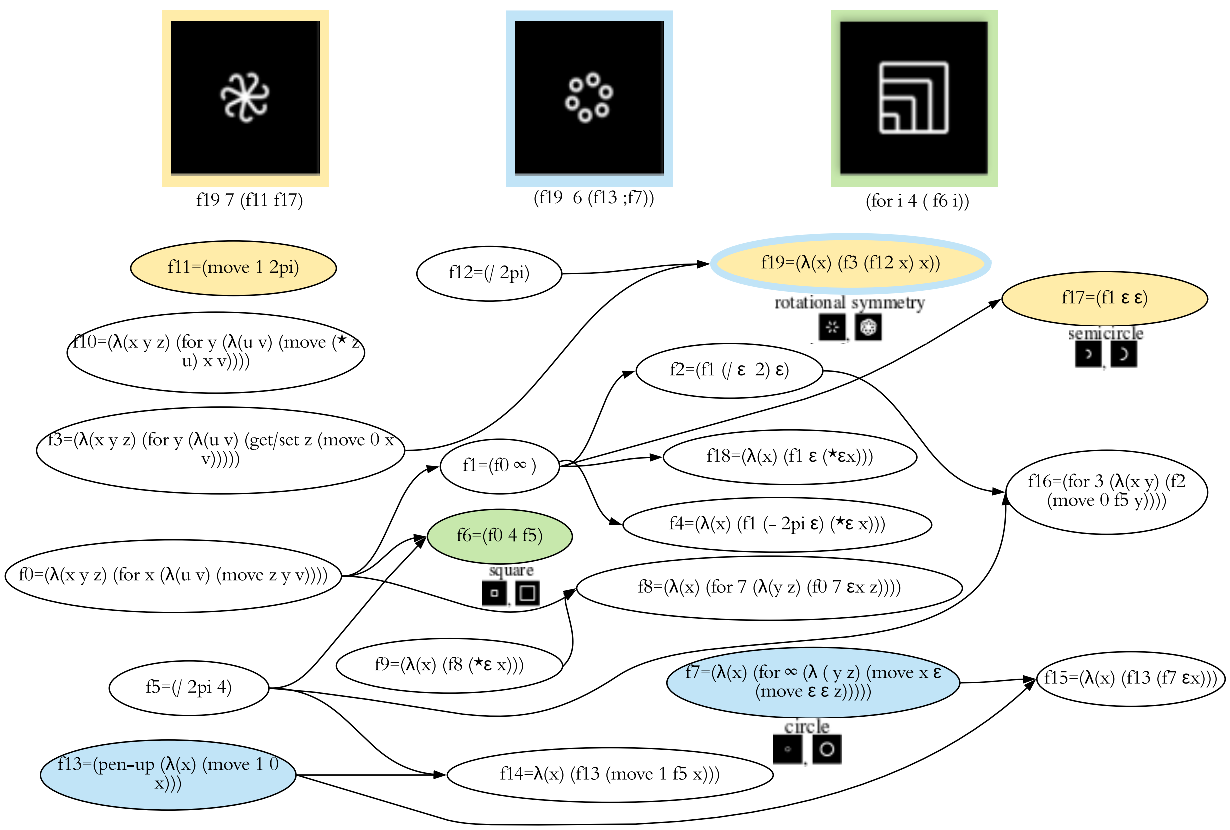
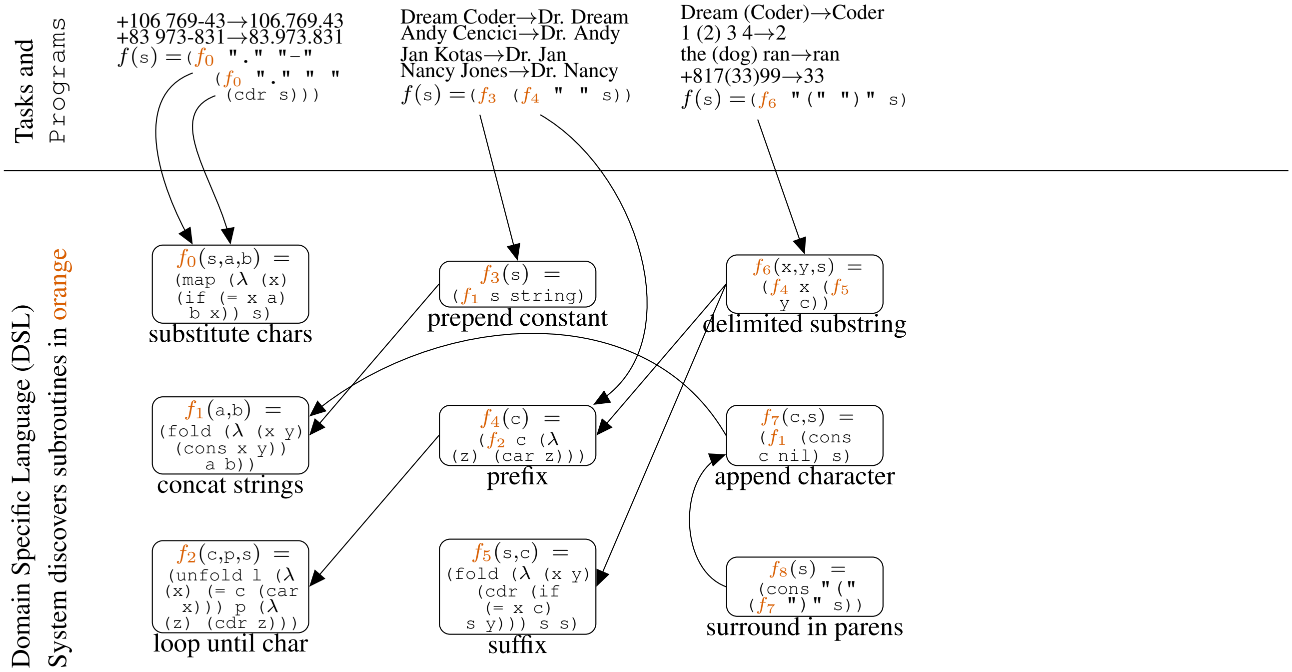
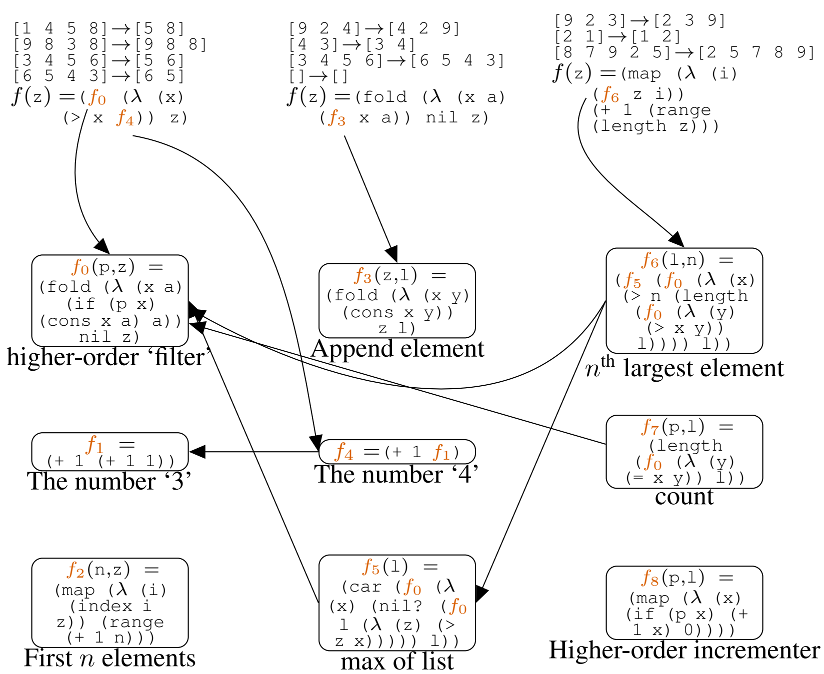



Figure 1: **Bottom**: Learned graphics programing DSL. Each circle encloses a different learned component; arrows point from a component to other components that use it. **Top**: 3 tasks, and below them the programs that solve them. Yellow/Blue/Green highlighting indicates which DSL components are used to solve which tasks.



Tasks and Programs
+106 769-43→106.769.43
+83 973-831→83.973.831
$f$(s) = ($f_0$ "." "-" ($f_0$ "." " " (cdr s)))
Dream Coder→Dr. Dream
Andy Cencici→Dr. Andy
Jan Kotas→Dr. Jan
Nancy Jones→Dr. Nancy
$f$(s) = ($f_3$ ($f_4$ " " s))
Dream (Coder)→Coder
1 (2) 3 4→2
the (dog) ran→ran
+817(33)99→33
$f$(s) = ($f_6$ "(" ")" s)
Domain Specific Language (DSL)
System discovers subroutines in orange
$f_0$(s,a,b) = (map (λ (x) (if (= x a) b x)) s)
substitute chars
$f_3$(s) = ($f_1$ s string)
prepend constant
$f_6$(x,y,s) = ($f_4$ x ($f_5$ y c))
delimited substring
$f_1$(a,b) = (fold (λ (x y) (cons x y)) a b))
concat strings
$f_4$(c) = ($f_2$ c (λ (z) (car z)))
prefix
$f_7$(c,s) = ($f_1$ (cons c nil) s)
append character
$f_2$(c,p,s) = (unfold l (λ (x) (= c (car x))) p (λ (z) (cdr z)))
loop until char
$f_5$(s,c) = (fold (λ (x y) (cdr (if (= x c) s y))) s s)
suffix
$f_8$(s) = (cons "(" ($f_7$ ")" s))
surround in parens

[1 4 5 8]→[5 8]
[9 8 3 8]→[9 8 8]
[3 4 5 6]→[5 6]
[6 5 4 3]→[6 5]
$f$(z) = ($f_0$ (λ (x) (> x $f_4$)) z)
[9 2 4]→[4 2 9]
[4 3]→[3 4]
[3 4 5 6]→[6 5 4 3]
[]→[]
$f$(z) = (fold (λ (x a) ($f_3$ x a)) nil z)
[9 2 3]→[2 3 9]
[2 1]→[1 2]
[8 7 9 2 5]→[2 5 7 8 9]
$f$(z) = (map (λ (i) ($f_6$ z i)) (+ 1 (range (length z)))
$f_0$(p,z) = (fold (λ (x a) (if (p x) (cons x a) a)) nil z)
higher-order 'filter'
$f_3$(z,l) = (fold (λ (x y) (cons x y)) z l)
Append element
$f_6$(l,n) = ($f_5$ ($f_0$ (λ (x) (> n (length ($f_0$ (λ (y) (> x y)) l)))) l))
$n^{\text{th}}$ largest element
$f_1$ = (+ 1 (+ 1 1))
The number '3'
$f_4$ = (+ 1 $f_1$)
The number '4'
$f_7$(p,l) = (length ($f_0$ (λ (y) (= x y)) l))
count
$f_2$(n,z) = (map (λ (i) (index i z)) (range (+ 1 n)))
First $n$ elements
$f_5$(l) = (car ($f_0$ (λ (x) (nil? ($f_0$ l (λ (z) (> z x)))))) l))
max of list
$f_8$(p,l) = (map (λ (x) (if (p x) (+ 1 x) 0)))
Higher-order incrementer

# 1 Introduction

An old dream within AI is a machine that learns and reasons by writing its own programs. This vision stretches back to the 1960's [35] and, if fully realized, could bring us much closer to machines that learn and think like humans. Computational models of cognition often explain the flexibility and richness of human thinking in terms of program learning: from everyday thinking and problem solving (motor program induction as an account of recognition and generation of handwriting and speech [20]; functional programs as a model of natural language semantics [**?**]) to learning problems that unfold over longer developmental time scales: the child's acquisition of intuitive theories (of kinship, taxonomy, etc.) [40] and natural language grammar [32], to name just a few. An outstanding challenge, however, is to engineer program-learners that display the same level of domain-generality as the humans they are meant to model.

Recent program-learning systems developed within the AI and machine learning community are impressive along many dimensions, authoring programs for problem domains like drawing pictures [**?**, 12], transforming text [14] and numerical sequences [3], robot motion planning [8], and reasoning over common sense knowledge bases [27]. These systems work in different ways, but typically hinge upon a carefully hand-engineered Domain Specific Language (DSL). The DSL restricts the space of programs to contain the kinds of concepts needed for one specific domain. For example, a picture-drawing DSL could include concepts like circles and spirals, and a DSL for numerical sequences could include sorting and reversing lists of numbers. Modern systems also learn how to efficiently deploy the DSL on new problems [9, 3, 17], but – unlike human learners – do not discover the underlying system of concepts needed to navigate the domain.

We contribute a program-induction system that learns the domain-specific concepts (DSL) while jointly learning how to use those concepts. This joint learning problem models two complementary notions of domain expertise: domain experts have at their disposal a powerful, yet specialized repertoire of concepts and abstractions (analogous to the DSL) while also having accurate intuitions about when and how to use those concepts to solve new problems. Representative domains, along with DSLs we learn for them, are shown in Figure 2.

We call our system 'DreamCoder' because it acquires these two kinds of expertise through a novel kind of wake/sleep or 'dream' learning [15], iterating through a wake cycle – where it solves problems by writing programs – and a pair of sleep cycles, both of which are loosely biologically inspired by actual sleep. The first sleep cycle, which we refer to as **consolidation**, grows the DSL by replying experiences from waking and consolidating them into new code abstractions. This cycle is inspired by the formation of abstractions during sleep memory consolidation [10]. The second sleep cycle, which we refer to as **dreaming**, improves the agents knowledge of how to write code by training a neural network to help search for programs. The neural net is trained on replayed experiences as well as 'fantasies', or samples, from the DSL. These two kinds of dreams are inspired by the distinct episodic replay and hallucination components of dream sleep [13].

Each wake/sleep cycle creates new DSL components that build on components learned in earlier sleep cycles, growing a DSL with nested hierarchies of code. We identify this cumulative nesting of abstractions as a variety of deep representation learning [23]. Figure 2 diagrams a subset of these learned networks (the DSL). For example, the system learns to sort sequences of numbers by invoking a DSL component 4 layers deep, or draws the leftmost pairs of images in Figure 2 using a depth-3 component. For this reason we refer to DREAMCODER as an instance of 'deep program learning'.

Our goal with DREAMCODER is to engineer a system that develops domain expertise in humanlike ways. This involves learning both declarative and procedural knowledge, like a DSL and a recognition model, but includes other features of the development of expertise: humans can become experts in many fields, and so we evaluate our algorithm across six different problem domains; a human expert doesn't become an expert overnight, and needs more than a handful of example problems to learn from, but doesn't need millions of examples; similarly our algorithm's learning trajectory unfolds over a series of wake/sleep cycles requiring around a hundred problems per domain.

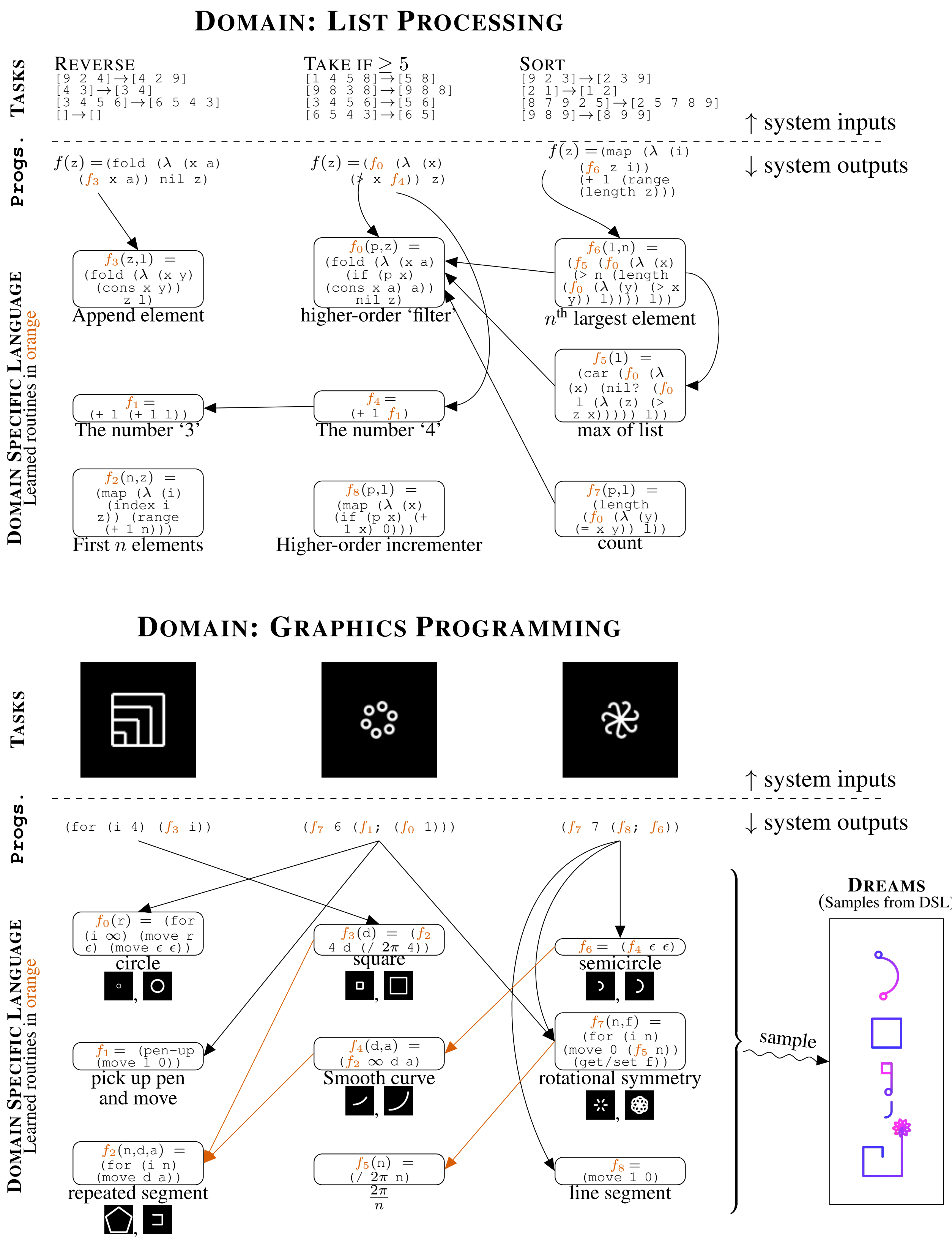



Figure 2: Two of the six domains we apply our system to. Agent observes tasks (top rows) which it solves by writing programs (middle rows) while jointly growing a library (DSL; bottom rows). Learned DSLs rediscover multiple higher-order functions (`filter` for list functions and rotational symmetry for generative graphics). Learned DSL components call each other (arrows).

## 2 Deep Wake/Sleep Program Induction

DREAMCODER takes as its goal to acquire domain-specific expertise, which it learns by solving a collection of programming **tasks**. It alternatingly finds programs that solve tasks (Wake – Figure 3 top); improves its DSL by analyzing programs found during waking (Consolidation – Figure 3 left); and trains a neural network that efficiently guides search for programs in the DSL (Dreaming – Figure 3 right). The learned DSL acts as a a prior on programs likely to solve tasks in the domain, while the neural net looks at a specific task and produces a "posterior" for programs likely to solve that specific task (Figure 3 middle). The neural network thus functions as a **recognition model** supporting a form of approximate Bayesian program induction, jointly trained with a **generative model** for programs encoded in the DSL, in the spirit of the Helmholtz machine [15]. The recognition model ensures that searching for programs remains tractable even as the DSL (and hence the search space for programs) expands. The generative model, or DSL, distills out common abstractions across programs discovered during waking, growing a network of increasingly deep and specialized domain-specific concepts (Figure 2, bottom rows).

