YANG-MILLS THEORY ON RIEMANN SURFACES

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1. Introduction

Gauge theories (which originated from the work of Hermann Weyl) are the most important Quantum Field Theories, and Maxwell's theory of electromagnetism is an example of a Gauge theory. In Maxwell's theory, we take the Gauge group, otherwise known as the group of gauge symmetries of the theory, to be the unitary group $U(1)$ which is abelian. Letting A be the $U(1)$-gauge connection thought of as a 1-form, we have that Maxwell's equations are precisely the two equations $dF_A = 0$ and $d \star F_A = 0$, where $F_A = dA$ is the curvature of A, and $\star$ is the Hodge star operator. Now, the Yang-Mills equations can be thought of as a generalisation of Maxwell's equations whereby we upgrade the $U(1)$ Gauge group to a (not necessarily abelian) compact group $\mathcal{G}$. In doing this, we now define the curvature of A (looking now over an arbitrary compact Lie group G) to be $F_A = dA + \frac{1}{2}[A \wedge A]$, and the Yang-Mills equations are now given by $d_A F_A = 0$ and $d_A \star F_A = 0$, where d_A is a covariant exterior derivative. In particular, we have that the Yang-Mills equations in the $U(1)$ case correspond precisely to Maxwell's equations. Yang-Mills theory is used extensively in particle physics to understand the standard model.

In this report, we will formalise this. After first providing some background on principal G-bundles, the Gauge group and connections and curvature on principal bundles, our first goal is to derive the (two-dimensional) Yang-Mills equations over a fixed Riemann surface Σ. We will then look at the moduli space of flat principal G-bundles over Σ (i.e. the space of principal G-bundles equipped with a flat connection), which can equivalently be thought of as the space of conjugacy classes of representations of the fundamental group of Σ, and we compute this moduli space for $SU(2)$-bundles over the torus $S^1 \times S^1$. Finally, we discuss the symplectic structure we can put on the affine space of connections on a principal G-bundle and a corresponding moment map presented by Atiyah and Bott in [1], which we can use to view the moduli space as an (infinite-dimensional) symplectic reduction.

2. The Gauge Group

Throughout this report, we let Σ be a closed connected Riemann surface (i.e. a complex manifold of complex dimension 1). We begin by recalling the definition of a (smooth) fiber bundle.

Definition 2.1. Suppose that E, F and B are smooth manifolds and that $\pi\colon E \to B$ is a smooth surjective submersion. We say that (F, E, π, B) is a *smooth fiber bundle* if for every $x \in B$ there is an open neighbourhood

$U \subset B$ containing x and a diffeomorphism $\psi\colon \pi^{-1}(U) \to U \times F$ (called a local trivialisation) such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\psi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

commutes, where $p_1\colon U \times F \to U$ is the projection map onto the first component. For each $x \in B$, we write E_x for the fiber of x under π.

Given open sets $U_\alpha, U_\beta \subset B$ and local trivialisations $\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$ and $\psi_\beta\colon \pi^{-1}(U_\beta) \to U_\beta \times F$, there is a unique map $g_{\beta\alpha}\colon U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$ (called a transition function) such that the map

$$\psi_\beta \circ \psi_\alpha^{-1}\colon (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$$

is given by $(\psi_\beta \circ \psi_\alpha^{-1})(x, u) = (x, g_{\beta\alpha}(x)u)$ for every $x \in U_\alpha \cap U_\beta$ and $u \in F$, where $\mathrm{Diff}(F)$ is the group of diffeomorphisms of the smooth manifold F to itself which is the *structure group* of the smooth fiber bundle (F, E, π, B). If there is an open cover $\{U_\alpha\}$ of B with local trivialisations ψ_α, we say that the structure group of the fiber bundle is reduced to a subgroup $G \subset \mathrm{Diff}(F)$ if the image of all the transition functions $g_{\beta\alpha}$ are contained in G.

Note that we say that a (smooth) fiber bundle is a (smooth) *vector bundle* if $F = V$ is a vector space and the image of the transition functions are linear i.e. the structure group G is contained in $GL(V)$. Next, we come to another special class of fiber bundles that will be our main object of study throughout this report.

Definition 2.2. Let G be a Lie group with Lie algebra $\mathfrak{g}$ (which we fix for the rest of Sections 2 and 3). We say that a (smooth) fiber bundle $\pi\colon P \to \Sigma$ is a (smooth) principal G-bundle if we have a fiber-preserving right action $P \times G \to P$ of G on P such that G acts free and transitively on the fibers (i.e. the action of G on each P_x is free and transitive where $x \in \Sigma$). Note that the right action $P \times G \to P$ being fiber-preserving means that for every $x \in \Sigma$ and $y \in P_x$, we have that $yg \in P_x$ for every $g \in G$. We have that the fibers are all diffeomorphic to G and the local trivialisations of the fiber bundle are equivariant (using the action of G on P).

Note that if $\pi\colon P \to \Sigma$ is a principal G-bundle, we have in particular that P/G is diffeomorphic to Σ. Building towards the definition of the Gauge group of a principal G-bundle P, we first require the notion of a fiber bundle that is obtained from a principal G-bundle $\pi\colon P \to \Sigma$ and a representation $\rho\colon G \to \mathrm{Diff}(F)$ of the Lie group G for some smooth manifold F.

Definition 2.3. Let $\pi\colon P \to \Sigma$ be a principal G-bundle, let F be a smooth manifold, and let $\rho\colon G \to \mathrm{Diff}(F)$ be a representation of G as a subgroup of the group of diffeomorphisms of F to itself. Then, we obtain a (smooth) fiber bundle $\pi'\colon P \times_\rho F \to \Sigma$ with fiber F and structure group G (the transition functions are given by the image under ρ of the transition functions of the principal G-bundle P) called the *associated fiber bundle*, where

$$P \times_\rho F = (P \times F)/G,$$

with G acting on $P \times F$ by $g(p, x) = (pg, \rho(g)^{-1}x)$, and we write elements of $P \times_\rho F$ as equivalence classes $[p, x]$ of elements $(p, x) \in P \times F$ under the action of G. The projection map $\pi'\colon P \times_\rho F \to \Sigma$ is given by $\pi'([p, x]) = \pi(p)$, which can be checked to be well-defined i.e. independent of the choice of representative of the class $[p, x]$, which is a consequence of the fact that $\pi\colon P \to \Sigma$ is a principal G-bundle.

We are now ready to define the Gauge group of a principal G-bundle.

Definition 2.4. Let $\pi\colon P \to \Sigma$ be a principal G-bundle, and let

$$\mathrm{Ad}(P) = P \times_G G$$

denote the associated bundle for the representation $\rho\colon G \to \mathrm{Diff}(G)$ given by conjugation (which is no longer a principal G-bundle), i.e. for all $g \in G$ we have that $\rho(g)h = ghg^{-1}$ for every $h \in G$. The *Gauge group* of

the principal G-bundle P is the group
$$\mathcal{G}(P) = \Gamma \mathrm{Ad}(P)$$
of sections of $\mathrm{Ad}(P)$, i.e. the group of (smooth) maps $f\colon P \to G$ satisfying
$$f(pg) = g^{-1} f(p) g,$$
with group operation given by point-wise multiplication.

To see why sections of $\mathrm{Ad}(P)$ can be viewed as equivariant maps $f\colon P \to G$ satisfying $f(pg) = g^{-1}f(p)g$, recall that the fiber of any $x \in \Sigma$ is given by the set of all $[p, h] \in \mathrm{Ad}(P)$ where $h \in G$ and p is an arbitrary point in the fiber of x in the principal bundle $\pi\colon P \to \Sigma$. Indeed, this is because the fiber $\pi^{-1}(x) = \{pg : g \in G\}$ as discussed before, and the points (p, h) and (pg, ghg^{-1}) are identified in $\mathrm{Ad}(P)$. Hence, since a section σ of $\mathrm{Ad}(P)$ must satisfy $\pi' \circ \sigma = \mathrm{id}$ where π' is the associated bundle projection defined earlier, it follows that for each $x \in \Sigma$, the first component of $\sigma(x) \in \mathrm{Ad}(P)$ is already determined (i.e. it is any element of $\pi^{-1}(x)$), and the second component can be any element of G, and thus σ is identified with a map $f\colon P \to G$. However, for this to be independent of the choice of representatives of classes in $\mathrm{Ad}(P)$, we require that $f(pg) = g^{-1}f(p)g$ for all $p \in P$ and $g \in G$ as $[p, h] = [pg, g^{-1}hg] \in \mathrm{Ad}(P)$.

Observe that the Gauge group $\mathcal{G}(P)$ can be identified with the group of equivariant automorphisms of P that cover the identity map $\mathrm{id}\colon \Sigma \to \Sigma$, i.e. the group of automorphisms $\psi\colon P \to P$ satisfying $\psi(pg) = \psi(p)g$ for all $g \in G$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d, "\pi"'] & P \arrow[d, "\pi"] \\ \Sigma \arrow[r, "\mathrm{id}"'] & \Sigma \end{tikzcd}$$

commutes. Indeed, suppose that $f\colon P \to G$ represents an element of $\mathcal{G}(P)$. Then, define the map $\psi\colon P \to P$ by for every $p \in P$, we have
$$\psi(p) = pf(p).$$
Notice that the condition $\pi \circ \psi = \mathrm{id} \circ \pi$ is satisfied since for every $p \in P$ we have that p and $\psi(p)$ are in the same fiber of the principal G-bundle $\pi\colon P \to \Sigma$ as the right-action of G on P is fiber-preserving by definition. Also, notice that ψ is G-equivariant. Indeed, letting $g \in G$ be arbitrary, we have that
$$\begin{aligned} \psi(pg) &= pgf(pg) \quad \text{(by definition)} \\ &= pgg^{-1}f(p)g \quad \text{(as } f(pg) = g^{-1}f(p)g \text{ for all } p \in P \text{ and } g \in G) \\ &= pf(p)g \\ &= \psi(p)g. \end{aligned}$$

Finally, note that ψ is an automorphism since the action of G is transitive on each of the fibers and $\psi(pg) = \psi(p)g$ for all $p \in P$ and $g \in G$. Conversely, suppose that $\psi\colon P \to P$ is an equivariant automorphism of P satisfying $\pi \circ \psi = \mathrm{id} \circ \pi$. So, in particular for each $p \in P$ we have that p and $\psi(p)$ are in the same fiber of the principal G-bundle. Thus, since the action of G on P is transitive on the fibers, for each $p \in P$ there is a $g_p \in G$ such that
$$\psi(p) = pg_p.$$
Now, define the map $f\colon P \to G$ by $f(p) = p_g$ so that for all $p \in P$ we have
$$\psi(p) = pf(p).$$
Then, by a similar calculation to before, we must have that $f(p) = g^{-1}pg$ for all $p \in P$ and $g \in G$ since $\psi(pg) = \psi(p)g$ for all $p \in P$ and $g \in G$. Thus, we have that $f\colon P \to G$ represents an element of the Gauge group $\mathcal{G}(P)$.

