INFINITE-DIMENSIONAL RAMSEY THEORY

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Abstract. The purpose of this essay is to first give the proof of a generalisation of the original (finite-dimensional) Ramsey theorem, and then to introduce infinite-dimensional Ramsey theory. We discuss in great detail the proof that Borel sets for the Ellentuck topology are completely Ramsey, mainly expanding on and filling the gaps of the proofs and discussion given in [1]. We end with a short sketch of how infinite-dimensional Ramsey theory is used in the proof of Rosenthal's ℓ_1 theorem.

1. Finite-Dimensional Ramsey Theory

First, we introduce some notation. We will denote by $\mathcal{P}\mathbb{N}$ the collection of all the subsets of the natural numbers $\mathbb{N}$, which we can identify with the Cantor set $\Delta := \{0,1\}^{\mathbb{N}}$ (i.e. the set of all functions $f\colon \mathbb{N} \to \{0,1\}$) via the map sending $A \in \mathcal{P}\mathbb{N} \mapsto \chi_A \in \Delta$, where $\chi_A\colon \mathbb{N} \to \{0,1\}$ is given by $\chi_A(n) = 1$ if $n \in A$ and $\chi_A(n) = 0$ if $n \notin A$. Next, we will denote by $\mathcal{P}_\infty\mathbb{N}$ the subset of $\mathcal{P}\mathbb{N}$ consisting of all the infinite subsets of $\mathbb{N}$, and we will denote by $\mathcal{F}\mathbb{N}$ the subset of $\mathcal{P}\mathbb{N}$ consisting of all the finite subsets of $\mathbb{N}$, i.e. $\mathcal{F}\mathbb{N} = \mathcal{P}\mathbb{N} \setminus \mathcal{P}_\infty\mathbb{N}$. If we are given some $M \in \mathcal{P}\mathbb{N}$, then we denote by $\mathcal{F}_r(M)$ the collection of all the finite subsets of M with r-many elements.

Now, given an infinite set $M \in \mathcal{P}_\infty\mathbb{N}$ and a function $f\colon \mathcal{F}_r(\mathbb{N}) \to \mathbb{R}$, we say that

$$\lim_{A \in \mathcal{F}_r(M)} f(A) = \alpha$$

if for every $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that if $A \in \mathcal{F}_r(M)$ and $A \subset [N, \infty)$ (noting that M is infinite), then we have that $|f(A) - \alpha| < \epsilon$. Now, a generalisation of the original Ramsey Theorem is a corollary of the following theorem.

Theorem 1.1. *Given an $r \in \mathbb{N}$ and a bounded function $f\colon \mathcal{F}_r(\mathbb{N}) \to \mathbb{R}$, there is an infinite set $M \in \mathcal{P}_\infty\mathbb{N}$ such that the limit $\lim_{A \in \mathcal{F}_r(M)} f(A)$ exists.*

Proof. As is common in many proofs in Ramsey theory, we will proceed by induction (on r). Note that the theorem is of course true in the case $r = 1$, as then a bounded function $f\colon \mathcal{F}_r(\mathbb{N}) \to \mathbb{R}$ is just a bounded sequence in $\mathbb{R}$, so by the Bolzano-Weierstrass theorem (i.e. the compactness of closed bounded subsets of $\mathbb{R}$), we have the existence of a convergent subsequence, i.e. a set $M \in \mathcal{P}_\infty\mathbb{N}$ such that $\lim_{A \in \mathcal{F}_1(M)} f(A)$ exists.

Now, assuming that the result holds for $r-1$, we want to show that it holds for r (where $r \geq 2$). So, we are given a bounded function $f\colon \mathcal{F}_r(\mathbb{N}) \to \mathbb{R}$, and to use the induction hypothesis we want to obtain a bounded function $g\colon \mathcal{F}_{r-1}(\mathbb{N}) \to \mathbb{R}$ from f. To do this, note that for each fixed and distinct $m_1, \dots, m_{r-1}$, we have that the sequence of all $f(\{m_1, \dots, m_{r-1}, m_r\})$ (where m_r runs over $\mathbb{N} \setminus \{m_1, \dots, m_{r-1}\}$) is a bounded sequence in $\mathbb{R}$ (as f is bounded), so there is an infinite set $M_{m_1,\dots,m_{r-1}} \in \mathcal{P}_\infty\mathbb{N}$ (thought of as a subsequence) depending on $m_1, \dots, m_{r-1}$ such that $\lim_{m_r \in M_{m_1,\dots,m_{r-1}}} f(\{m_1, \dots, m_{r-1}, m_r\})$ exists, using the Bolzano-Weierstrass theorem. Then, by a standard diagonal argument, we can obtain a subset $M_1 \in \mathcal{P}_\infty\mathbb{N}$ such that $\lim_{m_r \in M_1} f(\{m_1, \dots, m_{r-1}, m_r\})$ exists <u>for all</u> distinct $m_1, \dots, m_{r-1} \in \mathbb{N}$. Now, we define $g\colon \mathcal{F}_{r-1}(\mathbb{N}) \to \mathbb{R}$ by for all $\{m_1, \dots, m_{r-1}\} \in \mathcal{F}_{r-1}(\mathbb{N})$, we have

$$g(\{m_1, \dots, m_{r-1}\}) = \lim_{m_r \in M_1} f(\{m_1, \dots, m_{r-1}, m_r\}).$$

Note that g is of course bounded since f is bounded. We are now in a position to apply the induction hypothesis, but we will apply the induction hypothesis with respect to M_1 instead of $\mathbb{N}$ (which we can do as M_1 is in bijection with $\mathbb{N}$). Thus, we pick some infinite subset M_2 of M_1 such that $\lim_{A \in \mathcal{F}_{r-1}(M_2)} g(A)$ exists, and we call this limit α. However, since g was defined as a limit, we also have that for each finite set $A \in \mathcal{F}_{r-1}(M_2)$ and $\delta > 0$ there is a natural number $N(A, \delta)$ depending on A and δ such that for all $n \geq N(A, \delta)$ with $n \in M_2$ we have that $n \notin A$ (noting that this is possible since A is finite), and

$$|f(A \cup \{n\}) - g(A)| < \delta.$$

Note here that we're using both the definition of g and that M_2 is a subsequence (or more precisely a subset) of M_1. Now, we define our set M (which will be an infinite subset of M_2) in the statement of the theorem inductively. That is, after first choosing arbitrary natural numbers $m_1, \ldots, m_{r-1} \in M_2$ with $m_1 < m_2 < \cdots < m_{r-1}$, for each $n \geq r$ we choose $m_n \in M_2$ such that

$$m_n > \max_{A' \in \mathcal{F}_{r-1}(\{m_1, \ldots, m_{n-1}\})} N(A', 2^{-(n-1)}).$$

