha \cap U_\beta \to SU_+(2)$$

and

$$\rho_- \circ \tilde{g}_{\alpha\beta} \colon U_\alpha \cap U_\beta \to SU_-(2)$$

respectively.

Proposition 2.3. *The complexification of the tangent bundle TM is isomorphic to the Hom-bundle of W_+ and W_-. That is, we have that*

$$TM \otimes \mathbb{C} \cong Hom(W_+, W_-).$$

This proposition is analogous to the observation that we have an isomorphism of representations $V \otimes \mathbb{C} \cong \mathrm{Hom}(W_+, W_-)$ where here W_+ and W_- are two copies of $\mathbb{C}^2$.

Now, we let $W = W_+ \oplus W_-$ be the direct sum of the two vector bundles W_+ and W_-, which is therefore a four-dimensional complex vector bundle. Analogous to the construction of the bundles W_+ and W_- above, if we are given a fixed complex line bundle L with Hermitian metric and transition functions

$$h_{\alpha\beta} \colon U_\alpha \cap U_\beta \to U(1),$$

then we can put a spin^c structure on our spin manifold M via the transition functions

$$h_{\alpha\beta}\tilde{g}_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{Spin}^c(4),$$

where the $\tilde{g}_{\alpha\beta}$ are from the spin structure of M. Then, we obtain the two-dimensional complex vector bundles which we call $W_+ \otimes L$ and $W_- \otimes L$ by the transition functions

$$\rho_+^c \circ h_{\alpha\beta}\tilde{g}_{\alpha\beta} \colon U_\alpha \cap U_\beta \to U_+(2)$$

and

$$\rho_-^c \circ h_{\alpha\beta}\tilde{g}_{\alpha\beta} \colon U_\alpha \cap U_\beta \to U_-(2)$$

respectively.

Definition 2.4. The *spinor bundle* $W \otimes L$ is the four-dimensional complex vector bundle on M defined as the direct sum of the two-dimensional complex vector bundles $W_+ \otimes L$ and $W_- \otimes L$ as above. We call sections of the spinor bundle (i.e. elements of $C^\infty(W \otimes L)$) *spinor fields*. The bundles $W_+ \otimes L$ and $W_- \otimes L$ are often called the *chiral spinor bundles*.

It turns out that there is a close relationship between $SO(4)$-connections on the tangent bundle TM and so-called *spin connections* on the four-dimensional bundle W.

Definition 2.5. We say that a connection d_A on W is a *Spin*(4)*-connection* if in the local trivialisations we have that

$$(d_A\sigma)_\alpha = d\sigma_\alpha + \phi_\alpha\sigma_\alpha$$

where $\sigma \in C^\infty(W)$ and the one forms ϕ_α take values in the Lie algebra of Spin(4).

We in fact know that the Lie algebra of Spin(4) is six-dimensional and is generated by the products $e_i \cdot e_j$ for $i < j$, and thus by Definition 2.5 we have that a connection d_A on W is a Spin(4)-connection if the one-forms ϕ_α can be expressed as

$$\phi_\alpha = \sum_{i,j=1}^{4} \phi_{\alpha ij} e_i \cdot e_j,$$

the $\phi_{\alpha ij}$ being ordinary real-valued one-forms. By the Clifford relation $e_i \cdot e_j = -e_j \cdot e_i$ for $i \neq j$, we require that $\phi_{\alpha ij} = -\phi_{\alpha ji}$ for $i \neq j$. Now, the relationship we alluded to earlier between Spin(4)-connections on W and $SO(4)$-connections on TM is given by the following theorem.

Theorem 2.6. *Suppose that M is an oriented Riemannian 4-manifold with a spin structure. Then, there is a bijective correspondence between Spin*(4)*-connections on W and $SO(4)$-connections on TM. In particular, there is a unique Spin*(4)*-connection on W which induces the Levi-Civita connection on the Clifford bundle End*(W)*.*

Before we turn to the proof, first note that any Spin(4)-connection d_A on W induces a unique connection on the Clifford bundle End(W) satisfying the Leibniz rule, i.e.

$$d_A(\omega\sigma) = (d_A\omega)\sigma + \omega d_A\sigma$$

for all $\omega \in C^\infty(\mathrm{End}(W))$ and $\sigma \in C^\infty(W)$, but we want to know that if we start with the Levi-Civita connection on End(W), then there is a unique Spin(4)-connection d_A on W inducing this Levi-Civita connection. When we say the Levi-Civita connection on End(W), we mean the connection on End(W) induced by the Levi-Civita connection on the tangent bundle TM by the direct sum decomposition

$$\mathrm{End}(W) = \sum_{k=0}^{4} \bigwedge^k TM \otimes \mathbb{C}.$$

Proof. Let d_A be a Spin(4) connection on W, which by definition is given in a local trivialisation of W over an open set U in M by

$$d + \sum_{i,j=1}^{4} \phi_{ij} e_i \cdot e_j,$$

and let $\tilde{\psi}$ be this local trivialisation of W over U. Note that $\tilde{\psi}$ induces a local trivialisation of the Clifford bundle End(W) and in particular the subbundle corresponding to the subspace of the Clifford algebra spanned by the e_i. That is, we have that $\tilde{\psi}$ induces a local trivialisation ψ of the tangent bundle TM over $U \subset M$. We now let $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ be the canonical orthonormal basis of sections of W in the trivialisation $\tilde{\psi}$ over U defined by

$$\epsilon_1(p) = \left(p, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right), \ldots, \epsilon_4(p) = \left(p, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right),$$

and we let (e_1, e_2, e_3, e_4) be the canonical orthonormal basis of sections of TM in the trivialisation ψ over U defined by

$$e_1(p) = \left(p, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right), \ldots, e_4(p) = \left(p, \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}\right)$$

where $p \in U \subset M$. By simple matrix-vector multiplication, it follows that

$$e_k \epsilon_\lambda = \sum_{\mu=1}^{4} c_{k\lambda}^{\mu} \epsilon_\mu$$

where the $c_{k\lambda}^{\mu}$ are constants coming from the multiplication, and since they're constant it follows that

$$\begin{aligned} d_A(e_k \epsilon_\lambda) &= \left(d + \sum_{i,j=1}^{4} \phi_{ij} e_i e_j\right)(e_k \epsilon_\lambda) \\ &= \sum_{i,j=1}^{4} \phi_{ij} e_i e_j e_k \epsilon_\lambda, \end{aligned}$$

and similarly we have that $d_A(\epsilon_\lambda) = \sum_{i,j=1}^{4} \phi_{ij} e_i e_j \epsilon_\lambda$. Therefore, by substituting these into the Leibniz rule

$$d_A(e_k \epsilon_\lambda) = (d_A e_k)\epsilon_\lambda + e_k d_A(\epsilon_\lambda),$$

we have that

$$\sum_{i,j=1}^{4} \phi_{ij} e_i e_j e_k \epsilon_\lambda = (d_A e_k)\epsilon_\lambda + e_k \sum_{i,j=1}^{4} \phi_{ij} e_i e_j \epsilon_\lambda,$$

and therefore that

$$(d_A e_k)\epsilon_\lambda = \sum_{i,j=1}^{4} \phi_{ij}(e_i e_j e_k - e_k e_i e_j)\epsilon_\lambda.$$

By a simple calculation involving the Clifford relations $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for $i \neq j$, it follows that

$$d_A e_k = -4 \sum_{i=1}^{4} \phi_{ik} e_i,$$

which indeed is an $SO(4)$-connection on TM. Conversely, if we are given an $SO(4)$-connection

$$d_A e_k = \sum_{i=1}^{4} \omega_{ik} e_i$$

on TM, then by running through the previous argument we've just given, we see that the unique Spin(4)-connection on W inducing this $SO(4)$-connection on TM is given by

$$d - \frac{1}{4} \sum_{i,j=1}^{4} \omega_{ij} e_i \cdot e_j.$$

■

Now, if we denote by d_A a fixed $U(1)$-connection on our complex line bundle L on M, then we get a connection on the spinor bundle $W \otimes L$ by tensoring this $U(1)$-connection on L with the Spin(4)-connection on W induced by the Levi-Civita connection on TM given by Theorem 2.6. We will also denote this induced connection on the spinor bundle by d_A, and this connection turns out to be a *Spin^c(4)-connection*. That is, in the local trivialisations of $W \otimes L$ we have that

$$(d_A \sigma)_\alpha = d\sigma_\alpha + \phi_\alpha \sigma_\alpha$$

for $\sigma \in C^\infty(W \otimes L)$, and the one-forms ϕ_α take values in the Lie algebra of $\mathrm{Spin}^c(4)$. We are now ready to define the Dirac operator in general.