Finally, before we move onto the next section, we present the notion of the so-called Lie algebra of the Gauge group, noting that the Gauge group $\mathcal{G}(P)$ is an infinite-dimensional Lie group.

Definition 2.5. Let $\pi \colon P \to \Sigma$ be a principal G-bundle, and let

$$\mathrm{ad}(P) = P \times_{ad} \mathfrak{g}$$

denote the associated bundle for the adjoint action of G on its Lie algebra $\mathfrak{g}$. Then, we have that the *Lie algebra of the Gauge group* is given by

$$\mathfrak{g}(P) = \Gamma \mathrm{ad}(P),$$

where the Lie algebra structure comes from the fibers of $\mathrm{ad}(P)$ (which are each copies of $\mathfrak{g}$).

Note that Definition 2.5 should also technically be viewed as a proposition, as $\Gamma\mathrm{ad}(P)$ is indeed the Lie algebra of the Lie group $\mathcal{G}(P) = \Gamma\mathrm{Ad}(P)$.

3. Connections and Curvature for Principal Bundles

In this section, we will first discuss the notion of a connection on a principal G-bundle. Before, we do this however, we will give the definition of a connection (often called an Ehresmann connection) on a smooth fiber bundle. First, note that if $\pi \colon E \to B$ is a smooth fiber bundle, then the tangent bundle TE of the smooth manifold E contains a canonical subbundle

$$V = \ker(d\pi \colon TE \to TB)$$

called the vertical subbundle, which can be viewed as the tangent bundle along the fibers of $\pi \colon E \to B$. Here, $d\pi \colon TE \to TB$ is the induced bundle map by the differentials $d\pi_x \colon T_xE \to T_{\pi(x)}B$ at each $x \in E$, so that the diagram

$$\begin{tikzcd} TE \arrow[r, "d\pi"] \arrow[d, "\pi_1"'] & TB \arrow[d, "\pi_2"] \\ E \arrow[r, "\pi"'] & B \end{tikzcd}$$

commutes, where $\pi_1 \colon TE \to E$ and $\pi_2 \colon TB \to B$ are the tangent bundles of E and B respectively.

Definition 3.1. A *connection* on a smooth fiber bundle $\pi \colon E \to B$ is a smooth subbundle H of the tangent bundle TE (called the horizontal subbundle of the connection) such that

$$TE = V \oplus H,$$

where as before V is the vertical subbundle.

More precisely, to make sense of the above direct sum decomposition, the horizontal subbundle H satisfies $T_eE = V_e \oplus H_e$ as vector spaces for each $e \in E$, where $V_e \cap H_e = \{0\}$, noting that the fibers H_e depend smoothly on e where H_e is a subspace of the tangent space T_eE for each $e \in E$. Now, note that we can in fact view a connection on $\pi \colon E \to B$ as a V-valued 1-form (called the connection 1-form) i.e. as some element $A \in \Omega^1(E, V)$, recalling that a 1-form in $\Omega^1(E, V)$ is a map $TE \to V$. Indeed, we can identify H with the kernel of the projection map $\omega \colon TE \to V$, noting that this projection is determined by a direct sum decomposition of TE into vertical and horizontal subbundles. Then, ω of course satisfies $\omega^2 = \omega$ and $\omega|_V = \mathrm{id}_V$. Thus, the space of connections on a smooth fiber bundle $\pi \colon E \to B$ can just be viewed as the space of V-valued 1-forms in $\Omega^1(E, V)$ satisfying these two conditions, as we conversely have that the kernel of any bundle map $TE \to V$ satisfying these two properties is a horizontal subbundle of a connection on $\pi \colon E \to B$.

Now, using this discussion, it is not hard to generalise this definition to the notion of a connection on a principal G-bundle $\pi \colon P \to \Sigma$.

Definition 3.2. A *connection* on a principal G-bundle $\pi \colon P \to \Sigma$ is an equivariant horizontal subbundle H of TP such that

$$TP = V \oplus H,$$

where V is the vertical subbundle defined by $V = \ker(d\pi \colon TP \to T\Sigma)$.

Here, H being equivariant means that the fibers satisfy $H_{pg} = \mathrm{d}(R_g)_p(H_p)$ for all $p \in P$ and $g \in G$, where $R_g\colon P \to P$ is given by the right-action by g, i.e. $R_g(p) = pg$ for all $p \in P$, and $\mathrm{d}(R_g)_p$ is the differential of R_g at $p \in P$ (i.e. $\mathrm{d}(R_g)_p\colon T_pP \to T_{pg}P$), noting that indeed H_p is a subspace of T_pP as discussed earlier.

Now, note that we can identify the vertical subbundle V with $P \times \mathfrak{g}$ as follows. Given the smooth right G-action $P \times G \to P$ in the principal G-bundle $\pi\colon P \to \Sigma$, we have that the infinitesimal behaviour of this G-action on P is given by the so-called *fundamental vector fields*. More specifically, we have an infinitesimal right-action of the Lie algebra $\mathfrak{g}$ on P (where $\mathfrak{g}$ can be viewed as T_eG where e is the identity element of the Lie group G) given by for all $p \in P$ and $\xi \in \mathfrak{g}$,

$$p\xi = \frac{\mathrm{d}}{dt}\bigg|_{t=0} p\exp(t\xi),$$

which is an element of the tangent space T_pP, where as usual $\exp\colon \mathfrak{g} \to G$ is the exponential map sending ξ to $\gamma(1)$, where $\gamma\colon \mathbb{R} \to G$ is the unique one-parameter subgroup with tangent vector ξ at the identity $e \in G$, noting that $\exp(t\xi) = \gamma(t)$ for all $t \in \mathbb{R}$. Now, the vertical subbundle V is then simply given by

$$V = \ker(d\pi\colon TP \to T\Sigma) = \{(p, p\xi) : p \in P, \xi \in \mathfrak{g}\} \subset TP.$$

Indeed, by the definition of a principal G-bundle, we have that $\pi(pg) = p$ for all $p \in P$ and $g \in G$, so it follows directly from the definition of $d\pi\colon TP \to T\Sigma$ that for each $p \in P$, the elements of T_pP mapping to the zero tangent vector in $T_{\pi(p)}\Sigma$ are precisely all elements in the orbit of the infinitesimal action of $\mathfrak{g}$ at p. Using this identification of V with $P \times \mathfrak{g}$, we can in fact view a connection on $\pi\colon P \to \Sigma$ as a $\mathfrak{g}$-valued 1-form (called the connection 1-form) i.e. as some element $A \in \Omega^1(P, \mathfrak{g})$. Indeed, as before, we first identify the horizontal subbundle H with the projection map $\Pi\colon TP \to V$ (where $TP = V \oplus H$) which is determined by a direct sum decomposition of TP into V and a horizontal subbundle. Now, we define a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ via the equation

$$\Pi_p(\hat{p}) = pA_p(\hat{p})$$

for all $\hat{p} \in T_pP$, noting that for all $p \in P$ we have that $\Pi_p\colon T_pP \to V_p$ maps every $\hat{p} \in T_pP$ to $p\xi \in V_p$ for some $\xi \in \mathfrak{g}$ by the above characterisation of V. So, we indeed get a well-defined map $A\colon TP \to \mathfrak{g}$ i.e. a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ as each A_p maps T_pP into $\mathfrak{g}$. Now, observe that the 1-form $A \in \Omega^1(P, \mathfrak{g})$ satisfies the two relations

$$A_p(p\xi) = \xi \text{ and } A_{pg}(\hat{p}g) = g^{-1}A_p(\hat{p})g$$

for all $p \in P$, $\xi \in \mathfrak{g}$, $\hat{p} \in T_pP$ and $g \in G$. Here, when we right ξg for example where $\xi \in \mathfrak{g}$ and $g \in G$, we mean the image of ξ under the differential at the identity e of the right-multiplication map $G \to G$ given by multiplying on the right by g. Now, the first relation follows since if $\hat{p} = p\xi$ for some $\xi \in \mathfrak{g}$, then $\hat{p} \in V_p$, so (since Π_p is the projection of T_pP onto V_p)

$$\Pi_p(\hat{p}) = \hat{p} = p\xi,$$

and thus $A_p(\hat{p}) = A_p(p\xi) = \xi$. Next, the second relation holds since

$$\Pi_{pg}(\hat{p}g) = pgA_{pg}(\hat{p}g),$$

but by the equivariance condition in the definition of a connection on a principal G-bundle, we also have that

$$\Pi_{pg}(\hat{p}g) = \Pi_p(\hat{p})g = pA_p(\hat{p})g.$$

Thus, it follows that

$$pgA_{pg}(\hat{p}g) = pA_p(\hat{p})g,$$

and so

$$A_{pg}(\hat{p}g) = g^{-1}A_p(\hat{p})g,$$

which is precisely the second relation. Note here that for every $\hat{p} \in T_pP$ and $g \in G$, we have that $\hat{p}g = d(R_g)_p(\hat{p}) \in T_{pg}P$. Thus, we can view the space of connections on $\pi\colon P \to \Sigma$ as the space of $\mathfrak{g}$-valued 1-forms in $\Omega^1(P, \mathfrak{g})$ that satisfy these two relations, as we conversely have that every $\mathfrak{g}$-valued 1-form in $\Omega^1(P, \mathfrak{g})$ satisfying the above two relations has kernel an equivariant horizontal subbundle H of TP such that $TP = V \oplus H$. We will denote by $\mathcal{A}(P)$ the space of connections on the principal G-bundle $\pi\colon P \to \Sigma$, viewed as

$$\mathcal{A}(P) = \{A \in \Omega^1(P, \mathfrak{g}) : A_p(p\xi) = \xi \text{ and } A_{pg}(\hat{p}g) = g^{-1}A_p(\hat{p})g \text{ for all } p \in P, \xi \in \mathfrak{g}, \hat{p} \in T_pP \text{ and } g \in G\}.$$

Now, we have the following proposition.