We then define $M = \{m_j\}_{j=1}^{\infty} \in \mathcal{P}_\infty \mathbb{N}$ (which is also a subset of M_2). Now, we claim that $\lim_{A \in \mathcal{F}_r(M)} f(A)$ exists and is equal to α. Indeed, let $\epsilon > 0$ be arbitrary. Since $\lim_{A \in \mathcal{F}_{r-1}(M_2)} g(A) = \alpha$, we can choose an $n_1 \in \mathbb{N}$ such that whenever $A \in \mathcal{F}_{r-1}(M)$ with $A \subset [m_{n_1}, \infty)$, then

$$|g(A) - \alpha| < \frac{\epsilon}{2},$$

noting that this is possible since M is an infinite subset (or subsequence) of M_2. Now, choose an $n \in \mathbb{N}$ with $n \geq n_1$ and

$$2^{-n} < \frac{\epsilon}{2}.$$

Now, to show that $\lim_{A \in \mathcal{F}_r(M)} f(A) = \alpha$, let $A \in \mathcal{F}_r(M)$ be such that $A \subset [m_n, \infty)$. Then, if the largest natural number in A is m_k say (noting that A is finite), let $B = A \setminus \{m_k\}$. Then, since $m_k > \max_{A' \in \mathcal{F}_{r-1}(\{m_1, \ldots, m_{k-1}\})} N(A', 2^{-(k-1)})$, we have that

$$|f(A) - g(B)| = |f(B \cup \{m_k\}) - g(B)| < 2^{-(k-1)}.$$

However, as $A \subset [m_n, \infty)$ so that $n \leq k-1$, we have that $2^{-(k-1)} \leq 2^{-n} < \epsilon/2$. But since $B \subset [m_n, \infty)$ with $B \in \mathcal{F}_{r-1}(M)$, we have by above that $|g(B) - \alpha| < \epsilon/2$. Therefore, by the triangle inequality, it follows that

$$|f(A) - \alpha| \leq |f(A) - g(B)| + |g(B) - \alpha| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we therefore have the result. ■

Now, as almost an immediate corollary, we have the following generalisation of Ramsey's theorem (noting that the original Ramsey's theorem corresponds to the $r = 2$ case).

Corollary 1.2. *If $\mathcal{A} \subset \mathcal{F}_r(\mathbb{N})$, then there is an $M \in \mathcal{P}_\infty \mathbb{N}$ such that either $\mathcal{F}_r(M) \subset \mathcal{A}$ or $\mathcal{F}_r(M) \cap \mathcal{A} = \emptyset$.*

Proof. Consider the bounded function $\chi_{\mathcal{A}} \colon \mathcal{F}_r(\mathbb{N}) \to \mathbb{R}$ defined by $\chi_{\mathcal{A}}(A) = 1$ if $A \in \mathcal{A}$ and $\chi_{\mathcal{A}}(A) = 0$ otherwise. By Theorem 1.1, there is an $M \in \mathcal{P}_\infty \mathbb{N}$ such that $\lim_{A \in \mathcal{F}_r(M)} \chi_{\mathcal{A}}(A)$ exists, and call it α (noting that α is either 0 or 1). That is, taking $\epsilon = 1/2$, there is an $N \in \mathbb{N}$ such that if $A \in \mathcal{F}_r(M)$ and $A \subset [N, \infty)$, then

$$|\chi_{\mathcal{A}}(A) - \alpha| < \frac{1}{2},$$

so $\chi_{\mathcal{A}}(A) = \alpha$ (where again α is either 0 or 1). So, there exists an $N \in \mathbb{N}$ such that if $A \in \mathcal{F}_r(M)$ and $A \subset [N, \infty)$, then $A \in \mathcal{A}$ (or $A \notin \mathcal{A}$ depending on whether α is 0 or 1). So, taking $\tilde{M} = M \cap [N, \infty)$, we have that either $\mathcal{F}_r(\tilde{M}) \subset \mathcal{A}$ or $\mathcal{F}_r(\tilde{M}) \cap \mathcal{A} = \emptyset$. ■

2. Infinite-Dimensional Ramsey Theory

Now, we will introduce the so-called infinite-dimensional Ramsey theory. The motivating question in this section is for what infinite subsets $\mathcal{A} \subset \mathcal{P}_\infty \mathbb{N}$ does Corollary 1.2 remain true? We first discuss two different topologies we can put on $\mathcal{P}_\infty \mathbb{N}$. Recalling from our earlier discussion that $\mathcal{P}\mathbb{N}$ can be identified with the Cantor set Δ, we have the *Cantor topology* on $\mathcal{P}_\infty \mathbb{N}$, which is the topology inherited from the metric topology on Δ given by for any two sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$, the distance between them is given by

$$d((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = 2^{-k}$$

where k is the smallest natural number such that $x_k \neq y_k$. If the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are equal, then we define the distance between them to be zero. Observe that $\mathcal{P}_\infty\mathbb{N}$ is a closed subset of $\mathcal{P}\mathbb{N}$ in this metric, and therefore a G_δ-set i.e. a countable intersection of open sets (as in metrisable spaces every closed set is a G_δ-set). Indeed, we have that a limit of sequences in $\{0,1\}^\mathbb{N}$ (in this metric) with infinitely many 1s must of course also have infinitely many 1s. So, since we know that the Cantor set Δ is compact, we have in particular that the Cantor topology on $\mathcal{P}_\infty\mathbb{N}$ is given by a complete metric (as a closed subset of a complete space is complete).

