EXPOSITION INTO DE RHAM'S THEOREM

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Abstract. Homology and cohomology are prevalent methods of exploring a topological space. Homology groups are abelian groups derived in order to analyze the non-triviality of a topological space, for example holes in the space. de Rham cohomology groups are derived from the differential k-forms on smooth manifolds, which are generalizations of smooth functions. The relationship between homology and de Rham cohomology is not immediately clear by their construction. de Rham's Theorem proves that there is an isomorphism between the k^{th} de Rham cohomology group and the group of homomorphisms from the k^{th} homology group into $\mathbb{R}$. This paper explores the fundamentals of cubical homology and de Rham cohomology, and proves de Rham's theorem.

Contents

1. Homology

We formulate our understanding of homology using cubical homology, which begins with the definition of a cube.

Definition 1.1. A *k-cube* in a topological space X is a continuous map $\Gamma : I^k \to X$, where I^k is the k-dimensional cube,

$$I^k = [0,1] \times ... \times [0,1] \subset \mathbb{R}^k$$

A cube is *degenerate* if, for some i, $\Gamma(t_1, ..., t_k)$ does not depend on t_i.

Let $\mathcal{Q}_k(X)$ denote the free abelian group generated by all k-cubes in X, and let $\mathcal{D}_k(X)$ denote the (free) subgroup generated by degenerate k-cubes.

Let $\mathcal{C}_k(X)$ be the quotient group $\mathcal{Q}_k(X)/\mathcal{D}_k(X)$, called the group of (cubical) *k-chains* on X. Therefore, *k-chain* is a finite linear combination of non-degenerate k-cubes, $\Gamma = \sum_\alpha n_\alpha \Gamma_\alpha$ for integers n_α.

To motivate the definition of the boundary of a k-cube, consider a (k-1)-cube obtained by restricting the i^{th} coordinate of a k-cube to s, where $s = \{0, 1\}$. We can refer to these as "faces" of the cube, denoted as $\partial_i^s \Gamma$. In particular, for a k-cube Γ,

$$\partial_i^s(\Gamma) = \Gamma(t_1, \ldots, t_{i-1}, s, t_{i+1}, \ldots, t_k)$$

It is useful to introduce an additional function that defines the face of a cube, which will later be used in the definition of the orientation of a cube and Stokes Theorem for chains. We define $F^{k,i,s} : I^{k-1} \to I^k$

to be a diffeomorphism given by

$$F^{k,i,s}(t_1,\ldots,t_{k-1}) = (t_1,\ldots,t_{i-1},s,t_i,\ldots,t_{k-1})$$

Definition 1.2. The *boundary* of a k-cube is defined as

$$\partial\Gamma = \sum_{i=1}^{k}(-1)^i(\partial_i^0\Gamma - \partial_i^1\Gamma) = \sum_{i=1}^{k}(-1)^i(\Gamma\circ F^{i,k,0} - \Gamma\circ F^{i,k,1})$$

The boundary of a k-chain $\Gamma \in \mathcal{C}_k(X)$ is extended linearly from the boundary of each term, so

$$\partial(\sum n_\alpha\Gamma_\alpha) = \sum n_\alpha\partial\Gamma_\alpha$$

The boundary of a degenerate k-cube is a degenerate (k-1)-cube. So $\partial_k : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)$ is a homomorphism.

Theorem 1.3. *For any k+1-cube Γ, $\partial(\partial\Gamma) = 0$.*

Proof. Let Γ be a basis element of $\mathcal{C}_{k+1}(X)$. Denote $\Gamma_{\widetilde{t_i}} = \partial_i^0\Gamma - \partial_i^1\Gamma$, and $\Gamma_{\widetilde{t_i}\widetilde{t_j}} = \partial_j^0\Gamma_{\widetilde{t_i}} - \partial_j^1\Gamma_{\widetilde{t_i}}$. We can see that

$$\partial\Gamma = \sum_{i=1}^{k}(-1)^i\Gamma_{\widetilde{t_i}}$$

$$\partial(\partial\Gamma) = \sum_{j<i}(-1)^i(-1)^j\Gamma_{\widetilde{t_i}\widetilde{t_j}} + \sum_{j>i}(-1)^i(-1)^{j-1}\Gamma_{\widetilde{t_i}\widetilde{t_j}}$$

All the terms cancel, so $\partial(\partial\Gamma) = 0$. ☐

A k-chain is a *k-cycle* if its boundary is 0. We define the following subgroups of $\mathcal{C}_k(X)$, where $\mathcal{C}_{-1}(X) = 0$:

$$\mathcal{Z}_k(X) := \ker(\partial : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)) \text{ the set of all } k\text{-cycles}$$

$$\mathcal{B}_k(X) := \mathrm{im}(\partial : \mathcal{C}_{k+1}(X) \to \mathcal{C}_k(X)) \text{ the set of boundaries}$$

Notice $\mathcal{B}_k(X) \subseteq \mathcal{Z}_k(X)$ since $\partial\circ\partial = 0$

Definition 1.4. For a topological space X, the k^{th} cubical homology group is defined as

$$H_k(X) = \mathcal{Z}_k(X)/\mathcal{B}_k(X)$$

Example 1.5. *Calculating the homology of a point:*

Let p be the singleton set. Consider a k-cube $\Gamma : I^k \to p$. Γ is degenerate if $k \geq 1$, so $\mathcal{C}_k(p) = 0$ if $k \geq 1$. If $k = 0$, there exists a unique map $\Gamma : I^0 \to p$. Therefore, $\mathcal{C}_0(p) = \mathcal{Z}_0(p) = \mathbb{Z}$. $\mathcal{B}_0(p) = 0$ since $\mathcal{C}_1(p) = 0$. Therefore,

$$H_k(p) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 1.6. *$H_0(X)$ is isomorphic to the free abelian group generated by the set of path components on X.*

Proof. Let F be the the free abelian group generated by the path components of X. Consider a map ϵ that takes a point $p \in X$ to the path component containing it. $\mathcal{Z}_0(X) = \mathcal{C}_0(X)$, and we we will show that $\mathcal{B}_0(X)$ is the kernel of ϵ. Given a 1-chain $\sum n_i\gamma_i$, where $\gamma_i : [0,1] \to U$ is a continuous path with initial point P_i and terminal point Q_i. The boundary of this chain is $\partial(\sum n_i\gamma_i) = \sum n_i(Q_i - P_i)$. Then, $\epsilon(\sum n_i(Q_i - P_i)) = 0$, since the inital and terminal points of a path must be in the same connected component. Therefore, $\mathcal{B}_0(X) \subseteq ker(\epsilon)$. Conversely, if a 0-cycle is in the kernel of ϵ, the coefficients appearing in front of points in a component must sum to 0. Then, this cycle can be written as $\sum(Q_i - P_i)$ where P_i and Q_i are in the same component. Then, this cycle must be the boundary of a 1-cycle of paths with initial and terminal points P_i and Q_i. ☐

Lemma 1.7. *A continuous map $f : X \to Y$ induces a homomorphism $f_* : H_k(X) \to H_k(Y)$.*

Proof. We get the map $f_\sharp : \mathcal{C}_k(X) \to \mathcal{C}_k(Y)$ by composing the k-cube with f for a continuous map $f \circ \Gamma : I^k \to Y$. Extending $f_\sharp$ linearly, $f_\sharp(\sum n_\alpha \Gamma_\alpha) = \sum n_\alpha f_\sharp(\Gamma_\alpha)$. By a simple calculation, $f_\sharp(\partial_i^s \Gamma) = f \circ (\partial_i^s \Gamma) = \partial_i^s(f \circ \Gamma) = \partial_i^s(f_\sharp \Gamma)$. Then, we can show $f_\sharp$ commutes with the boundary operator.

$$\begin{aligned} f_\sharp(\partial\Gamma) &= \sum_{i=1}^{k}(-1)^i f_\sharp(\partial_i^0\Gamma - \partial_i^1\Gamma) \\ &= \sum_{i=1}^{k}(-1)^i(\partial_i^0(f_\sharp\Gamma) - \partial_i^1(f_\sharp\Gamma)) \\ &= \partial(f_\sharp\Gamma) \end{aligned}$$

If Γ is a boundary of a $k+1$-chain, then $f_\sharp\Gamma$ is also a boundary. If Γ is a cycle, then $\partial(f_\sharp\Gamma)) = f_\sharp(\partial\Gamma) = 0$, so $f_\sharp\Gamma$ is a cycle. Then, $f_\sharp$ induces a homomorphism $f_* : H_k(X) \to H_k(Y)$ that sends a homology class $[\Gamma]$ to $[f_\sharp\Gamma]$. □

Remark 1.8. For continuous maps between topological spaces, $g : X \to Y$ and $f : Y \to Z$, $(f \circ g)_* = f_* \circ g_*$. Additionally, $(id_X)_*$ is the identity map between homology groups on X.