Proposition 3.3. *The space of connections $\mathcal{A}(P)$ on a principal G-bundle $\pi\colon P \to \Sigma$ is an affine space with underlying linear space $\Omega^1(\Sigma, ad(P))$. That is, for all $A, A' \in \mathcal{A}(P)$, we have that $a := A - A'$ defines an $ad(P)$-valued 1-form in $\Omega^1(\Sigma, ad(P))$.*

Proof. Since $A, A' \in \mathcal{A}(P)$, we have that

$$\begin{aligned} a_p(p\xi) &= A_p(p\xi) - A'_p(p\xi) \\ &= p\xi - p\xi \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} a_{pg}(\hat{p}g) &= A_{pg}(\hat{p}g) - A'_{pg}(\hat{p}g) \\ &= g^{-1}A_p(\hat{p})g - g^{-1}A'_p(\hat{p})g \\ &= g^{-1}(A_p(\hat{p}) - A'_p(\hat{p}))g \\ &= g^{-1}a_p(\hat{p})g \end{aligned}$$

for all $p \in P$, $\xi \in \mathfrak{g}$, $\hat{p} \in T_pP$ and $g \in G$. With these two conditions, we say that a is a horizontal (i.e. it vanishes on sections of the vertical subbundle) and equivariant 1-form on P. Now, we define an $\mathrm{ad}(P)$-valued 1-form $\tilde{a} \in \Omega^1(\Sigma, \mathrm{ad}(P))$ by the equation

$$\tilde{a}_{\pi(p)}(d\pi_p(\hat{p})) = [p, a_p(\hat{p})]$$

for all $p \in P$ and $\hat{p} \in T_pP$. Here, $d\pi_p\colon T_pP \to T_{\pi(p)}\Sigma$ is the differential of $\pi\colon P \to \Sigma$ at $p \in P$. Note that since π is a smooth surjective submersion, we can view any element of Σ as $\pi(p)$ for some $p \in P$ and any element of $T_{\pi(p)}\Sigma$ as $d\pi_p(\hat{p})$ for some $\hat{p} \in T_pP$. Now, since we know as discussed before that $d\pi_p(p\xi) = 0$ for all $p \in P$ and $\xi \in \mathfrak{g}$, we require that $\tilde{a}_{\pi(p)}(d\pi_p(p\xi)) = [p, 0]$, but this is satisfied by the equation since we showed that $a_p(p\xi) = 0$ for all $p \in P$ and $\xi \in \mathfrak{g}$. Also, since $\pi(pg) = \pi(p)$ for all $p \in P$ and $g \in G$ by the definition of an principal G-bundle, we require that

$$\tilde{a}_{\pi(p)}(d\pi_p(\hat{p})) = \tilde{a}_{\pi(pg)}(d\pi_{pg}(\hat{p}g)),$$

i.e. that

$$[p, a_p(\hat{p})] = [pg, a_{pg}(\hat{p}g)]$$

for all $p \in P$, $g \in G$ and $\hat{p} \in T_pP$. Hence, by the definition of the associated bundle $\mathrm{ad}(P)$, we require that $a_{pg}(\hat{p}g) = g^{-1}a_p(\hat{p})g$, where $ga_p(\hat{p})g^{-1}$ denotes the adjoint action of G on its Lie algebra $\mathfrak{g}$. However, this is satisfied as we showed before that $a_{pg}(\hat{p}g) = g^{-1}a_p(\hat{p})g$ for all $p \in P$, $g \in G$ and $\hat{p} \in T_pP$. Hence, we have that $\tilde{a}$ is a well-defined $\mathrm{ad}(P)$-valued 1-form in $\Omega^1(\Sigma, \mathrm{ad}(P))$. Conversely, via pullback (looking at the differential of $\pi\colon P \to \Sigma$), we have that any $\mathrm{ad}(P)$-valued 1-form in $\Omega^1(\Sigma, \mathrm{ad}(P))$ defines an equivariant horizontal $\mathfrak{g}$-valued 1-form in $\Omega^1(P, \mathfrak{g})$. Hence, since $A, A' \in \mathcal{A}(P)$ were arbitrary, it follows that $\mathcal{A}(P)$ is an affine space with underlying linear space $\Omega^1(\Sigma, \mathrm{ad}(P))$, and thus in particular we can write $\mathcal{A}(P)$ as

$$\mathcal{A}(P) = \{A_0 + a : a \in \Omega^1(\Sigma, \mathrm{ad}(P))\},$$

where $A_0 \in \mathcal{A}(P)$ is a fixed reference connection. ■

Finally, before we move on to the Yang-Mills equations in the next section, we need the notion of the curvature F_A of a connection $A \in \mathcal{A}(P)$.

Definition 3.4. Let $A \in \mathcal{A}(P)$ be a connection on the principal G-bundle $\pi\colon P \to \Sigma$, viewed as a $\mathfrak{g}$-valued 1-form in $\Omega^1(P, \mathfrak{g})$. Then, the *curvature* of A is the $\mathfrak{g}$-valued 2-form $F_A \in \Omega^2(P, \mathfrak{g})$ defined by

$$F_A = dA + \frac{1}{2}[A \wedge A].$$

Here, d is the usual exterior derivative mapping $\mathfrak{g}$-valued k-forms to $\mathfrak{g}$-valued $(k+1)$-forms on P for each $k \geq 0$, and $[A \wedge A]$ is given by the usual formula for the wedge product but with multiplication replaced with the Lie bracket operation on $\mathfrak{g}$. That is, $[A \wedge A]$ is given by

$$[A \wedge A](\hat{p}, \hat{q}) = \frac{1}{2}([A(\hat{p}), A(\hat{q})] - [A(\hat{q}), A(\hat{p})]) = [A(\hat{p}), A(\hat{q})]$$

for tangent vectors $\hat{p}$ and $\hat{q}$ in T_pP. The addition and scalar multiplication in the definition of F_A is of course given point-wise. Now, using the notation discussed before, one can show that the curvature 2-form F_A of any connection $A \in \mathcal{A}(P)$ is an equivariant and horizontal 2-form, so by a similar argument to how we showed that $\mathcal{A}(P)$ is an affine space, we can equivalently view the curvature as an $\mathrm{ad}(P)$-valued 2-form

$$F_A \in \Omega^2(\Sigma, \mathrm{ad}(P)).$$

We will often move back and forth between viewing the curvature F_A as an element of $\Omega^2(\Sigma, \mathrm{ad}(P))$ and as an element of $\Omega^2(P, \mathfrak{g})$.

4. The Yang-Mills Equations

In this section, we will assume that G is a compact connected Lie group, and we again fix a principal G-bundle $\pi \colon P \to \Sigma$ where Σ is a closed connected Riemann surface. Before we begin, we first make the following definition.

Definition 4.1. Suppose that $\zeta \in \Omega^k(\Sigma)$ is a k-form. Then, the *Hodge dual* of ζ is defined to be the unique $(2-k)$-form $\star\zeta$ such that

$$\eta \wedge \star\zeta = \langle \eta, \zeta \rangle_\Sigma \, \omega$$

for all k-forms $\eta \in \Omega^k(\Sigma)$, where $\langle \cdot, \cdot \rangle_\Sigma$ is the inner product on forms induced by the Riemannian metric on Σ (recalling that the Riemannian metric is a family on positive-definite inner products on the tangent spaces of Σ, and we fix such a Riemannian metric on Σ), and ω is the volume form (which is a 2-form satisfying $\omega = \star 1$), which is also induced by the Riemannian metric on Σ, and we take it to have length 1, i.e. so that the integral of ω over Σ is equal to 1. The map defined by $\star$ is called the *Hodge star*.

Here, $1 \in \Omega^0(\Sigma)$ is the 1-form that maps everything in $T\Sigma$ to $1 \in \mathbb{R}$. Recall that we assume Σ to be a closed connected Riemann surface, so it does indeed come with a Riemannian metric which we fix. Also, noting that Σ being a Riemann surface means that Σ is an oriented manifold of real dimension 2, we have that the above definition generalises to Riemannian (or pseudo-Riemannian) manifolds of dimension n, where the Hodge dual $\star\zeta$ of a k-form ζ is instead an $(n-k)$-form.

Now, using this definition, we will define an L^2-inner product on $\Omega^*(\Sigma, \mathrm{ad}(P))$. Since G is a compact Lie group, we know that its Lie algebra $\mathfrak{g}$ admits an Ad-invariant (i.e. invariant under the adjoint action of G on $\mathfrak{g}$) inner product $\langle \cdot, \cdot \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. The inner product being Ad-invariant means that

$$\langle \mathrm{Ad}_g(\xi_1), \mathrm{Ad}_g(\xi_2) \rangle = \langle \xi_1, \xi_2 \rangle$$

for all $g \in G$ and $\xi_1, \xi_2 \in \mathfrak{g}$, where again for each $g \in G$ we have that Ad_g is the differential at the identity $e \in G$ of the conjugation by g map $G \to G$ given by $h \mapsto ghg^{-1}$ for all $h \in G$.

Since the fibers of $\mathrm{ad}(P)$ are simply copies of $\mathfrak{g}$ with the adjoint action, it follows that this Ad-invariant inner product $\langle \cdot, \cdot \rangle$ combined with the exterior multiplication $\wedge$ on forms then gives a multiplication

$$\Omega^k(\Sigma, \mathrm{ad}(P)) \otimes \Omega^\ell(\Sigma, \mathrm{ad}(P)) \to \Omega^{k+\ell}(\Sigma),$$

which we will denote by $\langle \zeta \wedge \eta \rangle$ where $\zeta \in \Omega^k(\Sigma, \mathrm{ad}(P))$ and $\eta \in \Omega^\ell(\Sigma, \mathrm{ad}(P))$. That is, ζ is a section of $\bigwedge^k T^*\Sigma \otimes \mathrm{ad}(P)$, η is a section of $\bigwedge^\ell T^*\Sigma \otimes \mathrm{ad}(P)$ and $\langle \zeta \wedge \eta \rangle$ is a section of $\bigwedge^{k+\ell} T^*\Sigma$. Indeed, we have that $\zeta \wedge \eta \in \Omega^{k+\ell}(\Sigma, \mathrm{ad}(P) \otimes \mathrm{ad}(P))$, but in a local trivialisation it takes values in $\mathfrak{g} \otimes \mathfrak{g}$ and this is where we apply the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Now, recall that since Σ is an oriented 2-dimensional real manifold, we are able to integrate 2-forms in $\Omega^2(\Sigma)$ over Σ. Thus, we get an inner product $(\cdot, \cdot)$ on $\Omega^*(\Sigma, \mathrm{ad}(P))$ defined by for all $\mathrm{ad}(P)$-valued k-forms $\zeta, \eta \in \Omega^k(\Sigma, \mathrm{ad}(P))$, we have

$$(\zeta, \eta) = \int_\Sigma \langle \zeta \wedge \star\eta \rangle.$$

Note that the Hodge star on $\Omega^*(\Sigma, \mathrm{ad}(P))$ is defined in exactly the same way as on $\Omega^*(\Sigma)$, so that $\star\eta$ is now an $\mathrm{ad}(P)$-valued $(2-k)$-form, meaning then that $\langle \zeta \wedge \star\eta \rangle$ is a 2-form in $\Omega^2(\Sigma)$, which can therefore be integrated over the oriented 2-manifold Σ. Now, building towards the Yang-Mills equations, we first make the following definition.