Now, we can put another topology on $\mathcal{P}_\infty\mathbb{N}$ called the *Ellentuck topology* defined as follows, which we'll use extensively in this essay. First, some notation. If $A \in \mathcal{F}\mathbb{N}$ is a finite subset of $\mathbb{N}$ and $E \in \mathcal{P}_\infty\mathbb{N}$ is an infinite subset of $\mathbb{N}$, we denote by $\mathcal{P}_\infty(A, E)$ the collection of all infinite subsets of $A \cup E$ that contain A. If $A = \emptyset$, we simply write $\mathcal{P}_\infty(E)$ for $\mathcal{P}_\infty(\emptyset, E)$. A useful fact that we'll use many times is that $\mathcal{P}_\infty(A, E) = \mathcal{P}_\infty(A, A \cup E)$ (which we'll call our 'favourite trick'). Now, we say that $\mathcal{U} \subset \mathcal{P}_\infty\mathbb{N}$ is *Ellentuck-open* if for all $E \in \mathcal{U}$, there is a finite subset $A \subset E$ such that $\mathcal{P}_\infty(A, E) \subset \mathcal{U}$. To get a feel for this somewhat confusing notation, we will show that this definition of an Ellentuck-open set indeed defines a topology on $\mathcal{P}_\infty\mathbb{N}$. To see this, note that the empty set $\emptyset$ is of course open (it is true vacuously), and that $\mathcal{P}_\infty\mathbb{N}$ is also open as if $E \in \mathcal{P}_\infty\mathbb{N}$ is arbitrary, then in particular $\mathcal{P}_\infty(\emptyset, E) = \mathcal{P}_\infty(E) \subset \mathcal{P}_\infty\mathbb{N}$. Next, suppose that $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is a sequence of Ellentuck-open subsets of $\mathcal{P}_\infty\mathbb{N}$. Then, let $\mathcal{U} = \bigcup_{i=1}^\infty \mathcal{U}_i$ and let $E \in \mathcal{U}$ be arbitrary. Then, we have in particular that $E \in \mathcal{U}_i$ for some i. But $\mathcal{U}_i$ is open, so there exists a finite set $A_i \subset E$ such that $\mathcal{P}_\infty(A_i, E) \subset \mathcal{U}_i$. But then $\mathcal{P}_\infty(A_i, E) \subset \mathcal{U}$ (as $\mathcal{U}_i \subset \mathcal{U}$), so $\mathcal{U}$ is open. Next, let $n \in \mathbb{N}$ be arbitrary, let $\mathcal{V} = \bigcap_{i=1}^n \mathcal{U}_i$ and let $E \in \mathcal{V}$ be arbitrary. Then, for each $1 \leq i \leq n$, there is a finite set $A_i \subset E$ so that $\mathcal{P}_\infty(A_i, E) \subset \mathcal{U}_i$. Letting $A = \bigcup_{i=1}^n A_i$, we have that A is a finite subset of E (as each A_i is a finite subset of E). Furthermore, for each $1 \leq i \leq n$, we have that $\mathcal{P}_\infty(A, E) \subset \mathcal{P}_\infty(A_i, E) \subset \mathcal{U}_i$, noting that any infinite subset of $A \cup E = E$ containing A is an infinite subset of $E = A_i \cup E$ containing A_i (as $A_i \subset A$). Thus, we have that $\mathcal{P}_\infty(A, E) \subset \bigcap_{i=1}^n \mathcal{U}_i = \mathcal{V}$, so $\mathcal{V}$ is open. Therefore, we get a topology on $\mathcal{P}_\infty\mathbb{N}$.

Observe that the Ellentuck topology on $\mathcal{P}_\infty\mathbb{N}$ is stronger than the Cantor topology on $\mathcal{P}_\infty\mathbb{N}$. Indeed, suppose that $\mathcal{U} \subset \mathcal{P}_\infty\mathbb{N}$ is Cantor-open. To show that $\mathcal{U}$ is Ellentuck-open, let $E \in \mathcal{U}$ be arbitrary. Since $\mathcal{U}$ is Cantor-open, there is a $k \in \mathbb{N}$ such that every sequence in $\{0,1\}^\mathbb{N}$ with infinitely many 1s that agree with the sequence corresponding to E up to the k^{th} component are also in $\mathcal{U}$. Letting $A \subset E$ be the finite set whose sequence agrees with the sequence corresponding to E up to component k but all further entries/components are 0, we therefore have in particular that every infinite subset of E containing A is in $\mathcal{U}$ (as they agree with A and E up to component k), i.e. that $\mathcal{P}_\infty(A, E) \subset \mathcal{U}$. Thus, $\mathcal{U}$ is Ellentuck-open.

Now, noting the parallel to the statement of Corollary 1.2, we say that $\mathcal{V} \subset \mathcal{P}_\infty\mathbb{N}$ is *Ramsey* if either there is some $M \in \mathcal{P}_\infty\mathbb{N}$ such that $\mathcal{P}_\infty(M) \subset \mathcal{V}$ or there is some $M \in \mathcal{P}_\infty\mathbb{N}$ such that $\mathcal{P}_\infty(M) \cap \mathcal{V} = \emptyset$. However, we say that $\mathcal{V} \subset \mathcal{P}_\infty\mathbb{N}$ is *completely Ramsey* if for <u>any</u> finite $A \in \mathcal{F}\mathbb{N}$ and infinite $E \in \mathcal{P}_\infty\mathbb{N}$, either there is some $M \in \mathcal{P}_\infty(E)$ with $\mathcal{P}_\infty(A, M) \subset \mathcal{V}$ or there is <u>some</u> $M \in \mathcal{P}_\infty(E)$ with $\mathcal{P}_\infty(A, M) \cap \mathcal{V} = \emptyset$. Observe that of course any completely Ramsey set is automatically Ramsey. Indeed, if $\mathcal{V} \subset \mathcal{P}_\infty\mathbb{N}$ is completely Ramsey, then taking $A = \emptyset$ and $E = \mathbb{N}$, we have that there exists $M \in \mathcal{P}_\infty\mathbb{N}$ such that $\mathcal{P}_\infty(M) \subset \mathcal{V}$ or there exists $M \in \mathcal{P}_\infty\mathbb{N}$ such that $\mathcal{P}_\infty(M) \cap \mathcal{V} = \emptyset$, which precisely means that $\mathcal{V}$ is Ramsey.

