UNIVERSAL C^*-ALGEBRAS, SEMIPROJECTIVITY AND GRAPH C^*-ALGEBRAS

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Abstract. In this report we begin by giving the definition of a universal C^*-algebra given sets of generators and relations. After giving some important examples of universal C^*-algebras including the concept of a free product of C^*-algebras, we discuss the notions of projective and semi-projective C^*-algebras, which can be viewed as non-commutative analogs of the topological notions of an absolute retract and an absolute neighbourhood retract respectively. We discuss what this means precisely, as well as some fairly recent results regarding this correspondence, and we then prove that the Toeplitz algebra and $C(S^1)$ are semi-projective. Finally, we give the definition of a graph C^*-algebra and some important examples, and discuss why graph C^*-algebras are useful, in particular stating a theorem that gives us a relatively easy way to compute K-theory groups of graph C^*-algebras.

Contents

1. Universal C^*-algebras

Many C^*-algebras that we often encounter can be expressed simply as universal C^*-algebras on sets of generators and relations. Before we give some examples of universal C^*-algebras on sets of generators and relations, we will first present the definition of a universal C^*-algebra that we will use in this report.

Let $\mathcal{G}$ be any set, which can be finite, countable or even uncountable. We will call the set $\mathcal{G}$ the *generators*. Now, by a *relation* on $\mathcal{G}$, we mean an expression of the form

$$\|p(x_1, x_1^*, \dots, x_n, x_n^*)\| \leqslant r,$$

where p is a polynomial in $2n$ non-commuting variables, $x_1, x_2, \dots, x_n \in \mathcal{G}$ and $r \in [0, \infty)$, noting that the above "norm" is formal i.e. is only realised in a representation, and similarly the x_i^* are formal adjoints of the x_i. Now, letting $\mathcal{G}$ be a set of generators and $\mathcal{R}$ a set of relations, a *representation* of $(\mathcal{G}, \mathcal{R})$ consists of a Hilbert space $\mathcal{H}$ and a set $\{a_x : x \in \mathcal{G}\} \subset \mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ indexed by elements of $\mathcal{G}$ such that

$$\|p(a_{x_1}, a_{x_1}^*, \dots, a_{x_n}, a_{x_n}^*)\| \leqslant r$$

is satisfied whenever $\|p(x_1, x_1^*, \dots, x_n, x_n^*)\| \leqslant r$ is a relation in $\mathcal{R}$. If $r = 0$, then we get an algebraic relation between the x_i and the x_i^*. Now, note that universal C^*-algebras don't necessarily exist for all sets $(\mathcal{G}, \mathcal{R})$ of generators and relations.

Definition 1.1. Given a set $\mathcal{G}$ of generators, we say that a set $\mathcal{R}$ of relations is *admissible*, if there exists a non-zero representation of $(\mathcal{G}, \mathcal{R})$ as bounded operators on some Hilbert space $\mathcal{H}$ (i.e. the representation must not send everything to the zero operator), and if for every $x \in \mathcal{G}$ there is a constant $r_x \in [0, \infty)$ such that for every representation $\{a_x : x \in \mathcal{G}\} \subset \mathcal{B}(\mathcal{H})$ of $(\mathcal{G}, \mathcal{R})$, we have that

$$\|a_x\| \leqslant r_x$$

for every $x \in \mathcal{G}$.

We have that universal C^*-algebras for admissible relations on a set of generators always exists (and is necessarily unique). By that we mean that if $\mathcal{G}$ is a set of generators and $\mathcal{R}$ is a set of admissible relations on $\mathcal{G}$, then there is a unique C^*-algebra $C^*\langle\mathcal{G}|\mathcal{R}\rangle$ called the *universal C^*-algebra* containing a set of generators $\{a_x : x \in \mathcal{G}\} \subset C^*\langle\mathcal{G}|\mathcal{R}\rangle$ satisfying the relations $\mathcal{R}$, and furthermore $C^*\langle\mathcal{G}|\mathcal{R}\rangle$ satisfies the universal property that if $\mathcal{B}$ is any C^*-algebra containing elements $\{b_x : x \in \mathcal{G}\}$ satisfying the relations $\mathcal{R}$, then there is a unique $*$-homomorphism $C^*\langle\mathcal{G}|\mathcal{R}\rangle \to \mathcal{B}$ sending $a_x \mapsto b_x$ for each $x \in \mathcal{G}$.

We can also consider the universal unital C^*-algebra for generators and relations $(\mathcal{G}, \mathcal{R})$ whereby we add an extra generator denoted by 1 to $(\mathcal{G}, \mathcal{R})$ which we force to satisfy the relations

$$1 = 1^* = 1^2 \text{ and } x1 = 1x = x$$

for all $x \in \mathcal{G}$. We will now give a few examples of universal C^*-algebras.

Example 1.2. Every C^*-algebra is a universal C^*-algebra on some set of generators and relations $(\mathcal{G}, \mathcal{R})$. Indeed, if we are given a C^*-algebra A, we can take the set of generators to be $\mathcal{G} = \{x_a : a \in A\}$ with relations $\mathcal{R}$ given by

$$\|x_a\| = \|a\|,\ x_a x_b = x_{ab} \text{ and } x_a^* = x_{a^*}$$

for all $a, b \in A$. That is, the set of relations is the set of all $*$-algebraic relations in A, and thus $C^*\langle \mathcal{G} | \mathcal{R} \rangle$ is isomorphic to A. However, expressing A as a universal C^*-algebra in this way is not very enlightening or useful.