Lemma 1.9. *If $f : X \to Y$ and $g : X \to Y$ are homotopic, then f_* and g_* are the same map.*

Proof. Suppose $H : X \times [0,1]$ is a homotopy from f to g and $\Gamma : I^k \to X$ is a k-cube, then we can define a $k+1$-cube $R(\Gamma)$ by

$$R(\Gamma)(s, t_1, ..., t_k) = H(\Gamma(t_1, ..., t_k) \times s)$$

If $\Gamma = \sum n_\alpha \Gamma_\alpha$ is a k-cycle representative of a homology class, then we can calculate $\partial(\sum n_\alpha R(\Gamma_\alpha)) = \sum n_\alpha(f \circ \Gamma_\alpha) - \sum n_\alpha(g \circ \Gamma_\alpha) = f(\Gamma) - g(\Gamma)$. In other words, $f(\Gamma)$ and $g(\Gamma)$ differ by a boundary, so $f_*([\Gamma]) = g_*([\Gamma])$ □

Lemma 1.10. *If f is a homotopy equivalence, then f_* is an isomorphism.*

Proof. If $f : X \to Y$ is a homotopy equivalence, then there exists a continuous map $g : Y \to X$ such that $f \circ g \cong^h id_Y$ and $g \circ f \cong^h id_X$, where $\cong^h$ denotes that the two maps are homotopic. From the above lemma, this implies that $(f \circ g)_* = (id_Y)_*$ and $(g \circ f)_* = (id_X)_*$. This implies that f_* and g_* are isomorphisms. More details can be found in [Hatcher(2002)] □

Example 1.11. Calculating $H_1(\mathbb{RP}^n)$ for $n \geq 2$.

Since (S^n, p) where $p : S^n \to \mathbb{RP}^n$ identifies antipodal points is a 2-sheeted universal covering space of $\mathbb{RP}^n$, the order of $\pi_1(\mathbb{RP}^n)$ is 2, and so must be $\mathbb{Z}_2$. Since $H_1(\mathbb{RP}^n)$ is the abelianization of $\pi_1(\mathbb{RP}^n)$, $H_1(\mathbb{RP}^n) = \mathbb{Z}_2$ [Massey(1991)].

Definition 1.12. A *complex* $A_\bullet$ is a sequence of abelian groups A_k and homomorphisms ∂_k such that $\partial_k(\partial_{k-1}) = 0$.

$$\cdots \longrightarrow A_{k+1} \xrightarrow{\partial_{k+1}} A_k \xrightarrow{\partial_k} A_{k-1} \longrightarrow \cdots$$

Denote the k^{th} *homology group of a complex* to be

$$H_k(A) = \frac{\ker(\partial : A_k \to A_{k-1})}{\operatorname{im}(\partial : A_{k+1} \to A_k)}$$

Definition 1.13. An *exact sequence* is sequence of abelian groups with homomorphisms between them, such that the $ker(\partial_k) = im(\partial_{k+1})$.

A *short exact sequence* is an exact sequence of 3 abelian groups:

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

An *short exact sequence of complexes* is 3 complexes $A_\bullet, B_\bullet$, and $C_\bullet$ with homomorphisms f, g between them.

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$$

such that the following sequence is exact for all k:

$$0 \to A_k \xrightarrow{f^k} B_k \xrightarrow{g^k} C_k \to 0$$

Which is equivalent to the following commutative diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \\
0 \arrow[r] & A_k \arrow[r, "f^k"] \arrow[d, "\partial"] & B_k \arrow[r, "g^k"] \arrow[d, "\partial"] & C_k \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{k-1} \arrow[r, "f^{k-1}"] \arrow[d, "\partial"] & B_{k-1} \arrow[r, "g^{k-1}"] \arrow[d, "\partial"] & C_{k-1} \arrow[r] \arrow[d, "\partial"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Lemma 1.14. *(Zig-Zag Lemma) Given an exact sequence of complexes*

$$0 \to \mathcal{A}_\bullet \xrightarrow{f} \mathcal{B}_\bullet \xrightarrow{g} \mathcal{C}_\bullet \to 0$$

There exists a connecting homomorphism,

$$\partial_* : H_k(C) \to H_{k-1}(A)$$

that makes the following sequence exact:

$$\begin{tikzcd}
\dots \arrow[r, "\partial_*"] & H_k(A) \arrow[r, "f^k_*"] & H_k(B) \arrow[r, "g^k_*"] & H_k(C) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- node[near end, above]{∂_*} ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
 & H_{k-1}(A) \arrow[r, "f^{k-1}_*"] & H_{k-1}(B) \arrow[r, "g^{k-1}_*"] & H_{k-1}(C) \arrow[r, "\partial_*"] & \dots
\end{tikzcd}$$

By exactness of the sequence, $\mathrm{im}(g^k)$ is equal to the kernel of the map $C_k \to 0$, so g^k must be surjective for any k. Similarly, since $\ker(f^k)$ is equal to the image of the map from $0 \to A_k$, f^k must be injective for any k.

For a chain $c_k \in C_k$, that represents a homology class of $H_k(C)$, there exists some $b_k \in B_k$ such that $g^k(b_k) = c_k$. Since the diagram commutes, $\partial(g^k(b_k)) = \partial c_k = 0 = g^{k-1}(\partial b_k)$. Then, since $\partial b_k \in \ker(g^{k-1})$, we get by exactness that, $\partial b_k \in \mathrm{im}(f^{k-1})$. So, there is some $a_{k-1} \in A_{k-1}$ such that $f^{k-1}(a_{k-1}) = \partial b_k$. Then, since $f^{k-1}(\partial a_{k-1}) = \partial(\partial b_k) = 0$ and f^{k-1} is injective, we know $\partial a_{k-1} = 0$, so a_{k-1} represents a homology class of $H_{k-1}(A)$.

We define ∂_* to be the map $\partial_*([c_k]) = [a_{k-1}]$. More details can be found in [Munkres(1984)].

Lemma 1.15. *Let $0 \to A_1 \to A_2 \to \dots \to A_n \to 0$ be an exact sequence of finitely generated abelian groups A_i, the rank of each A_i is finite. Then,*

$$\sum_{i=1}^{n} (-1)^i rk(A_i) = 0$$

Lemma 1.16. *(Five-Lemma) Given the commutative diagram of abelian groups and homomorphisms, where the horizontal sequences are exact*

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{i} & A_2 & \xrightarrow{j} & A_3 & \xrightarrow{k} & A_4 & \xrightarrow{l} & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \xrightarrow{i'} & B_2 & \xrightarrow{j'} & B_3 & \xrightarrow{k'} & B_4 & \xrightarrow{l'} & B_5 \end{array}$$

If f_1, f_2, f_4, and f_5 are isomorphisms, then f_3 is also an isomorphism.

(i.) f_3 is surjective since f_2 and f_4 are surjective and f_5 is injective.

Proof. For an element $b_3 \in B_3$, $k'(b_3) \in B_4$ is equal to $f_4(a_4)$ for some $a_4 \in A_4$, since f_4 is surjective. Since f_5 is injective and $f_5(\ell(a_4)) = \ell'(f_4(a_4)) = \ell'(k'(b_3)) = 0$, then $\ell(a_4) = 0$, which by exactness implies that $k(a_3) = a_4$. We can see that $k'(b_3 - f_3(a_3)) = k'(b_3) - k'(f_3(a_3)) = k'(b_3) - f_4(k(a_3)) = k'(b_3) - f_4(a_4) = 0$. Therefore, $b_3 - f_3(a_3)$ are in the image of j', so $b_3 - f_3(a_3) = j'(b_2)$. Since f_2 is surjective, $b_2 = f_2(a_2)$, so $f_3(a_3 + j(a_2)) = f_3(a_3) + f_3(j(a_2)) = f_3(a_3) + j'(f_2(a_2)) = b_3$, therefore f_3 is surjective. □

(ii.) f_3 is injective since f_2 and f_4 are injective, and f_1 is surjective.

Proof. Suppose $f_3(a_3) = 0$. Since f_4 is injective, $f_4(k(a_3)) = k'(f_3(a_3)) = 0$ implies $k(a_3) = 0$, which by exactness implies $a_3 = j(a_2)$ for some $a_2 \in B$. $j'(f_2(a_2)) = f_3(j(a_2)) = f_3(a_3) = 0$, therefore since $f_2(a_2) \in ker(j')$, $f_2(a_2) = i'(b_1)$ for some $b_1 \in B_1$. Since f_1 is surjective, $b_1 = f_1(a_1)$ for some $a_1 \in A_1$. Then, since f_2 is injective, $f_2(i(a_1) - a_2) = f_2(i(a_1)) - f_2(a_2) = i'(f_1(a_1)) - f_2(a_2) = i'(b_1) - f_2(b_2) = 0$, implies $i(a_1) = a_2$, so $a_3 = j(a_2) = j(i(a_1)) = 0$. Therefore, $f_3(a_3) = 0$ implies $a_3 = 0$, which shows f_3 is injective. □

Theorem 1.17. *(Mayer-Vietoris for Homology) For a topological space X with open cover U, V, the following sequence is exact:*

$$\cdots \xrightarrow{\partial_*} H_k(U \cap V) \xrightarrow{-} H_k(U) \oplus H_k(V) \xrightarrow{+} H_k(U \cup V) \xrightarrow{\partial_*} H_{k-1}(U \cap V) \xrightarrow{-} H_{k-1}(U) \oplus H_{k-1}(V) \xrightarrow{+} H_{k-1}(U \cup V) \xrightarrow{\partial_*} \cdots$$

Proof. There are many proofs of this theorem that utilize the Zig-Zag Lemma, but complicate the construction of the exact sequence of complexes to do so. This proof, from [Fulton(1995)] elegantly avoids these complications. We will use the following commutative diagram of homomorphisms between homology groups that are induced by inclusion maps:

$$\begin{array}{ccc} H_k(U) & \xrightarrow{k^*} & H_k(U \cup V) \\ {\scriptstyle i^*}\uparrow & & {\scriptstyle l^*}\uparrow \\ H_k(U \cap V) & \xrightarrow{j^*} & H_k(V) \end{array}$$