Theorem 2.1. *Every Ellentuck-open subset of $\mathcal{P}_\infty\mathbb{N}$ is completely Ramsey.*

Proof. Let $\mathcal{U} \subset \mathcal{P}_\infty\mathbb{N}$ be an arbitrary Ellentuck-open set. To make the proof slightly easier to comprehend, we will use the following notation. If $A \in \mathcal{F}\mathbb{N}$ is a finite set and $E \in \mathcal{P}_\infty\mathbb{N}$ is an infinite set, we will call (A, E) a *pair*. Given our fixed $\mathcal{U}$ at the start, we then say that a pair (A, E) is *good* if there is some infinite subset $M \in \mathcal{P}_\infty(E)$ such that $\mathcal{P}_\infty(A, M) \subset \mathcal{U}$. Notice that if every pair (A, E) is good for $\mathcal{U}$, then $\mathcal{U}$ is automatically completely Ramsey by definition. If a pair (A, E) is not good, then we say that the pair (A, E) is *bad*. We of course don't expect every pair (A, E) to necessarily be good, so we will show that if (A, E) is bad, then we get the other condition of a completely Ramsey set, namely we'll show that there is some $M \in \mathcal{P}_\infty(E)$ such that $\mathcal{P}_\infty(A, M) \cap \mathcal{U} = \emptyset$. Before we start the proof, we first make two observations about good and bad pairs. First, observe that if the pair (A, E) is bad and $F \in \mathcal{P}_\infty(E)$ is an infinite subset of E, then (A, F) is also bad. Indeed, if (A, E) is bad, then for all infinite subsets M of E, we have that $\mathcal{P}_\infty(A, M) \not\subset \mathcal{U}$. So, if $F \in \mathcal{P}_\infty(E)$, then for all $M \in \mathcal{P}_\infty(F)$, we also have that $\mathcal{P}_\infty(A, M) \not\subset \mathcal{U}$ as any infinite subset of F is also an infinite subset of E (where $M \subset F \subset E$). Our next observation is that if $E, F \in \mathcal{P}_\infty\mathbb{N}$ have finite symmetric difference $E \triangle F = (E \setminus F) \cup (F \setminus E)$, then we have that (A, E) is good if and only if (A, F)

is good. Indeed, suppose that (A, E) is good. Then by definition there is an infinite subset $M \in \mathcal{P}_\infty(E)$ such that $\mathcal{P}_\infty(A, M) \subset \mathcal{U}$. But since $E \triangle F = (E \setminus F) \cup (F \setminus E)$ is finite, M must have an infinite subset $N \in \mathcal{P}_\infty(M)$ such that $N \in \mathcal{P}_\infty(F)$. Then, we have that $\mathcal{P}_\infty(A, N) \subset \mathcal{P}_\infty(A, M) \subset \mathcal{U}$, as infinite subsets of $A \cup N$ containing A are automatically infinite subsets of $A \cup M$ containing A. Thus, we have by definition that (A, F) is good. A symmetric argument shows that if (A, F) is good, then (A, E) is good.

For the first step of the proof, suppose that we have a collection $(A_j)_{j=1}^m$ of finite subsets of $\mathbb{N}$ and an infinite set $E \in \mathcal{P}_\infty \mathbb{N}$ such that (A_j, E) is bad for every $1 \leq j \leq m$. We will show that there is some $n \in E \setminus \bigcup_{j=1}^m A_j$ and some $F \in \mathcal{P}_\infty(E)$ such that $(A_j \cup \{n\}, F)$ is bad for all $1 \leq j \leq m$. Indeed, suppose not. Then, for <u>every</u> $n \in E \setminus \bigcup_{j=1}^m A_j$ and <u>every</u> $F \in \mathcal{P}_\infty(E)$, we have that the pair $(A_j \cup \{n\}, F)$ is good for <u>some</u> $1 \leq j \leq m$. So, since E is infinite and $\bigcup_{j=1}^m A_j$ is finite, we can choose an increasing sequence of natural numbers $(n_k)_{k=1}^\infty$ in E and a decreasing set (i.e. $E_k \in \mathcal{P}_\infty(E_{k-1})$) of infinite subsets $(E_k)_{k=0}^\infty$ (where $E_0 := E$), and a sequence $(p(k))_{k=1}^\infty$ of natural numbers in $\{1, 2, \dots, m\}$ such that $n_k \in E_{k-1} \setminus \bigcup_{j=1}^m A_j$ for all $k \geq 1$ and $\mathcal{P}_\infty(A_{p(k)} \cup \{n_k\}, E_k) \subset \mathcal{U}$ by inductively using the definition of a good pair looking at each of the pairs $(\bigcup_{j=1}^m A_j, E_k)$ successively. Since $(p(k))_{k=1}^\infty$ is an infinite sequence in the finite set $\{1, \dots, m\}$, there must of course exist some $p \in \{1, 2, \dots, m\}$ such that $p(k) = p$ infinitely often. Now, we let $M = \{n_k : p(k) = p\}$, and suppose that $G \in \mathcal{P}_\infty(A_p, M)$ is arbitrary (we will show that (A_p, M) is good). Then, letting $\tilde{k}$ be the smallest natural number such that $n_{\tilde{k}} \in G$ (where $n_{\tilde{k}} \in M$ so $p(\tilde{k}) = p$), we have that $G \in \mathcal{P}_\infty(A_{p(\tilde{k})} \cup \{n_{\tilde{k}}\}, E_{\tilde{k}}) \subset \mathcal{U}$. Indeed, by definition G is an infinite subset of $M \cup A_p = \{n_k : p(k) = p\} \cup A_p$, so by construction in that $(E_k)_{k=0}^\infty$ is a decreasing sequence of infinite sets with $n_k \in E_{k-1} \setminus \bigcup_{j=1}^m A_j$, it follows that each $n_\ell \in G$ with $\ell > \tilde{k}$ is in $E_{\tilde{k}}$. That is, G is an infinite subset of $E_{\tilde{k}} \cup A_{p(\tilde{k})} \cup \{n_{\tilde{k}}\}$ containing $A_{p(\tilde{k})} \cup \{n_{\tilde{k}}\}$ i.e. $G \in \mathcal{P}_\infty(A_{p(\tilde{k})} \cup \{n_{\tilde{k}}\}, E_{\tilde{k}}) \subset \mathcal{U}$. But $G \in \mathcal{P}_\infty(A_p, M)$ was arbitrary, so we may now conclude that $\mathcal{P}_\infty(A_p, M) \subset \mathcal{U}$, i.e. by definition that the pair (A_p, M) is good. However, since $M \in \mathcal{P}_\infty(E)$ (as $M \in \mathcal{P}_\infty(E_{k-1})$ where n_k is the smallest element of M), we have by our earlier observation at the start that (A_p, E) must also be good, which contradicts our initial assumption, thereby concluding the first step of the proof.