These wake sleep/cycles function as an approximate inference algorithm that observes a collection of tasks, written $X$, and infers both a program solving each task, as well as a distribution over programs encoded by a DSL, written $\mathcal{D}$. We equip $\mathcal{D}$ with a learned weight vector $\theta$, and together $(\mathcal{D},\theta)$ define a generative model over programs (Appendix 2). Writing $Q(p|x)$ for the approximate posterior predicted by the recognition model, we iteratively (and approximately) solve for

$$p_x = \arg\max_p \mathbb{P}[x|p]\mathbb{P}[p|\mathcal{D}^*,\theta^*] \qquad \textit{Wake}$$

$$\begin{aligned}\mathcal{D}^* &= \arg\max_{\mathcal{D}} \int \mathbb{P}\left[\mathcal{D},\ \theta\right] \prod_{x\in X} \sum_{p \text{ found during waking}} \mathbb{P}[x|p]\mathbb{P}[p|\mathcal{D},\ \theta]\ \mathrm{d}\theta \\ \theta^* &= \arg\max_\theta \mathbb{P}\left[\mathcal{D}^*,\theta\right] \prod_{x\in X} \sum_{p \text{ found during waking}} \mathbb{P}[x|p]\mathbb{P}[p|\mathcal{D}^*,\theta]\end{aligned} \qquad \textit{Consolidation sleep}$$

$$Q(p|x) \approx \mathbb{P}[p|x,\mathcal{D}^*,\theta^*] \qquad \textit{Dream sleep}$$

which serves to maximize a lower bound on the posterior over $(\mathcal{D},\theta)$ given $X$ (Appendix A.1).

This 3-phase inference procedure works through two distinct kinds of bootstrapping. During each sleep cycle the next DSL bootstraps off the primitives learned during earlier cycles, growing an increasingly deep learned program representation. In tandem the DSL and recognition model bootstrap each other: a more finely tuned DSL yields richer dreams for the recognition model to learn from, while a more accurate recognition model solves more tasks during waking which then feed into the next DSL.

Waking consists of searching for task-specific programs with high posterior probability, or programs which are a priori likely and which solve a task. We find programs solving a task by enumerating programs from the DSL in decreasing order of their probability under the recognition model, and then check if a program $p$ assigns positive probability to a task ($\mathbb{P}[x|p] > 0$). We represent programs as polymorphicly typed $\lambda$-calculus expressions, an expressive formalism including conditionals, variables, higher-order recursive functions, and the ability to define new functions.

### 2.1 Consolidation-Sleep: Growing a Domain Specific Language

The DSL offers a set of abstractions that allow an agent to concisely express solutions to the tasks at hand. We automatically discover these new abstractions by combining two ideas. First, we build on techniques from the programming languages community to develop a new algorithm for automatically refractoring programs, where this refactoring exposes common reused subexpressions across the programs found during waking. Second, we use this automatic refactoring process to search for DSLs that maximally compress these programs by incorporating reused subexpressions into the DSL.

Mathematically this compression takes the form of finding the DSL maximizing $\int \mathbb{P}[\mathcal{D},\theta]\mathbb{P}[X|\mathcal{D},\theta]\ \mathrm{d}\theta$ (Sec. 2). We replace this marginal with an AIC approximation [1] and marginalize over refactorings of programs found during waking, minimizing the following expression, which can be interpreted as a kind of compression:

$$\underbrace{-\log\mathbb{P}[\mathcal{D}] + \min_\theta \Bigg( -\log\mathbb{P}[\theta|\mathcal{D}] + \|\theta\|_0}_{\text{Description length of }(\mathcal{D},\theta)} + \sum_{x\in X} \underbrace{-\log \sum_{\substack{p \text{ a refactoring of } p' \\ p' \text{ found during waking}}} \mathbb{P}[x|p]\mathbb{P}[p|\mathcal{D},\theta] \Bigg)}_{\text{Description length of programs for task } x} \tag{1}$$

But a program has infinitely many possible refactorings, rendering Eq. 1 intractable. Rather than consider every

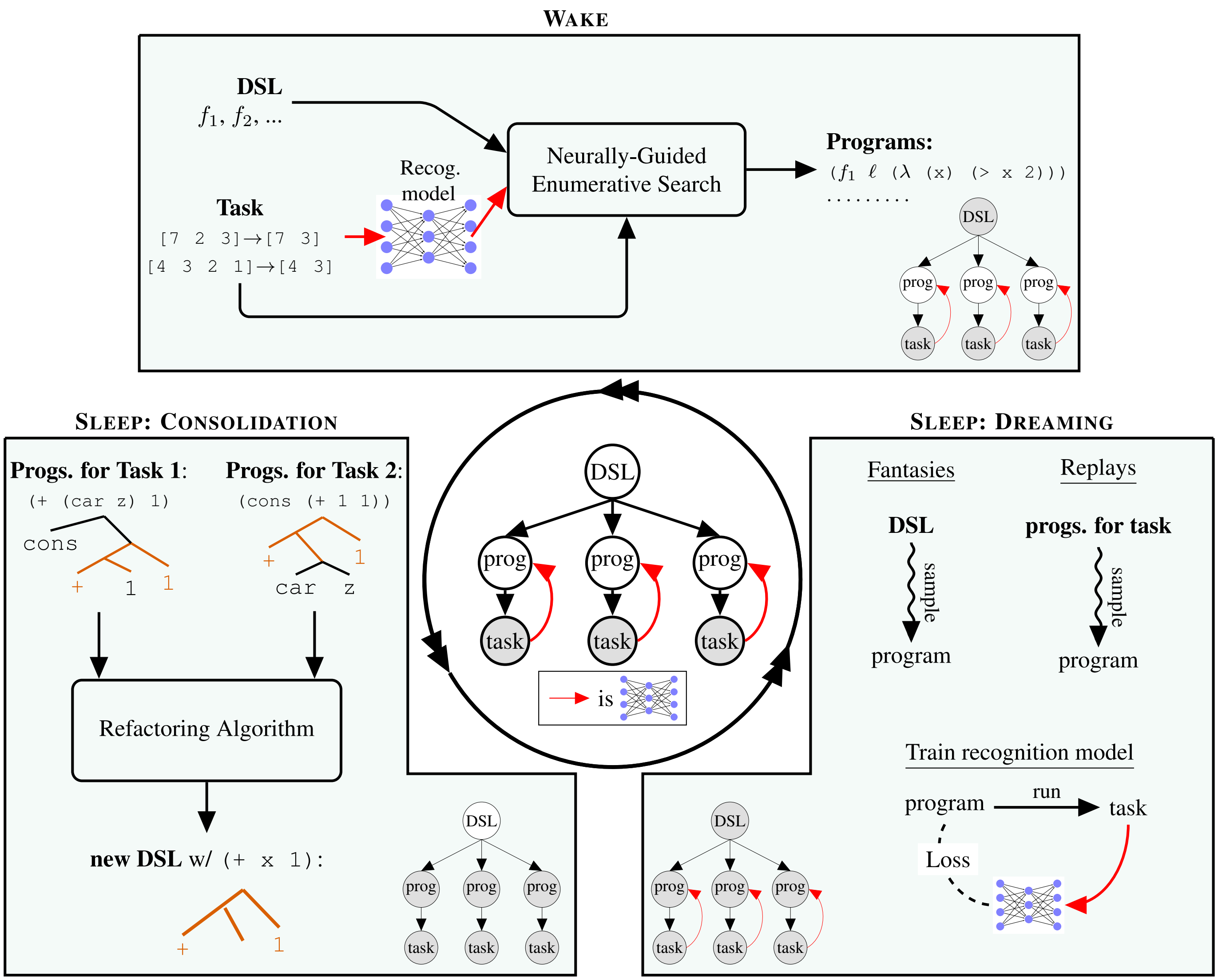



Figure 3: **Middle:** DreamCoder as a graphical model. Agent observes programming tasks (e.g., input/outputs for list processing or images for graphics programs), which it explains with latent programs, while jointly inferring a latent Domain Specific Language (DSL) capturing cross-program regularities. A neural network, called the *recognition model* (red arrows) is trained to quickly infer programs with high posterior probability. **Top**: Wake phase infers programs while holding the DSL and recognition model fixed. **Left**: Sleep (Consolidation) phase infers DSL while holding the programs fixed by refactoring programs found during waking and extracting common components. **Right**: Sleep (Dreaming) phase trains recognition model to predict approximate posterior over programs conditioned on task. Trained on 'Fantasies' (programs sampled from DSL) & 'Replays' (programs found during waking).

refactoring we bound the number of $\lambda$-calculus evaluation steps separating a program from its refactoring. Now the number of refactorings is finite but astronomically large: Figure 4A diagrams a problem where the agent rediscovers the higher-order function `map` starting from the basics of Lisp and the Y-combinator, but where there are approximately $10^{14}$ possible refactorings – a quantity that grows exponentially both as a function of program size and a function of the bound on evaluation steps. How can we tame this combinatorial explosion?

To resolve this exponential growth we introduce a new data structure combining ideas from version space algebras [21, 26, 29] and equivalence graphs [38]. A version space is a tree-shaped data structure that compactly represents a large set of programs and supports efficient set operations like union, intersection, and membership checking, while equivalence graphs are data structures that track semantic equivalences between program subexpressions. In Appendix A.5.1, we give a dynamic program that takes as input a program and then outputs a version space containing its refactorings while tracking semantically equivalent subexpressions. Figure 4B diagrams a subtree of a version space containing refactorings of a small program. Our technique is substantially more efficient than explicitly representing the space of possible refactorings: for the example in Figure 4A, we represent the space of refactorings using a version space with $10^6$ nodes, which encodes $10^{14}$ refactorings. Appendix A.5 specifies how we combine this probabilistic and symbolic machinery to update the DSL. At a high level, our approach is to search locally through the space of DSLs, proposing small changes until Eq. 1 fails to decrease.

## 2.2 Dream Sleep: Training a Neural Recognition Model

During "dreaming" the system learns a recognition model that guides program search. It learns from (program, task) pairs drawn from two sources of self-supervised data: *replays* of programs discovered during waking, and *fantasies*, or programs drawn from the DSL. Replays ensure that the recognition model is trained on the actual tasks it needs to solve, and does not forget how to solve them. Fantasies ensure that the recognition model has a large and highly varied corpus of (program, task) pairs to learn from.

Formally, the recognition model $Q(p|x)$ should approximate the posterior $\mathbb{P}[p|\mathcal{D},\theta,x]$. We can either train $Q$ to perform full posterior inference by minimizing the expected KL-divergence, $\mathbb{E}\left[\text{KL}\left(\mathbb{P}[p|x,\mathcal{D},\theta]\|Q(p|x)\right)\right]$, or we can train $Q$ to perform MAP inference by maximizing $\mathbb{E}\left[\max_{p \text{ maxing } \mathbb{P}[\cdot|x,\mathcal{D},\theta]} \log Q(p|x)\right]$, where in both cases the expectation is taken over tasks. Taking this expectation over the empirical distribution of tasks trains $Q$ on replays; taking it over samples from the generative model trains $Q$ on fantasies. We define a pair of alternative objectives for the recognition model, $\mathcal{L}^{\text{posterior}}$ and $\mathcal{L}^{\text{MAP}}$, which either train $Q$ to perform full posterior inference or MAP inference, respectively. These objectives combine replays and fantasies:

$$\mathcal{L}^{\text{posterior}} = \mathcal{L}^{\text{posterior}}_{\text{Replay}} + \mathcal{L}^{\text{posterior}}_{\text{Fantasy}} \qquad \mathcal{L}^{\text{MAP}} = \mathcal{L}^{\text{MAP}}_{\text{Replay}} + \mathcal{L}^{\text{MAP}}_{\text{Fantasy}}$$

$$\mathcal{L}^{\text{posterior}}_{\text{Replay}} = \mathbb{E}_{x\sim X}\left[\sum_{p\in\mathcal{B}_x}\frac{\mathbb{P}\left[x,p|\mathcal{D},\theta\right]\log Q(p|x)}{\sum_{p'\in\mathcal{B}_x}\mathbb{P}\left[x,p'|\mathcal{D},\theta\right]}\right] \qquad \mathcal{L}^{\text{MAP}}_{\text{Replay}} = \mathbb{E}_{x\sim X}\left[\max_{\substack{p\in\mathcal{B}_x \\ p \text{ maxing } \mathbb{P}[\cdot|x,\mathcal{D},\theta]}} \log Q(p|x)\right]$$

$$\mathcal{L}^{\text{posterior}}_{\text{Fantasy}} = \mathbb{E}_{(p,x)\sim(\mathcal{D},\theta)}\left[\log Q(p|x)\right] \qquad \mathcal{L}^{\text{MAP}}_{\text{Fantasy}} = \mathbb{E}_{x\sim(\mathcal{D},\theta)}\left[\max_{\substack{p \\ p \text{ maxing } \mathbb{P}[\cdot|x,\mathcal{D},\theta]}} \log Q(p)\right]$$

We maximize $\mathcal{L}^{\text{MAP}}$ rather than $\mathcal{L}^{\text{posterior}}$ for two reasons: $\mathcal{L}^{\text{MAP}}$ prioritizes the shortest program solving a task, thus more strongly accelerating enumerative search during waking; and, combined with our parameterization of $Q$, described next, we will show that $\mathcal{L}^{\text{MAP}}$ forces the recognition model to break symmetries in the space of programs.

**Parameterizing $Q$.** The recognition model predicts a fixed-dimensional tensor encoding a distribution over subroutines in the DSL, conditioned on the local context in the syntax tree of the program. This local context consists of the parent node in the syntax tree, as well as which argument is being generated, functioning as a kind of 'bigram' model over trees. Figure 5 (left) diagrams this generative process. This parameterization confers three main advantages: (1) it supports fast enumeration and sampling of programs, because the recognition model only runs once per task, like in [3, 11, 25] – thus we can fall back on fast enumeration if the target program is unlike the training programs; (2) the recognition model provides fine-grained information about the structure of the target program, similar to [9, 41]; and (3) in conjunction with $\mathcal{L}^{\text{MAP}}$ the recognition model learns to break symmetries in the space of programs.

**Symmetry breaking.** A good DSL not only exposes high-level building blocks, but also carefully restricts the ways in which those building blocks are allowed to compose. For example, a DSL for arithmetic should disallow adding zero, or force right-associative addition. A bigram parameterization of the recognition model, combined with the $\mathcal{L}^{\text{MAP}}$

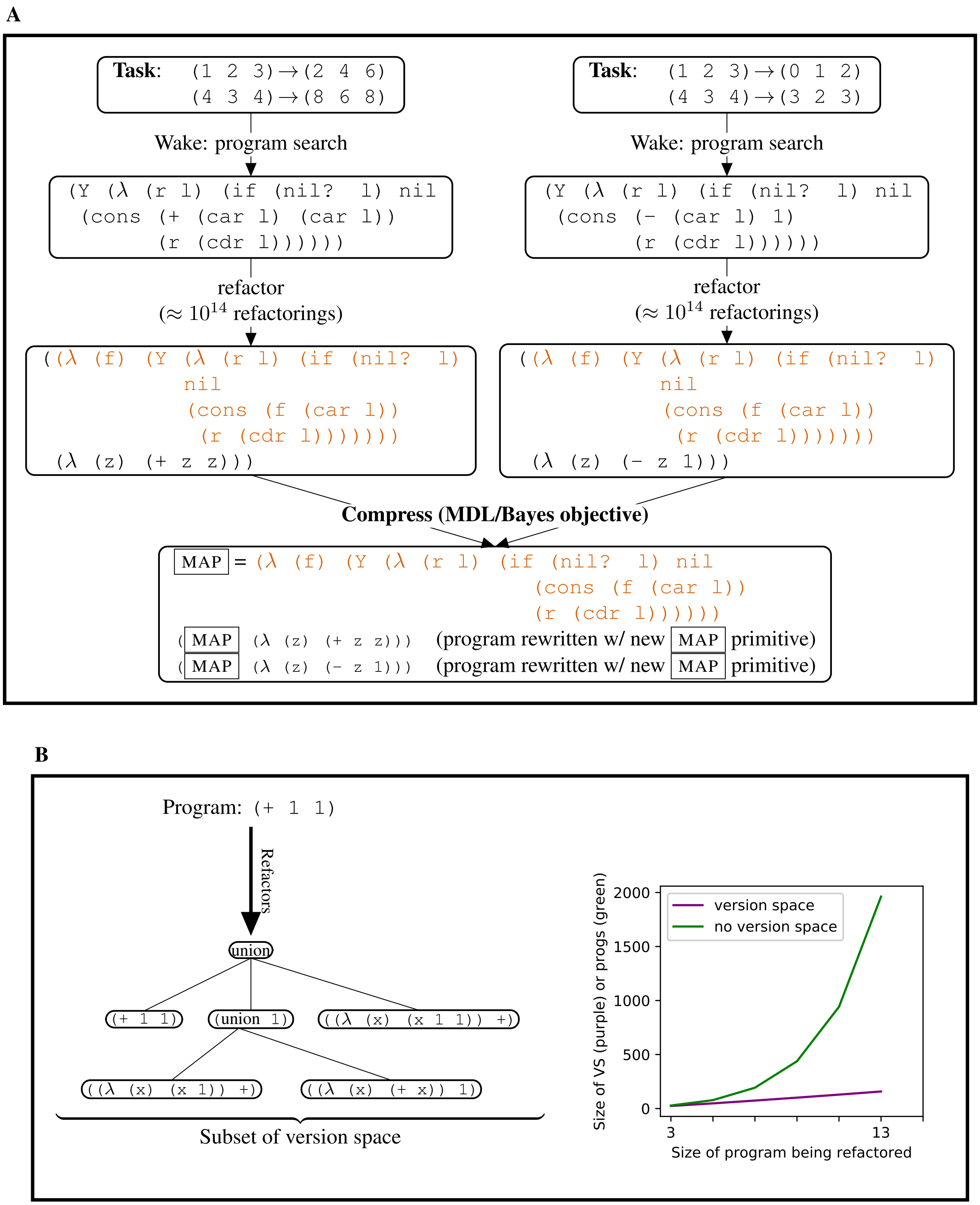
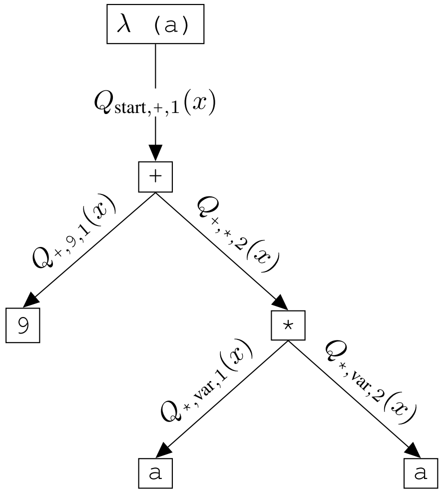



Figure 4: DSL learning as code refactoring. **Panel A:** During waking we discover programs for each task, then refactor the code from those programs to expose common subprograms (highlighted in orange). Common subprograms are incorporated into the DSL when they increase a Bayesian objective. Intuitively, these new DSL components best compress the programs found during waking. **Panel B:** # of possible refactorings grows exponentially with program size, so we represent refactorings using version spaces, which augment syntax trees with a *union* operator whose children are themselves version spaces. Right graph: version spaces are exponentially more efficient than explicitly constructing set of refactorings. In this graph, refactored programs are of the form $1 + 1 + \cdots + 1$.




| | Unigram | Bigram |
|---|---|---|
| $\mathcal{L}^{\text{posterior}}$ | *Three samples:*<br>`(+ 1 0)`<br>`(+ (+ 0 0)`<br>`   (+ 1 0))`<br>`(+ 1 1)`<br>63.0% right-associative<br>37.4% `+0`'s | *Three samples:*<br>`0`<br>`(+ (+ (+ 0 0)`<br>`      (+ 0 1)) 1)`<br>`1`<br>55.8% right-associative<br>31.9% `+0`'s |
| $\mathcal{L}^{\text{MAP}}$ | *Three samples:*<br>`1`<br>`(+ 1 (+ 1 (+ (+ 1`<br>`   (+ 1 1)) 1)))`<br>`(+ (+ 1 1) 1)`<br>48.6% right-associative<br>0.5% `+0`'s | *Three Samples:*<br>`(+ 1 (+ 1 (+ 1`<br>`     (+ 1 (+ 1 1)))))`<br>`0`<br>`(+ 1 (+ 1 (+ 1 1)))`<br>**97.9% right-associative**<br>2.5% `+0`'s |

Figure 5: **Left:** Bigram parameterization of distribution over programs predicted by recognition model. Here the program (syntax tree shown above) is `(λ (a) (+ 9 (* a a )))`. Each conditional distribution predicted by the recognition model is written $Q_{\text{parent,child,argument index}}(x)$, where $x$ is a task. **Right:** Agent learns to break symmetries in program space only when using both bigram parameterization and $\mathcal{L}^{\text{MAP}}$ objective, associating addition to the right and avoiding adding zero. % right-associative calculated by drawing 500 samples from $Q$. $\mathcal{L}^{\text{MAP}}$/Unigram agent incorrectly learns to never generate programs with `0`'s, while $\mathcal{L}^{\text{MAP}}$/Bigram agent correctly learns that `0` should only be disallowed as an argument of addition. Tasked with building programs from `+`, `1`, and `0`.

training objective, interact in a way that breaks symmetries like these, allowing the agent to more efficiently explore the space of programs. This interaction occurs because the bigram parameterization can disallow DSL primitives depending on their local syntactic context, while the $\mathcal{L}^{\text{MAP}}$ objective forces all probability mass onto a single member of a set of syntactically distinct but semantically equivalent expressions (Appendix A.6). We experimentally confirm this symmetry-breaking behavior by training recognition models that minimize either $\mathcal{L}^{\text{MAP}}/\mathcal{L}^{\text{posterior}}$ and which use either a bigram parameterization/unigram[1] parameterization. Figure 5 (right) shows the result of training $Q$ in these four regimes and then sampling programs. On this particular run, the combination of bigrams and $\mathcal{L}^{\text{MAP}}$ learns to avoid adding zero and associate addition to the right — different random initializations lead to either right or left association.