Then, for $[a] \in H_k(U \cap V)$, define $-([a]) = (i_*[a], -j_*[a])$ and for $(b, c) \in H_k(U) \oplus H_k(V)$, define $+(([b], [c])) = k_*[b] + l_*[c]$. The following steps are needed to define ∂_*:

i. Define the subdivision of a k-cube.
ii. Show that any class in $H_k(U \cup V)$ can be represented by a k-cycle z that is the sum $c_1 + c_2$, where c_1 and c_2 are k-chains contained in U and V respectively
iii. The k-1 chain $\partial c_1 = -\partial c_2$ is a k-1 cycle on $U \cap V$ and its homology class in $H_{k-1}(U \cap V)$ is independent of the choice of c_1 and c_2
iv. Define $\partial_* : H_k(U \cup V) \to H_{k-1}(U \cap V)$ by taking the homology class $[z]$ to the homology class $[\partial(c_1)]$

□

Example 1.18. Using Mayer-Vietoris Theorem, calculate $H_n(S^n)$:

For all $n \geq 0$, the n-sphere is defined to be $S^n = \{x \in \mathbb{R}^{n+1} || x| = 1\}$. $S^0 = \{\pm 1\}$. So,

$$H_k(S^0) = H_k(\{1\}) \oplus H_k(\{-1\}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $n \geq 1$, let $N = (0, 0, ..., 1) \in S^n$ and $S = (0, 0, ..., -1) \in S^n$. Then, we can write $S^n = U \cup V$ where $U = S^n \backslash \{N\}$ and $V = S^n \backslash \{S\}$. $U \cap V = S^n \backslash \{N, S\}$ is homotopy equivalent to S^{n-1}. By stereographic projection, U and V are both diffeomorphic to $\mathbb{R}^n$. Then, for $n \geq 1$, we get the exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_*"] & H_{k+1}(S^{n-1}) \arrow[r, "-"] & H_{k+1}(\mathbb{R}^n) \oplus H_{k+1}(\mathbb{R}^n) \arrow[r, "+"] & H_{k+1}(S^n) \arrow[dll, "\partial_*"] & \\
& H_k(S^{n-1}) \arrow[r, "-"] & H_k(\mathbb{R}^n) \oplus H_k(\mathbb{R}^n) \arrow[r, "+"] & H_k(S^n) \arrow[dll, dashed] & \\
& H_1(S^{n-1}) \arrow[r, "-"] & H_1(\mathbb{R}^n) \oplus H_1(\mathbb{R}^n) \arrow[r, "+"] & H_1(S^n) \arrow[dll, "\partial_*"] & \\
& H_0(S^{n-1}) \arrow[r, "-"] & H_0(\mathbb{R}^n) \oplus H_0(\mathbb{R}^n) \arrow[r, "+"] & H_0(S^n) \arrow[r, "\partial_*"] & 0
\end{tikzcd}$$

For $k > 0$, $H_k(\mathbb{R}^n) \oplus H_k(\mathbb{R}^n) = 0$. When $k = 0$, $H_0(\mathbb{R}^n) \oplus H_0(\mathbb{R}^n) = \mathbb{Z} \oplus \mathbb{Z}$. Since S^n has one connected component, $H_0(S^n) = \mathbb{Z}$ for all $n \geq 1$. For $n = 1$, $H_0(S^{n-1}) = \mathbb{Z}^2$ and $H_k(S^{n-1}) = 0$ for $k > 0$. So, we get the following two sequences:

$$0 \longrightarrow H_1(S^1) \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow H_k(S^1) \longrightarrow 0$$

By exactness, $H_1(S^1) \to \mathbb{Z}^2$ is injective, so see that $H_1(S^1)$ is a subgroup of $\mathbb{Z}^2$. By Lemma 1.6, we see that the rank of $H_1(S^1)$ must be 1, so $H_1(S^1) = \mathbb{Z}$. By exactness of the second sequence, $H_k(S^1) = 0$ for $k > 1$.

For $n > 1$, $H_0(S^{n-1}) = \mathbb{Z}$, so we get the following exact sequences:

$$0 \longrightarrow H_1(S^1) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow H_k(S^n) \longrightarrow H_{k-1}(S^{n-1}) \longrightarrow 0$$

Then, $H_1(S^n) \subset \mathbb{Z}$, so it has no torsion, and by Lemma 1.6, $rk(H_1(S^n)) = 0$. Therefore, $H_1(S^n) = 0$ for $n > 1$. From the second sequence, we see that $H_k(S^n) = H_{k-1}(S^{n-1})$ for $k > 1$. Therefore, for $n \geq 1$,

$$H_i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

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2. Manifolds

Definition 2.1. A topological space M is an *n-dimensional topological manifold* if it is Hausdorff, admits a countable basis, and is locally Euclidean in $\mathbb{R}^n$. The locally Euclidean condition means that for each $x \in M$, there exists a neighborhood U_x that is homeomorphic to an open subset $\widetilde{U} \subset \mathbb{R}^n$.

In order to define de Rham cohomology, topoligical manifolds must be endowed with a smooth structure so that calculus can be performed on the manifold.

Definition 2.2. Let M be a topological n-manifold. Suppose $U \subseteq M$ and $\widetilde{U} \subseteq \mathbb{R}^n$ are open sets with a homeomorphism $\varphi : U \to \widetilde{U}$. Then, the pair (U, φ) forms a *coordinate chart* on M, where U is the *domain* of the chart.

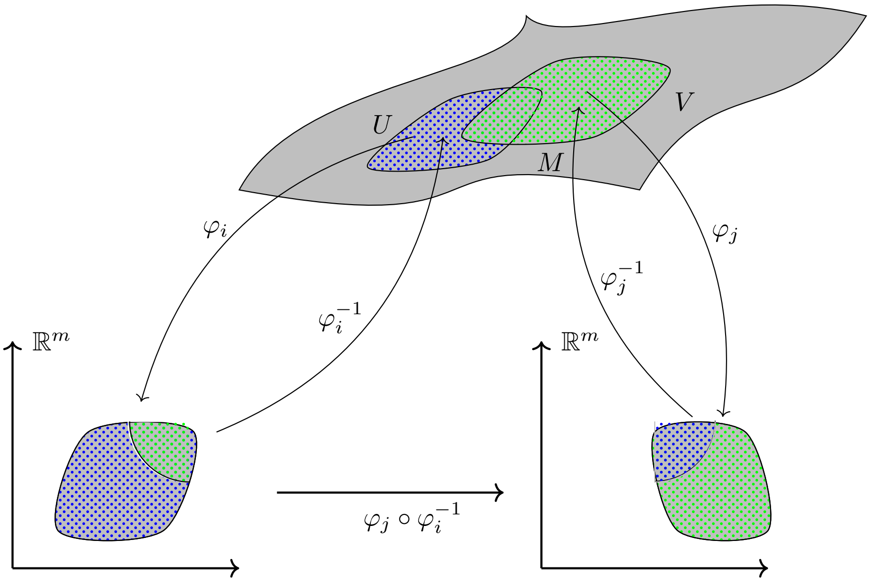


FIGURE 1. Smooth transition

Definition 2.3. Given two coordinate charts (U_i, φ_i) and (U_j, φ_j) on M such that $U_i \cap U_j \neq \emptyset$. The composition $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is the *transition map* from φ_i to φ_j [Figure 1]. If the transition map is a diffeomorphism, then the coordinate charts are *smoothly compatible.*

Definition 2.4. An *atlas* $\mathcal{A}$ for M is a collection of coordinate charts whose domains cover M.

Definition 2.5. If $\mathcal{A}$ is an atlas such that any two coordinate charts of $\mathcal{A}$ are smoothly-compatible, then $\mathcal{A}$ is a *smooth atlas.* A smooth atlas $\mathcal{A}$ for M is *maximal* if it is not properly contained in any larger smooth atlas. A maximal smooth atlas is known as a *smooth structure* on M.

Definition 2.6. A topological manifold M, together with a maximal smooth atlas $\mathcal{A}$, is a *smooth manifold.*

Example 2.7. Examples of smooth manifolds:

(1) *A 0-dimensional manifold*: Let M be a zero dimensional topological manifold. For every $p \in M$, a neighborhood U_p that is homeomorphic to an open subset of $\mathbb{R}^0$ must be the singleton set $\{p\}$. For each p, the coordinate map $\varphi : \{p\} \to \mathbb{R}^0$ is the same. Then, any two charts must be smoothly compatible, and thus M is endowed with a smooth structure

(2) *Finite-Dimensional Vector Spaces* Let V be a finite-dimensional real vector space with a topology induced by the norm. An ordered basis $\{E_1, ..., E_n\}$ defines a basis isomorphism $E : \mathbb{R}^n \to V$ by letting $E(x) = \sum\limits_{i=1}^{n} x^i E_i$. Since E is a homeomorphism, (V, E^{-1}) is a chart. If $\{\widetilde{E_1}, ...\widetilde{E_n}\}$ is another basis with isomorphism $\widetilde{E}(x) = \sum_j x^j \widetilde{E}_j$, then there is an invertible matrix (A_i^j) such that $E_i = \sum_j A_i^j \widetilde{E}_j$ for each i. The transition map between (V, E^{-1}) and $(V, \widetilde{E}^{-}1)$ is defined as $\widetilde{E}^{-1} \circ E(x) = \widetilde{x}$. Additionally, $\sum_j \widetilde{x}^j \widetilde{E}_j = \sum_i x^i E_i = \sum_{i,j} x^i A_j^i \widetilde{E}_j$. Therefore, $\widetilde{x}^j = \sum_i A_i^j x^i$. So, the transition map is invertible and linear, and hence is a diffeomorphism. It follows that any two charts are smoothly compatible, and so V is endowed with a smooth structure defined by these charts.