For the second step of the proof, we will show that in the case that (A, E) is bad, then there exists some infinite subset $M \in \mathcal{P}_\infty(E)$ such that if B is a finite set where $A \subset B \subset A \cup M$, then (B, M) is bad. Indeed, as usual, we proceed inductively using the first step of the proof. First, set $E_0 = E$. Then, we set E_1 to be the F in the statement of the first step of the proof which also gives an $n_1 \in E_0 \setminus A$ such that $(A \cup \{n_1\}, E_1)$ is bad. But we also know that (A, E_1) is bad since (A, E) is assumed to be bad and $E_1 \in \mathcal{P}_\infty(E)$. Therefore, we have that (B, E_1) is bad whenever $A \subset B \subset A \cup \{n_1\}$ (noting that B can only be A or $A \cup \{n_1\}$). Now, we inductively continue this process, whereby if we have already constructed sets $E_0 \supset E_1 \supset \cdots \supset E_k$ and natural numbers $n_1, \dots, n_k$ where $n_j \in E_{j-1}$ for all $1 \leq j \leq k$, such that (B, E_j) is bad whenever $A \subset B \subset A \cup \{n_1, \dots, n_j\}$ for all $1 \leq j \leq k$, then we construct an $E_{k+1} \subset E_k$ with $n_{k+1} > n_k$ and $n_{k+1} \in E_k$ as follows, so that $(B \cup \{n_{k+1}\}, E_{k+1})$ is bad for every $A \subset B \subset A \cup \{n_1, \dots, n_k\}$. Since (B, E_k) is bad for every $A \subset B \subset A \cup \{n_1, \dots, n_k\}$, we have by the first step of the proof that there is some

$$n_{k+1} \in E_k \setminus \bigcup \{\text{all (necessarily finite) } B \text{ with } A \subset B \subset A \cup \{n_1, \dots, n_k\}\}$$

and $E_{k+1} \in \mathcal{P}_\infty(E_k)$ such that $(B \cup \{n_{k+1}\}, E_{k+1})$ is bad whenever $A \subset B \subset A \cup \{n_1, \dots, n_k\}$. Now, let $M = \{n_1, n_2, \dots\}$. Noting that $M \in \mathcal{P}_\infty(E)$ by construction, we will show that M is the set we want, i.e. that it satisfies the property that (B, M) is bad for every finite set B with $A \subset B \subset A \cup M$. To see this, let B be a finite subset of $A \cup M$ containing A. If $B = A$, then we're done as (A, E) being bad implies that (A, M) is bad as $M \in \mathcal{P}_\infty(E)$. Noting that B contains only finitely many elements from M, let k be the largest natural number with $n_k \in B$. Then, we have that $A \subset B \subset A \cup \{n_1, \dots, n_k\}$ so that (B, E_k) is bad, which follows by our earlier construction that for each natural number ℓ, we have that (B, E_ℓ) is bad if $A \subset B \subset A \cup \{n_1, \dots, n_\ell\}$. Now, notice that $M \subset E_k \cup \{n_1, \dots, n_k\}$. Indeed, recall that $M = \{n_1, n_2, \dots\}$ where n_1 is in E_0 which contains E_1 as a subset, which contains n_2 as an element. But $E_1 \supset E_2 \ni n_3$, and so on, until we get to E_k which contains n_{k+1}, so we must have that $M \subset E_k \cup \{n_1, \dots, n_k\}$. Finally, since (B, E_k) is bad, we get that $(B, \{n_{k+1}, n_{k+2}, \dots\})$ is bad (as $\{n_{k+1}, n_{k+2}, \dots\} \in \mathcal{P}_\infty(E_k)$), which then implies that (B, M) is bad as $M = \{n_1, n_2, \dots\}$ and the <u>symmetric difference</u> $M \triangle \{n_{k+1}, n_{k+2}, \dots\} = \{n_1, \dots, n_k\}$ is of course finite, using our observation from the start. So, since $A \subset B \subset A \cup M$ was arbitrary, the second step of the proof is now complete.