Definition 2.7 (The Dirac Operator)**.** Let d_A be the $\mathrm{Spin}^c(4)$-connection on the spinor bundle $W \otimes L$ induced by the $U(1)$-connection on L and the $\mathrm{Spin}(4)$-connection on W from the Levi-Civita connection on TM. Then the *Dirac Operator* $D_A\colon C^\infty(W \otimes L) \to C^\infty(W \otimes L)$ with coefficients in L is given by

$$D_A\psi(p) = \sum_{i=1}^{4} e_i \cdot d_A\psi(e_i)(p) = \sum_{i=1}^{4} e_i \cdot \nabla^A_{e_i}\psi(p)$$

for all spinor fields ψ and $p \in M$, where $\{e_i\}$ is an orthonormal basis of sections for TM_p.

One can show (see [7, Lemma 3.3.1]) that the Dirac operator defined above is independent of the choice of orthonormal frame $\{e_i\}$. While we won't discuss it here, it is possible to extend the definition of the Dirac operator to have coefficients in an arbitrary complex vector bundle E with Hermitian metric and unitary connection, which allows a larger class of examples of Dirac operators which come up in other places in differential geometry.

Example 2.8. The Dirac operator with coefficients in W itself is the well-known operator

$$d + \delta\colon \sum_{k=0}^{4} \bigwedge^k TM \otimes \mathbb{C} \to \sum_{k=0}^{4} \bigwedge^k TM \otimes \mathbb{C}$$

from Hodge theory, where d is the exterior derivative and δ is the co-differential (which is the formal adjoint of d).

We now introduce Weitzenböck's formula, which can be thought of as a generalisation of the notion that the Dirac operator is a square root of the Laplacian. Weitzenböck's formula is also a crucial tool in the proof of the local index theorem, as it is used to arrive at the heat kernel of a generalised harmonic oscillator.

Theorem 2.9 (Weitzenböck's formula)**.** *Let $D_A\colon C^\infty(W \otimes L) \to C^\infty(W \otimes L)$ be the Dirac operator with coefficients in L. Then, the square of D_A (which we will call a "generalised Laplacian") is given by*

$$D_A^2\psi = \Delta^A\psi + \frac{s}{4}\psi - \sum_{i<j} F_A(e_i, e_j)(ie_i \cdot e_j \cdot \psi)$$

where here s is the scalar curvature of M, F_A is the curvature of the $U(1)$-connection on L and the "vector bundle Laplacian" $\Delta^A\colon C^\infty(W \otimes L) \to C^\infty(W \otimes L)$ is defined by

$$\Delta^A\psi = -\sum_{i=1}^{4} [\nabla^A_{e_i} \circ \nabla^A_{e_i}\psi - \nabla^A_{\nabla_{e_i} e_i}\psi],$$

where $\{e_1, e_2, e_3, e_4\}$ is a moving orthonormal frame for TM.

Proof. We choose a coordinate system at $p \in M$ so that $\nabla_{e_i} e_j(p) = 0$ (which can be done by parallel translation along geodesic rays from p), and thus at p we have that

$$\Delta^A\psi = -\sum_{i=1}^{4} \nabla^A_{e_i}\nabla^A_{e_i}\psi.$$

Then, computing at p and using the Clifford relations $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for $i \neq j$, we see that

$$\begin{aligned}
D_A^2 \psi &= \left(\sum_{i=1}^4 e_i \nabla^A_{e_i} \right) \circ \left(\sum_{j=1}^4 e_j \nabla^A_{e_j} \right) \psi \\
&= \sum_{i,j=1}^4 e_i e_j \nabla^A_{e_i} \nabla^A_{e_j} \psi \\
&= \sum_{i=1}^4 e_i e_i \nabla^A_{e_i} \nabla^A_{e_i} \psi + \sum_{i<j} e_i e_j \nabla^A_{e_i} \nabla^A_{e_j} \psi + \sum_{j<i} e_i e_j \nabla^A_{e_i} \nabla^A_{e_j} \psi \\
&= -\sum_{i=1}^4 \nabla^A_{e_i} \nabla^A_{e_i} \psi + \sum_{i<j} e_i e_j \nabla^A_{e_i} \nabla^A_{e_j} \psi - \sum_{j<i} e_j e_i \nabla^A_{e_i} \nabla^A_{e_j} \psi \\
&= -\sum_{i=1}^4 \nabla^A_{e_i} \nabla^A_{e_i} \psi + \sum_{i<j} e_i e_j [\nabla^A_{e_i} \nabla^A_{e_j} - \nabla^A_{e_j} \nabla^A_{e_i}] \psi \quad \text{(by re-labelling } i \text{ with } j \text{ and } j \text{ with } i \text{ in the third sum).}
\end{aligned}$$

But by re-labelling the second sum instead, we have that

$$-\sum_{i<j} e_j e_i \nabla^A_{e_i} \nabla^A_{e_j} \psi + \sum_{j<i} e_i e_j \nabla^A_{e_i} \nabla^A_{e_j} \psi = \sum_{j<i} e_i e_j [\nabla^A_{e_i} \nabla^A_{e_j} - \nabla^A_{e_j} \nabla^A_{e_i}] \psi,$$

and thus

$$\begin{aligned}
D_A^2 \psi &= -\sum_{i=1}^4 \nabla^A_{e_i} \nabla^A_{e_i} \psi + \frac{1}{2} \sum_{i \neq j} e_i e_j [\nabla^A_{e_i} \nabla^A_{e_j} - \nabla^A_{e_j} \nabla^A_{e_i}] \psi \\
&= -\sum_{i=1}^4 \nabla^A_{e_i} \nabla^A_{e_i} \psi + \frac{1}{2} \sum_{i,j=1}^4 e_i e_j [\nabla^A_{e_i} \nabla^A_{e_j} - \nabla^A_{e_j} \nabla^A_{e_i}] \psi \\
&= \Delta^A \psi + \frac{1}{2} \sum_{i,j=1}^4 e_i e_j [\nabla^A_{e_i} \nabla^A_{e_j} - \nabla^A_{e_j} \nabla^A_{e_i}] \psi \\
&= \Delta^A \psi + \frac{1}{2} \sum_{i,j=1}^4 e_i e_j \Omega_A(e_i, e_j) \cdot \psi
\end{aligned}$$

as $\nabla^A_{e_i} \nabla^A_{e_j} - \nabla^A_{e_j} \nabla^A_{e_i}$ is precisely the curvature 2-form Ω_A of the connection on $W \otimes L$ applied to (e_i, e_j). Now, recall from the proof of Theorem 2.6 that given an $SO(4)$-connection

$$d_A e_k = \sum_{i=1}^4 \omega_{ik} e_i$$

on TM, the unique Spin(4)-connection on W inducing this $SO(4)$-connection on TM is given by

$$d - \frac{1}{4} \sum_{i,j=1}^4 \omega_{ij} e_i \cdot e_j.$$

Then, by definition the curvature of this Spin(4)-connection on W is given by

$$\Omega = \left(d - \frac{1}{4} \sum_{i,j=1}^4 \omega_{ij} e_i \cdot e_j \right)^2.$$

Using that the orthonormal sections e_i are constant in the local trivialisations of the tangent bundle TM (i.e. they are simply constant matrices), by a direct computation making use again of the Clifford relation and that $d^2 = 0$, we see that

$$\Omega = -\frac{1}{4} \sum_{i,j=1}^{4} \Omega_{ij} e_i \cdot e_j,$$

where the 2-form

$$\Omega_{ij} = d\omega_{ij} + \sum_{k=1}^{4} \omega_{ik} \wedge \omega_{kj}.$$

Now, we know from Riemannian geometry that $R_{ijkl} = \Omega_{ij}(e_k, e_l)$ are precisely the components of the Riemann-curvature tensor for the Riemannian manifold M, which satisfy the symmetry relations