Definition 2.8. Suppose M and N are smooth manifolds and there exists a map $F : M \to N$. F is a *smooth map* if for every $x \in M$, there exists smooth charts (U_x, φ) and $(V_{F(x)}, \psi)$ such that $F(U_x) \subseteq V_{F(x)}$ and $\hat{F} := \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is a smooth function.

Definition 2.9. Suppose M is a smooth n-manifold and $f : M \to \mathbb{R}$ an arbitrary function. f is a *smooth function* if for every $x \in M$, there exists a smooth chart (U_x, φ) such that $f \circ \varphi^{-1}$ is smooth on the open subset $\widetilde{U} = \varphi(U) \subseteq \mathbb{R}^n$. The set of all smooth functions $f : M \to \mathbb{R}$ is denoted as $C^\infty(M)$.

A linear map $v : C^\infty(M) \to \mathbb{R}$ is called a *derivation at p* if it satisfies $v(fg) = f(p)vg + g(p)vf$ for all $f, g \in C^\infty(M)$. For a manifold M, the *tangent space* T_pM of M at the point p is defined to be the set of derivations of $C^\infty(M)$ at p.

Example 2.10. *Tangent space of embedded manifold*:

Suppose M is a smooth manifold embedded in $\mathbb{R}^n$. Then, for $p \in M$, define the geometric tangent space to be the set $\mathbb{R}^n_p := \{p\} \times \mathbb{R}^n = \{(p, v) : v \in \mathbb{R}^n\}$. $(p, v) \in \mathbb{R}^n_p$ is abbreviated as v_p. In terms of the coordinate basis, $v_p = \sum_i v^i e_i\big|_p$. We can define $D_v\big|_p : C^\infty(M) \to \mathbb{R}$ to be the directional derivative of a smooth function f in the direction v at p to be:

$$D_v\big|_p f = \sum_i v^i \frac{\partial f}{\partial x^i}(p)$$

$D_v\big|_a$ is linear over $\mathbb{R}$ and satisfies the product rule:

$$D_v\big|_p(fg) = f(p)D_v\big|_p g + g(p)D_v\big|_p f$$

which characterises the operation as a *derivation.* It is shown in [Lee(2013)] that the map $v_p \mapsto D_v\big|_a$ is an isomorphism between $\mathbb{R}^n_p$ and $T_p(\mathbb{R}^n)$, which has basis $\{\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p\}$.

Definition 2.11. If M and N are smooth manifolds and $F : M \to N$ is a smooth map, for each $p \in M$, we define the *differential of F at p* to be $dF_p : T_pM \to T_{F(p)}N$. For $v \in T_pM$, the derivation at $F(p)$ that acts on $f \in C^\infty(M)$ is given by

$$dF_p(v)(f) = v(f \circ F)$$

For a smooth manifold, the differential operator can be used to relate the T_pM to $T_{\varphi(p)}\mathbb{R}^n$, where φ is the homeomorphism of a coordinate chart (U, φ). In order to make this connection, it is important to note that tangent vectors act locally. In particular, for $v \in T_pM$, if $f, g \in C^\infty(M)$ agree on some neighborhood of p, then $vf = vg$

It can be shown that $d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism. $T_{\varphi(p)}\mathbb{R}^n$ has basis $\{\frac{\partial}{\partial x^1}\big|_{\varphi(p)}, \dots, \frac{\partial}{\partial x^n}\big|_{\varphi(p)}\}$, so the preimages of these vectors under the isomorphism $d\varphi_p$ forms a basis for T_pM, which will be denoted $\{\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p\}$ for consistency.

Definition 2.12. The *tangent bundle* TM is the disjoint union of the tangent spaces at all points $p \in M$, i.e. $TM := \coprod_{p \in M} T_pM$. TM can be endowed with a natural smooth structure, making it a 2n-dimensional manifold such that the map $\pi : TM \to M$ is smooth, where $\pi^{-1}(p) = T_p(M)$ for all $p \in M$

For a vector space V, its dual is denoted as $V^* := \{\omega : V \to \mathbb{R} \mid \omega \text{ is a linear map}\}$. For each $p \in M$, we define the *cotangent space* T_p^*M to be the dual space to T_pM. Given a chart on an n-dimensional manifold with coordinates $(x^1, \dots, x^n)$, the *cotangent space* T_p^*M has basis $\{dx^1, \dots, dx^n\}$ where $dx^i(\frac{\partial}{\partial x^j}\big|_p) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

Definition 2.13. The *cotangent bundle* T^*M is the disjoint union of the cotangent spaces at all points $p \in M$, i.e. $T^*M = \coprod_{p \in M} T_p^*M$. Similar to the tangent bundle, this is a 2n-dimensional manifold endowed with a smooth structure using natural projection map $\pi : T^*M \to M$, which is smooth.

Both TM and T^*M are *smooth vector bundles over M*

Definition 2.14. A *smooth covector field*, α, is a smooth map $\alpha : M \to T^*M$ such that $\pi \circ \alpha = id_M$. If (U, ϕ) is a chart with coordinates $(x^1, ..., x^n)$, then $\alpha = \sum_{i=1}^n \alpha_i dx^i$ where α_i is a smooth function on U for all i.

We can similarly define a smooth manifold $\bigwedge^k(T^*M)$ for $k = 0, 1, 2, ...$ by taking the disjoint union $\coprod_{p\in M} \bigwedge^k(T_p^*M)$, where $\alpha_p \in \bigwedge^k(T_p^*M)$ if and only if α_p is a real valued k-multilinear map on T_pM.

3. Differential Forms

Definition 3.1. A *differential k-form* on M is a smooth map $\omega : M \to \bigwedge^k(T^*M)$ such that $\pi\circ\omega = id_M$. We say that ω is a smooth section of the vector bundle $\pi : \bigwedge^k(T^*M) \to M$

A differential *0-form* is a smooth real valued function of a manifold. A differential *1-form* on a manifold with a smooth chart is constructed by assigning to each point a covector from the dual of the tangent space at that point, making a smooth covector field defined as $p \to \omega_p \in T_p^*M$.

The set of k-forms on a manifold form a vector space denoted as $\Omega^k(M)$.

1. $\Omega^0(M) = C^\infty(M)$ since $\bigwedge^0(T_p^*M) = \mathbb{R}$ by definition.
2. $\Omega^1(M) =$ smooth covector fields on M since $\bigwedge^1(T_p^*M) = T_p^*M$.
3. $\Omega^k(M) = 0$ if $k > dimM$ since $\bigwedge^k(T_p^*M) = 0$.

If $\omega \in \Omega^k(M)$ with $1 \leq k \leq n$, then on any chart (U, φ) with coordinates $(x^1, ..., x^n)$,

$$\omega := \sum_{1\leq j_1<...<j_k\leq n} P_{j_1 j_2 ... j_k} dx^{j_1} \wedge ... \wedge dx^{j_k}$$

where $P_{j_1 j_2 ... j_k}$ are smooth functions on U. This can be abbreviated with a multi-index J, where $\omega = \sum_J P_J dx^J$

Definition 3.2. The *wedge product* $\wedge$ of two 1-forms to be an alternating bilinear operation, $\omega^1 \wedge \omega^2 = -\omega^2 \wedge \omega^1$. For any 1-form, $\omega \wedge \omega = 0$. In particular, $dx^i \wedge dx^i = 0$.

A k-form assigns to each point a covariant $k-$tensor $\omega_p = \omega^1 \wedge ... \wedge \omega^k : (T_p^*M \times ... \times T_p^*M) \to \mathbb{R}$ such that

$$\omega^1 \wedge ... \wedge \omega^k(v_1, ..., v_k) = det\begin{pmatrix} \omega^1(v_1) \dots \omega^k(v_1) \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ \omega^1(v_k) \dots \omega^k(v_k) \end{pmatrix}$$

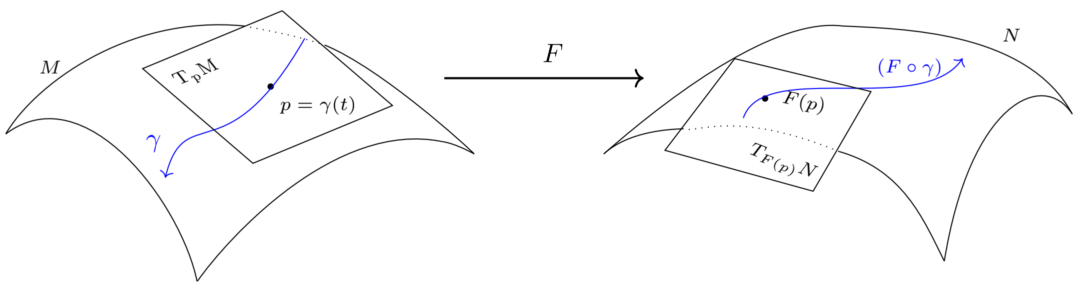
Remark 3.3. Let $F : M \to N$ be a smooth map between manifolds. It is useful to introduce an alternative view of the differential of F. Consider a path $\gamma : [0, 1] \to M$ parameterized by t, and $p = \gamma(t)$ for some t. Consider $v_p \in T_pM$, where v_p is the velocity of γ at p.