In the last step of the proof, we will finally use our assumption that $\mathcal{U}$ is an Ellentuck-open subset of $\mathcal{P}_\infty\mathbb{N}$. Recall that we want to show that if (A,E) is bad, then there is some $M \in \mathcal{P}_\infty(E)$ such that $\mathcal{P}_\infty(A,M)\cap\mathcal{U} = \emptyset$. Now, if (A,E) is bad, using the second step of the proof we can choose an $M \in \mathcal{P}_\infty(E)$ such that (B,M) is bad for all finite sets B with $A \subset B \subset A \cup M$, i.e. for all finite subsets of $A \cup M$ containing A. Now, suppose that $\mathcal{P}_\infty(A,M)\cap\mathcal{U} \neq \emptyset$. Then, choose some $G \in \mathcal{P}_\infty(A,M)\cap\mathcal{U}$. But $\mathcal{U}$ is Ellentuck-open, so G being an element of $\mathcal{U}$ means that there is some finite set $B \subset G$ such that $\mathcal{P}_\infty(B,G) \subset \mathcal{U}$. In particular, we may assume without loss of generality that B contains A. Indeed, $\mathcal{P}_\infty(B,G) \subset \mathcal{U}$ means that all infinite subsets of G containing B are in $\mathcal{U}$, but we also have that G contains A since $G \in \mathcal{P}_\infty(A,M)$. So, we have in particular that all infinite subsets of G containing $A \cup B$ are in $\mathcal{U}$. Thus, we have that $\mathcal{P}_\infty(B,G) \subset \mathcal{U}$ implies that $\mathcal{P}_\infty(A\cup B,G) \subset \mathcal{U}$ i.e. that $\mathcal{P}_\infty(D,G) \subset \mathcal{U}$, where the finite subset $D \subset G$ is defined by $D = A \cup B$. Thus, by replacing B with D if necessary, we may assume that B contains A. Now, let $G' = G \cap M$. Then, we have that $G' \in \mathcal{P}_\infty(M)$ (recalling that G was in $\mathcal{P}_\infty(A,M)$), and that $\mathcal{P}_\infty(B,G') = \mathcal{P}_\infty(B,G'\cup A) = \mathcal{P}_\infty(B,G) \subset \mathcal{U}$, where in the first equality we've used our favourite trick with this notation that we mentioned before. So, it follows that the pair (B,M) is good with respect to the infinite set $G' \in \mathcal{P}_\infty(M)$ (not with respect to G!), but this is a contradiction since we also know that (B,M) is bad (indeed, B is a finite set satisfying $A \subset B \subset A \cup M$). Hence, our initial assumption that $\mathcal{P}_\infty(A,M)\cap\mathcal{U} \neq \emptyset$ must be false, and we may now deduce that indeed $\mathcal{P}_\infty(A,M)\cap\mathcal{U} = \emptyset$. Thus if (A,E) is bad, then $\mathcal{U}$ is completely Ramsey. ■

Now, we come to the main theorem that we'll prove in this essay, but first a lemma (which we will not prove).

Lemma 2.2. *Suppose that X is a topological space and that $\mathcal{V}$ is a Borel set in X, i.e. an element of the σ-algebra generated by the open sets in X. Then, there is an open set $\mathcal{U}$ and a dense G_δ-set $\mathcal{G}$ such that*

$$\mathcal{V}\cap\mathcal{G} = \mathcal{U}\cap\mathcal{G}.$$

Given Lemma 2.2 and Theorem 2.1, we can now prove our main result, giving a partial answer to our motivating question.

Theorem 2.3. *Every Borel set for the Ellentuck topology on $\mathcal{P}_\infty\mathbb{N}$ is completely Ramsey.*

Proof. Again, the proof can be broken into three steps, and similarly to the proof of Theorem 2.1 we will not deal with arbitrary Borel sets until the third and final step of the proof. First, we let $\mathcal{U}$ be an open and dense set for the Ellentuck topology. By Theorem 2.1, we know that $\mathcal{U}$ is completely Ramsey, but since $\mathcal{U}$ is dense we in fact know which of the two conditions in the definition of a completely Ramsey set that $\mathcal{U}$ must satisfy. We know that for all pairs (A,E), there is an infinite set $M \in \mathcal{P}_\infty(E)$ such that $\mathcal{P}_\infty(A,M) \subset \mathcal{U}$. The reason for this is that there is no $M \in \mathcal{P}_\infty(E)$ satisfying $\mathcal{P}_\infty(A,M)\cap\mathcal{U} = \emptyset$. Indeed, since $\mathcal{U}$ is dense for the Ellentuck topology, we have that $\mathcal{U}$ has non-empty intersection with every non-empty open set in $\mathcal{P}_\infty\mathbb{N}$. Now, for every $M \in \mathcal{P}_\infty(E)$, we have that $\mathcal{P}_\infty(A,M)$ is open. Indeed, let $N \in \mathcal{P}_\infty(A,M)$ be arbitrary. Then, N is an infinite subset of $A \cup M$ containing A. Moreover, any infinite subset of $A \cup N$ containing A is an infinite subset of $A \cup M$ containing A (as $A \subset N' \subset N \cup A$ implies that $A \subset N' \subset M \cup A$ since $N \subset M \cup A$ implies that $N \cup A \subset M \cup A$). Thus, we have that $\mathcal{P}_\infty(A,N) \subset \mathcal{P}_\infty(A,M)$, so $\mathcal{P}_\infty(A,M)$ is open, meaning that $\mathcal{P}_\infty(A,M)\cap\mathcal{U} \neq \emptyset$.

