Rubiks Cube Project

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Figure 1: basically a dissertation

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The Rubiks Cube is made up out of 27 small cubes , referred to as cubies. These include the 8 corner cubies , 12 edge cubies and the 6 center ones with one non-visible 'cube' in the middle.

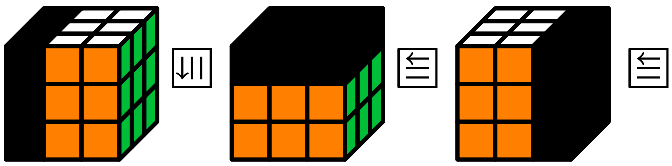
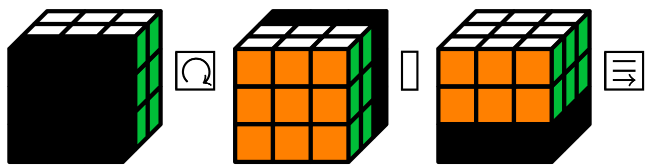


Figure 2: **Left (L), Up (U) and Right Moves (R)**

Figure 3: **Face (F) , Back (B) and Down (D) Moves**



Above shows the 6 basic moves of the cube, to which the black cubies are transformed by rotating the specific face 90 degrees clockwise. When applying one of these moves it is necessary to reference how to describe the change of the cubie. Fixing the cubies in the start configuration allows to reference cubbies in their correct (solved) positions. However a cubie could live in the correct position with a different orientation, in this way one refers to how a cubie lives in it's cubicle.

Introduction

History on the cube

(Limited, 2017) The cube was first created by a Hungarian Professor named Erno Rubik. The original version is one which mimics the cube seen today including the coloured stickers. To him it wasn't meant to be a hobby or game like element, instead it was created to aid him in explaining three dimensional geometry. Although being the creator ,he found as he mixed up the cube and the stickers became jumbled he was unable to solve it. *It was a code I myself had invented!" he wrote. "Yet I could not read it."* Thus creating the phenomenon seen today, with a staggering 400 million sold and 1 in 7 people believed to have played with the cube. After this amount of time the cube has still not fallen out of popularity.With 'SpeedCube' competitions still rife the world record has been broken once more in 2017 amounting to 4.69 seconds to solve the 3 by 3 cube by Patrick Ponce (USA) (record holders, 2017).
Singmasters (Singmaster, 1981) notation states that there are 6 basic moves of the rubiks cube (RLUFDB) which will be referenced throughout this project.

Possible and non possible moves

The cube has 8 corner cubies have 3 faces while the 12 edge cubies have 2 faces leaving the 6 center cubies with a singular face. When applying one of these basic moves to the cube it's impossible for these center cubies to stray from the cubicle in which they live in. So applying probability to the scenario , should allow to calculate the total number of configurations. There is a choice of 8!

positions for the corner cubies , and as there's 3 faces there are 3^8 possible orientations. Similarly with the 12 edge cubies , combining these results give a total of $2^{12} * 3^8 * 8! * 12! \approx 519$ quintillion configurations. However what isn't known yet is which of these are actually possible from a set of permutations from the starting configuration.

Goal of the project

This gives us motivation to see which of these configurations are not valid, why they're not valid and the mathematics behind the rubiks cube. This will require implementation of some of the key theorems and definitions from group theory. Firstly it will be shown that the Rubiks Cube can be expressed as a group by following the properties of the definition of a group from group theory.

Rubik's Cube Group

A group G is defined as a set with a binary operation $*$ such that , $* : G \ * \ G \rightarrow G$. Which follows these properties.

- It's closed under the operation $*$, $\forall f, g \in G, f * g\ is\ also in G$
- The operation is associative so $for\ any\ f, g\ and\ h in G$, $f * (g * h) = (f * g) * h$
- For every element in the group there exists an inverse, so for an element $f \ \in \ G\ exists\ g \in G\ such\ that\ f * g = e = g * f$
- The group contains an identity element $\epsilon\ such\ that\ for\ any\ f \in G\ f * \epsilon = \epsilon = \epsilon * f$

An example of a simple application of this is the integers under addition $(\mathbb{Z}, +)$. It's closed under its operation, as addition is closed e.g. any real integer added to another will give another integer. It's associative as multiplication itself is associative e.g. 4+(2+3) = 9 = (4+2)+3. The inetgers have the identity 0 following the defintion.Finally every element in this group has an inverse, namely the negative component of that integer e.g. 3 + (-3) = 0, 6 + (-6) = 0 etc. Also along with a group a rference to a subgroup should also be stated.

H is a subgroup of group G iff H is a subset of G and H is itself a group, for example this could be $(2\mathbb{Z}, +)$ group of even integers.