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$$

and

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

Next, we show that

$$\sum_{i,j=1}^{4} e_i e_j \Omega(e_i, e_j) = \frac{s}{2},$$

where the scalar curvature $s = \sum_{i,j=1}^{4} R_{ijij}$. Since we have as above that

$$\Omega = -\frac{1}{4} \sum_{k,l=1}^{4} \Omega_{kl} e_k \cdot e_l,$$

it follows that

$$\begin{aligned}
\sum_{i,j=1}^{4} e_i e_j \Omega(e_i, e_j) &= -\frac{1}{4} \sum_{i,j,k,l=1}^{4} e_i e_j e_k e_l \Omega_{kl}(e_i, e_j) \\
&= -\frac{1}{4} \sum_{i,j,k,l=1}^{4} e_i e_j e_k e_l R_{klij} \\
&= -\frac{1}{4} \sum_{i,j,k,l=1}^{4} e_i e_j e_k e_l R_{ijkl} \qquad (\text{as } R_{klij} = R_{ijkl}).
\end{aligned}$$

Making repeated use of the symmetry relations, we then get that

$$\begin{aligned}
\sum_{i,j=1}^{4} e_i e_j \Omega(e_i, e_j) &= -\frac{1}{4} \sum_{i,j,k,l=1}^{4} e_i e_j e_k e_l R_{ijkl} \\
&= -\frac{1}{4} \sum_{i,j=1}^{4} e_i e_j e_i e_j R_{ijij} - \frac{1}{4} \sum_{i,j=1}^{4} e_i e_j e_j e_i R_{ijji}.
\end{aligned}$$

Then, using the Clifford relation it follows that

$$\begin{aligned}\sum_{i,j=1}^{4} e_i e_j \Omega(e_i, e_j) &= \frac{1}{4}\sum_{i,j=1}^{4} e_i^2 e_j^2 R_{ijij} - \frac{1}{4}\sum_{i,j=1}^{4} e_i^2 e_j^2 \cdot -R_{ijij} \\ &= \frac{1}{4}\sum_{i,j=1}^{4} R_{ijij} + \frac{1}{4}\sum_{i,j=1}^{4} R_{ijij} \\ &= \frac{1}{2}\sum_{i,j=1}^{4} R_{ijij} \\ &= \frac{s}{2}.\end{aligned}$$

Now, one can show that the curvature 2-form Ω_A from before is given by

$$\Omega_A = -iF_A I + \Omega = -iF_A I - \frac{1}{4}\sum_{i,j=1}^{4} \Omega_{ij} e_i \cdot e_j,$$

where F_A is the curvature of the $U(1)$-connection on the complex line bundle L. So, since we had before that

$$D_A^2 \psi = \Delta^A \psi + \frac{1}{2}\sum_{i,j=1}^{4} e_i e_j \Omega_A(e_i, e_j) \cdot \psi,$$

substituting in the above gives

$$D_A^2 \psi = \Delta^A \psi - \frac{1}{2}\sum_{i,j=1}^{4} F_A(e_i, e_j) i e_i \cdot e_j \cdot \psi - \frac{1}{8}\sum_{i,j,k,l=1}^{4} \Omega_{kl}(e_i, e_j) e_i e_j e_k e_l \cdot \psi.$$

Therefore, since we showed that

$$\sum_{i,j=1}^{4} e_i e_j \Omega(e_i, e_j) = -\frac{1}{4}\sum_{i,j,k,l=1}^{4} e_i e_j e_k e_l \Omega_{kl}(e_i, e_j) = \frac{s}{2},$$

it follows that

$$\begin{aligned}D_A^2 \psi &= \Delta^A \psi - \frac{1}{2}\sum_{i,j=1}^{4} F_A(e_i, e_j) i e_i \cdot e_j \cdot \psi + \frac{s}{4}\psi \\ &= \Delta^A \psi - \sum_{i<j} F_A(e_i, e_j) i e_i \cdot e_j \cdot \psi + \frac{s}{4}\psi,\end{aligned}$$

where in this second equality we use a re-labelling argument and the Clifford relation (as well as that $F_A(e_i, e_i) = 0$) to deduce that

$$\begin{aligned}\sum_{i<j} F_A(e_i, e_j) i e_i \cdot e_j \cdot \psi &= \sum_{i<j} -F_A(e_j, e_i) i(-e_j \cdot e_i) \cdot \psi \\ &= \sum_{i<j} F_A(e_j, e_i) i e_j \cdot e_i \cdot \psi \\ &= \sum_{i>j} F_A(e_i, e_j) i e_i \cdot e_j \cdot \psi.\end{aligned}$$

■

One further useful property of Dirac operators is that they are self-adjoint with respect to the L^2 inner product $(\cdot,\cdot)_{L^2}$ defined by

$$(\psi,\eta)_{L^2} = \int_M \langle \psi,\eta\rangle \, dV,$$

for all spinor fields $\psi,\eta \in C^\infty(W\otimes L)$, where on the right-hand side we're taking the point-wise Hermitian inner product of the spinors.

Theorem 2.10. *The Dirac operator $D_A\colon C^\infty(W\otimes L)\to C^\infty(W\otimes L)$ is self-adjoint with respect to this L^2-inner product. That is, for all spinor fields ψ and η we have that*

$$(D_A\psi,\eta)_{L^2} = (\psi, D_A\eta)_{L^2},$$

i.e.

$$\int_M \langle D_A\psi,\eta\rangle \, dV = \int_M \langle \psi, D_A\eta\rangle \, dV.$$

Proof. As in the proof of Theorem 2.9, we choose a coordinate system at $p\in M$ so that $\nabla_{e_i}e_j(p)=0$ (which again can be done by parallel translation along geodesic rays from p). Now, computing at p we have that

$$\begin{aligned}
\langle D_A\psi,\eta\rangle &= \left\langle \sum_{i=1}^4 e_i\cdot\nabla^A_{e_i}\psi,\eta\right\rangle\\
&= \sum_{i=1}^4 \langle e_i\cdot\nabla^A_{e_i}\psi,\eta\rangle
\end{aligned}$$

as $\langle\cdot,\cdot\rangle$ is a Hermitian inner product. Furthermore, we have that multiplication by e_i is skew-Hermitian, which follows from the fact that multiplication by e_i is orthogonal i.e.

$$\langle e_i\cdot\eta_1, e_i\cdot\eta_2\rangle = \langle\eta_1,\eta_2\rangle,$$

as well as that $e_i^2=-1$, so that

$$\langle e_i\cdot\eta_1,\eta_2\rangle + \langle\eta_1, e_i\cdot\eta_2\rangle = 0.$$

Therefore, we have that

$$\begin{aligned}
\langle D_A\psi,\eta\rangle &= \sum_{i=1}^4 \langle e_i\cdot\nabla^A_{e_i}\psi,\eta\rangle\\
&= -\sum_{i=1}^4 \langle \nabla^A_{e_i}\psi, e_i\cdot\eta\rangle\\
&= -\sum_{i=1}^4 [e_i\langle\psi, e_i\cdot\eta\rangle - \langle\psi, e_i\cdot\nabla^A_{e_i}\eta\rangle]\\
&= \left\langle \psi, \sum_{i=1}^4 e_i\cdot\nabla^A_{e_i}\eta\right\rangle - \sum_{i=1}^4 e_i\langle\psi, e_i\cdot\eta\rangle\\
&= \langle\psi, D_A\eta\rangle - \sum_{i=1}^4 e_i\langle\psi,e_i\cdot\eta\rangle,
\end{aligned}$$

noting that in the third equality we've used that ∇^A is a unitary connection so that

$$\begin{aligned}
e_i\langle\psi,e_i\cdot\eta\rangle &= \langle\nabla^A_{e_i}\psi, e_i\cdot\eta\rangle + \langle\psi, \nabla^A_{e_i}(e_i\cdot\eta)\rangle\\
&= \langle\nabla^A_{e_i}\psi, e_i\cdot\eta\rangle + \langle\psi, (\nabla^A_{e_i}e_i)\cdot\eta\rangle + \langle\psi, e_i\cdot\nabla^A_{e_i}\eta\rangle\\
&= \langle\nabla^A_{e_i}\psi, e_i\cdot\eta\rangle + \langle\psi, e_i\cdot\nabla^A_{e_i}\eta\rangle \qquad \text{(as we're computing at } p\text{)}.
\end{aligned}$$

Finally, defining the one-form (or vector field) b on M by $\langle b, e_i\rangle = -\langle \psi, e_i \cdot \eta\rangle$, it follows that

$$\langle D_A\psi, \eta\rangle = \langle \psi, D_A\eta\rangle + \delta b$$

and thus that

$$\begin{aligned}\int_M \langle D_A\psi, \eta\rangle \, dV &= \int_M \langle \psi, D_A\eta\rangle \, dV + \int_M \delta b \, dV \\ &= \int_M \langle \psi, D_A\eta\rangle \, dV \qquad \text{(by the divergence theorem)},\end{aligned}$$

which is precisely the conclusion of the theorem. ■

3. The Atiyah-Singer Index Theorem and its Consequences

In this section we introduce the Atiyah-Singer index theorem in multiple forms and discuss some of its consequences. The Atiyah-Singer index theorem is an important and surprising theorem as it says that the index of a Dirac operator over our four-manifold M, which is a purely analytical concept, can be computed entirely through considering topological invariants associated to M. That is, it is a deep connection between analysis and topology in that the analytical index is equal to the topological index.

