THE ATIYAH-SEGAL COMPLETION THEOREM

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Abstract. After first introducing some necessary background on equivariant homotopy theory and equivariant K-theory, we prove the Atiyah-Segal completion theorem (following Adams, Haeberly, Jackowski and May [1]) relating the ordinary complex K-theory $K(BG)$ of the classifying space of a finite or compact Lie group G to the complex representation ring $R(G)$. We then apply the completion theorem to calculate the K-theory of both $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$.

Contents

1. Equivariant Homotopy Theory

We begin by giving some background on equivariant homotopy theory that we will need for the proof of the completion theorem. We let G be a compact Lie group.

Definition 1.1. A *G-space* X is a topological space together with a (continuous) left G-action. A *G-equivariant map* or *G-map* from a G-space X to a G-space Y is a continuous map $f\colon X \to Y$ such that $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. That is, the map f commutes with the G-actions on X and Y.

As in the non-equivariant case, we can consider the notion of a *based G-space*, which is simply a G-space X together with a distinguished basepoint $x_0 \in X$ where we require G to act trivially on x_0. Naturally, we then define a *based G-map* $f\colon (X, x_0) \to (Y, y_0)$ between based G-spaces to be a G-map sending x_0 to y_0. We will from now on simply write a based G-space (X, x_0) as X i.e. we will drop the x_0 for ease of notation.

As usual, we can always obtain a based G-space X_+ from a non-based G-space X by simply defining X_+ to be the disjoint union of X with a point $*$ which we define to be fixed by the G-action.

Example 1.2. The orbit space G/H for H a subgroup of G is naturally a G-space (noting that we equip G with the discrete topology if G is finite) with G-action given by $g(g'H) = (gg')H$ for all $g, g' \in G$.

We can define the smash product of two (based) G-spaces as in the non-equivariant case, where if X and Y are two (based) G-spaces, then the G-action on $X \wedge Y$ is given by $g[x, y] = [gx, gy]$ for all $g \in G$ and $[x, y] \in X \wedge Y$.

If H is a subgroup of G, then the *H-fixed point set* or *H-fixed point space* of the G-space X is the set

$$X^H = \{x \in X : hx = x \text{ for all } h \in H\},$$

which is also a topological space, and taking G-fixed points gives us a functor from the category of G-spaces to the category of spaces. In particular, if $f\colon X \to Y$ is a G-map of G-spaces, then for every subgroup $H < G$ we have that f maps X^H into Y^H.

Now, if H is a subgroup of G and we are given an H-space Y, then we can construct the G-space $G \times_H Y$ defined by

$$G \times_H Y = G \times Y / \sim$$

where the equivalence relation $\sim$ is defined by $(gh, y) \sim (g, hy)$ for all $g \in G$, $y \in Y$ and $h \in H$, and the action of G on this space $G \times_H Y$ is defined by $g'[g, y] = [g'g, y]$ for all $g' \in G$ and $[g, y] \in G \times_H Y$. We can similarly construct the based G-space $G_+ \wedge_H Y$ from a based H-space Y.

Proposition 1.3. *If X is a G-space, then as G-spaces we have an isomorphism*

$$G \times_H X \xrightarrow{\cong} G/H \times X.$$

Proof. We have a G-map $G \times_H X \to G/H \times X$ defined by $[g, x] \mapsto ([g], gx)$ with inverse the G-map $G/H \times X \to G \times_H X$ defined by $([g], x) \mapsto [g, g^{-1}x]$. ■

We again have a similar result for based G-spaces, where we now look at $G/H_+ \wedge X$. Equivariant homotopy theory with respect to a compact Lie group G is based on the representation theory of G, and the notion of a G-representation will also be needed when discussing the completion theorem. We will primarily be looking at complex representations.

Definition 1.4. A *G-representation* V is a (complex) vector space with an action of G such that the induced maps $V \to V$ by the G-action sending $v \in V$ to gv are linear. The *dimension* of a G-representation V is simply the dimension of the underlying complex vector space.

We can equivalently view a G-representation as a group homomorphism $\rho\colon G \to GL(V)$, where here $GL(V)$ is the group of invertible endomorphisms of V. Using this definition of a G-representation, we define the direct sum $\rho \oplus \rho'\colon G \to GL(V \oplus V')$ of two G-representations $\rho\colon G \to GL(V)$ and $\rho'\colon G \to GL(V')$ to be given by $(\rho \oplus \rho')(g)(v, v') = (\rho(g)(v), \rho'(g)(v'))$ for all $g \in G$ and $(v, v') \in V \oplus V'$. The tensor product of two G-representations is defined analogously.

Definition 1.5. A G-representation V (with $V \neq 0$) is called *irreducible* if it contains no proper subrepresentations, and is called *completely reducible* if it can be decomposed as a direct sum of irreducible G-representations.

Note here that a subrepresentation of a G-representation V is a subspace W of V that is invariant under the action of G on V, and W is a *proper subrepresentation* if $W \neq 0$ and $W \neq V$.

Definition 1.6. An *isomorphism* of two G-representations $\rho\colon G \to GL(V)$ and $\rho'\colon G \to GL(V')$ is an invertible linear map $\psi\colon V \to V'$ satisfying the 'intertwining' property that

$$\rho'(g) \circ \psi = \psi \circ \rho(g)$$

for all $g \in G$.

We have the following key theorem that is useful in classifying isomorphism classes of G-representations for finite G.

Theorem 1.7 (Maschke)**.** *A G-representation V over a field k is always completely reducible as long as the characteristic of k doesn't divide $|G|$.*

Example 1.8. Let $G = C_2$ be the cyclic group of order two. Then, we have precisely two (complex) irreducible representations up to isomorphism, namely the trivial one-dimensional representation $\mathbf{1}$ and the one-dimensional sign representation σ. Here the representation $\mathbf{1}$ is given by C_2 acting trivially on $\mathbb{C}$, and the representation σ is given by the non-trivial element of C_2 sending any point $z \in \mathbb{C}$ to $-z$. By Theorem 1.7, it follows that any complex representation of C_2 is a direct sum of some number of copies of the trivial representation $\mathbf{1}$ and some number of copies of the sign representation σ.

Analogous to the construction of the ordinary sphere S^n as the one-point compactification of $\mathbb{R}^n$, we have the similar notion of the so-called *representation spheres*.

Definition 1.9. The *representation sphere* S^V corresponding to a G-representation V is the one-point compactification of V, i.e. is given by $V \cup \{\infty\}$ where G is defined to act trivially on the point-at-infinity.

We now extend our previous notion of a CW-complex to capture G-equivariance. Intuitively, a CW complex X which is also a G-space is a *G-CW complex* if the action of G restricts to an action on cells of the same dimension, i.e. it sends n-cells to n-cells. However we have the following more formal definition.