Definition 4.2. The *Yang-Mills functional* is the map $L\colon \mathcal{A}(P) \to \mathbb{R}$ defined by

$$L(A) = \|F_A\|^2 = \int_\Sigma \langle F_A \wedge \star F_A \rangle,$$

where $A \in \mathcal{A}(P)$ and F_A is viewed as an $\mathrm{ad}(P)$-valued 2-form in $\Omega^2(\Sigma, \mathrm{ad}(P))$. A *Yang-Mills connection* is a connection $A \in \mathcal{A}(P)$ that is a stationary point for the Yang-Mills functional L.

Here $\|\cdot\|$ is the L^2 norm on $\Omega^*(\Sigma, \mathrm{ad}(P))$ defined by

$$\|\zeta\| = \sqrt{(\zeta, \zeta)} = \sqrt{\int_\Sigma \langle \zeta \wedge \star\zeta \rangle},$$

where $\zeta \in \Omega^k(\Sigma, \mathrm{ad}(P))$. Now, before we derive the Yang-Mills equations (whose solutions are the Yang-Mills connections), note that a connection $A \in \mathcal{A}(P)$ induces a covariant derivative on the associated bundle $\mathrm{ad}(P)$. Indeed, by similar reasoning to what we've discussed before, we have that $\mathrm{ad}(P)$-valued k-forms in $\Omega^k(\Sigma, \mathrm{ad}(P))$ can be viewed (via pullback) as G-equivariant horizontal $\mathfrak{g}$-valued k-forms in $\Omega^k(P, \mathfrak{g})$. Now, if $\zeta \in \Omega^k(P, \mathfrak{g})$ is an equivariant horizontal $\mathfrak{g}$-valued k-form, then we have that

$$d_A(\zeta) := d\zeta + [A \wedge \zeta] \in \Omega^{k+1}(P, \mathfrak{g})$$

is an equivariant horizontal $\mathfrak{g}$-valued $(k+1)$-form (note the similarity to the definition of the curvature $F_A \in \Omega^2(P, \mathfrak{g})$), and can therefore be viewed as an element of $\Omega^{k+1}(\Sigma, \mathrm{ad}(P))$. Therefore, the connection $A \in \mathcal{A}(P)$ defines the exterior covariant derivative

$$d_A\colon \Omega^k(\Sigma, \mathrm{ad}(P)) \to \Omega^{k+1}(\Sigma, \mathrm{ad}(P)),$$

which in the case $k = 0$ corresponds to a covariant derivative on $\mathrm{ad}(P)$ (i.e. a way of differentiating sections of $\mathrm{ad}(P)$). Using this, we will finally derive the Yang-Mills equations (in the 2-dimensional case as Σ is a Riemann surface), i.e. we will derive equations whose solutions are precisely the Yang-Mills connections in $\mathcal{A}(P)$.

Lemma 4.3. *For $t \in \mathbb{R}$, consider the connection $A_t \in \mathcal{A}(P)$ given by*

$$A_t = A + ta,$$

where $A \in \mathcal{A}(P)$ and $a \in \Omega^1(\Sigma, ad(P))$ are arbitrary. Then, the curvature F_{A_t} is given by

$$F_{A_t} = F_A + t d_A a + \frac{1}{2}t^2[a \wedge a].$$

Proof. By the definition of the curvature (Definition 3.4), we have that

$$F_{A_t} = dA_t + \frac{1}{2}[A_t \wedge A_t].$$

However, since $A_t = A + ta$, we have that (recalling that $a \in \Omega^1(\Sigma, \mathrm{ad}(P))$ can be viewed as a G-equivariant horizontal $\mathfrak{g}$-valued 1-form in $\Omega^1(P, \mathfrak{g})$)

$$\begin{aligned}
dA_t + \frac{1}{2}[A_t \wedge A_t] &= d(A + ta) + \frac{1}{2}[(A + ta) \wedge (A + ta)] \\
&= dA + tda + \frac{1}{2}([A \wedge A] + [A \wedge ta] + [ta \wedge A] + [ta \wedge ta]) \\
&= \left(dA + \frac{1}{2}[A \wedge A]\right) + t(da + [A \wedge a]) + \frac{1}{2}t^2[a \wedge a] \\
&= F_A + t d_A a + \frac{1}{2}t^2[a \wedge a],
\end{aligned}$$

where the last equality follows by the definition of the curvature F_A of A and the definition of the exterior covariant derivative d_A induced by A. Thus, we have that

$$F_{A_t} = F_A + t d_A a + \frac{1}{2} t^2 [a \wedge a],$$

as required. ■

Note that in the proof Lemma 4.3, we used that if $\zeta, \eta \in \Omega^1(P, \mathfrak{g})$, then $[\zeta \wedge \eta] = [\eta \wedge \zeta]$. To see this, note that for tangent vectors $\hat{p}$ and $\hat{q}$, we have by definition that

$$\begin{aligned}
[\zeta \wedge \eta](\hat{p}, \hat{q}) &= \frac{1}{2}([\zeta(\hat{p}), \eta(\hat{q})] - [\zeta(\hat{q}), \eta(\hat{p})]) \\
&= \frac{1}{2}(-[\eta(\hat{q}), \zeta(\hat{p})] + [\eta(\hat{p}), \zeta(\hat{q})]) \qquad \text{(as the Lie bracket } [\cdot,\cdot] \text{ on } \mathfrak{g} \text{ is anti-symmetric)} \\
&= \frac{1}{2}([\eta(\hat{p}), \zeta(\hat{q})] - [\eta(\hat{q}), \zeta(\hat{p})]) \\
&= [\eta \wedge \zeta](\hat{p}, \hat{q}).
\end{aligned}$$

Thus, since $\hat{p}$ and $\hat{q}$ were arbitrary tangent vectors, it follows that $[\zeta \wedge \eta] = [\eta \wedge \zeta] \in \Omega^2(P, \mathfrak{g})$. Also, we used in the proof of Lemma 4.3 the fact that if $t_1, t_2 \in \mathbb{R}$ and $\zeta, \eta \in \Omega^1(P, \mathfrak{g})$, then $[t_1\zeta \wedge t_2\eta] = t_1 t_2 [\zeta \wedge \eta]$. To see this, note that for tangent vectors $\hat{p}$ and $\hat{q}$, we have by definition that

$$\begin{aligned}
[t_1\zeta \wedge t_2\eta](\hat{p}, \hat{q}) &= \frac{1}{2}([t_1\zeta(\hat{p}), t_2\eta(\hat{q})] - [t_1\zeta(\hat{q}), t_2\eta(\hat{p})]) \\
&= \frac{1}{2}(t_1 t_2 [\zeta(\hat{p}), \eta(\hat{q})] - t_1 t_2 [\zeta(\hat{q}), \eta(\hat{p})]) \qquad \text{(as the Lie bracket } [\cdot,\cdot] \text{ on } \mathfrak{g} \text{ is bilinear)} \\
&= t_1 t_2 [\zeta \wedge \eta](\hat{p}, \hat{q}).
\end{aligned}$$

Thus, since $\hat{p}$ and $\hat{q}$ were arbitrary tangent vectors, it follows that $[t_1\zeta \wedge t_2\eta] = t_1 t_2 [\zeta \wedge \eta] \in \Omega^2(P, \mathfrak{g})$. Note that we have in general used that $[\cdot \wedge \cdot]$ is bilinear. We will need one more lemma before deriving the Yang-Mills equations.

Lemma 4.4. *Let $A \in \mathcal{A}(P)$, and let $d_A^* \colon \Omega^{k+1}(\Sigma, ad(P)) \to \Omega^k(\Sigma, ad(P))$ denote the formal adjoint of $d_A \colon \Omega^k(\Sigma, ad(P)) \to \Omega^{k+1}(\Sigma, ad(P))$ satisfying*

$$\int_\Sigma \langle d_A \zeta \wedge \star\eta \rangle = \int_\Sigma \langle \zeta \wedge \star d_A^* \eta \rangle,$$

i.e.

$$(d_A \zeta, \eta) = (\zeta, d_A^* \eta),$$

for all $\zeta \in \Omega^k(\Sigma, ad(P))$ and $\eta \in \Omega^{k+1}(\Sigma, ad(P))$. Then, d_A^* is given by

$$d_A^* = - \star d_A \star$$

on $\Omega^k(\Sigma, ad(P))$.

Proof. Let $\zeta \in \Omega^k(\Sigma, \mathrm{ad}(P))$ and $\eta \in \Omega^{k+1}(\Sigma, \mathrm{ad}(P))$ be arbitrary. Then, we have that

$$d\langle \zeta \wedge \star\eta \rangle = \langle d_A \zeta \wedge \star\eta \rangle + (-1)^k \langle \zeta \wedge d_A \star \eta \rangle,$$

which is an identity that we will not prove in this report, where d is the exterior derivative on $\Omega^*(\Sigma)$. Then, by integrating both sides, we have that

$$\begin{aligned}
\int_\Sigma d\langle \zeta \wedge \star\eta \rangle &= \int_\Sigma \langle d_A \zeta \wedge \star\eta \rangle + (-1)^k \int_\Sigma \langle \zeta \wedge d_A \star \eta \rangle \\
&= (d_A \zeta, \eta) + (-1)^k (\zeta, \star^{-1} d_A \star \eta).
\end{aligned}$$

But

$$\int_\Sigma d\langle \zeta \wedge \star\eta\rangle = 0,$$

so

$$(d_A\zeta, \eta) = (-1)^{k+1}(\zeta, \star^{-1} d_A \star \eta).$$

However, we know that the inverse of the Hodge star $\star^{-1}$ is given by

$$\star^{-1} = (-1)^{k(2-k)}\star,$$

as Σ is 2-dimensional. Hence, by the definition of the adjoint d_A^* (noting that $\zeta \in \Omega^k(\Sigma, \mathrm{ad}(P))$ and $\eta \in \Omega^{k+1}(\Sigma, \mathrm{ad}(P))$ were arbitrary), it follows that

$$\begin{aligned} d_A^* &= (-1)^{k+1}(-1)^{k(2-k)} \star d_A \star \\ &= (-1)^{k+1+2k-k^2} \star d_A \star \\ &= (-1)^{k(1-k)+2k+1} \star d_A \star \\ &= - \star d_A \star . \end{aligned}$$

■

We are finally ready to derive the (2-dimensional) Yang-Mills equations.