The group definition is very useful when understanding following theorems and lemmas about the cube and can be applied directly below. The group of the Rubiks Cube $(G, *)$ can be represented by the cube permutations and orientations of such cubes, composed of the 6 basic moves of the cube listed previously. This group will be a subset of size to the total configurations stated in the Introduction as here the group actions (which can create a configuration) will be defined explicitly.

- G is closed under *
  To prove this, one will need to show that under no operation does the rubiks cube break from it's group. When talking about under an operation this means applying one of the basic moves, and breaking from the group is equivalent to saying that after the opeartion it stays in the orbit. Orbits and operations will be explained later in the project.

- e is the empty move
  This is trivial for this group and is just no move at all.
- if there is a move M then inverse is just reversing what you did
  , Equivalently is two sets of moves which cancel out each other e.g. F2 F2, DR $R^{-1}D^{-1}$
- associativity between moves follows as each move is made on a sequential turn basis
  Thus for example R (L D) = R L D = (R L) D, moves are applied to the cube from left to right one after the other.

A homomorphism is a function ϕ from two groups G and H, such that ϕ preserves the group operation

$$\phi(a * b) = \phi(a) * \phi(b) \tag{1}$$

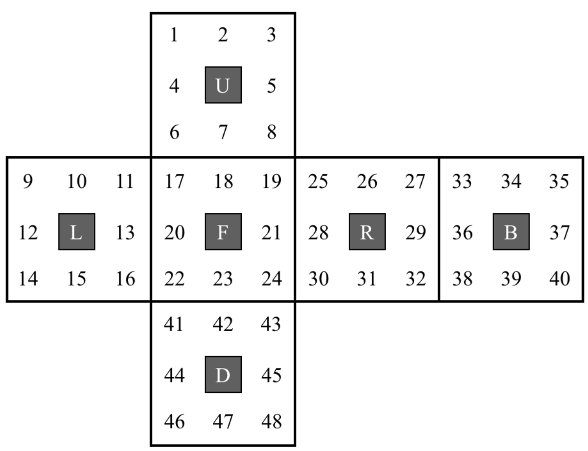
Automorphisms

This is just any isomorphism which is a mapping from one object to itself. The set of all of these automorphisms in an object form the automorphism group.

Basic Moves and Labelling

The Basic Moves M = $\{L, R, U, D, F, B\}$ have the following disjoint cycle notation on how they effect the cubies. This correspons with the labelled diagram**??**.

$$\begin{aligned}
U &= (1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)\\
L &= (9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)\\
F &= (17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)\\
R &= (25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)\\
B &= (33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)\\
D &= (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)
\end{aligned}$$



Alternating and Symmetric Groups

Let X be a set and let S(X) be the symmetric group on X; the elements of S(X) are the permutations of X (i.e. bijections $X \to X$) and the group operation is composition of maps. In contrast an Alternating group A(X) only contains the permutations to which are just even cycles, as opposed to the symmetric group above containing even and odd cycles. For notation purposes these two groups will be defined as $A_n$and S_n.

A useful map to consider is an operation called a group homomorphism between two groups defined by the following properties:

$$\begin{aligned}
&\phi : G \to H \ is \ a \ homomorphism \ if : \\
&\qquad The \ map \ is \ a \ well-defined \ mapping \\
&\qquad \phi \ (ab) \ = \ \phi \ (a) \ * \ \phi \ (b) \ \forall \ a, b \ \in \ G \\
&\qquad im(\phi) = \{\phi(a) | a \in G\} \\
&\qquad ker(\phi) = \{a \in G | \phi(a) = e'\}
\end{aligned}$$

In our case we can define a function as below which will map the elements of S_n to either 1 or -1 in the cases outlined in the definition below.

$$\phi : S_n \to \{-1, 1\} \qquad \phi(x) = \begin{cases} 1, & \text{if x is an even number of transpositions.} \\ -1, & \text{if x is an odd number of transpositions.} \end{cases} \tag{2}$$

When x $\in \phi(x)$ equals 1, it means it can be expressed as an even number of transpositions. Transpositions are defined as the 2 cycle permutations, but to show the well-defined property of this map we need to first show all permutations in S_n can be written as a product of transpositions.This can be shown by the following inductive argument.The base case for S_2 is trivial. Suppose any permutation of n takes less than n transpositions. Consider a permutation w of length n+1 (transpositions). Using the base case we can perform one transposition to swap the $n + 1^{st}$ element in the correct place. Thus by the inductive hypothesis we have less than n transpositions for the remaining n elements , making the total less than n+1 transpositions. Now this is known the map can be shown to be a homomorphism by the following.

Proof.

Our map can easily be verified to satisfy the conditions above. It is well defined , it can be expressed as some product of transpositions by the proof above.