Example 1.3. Consider the set of generators $\mathcal{G} = \{x, 1\}$ with relations $\mathcal{R}$ given by

$$x^*x = xx^*,\ \|x\| \leqslant 1,\ 1 = 1^* = 1^2 \text{ and } x1 = 1x = x.$$

Then, we have that the universal (unital) C^*-algebra $C^*\langle \mathcal{G} | \mathcal{R} \rangle$ exists (indeed, this follows because of the relation $\|x\| \leqslant 1$ and that $C(\Delta)$ i.e. the C^*-algebra of continuous functions on the unit disk Δ is a representation of $(\mathcal{G}, \mathcal{R})$ after viewing it as a C^*-algebra of bounded operators on some Hilbert space by the Gelfand-Naimark theorem) and is in fact isomorphic to $C(\Delta)$ using the functional calculus and that the spectrum of the generator (corresponding to x) of $C^*\langle \mathcal{G} | \mathcal{R} \rangle$ must be contained in the closed unit disk Δ, together with the fact that $C^*\langle \mathcal{G} | \mathcal{R} \rangle$ satisfies the universal property of a universal C^*-algebra. By similar reasoning, we have that the universal C^*-algebra $C^*\langle u | u^*u = uu^* = 1 \rangle$ exists and is isomorphic to $C(S^1)$, i.e. the C^*-algebra of continuous functions on the unit circle S^1. We call the universal C^*-algebra $C^*\langle v | v^*v = 1 \rangle$ the *Toeplitz algebra*, which is isomorphic to $C^*(u)$, where u is the unilateral shift operator on $\ell^2(\mathbb{N})$.

Example 1.4. For each $n \geqslant 2$, the *Cuntz algebra* $\mathcal{O}_n$ is the universal C^*-algebra for the set of generators $\mathcal{G} = \{s_1, \ldots, s_n\}$ satisfying the relations $s_i^* s_i = 1$ for all $1 \leqslant i \leqslant n$ (that is, the s_i are isometries) and $s_i^* s_j = 0$ for $i \neq j$, as well as the relation

$$\sum_{j=1}^{n} s_j s_j^* = 1.$$

We have that if $n \neq m$ with $n, m \geqslant 2$, then $\mathcal{O}_n$ is not isomorphic to $\mathcal{O}_m$.

Example 1.5. Suppose that we are given a collection of C^*-algebras $\{A_i\}_{i \in I}$, where I is some indexing set. Then, the *free product* $\circledast\{A_i\}_{i \in I}$ of the A_i is defined to be the universal C^*-algebra for the set of generators $\mathcal{G} = \bigcup_{i \in I} A_i$ and relations $\mathcal{R}$ given by all $*$-algebraic relations in all of the A_i. If the A_i are unital, then the *unital free product* $\circledast_{\mathbb{C}}\{A_i\}_{i \in I}$ of the A_i is the universal C^*-algebra now generated by unital copies of the A_i. To make a bit more sense of this, suppose that we are given a C^*-algebra D and injective $*$-homomorphisms $\varphi_i \colon D \to A_i$ for each $i \in I$. Then, we define the *amalgamated free product* $\circledast_D\{A_i\}_{i \in I}$ (or $\circledast_{\varphi_i}\{A_i\}_{i \in I}$) to be the universal C^*-algebra for the set of generators $\mathcal{G} = \bigcup_{i \in I} A_i$ and relations $\mathcal{R}$ given by all $*$-algebraic relations in all of the A_i as well as the set of relations $\{\varphi_i(d) = \varphi_j(d) \text{ for all } d \in D \text{ and } i, j \in I\}$ identifying the copies of D in all of the A_i. We have that if $D = \{0\}$ is the zero C^*-algebra, then the amalgamated free product $\circledast_D\{A_i\}_{i \in I}$ is precisely the free product $\circledast\{A_i\}_{i \in I}$, and if $D = \mathbb{C}$ then $\circledast_D\{A_i\}_{i \in I}$ is precisely the unital free product $\circledast_{\mathbb{C}}\{A_i\}_{i \in I}$. It is a fact that there is an embedding of each A_i into the amalgamated free product.

2. Projective and Semiprojective C^*-algebras

In order to motivate the definitions of projective and semi-projective C^*-algebras, we first introduce the topological notions of an absolute retract (AR) and an absolute neighbourhood retract (ANR). When giving these definitions, we will assume that we are working in the category of compact metrisable spaces (with morphisms continuous maps), and from now on we will simply call them 'spaces'. Furthermore, in this section we will assume that all C^*-algebras are separable.

Definition 2.1. A space X is called an *absolute retract* (or an *AR*) if whenever X can be embedded as a closed subspace of another space Y, then there is a retraction $r \colon Y \to X$, i.e. a (continuous) map $r \colon Y \to X$ that is the identity on X when restricted to X.