To state the index theorem, first consider a Dirac operator $D_A \colon C^\infty(W \otimes L) \to C^\infty(W \otimes L)$ with coefficients in the complex line bundle L. This Dirac operator splits into two chiral Dirac operators on the two corresponding chiral spinor bundles, i.e. we have that D_A splits into the two pieces

$$D_A^+ \colon C^\infty(W_+ \otimes L) \to C^\infty(W_+ \otimes L)$$

and

$$D_A^- \colon C^\infty(W_- \otimes L) \to C^\infty(W_- \otimes L),$$

and these two chiral Dirac operators are formal adjoints of each other, recalling as in Theorem 2.10 that D_A is self-adjoint. So, with respect to the splitting $W \otimes L = (W_+ \otimes L) \oplus (W_- \otimes L)$, we can view D_A as

$$D_A = \begin{pmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{pmatrix}.$$

The chiral Dirac operators D_A^+ and D_A^- are elliptic differential operators, so we know that $\ker(D_A^+)$ and $\ker(D_A^-)$ are finite-dimensional complex vector spaces.

Definition 3.1. The *index* of the chiral Dirac operator D_A^+ is defined to be

$$\operatorname{Ind}(D_A^+) = \dim\ker(D_A^+) - \dim\ker(D_A^-).$$

So, we of course have that the index of D_A^+ is an integer. Now, the following version of the Atiyah-Singer index theorem is not the most general form (and is not the version that we will prove in the next section) but highlights that the index D_A^+ can be purely expressed in terms of topological classes obtained from our 4-manifold M.

Theorem 3.2. *Suppose that M is a compact oriented 4-manifold and that D_A is the Dirac operator with coefficients in the line bundle L with unitary connection. Then, the index of the chiral Dirac operator D_A^+ is given by*

$$\mathit{Ind}(D_A^+) = -\frac{1}{8}\tau(M) + \frac{1}{2}\int_M c_1(L)^2,$$

where $\tau(M) = b_+ - b_-$ is the signature of M and $c_1(L)$ is the first Chern class of the line bundle L.

Note here that b_+ is the dimension of the space $\mathcal{H}^2_+(M)$ of self-dual harmonic 2-forms on M and b_- is the dimension of the space $\mathcal{H}^2_-(M)$ of anti-self-dual harmonic 2-forms on M. Observe in particular that if we take our line bundle L to be trivial (with trivial connection), then the first Chern class $c_1(L)$ vanishes, and we thus get by Theorem 3.2 that

$$\operatorname{Ind}(D^+) = -\frac{1}{8}\tau(M),$$

where we denote by $D\colon C^\infty(W) \to C^\infty(W)$ the Dirac operator for the spinor bundle W when the coefficient line bundle is trivial, with corresponding chiral Dirac operators $D^+\colon C^\infty(W_+) \to C^\infty(W_+)$ and $D^-\colon C^\infty(W_-) \to C^\infty(W_-)$. So, since the index $\mathrm{Ind}(D^+)$ is an integer, it follows that the signature $\tau(M)$ of M must be divisible by 8. However, we in fact know that it is divisible by 16 by Rochlin's theorem.

Theorem 3.3 (Rochlin)**.** *The signature of a compact oriented smooth spin 4-manifold satisfies*

$$\tau(M) \equiv 0 \,(mod\, 16),$$

i.e the signature of M must be divisible by 16.

The proof of this theorem leverages the fact that $\mathrm{Ind}(D^+) = -\frac{1}{8}\tau(M)$ (so that $\tau(M)$ is divisible by 8), and thus it suffices to show that $\mathrm{Ind}(D^+)$ is divisible by 2 in order to deduce that $\tau(M)$ is divisible by 16. That is, it suffices to show that $\dim\ker(D^+)$ and $\dim\ker(D^-)$ are each divisible by 2 as $\mathrm{Ind}(D^+) = \dim\ker(D^+) - \dim\ker(D^-)$, and this amounts to showing that the finite-dimensional vector spaces $\ker(D^+)$ and $\ker(D^-)$ have an even-number of basis elements given by $\psi_1, \dots, \psi_k, J(\psi_1), \dots, J(\psi_k)$ where $\psi_1, \dots, \psi_k$ are harmonic spinor fields (i.e. each $\psi_i \in C^\infty(W_+)$ is such that $D^+\psi = 0$ if we're looking at $\ker(D^+)$ and similarly for $\ker(D^-)$) and $J\colon W_+ \to W_+$ (respectively $J\colon W_- \to W_-$) is given by Clifford multiplication by the basis element $\mathbf{e}_4 \in V$ from Section 2 so that $J(\psi_i)$ is perpendicular to ψ_i.
Now that we know the signature of the 4-manifold M is divisible by 16, the following theorem tells us when in fact the signature is zero, and is another application of Weitzenböck's formula (i.e. Theorem 2.9).

Theorem 3.4 (Lichnerowicz)**.** *Suppose that M is a compact oriented smooth Riemannian spin 4-manifold with positive scalar curvature. Then, we have that*

$$\tau(M) = 0.$$

Proof. Consider again the Dirac operator $D\colon C^\infty(W) \to C^\infty(W)$ where our complex line bundle is trivial (with trivial connection). Since we know that

$$\mathrm{Ind}(D^+) = -\frac{1}{8}\tau(M),$$

it suffices to show that $\mathrm{Ind}(D^+) = 0$, i.e. that both $\ker(D^+)$ and $\ker(D^-)$ are zero-dimensional. To do this, consider an arbitrary harmonic spinor field $\psi \in C^\infty(W)$ so that $D\psi = 0$. Then, since $D^2\psi = 0$, we have by Weitzenböck's formula (noting that the curvature term vanishes as our line bundle is trivial) that

$$\Delta^A\psi + \frac{s}{4}\psi = 0.$$

As in Section 2, we choose a coordinate system at $p \in M$ so that $\nabla_{e_i}e_j(p) = 0$ (which again can be done by parallel translation along geodesic rays from p). Taking the L^2-inner product of the above equation with ψ, we get that

$$\int_M \langle \Delta^A\psi, \psi\rangle \, dV + \frac{1}{4}\int_M \langle s\psi, \psi\rangle \, dV = 0.$$

Now, computing at p and using that $\nabla_{e_i}e_j(p) = 0$ in the definition of the vector bundle Laplacian, it follows that

$$\begin{aligned}
\int_M \langle \Delta^A\psi, \psi\rangle \, dV &= -\int_M \sum_{i=1}^4 \langle \nabla^A_{e_i}\nabla^A_{e_i}\psi, \psi\rangle \, dV \qquad \text{(as } \langle\cdot,\cdot\rangle \text{ is a Hermitian inner product)} \\
&= -\int_M \sum_{i=1}^4 [e_i\langle \nabla^A_{e_i}\psi, \psi\rangle - \langle \nabla^A_{e_i}\psi, \nabla^A_{e_i}\psi\rangle] \, dV \\
&= \int_M \langle \nabla^A\psi, \nabla^A\psi\rangle \, dV,
\end{aligned}$$

noting that the second equality follows from the fact that ∇^A is a (unitary) connection, and thus

$$e_i\langle \nabla^A_{e_i}\psi, \psi\rangle = \langle \nabla^A_{e_i}\nabla^A_{e_i}\psi, \psi\rangle + \langle \nabla^A_{e_i}\psi, \nabla^A_{e_i}\psi\rangle,$$

and the third equality follows since

$$\int_M \sum_{i=1}^{4} e_i \langle \nabla^A_{e_i} \psi, \psi \rangle = 0$$

as we can define a one-form (or vector field) b on M by $\langle b, e_i \rangle = \langle \nabla^A_{e_i} \psi, \psi \rangle$, and then the above integral is just

$$\int_M \delta b \, dV$$

which is zero by the divergence theorem. Putting this all together, we therefore have that

$$\int_M \langle \nabla^A \psi, \nabla^A \psi \rangle \, dV + \frac{1}{4} \int_M \langle s\psi, \psi \rangle \, dV = 0,$$

i.e. that

$$\int_M |\nabla^A \psi|^2 \, dV + \frac{1}{4} \int_M s|\psi|^2 \, dV = 0.$$

So, since the scalar curvature $s > 0$ by assumption, in order for the left-hand side of the above equation to be zero, we must have that $\psi = 0$. Hence, since $\psi \in C^\infty(W)$ was an arbitrary spinor field satisfying $D\psi = 0$, it follows that both kernels $\ker(D^+)$ and $\ker(D^-)$ are zero, which as discussed earlier is precisely what we needed in order to deduce that $\tau(M) = 0$. ■