Now, the first step of the proof will be showing that if $\mathcal{U}$ is open and dense for the Ellentuck topology, then in fact for every pair (A,E) and finite set $B \subset E$, there is an $M \in \mathcal{P}_\infty(B,E)$ such that $\mathcal{P}_\infty(A,M) \subset \mathcal{U}$, noting that this is stronger than what we remarked above. To do this, we first write out all the subsets of the finite set B as a sequence $(B_j)_{j=1}^N$ where $N = 2^{|B|}$. Noting that $\mathcal{U}$ is open and dense, we have by our earlier remark that for the pair $(A\cup B_1,E)$, there is some $H_1 \in \mathcal{P}_\infty(E)$ with $\mathcal{P}_\infty(A\cup B_1,H_1) \subset \mathcal{U}$. Now, for the pair $(A\cup B_2,H_1)$, there is some $H_2 \in \mathcal{P}_\infty(H_1)$ with $\mathcal{P}_\infty(A\cup B_2,H_2) \subset \mathcal{U}$. Then, by inductively repeating this process, for all $1 \le j \le N$ (where $H_0 := E$) we have an $H_j \in \mathcal{P}_\infty(H_{j-1})$ such that $\mathcal{P}_\infty(A\cup B_j,H_j) \subset \mathcal{U}$. Then, we let $M \in \mathcal{P}_\infty(E)$ be defined by $M = H_N$. First, we will show that $\mathcal{P}_\infty(A,M) \subset \mathcal{U}$. To do this, let $G \in \mathcal{P}_\infty(A,M)$ be arbitrary. Now, let $1 \le j \le N$ be such that $B_j = G \cap B$, noting that $G \cap B$ is some subset of B (which could be empty!). Now, G is an infinite subset of $M \cup A$ containing A, so it must in particular be an infinite subset of $M\cup A\cup B_j$ containing $A \cup B_j$ (as $B_j = G\cap B \subset G$). So, we have that $G \in \mathcal{P}_\infty(A\cup B_j,M)$. Now, $M = H_N$ is an infinite subset of H_j by construction, so $\mathcal{P}_\infty(A\cup B_j,M) \subset \mathcal{P}_\infty(A\cup B_j,H_j)$. But $\mathcal{P}_\infty(A\cup B_j,H_j) \subset \mathcal{U}$ by above, so it follows that $G \in \mathcal{U}$, and thus $\mathcal{P}_\infty(A,M) \subset \mathcal{U}$. However, we do not necessarily have that $M \in \mathcal{P}_\infty(B, E)$ i.e. that M contains B (which was the whole point of this step of the proof!). Note however that for every $1 \leq j \leq N$, we have that $\mathcal{P}_\infty(A \cup B_j, M) \subset \mathcal{P}_\infty(A \cup B_j, H_j) \subset \mathcal{U}$ by the above reasoning, and so using our favourite trick we have in particular that $\mathcal{P}_\infty(A \cup B_j, M \cup B_j) \subset \mathcal{U}$ for every $1 \leq j \leq N$. Now, let $M' = M \cup B$, so that indeed $M' \in \mathcal{P}_\infty(B, E)$. Then, we see that $\mathcal{P}_\infty(A, M') \subset \mathcal{U}$. Indeed, let N' be an infinite subset of $M' \cup A$ containing A. Then, since $M' = M \cup B$, we have that N' is an infinite subset of $M \cup B_j \cup A$ containing $A \cup B_j$ for some $1 \leq j \leq N$, i.e. some subset of B, so $N' \in \mathcal{P}_\infty(A \cup B_j, M \cup B_j)$. But we showed above that $\mathcal{P}_\infty(A \cup B_j, M \cup B_j) \subset \mathcal{U}$ for every $1 \leq j \leq N$, and so $N' \in \mathcal{U}$. Hence, since $N' \in \mathcal{P}_\infty(A, M')$ was arbitrary, it follows that $\mathcal{P}_\infty(A, M') \subset \mathcal{U}$, and the first step of the proof is now complete.

In the second step of the proof (looking at the statement of Lemma 2.2), we will show the result for dense G_δ-sets in the Ellentuck topology. That is, letting $\mathcal{G} = \bigcap_{n=1}^\infty \mathcal{U}_n$ where each $\mathcal{U}_n$ is a dense open set in the Ellentuck topology, we will show that $\mathcal{G}$ is completely Ramsey by showing that it in fact always satisfies the first condition in the definition of a completely Ramsey set. That is, we will show that if (A, E) is a pair, then there is some $M \in \mathcal{P}_\infty(E)$ such that $\mathcal{P}_\infty(A, M) \subset \mathcal{U}$. This will just be a standard induction-type argument that we've seen many times before. First, we let $E_0 = E$ and let $n_1 \in E_0$ be arbitrary. Then, using the first step of the proof, since $\mathcal{U}_1$ is an open and dense set, we can choose an $E_1 \in \mathcal{P}_\infty(E_0)$ so that E_1 contains n_1 (i.e. $E_1 \in \mathcal{P}_\infty(\{n_1\}, E_0)$) and $\mathcal{P}_\infty(A, E_1) \subset \mathcal{U}_1$. Then, inductively, for each $k \geq 2$ we first let $n_k \in E_{k-1}$ be such that $n_k > n_{k-1}$ (noting that E_{k-1} is infinite). But then again by the first step of the proof, we can choose an $E_k \in \mathcal{P}_\infty(E_{k-1})$ such that E_k contains the finite set $\{n_1, \ldots, n_k\} \subset E_{k-1}$ (i.e. $E_k \in \mathcal{P}_\infty(\{n_1, \ldots, n_k\}, E_{k-1})$) and $\mathcal{P}_\infty(A, E_k) \subset \mathcal{U}_k$ (using that $\mathcal{U}_k$ is open and dense). Now, let $M = \{n_1, n_2, n_3, \ldots\} \in \mathcal{P}_\infty(E)$. We want to show that $\mathcal{P}_\infty(A, M) \subset \mathcal{G}$, so let $G \in \mathcal{P}_\infty(A, M)$ be arbitrary. Since M is an infinite subset of E_k for every k by construction, we have that $G \in \mathcal{P}_\infty(A, E_k)$ for every k. But this then implies that $G \in \mathcal{U}_k$ for every k as again $\mathcal{P}_\infty(A, E_k) \subset \mathcal{U}_k$ for every k by construction. Hence, it follows that $G \in \bigcap_{n=1}^\infty \mathcal{U}_n = \mathcal{G}$, and thus since $G \in \mathcal{P}_\infty(A, M)$ was arbitrary we have that $\mathcal{P}_\infty(A, M) \subset \mathcal{G}$, and the second step of the proof is now complete.