We also have the following slightly weaker notion of an absolute neighbourhood retract.

Definition 2.2. A space X is called an *absolute neighbourhood retract* (or an *ANR*) if whenever X can be embedded as a closed subspace of another space Y, then there is an open subset $U \subset Y$ containing X (i.e. an open neighbourhood of X in Y) and a retraction $r \colon U \to X$.

Now, the following two theorems (which we won't prove) and a corollary give equivalent definitions of an *AR* and an *ANR*, which will be the topological analogs of the definitions of projective and semi-projective C^*-algebras that we will soon give. When we say 'map', we mean a continuous map.

Theorem 2.3. *A space X is an AR if and only if for any space Y containing a closed subspace $Z \subset Y$, we have that any map $\phi \colon Z \to X$ lifts to a map $\tilde{\phi} \colon Y \to X$, so that the diagram*

$$\begin{tikzcd} Y \arrow[dr, "\tilde{\phi}"] & \\ Z \arrow[u, "i"] \arrow[r, "\phi"'] & X \end{tikzcd}$$

commutes, where $i\colon Z \to Y$ is the inclusion map.

Theorem 2.4. *A space X is an ANR if and only if for any space Y containing a closed subspace $Z \subset Y$, we have that any map $\phi\colon Z \to X$ lifts to a map $\tilde{\phi}\colon C \to X$ for some closed neighbourhood C of Z in Y.*

We didn't draw the diagram in Theorem 2.4 as the actual topological analog of a semi-projective C^*-algebra is given by the following corollary of Theorem 2.4.

Corollary 2.5. *A space X is an ANR if and only if for any space Y containing a decreasing sequence*

$$Z_1 \supset Z_2 \supset Z_3 \supset \dots$$

of closed subspaces with $Z = \bigcap_{n=1}^{\infty} Z_n$, we have that any map $\phi\colon Z \to X$ lifts to a map $\tilde{\phi}\colon Z_n \to X$ for some large enough n. That is, so that the diagram

$$\begin{tikzcd} Z_n \arrow[dr, "\tilde{\phi}"] & \\ Z \arrow[u, "i_n"] \arrow[r, "\phi"'] & X \end{tikzcd}$$

commutes, where $i_n\colon Z \to Z_n$ is the inclusion map.

Indeed, this just follows by a compactness argument, using that any closed neighbourhood C of Z in Y must eventually contain Z_j for all $j \geqslant n$ for some sufficiently large n.

We are now ready to give the definition of (semi-)projective C^*-algebras, recalling that we assume our C^*-algebras in this section to be separable.

Definition 2.6. A C^*-algebra A is *projective* if for any C^*-algebra B, closed 2-sided ideal $I \lhd B$ and $*$-homomorphism $\phi\colon A \to B/I$, there is a lift $\tilde{\phi}\colon A \to B$ (which is also a $*$-homomorphism) such that the diagram

$$\begin{tikzcd} & B \arrow[d, "\pi"] \\ A \arrow[ur, "\tilde{\phi}"] \arrow[r, "\phi"] & B/I \end{tikzcd}$$

commutes, where $\pi\colon B \to B/I$ is the natural quotient map.

Definition 2.7. A C^*-algebra A is *semi-projective* if for any C^*-algebra B and increasing sequence of closed 2-sided ideals

$$I_1 \lhd I_2 \lhd \dots \lhd I$$

with $I = \overline{\bigcup_{n=1}^{\infty} I_n}$, we have that any $*$-homomorphism $\phi\colon A \to B/I$ has a partial lift to a $*$-homomorphism $\tilde{\phi}\colon A \to B/I_n$ for some $n \in \mathbb{N}$, so that the diagram

$$\begin{tikzcd} & B/I_n \arrow[d, "\pi_n"] \\ A \arrow[ur, "\tilde{\phi}"] \arrow[r, "\phi"] & B/I \end{tikzcd}$$

commutes, where $\pi_n\colon B/I_n \to B/I$ is the natural quotient map.

Note that the definitions of projective and semi-projective C^*-algebras can be defined in many different subcategories of the category of C^*-algebras with $*$-homomorphisms. Already, we observe that Definition 2.6 and Definition 2.7 are essentially obtained from Theorem 2.3 and Corollary 2.5 (respectively) by 'turning the arrows around'. In fact, we have the following theorem.

Theorem 2.8. *Let $A = C(X)$ be a commutative C^*-algebra with unit, where X is a compact metrisable space. Then, we have that A is projective in the category of unital commutative C^*-algebras if and only if X is an AR. Furthermore, we have that A is semi-projective in the category of unital commutative C^*-algebras if and only if X is an ANR.*

Indeed, this follows almost immediately from the following observation. By [2, Theorem 2.2.4], any unital commutative C^*-algebra A is isometrically $*$-isomorphic to $C(X)$, where $X = \sigma(A)$ is the Gelfand spectrum of A, which is a compact metrisable space. Recall that the unital commutative C^*-algebra $C(X)$ has addition, multiplication and the involution defined pointwise (where the involution is pointwise complex conjugation), and is equipped with the supremum norm. Now, let **CMetr** denote the category of compact metrisable spaces with morphisms continuous maps, and let $\mathbf{AbC_1^*}$ denote the category of unital commutative C^*-algebras with morphisms unital $*$-homomorphisms.