We will now introduce the version of the Atiyah-Singer index theorem that we'll prove in the next section. Recall from the proof of Weitzenböck's formula the curvature Ω with corresponding 2-form

$$\Omega_{ij} = d\omega_{ij} + \sum_{k=1}^{4} \omega_{ik} \wedge \omega_{kj}.$$

Definition 3.5. The $\hat{A}$*-genus* is the cohomology class $\hat{A}(\Omega)$ from the differential form

$$\det{}^{1/2} \left(\frac{\Omega/2}{\sinh(\Omega/2)} \right),$$

where Ω is the curvature.

In fact, one can show that

$$\hat{A}(\Omega) = 1 - \frac{1}{24} p_1(TM),$$

where $p_1(TM) \in H^4(M; \mathbb{Z})$ is the first Pontryagin class of the tangent bundle TM. For higher-dimensional manifolds the above expression for the $\hat{A}$-genus contains more terms in higher Pontryagin classes, but our manifold M is of dimension 4 so that all higher Pontryagin classes are zero (as the cohomology of M is zero in all degrees above degree 4). Now, the version of the Atiyah-Singer index theorem that we'll prove in the next section is as follows, where we again consider the Dirac operator $D\colon C^\infty(W) \to C^\infty(W)$ where our complex line bundle L is trivial.

Theorem 3.6 (Atiyah-Singer Index Theorem for Four-Manifolds)**.** *Suppose that M is a compact oriented 4-manifold. Then, the index of the chiral Dirac operator $D^+\colon C^\infty(W_+) \to C^\infty(W_+)$ is given by*

$$\mathit{Ind}(D^+) = \int_M \hat{A}(\Omega).$$

In particular, as discussed above this tells us that

$$\mathrm{Ind}(D^+) = \int_M \left(1 - \frac{1}{24} p_1(TM) \right).$$

That is, although we know that $1 - \frac{1}{24} p_1(TM)$ is a rational class as the first Pontryagin class is an integral class, we in fact know that $\int_M (1 - \frac{1}{24} p_1(TM))$ is an integer (as the index $\mathrm{Ind}(D^+)$ is an integer). The formula for the index is slightly more complicated when our line bundle L is non-trivial in that we need to introduce the Chern character, but we do have a formula for it when we consider the Dirac operator $D_A\colon C^\infty(W \otimes E) \to C^\infty(W \otimes E)$ with coefficients in any complex vector bundle E on M with Hermitian metric and unitary connection.

Theorem 3.7 (Atiyah-Singer Index Theorem for Four-Manifolds with Coefficients in E)**.** *Suppose that M is a compact oriented 4-manifold. Then, the index of the chiral Dirac operator $D_A^+ \colon C^\infty(W_+ \otimes E) \to C^\infty(W_+ \otimes E)$ with coefficients in a complex vector bundle E with Hermitian metric and unitary connection is given by*

$$Ind(D_A^+) = \int_M \hat{A}(\Omega) \wedge ch\left(\frac{\sqrt{-1}F_A}{2\pi}\right),$$

where F_A is the curvature of the unitary connection on E (and the Chern character is defined as the trace of the exponential of $\frac{\sqrt{-1}F_A}{2\pi}$).

One can show that the Chern character $\mathrm{ch}(\frac{\sqrt{-1}F_A}{2\pi}) \in H^*(M;\mathbb{Q})$ can in fact be expressed as

$$\mathrm{ch}\left(\frac{\sqrt{-1}F_A}{2\pi}\right) = \dim E + c_1(E) + \frac{1}{2}(c_1^2(E)) - c_2(E) + \dots$$

For example, if we take E to just be a complex line bundle L, then it follows from the Atiyah-Singer index theorem (Theorem 3.7) that

$$\mathrm{Ind}(D_A^+) = -\frac{1}{24}\int_M p_1(TM) + \frac{1}{2}\int_M c_1(L)^2,$$

i.e. the analytical index can be expressed entirely in terms of the first Pontryagin class of the tangent bundle TM and the first Chern class of the complex line bundle L (which are purely topological invariants). However, recall that as stated in Theorem 3.2 we know that

$$\mathrm{Ind}(D_A^+) = -\frac{1}{8}\tau(M) + \frac{1}{2}\int_M c_1(L)^2,$$

which we said was a corollary of the more general Atiyah-Singer index theorem as stated in Theorem 3.7, and of course to derive this expression for the index in terms of the signature of M from the one above in terms of the first Pontryagin class of TM, we require that

$$\tau(M) = \frac{1}{3}\int_M p_1(TM),$$

and indeed this is precisely the content of the Hirzebruch Signature Theorem.

Theorem 3.8 (Hirzebruch Signature Theorem)**.** *Suppose that M is a compact oriented smooth 4-manifold. Then, the signature of M is given by*

$$\tau(M) = \frac{1}{3}\int_M p_1(TM).$$

The proof of this theorem considers the Dirac operator $D_A \colon C^\infty(W \otimes W) \to C^\infty(W \otimes W)$ with coefficients in W itself, and as discussed in Section 2 this is precisely

$$d + \delta \colon \sum_{k=0}^{4} \bigwedge^k TM \otimes \mathbb{C} \to \sum_{k=0}^{4} \bigwedge^k TM \otimes \mathbb{C}.$$

Then, using that the kernel of $d + \delta$ is precisely the space of complex-valued harmonic forms, one computes explicitly by the definition of the index that

$$\mathrm{Ind}(D_A^+) = b_+ - b_- = \tau(M).$$

So, by the Atiyah-Singer index theorem (Theorem 3.7), we have that

$$\begin{aligned}\tau(M) &= \int_M \left(-\frac{1}{24}\dim(W)p_1(TM) - c_2(W)\right)\\ &= \int_M \left(-\frac{1}{6}p_1(TM) - c_2(W)\right) \qquad \text{(as } W \text{ is four-dimensional).}\end{aligned}$$

However, by Proposition 2.3 we have that

$$TM \otimes \mathbb{C} \cong \mathrm{Hom}(W_+, W_-) \cong W_+^* \otimes W_-,$$

where W_+^* is the conjugate bundle of W_+ which is in fact isomorphic to W_+. Thus, we have that

$$TM \otimes \mathbb{C} \cong W_+ \otimes W_-,$$

which implies that we can re-write the second Chern class as

$$c_2(TM \otimes \mathbb{C}) = 2c_2(W_+) + 2c_2(W_-).$$

But using that $W = W_+ \oplus W_-$, this is precisely equal to $2c_2(W)$. Therefore, since the Pontryagin class $p_1(TM) = -c_2(TM \otimes \mathbb{C})$ by definition, it follows that

$$\begin{aligned}
\tau(M) &= \int_M \left(-\frac{1}{6} p_1(TM) - c_2(W) \right) \\
&= \int_M \left(-\frac{1}{6} p_1(TM) + \frac{1}{2} p_2(TM) \right) \\
&= \frac{1}{3} \int_M p_1(TM),
\end{aligned}$$

which is the conclusion of the Hirzebruch Signature Theorem.

4. Proof of the Local Index Theorem

In this section we will discuss the proof of the Atiyah-Singer index theorem for 4-manifolds as stated in Theorem 3.6 (i.e. we will be looking at the Dirac operator $D\colon C^\infty(W) \to C^\infty(W)$ where our complex line bundle L is trivial). The proof is very similar but more notationally-heavy if we want to introduce a non-trivial coefficient vector bundle E (as in the statement of Theorem 3.7). In fact, we will prove a slightly stronger version called the local Atiyah-Singer index theorem due to Getzler [3], and we will see how this implies Theorem 3.6. An outline of the main steps of the proof are as follows.