Proposition 4.5. *A connection $A \in \mathcal{A}(P)$ is a stationary point for the Yang-Mills functional L (i.e. a Yang-Mills connection) if and only if*

$$d_A \star F = 0.$$

Proof. First, recall that the Yang-Mills functional is given by $L(A) = \|F_A\|^2 = (F_A, F_A)$, so we want to determine the stationary points of the L^2 inner product (F_A, F_A) as A varies over $\mathcal{A}(P)$. To do this, consider the line of connections $A_t = A + ta$ where $A \in \mathcal{A}(P)$ and $a \in \Omega^1(\Sigma, \mathrm{ad}(P))$ are arbitrary. Then, by Lemma 4.3, we have that

$$F_{A_t} = F_A + t d_A a + \frac{1}{2}t^2[a \wedge a],$$

and so

$$\begin{aligned} (F_{A_t}, F_{A_t}) &= \left(F_A + t d_A a + \frac{1}{2}t^2[a \wedge a], F_A + t d_A a + \frac{1}{2}t^2[a \wedge a]\right) \\ &= (F_A, F_A) + 2t(d_A a, F_A) + t^2((d_A a, d_A a) + (F_A, [a \wedge a])) + \text{higher order terms.} \end{aligned}$$

Now, for $A \in \mathcal{A}(P)$ to be a stationary point for L i.e. a Yang-Mills connection, we require that $(F_{A_t}, F_{A_t}) = (F_A, F_A)$ for small t, and this needs to hold for all directions of the line of connections i.e. for all $a \in \Omega^1(\Sigma, \mathrm{ad}(P))$. Therefore, by above, we have that $A \in \mathcal{A}(P)$ is a Yang-Mills connection if and only if

$$(d_A a, F_A) = 0$$

for all $a \in \Omega^1(\Sigma, \mathrm{ad}(P))$ (noting that we can ignore the terms in t^2 or higher). That is, it is a Yang-Mills connection if and only if

$$(a, d_A^* F_A) = 0$$

for all $a \in \Omega^1(\Sigma, \mathrm{ad}(P))$, and thus if and only if

$$d_A^* F_A = 0,$$

i.e.

$$\star d_A \star F_A = 0,$$

using Lemma 4.4. But using the definition of the Hodge star and its inverse, we have that $\star d_A \star F_A = 0$ if and only if $d_A \star F_A = 0$ as elements of $\Omega^1(\Sigma, \mathrm{ad}(P))$. Here, $0 \in \Omega^1(\Sigma, \mathrm{ad}(P))$ represents the element of $\Omega^1(P, \mathfrak{g})$ that sends everything to $0 \in \mathfrak{g}$ (noting that $\mathfrak{g}$ is a vector space). So, we have that $A \in \mathcal{A}(P)$ is a Yang-Mills connection if and only if

$$d_A \star F_A = 0,$$

as required. ■

Note that the Bianchi identities give that $d_A F_A = 0$ for every $A \in \mathcal{A}(P)$, so by Proposition 4.5 we have that $A \in \mathcal{A}(P)$ is a Yang-Mills connection if and only if

$$d_A F_A = 0 \text{ and } d_A \star F_A = 0,$$

and these are called the Yang-Mills equations. Note that the derivation we've just given of the Yang-Mills equations where Σ was 2-dimensional is exactly the same in higher dimensions.

5. Representations of the Fundamental Group

In this section, we will assume that our Riemann surface Σ has genus g, but we no longer fix a principal G-bundle over Σ. First, we make the following definition.

Definition 5.1. Let $\pi\colon P \to \Sigma$ be a principal G-bundle. A connection $A \in \mathcal{A}(P)$ is *flat* if its curvature $F_A \in \Omega^2(P, \mathfrak{g})$ vanishes identically, i.e.

$$dA + \frac{1}{2}[A \wedge A] = 0.$$

We let $\mathcal{A}_F(P) \subset \mathcal{A}(P)$ denote the subspace of flat connections on $\pi\colon P \to \Sigma$. Now, first observe that the Gauge group $\mathcal{G}(P)$ acts on the space of connections $\mathcal{A}(P)$ via pullback on forms. More precisely, let $A \in \mathcal{A}(P)$ be an arbitrary connection on $\pi\colon P \to \Sigma$, and let ψ be an element of the Gauge group $\mathcal{G}(P)$, which as discussed earlier is an equivariant automorphism of P that covers the identity map $\mathrm{id}\colon \Sigma \to \Sigma$. Then, the action of ψ on $\mathcal{A}(P)$ gives the element $\psi^* A \in \Omega^1(P, \mathfrak{g})$ defined by

$$(\psi^* A)_p(\hat{p}) = A_{\psi(p)}(d\psi_p(\hat{p}))$$

for all $p \in P$ and $\hat{p} \in T_pP$, where $d\psi_p\colon T_pP \to T_{\psi(p)}P$ is the differential of $\psi\colon P \to P$ at $p \in P$. The $\mathfrak{g}$-valued 1-form $\psi^* A \in \Omega^1(P, \mathfrak{g})$ is indeed a connection on $\pi\colon P \to \Sigma$ as one can check that $(\psi^* A)_p(p\xi) = \xi$ and $(\psi^* A)_{pg}(\hat{p}g) = g^{-1}(\psi^* A)_p(\hat{p})g$ for all $p \in P$, $\xi \in \mathfrak{g}$, $\hat{p} \in T_pP$ and $g \in G$. We also have that if $A \in \mathcal{A}_F(P)$ is a flat connection on $\pi\colon P \to \Sigma$ and $\psi \in \mathcal{G}(P)$, then $\psi^* A$ is also a flat connection on $\pi\colon P \to \Sigma$. Hence, the pullback action of $\mathcal{G}(P)$ on the space of connections $\mathcal{A}(P)$ descends to an action of $\mathcal{G}(P)$ on the subspace of flat connections $\mathcal{A}_F(P)$.

Now, in this section we will mainly be working with the moduli space $\mathcal{M}(\Sigma, G) = \mathcal{A}_F/\mathcal{G}$, i.e. the space of principal G-bundles $\pi\colon P \to \Sigma$ equipped with a flat connection $A \in \mathcal{A}(P)$ up to gauge equivalence (i.e. modulo the action of the Gauge group $\mathcal{G}(P)$ on A). We will show that in fact we can equivalently view the moduli space $\mathcal{M}(\Sigma, G)$ as the space of conjugacy classes of representations $\pi_1(\Sigma) \to G$ of the fundamental group $\pi_1(\Sigma)$ of our Riemann surface Σ. Here, we take $\pi_1(\Sigma) \to G$ to be a group homomorphism, and we call it a representation as we assume that the Lie group G acts on some vector space (or some Hilbert space).

First, given a principal G-bundle $\pi\colon P \to \Sigma$, note that a connection in $\mathcal{A}(P)$ provides a way of doing parallel transport on the fibers of the principal G-bundle $\pi\colon P \to \Sigma$. More precisely, first fix a connection $A \in \mathcal{A}(P)$. Then, for any smooth curve $\gamma(t)$ in Σ and any point $p \in P$ in the fiber of $\gamma(0) = x$ (i.e. so that $\pi(p) = x$), there is a unique horizontal lift $\tilde{\gamma}$ of γ through $p \in P$ for small t. That is, $\tilde{\gamma}(t)$ is a curve in P with $\tilde{\gamma}(0) = p$ such that $\pi(\tilde{\gamma}(t)) = \gamma(t)$ and each tangent of $\tilde{\gamma}$ lies in the horizontal subbundle H of TP determined by the connection A, i.e. $\tilde{\gamma}'(t) \in H_{\tilde{\gamma}(t)} \subset T_{\tilde{\gamma}(t)}P$ for all t in this small interval. Thus, since this holds for any $p \in P_{\gamma(0)}$, it follows that the connection A provides (for all curves γ in Σ) isomorphisms

$$\Gamma(\gamma)_s^t\colon P_{\gamma(s)} \to P_{\gamma(t)}$$

of the fibers over $\gamma(s)$ and $\gamma(t)$ for all t and s that are furthermore equivariant, i.e. $\Gamma(\gamma)_s^t(pg) = g\Gamma(\gamma)_s^t(p)$ for all $p \in P_{\gamma(s)}$ and $g \in G$. Now, in general the parallel transport depends on both the connection $A \in \mathcal{A}(P)$ and the curve γ in Σ. However, if $A \in \mathcal{A}_F(P)$ is a flat connection, then the parallel transport only depends on the connection and the homotopy class of γ. Furthermore, if γ is a closed curve in Σ i.e. the curve γ starts and ends at $\gamma(0)$, and $\tilde{\gamma}(0) = p$ is the fixed starting point in the fiber $P_{\gamma(0)}$ for the parallel transport given by a connection $A \in \mathcal{A}(P)$, then by the definition of a principal G-bundle we have that the endpoint of the horizontal lift $\tilde{\gamma}$ given by the parallel transport is of the form pg for some $g \in G$. If we change the starting point from p to p' say (still in the fiber of $\gamma(0)$), then the endpoint will now be of the form $p'h$ where h is given by conjugating g with some element of G. Again, if $A \in \mathcal{A}_F(P)$ is a flat connection, then the parallel transport only depends on the homotopy class of γ, i.e. the element of the fundamental group $\pi_1(\Sigma)$ representing the closed curve γ. Thus, in the case that $A \in \mathcal{A}_F(P)$ is a flat connection, we have a map (up to conjugation by elements of G) $\pi_1(\Sigma) \to G$, which we will call the *holonomy map*, noting that the map is independent of the choice of representatives of the elements of $\pi_1(\Sigma)$ since as mentioned before the parallel transport only depends on the homotopy class of the loop γ (as A is flat).