Theorem 2.9. *The map $X \mapsto C(X)$ defines a contravariant equivalence of the categories* **CMetr** *and* $\mathbf{AbC_1^*}$.

Proof. First, suppose that $\phi\colon X \to Y$ is a continuous map, i.e. a morphism in **CMetr**. Then, via pullback, we get a unital $*$-homomorphism $\phi^*\colon C(Y) \to C(X)$ defined by $\phi^*(f) = f \circ \phi$ for all $f \in C(Y)$. Note that ϕ^* is injective if and only if ϕ is surjective, and that ϕ^* is surjective if and only if ϕ is injective. Now, conversely suppose that we are given a unital $*$-homomorphism $\varphi\colon C(Y) \to C(X)$, i.e. a morphism in $\mathbf{AbC_1^*}$. Then, we construct a continuous map $\varphi^*\colon X \to Y$ as follows. Let $x \in X$ be arbitrary. Recall that we can identify X with the Gelfand spectrum $\sigma(C(X))$ where $p \in X$ corresponds to the evaluation map $\mathrm{ev}_p \in \sigma(C(X))$ where $\mathrm{ev}_p(f) = f(p)$ for all $f \in C(X)$. So, given our $x \in X$, consider the homomorphism $\mathrm{ev}_x\colon C(X) \to \mathbb{C}$. Then, we have the pullback $\mathrm{ev}_x \circ \varphi\colon C(Y) \to \mathbb{C}$ which is also a homomorphism (as both ev_x and φ are homomorphisms), and thus must be given by evaluation at some $y \in Y$ as Y is identified with $\sigma(C(Y))$ in the same way. Finally, we define $\varphi^*(x) = y$. It can be easily checked that $\varphi^*\colon X \to Y$ is a continuous map, and that these two maps between the set of morphisms in **CMetr** and the set of morphisms in $\mathbf{AbC_1^*}$ are inverses of each other, therefore giving the result. ∎

Note that Theorem 2.8 now follows from Theorem 2.9 and that Definition 2.6 and Definition 2.7 are the corresponding notions of an AR and an ANR given in Theorem 2.3 and Corollary 2.5 respectively but now in the category $\mathbf{AbC_1^*}$ which is contravariantly equivalent to the category **CMetr**. Indeed, for the semi-projective case looking at Corollary 2.5 and Definition 2.7, under this contravariant equivalence of categories the quotient map $B/I_n \to B/I$ in $\mathbf{AbC_1^*}$ corresponds to the inclusion map $Z \to Z_n$ for each n where $B/I_n \cong C(Z_n)$ and $B/I \cong C(Z)$ (noting that $C(Z_n)$ surjects onto $C(Z)$ via restriction as $Z \subset Z_n$).

Example 2.10. The unit disk Δ is an AR, so by Theorem 2.8 we have that $C(\Delta)$ is projective in the category $\mathbf{AbC_1^*}$. Similarly, we know that $S^1 = \partial\Delta$ is an ANR, so by Theorem 2.8 we have that $C(S^1)$ is semi-projective in the category $\mathbf{AbC_1^*}$.

The following theorem proved fairly recently (in [9] for the projective case and [8] for the semi-projective case which was first conjectured by Blackadar) generalises Theorem 2.8 to the more general category $\mathbf{C_1^*}$ of unital C^*-algebras with morphisms unital $*$-homomorphisms.

Theorem 2.11. *Let $A = C(X)$ be a commutative C^*-algebra with unit, where X is a compact metrisable space. Then, we have that A is projective in $\mathbf{C_1^*}$ if and only if X is an AR and $\dim(X) \leqslant 1$. Furthermore, we have that A is semi-projective in $\mathbf{C_1^*}$ (or even the general category $\mathbf{C^*}$ of C^*-algebras) if and only if X is an ANR and $\dim(X) \leqslant 1$.*

Here, $\dim(X)$ is the covering dimension of X, and $\dim(X) \leqslant 1$ means that for every finite open cover $\mathcal{U}$ of the compact set X, there is a subcover $\mathcal{V}$ such that every element of X is contained in at most 2 open sets in the finite open cover $\mathcal{V}$.

We will now give some examples (and non-examples) of projective and semi-projective C^*-algebras in the category $\mathbf{C_1^*}$.