# 3 Experiments

## 3.1 Programs that manipulate sequences

We first apply DREAMCODER to two classic benchmark domains: list processing and text editing. In both cases we solve tasks specified by a input/output examples, starting with a generic functional programming basis: `foldr`, `unfold`, `if`, `map`, `length`, `index`, `=`, `+`, `-`, `0`, `1`, `cons`, `car`, `cdr`, `nil`, and `is-nil`.

### 3.1.1 List Processing

We took 218 list manipulation tasks from our previous work [11], each with 15 input/output examples. In solving these tasks, the system composed 16 new DSL components, and discovered multiple higher-order functions. Each round of memory consolidation built on components discovered in earlier sleep cycles — for example the agent first learns the higher-order function `filter`, uses `filter` to learn to take the maximum element of a list, then uses that routine to learn a new component for extracting the $n^{\text{th}}$ largest element of a list, which it finally uses to solve a task involving sorting a list of numbers (Figure **??**). This incremental, modular learning of deep hierarchies of DSL components occurs because of the alternation between code writing (during waking) and code refactoring (during the consolidation phase of sleep).

[1] In the unigram variant $Q$ predicts a $|\mathcal{D}|+1$-dimensional vector: $Q(p|x) = \mathbb{P}[p|\mathcal{D}, \theta_i = Q_i(x)]$, and was used in our prior work [11]

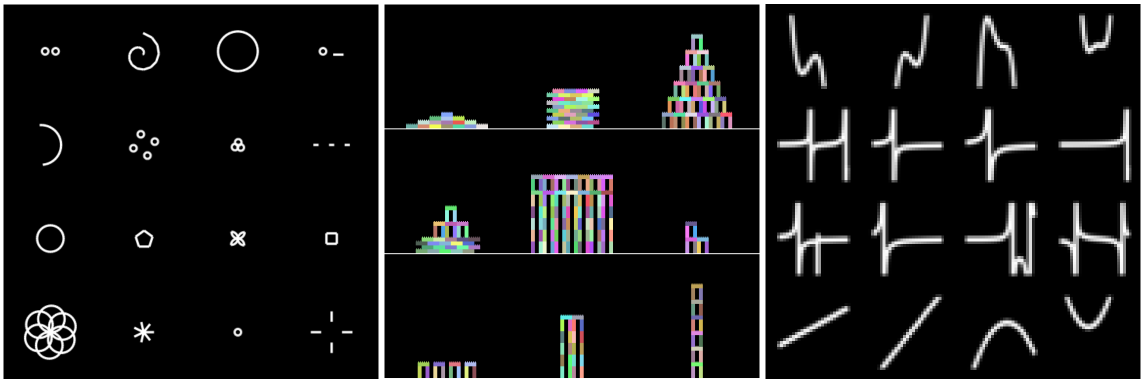


Figure 6: Three domains where the agent infers a program from visual input. **Left**: 16 (out of 160) LOGO graphics tasks. Agent writes a program controlling a ‘pen’ that draws the target picture. **Middle**: 9 (out of 112) tower building tasks. Agent writes a program controlling a ‘hand’ that builds the target tower. **Right**: 16 (out of 200) symbolic regression tasks. Agent writes a program containing continuous real numbers that fits the points along the curve.

### 3.1.2 Text Editing

Synthesizing programs that edit text is a classic problem in the programming languages and AI literatures [21], and algorithms that synthesize text editing programs ship in Microsoft Excel [**?**]. This prior work uses hand-engineered DSLs and hand-engineered search strategies. Here, we will show that we can jointly learn both these ingredients and surpass the state-of-the-art domain-general program synthesizers on a standard text editing benchmark.

We trained our system on 128 automatically-generated text editing tasks, with 4 input/output examples each. We tested, but did not train, on the 108 text editing problems from the SyGuS [2] program synthesis competition. Before any learning, DREAMCODER solves 3.7% of the problems within 10 minutes with an average search time of 235 seconds. After learning, it solves 79.6%, and does so much faster, solving them in an average of 40 seconds. As of the 2017 SyGuS competition, the best-performing synthesizer (CVC4) solves 82.4% of the problems — but here, the competition conditions are 1 hour & 8 CPUs per problem, and with this more generous compute budget we surpass this previous result and solve 84.3% of the problems. SyGuS additionally comes with a different hand-engineered DSL *for each text editing problem*. Here we learned a single DSL that applied generically to all of the tasks, and perform comparably to the best prior work.

## 3.2 Programs from visual input

We consider three domains where the agent must infer a program from an image (Figure 6). First we consider programs that make plans and take actions: drawing pictures and building towers out of blocks (Sec. 3.2.1-3.2.2).

### 3.2.1 Programs that draw pictures

Procedural visual concepts are studied across AI and cognitive science — Bongard problems [5], Raven’s progressive matrices [30], and Lake et al.’s BPL model of omniglot [20] are prominent examples. Here we take inspiration from LOGO Turtle graphics [39], tasking our agent with drawing a modest corpus of images while equipping it with control over a ‘pen’, along with arithmetic operations on angles and distances.

Inside its learned DSL we find interpretable parametric drawing routines corresponding to the families of visual objects in its training data, like polygons, circles, and spirals (Figure 7, left) – without supervision the agent has learned the basic types of objects in its visual world. It additionally learns more abstract visual relationships, like rotational symmetry, which it models by incorporating a higher-order function into its DSL (Figure 7, right).

What does DREAMCODER dream of? Prior to learning samples from the DSL are simple and largely unstructured (Figure 8, left). After training the samples become richly structured (Figure 8, right), compositionally recombining latent building blocks and motifs acquired from the training data. This offers a visual window into how the generative model bootstraps recognition model training: as the DSL grows more finely tuned to the domain, the neural net gets richer and more highly varied training data.

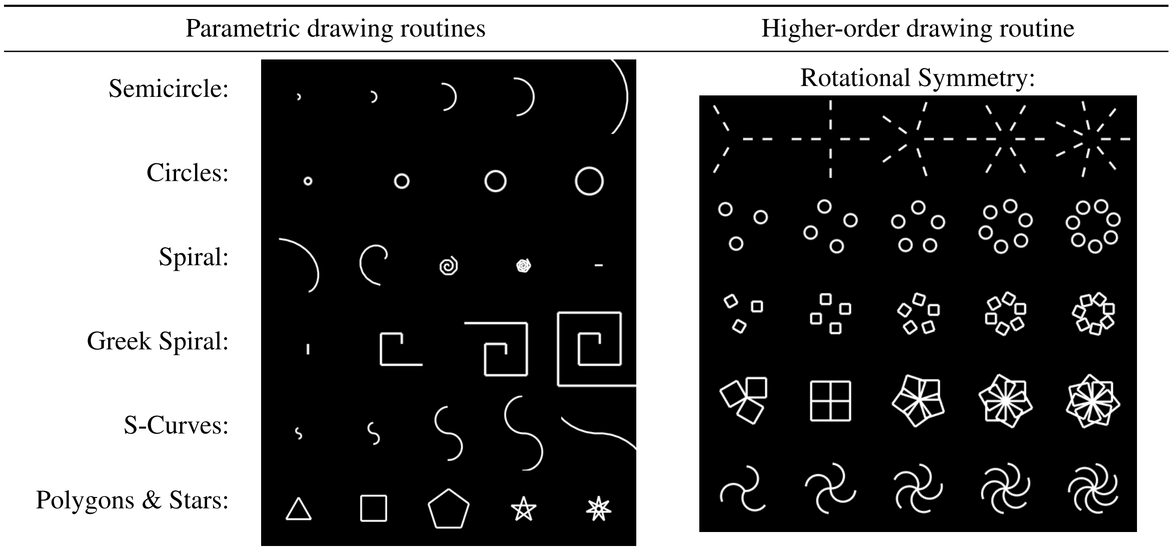



Figure 7: Example primitives learned by DREAMCODER when trained on tasks in Figure **??**. Agent learns parametric routines for drawing families of curves (left) as well as subroutines that take entire programs as input (right). Each row of images on the left is the same code executed with different parameters. Each image on the right is the same code executed with different parameters and with a different subprogram provided as input.

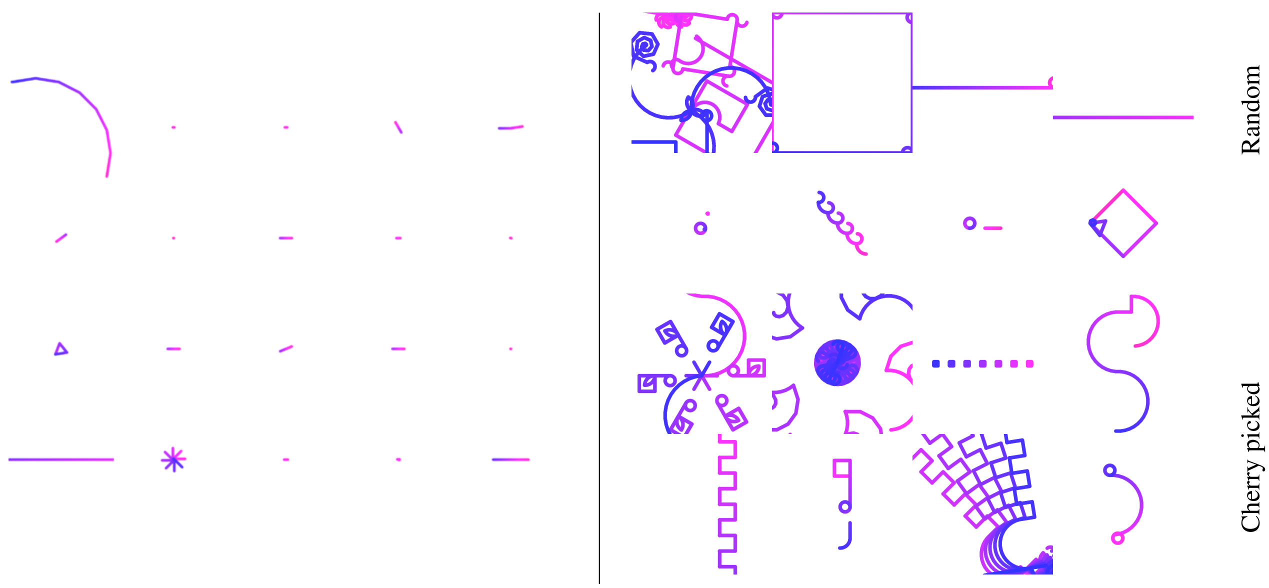



Figure 8: Sixteen dreams, or samples, from the DSL before (left) and after (right) training on tasks in Figure **??**. Blue: where the agent started drawing. Pink: where the agent ended drawing.

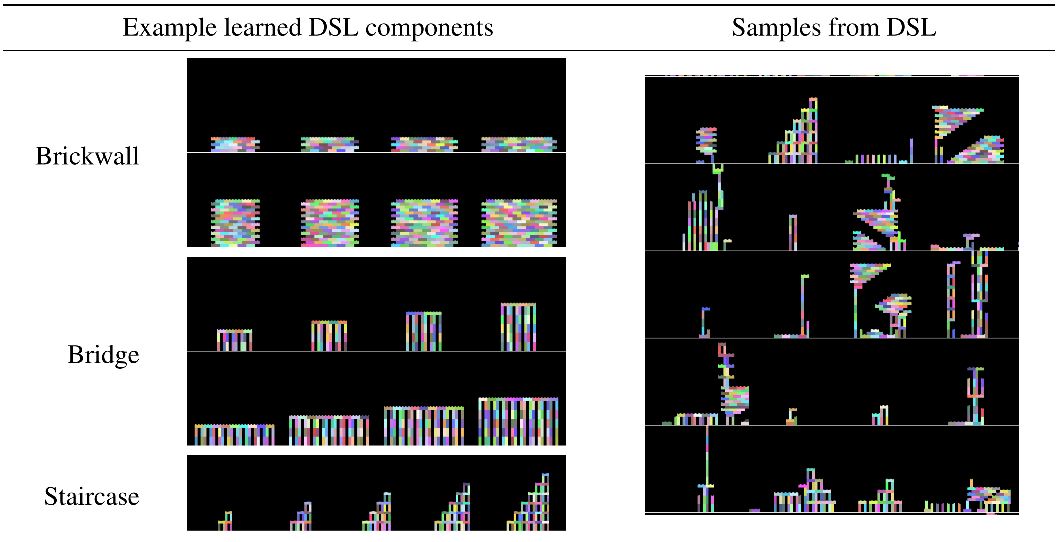



Figure 9: **Left**: Three (out of 19) learned DSL components for building towers out of Lego-style blocks. These components act like parametric options [37], giving higher-level building blocks that the agent can use to plan. **Right**: 16 random samples, or 'dreams', from learned DSL.

#### 3.2.2 Building towers out of 'Lego' blocks

Inspired by the classic AI 'copy task' — where an agent must look at an image of a tower made of toy blocks and re-create the tower [**?**] — we give DREAMCODER 112 tower 'copy tasks' (Figure **??**). Here the agent observes both an image of a tower and the locations of each of its blocks, and must write a program that plans how a simulated hand would build the tower. These towers are built from Lego-style blocks that snap together on a discrete grid. The system starts with the same control flow primitives as with LOGO graphics, and learns parametric 'options' for building blocks towers (Figure 9), inferring concepts like arches, staircases, bridges and brick walls.

#### 3.2.3 Symbolic Regression

Here, the agent observes points along the curve of a function, and must write a program that fits those points. We initially equip our learner with addition, multiplication, and division, and task it with solving 200 symbolic regression problems, each either a polynomial or rational function. The recognition model is a convnet that observes an image of the target function's graph (Fig. 6, rightmost) — visually, different kinds of polynomials and rational functions produce different kinds of graphs, and so the convnet can look at a graph and predict what kind of function best explains it. A key difficulty, however, is that these problems are best solved with programs containing real numbers. Our solution to this difficulty is to enumerate programs with real-valued parameters, and then fit those parameters by automatically differentiating through the programs the system writes and use gradient descent to fit the parameters. We define the likelihood model, $\mathbb{P}[x|p]$, by assuming a Gaussian noise model for the input/output examples, and penalize the use of real-valued parameters using the BIC [4].

We learn a DSL containing 13 new functions, mainly templates for different pieces of polynomials or ratios of polynomials. The model also learns to find programs minimizing the number of continuous parameters — for example, learning to represent linear functions with `(* real (+ x real))`. This phenomenon arises from our Bayesian framing: both the generative model's bias toward shorter programs, and the likelihood model's BIC penalty.

### 3.3 Quantitative Results on Held-Out Tasks

To evaluate the relative importance of DSL learning and recognition model training, we evaluate on held-out testing tasks for each of our domains, measuring both how many tasks are solved and how long it takes to solve them across successive wake/sleep iterations (Fig. 10). We always solve more held-out tasks – and generally solve them in less time – with both components combined. Why? One hypothesis is that some tasks are best solved by DSL learning and others

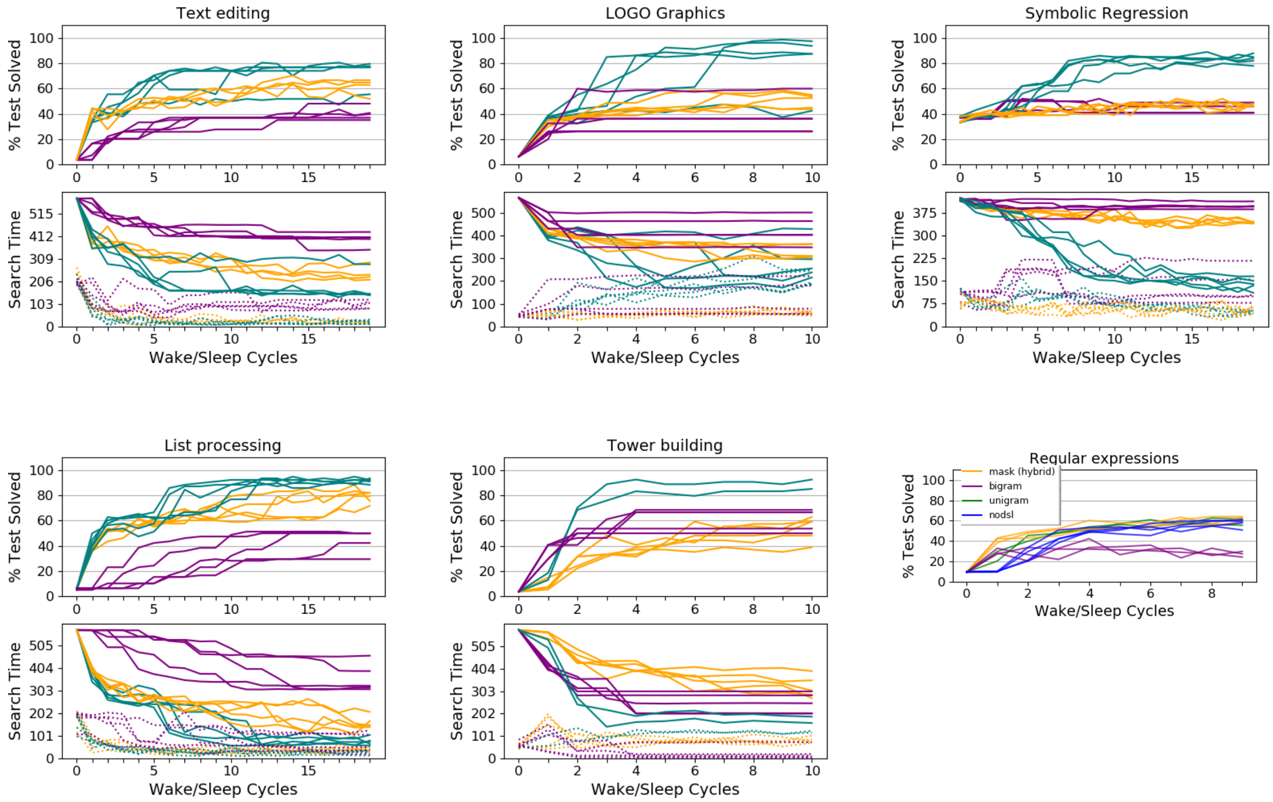



Figure 10: Test set performance across wake/sleep iterations. Each curve is a run with a different random seed. Teal: Full model. Orange: Dreaming only (no DSL learning). Purple: Consolidation only (no recognition model). Search time plots show solid lines (time averaged over all tasks) and dotted lines (time averaged over solved tasks).

by a neural network, and so including both of our sleep cycles takes the union of these sets of tasks. Another hypothesis is that the DSL and neural network interact synergistically, bootstrapping off each other to solve tasks for which neither alone suffice. We evaluate the relative weight of these interactions by computing the ratio of the tasks solved uniquely by the full model to

# 4 Discussion

## 4.1 Learning from Scratch

A long-standing dream within the program induction community is "learning from scratch": starting with a *minimal* Turing-complete programming language, and then learning to solve a wide swath of induction problems [35, 33, 16, 36]. All existing systems, including ours, fall far short of this dream, and it is unclear (and we believe unlikely) that this dream could ever be fully realized. How far can we push in this direction? "Learning from scratch" is subjective, but a reasonable starting point is the set of primitives provided in 1959 Lisp [24]: these include conditionals, recursion, arithmetic, and the list operators `cons`, `car`, `cdr`, and `nil`. A basic first goal is to start with these primitives, and then recover a DSL that more closely resembles modern functional languages like Haskell and OCaml. Recall (Sec. 3.1) that we initially provided our system with functional programming routines like `map` and `fold`.