(1) Derive the McKean-Singer formula for the index in terms of the supertrace of the heat kernel of the heat operator e^{-tD^2}.
(2) Use an asymptotic expansion of the heat kernel to localise our problem leveraging time-independence in the McKean-Singer formula.
(3) Apply Getzler's rescaling of the heat kernel and use Weitzenböck's formula to show that the re-scaled heat kernel converges to the heat kernel of a generalised harmonic oscillator.
(4) Use Mehler's formula to derive an explicit expression for the heat kernel of the generalised harmonic oscillator which gives us the conclusion of the index theorem.

The first ingredient to the proof of the index theorem as mentioned above is the McKean-Singer formula for the index. We know that the Dirac operator $D\colon C^\infty(W) \to C^\infty(W)$ of the spinor bundle W over our compact oriented 4-manifold M is an elliptic differential operator as discussed in Section 3, and by the theory of elliptic operators we in fact know (using slightly different language to that used in Section 3) that D is a *Fredholm operator* when viewed as an operator $D\colon L^2(W) \to L^2(W)$ (looking at L^2-sections of W).

Definition 4.1. An operator $P\colon H_1 \to H_2$ between Hilbert spaces H_1 and H_2 is called *Fredholm* if both $\ker(P)$ and $\operatorname{coker}(P)$ are finite-dimensional.

We will now briefly discuss some properties of Fredholm operators that we'll need, which apply to the Dirac operator D as well as the chiral Dirac operators $D^+\colon C^\infty(W_+) \to C^\infty(W_+)$ and $D^-\colon C^\infty(W_-) \to C^\infty(W_-)$ since they are also Fredholm as discussed in Section 3.

Definition 4.2. If P is a Fredholm operator of Hilbert spaces, then the *index* of P is defined to be

$$\operatorname{Ind}(P) = \dim\ker(P) - \dim\operatorname{coker}(P).$$

So, by Definition 4.2 we have in particular that

$$\operatorname{Ind}(D^+) = \dim\ker(D^+) - \dim\operatorname{coker}(D^+).$$

However, we defined in Section 3 (recall Definition 3.1) that

$$\operatorname{Ind}(D^+) = \dim\ker(D^+) - \dim\ker(D^-).$$

These two definitions indeed agree as D is self-adjoint (by Theorem 2.10) and D^- is the adjoint of D^+ (i.e. $D^- = (D^+)^*$), and so

$$\begin{aligned}\dim\ker(D^+) - \dim\ker(D^-) &= \dim\ker(D^+) - \dim\ker((D^+)^*)\\ &= \dim\ker(D^+) - \dim\operatorname{coker}(D^+).\end{aligned}$$

A useful property of Fredholm operators (with some additional assumptions) is that their index can be computed in terms of the traces of particular operators derived from the Fredholm operator using the *heat equation method*. Recall that for compact operators $T\colon H \to H$ on a Hilbert space H that are of *trace-class*, i.e.

$$\sum_{i=1}^{\infty} s_i^{\frac{1}{2}} < \infty$$

where the s_i are the eigenvalues of the compact self-adjoint operator T^*T (where T^* is the adjoint of T), then the trace is given by

$$\operatorname{Tr}(T) = \sum_{i=1}^{\infty} \langle Te_i, e_i\rangle.$$

With T a trace-class operator we can equivalently view the trace as the sum of the eigenvalues of T counted with multiplicity. Furthermore, we have the following theorem which gives us the trace when T is an operator on L^2-spaces given by a continuous kernel function.

Theorem 4.3 (Lidskii)**.** *Suppose that our Hilbert space is some L^2-space and that our operator T is given by*

$$(Tf)(x) = \int K(x,y) f(y)\, dy$$

where $K(x,y)$ is some continuous kernel function. Then the trace of T is given by integrating along the diagonal of the kernel function, i.e.

$$Tr(T) = \int K(x,x)\, dx.$$

Now, using this notion of the trace and as alluded to earlier, we have the following theorem telling us about the trace of Fredholm operators, and often called the *heat equation method.*

Theorem 4.4. *If P is a Fredholm operator and the operators P^*P and PP^* have discrete spectrum, then for every $t > 0$ the index of P can be calculated as*

$$Ind(P) = Tr(e^{-tP^*P}) - Tr(e^{-tPP^*}),$$

if the exponential operators e^{-tP^*P} and e^{-tPP^*} are trace-class.

However, we are also interested in the notion of the *supertrace*. Before we give the definition, we first introduce the concept of *super vector bundles*, of which we have already seen some examples.

Definition 4.5. A vector bundle $p\colon E \to M$ is called a *super vector bundle* if it is $\mathbb{Z}/2$-graded. That is, if we have a splitting of E into subbundles E^+ and E^-, i.e. so that $E = E^+ \oplus E^-$.

Now, if E is a super vector bundle, then we define the *supertrace* of a section $A \in C^\infty(\operatorname{End}(E))$ of the endomorphism bundle of E to be given by

$$\operatorname{Tr}_s(A) = \operatorname{Tr}(A_{11}) - \operatorname{Tr}(A_{22}),$$

where we view A as the matrix

$$A = \begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}$$

as we can view $\operatorname{End}(E)$ as $\operatorname{End}(E^+ \oplus E^-)$. Now, if we consider an operator $P\colon C^\infty(E) \to C^\infty(E)$ with E a super vector bundle on M, we know by the heat equation method that

$$\operatorname{Ind}(P) = \operatorname{Tr}(e^{-tP^*P}) - \operatorname{Tr}(e^{-tPP^*}),$$

but using the supertrace we can simplify this and write

$$\operatorname{Ind}(P) = \operatorname{Tr}_s(e^{-t\hat{P}^2})$$

where $\hat{P}$ is given by the matrix

$$\hat{P} = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}$$

with respect to the $\mathbb{Z}/2$-grading $E^+ \oplus E^-$ on the super vector bundle E, so that $\hat{P}^2$ has block form

$$\hat{P}^2 = \begin{pmatrix} P^*P & 0 \\ 0 & PP^* \end{pmatrix}.$$

For example, consider the Dirac operator $D\colon C^\infty(W) \to C^\infty(W)$. We know that the spinor bundle W is a super vector bundle as we do indeed have a splitting $W = W_+ \oplus W_-$ into the chiral spinor bundles as discussed in Section 2 (recall we in fact defined W to be the direct sum of the chiral spinor bundles W_+ and W_- we constructed explicitly), which gives us the two chiral Dirac operators $D^+\colon C^\infty(W_+) \to C^\infty(W_+)$ and $D^-\colon C^\infty(W_-) \to C^\infty(W_-)$ with D^- the adjoint of D^+. Therefore, as we can view the Dirac operator D in block form as

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

with respect to the splitting $W = W_+ \oplus W_-$, it follows by above that the index of D^+ is given by the supertrace

$$\operatorname{Ind}(D^+) = \operatorname{Tr}_s(e^{-tD^2}).$$

To derive the McKean-Singer formula for the index (and the index theorem in general), we analyse this operator $e^{-tD^2}\colon C^\infty(W) \to C^\infty(W)$ which we call the *heat operator*. Now, for each spinor field $\psi \in C^\infty(W)$, we have that

$$(e^{-tD^2}\psi)(x) = \int_M K(t,x,y)\psi(x)\, dV,$$

where the *heat kernel* $K(t,x,y)$ is given by

$$K(t,x,y) = \sum_j e^{-t\lambda_j} \varphi_j(x) \otimes \overline{\varphi_j(y)},$$

and the eigenvalues λ_j and eigenfunctions (or eigensections) $\varphi_j \in L^2(W)$ are given by the spectral theorem applied to the square of the Dirac operator D^2 (so $D^2\varphi_j = \lambda_j\varphi_j$ for all j). So, by our above expression for $e^{-tD^2}\psi$ for any spinor field ψ as an integral with continuous heat kernel $K(t,x,y)$, it follows by Lidskii's theorem (and the definition of the supertrace as the difference of the traces on the two graded pieces) that for each $t > 0$ the supertrace

$$\operatorname{Tr}_s(e^{-tD^2}) = \int_M \operatorname{Tr}_s(K(t,x,x))\, dV.$$

That is, since $\operatorname{Ind}(D^+) = \operatorname{Tr}_s(e^{-tD^2})$ as discussed earlier, we have that

$$\operatorname{Ind}(D^+) = \int_M \operatorname{Tr}_s(K(t,x,x))\, dV,$$

which is precisely the McKean-Singer formula for the index of the chiral Dirac operator D^+ in terms of the heat kernel. Now, we know that the heat kernel $K(t,x,y)$ is smooth and solves the heat equation for the generalised Laplacian D^2, i.e. we have that

$$\frac{\partial}{\partial t} K(t,x,y) + D^2 K(t,x,y) = 0,$$

where we can either apply D^2 to $K(t,-,y)$ or $K(t,x,-)$ (i.e. we can fix either x or y). Note that in the McKean-Singer formula, the left-hand side $\operatorname{Ind}(D^+)$ is independent of t whereas the right-hand side does involve t. Therefore, we may take t to be as small as we want and we will still get the index $\operatorname{Ind}(D^+)$, i.e. we have that

$$\operatorname{Ind}(D^+) = \int_M \lim_{t\to 0} \operatorname{Tr}_s(K(t,x,x))\, dV.$$

Now, as mentioned earlier we have a useful asymptotic expansion of the heat kernel $K(t,x,x)$ given by the following theorem.