In the third and final step of the proof, we finally let $\mathcal{V} \subset \mathcal{P}_\infty\mathbb{N}$ be an arbitrary Borel set for the Ellentuck topology. Then, by Lemma 2.2, there is an open set $\mathcal{U}$ and a dense G_δ-set $\mathcal{G}$ such that $\mathcal{V} \cap \mathcal{G} = \mathcal{U} \cap \mathcal{G}$. We want to show that $\mathcal{V}$ is completely Ramsey, so let (A, E) be a pair. First, since $\mathcal{G}$ is a dense G_δ-set, we have by the second step of the proof that there is a $G \in \mathcal{P}_\infty(E)$ such that $\mathcal{P}_\infty(A, G) \subset \mathcal{G}$. Next, looking at the pair (A, G), since $\mathcal{U}$ is open we have by Theorem 2.1 that $\mathcal{U}$ is completely Ramsey, and so there is some $M \in \mathcal{P}_\infty(G)$ such that $\mathcal{P}_\infty(A, M) \subset \mathcal{U}$ or $\mathcal{P}_\infty(A, M) \cap \mathcal{U} = \emptyset$. However, if $\mathcal{P}_\infty(A, M) \subset \mathcal{U}$, then $\mathcal{P}_\infty(A, M) \subset \mathcal{V}$. Indeed, we also have $\mathcal{P}_\infty(A, M) \subset \mathcal{P}_\infty(A, G) \subset \mathcal{G}$ (as $M \in \mathcal{P}_\infty(G)$), but $\mathcal{G} \cap \mathcal{V} = \mathcal{G} \cap \mathcal{U}$ by Lemma 2.2. Also, if $\mathcal{P}_\infty(A, M) \cap \mathcal{U} = \emptyset$, then $\mathcal{P}_\infty(A, M) \cap \mathcal{V} = \emptyset$ as $\mathcal{P}_\infty(A, M) \subset \mathcal{G}$ and $\mathcal{G} \cap \mathcal{V} = \mathcal{G} \cap \mathcal{U}$. Thus, we have found an $M \in \mathcal{P}_\infty(E)$ (as $M \in \mathcal{P}_\infty(G)$ and $G \in \mathcal{P}_\infty(E)$) such that $\mathcal{P}_\infty(A, M) \subset \mathcal{V}$ or $\mathcal{P}_\infty(A, M) \cap \mathcal{V} = \emptyset$. That is, $\mathcal{V}$ is completely Ramsey. ∎

3. Applications in Banach Space Theory

In this final section of the essay, we will discuss an application of this infinite-dimensional Ramsey theory in Banach spaces. This application is Rosenthal's ℓ_1 theorem, which provides us with necessary and sufficient conditions for an arbitrary Banach space X to contain a closed subspace isomorphic to ℓ_1.

Theorem 3.1 (Rosenthal)**.** *Let X be a Banach space. Then, either X contains a closed subspace isomorphic to ℓ_1, or every bounded sequence in X has a weakly Cauchy subsequence.*

In particular, since the canonical basis of ℓ_1 (which is bounded) does not contain any weakly Cauchy subsequences, Theorem 3.1 tells us that X contains a closed subspace isomorphic to ℓ_1 if and only if there is some bounded sequence in X with no weakly Cauchy subsequences. Here, a sequence $(x_n)_{n=1}^\infty$ in X is *weakly Cauchy* if $(x^*(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for every $x^* \in X^*$.

We will now sketch the proof of Theorem 3.1, and in particular highlight how Ramsey theory comes into it.

Proof Sketch. We suppose that $(x_n)_{n=1}^\infty$ is a bounded sequence in X with no weakly Cauchy subsequences, and for all $M \in \mathcal{P}_\infty\mathbb{N}$ we define

$$\operatorname{osc}(M) := \sup_{\|x^*\| \leq 1} \lim_{k\to\infty} \sup_{\substack{m,n>k \\ m,n\in M}} |x^*(x_m) - x^*(x_n)|,$$

which can be thought of as a measure of how far a subsequence of $(x_n)_{n=1}^{\infty}$ is from being weakly Cauchy. Then, after some reductions, we are able to assume that the sequence $(x_n)_{n=1}^{\infty}$ is basic, and furthermore we are able to assume that $\mathrm{osc}(M) = 4\delta > 0$ is constant for all $M \in \mathcal{P}_\infty\mathbb{N}$, and that there is some functional $u^* \in B_{X^*}$ in the unit ball of X^* such that $\lim_{n\to\infty} u^*(x_n) = \theta$ where $|\theta| > \delta$. We also get a bound on the norms of all the x_n^* so that $\sup_{n\in\mathbb{N}}\|x_n^*\| < \infty$.

Now, the following is where our earlier work with Ramsey theory comes into play. We let $\mathcal{V} \subset \mathcal{P}_\infty\mathbb{N}$ be the subset representing all subsequences $(x_{m_j})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ (by the set $M = \{m_1, m_2, \dots\} \in \mathcal{P}_\infty\mathbb{N}$ with $m_1 < m_2 < \dots$) for which $x^*(x_{m_j}) = (-1)^j$ (for all $j \in \mathbb{N}$) for some $x^* \in X^*$ with $\|x^*\| \le 1 + \delta^{-1} + \delta^{-2}$. Using the Banach-Alaoglu theorem, we get that $\mathcal{V}$ is closed in the Cantor topology on $\mathcal{P}_\infty\mathbb{N}$, which therefore implies that it is closed in the Ellentuck topology (which follows since we showed earlier that the Ellentuck topology is stronger than the Cantor topology). Hence, by Theorem 2.3, we have that $\mathcal{V}$ is completely Ramsey, and thus in particular is Ramsey. But using our earlier bounds and reductions, we can show that any $M \in \mathcal{P}_\infty\mathbb{N}$ contains some $M' \in \mathcal{P}_\infty(M)$ such that $M' \in \mathcal{V}$. Thus, there is some $M \in \mathcal{P}_\infty\mathbb{N}$ such that $\mathcal{P}_\infty(M) \subset \mathcal{V}$ by the definition of a Ramsey set. Letting this M be the set $\{m_1, m_2, \dots\}$ with $m_1 < m_2 < \dots$, we see that the set $\{m_{2j}\}_{j=1}^{\infty} \in \mathcal{P}_\infty(M)$ (which is therefore in $\mathcal{V}$) satisfies the property that for every sequence $(\epsilon_j)_{j=1}^{\infty} \in \{-1, 1\}^{\mathbb{N}}$, there is an $x^* \in X^*$ with $\|x^*\| \le 1 + \delta^{-1} + \delta^{-2}$ such that $x^*(x_{m_{2j}}) = \epsilon_j$ for all $j \in \mathbb{N}$. This can then be used to show that the closed subspace $\overline{\mathrm{span}}\{x_{m_{2j}} : j \in \mathbb{N}\} \subset X$ is isomorphic to ℓ_1. ■

Aside from Rosenthal's ℓ_1 theorem, Ramsey theory (which is essentially a combinatorial subject) is used quite extensively in the theory of infinite-dimensional Banach spaces. See [5] for example, and see [6] for a list of some open problems in Banach space theory that have a Ramsey-theoretic flavour to them.

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