The *differential* of F, $dF_p : T_pM \to T_{F(p)M}$ defined by

$$dF_p(v_p) = (F \circ \gamma)'(t) \in T_{F(p)}N$$

For a differential acting on $f \in C^\infty(N)$, we define:

$$dF_p(v_p)(f) := v_p(f \circ F) \;\forall f \in C^\infty(N)$$



Definition 3.4. For a differential k-form $\omega \in \Omega^k(N)$, define $F^* : \Omega^k(N) \to \Omega^k(M)$ to be the pullback of of ω evaluated at vectors $v_1, ..., v_k$ is defined by pushing these vectors forward

$$F^*(\omega)_p(v_1, ..., v_k) = \omega_{F(p)}(dF(v_1), ..., dF(v_k))$$

Example 3.5. *Calculating the pullback:* Given $F : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $F(u, v) = (u, v, u^2 - v^2)$, and let ω be the 2-form $ydx \wedge dz + xdy \wedge dz$ on $\mathbb{R}^3$.

$$\begin{aligned} F^*(\omega) &= F^*(ydx \wedge dz + xdy \wedge dz) \\ &= vdu \wedge d(u^2 - v^2) + udv \wedge d(u^2 - v^2) \\ &= 2u^2 + 2v^2 dv \wedge du \end{aligned}$$

Remark 3.6. : $(F \circ G)^* = G^* \circ F^*$ and $d(F \circ G) = dF \circ dG$

Remark 3.7. Recall from calculus, we can find the gradient of a smooth function on $\mathbb{R}^n$, which calculates a vector field in the direction of the function's steepest increase. But, the calculation relies on an understanding of the coordinates of $\mathbb{R}^n$. We can take away this dependence by considering the differential of a function on a manifold M, which calculates a covector field, defined by $df_p(v) = vf$ for $v \in T_pM$.

Definition 3.8. The *exterior derivative* $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is defined as

$$d\omega = \sum_J \sum_{i=1}^n \frac{\partial P_J}{\partial x_i} dx_i \wedge dx_J, \text{ where } n = dimM$$

The exterior derivative can be though of as a generalization of the differential of a function.

Theorem 3.9. *(Properties of the exterior derivative)*

(i) d is linear over $\mathbb{R}$
(ii) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
(iii) $d \circ d = 0$
(iv) For $f \in \Omega^0(M) = C^\infty(M), df$ is the differential of f, given by $df(X) = Xf$

Proof. We will prove (iii), and proofs of the other properties can be found in [Lee(2013)].

$$\begin{aligned} d(d\omega) &= \sum_J \sum_{i=1}^n d\left(\frac{\partial P_J}{\partial x_i} dx_i \wedge dx_J\right) = \sum_J \sum_{i=1}^n \sum_{k=1}^n \frac{\partial P_J}{\partial x_i \partial x_k} dx_i \wedge dx_k \wedge dx_J \\ &= \sum_J \sum_{i=1}^n \left(\sum_{k=1}^{i-1} \frac{\partial P_J}{\partial x_i \partial x_k} dx_i \wedge dx_k \wedge dx_J + \sum_{k=i+1}^n \frac{\partial P_J}{\partial x_i \partial x_k} dx_i \wedge dx_k \wedge dx_J \right) \\ &= \sum_J \sum_{i=1}^n \left(\sum_{k=1}^{i-1} \frac{\partial P_J}{\partial x_i \partial x_k} dx_i \wedge dx_k \wedge dx_J - \sum_{k=i+1}^n \frac{\partial P_J}{\partial x_i \partial x_k} dx_k \wedge dx_i \wedge dx_J \right) = 0 \end{aligned}$$

□

4. de Rham Cohomology

Define the following subspaces of $\Omega^k(M)$, where $\Omega^{-1}(M) = 0$

$$\mathcal{Z}^k(M) = \ker(d : \Omega^k(M) \to \Omega^{k+1}(M)), \text{ the subspace of closed forms}$$

$$\mathcal{B}^k(M) = im(d : \Omega^{k-1}(M) \to \Omega^k(M)), \text{ the subspace of exact forms}$$

Notice $\mathcal{B}^k(M) \subseteq \mathcal{Z}^k(M)$ since $d \circ d = 0$.

Definition 4.1. For a smooth n-dimensional manifold M, The k^{th} de Rham cohomology group is defined as

$$H^k(M) = \mathcal{Z}^k(M)/\mathcal{B}^k(M)$$

Example 4.2. *Zeroth cohomology of connected smooth manifolds:*

Let M be a smooth manifold and U be a connected component of M. Since $\Omega^{-1}(U) = 0$, $\mathcal{B}^0(U) = 0$. A closed 0-form f is a smooth real-valued function such that $df = 0$. Since U is connected, this implies that f is constant. Therefore, $H^0(U) = \mathbb{Z}^0(U) =$ constant functions. Therefore, if c is equal to the number of connected components of M, then

$$H^0(M) = \mathbb{R}^c$$

Example 4.3. *Dimension bound of cohomology:*

Let M be a smooth n-manifold. For $k > n$, $\bigwedge^k T^*M = 0$, so $\Omega^k(M) = 0$, which implies

$$H^k(M) = 0 \text{ for } k > dim(M)$$

Definition 4.4. For a smooth manifold M, we have the following family of vector spaces and linear maps d_i

$$0 \xrightarrow{d_0} \Omega^0(M) \xrightarrow{d_1} ... \xrightarrow{d_p} \Omega^p(M) \xrightarrow{d_{p+1}} \Omega^{p+1}(M) \xrightarrow{d} ... \xrightarrow{d_n} \Omega^n(M) \xrightarrow{d_{n+1}} 0$$

This is an example of a *cochain complex* $\Omega^\bullet(M)$, which is analogous to a chain complex.

For all smooth maps $F : M \to N$, we obtain the pullback map $F^* : \Omega^k(N) \to \Omega^k(M)$ such that $d \circ F^* = F^* \circ d$:

Theorem 4.5. *Let $F : M \to N$ be a smooth map. Its pullback $F^* : \Omega^k(N) \to \Omega^k(M)$ induces a linear map, also denoted as $F^* : H^k(N) \to H^k(M)$.*

Proof. If $\omega \in \Omega^k(N)$ is closed, then $d(F^*\omega) = F^*(d\omega) = 0$, so $F^*\omega$ is closed. If $\omega \in \Omega^k M$ is exact, then $F^*(\omega) = F^*(d\eta) = d(F^*\eta)$, so $F^*\omega$ is exact. Then, if F is a diffeomorphism, then F^* will be an isomorphism, which implies that diffeomorphic smooth manifolds have isomorphic de Rham cohomology groups. □

We will show that the de Rham cohomology group is invariant under homeomorphism, and introduce the Mayer-Vietoris sequence, which are useful in calculating these groups and necessary for the proof de Rham's Theorem.

Theorem 4.6. *(Whitney Approximation Theorem) If $F : M \to N$ is a continuous map between manifolds, then F is homotopic to a smooth map between M and N. More details in [1].*

Lemma 4.7. *Let M and N be smooth manifolds and $F, G : M \to N$ homotopic smooth maps. For every k, the pullback maps induce a natural map between cohomology groups such that $F^*, G^* : H^k(N) \to H^k(M)$ are equal.*

Lemma 4.8. *Suppose M and N are smooth manifolds. If $F, G : M \to N$ are homotopic smooth maps, then they are smoothly homotopic.*

Theorem 4.9. *If M and N are homeomorphic smooth manifolds, then $H^k(M)$ is isomorphic to $H^k(N)$ for all k.*

Proof. The outline of this proof is to first show that de Rham cohomology groups are invariant under homotopy, and Theorem 3.1 follows. Suppose M and N are homotopy equivalent smooth manifolds, i.e. there exists a continuous maps $F : M \to N$ with a continuous map $G : N \to M$ such that $F \circ G \cong^h id_N$ and $G \circ F \cong^h id_M$. By the Whitney Approximation Theorem, there are smooth maps $\widetilde{F} : M \to N$ and $\widetilde{G} : N \to M$ that are homotopic to F and G respectively. Homotopy is preserved by composition, so $\widetilde{F} \circ \widetilde{G} \cong^h F \circ G \cong^h id_N$ and $\widetilde{G} \circ \widetilde{F} \cong^h G \circ F \cong^h id_M$. Then, by Lemma 4.2, these are smoothly homotopic. By Lemma 4.1, $\widetilde{F}^* \circ \widetilde{G}^* = (\widetilde{G} \circ \widetilde{F})^* = (id_M)^* = id_{H^k(M)}$ and $\widetilde{G}^* \circ \widetilde{F}^* = (\widetilde{F} \circ \widetilde{G})^* = (id_N)^* = id_{H^k(N)}$. Therefore, $\widetilde{F}^*$ is an isomorphism between $H^k(N)$ and $H^k(M)$. Since every homeomorphism is a homotopy equivalence, homeomorphic manifolds must have isomorphic de Rham cohomology groups. □

Theorem 4.10. *(Cohomology of Contractible Manifolds) If M is a contractible smooth manifold, then $H^k(M) = 0$ for $k \geq 1$.*

Proof. By the definition of contractible, there is some point $q \in M$ such that the identity map of M is homotopic to the constant map $c_q : M \to M$ sending all of M to q. Let $\iota_q : \{q\} \to M$ denote the inclusion map, so $c_q \circ \iota_q = id_q$ and $\iota_q \circ c_q \cong^h id_M$. Then, ι_q is a homotopy equivalence. $H^k(\{q\}) = 0$ for $p \geq 1$ since q is a 0-manifold. Therefore, $H^k(M) = 0$. □

Example 4.11. *Cohomology of star-shaped subsets:* If U is a star-shaped open subset of $\mathbb{R}^n$, then $H^p(U) = 0$ for $p \geq 1$. Suppose U is star shaped with respect to c. Then, U is contractible by the homotopy: $H(x,t) = c + t(x - c)$.