Definition 1.10. A *G-CW complex* is a G-space X which can be expressed as a union $X = \bigcup_{n\geq 0} X_n$ where the spaces X_n are defined inductively as follows. We require that X_0 is a disjoint union of orbit spaces G/H_α for various $H_\alpha < G$, and that the space X_n is obtained inductively from X_{n-1} by equivariant attaching maps $\varphi_\alpha \colon G/H_\alpha \times S^{n-1} \to X_{n-1}$ so that X_n is the pushout of the diagram

$$\begin{tikzcd}
\coprod_\alpha G/H_\alpha \times S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_\alpha G/H_\alpha \times D^n \arrow[r] & X_n
\end{tikzcd}$$

noting that the upper horizontal map is simply the disjoint union of the equivariant attaching maps. As usual, we say that the G-CW complex is *finite* if $\bigcup_{n\geq 0} X_n$ is a finite union. If we are working with based G-spaces, then our equivariant cells are instead of the form $G/H_+ \wedge S^n$.

We can also extend our usual notion of a homotopy between two maps of spaces to capture G-equivariance when our spaces are G-spaces.

Definition 1.11. A *G-homotopy* between two based G-maps $f, g \colon X \to Y$ is just an ordinary homotopy $F \colon X \wedge [0,1]_+ \to Y$, i.e. we have that $F([x,0]) = f(x)$ and $F([x,1]) = g(x)$ for all $x \in X$, but we furthermore require our homotopy F to be a G-map, where G acts diagonally on $X \wedge [0,1]_+$ with G acting trivially on the unit interval $[0,1]$.

Using this definition, we can then define the notions of null-G-homotopies, G-homotopy equivalence and G-contractibility of G-spaces as in the non-equivariant case. We also have the following equivariant analogue of the Whitehead theorem from ordinary homotopy theory.

Theorem 1.12 (Equivariant Whitehead theorem)**.** *Suppose that $f \colon X \to Y$ is a G-map of G-CW complexes. If the induced map $f^H \colon X^H \to Y^H$ on the fixed point spaces for every subgroup H of G is an ordinary homotopy equivalence, then f is a G-homotopy equivalence.*

2. Equivariant K-theory

Let G be a compact Lie group, which we fix throughout this section. When we talk about subgroups H of G, we consider our subgroups to be closed. We will mainly focus on complex K-theory, but most of what we say also works for real K-theory. The first concept we'll need is that of a G-vector bundle.

Definition 2.1. If X is a G-space, then we say that a (complex) vector bundle $p \colon E \to X$ is a *G-vector bundle* if G acts on E such that p is G-equivariant and the action of G on the fibres $g \colon E_x \to E_{gx}$ are linear maps of vector spaces.

If X is compact, then we define $K_G(X)$ to be the Grothendieck group of the monoid $\mathrm{Vect}_G(X)$ of finite-dimensional G-vector bundles on X, noting that given G-vector bundles E and F on X we can as usual form the direct sum $E \oplus F$ which can be naturally made into a G-vector bundle (and similarly for the tensor product). As in non-equivariant K-theory we can make $K_G(X)$ into a ring with respect to the tensor product of G-vector bundles.

Remark 2.2. For the remainder of this section we will assume that our G-spaces are compact. However, we will encounter G-spaces that are not compact in later sections, and we define equivariant K-theory for non-compact spaces as in [2, Section 4].

We can take the pullback of a G-vector bundle on Y along a G-map $f \colon X \to Y$ to obtain a G-vector bundle on X, and one can show that the pullback of G-vector bundles commutes with direct sums and tensor products. So, we have that $K_G(-)$ is a contravariant functor from the category of (compact) G-spaces to the category of (commutative) rings.

If the G-space X has a disjoint basepoint x_0 fixed by the action of G, then we define the reduced equivariant K-theory $\widetilde{K}_G(X)$ of X to be the kernel of the map $K_G(X) \to K_G(x_0)$ induced by the inclusion of the basepoint x_0 into X.

Example 2.3. If $X = *$ with of course trivial G-action, then since a vector bundle on a point is just some finite-dimensional complex vector space, it follows that a G-vector bundle on a point is just some finite-dimensional G-representation. So, we have that $K_G(*) \cong R(G)$, where $R(G)$ is the (complex) *representation ring* of G, i.e. the free abelian group on the set of isomorphism classes of (complex) representations of G.

Thus, by looking at the induced map in equivariant K-theory by the G-map $X \to *$, it follows that $K_G(X)$ is a module over $R(G)$ for any compact G-space X.

Note that if H is a subgroup of G, then we have a restriction map $r^G_H\colon R(G) \to R(H)$ as every G-representation can be viewed as an H-representation. We define I^G_H to be the kernel of r^G_H, and we let $I^G_1 = I$ (where 1 denotes the trivial subgroup) and call it the *augmentation ideal*, which explicitly is the kernel of the *augmentation map* $R(G) \to \mathbb{Z}$ sending an irreducible G-representation to its dimension and extending linearly to the whole of $R(G)$.

Again if H is a subgroup of G, then we also have a restriction map $K_G(X) \to K_H(X)$ for any compact G-space which comes from the fact that we can view any G-vector bundle as an H-vector bundle. We also have the following result.

Proposition 2.4. *If Y is an H-space, then we have an isomorphism*

$$K_G(G \times_H Y) \cong K_H(Y).$$

Proof. This follows from the fact that we can identify the monoid $\mathrm{Vect}_G(G \times_H Y)$ with $\mathrm{Vect}_H(Y)$ as every G-vector bundle on $G \times_H Y$ is completely determined by a unique H-vector bundle on Y by the construction of the G-space $G \times_H Y$, i.e. we can identify Y with the H-space $H \times_H Y \subset G \times_H Y$. ∎

In particular, taking Y to be a point with trivial H-action we get that $K_G(G/H) \cong R(H)$. Now, if X is a G-space, then we can alternatively view the restriction map $K_G(X) \to K_H(X)$ as the composite

$$K_G(X) \to K_G(G/H \times X) \xrightarrow{\cong} K_H(X)$$

where the first map is induced by the projection $G/H \times X \to X$ onto the second component and the second map is given by Proposition 2.4 together with the isomorphism of G-spaces $G \times_H X \cong G/H \times X$ as discussed in Section 1 since X is a G-space.

Of course, if G is trivial then $K_G(-)$ is just ordinary complex K-theory $K(-)$, but there is also a deeper connection between equivariant and non-equivariant K-theory.