Theorem 4.6. *We have the asymptotic expansion of the heat kernel $K(t,x,x)$ along the diagonal as t becomes small given by*

$$K(t,x,x) \sim (4\pi t)^{-2} \sum_{i=0}^{\infty} t^i a_i(x)$$

where the a_i are polynomials which are smooth kernels for the spinor bundle W.

Observe the similarities between the heat kernel $K(t,x,x)$ for small t when considering the generalised Laplacian D^2 for our 4-manifold M compared to the heat kernel when our manifold is just $\mathbb{R}^4$. Indeed, if we take our manifold to be $\mathbb{R}^4$, then D^2 is just the Euclidean Laplacian Δ, and the heat kernel is given by

$$K(t,x,y) = (4\pi t)^{-2} e^{-|x-y|^2/4t}$$

which solves the heat equation

$$\frac{\partial}{\partial t} K(t,x,y) + \Delta_x K(t,x,y) = 0.$$

In particular, the heat kernel along the diagonal is given by

$$K(t,x,x) = (4\pi t)^{-2},$$

so for small time t the heat kernel with asymptotic expansion given by Theorem 4.6 looks like the Euclidean heat kernel. Note that $K(t,x,x)$ diverges as $t \to 0$, i.e. it becomes singular, but intuitively in taking the supertrace $\mathrm{Tr}_s(K(t,x,x))$ as $t \to 0$ we are subtracting two terms involving the heat kernel along the diagonal, and the point is that marvellously the singular terms cancel, which will give us the local index theorem. That is, we end up simply with the purely topological class $\hat{A}(\Omega)$.

Now, to prove the local index theorem for the Dirac operator D on our compact oriented 4-manifold M, we use the McKean-Singer formula

$$\mathrm{Ind}(D^+) = \int_M \mathrm{Tr}_s(K(t,x,x))\, dV$$

and analyse the right-hand side when $t \to 0$ which we can do again as the left-hand side is independent of t, and to do this we can use the asymptotic expansion of the heat kernel given by Theorem 4.6.

More specifically, we are going to use 'Getzler's re-scaling' to re-scale our heat kernel $K(t,x,x)$ so that after re-scaling it looks approximately like the heat kernel of a generalised harmonic oscillator (which is something we will be able to write down explicitly). Now, by the asymptotic expansion of the heat kernel given by Theorem 4.6, we see that for small time t the supertrace $\mathrm{Tr}_s(K(t,x,x))$ depends only on the generalised Laplacian D^2 locally at x. So, our problem is in fact local (hence why this is known as the local Atiyah-Singer index theorem), and we fix a point $p \in M$ and work locally at this point in normal coordinates so that $x = 0$ at this point (and the exponential map is an isomorphism). Furthermore, we can trivialise our spinor bundle W in this normal neighbourhood via parallel translation along geodesic rays to our fixed point $p \in M$. So, we can view our 4-dimensional Riemannian manifold M simply as $\mathbb{R}^4$ where we're working in local coordinates (x_1, x_2, x_3, x_4) at the origin, and we can also take our metric to be (up to a small perturbation) the Euclidean metric, i.e.

$$g_{ij} = \delta_{ij} + O(|x|^2)$$

where δ_{ij} is the Euclidean metric. We now introduce Getzler's re-scaling of our heat kernel. Consider a section $a(t,x)$ (thought of as an endomorphism of the fibre W_x of the spinor bundle), which we can write as

$$a(t,x) = \sum_I a_I(t,x) e_I,$$

since we recall that the endomorphism algebra of the fibre W_x is precisely the Clifford algebra $Cl(V) = \mathrm{End}(W)$ (where here W is just a copy of $\mathbb{C}^4$), and e_I is the Clifford multiplication $e_I = e_{i_1} \cdot \ldots \cdot e_{i_p}$ where $|I| = p$, i.e. the multi-index $I = (i_1, \ldots, i_p)$. Then, for a fixed small $\epsilon > 0$ the Getzler's rescaling $(\delta_\epsilon a)(t,x)$ is given by

$$(\delta_\epsilon a)(t,x) = \sum_I a_I(\epsilon t, \epsilon^{\frac{1}{2}} x) \epsilon^{-\frac{|I|}{2}} e_I,$$

where for a given multi-index $I = (i_1, \ldots, i_p)$ we have that

$$\epsilon^{-\frac{|I|}{2}} e_I = \epsilon^{-\frac{p}{2}} e_{i_1} \cdot \ldots \cdot e_{i_p}.$$

We can use this re-scaling to approximate the supertrace of the section $a(t,x)$, given by the following lemma.

Lemma 4.7. *With $a(t,x)$ as above, we have that*

$$\lim_{t\to 0} Tr_s(a(t,0)) = -4[\lim_{\epsilon\to 0} \epsilon^2(\delta_\epsilon a)(t,0)]_{(4)}.$$

On the right-hand side, note that we are taking the top degree term of $\lim_{\epsilon\to 0}\epsilon^2(\delta_\epsilon a)(t,0)$. The proof of Lemma 4.7 just follows from the fact that the supertrace of the Clifford multiplication e_I is always zero unless $I = (1,2,3,4)$ in which case it is equal to -4, and so

$$\lim_{t\to 0} \mathrm{Tr}_s(a(t,0)) = -4a_{(1,2,3,4)}(0,0),$$

which is precisely what we get when we multiply the top order of $\lim_{\epsilon\to 0}\epsilon^2(\delta_\epsilon a)(t,0)$ by -4 following from the definition of the rescaling. So, applying Lemma 4.7 to our heat kernel $a(t,x) = K(t,x,0)$ for D^2 (noting that we can express $K(t,x,0)$ as a linear combination of elements in the Clifford algebra as we are working in a trivialising neighbourhood of the spinor bundle), we have in particular that

$$\lim_{t\to 0} \mathrm{Tr}_s(K(t,0,0)) = -4[\lim_{\epsilon\to 0} \epsilon^2(\delta_\epsilon K)(t,0,0)]_{(4)}.$$

We now analyse the re-scaled heat kernel and see what it converges to as $\epsilon \to 0$, which we define to be given by

$$K_\epsilon(t,x,y) = \epsilon^2(\delta_\epsilon K)(t,x,y).$$

This re-scaled or modified heat kernel no longer satisfies our original heat equation for D^2, but it does satisfy the re-scaled heat equation for the operator D_ϵ^2 defined below, i.e. we have that

$$\frac{\partial}{\partial t} K_\epsilon(t,x,y) + D_\epsilon^2 K_\epsilon(t,x,y) = 0,$$

where here

$$D_\epsilon^2 = \epsilon\delta_\epsilon D^2 \delta_\epsilon^{-1}$$

noting that in our local coordinates (x_1,x_2,x_3,x_4) we have that $\delta_\epsilon \partial_{x_i}\delta_\epsilon^{-1} = \epsilon^{-\frac{1}{2}}\partial_{x_i}$. Now, the key point is that as $\epsilon \to 0$, our re-scaled heat kernel $K_\epsilon(t,x,y)$ converges to the heat kernel $K_0(t,x,y)$ of a generalised harmonic oscillator, and we have an explicit expression for this heat kernel given by Mehler's formula.