Proposition 2.12. *The Toeplitz algebra $C^*\langle v | v^*v = 1\rangle$ is semi-projective.*

Proof. Let $A = C^*\langle v | v^*v = 1\rangle$. Abusing notation slightly, we will say that A is generated by the element v, which satisfies the relation $v^*v = 1$. Now, suppose that we are given a C^*-algebra B, an increasing sequence of closed 2-sided ideals $I_1 \lhd I_2 \lhd \cdots \lhd I$ with $I = \overline{\bigcup_{n=1}^\infty I_n}$ and a $*$-homomorphism $\phi\colon A \to B/I$. We want to find an $n \in \mathbb{N}$ and a $*$-homomorphism $\tilde{\phi}\colon A \to B/I_n$ such that $\phi = \pi_n \circ \tilde{\phi}$, where $\pi_n\colon B/I_n \to B/I$ is the quotient map. Now, since ϕ is a $*$-homomorphism, we know that there exists some $x \in B/I$ (namely $\phi(v)$) satisfying $x^*x = 1$. Now, let $y \in B$ be such that $\pi(y) = x$, where $\pi\colon B \to B/I$ is the natural quotient map, i.e. y is a lift of x in B. For each $n \in \mathbb{N}$, we have by the definition of the norm on the quotient B/I_n that

$$\|(y^*y - 1) + I_n\| = \inf_{z_n \in I_n} \|y^*y - 1 + z_n\|.$$

So, since $I = \overline{\bigcup_{n=1}^\infty I_n}$, we have that

$$\begin{aligned}\lim_{n\to\infty} \|(y^*y - 1) + I_n\| &= \lim_{n\to\infty} \inf_{z_n \in I_n} \|y^*y - 1 + z_n\| \\ &= \inf_{z \in I} \|y^*y - 1 + z\| \\ &= 0.\end{aligned}$$

Indeed, the fact that $\inf_{z\in I}\|y^*y-1+z\|=0$ follows since $y^*y-1\in I$ (and therefore $-(y^*y-1)\in I$). To see why $y^*y-1\in I$, it suffices to show that $\pi(y^*y-1)=0$ where again $\pi\colon B\to B/I$ is the quotient map. But this follows immediately from the fact that π is a $*$-homomorphism, as

$$\pi(y^*y-1)=\pi(y)^*\pi(y)-1=x^*x-1=0.$$

By 0 we of course mean $0+I\in B/I$. So, since $\lim_{n\to\infty}\|(y^*y-1)+I_n\|=0$, we can choose an $N\in\mathbb{N}$ such that $\|(y^*y-1)+I_N\|<1$. Now, consider the element $y_N:=y+I_N\in B/I_N$. Then, we have that

$$\begin{aligned}\|y_N^*y_N-1\|&=\|(y^*+I_N)(y+I_N)-(1+I_N)\|\\&=\|(y^*y-1)+I_N\|\\&<1.\end{aligned}$$

Therefore, by [2, Theorem 1.5.2], we have that $1-(y_N^*y_N-1)=-y_N^*y_N$ is invertible, and thus $y_N^*y_N$ is also invertible. Now, note that $y_N^*y_N$ is a positive element of the C^*-algebra B/I_N as it is of course of the form s^*s for some $s\in B/I_N$ (take $s=y_N$). Therefore, we have that $y_N^*y_N$ has a unique positive square root, call it $(y_N^*y_N)^{\frac{1}{2}}$. Since $y_N^*y_N$ is invertible, we must also have that $(y_N^*y_N)^{\frac{1}{2}}$ is invertible. Indeed, we know that the spectrum $\sigma((y_N^*y_N)^{\frac{1}{2}})=\sqrt{\sigma(y_N^*y_N)}$, where $\sqrt{\sigma(y_N^*y_N)}=\{\sqrt{\lambda}:\lambda\in\sigma(y_N^*y_N)\}$ since $\sigma(y_N^*y_N)\subset[0,\infty)$ (as $y_N^*y_N$ is positive). Thus, since $0\notin\sigma(y_N^*y_N)$ as $y_N^*y_N$ is invertible, it follows that $0\notin\sigma((y_N^*y_N)^{\frac{1}{2}})$, and thus $(y_N^*y_N)^{\frac{1}{2}}$ is invertible. We will denote by $(y_N^*y_N)^{-\frac{1}{2}}$ the inverse of $(y_N^*y_N)^{\frac{1}{2}}$ in B/I_N. Finally, we let $z=y_N(y_N^*y_N)^{-\frac{1}{2}}\in B/I_N$. Again letting $\pi_N\colon B/I_N\to B/I$ be the natural quotient map, we have that

$$\begin{aligned}\pi_N(z)&=\pi_N(y_N(y_N^*y_N)^{-\frac{1}{2}})\\&=\pi_N(y_N)(\pi_N(y_N^*y_N))^{-\frac{1}{2}}\qquad\text{(as }\pi_N\text{ is a }*\text{-homomorphism)}\\&=\pi_N(y+I_N)(\pi_N(y^*y+I_N))^{-\frac{1}{2}}\\&=x(x^*x)^{-\frac{1}{2}}\\&=x\qquad\text{(as }x^*x=1\text{)},\end{aligned}$$

and that

$$\begin{aligned}z^*z&=(y_N^*y_N)^{-\frac{1}{2}}y_N^*y_N(y_N^*y_N)^{-\frac{1}{2}}\\&=((y_N^*y_N)^{\frac{1}{2}})^{-1}((y_N^*y_N)^{\frac{1}{2}})^2((y_N^*y_N)^{\frac{1}{2}})^{-1}\\&=1,\end{aligned}$$

where we have used that $((y_N^*y_N)^{-\frac{1}{2}})^*=(y_N^*y_N)^{-\frac{1}{2}}$ since we know in general that the inverse of a self-adjoint element in a C^*-algebra is also self-adjoint. Since $z\in B/I_N$ satisfies the relation $z^*z=1$, we have by the universal property of $A=C^*\langle v|v^*v=1\rangle$ that there is a unique $*$-homomorphism $\tilde{\phi}\colon A\to B/I_N$ sending $v\mapsto z$. But since $\phi(v)=x$ and $\pi_N(z)=x$, we therefore have that $\phi=\pi_N\circ\tilde{\phi}$ (as v generates A), and thus we may now conclude that A is semi-projective. ∎

We will now use Proposition 2.12 to show that the universal C^*-algebra $C^*\langle u|u^*u=uu^*=1\rangle$ is also semi-projective (recalling that by the functional calculus this universal C^*-algebra is identified with $C(S^1)$).