We ran the following experiment: DREAMCODER was given a subset of the 1959 Lisp primitives, and tasked with solving 18 programming exercises. A key difference between this setup and our previous experiments is that, for this experiment, the system is given primitive recursion, whereas previously we had sequestered recursion within higher-order functions like `map`, `fold`, and `unfold`.

After running for 93 hours on 48 CPUs, our algorithm solves these 18 exercises, along the way assembling a DSL with a modern repertoire of functional programming idioms and subroutines, including `map`, `fold`, `unfold`, `index`, `length`, and arithmetic operations like building lists of natural numbers between an interval (see Appendix A.9).

We believe that program learners should *not* start from scratch, but instead should start from a rich, domain-agnostic basis like those embodied in the standard libraries of modern languages. What this experiment shows is that DREAMCODER doesn't *need* to start from a rich basis, and can in principle recover many of the amenities of modern programming systems, provided it is given enough computational power and a suitable spectrum of tasks.

### 4.2 DREAMCODER and the Exploration-Compression family of algorithms

Our work sits within the Exploration-Compression (EC) family of algorithms. EC [7] is a program induction framework where an agent alternates between searching, or 'exploring', the space of programs, and then updating its search procedure by compressing programs found during exploration. DREAMCODER grew directly out of research on EC-style systems and both of our sleep phases can be interpreted as a kind of compression: consolidation aims to compactly refactor code, while the recognition model aims to encode a program in as few bits as possible, conditioned on a task. Other offshoots of EC include our previous work, $EC^2$ [11], the direct predecessor of DREAMCODER; and the neurosymbolic framework in [22]. Closely aligned ideas go much further back [36, 33].

### 4.3 Acquiring Domain Expertise

One interpretation of our system is as a model of the acquisition of domain expertise. Humans can acquire expertise across many domains – cooking, coding, music, architecture, painting, tennis, or calculus, to name a handful of examples, and every child develops expertise in natural language, intuitive physics, motor control, kinship relationships, and more. Domain experts learn domain-specific abstractions, similar to a DSL: for example, a expert chef knows what combinations of seasonings go together, or an expert mathematician knows a wide set of useful theorems and lemmas. Jointly, experts learn how to recognize when to use these abstractions to solve problems. Becoming an expert involves a learning trajectory that unfolds over relatively long time scales, but has modest data requirements relative to the dominant machine learning paradigms. For example, basic competency in cooking or coding might require on the order of learning a hundred recipes or solving a hundred programming exercises. It is this learning regime that we have targeted with DREAMCODER: Learning about a domain from at most several hundred tasks, but where the learning unfolds over many wake/sleep cycles.

### 4.4 Prospects for program Induction as part of the generic AI toolkit

Our aim with DREAMCODER is to chart a path by which program induction can become more broadly useful for AI. This means viewing the AI landscape through the lens of program learning, including the terrain considered here — simple kinds of generative modeling, inverse graphics, planning, and programming by example — but also many others like reinforcement learning, commonsense reasoning, natural language understanding, and causal inference. Can program induction rise to the challenge? We believe it can, provided we push jointly along many different axes of AI research; and provided we continue to integrate learning algorithms – both symbolic and neural, both top-down and bottom-up – into our artificial agents.

Our system, with its learned DSL and neural recognition model, is one embodiment of this hybrid symbolic/neural approach, and enjoys some success across small-scale problems in the different domains considered here. Scaling to larger problems, such as inferring 3-D object models (vs LOGO/Turtle), learning natural image grammars (vs ),

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# A Appendix

## A.1 Probabilistic Formulation of DREAMCODER

Our objective is to infer the maximum a posteriori DSL $\mathcal{D}$ and parameters $\theta$. Writing $J$ for the joint probability of $(\mathcal{D}, \theta)$, this corresponds to solving

$$J(\mathcal{D},\theta) \triangleq \mathbb{P}[\mathcal{D},\theta] \prod_{x\in X} \sum_{p} \mathbb{P}[x|p]\mathbb{P}[p|\mathcal{D},\theta]$$

$$\mathcal{D}^* = \arg\max_{\mathcal{D}} \int J(\mathcal{D},\theta)\ \mathrm{d}\theta \qquad \theta^* = \arg\max_{\theta} J(\mathcal{D}^*,\theta) \tag{2}$$

where $\mathbb{P}[x|p]$ scores the likelihood of a task $x \in X$ given a program $p$.[2]

[2]For example, for list processing, the likelihood is 1 if the program predicts the observed outputs on the observed inputs, and 0 otherwise; when learning a generative model or probabilistic program, the likelihood is the probability of the program sampling the observation.

Evaluating Eq. 2 entails marginalizing over the infinite set of all programs – which is impossible. We make a particle-based approximation to Eq. 2 and instead marginalize over a finite **beam** of programs, with one beam per task, collectively written $\{\mathcal{B}_x\}_{x\in X}$. This particle-based approximation is written $\mathscr{L}(\mathcal{D},\theta,\{\mathcal{B}_x\})$ and acts as a lower bound on the joint density:

$$J(\mathcal{D},\theta) \geq \mathscr{L}(\mathcal{D},\theta,\{\mathcal{B}_x\}) \triangleq \mathbb{P}[\mathcal{D},\theta] \prod_{x\in X} \sum_{p\in\mathcal{B}_x} \mathbb{P}[x|p]\mathbb{P}[p|\mathcal{D},\theta], \text{ where } |\mathcal{B}_x| \text{ is small} \tag{3}$$

In all of our experiments we set the maximum beam size $|\mathcal{B}_x|$ to 5.

Wake and sleep cycles correspond to alternate maximization of $\mathscr{L}$ w.r.t. $\{\mathcal{B}_x\}_{x\in X}$ (**Wake**) and $(\mathcal{D},\theta)$ (**Consolidation**):

**Wake: Maxing $\mathscr{L}$ w.r.t. the beams.** Here $(\mathcal{D},\theta)$ is fixed and we want to find new programs to add to the beams so that $\mathscr{L}$ increases the most. $\mathscr{L}$ most increases by finding programs where $\mathbb{P}[x|p]\mathbb{P}[p|\mathcal{D},\theta] \propto \mathbb{P}[p|x,\mathcal{D},\theta]$ is large, i.e., programs with high posterior probability, which is the search objective during waking.

**Sleep (Consolidation): Maxing $\mathscr{L}$ w.r.t. the DSL.** Here $\{\mathcal{B}_x\}_{x\in X}$ is held fixed and the problem is to search the discrete space of DSLs and find one maximizing $\int \mathscr{L}\ \mathrm{d}\theta$, and then update $\theta$ to $\arg\max_\theta \mathscr{L}(\mathcal{D},\theta,\{\mathcal{B}_x\})$.

Finding programs solving tasks is difficult because of the infinitely large, combinatorial search landscape. We ease this difficulty by training a neural recognition model, $Q(p|x)$, during the **Dreaming** phase: $Q$ is trained to approximate the posterior over programs, $Q(p|x) \approx \mathbb{P}[p|x,\mathcal{D}] \propto \mathbb{P}[x|p]\mathbb{P}[p|\mathcal{D}]$. Thus training the neural network amortizes the cost of finding programs with high posterior probability.

**Sleep (Dreaming): tractably maxing $\mathscr{L}$ w.r.t. the beams.** Here we train $Q(p|x)$ to assign high probability to programs $p$ where $\mathbb{P}[x|p]\mathbb{P}[p|\mathcal{D},\theta]$ is large, because incorporating those programs into the beams will most increase $\mathscr{L}$.

## A.2 DREAMCODER pseudocode

Algorithm 1 specifies how we integrate wake and sleep cycles.

**Algorithm 1** Full DREAMCODER algorithm

**function** DREAMCODER$(\mathcal{D}, X)$:
**Input:** Initial DSL $\mathcal{D}$, tasks $X$
**Output:** Infinite stream of DSLs, recognition models, and beams
**Hyperparameters:** Batch size $B$, enumeration timeout $T$, maximum beam size $F$
$\theta \leftarrow$uniform distribution
$\mathcal{F}_x \leftarrow \varnothing, \forall x \in X$ ▷ Initialize beams to be empty
**while** true **do** ▷ Loop over epochs
  shuffle $\leftarrow$ random permutation of $X$ ▷ Randomize minibatches
  **while** shuffle is not empty **do** ▷ Loop over minibatches
    batch $\leftarrow$ first $B$ elements of shuffle ▷ Next minibatch of tasks
    shuffle $\leftarrow$ shuffle with first $B$ elements removed
    $\forall x \in$ batch: $\mathcal{F}_x \leftarrow \mathcal{F}_x \cup \{p \mid p \in \text{enumerate}(\mathbb{P}[\cdot|\mathcal{D},\theta], T) \text{ if } \mathbb{P}[x|p] > 0\}$ ▷ Wake
    Train $Q(\cdot|\cdot)$ to minimize $\mathcal{L}^{\text{MAP}}$ across all $\{\mathcal{F}_x\}_{x\in X}$ ▷ Dream Sleep
    $\forall x \in$ batch: $\mathcal{F}_x \leftarrow \mathcal{F}_x \cup \{p \mid p \in \text{enumerate}(Q(\cdot|x), T) \text{ if } \mathbb{P}[x|p] > 0\}$ ▷ Wake
    $\forall x \in$ batch: $\mathcal{F}_x \leftarrow$ top $F$ elements of $\mathcal{F}_x$ as measured by $\mathbb{P}[\cdot|x,\mathcal{D},\theta]$ ▷ Keep top $F$ programs
    $\mathcal{D},\theta,\{\mathcal{F}_x\}_{x\in X} \leftarrow$CONSOLIDATE$(\mathcal{D},\theta,\{\mathcal{F}_x\}_{x\in X})$ ▷ Consolidation Sleep
    **yield** $(\mathcal{D},\theta)$, $Q$, $\{\mathcal{F}_x\}_{x\in X}$ ▷ Yield the updated DSL, recognition model, and solutions found to tasks
  **end while**
**end while**

## A.3 Generative model over programs

Algorithm 2 gives a stochastic procedure for drawing samples from $\mathbb{P}[\cdot|\mathcal{D},\theta]$. It takes as input the desired type of the unknown program, and performs type inference during sampling to ensure that the program has the desired type. It also maintains a *environment* mapping variables to types, which ensures that lexical scoping rules are obeyed.

**Algorithm 2** Generative model over programs

**function** sample$(\mathcal{D}, \theta, \tau)$:
**Input:** DSL $(\mathcal{D}, \theta)$, type $\tau$
**Output:** a program whose type unifies with $\tau$
**return** sample$'(\mathcal{D}, \theta, \varnothing, \tau)$

**function** sample$'(\mathcal{D}, \theta, \mathcal{E}, \tau)$:
**Input:** DSL $(\mathcal{D}, \theta)$, environment $\mathcal{E}$, type $\tau$ — ▷ Environment $\mathcal{E}$ starts out as $\varnothing$
**Output:** a program whose type unifies with $\tau$
**if** $\tau = \alpha \rightarrow \beta$ **then** — ▷ Function type — start with a lambda
  var $\leftarrow$ an unused variable name
  body $\sim$ sample$'(\mathcal{D}, \theta, \{\text{var} : \alpha\} \cup \mathcal{E}, \beta)$ — ▷ Recursively sample function body
  **return** `(lambda (`var`)` body`)`
**else** — ▷ Build an application to give something w/ type $\tau$
  primitives $\leftarrow \{p | p : \tau' \in \mathcal{D} \cup \mathcal{E}$ if $\tau$ can unify with yield$(\tau')\}$ — ▷ Everything in scope w/ type $\tau$
  variables $\leftarrow \{p \mid p \in$ primitives and $p$ a variable$\}$
  Draw $e \sim$ primitives, w.p. $\propto \begin{cases} \theta_e & \text{if } e \in \mathcal{D} \\ \theta_{var}/|\text{variables}| & \text{if } e \in \mathcal{E} \end{cases}$
  Unify $\tau$ with yield$(\tau')$. — ▷ Ensure well-typed program
  $\{\alpha_k\}_{k=1}^{K} \leftarrow$ args$(\tau')$
  **for** $k = 1$ **to** $K$ **do** — ▷ Recursively sample arguments
    $a_k \sim$ sample$'(\mathcal{D}, \theta, \mathcal{E}, \alpha_k)$
  **end for**
  **return** `(`$e\ a_1\ a_2\ \cdots\ a_K$`)`
**end if**

**where:**
yield$(\tau) = \begin{cases} \text{yield}(\beta) & \text{if } \tau = \alpha \rightarrow \beta \\ \tau & \text{otherwise.} \end{cases}$ — ▷ Final return type of $\tau$
args$(\tau) = \begin{cases} [\alpha] + \text{args}(\beta) & \text{if } \tau = \alpha \rightarrow \beta \\ [] & \text{otherwise.} \end{cases}$ — ▷ Types of arguments needed to get something w/ type $\tau$

## A.4 Enumerative program search

Our current implementation of DREAMCODER takes the simple and generic strategy of enumerating programs in descending order of their probability under either $(\mathcal{D}, \theta)$ or $Q(p|x)$. Algorithm 2 and 5 specify procedures for sampling from these distributions, but not for enumerating from them. We combine two different enumeration strategies, which allowed us to build a massively parallel program enumerator:

- **Best-first search:** Best-first search maintains a heap of partial programs ordered by their probability — here a partial program means a program whose syntax tree may contain unspecified 'holes'. Best-first search is guaranteed to enumerate programs in decreasing order of their probability, and has memory requirements that in general grow exponentially as a function of the description length of programs in the heap (thus linearly as a function of run time).
- **Depth-first search:** Depth first search recursively explores the space of execution traces through Algorithm 2 and 5, equivalently maintaining a stack of partial programs. In general it does not enumerate programs in decreasing order of probability, but has memory requirements that grow linearly as a function of the description length of the programs in the stack (thus logarithmically as a function of run time).

Our parallel enumeration algorithm (Algorithm 3) first performs a best-first search until the best-first heap is much larger than the number of CPUs. At this point, it switches to performing many depth-first searches in parallel, initializing a depth first search with one of the entries in the best-first heap. Because depth-first search does not produce programs in decreasing order of their probability, we wrap this entire procedure up into an outer loop that first enumerates programs whose description length is between 0 to $\Delta$, then programs with description length between $\Delta$ and $2\Delta$, then $2\Delta$ to $3\Delta$, etc., until a timeout is reached. This is similar in spirit to iterative deepening depth first search [31].

## A.5 Details of DSL learning

Algorithm 4 specifies our DSL learning procedure. This integrates two toolkits: the machinery of version spaces and equivalence graphs (Appendix A.5.1) along with a probabilistic objective favoring compressive DSLs. The functions $I\beta(\cdot)$ and REFACTOR construct a version space from a program and extract the shortest program from a version space, respectively (Algorithm 4, lines 5-6, 14; Appendix A.5.1). To define the prior distribution over $(\mathcal{D}, \theta)$ (Algorithm 4, lines 7-8), we penalize the syntactic complexity of the $\lambda$-calculus expressions in the DSL, defining $\mathbb{P}[\mathcal{D}] \propto \exp(-\lambda \sum_{p \in \mathcal{D}} \text{size}(p))$ where $\text{size}(p)$ measures the size of the syntax tree of program $p$, and $\lambda$ controls how strongly we regularize the size of the DSL. We place a symmetric Dirichlet prior over the weight vector $\theta$.

To appropriately score each proposed $\mathcal{D}$ we must reestimate the weight vector $\theta$ (Algorithm 4, line 7). Although this may seem very similar to estimating the parameters of a probabilistic context free grammar, for which we have effective approaches like the Inside/Outside algorithm [19], our DSLs are context-sensitive due to the presence of variables in the programs and also due to the polymorphic typing system. Appendix A.5.4 derives a tractable MAP estimator for $\theta$.

### A.5.1 Refactoring code with version spaces

Formally, a version space is either:

- A deBuijn[3] index: written $\$i$, where $i$ is a natural number
- An abstraction: written $\lambda v$, where $v$ is a version space
- An application: written $(f\ x)$, where both $f$ and $x$ are version spaces
- A union: $\uplus V$, where $V$ is a set of version spaces
- The empty set, $\varnothing$
- The set of all $\lambda$-calculus expressions, $\Lambda$

[3]deBuijn indices are an alternative way of naming variables in $\lambda$-calculus. When using deBuijn indices, $\lambda$-abstractions are written *without* a variable name, and variables are written as the count of the number of $\lambda$-abstractions up in the syntax tree the variable is bound to. For example, $\lambda x.\lambda y.(x\ y)$ is written $\lambda\lambda(\$1\ \$0)$ using deBuijn indices. See [28] for more details.