Lemma 4.12. *(Zig-Zag Lemma for Cohomology) Given an exact sequence of cochain complexes*

$$0 \to \Omega^\bullet(M) \to \Omega^\bullet(N) \to \Omega^\bullet(R) \to 0$$

There exists a connecting homomorphism $\delta : H^k(R) \to H^{k+1}(M)$ that makes the following sequence exact:

$$\begin{tikzcd}
\dots \arrow[r, "\delta"] & H^k(M) \arrow[r, "F_k^*"] & H^k(N) \arrow[r, "G_k^*"] & H^k(R) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- node[pos=0.75, fill=white]{δ} ([yshift=-2ex, xshift=-2ex]\tikztotarget.north west) |- (\tikztotarget.west)}] & \\
& H^{k+1}(M) \arrow[r, "F_{k-1}^*"] & H^{k+1}(N) \arrow[r, "G_{k-1}^*"] & H^{k+1}(R) \arrow[r, "\delta"] & \dots
\end{tikzcd}$$

Theorem 4.13. *(Mayer-Vietoris) A smooth manifold M with an open cover $\{U, V\}$, we have the following inclusion maps:*

$$\begin{tikzcd}
U \arrow[r, "i"] & M \\
U \cap V \arrow[u, "l"] \arrow[r, "k"] & V \arrow[u, "j"]
\end{tikzcd}$$

Which have pullback maps, with corresponding maps between cohomology groups. $i^*(\omega) = (\omega)\big|_U$, and similarly for j^*, k^*,and *

$$\begin{tikzcd}
\Omega^k(U) \arrow[d, "l^*"] & \Omega^k(M) \arrow[l, "i^*"'] \arrow[d, "j^*"] \\
\Omega^k(U \cap V) & \Omega^k(V) \arrow[l, "k^*"]
\end{tikzcd}$$

Then, consider the sequence

$$\begin{tikzcd}
\dots \arrow[r, "\delta"] & H^k(M) \arrow[r, "i^* \oplus j^*"] & H^k(U) \oplus H^k(V) \arrow[r, "l^* - k^*"] & H^k(U \cap V) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- node[pos=0.75, fill=white]{δ} ([yshift=-2ex, xshift=-2ex]\tikztotarget.north west) |- (\tikztotarget.west)}] & \\
& H^{k+1}(M) \arrow[r, "i^* \oplus j^*"] & H^{k+1}(U) \oplus H^{k+1}(V) \arrow[r, "l^* - k^*"] & H^{k+1}(U \cap V) \arrow[r, "\delta"] & \dots
\end{tikzcd}$$

We can verify that this is exact:

i. $\ker(i^* \oplus j^*) = 0$. Suppose $\omega_M \in \Omega^k(M)$ such that $(i^* \oplus j^*)(\omega_M) = (i^*(\omega_M), j^*(\omega_M)) = (\omega_M\big|_U, \omega_M\big|_V) = (0,0)$. Since U and V cover M, this implies that $\omega_M = 0$

ii. $\operatorname{im}(i^* \oplus j^*) = \ker(l^* - k^*)$. Suppose $(\omega_U, \omega_V) \in \Omega^k U \oplus \Omega^k V$ such that $(l^* - k^*)(\omega_U, \omega_V) = 0$. Then, $l^*(\omega_U) = k^*(\omega_V)$. This implies that ω_U and ω_V can be extended to a differential form on M, which implies that $(\omega_U, \omega_V) \in \operatorname{im}(i^* \oplus j^*)$. Conversely, $(l^* - k^*)(i^*(\omega_M), j^*(\omega_M)) = l^*(i^*(\omega_M)) - k^*(j^*(\omega_M)) = (\omega_M\big|_U)\big|_{U\cap V} - (\omega_M\big|_V)\big|_{U\cap V} = 0$

iii. $\operatorname{im}(l^* - k^*) = \Omega^k(U \cap V)$ is nontrivial, see [1] for details.

iv. δ is exact by the Zig-Zag Lemma.

Example 4.14. *Cohomology of S^n*

Since S^n is connected and smooth, $H^0(S^n) = \mathbb{R}$ for all n. As the base case for induction, we will calculate $H^1(S^1)$ using Mayer-Vietoris. Consider $S^1 = U \cup V$ for intervals U, V. $H^0(U) \oplus H^0(V) = \mathbb{R} \oplus \mathbb{R}$ and $H^0(U \cap V) = \mathbb{R} \oplus \mathbb{R}$. $H^1(U) = H^1(V) = 0$. Then, from Mayer-Vietoris, we get the following exact sequence:

$$0 \longrightarrow \mathbb{R} \overset{i^* \oplus j^*}{\longrightarrow} \mathbb{R} \oplus \mathbb{R} \overset{k^* - \ell^*}{\longrightarrow} \mathbb{R} \oplus \mathbb{R} \overset{\delta}{\longrightarrow} H^1(S^1) \overset{i^* \oplus j^*}{\longrightarrow} 0$$

Which by exactness and the rank lemma, $H^1(S^1) = \mathbb{R}$. For the inductive hypothesis, assume that this is true for S^{n-1}. As above, let $U = S^n \backslash \{N\}$ and $V = S^n \backslash \{S\}$. Then, by stereographic projection, U and V are both diffeomorphic to $\mathbb{R}^n$. $U \cap V$ is homotopy equivalent to S^{n-1}, which implies $H^i(U \cap V) = H^i(S^{n-1})$. Then, from Mayer-Vietoris, we get the following exact sequence:

$$\begin{aligned} H^{i-1}(U) \oplus H^{i-1}(V) \to H^{i-1}(U \cap V) \to H^i(S^n) \to H^i(U) \oplus H^i(V) \\ = 0 \to H^{i-1}(S^{n-1}) \to H^i(S^n) \to 0 \end{aligned}$$

Which implies that $H^i(S^n) \cong H^{i-1}(S^{n-1})$. Therefore, $H^i(S^n) = \begin{cases} \mathbb{R} & \text{if } i = 0, n \\ 0 & \text{otherwise} \end{cases}$.

5. Orientation

We need to formalize the definition of orientability of a smooth cube in order to define integration over the cube.

Definition 5.1. An *orientation* on M is a choice of an equivalence class $[\omega]$ where $\omega \in \Omega^n(M)$ such that $\omega_p \neq 0$ for all $p \in M$ and $\omega \sim \widetilde{\omega}$ iff $\widetilde{\omega} = f\omega$ where $f \in C^\infty(M)$ such that $f(p) > 0$ for all $p \in M$.

Remark 5.2. If every $\omega \in \Omega^n(M)$ is zero at some point of M, then M is *nonorientable.*

Lemma 5.3. *If M is orientable and connected then M has two orientations.*

Consider $\mathbb{R}^n$ with standard coordinates $(x^1, ..., x^n)$. The differential n-form $[\omega^n] = [dx^1 \wedge ... \wedge dx^n]$ is the standard orientation on $\mathbb{R}^n$.

Let $I^k = [0,1] \times [0,1], ..., \times [0,1]$. The restriction of $\omega^k = dx^1 \wedge ... \wedge dx^k$ to $I^k \subseteq \mathbb{R}^k$ induces the *standard orientation on* I^k, also denoted by $[dx^1 \wedge ... \wedge dx^k]$.

Let $I^{k,i,s} = \{(x^1, ..., x^k) \in I^k \big| x_i = s\}$ where $i = 1, ..., k$ and $s = \{0, 1\}$ denote the faces of the cube. Then, the standard orientation on I^k induces an orientation on $I^{k,i,s}$ as follows: For all $p \in I^{k,i,s}$, consider the *outward pointing unit normal at p*

$$N_p^{i,s} = (-1)^{s-1} \frac{\partial}{\partial x_i}\bigg|_p$$

Definition 5.4. The *induced orientation of the face* $I^{k,i,s}$ is $[\omega^{k,i,s}]$, where $\omega^{k,i,s} \in \Omega^{k-1}(I^{k,i,s})$ is

$$\begin{aligned} \omega^{k,i,s}(v_1, ..., v_k) &= (dx^1 \wedge \ldots \wedge dx^k)(N^{i,s}, v_1, \ldots v_k) \\ &= (dx^1 \wedge \ldots \wedge dx^k)((-1)^{s-1}\partial_{x^i}, v_1, \ldots, v_k) \\ &= (-1)^{s-1}(dx^1 \wedge \ldots \wedge dx^k)(\partial_{x^i}, v_1, \ldots, v_k) \\ &= (-1)^{s-1}(-1)^{i-1}(dx^1 \wedge ... \wedge dx^{i-1} \wedge dx^{i+1} \wedge ... \wedge dx^k)(v_1, \ldots, v_k) \end{aligned}$$

Therefore, $\omega^{k,i,s} = (-1)^{s+i} dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^k$, where $\widehat{dx^i}$ indicates that the dx^i has been removed. Therefore, $\omega^{k,i,s} \in \Omega^{k-1}(I^{k,i,s})$.