Proposition 2.5. *Suppose that X is a free G-space, i.e. the action of G on X is free. Then, we have a canonical isomorphism of rings*

$$K_G(X) \cong K(X/G).$$

Proof. Consider the map $p\colon X \to X/G$ given by projecting X onto its orbit space X/G. This induces a map of rings $p^*\colon K(X/G) \to K_G(X)$, and p^* has an inverse given by sending a G-vector bundle $E \to X$ to the non-equivariant vector bundle $E/G \to X/G$ which is indeed a vector bundle following from the fact that G is a compact Lie group (see [12, Chapter 7]). ∎

Now, we define the higher equivariant K-groups in a similar way to how we defined the higher K-groups for ordinary complex K-theory. If X is a based G-space, we define $\widetilde{K}^{-q}_G(X) := \widetilde{K}_G(\Sigma^q X)$ for $q \geq 0$. Then, we can use the following equivariant Bott periodicity theorem to define $\widetilde{K}^n_G(X)$ for all $n \in \mathbb{Z}$ so that we obtain a $\mathbb{Z}$-graded ring, and we get that equivariant K-theory is an *equivariant cohomology theory.* We won't go into detail about what a G-equivariant cohomology theory is, but we require our functors to be G-homotopy invariant, to satisfy the suspension isomorphism and among others we require a long exact sequence induced by G-cofiber sequences (see [6], [4] and [10]).

Theorem 2.6. *We have that $\widetilde{K}^{-q}_G(X)$ is naturally isomorphic to $\widetilde{K}^{-q-2}_G(X)$.*

We will generally from now take our G-space X to be a finite G-CW complex, and we let $K_G^*(X)$ be the reduced theory where if X is a based finite G-CW complex then $K_G^0(X) = \widetilde{K}_G(X)$ and if X is not based then $K_G^0(X_+) = K_G(X)$.

We also have the following equivariant Thom isomorphism which will be useful in the proof of the completion theorem and can also be viewed as a generalisation of equivariant Bott periodicity [8].

Theorem 2.7. *Suppose that X is a compact G-space and that V is a G-representation. Then, there is an isomorphism*

$$\widetilde{K}_G(X_+) \xrightarrow{\cong} \widetilde{K}_G(S^V \wedge X_+)$$

given by multiplication by a class $b_V \in \widetilde{K}_G(S^V)$ which we call the Bott class.

Furthermore, if we let $e_V\colon S^0 \to S^V$ be the based map which sends the non-basepoint in S^0 to the zero vector $0 \in S^V = V \cup \{\infty\}$, then looking at the induced map $e_V^*\colon \widetilde{K}_G(S^V) \to \widetilde{K}_G(S^0) \cong R(G)$, we have that $e_V^*(b_V) = \lambda_V$, where λ_V is the *Euler class* defined as the alternating sum of the exterior powers of V. That is,

$$\lambda_V = 1 - V + \bigwedge^2 V - \cdots + (-1)^{\dim(V)} \bigwedge^{\dim(V)} V \in R(G).$$

3. Pro-Groups and Completion

For G a compact Lie group, our goal is to compute the ordinary (complex) K-theory of the classifying space BG of G. Recall that if EG is a weakly contractible space (i.e. all of its higher homotopy groups are trivial) with a proper free action of G, then the classifying space BG is given by the orbit space

$$BG = EG/G.$$

Now, if we are given a G-representation V, then we can construct the (complex) vector bundle

$$p\colon EG \times_G V \to BG$$

from the universal principal G-bundle

$$\pi\colon EG \to BG,$$

where $EG \times_G V$ is given by

$$EG \times_G V = EG \times V \Big/ (eg, v) \sim (e, gv) \text{ for } g \in G$$

and the projection map in the vector bundle $EG \times_G V \to BG$ is given by the composite $\pi \circ p_1$ where p_1 is the projection onto the first component. That is, we have that $p[e, v] = \pi(e)$ for all $[e, v] \in EG \times_G V$. So, since we started with a G-representation and obtained a (complex) vector bundle on BG, it follows that we have a functor $\mathrm{Rep}(G) \to \mathrm{Vect}(BG)$ which then induces a map $R(G) \to K(BG)$ where again $R(G)$ is the (complex) representation ring of G and $K(BG)$ is the ordinary (complex) K-theory of the classifying space BG.

However, the map $R(G) \to K(BG)$ is not an isomorphism. Indeed, since BG is an infinite complex (unless G is trivial) we would expect the K-theory $K(BG)$ to admit the structure of a complete topological group, which is not the case for the representation ring of G.

Using Proposition 2.5 and the definition of the classifying space BG, we can view this map $R(G) \to K(BG)$ as a map $R(G) \to K_G(EG)$, and this is in fact induced by the projection $EG \to *$. The Atiyah-Segal completion theorem tells us that this induced map in equivariant K-theory by the projection $EG \to *$ is an isomorphism after *I-adic completion*, where as before $I \subset R(G)$ is the augmentation ideal.

3.1. Completion.

If R is a commutative ring and M is an R-module, then given an ideal $J \subset R$ the *J-adic completion* of M, denoted by $M_J^\wedge$, is the inverse limit

$$M_J^\wedge = \varprojlim M/J^n M.$$

We can equivalently view the J-adic completion $M_J^\wedge$ as follows. The *J-adic topology* on M is defined by giving a basis of open neighbourhoods of 0 as the submodules $J^n M$, and then translation gives bases of open neighbourhoods around any other point in M. Then, we have that $M_J^\wedge$ is just the usual Hausdorff completion of M with respect to the J-adic topology on M.

In our case, we will take R to be the complex representation ring $R(G)$, and the module M to be the equivariant K-theory $K_G^*(X)$ for some compact G-space X. Indeed, as discussed before (which holds for any cohomology theory by considering the projection $X \to *$), we know that $K_G^*(X)$ is a module over $K_G^*(*) \cong R(G)$. So, letting $I \subset R(G)$ be the augmentation ideal, we have that the I-adic completion $K_G^*(X)_I^\wedge$ is given by

$$K_G^*(X)_I^\wedge = \varprojlim K_G^*(X)/I^n K_G^*(X).$$

Now, in order to give the proof of the Atiyah-Segal completion theorem, we need the notion of a *pro-group.*

3.2. **Pro-objects.** First, we need to recall the definition of an *inverse system* in an arbitrary category $\mathcal{C}$.

Definition 3.1. An *inverse system* $\{A_\alpha\}_{\alpha\in S}$ in a category $\mathcal{C}$ is a collection of objects A_α in $\mathcal{C}$ running over a direct set S together with morphisms $f_{\alpha,\beta}\colon A_\beta \to A_\alpha$ (called the *structure maps*) whenever $\alpha \leq \beta$ in the directed set S, such that $f_{\alpha,\alpha}\colon A_\alpha \to A_\alpha$ is the identity and for every $\alpha \leq \beta \leq \gamma$ in S, we have that $f_{\alpha,\gamma} = f_{\alpha,\beta} \circ f_{\beta,\gamma}$.

Given the definition of an inverse system, we can now introduce the category $\mathrm{Pro}(\mathcal{C})$ of *pro-objects* associated to an arbitrary category $\mathcal{C}$.