**Algorithm 3** Parallel enumerative program search algorithm

```
function enumerate(μ, T, CPUs):
Input: Distribution over programs μ, timeout T, CPU count
Output: stream of programs in approximately descending order of probability under μ
Hyperparameter: nat increase rate Δ                                          ▷ We set Δ = 1.5
lowerBound ← 0
while total elapsed time < T do
    heap ← newMaxHeap()                                                      ▷ Heap for best-first search
    heap.insert(priority = 0, value = empty syntax tree)                     ▷ Initialize heap with start state of search space
    while 0 < |heap| ≤ 10 × CPUs do                                          ▷ Each CPU will get approximately 10 jobs (a partial program)
        priority, partialProgram ← heap.popMaximum()
        if partialProgram is finished then                                   ▷ Nothing more to fill in in the syntax tree
            if lowerBound ≤ −priority < lowerBound + Δ then
                yield partialProgram
            end if
        else
            for child ∈ children(partialProgram) do                          ▷ children(·) fills in next random choice in syntax tree.
                if − log μ(child) < lowerBound + Δ then                      ▷ Child's description length small enough
                    heap.insert(priority = log μ(child), value = child)
                end if
            end for
        end if
    end while
    yield from ParallelMap_CPUs(depthFirst(μ, T − elapsed time, lowerBound, ·), heap.values())
    lowerBound ← lowerBound + Δ                                              ▷ Push up lower bound on MDL by Δ
end while

function depthFirst(μ, T, lowerBound, partialProgram):                       ▷ Each worker does a depth first search. Enumerates
completions of partialProgram whose MDL is between lowerBound and lowerBound + Δ
stack ← [partialProgram]
while total elapsed time < T and stack is not empty do
    partialProgram ← stack.pop()
    if partialProgram is finished then
        if lowerBound ≤ − log μ(partialProgram) < lowerBound + Δ then
            yield partialProgram
        end if
    else
        for child ∈ children(partialProgram) do
            if − log μ(child) < lowerBound + Δ then                          ▷ Child's description length small enough
                stack.push(child)
            end if
        end for
    end if
end while
```

**Algorithm 4** DSL Induction Algorithm

**Input:** Set of beams $\{\mathcal{B}_x\}$
**Output:** DSL $\mathcal{D}$, weight vector $\theta$
$\mathcal{D} \leftarrow$ every primitive in $\{\mathcal{B}_x\}$
**while** true **do**
  $\forall p \in \bigcup_x \mathcal{B}_x : v_p \leftarrow I\beta(p)$ ▷ Construct a version space for each program
  Define $L(\mathcal{D}', \theta) = \prod_x \sum_{p \in \mathcal{B}_x} \mathbb{P}[x|p]\mathbb{P}[\text{REFACTOR}(v_p|\mathcal{D}')|\mathcal{D}', \theta]$ ▷ Likelihood if $(\mathcal{D}', \theta)$ were the DSL
  Define $\theta^*(\mathcal{D}') = \arg\max_\theta \mathbb{P}[\theta|\mathcal{D}']L(\mathcal{D}', \theta)$ ▷ MAP estimate of $\theta$
  Define score$(\mathcal{D}') = \log \mathbb{P}[\mathcal{D}'] + L(\mathcal{D}', \theta^*(\mathcal{D})) - \|\theta^*(\mathcal{D})\|_0$ ▷ objective function
  components $\leftarrow \{\text{REFACTOR}(v|\mathcal{D}) \ : \ \forall x, \forall p \in \mathcal{B}_x, \forall v \in \text{children}(v_p)\}$ ▷ Propose many new DSL components
  proposals $\leftarrow \{\mathcal{D} \cup \{c\} \ : \ \forall c \in \text{components}\}$ ▷ Propose many new DSLs
  $\mathcal{D}' \leftarrow \arg\max_{\mathcal{D}' \in \text{proposals}} \text{score}(\mathcal{D}')$ ▷ Get highest scoring new DSL
  **if** score$(\mathcal{D}') <$ score$(\mathcal{D})$ **return** $\mathcal{D}, \theta^*(\mathcal{D})$ ▷ No changes to DSL led to a better score
  $\mathcal{D} \leftarrow \mathcal{D}'$ ▷ Found better DSL. Update DSL.
  $\forall x \ : \ \mathcal{B}_x \leftarrow \{\text{REFACTOR}(v_p|\mathcal{D}) \ : \ p \in \mathcal{B}_x\}$ ▷ Refactor beams in terms of new DSL
**end while**

The purpose of a version space to compactly represent a set of programs. We refer to this set as the **extension** of the version space:

**Definition 1.** *The **extension** of a version space $v$ is written $[\![v]\!]$ and is defined recursively as:*

$$[\![\$i]\!] = \{\$i\} \qquad [\![\lambda v]\!] = \{\lambda e : e \in [\![v]\!]\} \qquad [\![(v_1 \ v_2)]\!] = \{(e_1 \ e_2) : e_1 \in [\![v_1]\!], \ e_2 \in [\![v_2]\!]\}$$
$$[\![\uplus V]\!] = \{e : v \in V, \ e \in [\![v]\!]\} \qquad [\![\varnothing]\!] = \varnothing \qquad [\![\Lambda]\!] = \Lambda$$

Version spaces also support efficient membership checking, which we write as $e \in [\![v]\!]$. Important for our purposes, it is also efficient to refactor the members of a version space's extension in terms of a new DSL. We define $\text{REFACTOR}(v|\mathcal{D})$ inductively as:

$$\text{REFACTOR}(v|\mathcal{D}) = \begin{cases} e, \text{ if } e \in \mathcal{D} \text{ and } e \in [\![v]\!]. \text{ Exploits the fact that } [\![e]\!] \in v \text{ can be efficiently computed.} \\ \text{REFACTOR}'(v|\mathcal{D}), \text{ otherwise.} \end{cases}$$

$$\text{REFACTOR}'(e|\mathcal{D}) = e, \text{ if } e \text{ is a leaf} \qquad \text{REFACTOR}'(\lambda b|\mathcal{D}) = \lambda \text{REFACTOR}(b|\mathcal{D})$$
$$\text{REFACTOR}'(f \ x|\mathcal{D}) = \text{REFACTOR}(f|\mathcal{D}) \ \text{REFACTOR}(x|\mathcal{D}) \qquad \text{REFACTOR}'(\uplus V|\mathcal{D}) = \underset{e \in \{\text{REFACTOR}(v|\mathcal{D}) \ : \ v \in V\}}{\arg\min} \text{size}(e|\mathcal{D})$$

where size$(e|\mathcal{D})$ for program $e$ and DSL $\mathcal{D}$ is the size of the syntax tree of $e$, when members of $\mathcal{D}$ are counted as having size 1. Concretely, $\text{REFACTOR}(v|\mathcal{D})$ calculates $\arg\min_{p \in [\![v]\!]} \text{size}(p|\mathcal{D})$.

Recall that our goal is to define an operator over version spaces, $I\beta_n$, which calculates the set of $n$-step refactorings of a program $p$, e.g., the set of all programs $p'$ where $p' \underbrace{\longrightarrow q \longrightarrow \cdots \longrightarrow}_{\leq n \text{ times}} p$, where $a \longrightarrow b$ is the standard notation for $a$ rewriting to $b$ according to the standard rewrite rules of $\lambda$-calculus [28].

We define this operator in terms of another operator, $I\beta'$, which performs a single step of refactoring:

$$I\beta_n(v) = \uplus \left\{ \underbrace{I\beta'(I\beta'(I\beta'(\cdots v)))}_{i \text{ times}} \ : \ 0 \leq i \leq n \right\}$$

where

$$I\beta'(u) = \uplus \{(\lambda b)v \ : \ v \mapsto b \in S_0(u)\} \cup \begin{cases} \text{if } u \text{ is a primitive or index or } \varnothing\text{:} & \varnothing \\ \text{if } u \text{ is } \Lambda\text{:} & \{\Lambda\} \\ \text{if } u = \lambda b\text{:} & \{\lambda I\beta'(b)\} \\ \text{if } u = (f \ x)\text{:} & \{(I\beta'(f) \ x), \ (f \ I\beta'(x))\} \\ \text{if } u = \uplus V\text{:} & \{I\beta'(u') \mid u' \in V\} \end{cases}$$

where we have defined $I\beta'$ in terms of another operator, $S_k : \text{VS} \to 2^{\text{VS}\times\text{VS}}$, whose purpose is to construct the set of substitutions that are refactorings of a program in a version space. We define $S$ as:

$$S_k(v) = \left\{\downarrow_0^k v \mapsto \$k\right\} \cup \begin{cases} \text{if } v \text{ is primitive:} & \{\Lambda \mapsto v\} \\ \text{if } v = \$i \text{ and } i < k\text{:} & \{\Lambda \mapsto \$i\} \\ \text{if } v = \$i \text{ and } i \geq k\text{:} & \{\Lambda \mapsto \$(i+1)\} \\ \text{if } v = \lambda b\text{:} & \{v' \mapsto \lambda b' \;:\; v' \mapsto b' \in S_{k+1}(b)\} \\ \text{if } v = (f\ x)\text{:} & \{v_1 \cap v_2 \mapsto (f'\ x') \;:\; v_1 \mapsto f' \in S_k(f),\ v_2 \mapsto x' \in S_k(x)\} \\ \text{if } v = \uplus V\text{:} & \bigcup_{v' \in V} S_n(v') \\ \text{if } v \text{ is } \varnothing\text{:} & \varnothing \\ \text{if } v \text{ is } \Lambda\text{:} & \{\Lambda \mapsto \Lambda\} \end{cases}$$

$$\begin{aligned} \downarrow_c^k \$i &= \$i, \text{ when } i < c \\ \downarrow_c^k \$i &= \$(i-k), \text{ when } i \geq c+k \\ \downarrow_c^k \$i &= \varnothing, \text{ when } c \leq i < c+k \\ \downarrow_c^k \lambda b &= \lambda \downarrow_{c+1}^k b \\ \downarrow_c^k (f\ x) &= (\downarrow_c^k f\ \ \downarrow_c^k x) \\ \downarrow_c^k \uplus V &= \uplus \left\{\downarrow_c^k v \mid v \in V\right\} \\ \downarrow_c^k v &= v, \text{ when } v \text{ is a primitive or } \varnothing \text{ or } \Lambda \end{aligned}$$

where $\uparrow^k$ is the shifting operator [28], which adds $k$ to all of the free variables in a $\lambda$-expression or version space, and we have defined a new operator, $\downarrow$, whose purpose is to undo the action of $\uparrow$. We have written definitions recursively, but implement them using a dynamic program: we hash cons each version space, and only calculate the operators $I\beta_n$, $I\beta'$, and $S_k$ once per each version space.

We now formally prove that $I\beta$ exhaustively enumerates the space of possible refactorings. Our approach is to first prove that $S_k$ exhaustively enumerates the space of possible substitutions that could give rise to a program. The following pair of technical lemmas are useful; both are easily proven by structural induction.

**Lemma 1.** *Let $e$ be a program or version space and $n$, $c$ be natural numbers. Then $\uparrow_{n+c}^{-1}\uparrow_c^{n+1} e = \uparrow_c^n e$, and in particular $\uparrow_n^{-1}\uparrow^{n+1} e = \uparrow^n e$.*

**Lemma 2.** *Let $e$ be a program or version space and $n$, $c$ be natural numbers. Then $\downarrow_c^n\uparrow_c^n e = e$, and in particular $\downarrow^n\uparrow^n e = e$.*

**Theorem 1.** ***Consistency of $S_n$.***
*If $(v \mapsto b) \in S_n(u)$ then for every $v' \in v$ and $b' \in b$ we have $\uparrow_n^{-1} [\$n \mapsto \uparrow^{1+n} v']b' \in u$.*

*Proof.* Suppose $b = \$n$ and therefore, by the definition of $S_n$, also $v = \downarrow_0^n u$. Invoking Lemmas 1 and 2 we know that $u = \uparrow_n^{-1}\uparrow^{n+1} v$ and so for every $v' \in v$ we have $\uparrow_n^{-1}\uparrow^{n+1} v' \in u$. Because $b = \$n = b'$ we can rewrite this to $\uparrow_n^{-1} [\$n \mapsto \uparrow^{n+1} v']b' \in u$.

Otherwise assume $b \neq \$n$ and proceed by structural induction on $u$:

- If $u = \$i < n$ then we have to consider the case that $v = \Lambda$ and $b = u = \$i = b'$. Pick $v' \in \Lambda$ arbitrarily. Then $\uparrow_n^{-1} [\$n \mapsto \uparrow^{1+n} v']b' = \uparrow_n^{-1} \$i = \$i \in u$.
- If $u = \$i \geq n$ then we have consider the case that $v = \Lambda$ and $b = \$(i+1) = b'$. Pick $v' \in \Lambda$ arbitrarily. Then $\uparrow_n^{-1} [\$n \mapsto \uparrow^{1+n} v']b' = \uparrow_n^{-1} \$(i+1) = \$i \in u$.
- If $u$ is primitive then we have to consider the case that $v = \Lambda$ and $b = u = b'$. Pick $v' \in \Lambda$ arbitrarily. Then $\uparrow_n^{-1} [\$n \mapsto \uparrow^{1+n} v']b' = \uparrow_n^{-1} u = u \in u$.
- If $u$ is of the form $\lambda a$, then $S_n(u) \subset \{v \mapsto \lambda b \mid (v \mapsto b) \in S_{n+1}(a)\}$. Let $v \mapsto \lambda b \in S_n(u)$. By induction for every $v' \in v$ and $b' \in b$ we have $\uparrow_{n+1}^{-1} [\$n \mapsto \uparrow^{2+n} v']b' \in a$, which we can rewrite to $\uparrow_n^{-1} [\$n \mapsto \uparrow^{1+n} v']\lambda b' \in \lambda a = u$.

- If $u$ is of the form $(f\ x)$ then then $S_n(u) \subset \{v_f \cap v_x \mapsto (b_f\ b_x) \mid (v_f \mapsto b_f) \in S_n(f),\ (v_x \mapsto b_x) \in S_n(x)\}$. Pick $v' \in v_f \cap v_x$ arbitrarily. By induction for every $v'_f \in v_f$, $v'_x \in v_x$, $b'_f \in b_f$, $b'_x \in b_x$ we have $\uparrow_n^{-1} [\$n \mapsto \uparrow^{1+n} v'_f]b'_f \in f$ and $\uparrow_n^{-1} [\$n \mapsto \uparrow^{1+n} v'_x]b'_x \in x$. Combining these facts gives $\uparrow_n^{-1} [\$n \mapsto \uparrow^{1+n} v'](b'_f\ b'_x) \in (f\ x) = u$.

- If $u$ is of the form $\uplus U$ then pick $(v \mapsto b) \in S_n(u)$ arbitrarily. By the definition of $S_n$ there is a $z$ such that $(v \mapsto b) \in S_n(z)$, and the theorem holds immediately by induction.

- If $u$ is $\varnothing$ or $\Lambda$ then the theorem holds vacuously.

$\square$

**Theorem 2.** ***Completeness of*** $S_n$.
*If there exists programs $v'$ and $b'$, and a version space $u$, such that $\uparrow_n^{-1} [\$n \mapsto \uparrow^{1+n} v']b' \in u$, then there also exists $(v \mapsto b) \in S_n(u)$ such that $v' \in v$ and $b' \in b$.*

*Proof.* As before we first consider the case that $b' = \$n$. If so then $\uparrow_n^{-1}\uparrow^{1+n} v' \in u$ or (invoking Lemma 1) that $\uparrow^n v' \in u$ and (invoking Lemma 2) that $v' \in \downarrow^n u$. From the definition of $S_n$ we know that $(\downarrow^n u \mapsto \$n) \in S_n(u)$ which is what was to be shown.

Otherwise assume that $b' \neq \$n$. Proceeding by structural induction on $u$:

- If $u = \$i$ then, because $b'$ is not $\$n$, we have $\uparrow_n^{-1} b' = \$i$. Let $b' = \$j$, and so

$$i = \begin{cases} j & \text{if } j < n \\ j-1 & \text{if } j > n \end{cases}$$

  where $j = n$ is impossible because by assumption $b' \neq \$n$.

  If $j < n$ then $i = j$ and so $u = b'$. By the definition of $S_n$ we have $(\Lambda \mapsto \$i) \in S_n(u)$, completing this inductive step because $v' \in \Lambda$ and $b' \in \$i$. Otherwise assume $j > n$ and so $\$i = \$(j-1) = u$. By the definition of $S_n$ we have $(\Lambda \mapsto \$(i+1)) \in S_n(u)$, completing this inductive step because $v' \in \Lambda$ and $b' = \$j = \$(i+1)$.

- If $u$ is a primitive then, because $b'$ is not $\$n$, we have $\uparrow_n^{-1} b' = u$, and so $b' = u$. By the definition of $S_n$ we have $(\Lambda \mapsto u) \in S_n(u)$ completing this inductive step because $v' \in \Lambda$ and $b' = u$.

- If $u$ is of the form $\lambda a$ then, because of the assumption that $b' \neq \$n$, we know that $b'$ is of the form $\lambda c'$ and that $\lambda \uparrow_{n+1}^{-1} [\$(n+1) \mapsto \uparrow^{2+n} v']c' \in \lambda a$. By induction this means that there is a $(v \mapsto c) \in S_{n+1}(a)$ satisfying $v' \in v$ and $c' \in c$. By the definition of $S_n$ we also know that $(v \mapsto \lambda c) \in S_n(u)$, completing this inductive step because $b' = \lambda c' \in \lambda c$.

- If $u$ is of the form $(f\ x)$ then, because of the assumption that $b' \neq \$n$, we know that $b'$ is of the form $(b'_f\ b'_x)$ and that both $\uparrow_n^{-1} [\$n \mapsto \uparrow^{1+n} v']b'_f \in f$ and $\uparrow_n^{-1} [\$n \mapsto \uparrow^{1+n} v']b'_x \in x$. Invoking the inductive hypothesis twice gives a $(v_f \mapsto b_f) \in S_n(f)$ satisfying $v' \in v_f$, $b'_f \in b_f$ and a $(v_x \mapsto b_x) \in S_n(x)$ satisfying $v' \in v_x$, $b'_x \in b_x$. By the definition of $S_n$ we know that $(v_f \cap v_x \mapsto b_f\ b_x) \in S_n(u)$ completing the inductive step because $v'$ is guaranteed to be in both $v_f$ and $v_x$ and we know that $b' = (b'_f\ b'_x) \in (b_f\ b_x)$.

- If $u$ is of the form $\uplus U$ then there must be a $z \in U$ such that $\uparrow_n^{-1} [\$n \mapsto \uparrow^{1+n} v']b' \in z$. By induction there is a $(v \mapsto b) \in S_n(z)$ such that $v' \in v$ and $b' \in v$. By the definition of $S_n$ we know that $(v \mapsto b)$ is also in $S_n(u)$ completing the inductive step.

- If $u$ is $\varnothing$ or $\Lambda$ then the theorem holds vacuously.

$\square$

From these results the consistency and completeness of $I\beta$ follows:

**Theorem 3.** ***Consistency of*** $I\beta'$.
*If $p \in [\![I\beta'(u)]\!]$ then there exists $p' \in [\![u]\!]$ such that $p \longrightarrow p'$.*

*Proof.* Proceed by structural induction on $u$. If $p \in [\![I\beta'(u)]\!]$ then, from the definition of $I\beta'$ and $[\![\cdot]\!]$, at least one of the following holds:

- Case $p = (\lambda b')v'$ where $v' \in v$, $b' \in b$, and $v \mapsto b \in S_0(u)$: From the definition of $\beta$-reduction we know that $p \longrightarrow \uparrow^{-1} [\$0 \mapsto \uparrow^1 v']b'$. From the consistency of $S_n$ we know that $\uparrow^{-1} [\$0 \mapsto \uparrow^1 v']b' \in u$. Identify $p' = \uparrow^{-1} [\$0 \mapsto \uparrow^1 v']b'$.
- Case $u = \lambda b$ and $p = \lambda b'$ where $b' \in [\![I\beta'(b)]\!]$: By induction there exists $b'' \in [\![b]\!]$ such that $b' \longrightarrow b''$. So $p \longrightarrow \lambda b''$. But $\lambda b'' \in [\![\lambda b]\!] = [\![u]\!]$, so identify $p' = \lambda b''$.
- Case $u = (f\ x)$ and $p = (f'\ x')$ where $f' \in [\![I\beta'(f)]\!]$ and $x' \in [\![x]\!]$: By induction there exists $f'' \in [\![f]\!]$ such that $f' \longrightarrow f''$. So $(f'\ x') \longrightarrow (f''\ x')$. But $(f''\ x') \in [\![(f\ x)]\!] = [\![u]\!]$, so identify $p' = (f''\ x')$.
- Case $u = (f\ x)$ and $p = (f'\ x')$ where $x' \in [\![I\beta'(x)]\!]$ and $f' \in [\![f]\!]$: Symmetric to the previous case.
- Case $u = \uplus U$ and $p \in [\![I\beta'(u')]\!]$ where $u' \in U$: By induction there is a $p' \in [\![u']\!]$ satisfying $p' \longrightarrow p$. But $[\![u']\!] \subseteq [\![u]\!]$, so also $p' \in [\![u]\!]$.
- Case $u$ is an index, primitive, $\varnothing$, or $\Lambda$: The theorem holds vacuously.

$\square$

**Theorem 4.** ***Completeness of*** $I\beta'$**.**
*Let* $p \longrightarrow p'$ *and* $p' \in [\![u]\!]$*. Then* $p \in [\![I\beta'(u)]\!]$*.*

*Proof.* Structural induction on $u$. If $u = \uplus V$ then there is a $v \in V$ such that $p' \in [\![v]\!]$; by induction on $v$ combined with the definition of $I\beta'$ we have $p \in [\![I\beta'(v)]\!] \subseteq [\![I\beta'(u)]\!]$, which is what we were to show. Otherwise assume that $u \neq \uplus V$.