Proposition 2.13. *The universal C^*-algebra $C^*\langle u|u^*u=uu^*=1\rangle$ is semi-projective. That is, the C^*-algebra $C(S^1)$ of continuous functions on the unit circle S^1 is semi-projective.*

Proof. We let $A=C^*\langle u|u^*u=uu^*=1\rangle$, and again we slightly abuse notation and say that A is generated by u, which satisfies the relation $u^*u=uu^*=1$. As before, suppose that we are given a C^*-algebra B, an increasing sequence of closed 2-sided ideals $I_1\lhd I_2\lhd\cdots\lhd I$ with $I=\overline{\bigcup_{n=1}^\infty I_n}$ and a $*$-homomorphism $\phi\colon A\to B/I$. We want to find an $n\in\mathbb{N}$ and a $*$-homomorphism $\tilde{\phi}\colon A\to B/I_n$ such that $\phi=\pi_n\circ\tilde{\phi}$, where $\pi_n\colon B/I_n\to B/I$ is the quotient map. First, let $x\in B/I$ be such that $x^*x=xx^*=1$ (we can take $x=\phi(u)$ for example). Now, by Proposition 2.12, we know that there is some $N\in\mathbb{N}$ and $z\in B/I_N$ such that $\pi_N(z)=x$ and $z^*z=1$. Observe that zz^* is a projection. Indeed, we have that $(zz^*)^*=zz^*$ and $(zz^*)^2=z(z^*z)z^*=zz^*$. Now, let $\tilde{z}\in B$ be some lift of $z\in B/I_N$. Letting $\pi^N\colon B\to B/I_N$ be the natural quotient map, we must have then that $\tilde{z}^*\tilde{z}=1$ and that $\tilde{z}\tilde{z}^*$ is a projection since π^N is a $*$-homomorphism. Now, we also have that $\tilde{z}\tilde{z}^*-1\in I$. Indeed, this is because

$$\pi(\tilde{z}\tilde{z}^*-1)=\pi_N(zz^*-1)=xx^*-1=0.$$

Thus, as in the proof of Proposition 2.12, we have that

$$\lim_{k\to\infty}\|\tilde{z}\tilde{z}^*-1+I_k\|=\|\tilde{z}\tilde{z}^*-1+I\|=0.$$

So, we can choose some $K \geqslant N$ such that

$$\|\tilde{z}\tilde{z}^* - 1 + I_K\| < 1.$$

Then, we let $w = \tilde{z} + I_K \in B/I_K$. Notice that $w^*w = 1$ as

$$w^*w = (\tilde{z} + I_K)^*(\tilde{z} + I_K) = \tilde{z}^*\tilde{z} + I_K = 1 + I_K,$$

and we furthermore have that

$$\|ww^* - 1\| < 1$$

by construction. So, by [2, Theorem 1.5.2], we have that $1 - (ww^* - 1) = -ww^*$ is invertible, and therefore ww^* is invertible. But then ww^* is an invertible projection (it is a projection since $\tilde{z}\tilde{z}^*$ is a projection), which therefore implies that $ww^* = 1$. Indeed, we have that

$$ww^* = (ww^*)^{-1}(ww^*)^2 = (ww^*)^{-1}ww^* = 1,$$

where in the second equality we used that ww^* is a projection. Finally, again letting $\pi_K \colon B/I_K \to B/I$ be the natural quotient map, we have that $\pi_K(w) = x$ since $K \geqslant N$ and $w = \tilde{z} + I_K$ so $\pi_K(w) = \pi_N(z) = x$. Therefore, by the universal property of the universal C^*-algebra A, we have that there is a unique $*$-homomorphism $\tilde{\phi} \colon A \to B/I_K$ sending $u \mapsto w$ (as w satisfies $w^*w = ww^* = 1$). Moreover, since $\pi_K(w) = x$ and u generates A, we have that $\phi = \pi_K \circ \tilde{\phi}$ and so we may now conclude that A is semi-projective. ■

An example of a projective C^*-algebra in $\mathbf{C}_1^*$ (which we will not prove) is the C^*-algebra $C([0,1])$ of continuous functions on the unit interval $[0,1]$.

3. Graph C^*-algebras

A rich class of universal C^*-algebras is the class of *graph C^*-algebras*, which can be thought of as a generalisation of the Cuntz algebra, which can be seen from their respective definitions. Graph C^*-algebras play an important role in the classification theory of C^*-algebras, as their invariants are very computable. In particular, we will see that there is an easy way to compute the K_0 and K_1 groups of a graph C^*-algebra.