Lemma 4.8. *We have that*

$$\lim_{\epsilon\to 0} K_\epsilon(t,x,y) = K_0(t,x,y)$$

where $K_0(t,x,y)$ is the heat kernel of a generalised harmonic oscillator. That is, we have that

$$\lim_{\epsilon\to 0} D_\epsilon^2 = -\sum_i \left(\partial_{x_i} - \frac{1}{4}\Omega_{ij}x_j\right)^2$$

where $\Omega_{ij} = \frac{1}{2}R_{ijab}e^a \wedge e^b$ is the curvature 2-form.

Proof. This is a computation using Weitzenböck's formula (noting that the curvature term vanishes as our complex line bundle is trivial)

$$\begin{aligned} D^2 &= -\sum_{i=1}^4 [\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i}] + \frac{s}{4} \\ &= -\sum_{i,j,k} g^{ij}(\nabla_{\partial_{x_i}}\nabla_{\partial_{x_j}} - \Gamma_{ij}^k \nabla_{\partial_{x_k}}) + \frac{s}{4}, \end{aligned}$$

where we choice the orthonormal frame $e_1, \ldots, e_4$ for TM to be obtained from $\partial_{x_1}, \ldots, \partial_{x_4}$ by parallel translation along geodesic rays from the origin. The key idea is that since our metric is a small perturbation of the Euclidean metric i.e.

$$g^{ij} = \delta^{ij} + O(|x|^2)$$

recalling that we're working in the local coordinates (x_1, x_2, x_3, x_4), we have by techniques from Riemannian geometry that

$$\nabla_{\partial_{x_i}} = \partial_{x_i} - \frac{1}{2}\sum_{i,j,a,b} R_{ijab} x_j e^a \wedge e^b + O(|x|^2)$$

where $e^1, \ldots, e^4$ is the dual basis obtained from our original orthonormal frame $e_1, \ldots, e_4$ from Weitzenböck's formula. When we introduce our ϵ into Weitzenböck's formula, we end up precisely with

$$-\sum_i \left(\partial_{x_i} - \frac{1}{8} R_{ijab} x_j e^a \wedge e^b\right)^2$$

plus terms involving ϵ which vanish when we let $\epsilon \to 0$. ■

Now, we call $-\sum_i \left(\partial_{x_i} - \frac{1}{4}\Omega_{ij} x_j\right)^2$ a *generalised harmonic oscillator*. Looking at $\mathbb{R}$, recall that the standard one-dimensional harmonic oscillator is given by

$$H = -\frac{d^2}{dx^2} + \lambda^2 x^2.$$

If we take $\lambda = 1$ for simplicity, then Mehler's formula gives us the heat kernel of the harmonic oscillator $H = -\frac{d^2}{dx^2} + x^2$ explicitly.

Lemma 4.9 (Mehler's formula)**.** *The heat kernel $K(t,x,y)$ of the one-dimensional harmonic oscillator $H = -\frac{d^2}{dx^2} + x^2$ is given by*

$$K(t,x,y) = \frac{1}{(4\pi t)^{\frac{1}{2}}}\left(\frac{2t}{\sinh(2t)}\right)^{\frac{1}{2}} \exp\left(-\frac{x^2+y^2}{2\tanh(2t)} + \frac{xy}{\sinh(2t)}\right).$$

The heat kernel given by Mehler's formula can simply be obtained by a computation in solving the corresponding heat equation

$$\left(\frac{\partial}{\partial t} + H\right) K(t,x,y) = 0.$$

By applying a change of variables, we can use Mehler's formula to write down the heat kernel for the more general one-dimensional harmonic oscillator $H = -\frac{d^2}{dx^2} + \lambda^2 x^2$, and what we get (just looking at $K(t,x,0)$) is as follows.

Corollary 4.10. *The heat kernel of the general one-dimensional harmonic oscillator $H = -\frac{d^2}{dx^2} + \lambda^2 x^2$ is given by*

$$K(t,x,0) = \frac{1}{(4\pi t)^{\frac{1}{2}}}\left(\frac{2\lambda t}{\sinh(2\lambda t)}\right)^{\frac{1}{2}} \exp\left(-\frac{\lambda x^2}{2\tanh(2\lambda t)}\right).$$

Now, using Mehler's formula we get the following expression for the heat kernel of the generalised harmonic oscillator.

Lemma 4.11. *The heat kernel of the generalised harmonic oscillator $H = -\sum_i \left(\partial_{x_i} - \frac{1}{4}\Omega_{ij} x_j\right)^2$ is given by*

$$K_0(t,x,0) = \frac{1}{(4\pi t)^2}\det^{\frac{1}{2}}\left(\frac{t\Omega/2}{\sinh(t\Omega/2)}\right) \exp\left(-\frac{1}{4t}\left(\frac{t\Omega/2}{\tanh(t\Omega/2)}\right)_{ij} x_i x_j\right).$$

The idea of the proof of Lemma 4.11 is again to look at the corresponding heat equation, but to also observe that it is enough to prove the result when the matrix of 2-forms Ω is just a matrix with real-values since both sides of the heat equation are analytic in the Ω_{ij}. Then, after choosing the right orthonormal basis it suffices to prove the result in 2-dimensions and from here it is just a computation using Mehler's formula.

Notice that the heat kernel given by Lemma 4.11 looks very similar to the $\hat{A}$-genus in Definition 3.5, especially when we ignore the exponential term in the heat kernel (i.e. when the term inside the exponential is zero). In particular, if we take x to be 0, then we have that

$$K_0(t,0,0) = \frac{1}{(4\pi t)^2}\det^{\frac{1}{2}}\left(\frac{t\Omega/2}{\sinh(t\Omega/2)}\right).$$

Now, as discussed earlier, we have that

$$\lim_{t\to 0} \operatorname{Tr}_s(K(t,0,0)) = -4[\lim_{\epsilon\to 0} \epsilon^2(\delta_\epsilon K)(t,0,0)]_{(4)}.$$

But by the definition of the re-scaled heat kernel, we have that

$$\lim_{\epsilon\to 0} \epsilon^2(\delta_\epsilon K)(t,0,0) = \lim_{\epsilon\to 0} K_\epsilon(t,0,0),$$

and thus by Lemma 4.8 that

$$\lim_{\epsilon\to 0} \epsilon^2(\delta_\epsilon K)(t,0,0) = K_0(t,0,0).$$

Therefore, by Lemma 4.11 (i.e. our above explicit expression $K_0(t,0,0)$) and taking $t=1$ on the right-hand side, it follows that

$$\begin{aligned}\lim_{t\to 0} \operatorname{Tr}_s(K(t,0,0)) &= -4\left(\frac{1}{(4\pi)^2}\det{}^{\frac{1}{2}}\left(\frac{\Omega/2}{\sinh(\Omega/2)}\right)\right)\\ &= -\frac{1}{4\pi^2}\det{}^{\frac{1}{2}}\left(\frac{\Omega/2}{\sinh(\Omega/2)}\right)\\ &= -\frac{1}{4\pi^2}\hat{A}(\Omega),\end{aligned}$$

which is precisely the conclusion of the local index theorem. To see why this implies the Atiyah-Singer index theorem as stated in Theorem 3.6, recall as discussed earlier that to compute $\lim_{t\to 0}\operatorname{Tr}_s(K(t,x,x))$, it sufficed by the asymptotic expansion to localise the problem and work in normal coordinates in a neighbourhood around our fixed point $p \in M$ (as the asymptotic expansion told us that the heat kernel only depended on D^2 locally) so that $x = 0$ at this point, which implies that it is enough to compute $\lim_{t\to 0}\operatorname{Tr}_s(K(t,0,0))$. Hence, we may conclude by the McKean-Singer formula for the index that

$$\begin{aligned}\operatorname{Ind}(D^+) &= \int_M \lim_{t\to 0}\operatorname{Tr}_s(K(t,x,x))\\ &= -\frac{1}{4\pi^2}\int_M \hat{A}(\Omega) \qquad \text{(by above)},\end{aligned}$$

which is the Atiyah-Singer index theorem. Note however that that we have this extra factor of $-1/4\pi^2$, but we could have added in this factor into the statement of Theorem 3.6 and the definition of the $\hat{A}$-genus in Definition 3.5 as this normalising factor (which for an n-dimensional manifold is $(2\pi i)^{-n/2}$) is usually encompassed in the definition of the characteristic class $\hat{A}(\Omega)$ from the point-of-view of topology.

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