Definition 3.2. The category $\mathrm{Pro}(\mathcal{C})$ has objects inverse systems $\{A_\alpha\}_{\alpha\in S}$ with morphisms defined as follows. Suppose that $\{A_\alpha\}_{\alpha\in S}$ and $\{B_\beta\}_{\beta\in T}$ are two inverse systems where S and T are directed sets. Then, a morphism (up to an equivalence relation) from $\{A_\alpha\}_{\alpha\in S}$ to $\{B_\beta\}_{\beta\in T}$ in $\mathrm{Pro}(\mathcal{C})$ is given by a collection of morphisms $\Phi_\beta\colon A_{\theta(\beta)} \to B_\beta$ in $\mathcal{C}$ for each $\beta \in T$ where $\theta\colon T \to S$ is some fixed function, and we require these morphisms Φ_β to behave well with respect to the structure maps. More precisely, we require them to satisfy the property that if $\beta \leq \beta'$ in the directed set T, then we can choose an $\alpha \in S$ such that the diagram

$$\begin{tikzcd}
& A_{\theta(\beta)} \arrow[r, "\Phi_\beta"] & B_\beta \\
A_\alpha \arrow[ur, "f_{\theta(\beta),\alpha}"] \arrow[dr, "f_{\theta(\beta'),\alpha}"] & & \\
& A_{\theta(\beta')} \arrow[r, "\Phi_{\beta'}"] & B_{\beta'} \arrow[uu, "g_{\beta,\beta'}"']
\end{tikzcd}$$

commutes, where of course for these maps to make sense we require that $\theta(\beta) \leq \alpha$ and $\theta(\beta') \leq \alpha$, and the $f_{\gamma,\gamma'}$ and $g_{\delta,\delta'}$ are the structure maps in the inverse systems $\{A_\alpha\}_{\alpha\in S}$ and $\{B_\beta\}_{\beta\in T}$ respectively. However, as mentioned before, these morphisms are defined up to an equivalence relation, where we identify two such morphisms thought of as pairs (θ, Φ_β) and (θ', Φ'_β) if their corresponding upper composites in the above diagram are equal. That is, we identify them under this equivalence relation if for every $\beta \in T$ there is an $\alpha \in S$ with $\theta(\beta) \leq \alpha$ and $\theta'(\beta) \leq \alpha$ such that $\Phi_\beta \circ f_{\theta(\beta),\alpha} = \Phi'_\beta \circ f_{\theta'(\beta),\alpha}$.

We will primary be looking at the category of pro-groups whose objects are inverse systems of groups, or at the category of pro-$R(G)$-modules whose objects are inverse systems of $R(G)$-modules. Something to note is that if $\mathcal{C}$ is an abelian category, then $\mathrm{Pro}(\mathcal{C})$ is also an abelian category.

Another concept that we'll need in the proof of the completion theorem is that of the *Mittag-Leffler condition.*

Definition 3.3. A pro-object $\{A_\alpha\}_{\alpha\in S}$ is said to satisfy the *Mittag-Leffler condition* if for every $\alpha \in S$ there exists some $\beta \in S$ with $\alpha \leq \beta$ such that the images of the structure maps $f_{\alpha,\beta}$ and $f_{\alpha,\gamma}$ are equal for all $\gamma \geq \beta$.

A useful fact is that if all the structure maps in a pro-object $\{A_\alpha\}_{\alpha\in S}$ are epimorphisms, then the pro-object satisfies the Mittag-Leffler condition. The main reason why we are interested in the Mittag-Leffler condition in the context of the completion theorem is due to the following proposition.

Proposition 3.4. *If a pro-group satisfies the Mittag-Leffler condition, then it's* $\lim^1$ *term vanishes.*

Here $\lim^1$ is the first right-derived functor of the limit functor, and by the Milnor exact sequence (see [2, Proposition 4.1]) the vanishing of $\lim^1$ allows our functors (such as equivariant K-theory) to behave well with inverse limits. We now introduce a pro-group valued analogue of equivariant K-theory.

Definition 3.5. For every $n \in \mathbb{Z}$, we define $\mathcal{K}_G^n(-)$ to be the functor from the category of G-CW complexes to the category of pro-groups that sends a G-CW complex X to the inverse system $\{K_G^n(X_\alpha)\}$ running over the finite-subcomplexes X_α of X, with structure maps given by the induced maps in equivariant K-theory by the inclusions of finite-subcomplexes.

More generally, we can define pro-group valued analogues $\mathcal{H}$ of all our previous abelian group valued functors H on the category of G-CW complexes by defining $\mathcal{H}(X)$ to be the pro-group $\{H(X_\alpha)\}$ running over the finite-subcomplexes X_α of X.

We have that $\mathcal{K}_G^*(-)$ (which as before we use to denote the reduced theory) defines a pro-group valued equivariant cohomology theory. Since each $K_G^*(X)$ can be instead viewed as an $R(G)$-module, we can equivalently view the cohomology theory $\mathcal{K}_G^*(-)$ as taking values in the category of pro-$R(G)$-modules.

Again letting $I \subset R(G)$ be the augmentation ideal, we have the very useful fact that $\mathcal{K}_G^*(-)^\wedge_I$ is also a pro-group valued (or pro-$R(G)$-module valued) equivariant cohomology theory, where for a pro-$R(G)$-module $\{M_\alpha\}$ the I-adic completion is the inverse system $\{M_\alpha/I^n M_\alpha\}$. A useful property (see [3, Proposition 5.10]) of the category of pro-$R(G)$-modules is that in this category we have that

$$\mathcal{K}_G^*(X)^\wedge_I = \varprojlim \mathcal{K}_G^*(X)/I^n\mathcal{K}_G^*(X)$$

where $R(G)$ acts on $\mathcal{K}_G^*(X)$ object-wise in the inverse system. Note that I-adically completed equivariant K-theory $K_G^*(-)^\wedge_I$ is not a cohomology theory, which is one of the reasons why we need our functors to take values in the category of pro-groups for the proof of the completion theorem.

4. The Completion Theorem

We are now ready to prove the Atiyah-Segal completion theorem for compact Lie groups G. The proof is somewhat simpler when the compact Lie group G is a finite group (see [4, Chapter 14]), and the theorem in the case of finite groups was first given in [9].

Theorem 4.1. *Let X be a finite G-CW complex and let $\pi\colon EG_+ \wedge X \to X$ be the projection onto the second component. Then, the induced map $\pi^*\colon K_G^*(X) \to K_G^*(EG_+ \wedge X)$ is completion at the augmentation ideal I, i.e. we have that*

$$K_G^*(EG_+ \wedge X) \cong K_G^*(X)^\wedge_I.$$

In particular, if we take $X = *$, then the theorem tells us that $K_G^0(EG_+) \cong K^0(BG_+) \cong R(G)^\wedge_I$ and $K_G^1(EG_+) \cong K^1(BG_+) = 0$ (note we're using here that $K_G^1(*) = 0$). We will now give the proof by Adams, Haeberly, Jackowski and May [1] who in fact proved a slightly more general result which we will mention later.