Definition 3.1. Let G denote a directed graph with vertex set G^0 and edge set G^1. Furthermore, we let $r \colon G^1 \to G^0$ be the function sending an edge to its range vertex, and $s \colon G^1 \to G^0$ be the function sending an edge to its source vertex. Then, the graph C^*-algebra $C^*(G)$ associated to G is the universal C^*-algebra for the set of generators

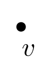
$$\mathcal{G} = \{p_v : v \in G^0\} \cup \{s_e : e \in G^1\}$$

with relations $\mathcal{R}$ given by the following:

(1) The p_v are mutually orthogonal projections. That is, for all $v \in G^0$, we have that $p_v^2 = p_v^* = p_v$ and $p_v p_w = 0$ for all $w \neq v$.
(2) For all $e \in G^1$, we have that $s_e^* s_e = p_{r(e)}$ and $s_e^* s_f = 0$ for all $f \neq e$. Furthermore, the s_e are partial isometries i.e. $s_e s_e^* s_e = s_e$ for all $e \in G^1$.
(3) If $v \in G^0$ is not a sink, that is to say the set $\{e \in G^1 : s(e) = v\}$ is non-empty, but that furthermore this set is finite, then we have that
$$p_v = \sum_{\{e : s(e) = v\}} s_e s_e^*.$$
(4) For all $e \in G^1$, we have that $s_e s_e^* \leqslant p_{s(e)}$.

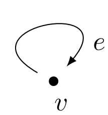
We will now give some examples of graph C^*-algebras.

Example 3.2. Consider the directed graph G given by



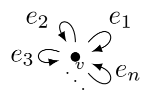
i.e. where $G^0 = \{v\}$ and $G^1 = \varnothing$. Then, by Definition 3.1, we have that the graph C^*-algebra $C^*(G)$ is the universal C^*-algebra for the set of generators $\mathcal{G} = \{p_v\}$ and relations $\mathcal{R}$ given by $p_v = p_v^2 = p_v^*$ (by condition (1) in Definition 3.1). Therefore, it immediately follows that $C^*(G)$ is isomorphic to $\mathbb{C}$ (as $\mathbb{C}$ is the universal C^*-algebra for the single generator a unit 1 satisfying $1 = 1^2 = 1^*$).

Example 3.3. Consider the directed graph G given by



i.e. where $G^0 = \{v\}$ and $G^1 = \{e\}$. Then, by Definition 3.1, we have that the graph C^*-algebra $C^*(G)$ is the universal C^*-algebra for the set of generators $\mathcal{G} = \{p_v, s_e\}$ and relations $\mathcal{R}$ given by $p_v = p_v^2 = p_v^*$ (by condition (1) in Definition 3.1), and furthermore $s_e^* s_e = p_v$ and $s_e = s_e s_e^* s_e$ (by condition (2) in Definition 3.1), and $p_v = s_e s_e^*$ (by condition (3) in Definition 3.1, noting that condition (4) is implied by condition (3) in this case). Note that $s_e^* s_e = p_v = s_e s_e^*$ implies that $s_e p_v = p_v s_e$ as $s_e p_v = s_e s_e^* s_e = p_v s_e$, and furthermore $s_e = s_e s_e^* s_e$ implies that $s_e = s_e p_v = p_v s_e$. Therefore, we have that $C^*(G)$ is the universal C^*-algebra for the generators $\mathcal{G} = \{p_v, s_e\}$ with relations $p_v = p_v^2 = p_v^*$, $s_e^* s_e = p_v = s_e s_e^*$ and $s_e = s_e p_v = p_v s_e$. Thus, it immediately follows that $C^*(G)$ is isomorphic to $C(S^1)$ i.e. the C^*-algebra of continuous functions on the unit circle S^1 (as we know that $C(S^1)$ is the universal C^*-algebra for the generators $\{1, u\}$ where $u^*u = 1 = uu^*$, $1 = 1^2 = 1^*$ and $u = 1u = u1$).

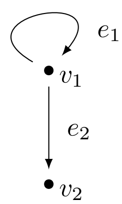
Example 3.4. Observe that Definition 3.1 is very similar to the definition of the Cuntz algebra $\mathcal{O}_n$ given in Example 1.4. Indeed, we have that the Cuntz algebra $\mathcal{O}_n$ is the graph C^*-algebra $C^*(G)$ for the directed graph G given by



so $C^*(G)$ is the universal C^*-algebra for the set of generators $\mathcal{G} = \{p_v, s_{e_1}, \ldots, s_{e_n}\}$ with relations $\mathcal{R}$ given by $p_v = p_v^2 = p_v^*$ (by condition (1) in Definition 3.1), $s_{e_i}^* s_{e_i} = p_v$ and $s_{e_i} = s_{e_i} s_{e_i}^* s_{e_i}$ for all $1 \leqslant i \leqslant n$, $s_{e_i}^* s_{e_j} = 0$ for all $j \neq i$ (all by condition (2) in Definition 3.1) and $p_v = \sum_{j=1}^n s_{e_j} s_{e_j}^*$ (by condition (3) in Definition 3.1). Note then that for every $1 \leqslant i \leqslant n$ we have that $s_{e_i} p_v = s_{e_i} s_{e_i}^* s_{e_i} = s_{e_i}$, and $p_v s_{e_i} = \sum_{j=1}^n s_{e_j} s_{e_j}^* s_{e_i} = s_{e_i} s_{e_i}^* s_{e_i} = s_{e_i} = s_{e_i} p_v$. Thus, we have that p_v acts as a unit, and we see by definition that $C^*(G)$ is isomorphic to the Cuntz algebra $\mathcal{O}_n$.