Proposition 5.2. *There is a bijective (mutually inverse) correspondence*

$$\mathcal{A}_F/\mathcal{G} \longleftrightarrow Hom(\pi_1(\Sigma), G)/G$$

between the space of flat principal G-bundles (i.e. principal G-bundles equipped with a flat connection) modulo gauge equivalence and the space of conjugacy classes of representations $\pi_1(\Sigma) \to G$.

Proof. If we are given a principal G-bundle $\pi\colon P \to \Sigma$ and a flat connection $A \in \mathcal{A}_F(P)$, then we have already discussed that the holonomy map then gives a representation $\pi_1(\Sigma) \to G$, but that this map taking $A \in \mathcal{A}_F(P)$ to $\pi_1(\Sigma) \to G$ is only defined up to conjugation by elements of G (depending on the choice of base point in P for the parallel transport). Then, we have that this holonomy map descends to a map $\mathcal{A}_F/\mathcal{G} \to \mathrm{Hom}(\pi_1(\Sigma), G)/G$.

For the other direction, suppose first that we are given a representation $\rho\colon \pi_1(\Sigma) \to G$. Now, let $q\colon \tilde{\Sigma} \to \Sigma$ be the universal cover of Σ. Fixing a basepoint $z_0 \in \tilde{\Sigma}$ with $x_0 = q(z_0)$, we can identify $\tilde{\Sigma}$ with the space of homotopy classes of paths in Σ starting at x_0. Using this identification, it is now easy to see that the fundamental group $\pi_1(\Sigma, x_0)$ acts on $\tilde{\Sigma}$ on the right by pre-composing a path starting at x_0 with a loop at x_0 representing an element of $\pi_1(\Sigma, x_0)$, with both the path and loop being defined up to homotopy. This action of Σ on $\tilde{\Sigma}$ is called monodromy. Now, we obtain a principal G-bundle over Σ as follows. Note that the fundamental group $\pi_1(\Sigma)$ acts on the product $\tilde{\Sigma} \times G$ via for all $[\gamma] \in \pi_1(\Sigma)$, we have that

$$[\gamma](x, h) := (x[\gamma], \rho([\gamma])^{-1}h)$$

for all $(x, h) \in \tilde{\Sigma} \times G$, where the action in the first component is the monodromy action discussed above. Now, we have that this action of $\pi_1(\Sigma)$ on $\tilde{\Sigma} \times G$ is proper and free, where proper means that the corresponding map $\pi_1(\Sigma) \times (\tilde{\Sigma} \times G) \to (\tilde{\Sigma} \times G) \times (\tilde{\Sigma} \times G)$ (where the first factor $\tilde{\Sigma} \times G$ in the codomain corresponds to the image of the action) is a proper map i.e. the inverse of compacts sets are compact, so that the quotient map

$$\tilde{\Sigma} \times G \to P = (\tilde{\Sigma} \times G)/\pi_1(\Sigma)$$

is a smooth submersion, and P is a smooth manifold. Then, we get that $\pi\colon P \to \Sigma$ is a principal G-bundle, where $\pi([x, h]) = q(x)$ for $[x, h] \in P$, which can be checked to be well-defined. Furthermore, the action of G on P is given by $[x, h]g = [x, hg]$, which can also be checked to be well-defined. We also obtain a flat connection on $\pi\colon P \to \Sigma$ as follows. We first have a connection on the product bundle $\tilde{\Sigma} \times G$ (over $\tilde{\Sigma}$), whereby we say that a tangent vector to $\tilde{\Sigma} \times G$ is horizontal if it has no component in the G-direction, i.e. if it is of the form $(v, 0)$ where v is a tangent vector to $\tilde{\Sigma}$. This determines the horizontal subspaces at each point in $\tilde{\Sigma} \times G$, and therefore a connection on $\tilde{\Sigma} \times G$. Now, we have that the action of $\pi_1(\Sigma)$ preserves horizontal tangent vectors (i.e. it takes a horizontal tangent vector to another horizontal tangent vector), so when we pass to the quotient $P = (\tilde{\Sigma} \times G)/\pi_1(\Sigma)$ by the action of $\pi_1(\Sigma)$, the connection on $\tilde{\Sigma} \times G$ induces a connection on P, and it turns out that this connection on P is flat. Thus, given a representation $\pi_1(\Sigma) \to G$, we have constructed a principal G-bundle $\pi\colon P \to \Sigma$ and a flat connection on it, and we have that this map $\mathrm{Hom}(\pi_1(\Sigma), G) \to \mathcal{A}_F$ descends to a map $\mathrm{Hom}(\pi_1(\Sigma), G)/G \to \mathcal{A}_F/\mathcal{G}$.

It can be checked that the maps $\mathcal{A}_F/\mathcal{G} \to \mathrm{Hom}(\pi_1(\Sigma), G)/G$ and $\mathrm{Hom}(\pi_1(\Sigma), G)/G \to \mathcal{A}_F/\mathcal{G}$ that we've constructed are inverses of each other, so we therefore have the result. ■

We will now look at an example of this moduli space $\mathcal{M}(\Sigma, G)$ for certain Σ and G, and we will primarily view this moduli space as the space of conjugacy classes of representations $\pi_1(\Sigma) \to G$. Now, note that since Σ is an oriented 2-manifold of genus g, we know that the fundamental group $\pi_1(\Sigma)$ is given through generators and relations by

$$\pi_1(\Sigma) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \;\middle|\; \prod_{j=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1 \right\rangle.$$

That is, $\pi_1(\Sigma)$ is given by $2g$-many generators with a single relation. In the following example, we will look at the case $g = 2$.

Example 5.3. Suppose that $\Sigma = S^1 \times S^1$ is the torus (i.e. an oriented 2-manifold of genus $g = 1$) and $G = SU(2)$ is the Lie group of 2×2 unitary matrices with determinant 1. We want to compute the moduli space $\mathcal{M}(S^1 \times S^1, SU(2))$ of flat principal $SU(2)$-bundles (over $S^1 \times S^1$) modulo gauge transformations, but using Proposition 5.2 we will do this by computing the space of conjugacy classes of representations $\pi_1(S^1 \times S^1) \to SU(2)$. First, by above, we have that the fundamental group of the torus is given by

$$\pi_1(S^1 \times S^1) = \langle a, b \,|\, aba^{-1}b^{-1} = 1 \rangle.$$

That is, the generators a and b of $\pi_1(S^1 \times S^1)$ commute (i.e. $ab = ba$), and so $\pi_1(S^1 \times S^1)$ is abelian. Thus, to find the space of conjugacy classes of representations $\pi_1(S^1 \times S^1) \to SU(2)$, we need to find the space of matrices $A, B \in SU(2)$ (up to conjugation by elements of $SU(2)$) such that $AB = BA$, thereby giving the representation $\pi_1(S^1 \times S^1) \to SU(2)$ whereby $a \mapsto A$ and $b \mapsto B$.

First, observe that any matrix in $M \in SU(2)$ is conjugate to a diagonal matrix $D \in SU(2)$ via a matrix in $SU(2)$, i.e. there is an $S \in SU(2)$ such that $M = SDS^{-1}$. Indeed, since $M \in SU(2)$, we have in particular that $MM^* = M^*M$ i.e. that M is normal, so by the spectral theorem there is a unitary matrix $U \in U(2)$ such that $M = UDU^{-1}$ for some diagonal matrix D which is necessarily in $SU(2)$ as

$$1 = \det M = \det U \det D \det U^{-1},$$

and U being unitary implies that $\det U = \pm 1$, so $\det D = 1$, and D is also unitary as $U(2)$ is a group. However, letting $S = iU$, we have that $S \in SU(2)$, and also that $M = SDS^{-1}$. Thus, we have that any matrix in $SU(2)$ is conjugate (by an element of $SU(2)$) to a diagonal matrix in $SU(2)$. However, we are looking at homomorphisms $\pi_1(S^1 \times S^1) \to SU(2)$ up to conjugation. Thus, up to simultaneous conjugation of (A, B) by an element of $SU(2)$, we have that the set of matrices $A, B \in SU(2)$ satisfying $AB = BA$ is just the set $T \times T$ where

$$T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in SU(2) : \theta \in [0, 2\pi) \right\},$$

which is diffeomorphic to the unit circle S^1, so that $T \times T$ is diffeomorphic to the torus $S^1 \times S^1$. Indeed, after conjugation, we can take A to be diagonal, but then (after conjugating by the same matrix in $SU(2)$ that diagonalised A) we must have that B commutes with the diagonal matrix A. If the diagonal matrix $A = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ where θ is not equal to 0, π or 2π, then the only matrices that commute with A are the diagonal matrices, so B itself must be diagonal. However, if $A = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ where θ is equal to 0, π or 2π, then we can simply diagonalise B by an element of $SU(2)$, noting that conjugating A by this element of $SU(2)$ used to diagonalise B will not change A. Finally, we note that all that is left of the conjugation by $SU(2)$ action on $T \times T$ is the element $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in SU(2)$, i.e. this is the only element of $SU(2)$ that acts on an element of $T \times T$ to give a another element of $T \times T$, and it does this by simultaneously switching the two diagonal elements of each pair of matrices $(A, B) \in T \times T$. Indeed, we have that

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^{-1} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

for all $\theta \in [0, 2\pi)$, so we can think of this element of $SU(2)$ as acting on $T \times T$ by simultaneous complex conjugation entry-wise on each matrix in a pair $(A, B) \in T \times T$. In fact, this action is precisely the simultaneous action of the Weyl group $W = N(T)/T \cong \mathbb{Z}/2$ on $T \times T$, where $N(T) = \{S \in SU(2) : STS^{-1} = T\}$ is the normaliser of $T \subset SU(2)$. So, putting this all together, we have therefore shown that the moduli space $\mathcal{M}(S^1 \times S^1, SU(2))$ of flat principal $SU(2)$-bundles is homeomorphic to

$$\frac{S^1 \times S^1}{\mathbb{Z}/2},$$

where $\mathbb{Z}/2$ acts on $S^1 \times S^1$ via simultaneous complex conjugation on both copies of S^1.