Take two elements $\sigma, \tau \in S_n$. The goal is to show $S_{\tau\sigma} = S_\tau * S\sigma$

Either of these can be written like so $\sigma = \alpha_1 ... \alpha_a$,$\tau = \beta_1 ... \beta_b$, with some integers a and b factorisations of α and β as a product of transpositions.

$\sigma\tau = \alpha_1 ... \alpha_a \beta_1 ... \beta_b$ now expresses $\sigma\tau$ as a product of a+b transpositions.

$\phi(\sigma\tau) \ = \ (-1)^{a+b} \ = \ (-1)^a(-1)^b \ = \ \phi(\sigma)\phi(\tau)$

thus conditions of homomorphisms are satisfied. $\square$

As the map is known to be a homomorphism the kernel $ker(\phi)$ is of the form $\{a \in g|\phi(a) = e'\}$ where e' is the identity element of the set -1,1. For this case as the identity of this set cannot be 0 as a norm for groups such as $(\mathbb{Z}, +)$,$(\mathbb{N}, +)$ as the 0 element does not exist thus shall be defined at the elements of even cycles (odd transpositions). So we have $A_n \subset ker(\phi)$ but also it is the entire kernel itself so $A_n = ker(\phi)$, if it wasn't then for some $S_n \notin A_n \in ker(\phi)$. However this cannot be the case as only even cycles are mapped to -1 , and any even cycles in S_n are in A_n by definition. Taking into consideration the Group Homomorphism Theorem, or alternatively the First Isomorphism Theorem

Theorem 0.1 (Group Homomorphism Theorem)**.** $G/ker(\phi) \;\cong\; im(\phi)$

Showing that the group G / by the kernel is isomorphic to the image of the group. Being isomorphic means there exists a homomorphism between the two groups which is also injective (one to one). Substituting the values above defined for the kernel and the image of the group , it can be concluded that,

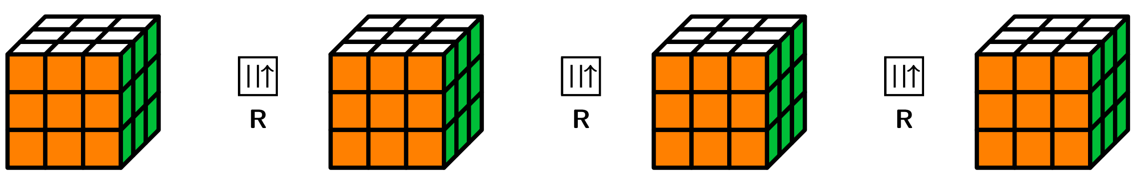
$$S_n/A_n \cong \mathbb{Z}_2 \tag{3}$$

The cyclic group $\mathbb{Z}_2$ is placed at the end as it is equivalent to the set of $\{1, -1\}$ with just two elements, x and the identity (as x is the 1 and the identity has already been defined to be -1).

From the Group Homomorphism Theorem there is now a new collection called a quotient group , but to understand what a quotient group is to groups one must first consider the left and right cosets formed by each corresponding group. With G a group and H a subgroup of G the left and right cosets are as follows:

- gH = $\{gh \mid h \in H\}$ is the left coset
- Hg = $\{hg \mid h \in H\}$ is the right coset

The subgroup H is just the smallest group contained in G which generates all the elements of H. Taking H = $\{R\}$ (the right turn) as an example, H is now the subgroup of $\mathcal{G}_{RC}$ generated by all right turns as shown.



This is one instance of a right coset purely by definition on the original group $\mathcal{G}_{RC}$. It is such that whatever the size of the subgroup H it is identical to size of the coset. This is by construction as there is one element in the coset for each element in the subgroup. It can also be said that any two coset from a subgroup H constructing from the group G, must be either distinct or the same. So any subgroup must have disjoint cosets , making them unique in definition. This can be shown by a simple proof by contradiction:

Proof.

Suppose there were two right cosets Hx and Hy which share elements.

$h_1x = h_2y$ for some $h_1, h_2 \in H$

taking inverses x $= h_1^{-1}h_2y$, $h_3 = h_1^{-1}h_2$

thus x $= h_3y$ and hx $= hh_3y$

so every element of Hx can be written as one of Hy

so every element of Hx is in Hy, thus Hx= Hy □

These such right cosets will partition the rubiks cube group into equal sized disjoint sets. As the kernel of $\mathcal{G}_{RC}$ is a group itself, the number of distinct cosets is the size of the quotient $|G|/|ker(\phi)$. This is by Lagrange's Theorem.