Proof. First, we will show that $\pi\colon EG_+ \wedge X \to X$ induces an isomorphism $\pi^*\colon \mathcal{K}_G^*(X)^\wedge_I \to \mathcal{K}_G^*(EG_+ \wedge X)^\wedge_I$ in $(\mathcal{K}_G^*)^\wedge_I$ i.e. in I-adically completed pro-group valued equivariant K-theory. Consider the cofiber sequence (often called the isotropy separation sequence)

$$EG_+ \to S^0 \to \tilde{E}G,$$

where the first map collapses EG to a point and sends the disjoint basepoint of EG_+ to the basepoint of S^0, and we define $\tilde{E}G$ to be the cofiber (or mapping cone) of this map. Then, by smashing this cofiber sequence with X, we get the cofiber sequence

$$EG_+ \wedge X \to S^0 \wedge X \to \tilde{E}G \wedge X,$$

i.e. the cofiber sequence

$$EG_+ \wedge X \to X \to \tilde{E}G \wedge X,$$

noting that the first map $EG_+ \wedge X \to X$ is just our map π. Now, since $(\mathcal{K}_G^*)^\wedge_I$ is a (pro-group valued) equivariant cohomology theory, it is in particular exact on cofiber sequences, so since the first map is π, in order to show that $\pi^*\colon \mathcal{K}_G^*(X)^\wedge_I \to \mathcal{K}_G^*(EG_+ \wedge X)^\wedge_I$ is an isomorphism it suffices to show that $\mathcal{K}_G^*(\tilde{E}G \wedge X)^\wedge_I$ is pro-zero (i.e. the zero object in the abelian category of pro-groups).

Now, we construct a space U (often called a G-universe [7, Section 2.3.2]) as follows. Let $\{V_i\}$ be a countable collection of non-trivial (complex) G-representations with the property that $V_i^G = 0$ for every vector space (or G-representation) V_i in the collection, and furthermore for every proper subgroup $H < G$ there is some G-representation V_j in the collection such that $V_j^H \neq 0$. Then, we define U to be the infinite-dimensional G-representation formed by the direct sum of countably many copies of each of the G-representations V_i in our countable collection. We now define Y to be the colimit of the representation spheres S^V running over the directed system of the finite-dimensional subrepresentations V of the G-universe U.

Observe that $Y^G \simeq S^0$ and $Y^H \simeq *$ for all proper subgroups $H < G$. Indeed, the fact that $Y^G \simeq S^0$ is immediate by definition, as Y is defined to be the colimit of the representation spheres S^V (where each V is a finite-dimensional subrepresentation of U), but each such V satisfies $V^G = 0$ as $V_i^G = 0$ for all V_i in our countable collection of non-trivial G-representations used to construct U. Now, we fix H to be a proper subgroup of G, and we want to show that Y^H is contractible i.e. that $Y^H \simeq *$. Again by the definition of our countable collection $\{V_i\}$, we know that there is some G-representation V_j in this collection such that $V_j^H \neq 0$. Since by definition V_j occurs countably many times as a direct summand of U and for every finite dimensional subrepresentation V of V_j we have that $V^H \neq 0$, it follows that there is an ascending sequence

$$V_1 \subset V_2 \subset V_3 \subset \cdots \subset U$$

of finite-dimensional subrepresentations of U such that $(V_{k+1} - V_k)^H \neq 0$ for every $k \geq 1$, where $V_{k+1} - V_k$ is the complement of V_k in V_{k+1} (which is isomorphic to the quotient vector space V_{k+1}/V_k). However, since $(V_{k+1} - V_k)^H \neq 0$ for every $k \geq 1$, it follows that the inclusion $S^{V_k} \to S^{V_{k+1}}$ is null-H-homotopic. Indeed, we can give an explicit null-H-homotopy as follows. Since $(V_{k+1} - V_k)^H \neq 0$, let $u \in V_{k+1} - V_k$ be such that $u \neq 0$ and u is fixed by the action of H on the vector space $V_{k+1} - V_k$. Then, viewing S^{V_k} and $S^{V_{k+1}}$ as $V_k \cup \{\infty\}$ and $V_{k+1} \cup \{\infty\}$ respectively, we can define a null-H-homotopy $F\colon S^{V_k} \wedge [0,1]_+ \to S^{V_{k+1}}$ from the inclusion $S^{V_k} \to S^{V_{k+1}}$ (sending the point-at-infinity in S^{V_k} to the point-at-infinity in $S^{V_{k+1}}$) to the constant map at the point-at-infinity in $S^{V_{k+1}}$ by

$$F(v,t) = \begin{cases} v + \frac{t}{1-t}u & \text{if } v \in V_k, \\ \infty & \text{if } v = \infty. \end{cases}$$

Note that indeed $F(v,0) = v$ for every $v \in V_k$ with $F(\infty, 0) = \infty$, and that $F(v,1) = \infty$ for every $v \in S^{V_k}$. Furthermore, observe that F is continuous and is of course H-equivariant since H acts trivially on u (and we define H to act trivially on the unit interval $[0,1]$ as in Definition 1.11). However, following from the definition of Y and using that $V_1 \subset V_2 \subset V_3 \subset \dots$ is an ascending sequence of finite-dimensional subrepresentations of U, it follows that Y is the colimit of the sequence

$$S^{V_1} \to S^{V_2} \to S^{V_3} \to \dots$$

where each map is the inclusion which is null-H-homotopic as showed above. Hence, it follows that Y is H-contractible i.e. that $Y^H \simeq *$, which is what we wanted to show.

Now, since $Y^G \simeq S^0$, we have that $(Y/S^0)^G \simeq *$, and thus the G-CW complex Y/S^0 can be constructed purely out of cells of the form $G/H_+ \wedge S^n$ for H a proper subgroup of G. Recall that we're trying to show that $\mathcal{K}_G^*(\tilde{E}G \wedge X)^\wedge_I$ is pro-zero, and in fact it suffices to prove it assuming that it holds for all proper subgroups of G. More precisely, since G is a compact Lie group, we know that it satisfies the descending chain condition, i.e. every infinite descending chain

$$G > H_1 > H_2 > H_3 > \dots$$

of (closed) subgroups of G stabilises. So, this means that the partially ordered set of (closed) subgroups of G (where the partial order is given by whether one subgroup of G contains another subgroup of G) is well-founded, which allows us to use the principle of well-founded induction i.e. in order to prove some property of G it suffices to prove the property for all proper subgroups of G. Thus, by well-founded induction we may assume that $\mathcal{K}_H^*(Z \wedge X)^\wedge_{I_H}$ is pro-zero for any (non-equivariantly) contractible space Z and proper subgroup $H < G$, where here I_H is the augmentation ideal in $R(H)$. However, as pro-$R(G)$-modules (using Proposition 2.4 for based spaces) we have an isomorphism