Note that if Σ is now an arbitrary oriented 2-manifold of genus g, then one can show that the moduli space $\mathcal{M}(\Sigma, SU(2))$ has dimension $6g-6$. Indeed, to see why this is true intuitively, consider the map

$$\mu \colon SU(2)^{2g} \to SU(2)$$

defined by

$$(A_1, \ldots, A_g, B_1, \ldots, B_g) \mapsto \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1}.$$

Then, since defining a representation $\pi_1(\Sigma) \to SU(2)$ means finding elements $A_1, \ldots, A_g, B_1, \ldots, B_g \in SU(2)$ satisfying the relation $\prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = 1$ (where 1 denotes the identity matrix in $SU(2)$), it follows that

$$\mathcal{M}(\Sigma, SU(2)) = \mu^{-1}(1)/SU(2),$$

as we must quotient out by the conjugation action of $SU(2)$ on the space of representations $\pi_1(\Sigma) \to SU(2)$. Now, we expect the dimension of the fiber $\mu^{-1}(1)$ to be the dimension of the domain $SU(2)^{2g}$ minus the dimension of the codomain $SU(2)$. But we know that $SU(2)$ is 3-dimensional, so we expect the dimension of $\mu^{-1}(1)$ to be

$$\dim(\mu^{-1}(1)) = 2g(3) - 3 = 6g - 3.$$

Furthermore, the dimension of the quotient $\mu^{-1}(1)/SU(2)$ is then the dimension of $\mu^{-1}(1)$ minus the dimension of $SU(2)$, so that

$$\dim(\mathcal{M}(\Sigma, SU(2))) = 6g - 3 - 3 = 6g - 6.$$

So, in particular for our above example, we have that the dimension of the moduli space $\mathcal{M}(S^1 \times S^1, SU(2))$ is $6(1) - 6 = 0$.

Note that so far in this section we have been dealing with flat connections on principal G-bundles $\pi \colon P \to \Sigma$, i.e. with connections $A \in \mathcal{A}(P)$ whose curvature 2-form $F_A = 0$ vanishes identically. That is, we have been dealing with trivial solutions to the Yang-Mills equations

$$d_A F_A = 0 \text{ and } d_A \star F_A = 0.$$

One might ask if there is a similar correspondence to the one given in Proposition 5.2 if we replace flat connections with arbitrary Yang-Mills connections. In fact, there is such a generalisation, which we will state but not prove. First, we consider a central extension

$$1 \to \mathbb{Z} \to \Gamma \to \pi_1(\Sigma) \to 1,$$

which is universal. To explain what this means, first note that the above short exact sequence of groups being central means that $\mathbb{Z}$ is contained in the center of the group Γ. Now, we have that the set of isomorphism classes of central extensions of $\pi_1(\Sigma)$ by $\mathbb{Z}$ is in bijective correspondence with the singular cohomology group $H^1(\pi_1(\Sigma), \mathbb{Z}) \cong \mathbb{Z}$, and the universal central extension is the central extension of $\pi_1(\Sigma)$ by $\mathbb{Z}$ corresponding to the generator 1 of $\mathbb{Z}$. Explicitly, this universal central extension is given by the group Γ, which is the largest quotient of the free group $\langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle$ (i.e. with the least number of relations) such that the element $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}$ in the free group is central (i.e. commutes with everything). Then, we extend this universal central extension to a central extension

$$1 \to \mathbb{R} \to \Gamma_{\mathbb{R}} \to \pi_1(\Sigma) \to 1,$$

where we have extended Γ to $\Gamma_{\mathbb{R}}$ so that now $\mathbb{R}$ is contained in the center of $\Gamma_{\mathbb{R}}$, not just $\mathbb{Z}$. However, observe that since the above short exact sequence tells us that we have an isomorphism $\pi_1(\Sigma) \cong \Gamma_{\mathbb{R}}/\mathbb{R}$, we therefore have that

$$\Gamma_{\mathbb{R}}/\mathbb{Z} \cong \Gamma_{\mathbb{R}}/\mathbb{R} \times \mathbb{R}/\mathbb{Z} \cong \pi_1(\Sigma) \times U(1),$$

noting that $U(1) = S^1 \cong \mathbb{R}/\mathbb{Z}$, and thus we have the short exact sequence

$$1 \to \mathbb{Z} \to \Gamma_{\mathbb{R}} \to U(1) \times \pi_1(\Sigma) \to 1.$$

Using this, one can show that given a representation $\Gamma_{\mathbb{R}} \to G$ of $\Gamma_{\mathbb{R}}$ (not a representation of $\pi_1(\Sigma)$ like before), we obtain a principal G-bundle equipped with a Yang-Mills connection A. This process is somewhat similar to what we did before given a homomorphism $\pi_1(\Sigma) \to G$ to then construct a flat principal G-bundle, as again this time we look at the universal cover $\tilde{\Sigma} \to \Sigma$, but we instead construct a principal $(U(1) \times \pi_1(\Sigma))$-bundle (rather than a $\pi_1(\Sigma)$-bundle like before), which we then lift to a principal $\Gamma_{\mathbb{R}}$-bundle over Σ using the above short exact sequence. Now, the bijective correspondence that is a generalisation of Proposition 5.2 from flat connections to arbitrary Yang-Mills connections is as follows.

Theorem 5.4. *There is a bijective correspondence between the space of principal G-bundles over Σ equipped with a Yang-Mills connection modulo gauge equivalence and the space of conjugacy classes of representations $\Gamma_{\mathbb{R}} \to G$ of $\Gamma_{\mathbb{R}}$.*

6. Symplectic Structure and the Moment Map

In this final section, we fix Σ to be a closed connected Riemann surface, and we again fix G to be a compact connected Lie group. Our goal in this final section is to explain how we can view the moduli space $\mathcal{M}(\Sigma, G)$ as an infinite-dimensional symplectic reduction. We will explain the various terminology from symplectic geometry as we work through showing that the moduli space is a symplectic reduction.

Let $\pi\colon P \to \Sigma$ be a principal G-bundle. Recall from Proposition 3.3 that the space of connections $\mathcal{A}(P)$ is an affine space, so that

$$\mathcal{A}(P) = \{A + a : a \in \Omega^1(\Sigma, \mathrm{ad}(P))\},$$

where $A \in \mathcal{A}(P)$ is some reference connection. We will first put a symplectic structure on $\mathcal{A}(P)$, and by this we mean that we can equip $\mathcal{A}(P)$ with a *symplectic form*, i.e. a closed 2-form that induces a non-degenerate and skew-symmetric pairing on all the tangent spaces. Now, since for each $A \in \mathcal{A}(P)$ we have by above that $\mathcal{A}(P) = \{A + a : a \in \Omega^1(\Sigma, \mathrm{ad}(P))\}$, it follows that for each $A \in \mathcal{A}(P)$ we can express the tangent space $T_A\mathcal{A}(P)$ at $A \in \mathcal{A}(P)$ as

$$T_A\mathcal{A}(P) = \Omega^1(\Sigma, \mathrm{ad}(P)).$$

Now, as discussed in Section 4, we have the multiplication

$$\Omega^k(\Sigma, \mathrm{ad}(P)) \otimes \Omega^\ell(\Sigma, \mathrm{ad}(P)) \to \Omega^{k+\ell}(\Sigma)$$

denoted by $\langle \zeta \wedge \eta \rangle$ where $\zeta \in \Omega^k(\Sigma, \mathrm{ad}(P))$ and $\eta \in \Omega^\ell(\Sigma, \mathrm{ad}(P))$, which combines a fixed Ad-invariant inner product on the Lie algebra $\mathfrak{g}$ with the exterior multiplication $\wedge$ on forms. Using this, we therefore obtain the *Atiyah-Bott 2-form* ω_{AB} defined by for all $a, b \in \Omega^1(\Sigma, \mathrm{ad}(P))$ (thought of as tangent vectors), we have

$$\omega_{AB}(a, b) = \int_\Sigma \langle a \wedge b \rangle.$$

That is, we integrate the 2-form $\langle a \wedge b \rangle \in \Omega^2(\Sigma)$ over the 2-dimensional manifold Σ, which gives a real number. Hence, since $T_A\mathcal{A}(P) = \Omega^1(\Sigma, \mathrm{ad}(P))$ for every $A \in \mathcal{A}(P)$, we therefore have that

$$\omega_{AB}\colon T_A\mathcal{A}(P) \times T_A\mathcal{A}(P) \to \mathbb{R}$$

is a skew-symmetric pairing on all the tangent spaces of $\mathcal{A}(P)$, noting that it is skew-symmetric since $\langle a \wedge b \rangle = -\langle b \wedge a \rangle$ for all $a, b \in \Omega^1(\Sigma, \mathrm{ad}(P))$. One can also show that the pairing ω_{AB} is also closed and non-degenerate, so we have that ω_{AB} is a symplectic form on the affine space $\mathcal{A}(P)$ of connections on $\pi\colon P \to \Sigma$.

Next, recall from Section 2 that the Lie algebra of the Gauge group $\mathcal{G}(P)$ is given by

$$\mathfrak{g}(P) = \mathrm{Lie}(\mathcal{G}(P)) = \Gamma\mathrm{ad}(P),$$

i.e. the space of sections of $\mathrm{ad}(P)$, which can equivalently be viewed as the space of $\mathrm{ad}(P)$-valued 0-forms $\Omega^0(\Sigma, \mathrm{ad}(P))$. Now, we can view $\Omega^2(\Sigma, \mathrm{ad}(P))$ as the dual space of $\Omega^0(\Sigma, \mathrm{ad}(P))$, i.e. that

$$\Omega^2(\Sigma, \mathrm{ad}(P)) = \mathrm{Lie}(\mathcal{G}(P))^*.$$

The way we make this identification is that we view an $H \in \Omega^2(\Sigma, \mathrm{ad}(P))$ as acting on an $f \in \Omega^0(\Sigma, \mathrm{ad}(P))$ via the map

$$(\cdot, \cdot)\colon \Omega^2(\Sigma, \mathrm{ad}(P)) \times \Omega^0(\Sigma, \mathrm{ad}(P)) \to \mathbb{R}$$

defined by

$$(H, f) = \int_\Sigma \langle H \wedge f \rangle.$$

One can check that this map is a non-degenerate bilinear pairing, and thus we can view $\Omega^2(\Sigma, \mathrm{ad}(P))$ as the space of 'linear functionals' on $\Omega^0(\Sigma, \mathrm{ad}(P))$. Now, we claim that the map