One final example of a graph C^*-algebra that we will give (but not flesh out the details this time) is the following.

Example 3.5. The graph C^*-algebra $C^*(G)$ for the directed graph given by



is isomorphic to the Toeplitz algebra, i.e. the universal C^*-algebra $C^*\langle v | v^*v = 1\rangle$.

We mentioned earlier that graph C^*-algebras are useful as their basic structure is understood well, and we have for example the following theorem which we will not prove (see [11]) that is very useful in computing the K-theory groups K_0 and K_1 of graph C^*-algebras. We won't discuss K-theory for C^*-algebras in this report, but see [1] for more on operator K-theory.

Theorem 3.6. *Suppose that G is a directed graph with vertex set G^0, edge set G^1 and range and source functions $r, s\colon G^1 \to G^0$. Furthermore, let $G^0_+ \subset G^0$ denote the subset of vertices in G that emit finitely many edges but at least one edge. Now, let $\mathbb{Z}G^0_+$ denote the free abelian group on G^0_+ and let $\mathbb{Z}G^0$ denote the free abelian group on $\mathbb{Z}G^0$. Then, consider the map $A_G\colon \mathbb{Z}G^0_+ \to \mathbb{Z}G^0$ defined (on the generators of $\mathbb{Z}G^0_+$) by*

$$A_G(v) = \left(\sum_{s(e)=v} r(e) \right) - v.$$

Then, we have that the K_0 and K_1 groups of the graph C^*-algebra $C^*(G)$ are isomorphic to coker(A_G) and ker(A_G) respectively.

So, for example, since we know from Example 3.3 that the graph C^*-algebra $C(S^1)$ has directed graph G with $G^0 = \{v\}$ and $G^1 = \{e\}$, it follows that $\mathbb{Z}G^0 = \mathbb{Z}G^0_+$ is the free abelian group $\langle v\rangle$ generated by v, and $A_G(v) = v - v = 0$, so in this case $A_G\colon \langle v\rangle \cong \mathbb{Z} \to \langle v\rangle \cong \mathbb{Z}$ is the zero map, which has cokernel $\mathbb{Z}$ and kernel $\mathbb{Z}$. Thus, it follows that the K_0 and K_1 groups of the C^*-algebra $C(S^1)$ are given by $K_0(C(S^1)) \cong K_1(C(S^1)) \cong \mathbb{Z}$.

Theorem 3.6 is just one example of how useful graph C^*-algebras are in practice and why they are used extensively in current research.

References

[1] B. Blackadar. *Operator Algebras: Theory of C^*-Algebras and von Neumann Algebras.* Encyclopaedia of Mathematical Sciences. Springer, Berlin. 2006.
[2] W. Arveson. *A Short Course on Spectral Theory.* Graduate Texts in Mathematics 209, Springer-Verlag, New York. 2002.
[3] B. Blackadar. *Shape Theory for C^*-algebras.* Math. Scand., 56(2):249–275. 1985.

[4] B. Blackadar. *Noncommutative Shape Theory*. Operator Algebras and their Connections with Topology and Ergodic Theory. Buşteni, 1983. Lecture Notes in Math. 1132, Springer, Berlin. 1985. pp. 38–45. MR 799562. Zbl 0618.46058.
[5] E. Gardella. *Crossed Products and C^*-Dynamics*. April 2019. Available at https://ivv5hpp.uni-muenster.de/u/gardella/Docs/DynamicsCourse/C%5e-dynamics.pdf.
[6] H. Thiel. *Semiprojectivity and Semiinjectivity in Different Categories*. arXiv:1802.05037. February 2018.
[7] J. Spielberg. *Fundamental Results about Semiprojective C^*-algebras*. Masterclass on Semiprojectivity, Copenhagen. August 2012. Available at http://web.math.ku.dk/ eilers/sprepo/spielberg_slides_final.pdf.
[8] A. P. W. Sørensen and H. Thiel. *A Characterization of Semiprojectivity for Commutative C^*-algebras*. arXiv:1101.1856. January 2011.
[9] A. Chigogidze and A. N. Dranishnikov. *Which Compacta are Noncommutative ARs?* arXiv:0902.3020. February 2009.
[10] A. Kumjian. *Notes on C^*-algebras of Graphs*. Operator Algebras and Operator Theory. Shanghai. 1997. 189-200; Contemp. Math., 228, AMS, Providence, RI. 1998.
[11] R. Matthes and W. Szymański. *Graph C^*-algebras*. Notes taken by P. Witkowski. August 2005. Available at https://www.impan.pl/swiat-matematyki/notatki-z-wyklado /rainer_kgca.pdf.