$$\mathcal{K}_G^*(G/H_+ \wedge Z \wedge X) \cong \mathcal{K}_H^*(Z \wedge X)$$

where $R(G)$ acts object-wise in the inverse system $\mathcal{K}_H^*(Z \wedge X)$ via the restriction homomorphism $R(G) \to R(H)$, which implies that

$$\mathcal{K}_G^*(G/H_+ \wedge Z \wedge X)^\wedge_I \cong \mathcal{K}_H^*(Z \wedge X)^\wedge_{I_H}$$

using the non-trivial fact that the completion of an $R(H)$-module with respect to the I_H-adic topology coincides with the completion with respect to the I-adic topology (see [5] and [1]). So, by well-founded induction we may assume that for all proper subgroups $H < G$ we have that $\mathcal{K}_G^*(G/H_+ \wedge Z \wedge X)_I^\wedge$ is pro-zero for any (non-equivariantly) contractible space Z. Therefore, as $\mathcal{K}_G^*(-)_I^\wedge$ behaves well with respect to cofibers, wedges and colimits, we can assume that $\mathcal{K}_G^*(E \wedge Z \wedge X)_I^\wedge$ is pro-zero whenever E is a G-CW complex that can be constructed purely out of equivariant cells of the form $G/H_+ \wedge S^n$ for H a proper subgroup of G, and again where Z is contractible.

So, recalling that the G-CW complex Y/S^0 can be constructed purely out of cells of the form $G/H_+ \wedge S^n$ for H a proper subgroup of G which followed from the fact that $(Y/S^0)^G \simeq *$, we may assume by well-founded induction that $\mathcal{K}_G^*(Y/S^0 \wedge \tilde{E}G \wedge X)_I^\wedge$ is pro-zero, as $\tilde{E}G$ is (non-equivariantly) contractible. In fact, we have that $(\tilde{E}G)^G \simeq S^0$ and $(\tilde{E}G)^H \simeq *$ for every proper subgroup $H < G$, i.e. we have that $\tilde{E}G$ and Y have the same fixed-point spaces (up to homotopy equivalence). Now, consider the cofiber sequence

$$S^0 \to Y \to Y/S^0$$

where the first map is the inclusion (using that $Y^G \simeq S^0$). Smashing this cofiber sequence with $\tilde{E}G \wedge X$ yields the cofiber sequence

$$\tilde{E}G \wedge X \to Y \wedge \tilde{E}G \wedge X \to Y/S^0 \wedge \tilde{E}G \wedge X.$$

Since $(\mathcal{K}_G^*)_I^\wedge$ is a (pro-group valued) cohomology theory, it is exact on cofiber sequences and thus since we may assume that $\mathcal{K}_G^*(Y/S^0 \wedge \tilde{E}G \wedge X)_I^\wedge$ is pro-zero, it follows that $\mathcal{K}_G^*(Y \wedge \tilde{E}G \wedge X)_I^\wedge \cong \mathcal{K}_G^*(\tilde{E}G \wedge X)_I^\wedge$. So, recalling that we want to show that $\mathcal{K}_G^*(\tilde{E}G \wedge X)_I^\wedge$ is pro-zero, it suffices to show that $\mathcal{K}_G^*(Y \wedge \tilde{E}G \wedge X)_I^\wedge$ is pro-zero. However, we will take this one-step further, and note that the inclusion

$$Y \wedge X = Y \wedge S^0 \wedge X \to Y \wedge \tilde{E}G \wedge X$$

is a G-homotopy equivalence. Indeed, this follows from the Equivariant Whitehead Theorem (i.e. Theorem 1.12) as we know that this inclusion is a homotopy equivalence on all the fixed-point spaces which follows from the fact that this map is an inclusion and that Y and $\tilde{E}G$ have the same (or homotopy-equivalent) fixed-point spaces. So, since $(\mathcal{K}_G^*)_I^\wedge$ is a (pro-group valued) G-equivariant cohomology theory it follows that $\mathcal{K}_G^*(Y \wedge X)_I^\wedge \cong \mathcal{K}_G^*(Y \wedge \tilde{E}G \wedge X)_I^\wedge$ and thus it suffices to show that $\mathcal{K}_G^*(Y \wedge X)_I^\wedge$ is pro-zero.

To do this, we first show that $\mathcal{K}_G^*(Y)_I^\wedge$ is pro-zero. We know as discussed in Section 3 that $\mathcal{K}_G^*(Y)_I^\wedge$ is the inverse limit in the category of pro-$R(G)$-modules of the inverse system $\mathcal{K}_G^*(Y)/I^n\mathcal{K}_G^*(Y)$ running over powers of the augmentation ideal. So, in order to show that $\mathcal{K}_G^*(Y)_I^\wedge$ is pro-zero it suffices to show that each $\mathcal{K}_G^*(Y)/I^n\mathcal{K}_G^*(Y)$ is pro-zero. Hence, by the definition of Y as the colimit of the representation spheres S^V running over the finite-dimensional subrepresentations V of U, it suffices to show that for any finite-dimensional subrepresentation V of U then if W is a finite-dimensional subrepresentation of U containing V, then the induced map

$$\mathcal{K}_G^*(S^W)/I^n\mathcal{K}_G^*(S^W) \to \mathcal{K}_G^*(S^V)/I^n\mathcal{K}_G^*(S^V)$$

by the inclusion $S^V \to S^W$ (where again we include V into W and send the point-at-infinity to the point-at-infinity) is zero. So, fix a finite-dimensional subrepresentation V of U, and let W be a finite-dimensional subrepresentation of U containing V. First, we claim that the induced map

$$K_G^*(S^W) \to K_G^*(S^V)$$

in equivariant K-theory by the inclusion $S^V \to S^W$ is given by multiplication by the Euler class $\lambda_{W-V} \in K_G(*) \cong R(G)$. Indeed, we know that the Bott classes $b_V \in K_G^0(S^V)$, $b_W \in K_G^0(S^W)$ and $b_{W-V} \in K_G^0(S^{W-V})$ satisfy the multiplicative property (see [8]) that

$$b_W = b_{W-V}b_V.$$

By Theorem 2.7 (i.e. the equivariant Thom isomorphism or generalised equivariant Bott periodicity), we know that $K_G^*(S^W)$ is a free module over $K_G^*(S^0)$ generated by the Bott class $b_W \in K_G^0(S^W)$ and similarly for $K_G^*(S^V)$. So, we can write an arbitrary element of $K_G^*(S^W)$ as xb_W for some $x \in K_G^*(S^0)$, and so by the above multiplicative relation we can re-write it as $xb_{W-V}b_V$. So, since $K_G^*(S^V)$ is also a free module over $K_G^*(S^0)$ generated by the Bott class $b_V \in K_G^0(S^V)$ i.e. the induced map $K_G^*(S^W) \to K_G^*(S^V)$ is a map of free-modules over $K_G^*(S^0)$, it follows that this induced map sends xb_W to $x\lambda_{W-V}b_V$. Note here we're using that the inclusion $S^V \to S^W$ is just the smash product of the inclusion $e_{W-V} \colon S^0 \to S^{W-V}$ with S^V and that

$e^*_{W-V}(b_{W-V}) = \lambda_{W-V}$ as discussed in Section 2. So, it follows that the induced map $K^*_G(S^W) \to K^*_G(S^V)$ in equivariant K-theory is given by multiplication by the Euler class λ_{W-V}.