$$\mu\colon \mathcal{A}(P) \to \mathrm{Lie}(\mathcal{G}(P))^*$$

defined by $A \mapsto -F_A$ is a *moment map* for the action of the Gauge group $\mathcal{G}(P)$ on the affine space of connections $\mathcal{A}(P)$. To see this (we will explain the definition of a moment map simultaneously while showing that it is a moment map), first observe that the pullback action of the Gauge group $\mathcal{G}(P)$ on $\mathcal{A}(P)$ preserves the symplectic form ω_{AB}. That is, for each $g \in \mathcal{G}(P)$ the corresponding map $\mathcal{G}_g \colon \mathcal{A}(P) \to \mathcal{A}(P)$ (given by the action by g) is a symplectomorphism, i.e. $\mathcal{G}_g^*\omega_{AB} = \omega_{AB}$, where $\mathcal{G}_g^*$ is the pullback of $\mathcal{G}_g$, which is defined as it is for a smooth map between smooth manifolds. Now, let $f \in \mathrm{Lie}(\mathcal{G}(P)) = \Omega^0(\Sigma, \mathrm{ad}(P))$ be arbitrary. Then, consider the map

$$(\mu, f) \colon \mathcal{A}(P) \to \mathbb{R}$$

defined by

$$A \mapsto (\mu(A), f) = \int_\Sigma \langle -F_A \wedge f\rangle.$$

Now, the *Hamiltonian vector field* generated by the smooth function $(\mu, f) \colon \mathcal{A}(P) \to \mathbb{R}$ is the unique vector field $X_{(\mu,f)}$ satisfying

$$d(\mu, f)(Y) = \omega_{AB}(X_{(\mu,f)}, Y)$$

for all vector fields Y on $\mathcal{A}(P)$, where here we are using that ω_{AB} is non-degenerate. Also, given our element $f \in \mathrm{Lie}(\mathcal{G}(P))$, we have the vector field X_f defined by the infinitesimal action of f on $\mathcal{A}(P)$ given the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$, which as we've discussed before is given by

$$X_f(A) = \frac{d}{dt}\bigg|_{t=0} \exp(tf) \cdot A \in T_A\mathcal{A}(P),$$

where for each t we have that $\exp(tf) \cdot A$ is the pullback action of the gauge transformation $\exp(tf) \in \mathcal{G}(P)$ on the connection $A \in \mathcal{A}(P)$. Now, by definition, we have that μ is a *moment map* if the vector fields $X_{(\mu,f)}$ and X_f are equal (this needs to hold for all $f \in \mathrm{Lie}(\mathcal{G}(P))$, but we took f to be arbitrary anyway). To see why μ is indeed a moment map, note that it suffices to check that

$$d(\mu, f)(Y) = \omega_{AB}(X_f, Y)$$

for all vector fields Y on $\mathcal{A}(P)$, as this will then imply that $X_{(\mu,f)} = X_f$ since $X_{(\mu,f)}$ is defined as the unique vector field satisfying $d(\mu, f)(Y) = \omega_{AB}(X_{(\mu,f)}, Y)$ for all vector fields Y on $\mathcal{A}(P)$. We won't flesh out the details of this argument, but the essential ideas are as follows. Given a connection $A \in \mathcal{A}(P)$, we can write the curvature F_A as $F_A = dA \circ h$, where h is the projection $TP \to H$ onto the horizontal subbundle defined by the connection A, and dA is the differential of A, where here A is viewed as a $\mathfrak{g}$-valued 1-form in $\Omega^1(P, \mathfrak{g})$. Also, recalling the similarity between the definitions of the curvature F_A and the exterior covariant derivative d_A that we gave earlier, we can similarly define

$$d_A \colon \Omega^k(\Sigma, \mathrm{ad}(P)) \to \Omega^{k+1}(\Sigma, \mathrm{ad}(P))$$

by $d_A(\zeta) = d\zeta \circ h$ for all $\zeta \in \Omega^k(\Sigma, \mathrm{ad}(P))$, where now we are projecting collections of $(k+1)$-many tangent vectors in each tangent space onto the corresponding horizontal subspace. Using this, we then see that for a tangent vector Y on $\mathcal{A}(P)$ (viewed as an element of $\Omega^1(\Sigma, \mathrm{ad}(P))$), we have that

$$\mu(A + tY) = -F_{A+tY} = -d(A + tY) \circ h = -dA \circ h - tdY \circ h = -F_A - td_AY.$$

Thus, it follows that the differential of the map $(\mu, f) \colon A \mapsto (\mu(A), f)$ is given

$$d(\mu, f)(Y) = -\int_\Sigma \langle d_AY \wedge f\rangle$$

for all vector fields Y on $\mathcal{A}(P)$. The next key observation is that the vector field X_f corresponding to the infinitesimal action of f on $\mathcal{A}(P)$ is in fact given by

$$X_f = d_Af,$$

from which it immediately follows by definition that

$$\omega_{AB}(X_f, Y) = \int_\Sigma \langle d_Af \wedge Y\rangle$$

for all vector fields Y on $\mathcal{A}(P)$. So, in order to show that μ is a moment map, it suffices to show that

$$-\int_\Sigma \langle d_AY \wedge f\rangle = \int_\Sigma \langle d_Af \wedge Y\rangle$$

for all vector fields Y on $\mathcal{A}(P)$. However, since each $Y \in \Omega^1(\Sigma, \mathrm{ad}(P))$, we have the relation (as mentioned in Section 4)

$$d\langle Y \wedge f\rangle = \langle d_A Y \wedge f\rangle - \langle Y \wedge d_A f\rangle,$$

and so by integrating both sides, we have that

$$\int_\Sigma d\langle Y \wedge f\rangle = \int_\Sigma \langle d_A Y \wedge f\rangle - \int_\Sigma \langle Y \wedge d_A f\rangle.$$

But

$$\int_\Sigma d\langle Y \wedge f\rangle = 0,$$

so we therefore have that

$$-\int_\Sigma \langle d_A Y \wedge f\rangle = -\int_\Sigma \langle Y \wedge d_A f\rangle = \int_\Sigma \langle d_A f \wedge Y\rangle,$$

so we may now conclude that μ is a moment map. One can also show that μ is $\mathcal{G}(P)$-equivariant, but we won't discuss that here.

Since the moment map

$$\mu\colon \mathcal{A}(P) \to \mathrm{Lie}(\mathcal{G}(P))$$

is given by $\mu(A) = -F_A$ for all $A \in \mathcal{A}(P)$ and we of course have that F_A vanishes identically if and only if $-F_A$ vanishes identically, it follows that the the space of flat connections $\mathcal{A}_F(P)$ on $\pi\colon P \to \Sigma$ is given by

$$\mathcal{A}_F(P) = \mu^{-1}(0),$$

and therefore the moduli space $\mathcal{M}_P(\Sigma, G)$ of flat connections on $\pi\colon P \to \Sigma$ modulo gauge transformations is given by

$$\mathcal{M}_P(\Sigma, G) = \mu^{-1}(0)/\mathcal{G}(P).$$

The quotient $\mu^{-1}(0)/\mathcal{G}(P)$ is called the *Marsden-Weinstein quotient*, and under some mild non-degeneracy conditions we get that it is a symplectic manifold, inheriting its symplectic structure from $\mathcal{A}(P)$. The quotient is also known as a *symplectic reduction* of $\mathcal{A}(P)$ by $\mathcal{G}(P)$, and it is infinite-dimensional as the Gauge group $\mathcal{G}(P)$ is infinite-dimensional (and the dimension of the symplectic reduction is given by the dimension of $\mathcal{A}(P)$ minus twice the dimension of $\mathcal{G}(P)$).

References

[1] M. F. Atiyah and R. Bott. *The Yang-Mills Equations over Riemann Surfaces*. Phil. Trans. R. Soc. Lond. Series A, Mathematical and Physical Sciences, Vol. 308. 1982. pp. 523-615.

[2] S. Trautwein. *A Survey of the GIT Picture for the Yang-Mills Equation over Riemann Surfaces*. arXiv:1511.08122. 2015.

[3] L. C. Jeffrey. *Flat Connections on Oriented 2-Manifolds*. Bull. London Math. Soc. 37. 2005. pp. 1-14.

[4] J. Blaavand. *Computations of Moduli Spaces*. 2011. Available at http://blaavand.us/download/moduli_spaces.pdf.

[5] D. Michiels. *Moduli Spaces of Flat Connections*. Masters Thesis, KU Leuven. 2013. Available at https://faculty.math.illinois.edu/ michiel2/docs/thesis.pdf.

[6] J. François, S. Lazzarini and T. Masson. *Gauge Field Theories: Various Mathematical Approaches*. arXiv:1404.4604. 2014.

[7] Department of Mathematics - University of Toronto. *Chapter I: Principal Bundles and Connections*. Available at http://www.math.toronto.edu/mein/teaching/moduli.pdf.

[8] S. Morrison. *Connections on Principal Fibre Bundles*. Research Project Report. July 2000. Available at https://tqft.net/papers/ConnectionsAndBundles.pdf.

[9] A. Jaffe and E. Witten. *Quantum Yang-Mills Theory*. The Millenium Prize Problems. American Mathematical Society. 2006. 129-152.

[10] S. Picard. *Introduction to the Yang-Mills Equations*. Final Project Report. May 2012. Available at http://www.math.mcgill.ca/gantumur/math581w12/downloads/picard_YM.pdf.

[11] Y. Loizides. *Notes on Moduli Spaces of Flat Connections with Boundary Conditions*. Expository Notes. Spring 2012. Available at https://e.math.cornell.edu/people/Yiannis_Loizides/Moduli%20of%20flat%20connections.pdf.

[12] A. Haydys. *Introduction to Gauge Theory*. arXiv:1910.10436. October 2019.

[13] J. Evans. *Principal Bundles*. Lecture Notes. October 2010. Available at http://www.homepages.ucl.ac.uk/ ucahjde/YM-lectures/lecture6.pdf.

[14] J. Evans. *Yang-Mills Functional*. Lecture Notes. October 2010. Available at http://www.homepages.ucl.ac.uk/ ucahjde/YM-lectures/lecture7.pdf.

[15] J. Evans. *Moment Map in Yang-Mills Theory*. Lecture Notes. October 2010. Available at http://www.homepages.ucl.ac.uk/ ucahjde/YM-lectures/lecture9.pdf.

[16] A. Cannas da Silva. *Lectures on Symplectic Geometry*. Lecture Notes in Math. 1764 Springer, New York. 2001.