Let $F^{k,i,s} : I^k \to I^{k,i,s}$ be a diffeomorphism we defined in Section (1) that maps a k-cube onto the face of a $k+1$-cube. Notice that $I^{k,i,s} = F^{k,i,s}(I^{k-1})$. We can see

$$(F^{k,i,s})^* \omega^{k,i,s} = (-1)^{s+i}(F^{k,i,s})^*(dx^1 \wedge ... \wedge dx^{i-1} \wedge dx^{i+1} \wedge ... \wedge dx^k)$$

$$= (-1)^{s+i} dt^1 \wedge ... \wedge dt^{i-1} \wedge dt^i \wedge dt^{i+1} \wedge ... \wedge dt^k$$
$$= (-1)^{s+i} \omega^{k-1}$$

Therefore, $[(F^{k,i,s})^* \omega^{k,i,s}] = (-1)^{s+i}[\omega^{k-1}]$.

Theorem 5.5. *$F^{k,i,s}$ is orientation preserving iff $s+i$ is even and orientation reversing iff $s+i$ is odd.*

6. Smooth cubes, and Integration

Definition 6.1. A cube $\Gamma : I^k \to X$ is a *smooth cube* if it extends to a smooth mapping on some neighborhood of the cube $I^k \subseteq \mathbb{R}^k$.

The subgroup of $\mathcal{Q}_k(X)$ generated by smooth cubes is denoted as $\mathcal{Q}_k^\infty$ and the subgroup of $\mathcal{D}_k(X)$ generated by degenerate smooth cubes denoted as $\mathcal{D}_k^\infty(X)$.

Definition 6.2. Define $C_k^\infty(X) = \mathcal{Q}_k^\infty / \mathcal{D}_k^\infty(X)$ to be the group of *smooth k-chains* of X.

Definition 6.3.

$$\mathcal{Z}_k^\infty(X) \coloneqq \ker(\partial : \mathcal{C}_k^\infty(X) \mapsto \mathcal{C}_{k-1}^\infty(X)) \text{ the set of all smooth } k\text{-cycles}$$

$$\mathcal{B}_k^\infty(X) \coloneqq \operatorname{im}(\partial : \mathcal{C}_{k+1}^\infty(X) \mapsto \mathcal{C}_k(X)) \text{ the set of all smooth } k\text{-boundaries}$$

So we can define the *smooth homology group* of X to be

$$H_k^\infty(X) \coloneqq \mathcal{Z}_k^\infty(X) / \mathcal{B}_k^\infty(X)$$

The inclusion map $\iota : \mathcal{C}_k^\infty(X) \to \mathcal{C}_k(X)$ commutes with the boundary operators, so it induces a map on the homology $\iota_* : H_k^\infty(X) \to H_k(X)$ by $\iota_*[c] = [\iota[c]]$.

Theorem 6.4. $H_k(M) \cong H_k^\infty(M)$

Definition 6.5. If ω is a k-form on M and $\Gamma : I^k \to M$ is a smooth k-cube, define the *integral* of ω over Γ by

$$\int_\Gamma \omega = \int_{I^k} \Gamma^*(\omega)$$

where $\Gamma^*(\omega)$ is the pull-back of the map. This definition follows naturally from the standard definition of the integral of a k-form over a smooth manifold (with corners), which can be found in Chapter 16 of Lee.

For a smooth k-chain $\gamma = \sum n_\alpha \Gamma_\alpha$, the integral of ω over γ is defined as

$$\int_\gamma \omega = \sum_\alpha n_\alpha \int_{\Gamma_\alpha} \omega$$

Theorem 6.6. *(Stokes' Theorem) Let M be a compact, oriented n-manifold with boundary, and let ω be a compactly supported smooth (n-1)-form on M. Then,*

$$\int_M d\omega = \int_{\partial M} \omega$$

Theorem 6.7. *(Stokes Theorem for chains) For a smooth manifold M and $\Gamma \in \mathcal{C}_k^\infty(M)$ and $\omega \in \Omega^{k-1}(M)$*

$$\int_{\partial\Gamma} \omega = \int_\Gamma d\omega$$

Proof. It suffices to prove this when Γ is a cube, and the case when Γ is a chain will follow. Since Γ is a manifold with corners, use the standard Stokes' theorem to see that

$$\int_\Gamma d\omega = \int_{I^k} \Gamma^*(d\omega) = \int_{I^k} d\Gamma^*(\omega) = \int_{\partial I^k} \Gamma^*(\omega)$$

Then, since $F^{n,i,s}$ is orientation preserving if $s+i$ is even and orientation reversing if $s+i$ is odd, we get

$$\begin{aligned}
\int_{\partial I^k} \Gamma^*(\omega) &= \sum_{i=1}^{k} (-1)^i \int_{I^{k-1}} (F^{k,i,0})^*\Gamma^*(\omega) - (F^{k,i,1})^*\Gamma^*(\omega) \\
&= \sum_{i=1}^{k} (-1)^i \int_{I^{k-1}} (\Gamma \circ F^{k,i,0})^*(\omega) - (\Gamma \circ F^{k,i,1})^*(\omega) \\
&= \sum_{i=1}^{k} (-1)^i \int_{\Gamma \circ F^{k,i,0}} \omega - \int_{\Gamma \circ F^{k,i,1}} \omega \\
&= \int_{\partial \Gamma} \omega
\end{aligned}$$

□

7. Statement of de Rham's Theorem

Definition 7.1. The *de Rham homomorphism* is a linear map $\ell : H^k(M) \to Hom(H_k^\infty(M), \mathbb{R})$ where $\ell[\omega]$ is given by

$$[\gamma] \mapsto \int_\gamma \omega$$

We should verify that the homomorphism $[\gamma] \mapsto \int_\gamma \omega$ is well defined:

If $[\gamma] \in H_k(M) \cong H_k^\infty(M)$, let $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ be smooth representatives of the homology class $[\gamma]$. Then, $\widetilde{\gamma} - \widetilde{\gamma}' = \partial(\widetilde{\beta})$ for some smooth (k+1)-chain $\widetilde{\beta}$, which implies

$$\int_{\widetilde{\gamma}} \omega - \int_{\widetilde{\gamma}'} \omega = \int_{\partial \widetilde{\beta}} \omega = \int_{\widetilde{\beta}} d\omega = 0$$

If $\omega = d\eta$ is exact, then

$$\int_{\widetilde{\gamma}} \omega = \int_{\widetilde{\gamma}} d\eta = \int_{\partial \widetilde{\gamma}} \eta = 0$$

Lemma 7.2. *The following diagram commutes it $F : M \to N$ is a smooth map*

$$\begin{array}{ccc}
H^k(N) & \xrightarrow{F^*} & H^k(M) \\
\downarrow \ell & & \downarrow \ell \\
Hom(H_k(N), \mathbb{R}) & \xrightarrow{F^*} & Hom(H_k(M), \mathbb{R})
\end{array}$$

Proof. If $\Gamma \in C_k^\infty(M)$ and $\omega \in \Omega^p(N)$,

$$\int_\Gamma F^*\omega = \int_{I^k} \Gamma^* F^* \omega = \int_{I^l} (F \circ \Gamma)^* \omega = \int_{F \circ \Gamma} \omega$$

Then, $\ell(F^*[\omega])[\Gamma] = \ell[\omega][F \circ \Gamma] = \ell[\omega](F_*[\Gamma]) = F^*(\ell[\omega])[\Gamma]$ □

Lemma 7.3. *The following diagram commutes for $M = U \cup V$,*

$$\begin{array}{ccc}
H^{k-1}(U \cap V) & \xrightarrow{\delta} & H^k(M) \\
\downarrow \ell & & \downarrow \ell \\
Hom(H_{k-1}(U \cap V), \mathbb{R}) & \xrightarrow{\partial^*} & Hom(H_k(M), \mathbb{R})
\end{array}$$

where δ is the connecting homomorphism of the Mayer-Vietoris sequence for de Rham cohomology, and $\partial^*(\gamma) = \gamma \circ \partial_*$, where ∂_* is the connecting homomorphism for M-V sequence for homology.