However, notice that the Euler class λ_{W-V} is in the augmentation ideal I. Indeed, recall that by definition $\lambda_{W-V} \in R(G)$ is given by

$$\lambda_{W-V} = 1 - (W-V) + \bigwedge^2 (W-V) - \cdots + (-1)^{\dim(W-V)} \bigwedge^{\dim(W-V)} (W-V),$$

and thus the augmentation map $R(G) \to \mathbb{Z}$ sends λ_{W-V} to 0 as we have the simple combinatorial result (following from the binomial expansion) that

$$\sum_{k=0}^{\dim(W-V)} (-1)^k \binom{\dim(W-V)}{k} = 0,$$

recalling that the exterior power $\bigwedge^k(W-V)$ has dimension $\binom{\dim(W-V)}{k}$. Therefore, since the induced map $K^*_G(S^W) \to K^*_G(S^V)$ in equivariant K-theory by the inclusion $S^V \to S^W$ is multiplication by the Euler class λ_{W-V} which is in I (and therefore in I^n for every $n > 1$), it follows by the definition of the pro-group valued K-theory $\mathcal{K}^*_G(-)$ that the induced map

$$\mathcal{K}^*_G(S^W)/I^n\mathcal{K}^*_G(S^W) \to \mathcal{K}^*_G(S^V)/I^n\mathcal{K}^*_G(S^V)$$

by the inclusion $S^V \to S^W$ is zero. Hence, as discussed earlier, we may now conclude that $\mathcal{K}^*_G(Y)^\wedge_I$ is pro-zero.

Note that this then implies that $\mathcal{K}^*_G(Y \wedge X)^\wedge_I$ is pro-zero. Indeed, since X is a finite G-CW complex, it suffices to show that $\mathcal{K}^*_G(Y \wedge G/H_+ \wedge S^n)^\wedge_I$ is pro-zero for every n and H a subgroup of G, and thus by the suspension isomorphism we in fact just have to show that $\mathcal{K}^*_G(G/H_+ \wedge Y)^\wedge_I$ is pro-zero for every subgroup H of G. If $H = G$, then we just get $\mathcal{K}^*_G(Y)^\wedge_I$ which we already know is pro-zero. Now, suppose that H is a proper subgroup of G. Then, recall as seen earlier that we have an isomorphism

$$\mathcal{K}^*_G(G/H_+ \wedge Y)^\wedge_I \cong \mathcal{K}^*_H(Y)^\wedge_{I_H}.$$

However, since H is a proper subgroup of G, we also know as discussed earlier that $Y^H \simeq *$ i.e. that Y is H-contractible. So, since $\mathcal{K}^*_H(-)^\wedge_{I_H}$ is an H-equivariant cohomology theory it follows that $\mathcal{K}^*_H(Y)^\wedge_{I_H}$ is pro-zero and thus that $\mathcal{K}^*_G(G/H_+ \wedge Y)^\wedge_I$ is pro-zero. Hence, we have that $\mathcal{K}^*_G(Y \wedge X)^\wedge_I$ is pro-zero, which as discussed earlier is precisely what we needed to show in order to deduce that $\mathcal{K}^*_G(\tilde{E}G \wedge X)^\wedge_I$ is pro-zero, i.e. that $\pi\colon EG_+ \wedge X \to X$ induces an isomorphism $\pi^*\colon \mathcal{K}^*_G(X)^\wedge_I \to \mathcal{K}^*_G(EG_+ \wedge X)^\wedge_I$.

We now use this to show that $K^*_G(EG_+ \wedge X) \cong K^*_G(X)^\wedge_I$. Since X is a finite G-CW complex, we know that $\mathcal{K}^*_G(X)^\wedge_I$ satisfies the Mittag-Leffler condition (recall that any inverse system with structure maps that are epimorphisms satsifes the Mittag-Leffler condition), and thus it's $\lim^1$ term vanishes by Proposition 3.4. So, since we have that $\mathcal{K}^*_G(X)^\wedge_I \cong \mathcal{K}^*_G(EG_+ \wedge X)^\wedge_I$, it follows that $\mathcal{K}^*_G(EG_+ \wedge X)^\wedge_I$ also has vanishing $\lim^1$ term. But in fact $\mathcal{K}^*_G(EG_+ \wedge X)$ is already I-adically complete, which follows since all of the groups in the pro-group $K^*_G(EG_+ \wedge X)$ (each of which is the equivariant K-theory of some finite-subcomplex of $EG_+ \wedge X$) are I-adically complete. So, we in fact have that $\mathcal{K}^*_G(X)^\wedge_I \cong \mathcal{K}^*_G(EG_+ \wedge X)$ where $\mathcal{K}^*_G(EG_+ \wedge X)$ has vanishing $\lim^1$ term.

By the Milnor exact sequence [2, Proposition 4.1], the fact that $\mathcal{K}^*_G(EG_+ \wedge X)$ has vanishing $\lim^1$ term tells us that the equivariant K-theory $K^*_G(EG_+ \wedge X)$ is the inverse limit of the equivariant K-theories of the finite-subcomplexes of $EG_+ \wedge X$, and similarly for $\mathcal{K}^*_G(X)^\wedge_I$. So, since the inverse systems (or pro-groups) $\mathcal{K}^*_G(X)^\wedge_I$ and $\mathcal{K}^*_G(EG_+ \wedge X)$ are isomorphic, by taking inverse limits and using the above we may now conclude that $K^*_G(EG_+ \wedge X) \cong K^*_G(X)^\wedge_I$, as required. ■

As given in the paper by Adams, Haeberly, Jackowski and May [1], it is possible to generalise the Atiyah-Segal completion theorem (without too many adjustments to the above proof) to instead consider *families* of subgroups of G, rather than simply G itself.

Definition 4.2. A collection $\mathcal{F}$ of subgroups of G is called a *family* if the collection $\mathcal{F}$ is closed under taking subgroups (i.e. if $H_2 \in \mathcal{F}$ and $H_1 < H_2$ then $H_1 \in \mathcal{F}$) and closed under conjugation (i.e. if $H_2 \in \mathcal{F}$ and $H_1 = gH_2g^{-1}$ for some $g \in G$ then $H_1 \in \mathcal{F}$).