From the definition of $p \longrightarrow p'$ at least one of these cases must hold:

- Case $p = (\lambda b')v'$ and $p' = \uparrow^{-1} [\$0 \mapsto \uparrow^1 v']b'$: Using the fact that $\uparrow^{-1} [\$0 \mapsto \uparrow^1 v']b' \in [\![u]\!]$, we can invoke the completeness of $S_n$ to construct a $(v \mapsto b) \in S_0(u)$ such that $v' \in [\![v]\!]$ and $b' \in [\![b]\!]$. Combine these facts with the definition of $I\beta'$ to get $p = (\lambda b')v' \in [\![(\lambda b)v]\!] \subseteq I\beta'(u)$.
- Case $p = \lambda b$ and $p' = \lambda b'$ where $b \longrightarrow b'$: Because $p' = \lambda b' \in [\![u]\!]$ and by assumption $u \neq \uplus V$, we know that $u = \lambda v$ and $b' \in [\![v]\!]$. By induction $b \in [\![I\beta'(v)]\!]$. Combine with the definition of $I\beta'$ to get $p = \lambda b \in [\![\lambda I\beta'(v)]\!] \subseteq [\![I\beta'(u)]\!]$.
- Case $p = (f\ x)$ and $p' = (f'\ x)$ where $f \longrightarrow f'$: Because $p' = (f'\ x) \in [\![u]\!]$ and by assumption $u \neq \uplus V$ we know that $u = (a\ b)$ where $f' \in [\![a]\!]$ and $x \in [\![b]\!]$. By induction on $a$ we know $f \in [\![I\beta'(a)]\!]$. Therefore $p = (f\ x) \in [\![(I\beta'(a)\ b)]\!] \subseteq [\![I\beta'((a\ b))]\!] \subseteq [\![I\beta'(u)]\!]$.
- Case $p = (f\ x)$ and $p' = (f\ x')$ where $x \longrightarrow x'$: Symmetric to the previous case.

$\square$

Finally we have our main result:

**Theorem 5.** ***Consistency and completeness of*** $I\beta_n$*. Let $p$ and $p'$ be programs. Then* $p \underbrace{\longrightarrow q \longrightarrow \cdots \longrightarrow}_{\leq n \text{ times}} p'$ *if and only if* $p \in [\![I\beta_n(p')]\!]$*.*

*Proof.* Induction on $n$.

If $n = 0$ then $[\![I\beta_n(p')]\!] = \{p\}$ and $p = p'$; the theorem holds immediately. Assume $n > 0$.

If $p \underbrace{\longrightarrow q \longrightarrow \cdots \longrightarrow}_{\leq n \text{ times}} p'$ then $q \underbrace{\longrightarrow \cdots \longrightarrow}_{\leq n-1 \text{ times}} p'$; induction on $n$ gives $q \in [\![I\beta_{n-1}(p')]\!]$. Combined with $p \longrightarrow q$ we can invoke the completeness of $I\beta'$ to get $p \in [\![I\beta'(I\beta_{n-1}(p'))]\!] \subset [\![I\beta_n(p')]\!]$.

If $p \in [\![I\beta_n(p')]\!]$ then there exists a $i \leq n$ such that $p \in [\![\underbrace{I\beta'(I\beta'(I\beta'(\cdots}_{i \text{ times}} p')))]\!]$. If $i = 0$ then $p = p'$ and $p$ reduces to $p'$ in $0 \leq n$ steps. Otherwise $i > 0$ and $p \in [\![I\beta'(\underbrace{I\beta'(I\beta'(\cdots}_{i-1 \text{ times}} p'))]\!]$. Invoking the consistency of $I\beta'$ we know that

$p \longrightarrow q$ for a program $q \in [\![\underbrace{I\beta'(I\beta'(\cdots p'))}_{i-1 \text{ times}}]\!] \subseteq [\![I\beta_{i-1}(p')]\!]$. By induction $q \underbrace{\longrightarrow \cdots \longrightarrow}_{\leq i-1 \text{ times}} p'$, which combined with $p \longrightarrow q$ gives $p \underbrace{\longrightarrow q \longrightarrow \cdots \longrightarrow}_{\leq i \leq n \text{ times}} p'$. □

#### A.5.2 Tracking equivalences

#### A.5.3 Computational complexity of DSL learning

How long does each update to the DSL in Algorithm 4 take? Constructing the version spaces takes time linear in the number of programs (written $P$) in the beams (Algorithm 4, line 5), and, in the worst case, exponential time as a function of the number of refactoring steps $n$ — but we bound the number of steps to be a small number (typically $n = 3$). Writing $V$ for the number of version spaces, this means that $V$ is $O(P2^n)$. The number of proposals (line 10) is linear in the number of distinct version spaces, so is $O(V)$. For each proposal we have to refactor every program (line 6), so this means we spend $O(V^2) = O(P^2 2^n)$ per DSL update. In practice this quadratic dependence on $P$ (the number of programs) is prohibitively slow. We now describe a linear time approximation to the refactor step in Algorithm 4 based on beam search.

For each version space $v$ we calculate a *beam*, which is a function from a DSL $\mathcal{D}$ to a shortest program in $[\![v]\!]$ using primitives in $\mathcal{D}$. Our strategy will be to only maintain the top $B$ shortest programs in the beam; throughout all of the experiments in this paper, we set $B = 10^6$, and in the limit $B \to \infty$ we recover the exact behavior of REFACTOR. The following recursive equations define how we calculate these beams; the set 'proposals' is defined in line 10 of Algorithm 4, and $\mathcal{D}$ is the current DSL:

$$\text{beam}_v(\mathcal{D}') = \begin{cases} \text{if } \mathcal{D}' \in \text{dom}(b_v): & b_v(\mathcal{D}') \\ \text{if } \mathcal{D}' \not\in \text{dom}(b_v): & \text{REFACTOR}(v, \mathcal{D}) \end{cases}$$

$$b_v = \text{the } B \text{ pairs } (\mathcal{D}' \mapsto p) \text{ in } b'_v \text{ where the syntax tree of } p \text{ is smallest}$$

$$b'_v(\mathcal{D}') = \begin{cases} \text{if } \mathcal{D}' \in \text{proposals and } e \in \mathcal{D}' \text{ and } e \in v\text{: } e \\ \text{otherwise if } v \text{ is a primitive or index:} v \text{otherwise if } v = \lambda b\text{: } \lambda \text{beam}_b(\mathcal{D}') \\ \text{otherwise if } v = (f\ x)\text{: } (\text{beam}_f(\mathcal{D}')\ \text{beam}_x \mathcal{D}')) \\ \text{otherwise if } v = \uplus V\text{: } \arg\min_{e \in \left\{b'_{v'}(\mathcal{D}')\ :\ v' \in V\right\}} \text{size}(e|\mathcal{D}') \end{cases}$$

We calculate $\text{beam}_v(\cdot)$ for each version space using dynamic programming. Using a minheap to represent $\text{beam}_v(\cdot)$, this takes time $O(VB \log B)$, replacing the quadratic dependence on $V$ (and therefore the number of programs, $P$) with a $B \log B$ term, where the parameter $B$ can be chosen freely, but at the cost of a less accurate beam search.

After performing this beam search, we take only the top $I$ proposals as measured by $-\sum_x \min_{p \in \mathcal{F}_x} \text{beam}_{v_p}(\mathcal{D}')$. We set $I = 300$ in all of our experiments, so $I \ll B$. The reason why we only take the top $I$ proposals (rather than take the top $B$) is because parameter estimation (estimating $\theta$ for each proposal) is much more expensive than performing the beam search — so we perform a very wide beam search and then at the very end tim the beam down to only $I = 300$ proposals. Next, we describe our MAP estimator for the continuous parameters ($\theta$) of the DSL.

#### A.5.4 Estimating the continuous parameters $\theta$ of a DSL

We use an EM algorithm to estimate the continuous parameters of the DSL, i.e. $\theta$. Suppressing dependencies on $\mathcal{D}$, the EM updates are

$$\theta = \arg\max_\theta \log P(\theta) + \sum_x \mathbb{E}_{q_x} [\log \mathbb{P}[p|\theta]] \tag{4}$$

$$q_x(p) \propto \mathbb{P}[x|p]\mathbb{P}[p|\theta]\mathbb{1}[p \in \mathcal{F}_x] \tag{5}$$

In the M step of EM we will update $\theta$ by instead maximizing a lower bound on $\log \mathbb{P}[p|\theta]$, making our approach an instance of Generalized EM.

We write $c(e, p)$ to mean the number of times that primitive $e$ was used in program $p$; $c(p) = \sum_{e \in \mathcal{D}} c(e, p)$ to mean the total number of primitives used in program $p$; $c(\tau, p)$ to mean the number of times that type $\tau$ was the input to sample in Algorithm 2 while sampling program $p$. Jensen's inequality gives a lower bound on the likelihood:

$$\begin{aligned}
&\sum_x \mathbb{E}_{q_x}\left[\log \mathbb{P}[p|\theta]\right] = \\
&\sum_{e\in\mathcal{D}} \log\theta_e \sum_x \mathbb{E}_{q_x}\left[c(e,p_x)\right] - \sum_\tau \mathbb{E}_{q_x}\left[\sum_x c(\tau,p_x)\right] \log \sum_{\substack{e:\tau'\in\mathcal{D}\\ \text{unify}(\tau,\tau')}} \theta_e \\
=&\sum_e C(e)\log\theta_e - \beta\sum_\tau \frac{\mathbb{E}_{q_x}\left[\sum_x c(\tau,p_x)\right]}{\beta} \log \sum_{\substack{e:\tau'\in\mathcal{D}\\ \text{unify}(\tau,\tau')}} \theta_e \\
\geq&\sum_e C(e)\log\theta_e - \beta\log\sum_\tau \frac{\mathbb{E}_{q_x}\left[\sum_x c(\tau,p_x)\right]}{\beta} \sum_{\substack{e:\tau'\in\mathcal{D}\\ \text{unify}(\tau,\tau')}} \theta_e \\
=&\sum_e C(e)\log\theta_e - \beta\log\sum_\tau \frac{R(\tau)}{\beta} \sum_{\substack{e:\tau'\in\mathcal{D}\\ \text{unify}(\tau,\tau')}} \theta_e
\end{aligned}$$

where we have defined

$$\begin{aligned}
C(e) &\triangleq \sum_x \mathbb{E}_{q_x}\left[c(e,p_x)\right] \\
R(\tau) &\triangleq \mathbb{E}_{q_x}\left[\sum_x c(\tau,p_x)\right] \\
\beta &\triangleq \sum_\tau \mathbb{E}_{q_x}\left[\sum_x c(\tau,p_x)\right]
\end{aligned}$$

Crucially it was defining $\beta$ that let us use Jensen's inequality. Recalling from the main paper that $P(\theta) \triangleq \text{Dir}(\alpha)$, we have the following lower bound on M-step objective:

$$\sum_e (C(e)+\alpha)\log\theta_e - \beta\log\sum_\tau \frac{R(\tau)}{\beta} \sum_{\substack{e:\tau'\in\mathcal{D}\\ \text{unify}(\tau,\tau')}} \theta_e \tag{6}$$

Differentiate with respect to $\theta_e$, where $e:\tau$, and set to zero to obtain:

$$\frac{C(e)+\alpha}{\theta_e} \propto \sum_{\tau'} \mathbb{1}\left[\text{unify}(\tau,\tau')\right] R(\tau') \tag{7}$$

$$\theta_e \propto \frac{C(e)+\alpha}{\sum_{\tau'} \mathbb{1}\left[\text{unify}(\tau,\tau')\right] R(\tau')} \tag{8}$$

The above is our estimator for $\theta_e$. The above estimator has an intuitive interpretation. The quantity $C(e)$ is the expected number of times that we used $e$. The quantity $\sum_{\tau'} \mathbb{1}\left[\text{unify}(\tau,\tau')\right] R(\tau')$ is the expected number of times that we *could have* used $e$. The hyperparameter $\alpha$ acts as pseudocounts that are added to the number of times that we used each primitive, and are not added to the number of times that we could have used each primitive.

We are only maximizing a lower bound on the log posterior; when is this lower bound tight? This lower bound is tight whenever all of the types of the expressions in the DSL are not polymorphic, in which case our DSL is equivalent to a PCFG and this estimator is equivalent to the inside/outside algorithm. Polymorphism introduces context-sensitivity to the DSL, and exactly maximizing the likelihood with respect to $\theta$ becomes intractable, so for domains with polymorphic types we use this estimator.

## A.6 Recognition model training

Recall that our goal is to maximize either $\mathcal{L}^{posterior}$ or $\mathcal{L}^{MAP}$, defined as:

$$\mathcal{L}^{\text{posterior}} = \mathcal{L}^{\text{posterior}}_{\text{Replay}} + \mathcal{L}^{\text{posterior}}_{\text{Fantasy}} \qquad \mathcal{L}^{\text{MAP}} = \mathcal{L}^{\text{MAP}}_{\text{Replay}} + \mathcal{L}^{\text{MAP}}_{\text{Fantasy}}$$

$$\mathcal{L}^{\text{posterior}}_{\text{Replay}} = \mathbb{E}_{x\sim X}\left[\sum_{p\in\mathcal{F}_x}\frac{\mathbb{P}\left[x,p|\mathcal{D},\theta\right]\log Q(p|x)}{\sum_{p'\in\mathcal{F}_x}\mathbb{P}\left[x,p'|\mathcal{D},\theta\right]}\right] \qquad \mathcal{L}^{\text{MAP}}_{\text{Replay}} = \mathbb{E}_{x\sim X}\left[\max_{\substack{p\in\mathcal{F}_x\\ p \text{ maxing } \mathbb{P}[\cdot|x,\mathcal{D},\theta]}}\log Q(p|x)\right]$$

$$\mathcal{L}^{\text{posterior}}_{\text{Fantasy}} = \mathbb{E}_{(p,x)\sim(\mathcal{D},\theta)}\left[\log Q(p|x)\right] \qquad \mathcal{L}^{\text{MAP}}_{\text{Fantasy}} = \mathbb{E}_{x\sim(\mathcal{D},\theta)}\left[\max_{\substack{p\\ p \text{ maxing } \mathbb{P}[\cdot|x,\mathcal{D},\theta]}}\log Q(p)\right]$$

The fantasy objectives are essential for data efficiency: all of our experiments train DREAMCODER on only a few hundred tasks, which is too little for a high-capacity neural network. Once we bootstrap a $(\mathcal{D},\theta)$, we can draw unlimited samples from $(\mathcal{D},\theta)$ and train $Q$ on those samples. But, evaluating $\mathcal{L}_{\text{Fantasy}}$ involves drawing programs from the current DSL, running them to get their outputs, and then training $Q$ to regress from the input/outputs to the program. Since these programs map inputs to outputs, we need to sample the inputs as well. Our solution is to sample the inputs from the empirical observed distribution of inputs in $X$.

The $\mathcal{L}^{\text{MAP}}_{\text{Fantasy}}$ objective involves finding the MAP program solving a task drawn from the DSL. To make this tractable, rather than *sample* programs as training data for $\mathcal{L}^{\text{MAP}}_{\text{Fantasy}}$, we *enumerate* programs in decreasing order of their prior probability, tracking, for each dreamed task $x$, the set of enumerated programs maximizing $\mathbb{P}[x,p|\mathcal{D},\theta]$.

We parameterize $Q$ using a bigram model over syntax trees. Formally, $Q$ predicts a $(|\mathcal{D}|+2)\times(|\mathcal{D}|+1)\times A$-dimensional tensor, where $A$ is the maximum arity[4] of any primitive in the DSL. Slightly abusing notation, we write this tensor as $Q_{ijk}(x)$, where $x$ is a task, $i\in\mathcal{D}\cup\{\text{start},\text{var}\}$, $j\in\mathcal{D}\cup\{\text{var}\}$, and $k\in\{1,2,\cdots,A\}$. The output $Q_{ijk}(x)$ controls the probability of sampling primitive $j$ given that $i$ is the parent node in the syntax tree and we are sampling the $k^{\text{th}}$ argument. Algorithm 5 specifies a procedure for drawing samples from $Q(\cdot|X)$.

**Symmetry breaking.** Why does the combination of $\mathcal{L}^{\text{MAP}}$ and the bigram parameterization lead to symmetry breaking? The reason is twofold: (1) the objective $\mathcal{L}^{\text{MAP}}$ prefers symmetry breaking recognition models; and (2) the bigram parameterization permits certain kinds of symmetry breaking. To sharpen these intuitions, we prove (Theorem 6) that any global optimizer of $\mathcal{L}^{\text{MAP}}$ breaks symmetries, and then give a concrete worked out example contrasting the behavior of $\mathcal{L}^{\text{MAP}}$ and $\mathcal{L}^{\text{posterior}}$.

[4]The arity of a function is the number of arguments that it takes as input.

**Algorithm 5** Drawing from distribution over programs predicted by recognition model. Compare w/ Algorithm 2

**function** recognitionSample$(Q, x, \mathcal{D}, \tau)$:
**Input:** recognition model $Q$, task $x$, DSL $\mathcal{D}$, type $\tau$
**Output:** a program whose type unifies with $\tau$
**return** recognitionSample$'(Q, x, \text{start}, 1, \mathcal{D}, \varnothing, \tau)$

**function** recognitionSample$'(Q, x, \text{parent}, \text{argumentIndex}, \mathcal{D}, \mathcal{E}, \tau)$:
**Input:** recognition model $Q$, task $x$, DSL $\mathcal{D}$, parent $\in \mathcal{D} \cup \{\text{start}, \text{var}\}$, argumentIndex $\in \mathbb{N}$, environment $\mathcal{E}$, type $\tau$
**Output:** a program whose type unifies with $\tau$
**if** $\tau = \alpha \to \beta$ **then** ▷ Function type — start with a lambda
    var $\leftarrow$ an unused variable name
    body $\sim$ recognitionSample$'(Q, x, \text{parent}, \text{argumentIndex}, \mathcal{D}, \{\text{var} : \alpha\} \cup \mathcal{E}, \beta)$
    **return** `(lambda (var) body)`
**else** ▷ Build an application to give something w/ type $\tau$
    primitives $\leftarrow \{p | p : \tau' \in \mathcal{D} \cup \mathcal{E}$ if $\tau$ can unify with yield$(\tau')\}$ ▷ Everything in scope w/ type $\tau$
    variables $\leftarrow \{p \mid p \in$ primitives and $p$ a variable$\}$
    Draw $e \sim$ primitives, w.p. $\propto \begin{cases} Q_{\text{parent},e,\text{argumentIndex}}(x) & \text{if } e \in \mathcal{D} \\ Q_{\text{parent},\text{var},\text{argumentIndex}}(x)/|\text{variables}| & \text{if } e \in \mathcal{E} \end{cases}$
    Unify $\tau$ with yield$(\tau')$. ▷ Ensure well-typed program
    newParent$\leftarrow \begin{cases} e & \text{if } e \in \mathcal{D} \\ \text{var} & \text{if } e \in \mathcal{E} \end{cases}$
    $\{\alpha_k\}_{k=1}^{K} \leftarrow$ args$(\tau')$
    **for** $k = 1$ **to** $K$ **do** ▷ Recursively sample arguments
        $a_k \sim$ recognitionSample$'(Q, x, \text{newParent}, k, \mathcal{D}, \mathcal{E}, \alpha_k)$
    **end for**
    **return** $(e\ a_1\ a_2\ \cdots\ a_K)$
**end if**

**Theorem 6.** *Let $\mu(\cdot)$ be a distribution over tasks and let $Q^*(\cdot|\cdot)$ be a task-conditional distribution over programs satisfying*

$$Q^* = \arg\max_Q \; \mathbb{E}_\mu \left[ \max_{\substack{p \\ p \text{ maxing } \mathbb{P}[\cdot|x,\mathcal{D},\theta]}} \log Q(p|x) \right]$$

*where $(\mathcal{D}, \theta)$ is a generative model over programs. Pick a task $x$ where $\mu(x) > 0$. Partition $\Lambda$ into expressions that are observationally equivalent under $x$:*

$$\Lambda = \bigcup_i \mathcal{E}_i^x \text{ where for any } p_1 \in \mathcal{E}_i^x \text{ and } p_2 \in \mathcal{E}_j^x\text{: } \mathbb{P}[x|p_1] = \mathbb{P}[x|p_2] \iff i = j$$

*Then there exists an equivalence class $\mathcal{E}_i^x$ that gets all the probability mass of $Q^*$ – e.g., $Q^*(p|x) = 0$ whenever $p \notin \mathcal{E}_i^x$ – and there exists a program in that equivalence class which gets all of the probability mass assigned by $Q^*(\cdot|x)$ – e.g., there is a $p \in \mathcal{E}_i^x$ such that $Q^*(p|x) = 1$ – and that program maximizes $\mathbb{P}[\cdot|x, \mathcal{D}, \theta]$.*

*Proof.* We proceed by defining the set of "best programs" – programs maximizing the posterior $\mathbb{P}[\cdot|x, \mathcal{D}, \theta]$ – and then showing that a best program satisfies $Q^*(p|x) = 1$. Define the set of best programs $\mathcal{B}_x$ for the task $x$ by

$$\mathcal{B}_x = \left\{ p \mid \mathbb{P}[p|x, \mathcal{D}, \theta] = \max_{p' \in \Lambda} \mathbb{P}[p'|x, \mathcal{D}, \theta] \right\}$$

For convenience define

$$f(Q) = \mathbb{E}_\mu \left[ \max_{p \in \mathcal{B}_x} \log Q(p|x) \right]$$

and observe that $Q^* = \arg\max_Q f(Q)$.