Proof. We want to show that $\ell(\delta[\omega])[\gamma] = (\partial^* \ell[\omega])[\gamma]$ for any $[\omega] \in H^{p-1}(U \cap V)$ and any $[\gamma] \in H_p(M)$. Let ν be a smooth k-form representing $\delta[\omega]$ and c a smooth $(k-1)$-chain representing $\partial_*[\gamma]$. Then, it would suffice to show that $\int_\gamma \nu = \int_c \omega$. Let $c = \partial f$ where f, f' are smooth k-chains in U and V such that $[f + f'] = [\gamma]$. Similarly, we can choose $\eta \in \Omega^{p-1}(U)$ and $\eta' \in \Omega^{p-1}(V)$ such that $\omega = \eta\big|_{U\cap V} - \eta\big|_{U\cap V}$ such that $\nu = d\eta$ on U and $d\eta'$ on V. Then, because $\partial f + \partial f' = \partial\gamma = 0$ and $d\eta\big|_{U\cap V} - d\eta'\big|_{U\cap V} = d\omega = 0$, we have

$$\begin{aligned}\int_c \omega = \int_{\partial f} \omega = \int_{\partial f} \eta - \int_{\partial f} \eta' = \int_{\partial f} \eta + \int_{\partial f'} \eta' \\ = \int_f d\eta + \int_{f'} d\eta' = \int_f \nu + \int_{f'} \nu = \int_\gamma \nu\end{aligned}$$

□

In order to show that ℓ is an isomorphism, we will need the following lemmas:

Lemma 7.4. *If U is an open set in $\mathbb{R}^n$, then there is sequence of compact subsets $K_1, K_2, ...$ whose union is U, and so that $K_1 \subset Int(K_2) \subset K_2 \subset ... \subset K_n \subset Int(K_{n+1}) \subset ...$*

Proof. Start with a countable collection of open sets U_i that cover U such that $\bar{U_i}$ is compact and contained in U. For example, let each U_i be a ball that is centered at a rational coordinate with a rational radii. Let $K_1 = \bar{U_1}$ and $K_2 = \bar{U_1} \cup \bar{U_2} \cup ...\bar{U_m}$ where m is minimal such that $K_1 \subseteq U_1 \cup ... \cup U_m$. Continue so that $K_n = \bar{U_1} \cup ... \cup \bar{U_i}$ and $K_{n+1} = \bar{U_1} \cup ... \cup \bar{U_j}$ where j is minimal such that $K_n \subseteq U_1 \cup ...U_j$. □

The same argument can be used to show that for a manifold M whose topology has a countable basis of open sets, there is a sequence of compact subsets $K_1, K_2, ...$ whose union is M and such that $K_1 \subset Int(K_2) \subset ...$

Lemma 7.5. *If X is an open set in $\mathbb{R}^n$, then X can be written as the union of two open sets U and V such that U, V, and $U \cap V$ can be written as the disjoint union of open sets, each of which is a finite union of open rectangles.*

Proof. Take a sequence of compact sets $K_1 \subset K_2 \subset ...$ as in the above lemma. Construct a sequence of open sets U_i as follows. Let U_1 be a finite union of rectangles covering K_1, with the closure of each contained in the interior of K_2. Let U_2 be the finite union of rectangles covering $K_2 \backslash Int(K_1)$, with the closure of each contained in the interior of K_3. Continue such that U_i is the finite union of rectangles covering $K_i \backslash Int(K_{i-1})$, the closure of each contained in the interior of K_{i+1}, as well as in the complement of K_{i-2}, and disjoint from U_{k-2}. Now, let U be the union of all U_i where i is even, and V be the union of all U_i where i is odd. □

Lemma 7.6. *If M is a smooth manifold, then M may be written as the union of two sets U and V such that U, V, and $U \cap V$ are disjoint unions of open sets, each of which is a finite union of open sets diffeomorphic to open sets in $\mathbb{R}^n$.*

Proof. Replace "rectangle" with "open set diffeomorphic to an open set in $\mathbb{R}^n$" in the proof above. □

Theorem 7.7. *(ℓ is an isomorphism)*

Proof. Let "T(X)" be the statement that $\ell : H^k(X) \to Hom(H_k(X), \mathbb{R})$ is an isomorphism for all k. We will construct the argument using the following:

i. T(U) is true where U is an open rectangle in $\mathbb{R}^n$
   Proof. $H^k(U)$ and $H_k(U)$ vanish for $k > 0$, and $H^0(U) = \mathbb{R}$ and $H_0^\infty(U) = \mathbb{Z}$. Then, any element $f \in Hom(\mathbb{Z}, \mathbb{R})$ is determined by $f(1)$, therefore $Hom(\mathbb{Z}, \mathbb{R}) = \mathbb{R}$. So, for an open rectangle U in $\mathbb{R}^n$, ℓ is an isomorphism
ii. If U and V are open in a manifold, and if T(U), T(V), and T($U \cap V$) are true, then T($U \cup V$) is true.

Proof. We can construct the following maps between Mayer Vietoris sequences for cohomology and homology respectively, and apply the five-lemma. Let $\widetilde{H}^{k-1}(U)$ denote $Hom(H_{k-1}(U), \mathbb{R})$.

$$\begin{tikzcd} H^{k-1}(U) \oplus H^{k-1}(V) \arrow[r] \arrow[d, "\ell"] & H^{k-1}(U \cap V) \arrow[r] \arrow[d, "\ell"] & H^k(U \cup V) \arrow[r] \arrow[d, "\ell"] & H^k(U) \oplus H^k(V) \arrow[r] \arrow[d, "\ell"] & H^k(U \cap V) \arrow[d, "\ell"] \\ \widetilde{H}^{k-1}(U) \oplus \widetilde{H}^{k-1}(V) \arrow[r] & \widetilde{H}^{k-1}(U \cap V) \arrow[r] & \widetilde{H}^k(U \cup V) \arrow[r] & \widetilde{H}^k(U) \oplus \widetilde{H}^k(V) \arrow[r] & \widetilde{H}^k(U \cap V) \end{tikzcd}$$

We know that this diagram commutes by Lemmas (7.2) and (7.3) shown above. Then, we can apply the five lemma (1.16) to see that $\ell : H^k(U \cup V) \to Hom(H_k(U \cup V), \mathbb{R})$ is an isomorphism.

iii. If X is a disjoint union of open manifolds X_α and each $\mathrm{T}(X_\alpha)$ is true, then $\mathrm{T}(X)$ is true.

Proof: Identifying a class of $H^k(X)$ is the same as identifying a class on each X_α. In other words, $H^k(X) = \prod_\alpha H^k(X_\alpha)$. Similarly, $H_k^\infty(X)$ is the same as identifying a class on each X_α such that all but a finite number of these classes are 0. So, $H_k^\infty(X) = \bigoplus_\alpha H_k^\infty(X_\alpha)$.

Then, $Hom(H_k^\infty(X), \mathbb{R}) = Hom(\bigoplus_\alpha H_k^\infty(X_\alpha), \mathbb{R}) = \prod_\alpha Hom(H_k^\infty(X_\alpha), \mathbb{R})$.

With (i), (ii), and (iii), we can prove $T(X)$ for any smooth manifold X:

First, we will show it is true when $X \subseteq \mathbb{R}^n$ is the finite union of p open rectangles. Let U be the union of $p-1$ rectangles and V be the other. Then $T(U)$ and $T(V)$ are true by (i) and (iii), and $T(U \cap V)$ is true since $U \cap V$ is also a finite union of rectangles. By (ii), $T(X)$ is true.

When $X \subset \mathbb{R}^n$ is an open set, use (7.5) to write X as $U \cup V$ such that U, V, and $U \cap V$ are disjoint unions of open sets, and each open set is the finite union of open rectangles. Using the previous step and (iii), $T(U)$, $T(V)$, and $T(U \cap V)$ are true, therefore $T(X)$ is true.

Since diffeomorphisms between manifolds determine isomorphisms between the cohomology and homology groups, $T(X)$ is true for any manifold diffeomorphic to an open set in $\mathbb{R}^n$. When X is a finite union of p open sets, each diffeomorphic to an open set in $\mathbb{R}^n$, let U be the union of $p-1$ of these sets, and V be the remaining set, and $T(X)$ follows by (ii) and (iii). By (7.6), any smooth manifold M can be written as the union of two open sets U and V such that each U, V, and $U \cap V$ is a disjoint union of open sets, each of which is diffeomorphic to a finite union of open sets in $\mathbb{R}^n$. By (iii), $T(U)$, $T(V)$ and $T(U \cap V)$ are true. Then, by (ii), $T(M)$ is true. Therefore, for any smooth manifold, ℓ is an isomorphism. □

Example 7.8. Use de Rham's Theorem to find $H^i(S^n)$, $n \geq 1$.

We showed in Example 1.18 that $H_i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ 0 & \text{otherwise} \end{cases}$

So, $Hom(H_i^\infty(S^n), \mathbb{R}) = \begin{cases} Hom(\mathbb{Z}, \mathbb{R}) & \text{if } i = 0, n \\ Hom(0, \mathbb{R}) & \text{otherwise} \end{cases}$

By de Rham's theorem, $H^i(S^n) = \begin{cases} \mathbb{R} & \text{if } i = 0, n \\ 0 & \text{otherwise} \end{cases}$

Example 7.9. Use de Rham's Theorem to find $H^1(\mathbb{RP}^n)$, $n \geq 2$

We showed in Example 1.11 that $H_1(\mathbb{RP}^n) = \mathbb{Z}_2$. Then,
$H^1(\mathbb{RP}^n) = Hom(\mathbb{Z}_2, \mathbb{R}) = 0$

References

[Fulton(1995)] William Fulton. *Algebraic Topology: A First Course.* Graduate Texts in Mathematics, vol 153. Springer, New York, NY, 1995. ISBN 978-0-387-94327-5.

[Hatcher(2002)] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, UK, 2002.

[Lee(2013)] John M. Lee. *Introduction to Smooth Manifolds.* Graduate Texts in Mathematics, vol 218. Springer, New York, NY, second edition edition, 2013. ISBN 978-1-4419-9981-8.

[Massey(1991)] William S. Massey. *A Basic Course in Algebraic Topology.* Springer Verlag, New York, NY, 1991.

[Munkres(1984)] James R. Munkres. *Elements of Algebraic Topology.* Addison-Wesley Publishing Company, Menlo Park, CA, 1984.