Given a family $\mathcal{F}$ of subgroups of G, we now consider the space $E\mathcal{F}$ which has the property that $(E\mathcal{F})^H \simeq *$ if $H \in \mathcal{F}$ and $(E\mathcal{F})^H = \emptyset$ if $H \notin \mathcal{F}$. Furthermore, rather than completing with respect to the augmentation ideal I, we define the *$\mathcal{F}$-adic completion* of $K_G^n(X)$ to be given by

$$K_G^n(X)^\wedge_{\mathcal{F}} = \varprojlim K_G^n(X)/JK_G^n(X)$$

where now J runs over finite products of the ideals I_H^G (which we recall is the kernel of the restriction map $r_H^G \colon R(G) \to R(H)$) where $H \in \mathcal{F}$, noting that before we were just running over powers of the augmentation ideal I. Now, the generalised Atiyah-Segal completion theorem for families of subgroups of G is as follows.

Theorem 4.3. *Let X be a finite G-CW complex and let $\pi \colon E\mathcal{F}_+ \wedge X \to X$ be the projection onto the second component. Then, the induced map $\pi^* \colon K_G^*(X) \to K_G^*(E\mathcal{F}_+ \wedge X)$ is $\mathcal{F}$-adic completion, i.e. we have that*

$$K_G^*(E\mathcal{F}_+ \wedge X) \cong K_G^*(X)^\wedge_{\mathcal{F}}.$$

Notice that if we take $\mathcal{F}$ to be the trivial family $\mathcal{F} = \{1\}$, then we recover the original Atiyah-Segal completion theorem as stated in Theorem 4.1.

5. Examples

In this section we give a couple of examples of how we can apply the completion theorem in computations, namely we will use it to compute $K(\mathbb{R}P^\infty)$ and $K(\mathbb{C}P^\infty)$. First, we have the following useful theorem (see [3, Section 6] and [11, Chapter 7])

Theorem 5.1. *Suppose that R is a Noetherian commutative ring, and let $J = (a_1, \dots, a_n)$ be an ideal in R. Then, the J-adic completion $R^\wedge_J$ is given by the power series ring*

$$R^\wedge_J \cong R[[x_1, \dots, x_n]]/(x_1 - a_1, \dots, x_n - a_n).$$

We will first give an example of applying the completion theorem when our group G is finite.

Example 5.2. Let $G = C_2$ be the cyclic group of order 2. Then, we know that the representation ring $R(C_2)$ is given by $\mathbb{Z}[x]/(x^2-1)$, where we view x here as the one-dimensional complex sign representation (noting that the relation $x^2 - 1 = 0$ comes from the fact that the tensor product of the sign representation with itself gives the trivial one-dimensional representation). Note that of course $R(C_2) \cong \mathbb{Z}[x]/(x^2-1)$ is Noetherian as we know that $\mathbb{Z}[x]$ is Noetherian by Hilbert's basis theorem which then implies that $\mathbb{Z}[x]/(x^2-1)$ is Noetherian as the quotient of a Noetherian ring by an ideal is also Noetherian. In fact, we know that the representation ring $R(G)$ is Noetherian for any G (see [5, Corollary 3.3]). So, we can apply the result of Theorem 5.1.

In this case, the augmentation ideal i.e. the kernel of the augmentation map $R(C_2) \to \mathbb{Z}$ is the ideal $(x-1)$. Now, applying a change-of-variables where we let $t = x - 1$, we have that

$$\begin{aligned} R(C_2)^\wedge_{(x-1)} &\cong (\mathbb{Z}[x]/(x^2-1))^\wedge_{(x-1)} \\ &\cong (\mathbb{Z}[t+1]/(t^2+2t))^\wedge_{(t)} \\ &\cong (\mathbb{Z}[t]/(t^2+2t))^\wedge_{(t)}. \end{aligned}$$

However, by Theorem 5.1, we have that

$$\begin{aligned} (\mathbb{Z}[t]/(t^2+2t))^\wedge_{(t)} &= (\mathbb{Z}[t]/(t^2+2t))[[y]]/(y-t) \\ &\cong \mathbb{Z}[[t]]/(t^2+2t). \end{aligned}$$

Now, we know that a classifying space for C_2 is $\mathbb{R}P^\infty$, so by the Atiyah-Segal completion theorem (Theorem 4.1) it follows that

$$\begin{aligned} K(\mathbb{R}P^\infty) &\cong \mathbb{Z}[[t]]/(t^2+2t) \\ &\cong \mathbb{Z}[[t]]/(t(t+2)) \\ &\cong \mathbb{Z} \oplus \mathbb{Z}_2, \end{aligned}$$

where here $\mathbb{Z}_2$ is the 2-adic integers. So, the completion theorem has allowed us to compute the complex K-theory of $\mathbb{R}P^\infty$ relatively easily using the representation ring of C_2, which would otherwise be a difficult calculation involving methods such as the Atiyah-Hirzebruch spectral sequence.

We now give an example of applying the completion theorem when our group is infinite (and a compact Lie group).

Example 5.3. Consider the compact Lie group $G = S^1$, thought of as the unit circle in $\mathbb{C}$. In this case, we have that the representation ring $R(S^1)$ is given by $\mathbb{Z}[x, x^{-1}]$, where we view x here as the one-dimensional complex irreducible representation where we simply include S^1 into $GL_1(\mathbb{C}) = \mathbb{C}^\times$. As mentioned previously, we know that $R(S^1)$ is Noetherian so we can again apply the result of Theorem 5.1.

Following almost immediately by definition, the augmentation ideal in this case is again given by the ideal $(x-1)$. Now, applying a change-of-variables where we let $t = x - 1$, we have that

$$\begin{aligned} R(S^1)^\wedge_{(x-1)} &\cong \mathbb{Z}[x, x^{-1}]^\wedge_{(x-1)} \\ &\cong \mathbb{Z}[t+1, (t+1)^{-1}]^\wedge_{(t)} \\ &\cong \mathbb{Z}[t, (t+1)^{-1}]^\wedge_{(t)}. \end{aligned}$$

However, by Theorem 5.1, we have that

$$\begin{aligned} \mathbb{Z}[t, (t+1)^{-1}]^\wedge_{(t)} &\cong (\mathbb{Z}[t, (t+1)^{-1}])[[y]]/(y-t) \\ &\cong (\mathbb{Z}[(t+1)^{-1}])[[t]] \\ &\cong \mathbb{Z}[[t]] \quad \text{(as } (t+1)^{-1} \in \mathbb{Z}[[t]]). \end{aligned}$$

We know that a classifying space for S^1 is $\mathbb{C}P^\infty$, so by the Atiyah-Segal completion theorem (Theorem 4.1) it follows that

$$K(\mathbb{C}P^\infty) \cong \mathbb{Z}[[t]].$$

Thus, the completion theorem has allowed us to relatively easily calculate the complex K-theory of $\mathbb{C}P^\infty$ just using knowledge of the complex representation ring of S^1, which again would otherwise be a difficult computation.

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