Suppose by way of contradiction that there is a $q \notin \mathcal{B}_x$ where $Q^*(q|x) = \epsilon > 0$. Let $p^* = \arg\max_{p \in \mathcal{B}_x} \log Q^*(p|x)$. Define

$$Q'(p|x) = \begin{cases} 0 & \text{if } p = q \\ Q^*(p|x) + \epsilon & \text{if } p = p^* \\ Q^*(p|x) & \text{otherwise.} \end{cases}$$

Then

$$f(Q') - f(Q^*) = \mu(x) \left( \max_{p \in \mathcal{B}_x} \log Q'(p|x) - \max_{p \in \mathcal{B}_x} \log Q^*(p|x) \right) = \mu(x) \left( \log \left( Q^*(p^*|x) + \epsilon \right) - \log Q^*(p^*|x) \right) > 0$$

which contradicts the assumption that $Q^*$ maximizes $f(\cdot)$. Therefore for any $p \notin \mathcal{B}_x$ we have $Q^*(p|x) = 0$.

Suppose by way of contradiction that there are two distinct programs, $q$ and $r$, both members of $\mathcal{B}_x$, where $Q^*(q|x) = \alpha > 0$ and $Q^*(r|x) = \beta > 0$. Let $p^* = \arg\max_{p \in \mathcal{B}_x} \log Q^*(p|x)$. If $p^* \notin \{q, r\}$ then define

$$Q'(p|x) = \begin{cases} 0 & \text{if } p \in \{q, r\} \\ Q^*(p|x) + \alpha + \beta & \text{if } p = p^* \\ Q^*(p|x) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} f(Q') - f(Q^*) &= \mu(x) \left( \max_{p \in \mathcal{B}_x} \log Q'(p|x) - \max_{p \in \mathcal{B}_x} \log Q^*(p|x) \right) \\ &= \mu(x) \left( \log \left( Q^*(p^*|x) + \alpha + \beta \right) - \log Q^*(p^*|x) \right) > 0 \end{aligned}$$

which contradicts the assumption that $Q^*$ maximizes $f(\cdot)$. Otherwise assume $p^* \in \{q, r\}$. Without loss of generality let $p^* = q$. Define

$$Q'(p|x) = \begin{cases} 0 & \text{if } p = r \\ Q^*(p|x) + \beta & \text{if } p = p^* \\ Q^*(p|x) & \text{otherwise.} \end{cases}$$

Then

$$f(Q') - f(Q^*) = \mu(x)\left(\max_{p\in\mathcal{B}_x} \log Q'(p|x) - \max_{p\in\mathcal{B}_x} \log Q^*(p|x)\right) = \mu(x)\left(\log\left(Q^*(p^*|x)+\beta\right) - \log Q^*(p^*|x)\right) > 0$$

which contradicts the assumption that $Q^*$ maximizes $f(\cdot)$. Therefore $Q^*(p|x) > 0$ for at most one $p \in \mathcal{B}_x$. But we already know that $Q^*(p|x) = 0$ for any $p \notin \mathcal{B}_x$, so it must be the case that $Q^*(\cdot|x)$ places all of its probability mass on exactly one $p \in \mathcal{B}_x$. Call that program $p^*$.

Because the equivalence classes $\{\mathcal{E}_i^x\}$ form a partition of $\Lambda$ we know that $p^*$ is a member of exactly one equivalence class; call it $\mathcal{E}_i^x$. Let $q \in \mathcal{E}_j^x \neq \mathcal{E}_i^x$. Then because the equivalence classes form a partition we know that $q \neq p^*$ and so $Q^*(q|x) = 0$, which was our first goal: *any* program not in $\mathcal{E}_i^x$ gets no probability mass.

Our second goal — that there is a member of $\mathcal{E}_i^x$ which gets all the probability mass assigned by $Q^*(\cdot|x)$ — is immediate from $Q^*(p^*|x) = 1$.

Our final goal — that $p^*$ maximizes $\mathbb{P}[\cdot|x, \mathcal{D}, \theta]$ — follows from the fact that $p^* \in \mathcal{B}_x$. □

Notice that Theorem 6 makes no guarantees as to the cross-task systematicity of the symmetry breaking; for example, an optimal recognition model could associate addition to the right for one task and associate addition to the left on another task. *Systematic* breaking of symmetries must arise only as a consequence as the network architecture (i.e., it is more parsimonious to break symmetries the same way for every task than it is to break them differently for each task).

As a concrete example of symmetry breaking, consider an agent tasked with writing programs built from addition and the constants zero and one. A bigram parameterization of $Q$ allows it to represent the fact that it should never add zero ($Q_{+,0,0} = Q_{+,0,1} = 0$) or that addition should always associate to the right ($Q_{+,+,0} = 0$). The $\mathcal{L}^{\text{MAP}}$ training objective encourages learning these canonical forms. Consider two recognition models, $Q_1$ and $Q_2$, and two programs in frontier $\mathcal{F}_x$, $p_1 =$ `(+ (+ 1 1) 1)` and $p_2 =$ `(+ 1 (+ 1 1))`, where

$$\begin{aligned} Q_1(p_1|x) &= \frac{\epsilon}{2} \qquad Q_1(p_2|x) = \frac{\epsilon}{2} \\ Q_2(p_1|x) &= 0 \qquad Q_2(p_2|x) = \epsilon \end{aligned}$$

i.e., $Q_2$ breaks a symmetry by forcing right associative addition, but $Q_1$ does not, instead splitting its probability mass equally between $p_1$ and $p_2$. Now because $\mathbb{P}[p_1|\mathcal{D},\theta] = \mathbb{P}[p_2|\mathcal{D},\theta]$ (Algorithm 2), we have

$$\begin{aligned} \mathcal{L}_{\text{real}}^{\text{posterior}}(Q_1) &= \frac{\mathbb{P}[p_1|\mathcal{D},\theta]\log\frac{\epsilon}{2} + \mathbb{P}[p_2|\mathcal{D},\theta]\log\frac{\epsilon}{2}}{\mathbb{P}[p_1|\mathcal{D},\theta] + \mathbb{P}[p_2|\mathcal{D},\theta]} = \log\frac{\epsilon}{2} \\ \mathcal{L}_{\text{real}}^{\text{posterior}}(Q_2) &= \frac{\mathbb{P}[p_1|\mathcal{D},\theta]\log 0 + \mathbb{P}[p_2|\mathcal{D},\theta]\log\epsilon}{\mathbb{P}[p_1|\mathcal{D},\theta] + \mathbb{P}[p_2|\mathcal{D},\theta]} = +\infty \\ \mathcal{L}_{\text{real}}^{\text{MAP}}(Q_1) &= \log Q_1(p_1) = \log Q_1(p_2) = \log\frac{\epsilon}{2} \\ \mathcal{L}_{\text{real}}^{\text{MAP}}(Q_2) &= \log Q_2(p_2) \qquad\qquad\qquad = \log\epsilon \end{aligned}$$

So $\mathcal{L}^{\text{MAP}}$ prefers $Q_2$ (the symmetry breaking recognition model), while $\mathcal{L}^{\text{posterior}}$ reverses this preference.

How would this example work out if we did not have a bigram parameterization of $Q$? With a unigram parameterization, $Q_2$ would be impossible to express, because it depends on local context within the syntax tree of a program. So even though the objective function would prefer symmetry breaking, a simple unigram model lacks the expressive power to encode it.

To be clear, our recognition model does not learn to break *every* possible symmetry in every possible DSL. But in practice we found that a bigrams combined with $\mathcal{L}^{\text{MAP}}$ works well, and we use with this combination throughout the paper.

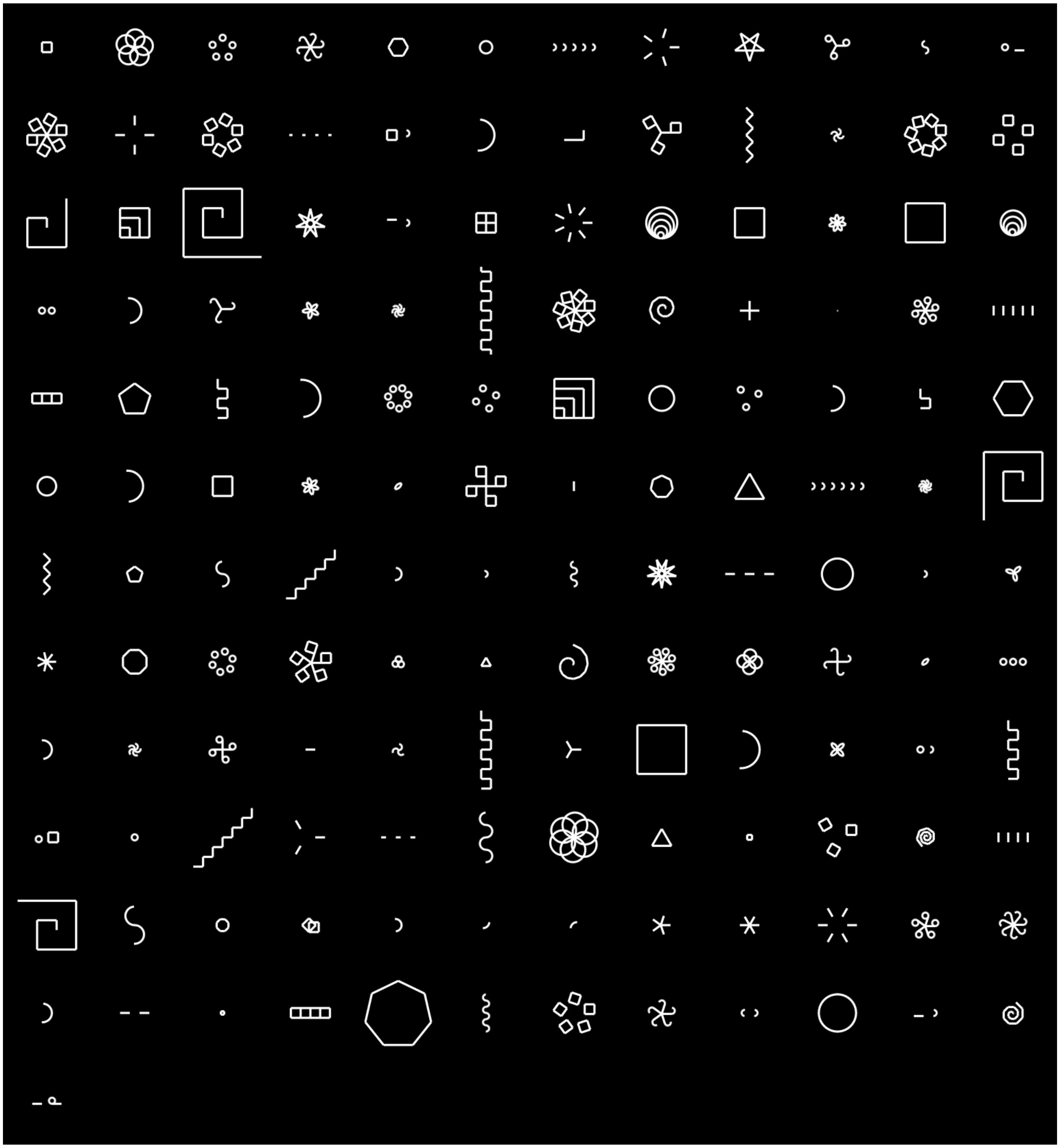

Figure 11: Full set of LOGO graphics tasks that we apply our system to

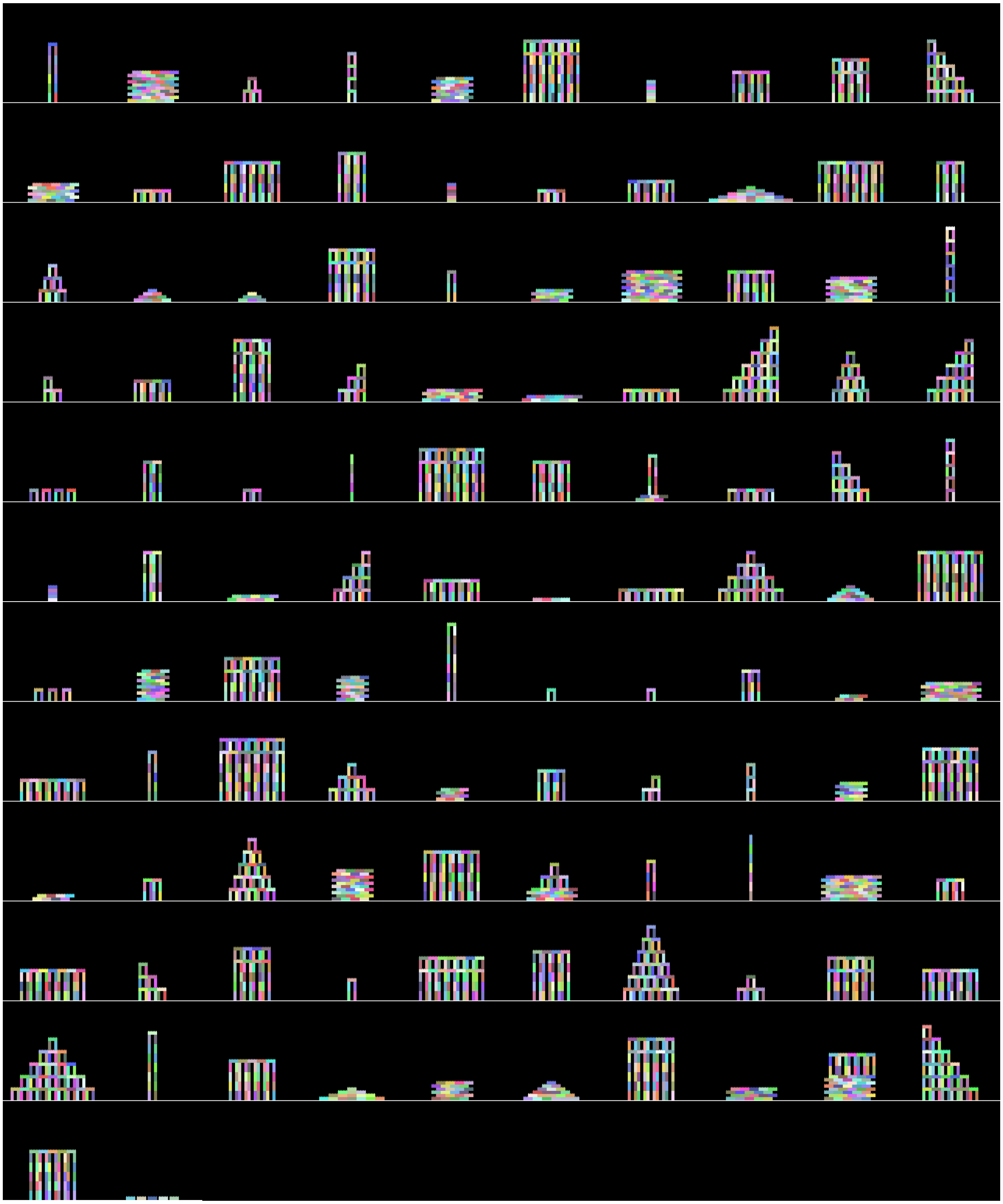

Figure 12: Full set of tower building tasks that we apply our system to

## A.7 Full set of LOGO tasks

## A.8 Full set of tower tasks

## A.9 Learning from Scratch: Tasks and DSL

We gave our system the following primitives: `if`, `=`, `>`, `+`, `-`, `0`, `1`, `cons`, `car`, `cdr`, `nil`, and `is-nil`, all of which are present in some form in McCarthy's 1959 Lisp [24].[5] We furthermore allowed functions to call themselves, which we modeled using the Y combinator. We did not use the recognition model for this experiment: a bottom-up pattern recognizer is of little use for acquiring this abstract knowledge from less than two dozen problems.

Figure 13 shows the full set of tasks and the learned DSL.

## A.10 Learned DSLs

Here we present representative DSLs learned by our model. DSL primitives discovered by the algorithm are prefixed with #. Variables are prefixed with $, and we adopt De Bruijn indices to model bound variables [28].

### A.10.1 List processing

```
#(λ (λ (map (λ (index $0 $2)) (range ($0 (+ 1))))))
#(λ (λ (fold $1 empty (λ (λ (if ($2 $1) (cons $1 $0) $0))))))
#(+ 1 (+ 1 1))
#(λ (λ (fold $1 (cons $0 empty) (λ (λ (cons $1 $0))))))
#(+ 1 #(+ 1 (+ 1 1)))
#(λ (map (λ (if ($1 $0) (+ $0 1) 0))))
#(λ (cdr (cdr $0)))
#(λ (λ (fold (#(λ (cdr (cdr $0))) (#(λ (cdr (cdr $0))) $1)) $0 (λ (λ (cdr (#(λ (λ (fold $1 (cons
    ↪ $0 empty) (λ (λ (cons $1 $0)))))) $0 (car $0))))))))
#(λ (car (#(λ (λ (fold $1 empty (λ (λ (if ($2 $1) (cons $1 $0) $0)))))) $0 (λ (empty? (#(λ (λ
    ↪ (fold $1 empty (λ (λ (if ($2 $1) (cons $1 $0) $0)))))) $1 (λ (gt? $0 $1))))))))
#(λ (λ (length (#(λ (λ (fold $1 empty (λ (λ (if ($2 $1) (cons $1 $0) $0)))))) $1 (λ (eq? $0
    ↪ $1))))))
#(λ (λ (#(λ (λ (fold $1 empty (λ (λ (if ($2 $1) (cons $1 $0) $0)))))) $1 (λ (is-prime (+ $1 (mod
    ↪ $0 #(+ 1 #(+ 1 (+ 1 1))))))))))
#(λ (λ (fold $1 $0 (λ (λ (cons $1 $0))))))
#(λ (map (λ (mod $0 $1))))
#(λ (map (λ (eq? (mod $0 $1) 0))))
#(λ (gt? (mod $0 #(+ 1 (+ 1 1))) 0))
#(λ (λ (#(λ (car (#(λ (λ (fold $1 empty (λ (λ (if ($2 $1) (cons $1 $0) $0)))))) $0 (λ (empty?
    ↪ (#(λ (λ (fold $1 empty (λ (λ (if ($2 $1) (cons $1 $0) $0)))))) $1 (λ (gt? $0 $1))))))))
    ↪ (#(λ (λ (fold $1 empty (λ (λ (if ($2 $1) (cons $1 $0) $0)))))) $1 (λ (gt? $1 (length (#(λ
    ↪ (λ (fold $1 empty (λ (λ (if ($2 $1) (cons $1 $0) $0)))))) $2 (λ (gt? $1 $0))))))))))
```

### A.10.2 Text editing

```
#(λ (λ (fold $1 $0 (λ (λ (cons $1 $0))))))
#(λ (λ (cons '.' (cons $0 $1))))
#(λ (λ (map (λ (index $0 $2)) (range $0))))
#(+ 1)
#(λ (λ (#(λ (λ (map (λ (index $0 $2)) (range $0)))) $1 (fold (cdr $1) 0 (λ (λ (#(+ 1) (if
    ↪ (char-eq? $2 $1) 0 $0))))))))
#(λ (λ (cons (car $0) (#(λ (λ (cons '.' (cons $0 $1)))) (cons '.' empty) (car $1)))))
#(λ (λ (map (λ (if (char-eq? $2 $0) $1 $0)))))
#(λ (#(λ (λ (fold $1 $0 (λ (λ (cons $1 $0)))))) (cons LPAREN $0) (cons RPAREN empty)))
#(λ (λ (#(λ (λ (map (λ (index $0 $2)) (range $0)))) $0 (length (cdr (cdr $1))))))
#(λ (λ (cdr (fold (#(λ (λ (#(λ (λ (map (λ (index $0 $2)) (range $0)))) $1 (fold (cdr $1) 0 (λ (λ
    ↪ (#(+ 1) (if (char-eq? $2 $1) 0 $0)))))))) $1 $0) $1 (λ (λ (cdr $0)))))))
#(λ (#(λ (λ (#(λ (λ (map (λ (index $0 $2)) (range $0)))) $0 (length (cdr (cdr $1)))))) (cdr (cdr
    ↪ $0))))
#(#(+ 1) 1)
#(λ (λ (cons (car $1) (cons $0 empty))))
```

[5] McCarthy's first version of Lisp used `cond` instead of `if`. Because we are using a typed language, we instead used `if`, because Lisp-style `cond` is unwieldy to express as a function in typed languages.

Programs & Tasks

[1 9]→2
[5 3 8]→3
$f(\ell) = (f_5\ \ell)$

[0 1 1 0 0]→[1 1]
[9 0 8]→[9 8]
$f(\ell) = (f_3$ (eq? 0) $\ell)$

[2 1 4]→[2 1 4 0]
[9 8]→[9 8 0]
$f(\ell) = (f_0$ cons $\ell$ (cons 0 nil))

[2 5 6 0 6]→19
[9 2 7 6 3]→27
$f(\ell) = (f_0$ + $\ell$ 0)

[4 2 6 4]→[-4 -2 -6 -4]
[2 3 0 7]→[-2 -3 -0 -7]
$f(\ell) = (f_2$ (- 0) $\ell)$

[1 5 2 9]→[1 2]
[3 8 1 3 1 2]→[3 1 1]
$f(\ell) = (f_6$ ($\lambda$ (l) (cdr (cdr l))) empty? $\ell$

3→[0 1 2 3]
2→[0 1 2]
$f(n) = (f_9$ (+ 1 $n$))

0, [9 2 3]→9
3, [0 2 8 4 5 6]→4
$f$(n,l) = ($f_{10}$ l n)

3→[-3 -2 -1]
4→[-4 -3 -2 -1]
$f$(n) = ($f_8$ 0 n)

[[2 1] []]→[1 0]
[[] [] [9 8 9 9]]→[0 0 4]
$f(\ell) = (f_2\ f_4\ \ell)$

[1 -1 0 2]→[1 2]
[9 -5 5 0 8]→[9 5 8]
$f(\ell) = (f_3$ (gt? 1) $\ell)$

[2 1 4]→[2 1]
[9 8]→[9]
$f(\ell) = (f_6$ cdr ($\lambda$ (z) (empty? (cdr z))) $\ell)$

[4 2 6 4]→[8 4 12 8]
[2 3 0 7]→[4 6 0 14]
$f(\ell) = (f_2$ ($\lambda$ (x) (+ x x)) $\ell)$

[4 2 6 4]→[5 3 7 5]
[2 3 0 7]→[3 4 1 8]
$f(\ell) = (f_2$ (+ 1) $\ell)$

3→[0 1 2]
2→[0 1]
$f(n) = (f_9\ n)$

[9 2]→[9 9 2 2]
[1 2 3 4]→[1 1 2 2 3 3 4 4]
$f(l) = (f_0$ ($\lambda$ (a x) (cons x (cons x a))) l nil)

1, [9 2 3]→9
4, [0 2 8 1 5 6]→1
$f$(n,l) = ($f_{10}$ l (+ 1 n))

2→[2 1]
4→[4 3 2 1]
$f$(n) = ($f_8$ 0 (- 0 n))

DSL

$f_0$(f,l,x) = (if (empty? l) x
(f (car l) ($f_0$ (cdr l))))
($f_0$: *fold*)

$f_1$(p,f,n,x) = (if (p x) nil
(cons (f x) ($f_1$ (n x))))
($f_1$: *unfold*)

$f_2$(f,l) = ($f_0$ nil l ($\lambda$ (x a) (cons (f x) a)))
($f_2$: *map*)

$f_3$(f,l) = ($f_0$ nil l ($\lambda$ (x a) (if (f x) a (cons x a))))
($f_3$: *filter*)

$f_4$(f,p,n) = ($f_1$ p f (+ 1) n)
($f_4$: *generalization of range*)

$f_5$(l) = ($f_0$ ($\lambda$ (a x) (+ 1 a)) l 0)
($f_5$: *length*)

$f_6$(n,p,l) = ($f_1$ p n car l)
($f_6$: *specialization of unfold*)

$f_7$(p) = ($f_4$ ($\lambda$ (x) x) p 0)
($f_7$: *another generalization of range*)

$f_8$(n,m) = ($f_4$ ($\lambda$ (x) (- n x)) (eq? 0) m)
($f_7$: *count downwards*)

$f_9$(n) = ($f_7$ (eq? n))
($f_9$: *range*)

$f_{10}$(l,n) = (car ($f_0$ ($\lambda$ (a x) (cdr a)) ($f_9$ n) l))
($f_{10}$: *index*)

Figure 13: Bootstrapping a standard library of functional programming routines, starting from recursion along with primitive operations found in 1959 Lisp. Complete set of tasks and learned DSL shown. Learned DSL components are numbered in the order that they are learned, *i.e.*, the agent first learns fold, then unfold, then uses fold to define map, etc.

```
#(λ (λ (#(λ (λ (#(λ (λ (map (λ (index $0 $2)) (range $0)))) $1 (fold (cdr $1) 0 (λ (λ (#(+ 1)
    ↪ (if (char-eq? $2 $1) 0 $0)))))))) (#(λ (λ (cdr (fold (#(λ (λ (#(λ (λ (map (λ (index $0
    ↪ $2)) (range $0)))) $1 (fold (cdr $1) 0 (λ (λ (#(+ 1) (if (char-eq? $2 $1) 0 $0)))))))) $1
    ↪ $0) $1 (λ (λ (cdr $0))))))) (#(λ (#(λ (λ (fold $1 $0 (λ (λ (cons $1 $0)))))) (cons LPAREN
    ↪ $0) (cons RPAREN empty))) $1) $0) $0)))
#(λ (λ (λ (#(λ (λ (fold $1 $0 (λ (λ (cons $1 $0)))))) $1 (cons $0 $2)))))
#(λ (#(λ (λ (fold $1 $0 (λ (λ (cons $1 $0)))))) $0 STRING))
#(#(+ 1) #(#(+ 1) 1))
#(λ (#(λ (λ (#(λ (λ (map (λ (index $0 $2)) (range $0)))) $1 (fold (cdr $1) 0 (λ (λ (#(+ 1) (if
    ↪ (char-eq? $2 $1) 0 $0)))))))) (#(λ (λ (fold $1 $0 (λ (λ (cons $1 $0)))))) STRING $0)
    ↪ SPACE))
#(λ (λ (#(λ (λ (λ (#(λ (λ (fold $1 $0 (λ (λ (cons $1 $0)))))) $1 (cons $0 $2))))) (cons $0 $1))))
#(#(λ (λ (fold $1 $0 (λ (λ (cons $1 $0)))))) STRING)
#(λ (λ (#(λ (λ (cons (car $0) (#(λ (λ (cons '.' (cons $0 $1)))) (cons '.' empty) (car $1)))))
    ↪ (#(λ (λ (cdr (fold (#(λ (λ (#(λ (λ (map (λ (index $0 $2)) (range $0)))) $1 (fold (cdr $1)
    ↪ 0 (λ (λ (#(+ 1) (if (char-eq? $2 $1) 0 $0)))))))) $1 $0) $1 (λ (λ (cdr $0))))))) $1 $0)
    ↪ $1)))
#(#(+ 1) #(#(+ 1) #(#(+ 1) 1)))
#(λ (λ (λ (#(λ (λ (fold $1 $0 (λ (λ (cons $1 $0)))))) $1 (#(λ (λ (cdr (fold (#(λ (λ (#(λ (λ (map
    ↪ (λ (index $0 $2)) (range $0)))) $1 (fold (cdr $1) 0 (λ (λ (#(+ 1) (if (char-eq? $2 $1) 0
    ↪ $0)))))))) $1 $0) $1 (λ (λ (cdr $0))))))) (#(λ (#(λ (λ (fold $1 $0 (λ (λ (cons $1
    ↪ $0)))))) $0 STRING)) (#(λ (λ (map (λ (index $0 $2)) (range $0)))) $2 #(#(+ 1) #(#(+ 1)
    ↪ #(#(+ 1) 1))))) $0)))))
```

### A.10.3 Graphics

```
  #(λ (λ (λ (logo_forLoop $2 (λ (λ (logo_FWRT $2 $3 $0)))))))
#(logo_DIVA logo_UA 4)
#(#(λ (λ (λ (logo_forLoop $2 (λ (λ (logo_FWRT $2 $3 $0))))))) logo_IFTY)
#(logo_PT (λ (logo_FWRT logo_UL logo_ZA $0)))
#(λ (λ (λ (logo_forLoop $1 (λ (λ (logo_FWRT (logo_MULL $2 $1) $4 $0)))))))
#(λ (logo_forLoop 7 (λ (λ (#(λ (λ (λ (logo_forLoop $2 (λ (λ (logo_FWRT $2 $3 $0))))))) 7
    ↪ logo_epsA $2 $0)))))
#(λ (#(#(λ (λ (λ (logo_forLoop $2 (λ (λ (logo_FWRT $2 $3 $0))))))) logo_IFTY) (logo_SUBA logo_UA
    ↪ logo_epsA) (logo_MULL logo_epsL $0)))
#(λ (logo_forLoop logo_IFTY (λ (λ (logo_FWRT $2 logo_epsA (logo_FWRT logo_epsL logo_epsA $0))))))
#(#(λ (λ (λ (logo_forLoop $2 (λ (λ (logo_FWRT $2 $3 $0))))))) 4 #(logo_DIVA logo_UA 4))
#(logo_DIVA logo_UA)
#(λ (λ (λ (logo_forLoop $1 (λ (λ (logo_GETSET $2 (logo_FWRT logo_ZL $4 $0))))))))
#(#(#(λ (λ (λ (logo_forLoop $2 (λ (λ (logo_FWRT $2 $3 $0))))))) logo_IFTY) (logo_DIVA logo_epsA
    ↪ 2) logo_epsL)
#(#(#(λ (λ (λ (logo_forLoop $2 (λ (λ (logo_FWRT $2 $3 $0))))))) logo_IFTY) logo_epsA logo_epsL)
#(logo_forLoop 3 (λ (λ (#(#(#(λ (λ (λ (logo_forLoop $2 (λ (λ (logo_FWRT $2 $3 $0)))))))
    ↪ logo_IFTY) (logo_DIVA logo_epsA 2) logo_epsL) (logo_FWRT logo_ZL #(logo_DIVA logo_UA 4)
    ↪ $0)))))
#(λ (#(λ (λ (λ (logo_forLoop $1 (λ (λ (logo_GETSET $2 (logo_FWRT logo_ZL $4 $0))))))))
    ↪ (#(logo_DIVA logo_UA) $0) $0))
#(λ (#(logo_PT (λ (logo_FWRT logo_UL logo_ZA $0))) (#(λ (logo_forLoop logo_IFTY (λ (λ (logo_FWRT
    ↪ $2 logo_epsA (logo_FWRT logo_epsL logo_epsA $0)))))) logo_epsL $0)))
#(λ (#(logo_PT (λ (logo_FWRT logo_UL logo_ZA $0))) (logo_FWRT logo_UL #(logo_DIVA logo_UA 4)
    ↪ $0)))
#(λ (#(#(λ (λ (λ (logo_forLoop $2 (λ (λ (logo_FWRT $2 $3 $0))))))) logo_IFTY) logo_epsA
    ↪ (logo_MULL logo_epsL $0)))
#(logo_FWRT logo_UL logo_UA)
#(λ (#(λ (logo_forLoop 7 (λ (λ (#(λ (λ (λ (logo_forLoop $2 (λ (λ (logo_FWRT $2 $3 $0))))))) 7
    ↪ logo_epsA $2 $0))))) (logo_MULL logo_epsL $0)))
```

### A.10.4 Towers

```
  #(λ (1x3 (left 4 (1x3 (right 2 (3x1 $0))))))
#(λ (1x3 (1x3 (right 4 (1x3 (1x3 $0))))))
#(λ (tower_loopM $0 (λ (λ (left 4 (#(λ (1x3 (left 4 (1x3 (right 2 (3x1 $0)))))) $0))))))
#(λ (tower_loopM $0 (λ (λ (left 8 (#(λ (1x3 (1x3 (right 4 (1x3 (1x3 $0)))))) (left 4 (#(λ (1x3
    ↪ (1x3 (right 4 (1x3 (1x3 $0)))))) (#(λ (1x3 (left 4 (1x3 (right 2 (3x1 $0)))))) $0)))))))))
#(λ (λ (λ (tower_loopM $0 (λ (λ (left $3 (3x1 (right 3 (3x1 (left $4 $0)))))))))))
#(λ (tower_loopM $0 (λ (λ (left 8 (#(λ (1x3 (1x3 (right 4 (1x3 (1x3 $0)))))) (left 2 (3x1
    ↪ $0))))))))
```

```
#(λ (1x3 (#(λ (1x3 (1x3 (right 4 (1x3 (1x3 $0)))))) (1x3 (#(λ (1x3 (left 4 (1x3 (right 2 (3x1
    ↪ $0)))))) (right 4 $0))))))
#(λ (tower_loopM $0 (λ (λ (1x3 $0)))))
#(tower_loopM 4 (λ (λ (right 6 (3x1 $0)))))
#(λ (tower_loopM $0 (λ (λ (#(λ (1x3 (#(λ (1x3 (1x3 (right 4 (1x3 (1x3 $0)))))) (1x3 (#(λ (1x3
    ↪ (left 4 (1x3 (right 2 (3x1 $0)))))) (right 4 $0)))))) $0)))))
#(λ (right 4 (#(λ (1x3 (1x3 (right 4 (1x3 (1x3 $0)))))) (#(λ (1x3 (left 4 (1x3 (right 2 (3x1
    ↪ $0)))))) $0))))
#(λ (right 2 (#(λ (1x3 (left 4 (1x3 (right 2 (3x1 $0)))))) $0)))
#(λ (λ (#(λ (tower_loopM $0 (λ (λ (#(λ (1x3 (#(λ (1x3 (1x3 (right 4 (1x3 (1x3 $0)))))) (1x3 (#(λ
    ↪ (1x3 (left 4 (1x3 (right 2 (3x1 $0)))))) (right 4 $0)))))) $0))))) $0 (right 2 (#(λ
    ↪ (tower_loopM $0 (λ (λ (left 4 (#(λ (1x3 (left 4 (1x3 (right 2 (3x1 $0)))))) $0)))))) $0
    ↪ $1)))))
#(tower_loopM 5 (λ (λ (#(λ (right 2 (#(λ (1x3 (left 4 (1x3 (right 2 (3x1 $0)))))) $0))) $0))))
#(λ (tower_loopM $0 (λ (λ (tower_embed (λ (#(λ (λ (λ (tower_loopM $0 (λ (λ (left $3 (3x1 (right
    ↪ 3 (3x1 (left $4 $0)))))))))))) 3 6 3 $0)) $0)))))
#(λ (λ (tower_loopM $0 (λ (λ (tower_embed (λ (#(λ (λ (λ (tower_loopM $0 (λ (λ (left $3 (3x1
    ↪ (right 3 (3x1 (left $4 $0)))))))))))) 4 5 $4 $0)) $0))))))
#(#(λ (λ (λ (tower_loopM $0 (λ (λ (left $3 (3x1 (right 3 (3x1 (left $4 $0)))))))))))) 4 5)
#(λ (λ (right 6 (tower_embed $0 $1))))
#(λ (tower_loopM $0 (λ (λ (#(λ (right 4 (#(λ (1x3 (1x3 (right 4 (1x3 (1x3 $0)))))) (#(λ (1x3
    ↪ (left 4 (1x3 (right 2 (3x1 $0)))))) $0)))) $0)))))
```

### A.10.5 Symbolic regression

```
  #(λ (+. $0 REAL))
#(λ (#(λ (+. $0 REAL)) (*. $0 REAL)))
#(/. REAL)
#(λ (#(λ (#(λ (+. $0 REAL)) (*. $0 REAL))) (*. (#(λ (+. $0 REAL)) $0) $0)))
#(λ (/. (#(/. REAL) $0) $0))
#(λ (#(λ (#(λ (+. $0 REAL)) (*. $0 REAL))) (*. $0 (#(λ (#(λ (#(λ (+. $0 REAL)) (*. $0 REAL)))
    ↪ (*. (#(λ (+. $0 REAL)) $0) $0))) $0))))
#(λ (#(/. REAL) (#(λ (+. $0 REAL)) $0)))
#(λ (#(λ (+. $0 REAL)) (*. (#(λ (#(λ (#(λ (+. $0 REAL)) (*. $0 REAL))) (*. (#(λ (+. $0 REAL))
    ↪ $0) $0))) $0) $0)))
```

## A.11 Hyperparameters and training details

**Neural net architecture** The recognition model for domains with sequential structure (list processing, text editing, regular expressions) is a recurrent neural network. We use a bidirectional GRU [6] with 64 hidden units that reads each input/output pair; we concatenate the input and output along with a special delimiter symbol between them. We use a 64-dimensional vectors to embed symbols in the input/output. We MaxPool the final hidden unit activations in the GRU along both passes of the bidirectional GRU.

The recognition model for domains with 2D visual structure (LOGO graphics, tower building, and symbolic regression) is a convolutional neural network. We take our convolutional architecture from [34].

We follow the RNN/CNN by an MLP with 128 hidden units and a ReLU activation which then outputs the $Q_{ijk}$ matrix described in A.6.

**Neural net training** We train our recognition models using Adam [18] with a learning rate of 0.001.

**DREAMCODER Hyperparameters** Due to the high computational cost we performed only an informal coarse hyper-parameter search. The most important parameter is the enumeration timeout during the wake phase; domains that present more challenging program synthesis problems require either longer timeouts, more CPUs, or both.

| Domain | Timeout | CPUs | Batch size | $\lambda$ (A.5) | $\alpha$ (A.5.4) | Max frontier (A.2) |
|---|---|---|---|---|---|---|
| Lists | 7m | 64 | 10 | 1 | 30 | 5 |
| Text | 7m | 64 | 10 | 1 | 30 | 5 |
| Graphics | 1h | 128 | 40 | 1.5 | 30 | 5 |
| Symbolic regression | 2m | 40 | 10 | 1 | 30 | 5 |
| Towers | 2m | 64 | 10 | 1 | 30 | 5 |
| Regexes | 30m | 64 | 40 | 1 | 30 | 5 |