Theorem 0.2 (Lagrange's Theorem)**.** *For H a subgroup of a group G , the size of H must be a divisor of the size of G. So* $x|H| = |G|$ *for some* $x \geq 1$

So now there is the quotient group S_n/A_n is isomorphic to $\mathbb{Z}_2$ and of order dividing the rubiks cube group $\mathcal{G}_{RC}$ by Lagrange, now whats left is to show is that the map $\phi : S_n/A_n \rightarrow \{1, -1\}$ preserves an homomorphism as above. As the alternating group has been defined as the kernel above , the group operation (multiplication) will hold in a similar way to the $\phi : S_n \rightarrow \{1, -1\}$ map. There are 4 possible occurrences of multiplication in this map defined below.

$$x \text{ is even, } y \text{ is even} \qquad \phi(xy) = \phi(x)\phi(y) = 1 * 1 = 1 \tag{4}$$

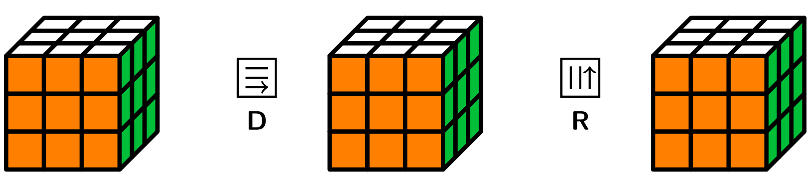
$$x \text{ is even, } y \text{ is odd} \qquad \phi(xy) = \phi(x)\phi(y) = 1 * (-1) = -1 \tag{5}$$

$$x \text{ is odd, } y \text{ is even} \qquad \phi(xy) = \phi(x)\phi(y) = (-1) * 1 = -1 \tag{6}$$

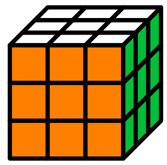
$$x \text{ is odd ,} y \text{ is odd} \qquad \phi(xy) = \phi(x)\phi(y) = (-1) * (-1*) = 1 \tag{7}$$

Note: By x is even , y is odd , it means for x to be defined as even number of transpositions , y to be odd number of transpositions.So homomorphic property of multiplication holds, thus map must be homomorphic.

While presently this may not seem that important this conclusion will be paramount in defining the analysis of the cube. This will be referenced later on in the project after valid configurations have been defined. Now this theory can be related to the example of the 3 by 3 cube. Consider the move DR^{-1} on the start configuration of the cube.



Only considering the corner cubies for now, labeling them appropriately will allow for the moves to be expressed as a combination of disjoint cycles. The group considered when making these permutations is the group of all corner cubies Z_3^8. The labeling of the cube is as such ***INSERT LABELLING***



$$D(\theta) = (1)(2)(3)(4)(5\ 6\ 7\ 8) \qquad R^{-1}(\theta) = (1\ 7\ 8\ 4)(2)(3)(5)(6) \tag{8}$$

Are the cycles of the Down and Right inverse moves respectively which can be combined to obtain.

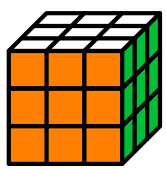
$$DR^{-1}(\theta) = (1\ 8\ 4)(5\ 6\ 7) \tag{9}$$

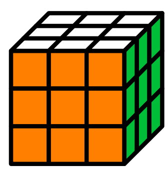
Now there is a cycle which alternates the position of the 6 cubies listed above. However this doesn't really achieve anything when looking to solve the cube, as far as the user is concerned this is just any random move $M \in G$. Aiming for a set of moves which oriented 2 cubies for instance would be far more useful in proving properties of the corner cubies and it's corresponding group Z_3^8. Taking the inverse of the move above, $D^{-1}R$, one obtains a similar disjoint cycle set of ,

$$D^{-1}R(\theta) = (1\ 7)(5\ 8) \tag{10}$$

These cycles are in fact isomorphic , which there is a combination which actually achieves the aim. Through inspection applying the moves together the combination is as follows

$$(DR^{-1}D^{-1}R)^2U^{-1}(DR^{-1}D^{-1}R)^4U \tag{11}$$





But this inspection doesn't seem very trivial and leaves the question why is this move above like so. For this explanation it's useful to turn to commutators from Group Theory.

$$\textit{The commutator of two elements , g and h of a group G , is the element} : ghg^{-1}h^{-1} \tag{12}$$

For the cube this can be translated by setting g,h to be a specific set of moves. So really what's done above is to find the commutator which orients the two corner cubies by using the definition of a commutator. Where instead of considering disjoint cycles , defining g and h like so

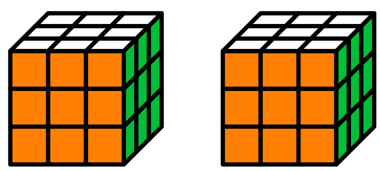
$$g = R^{-1}DRD^{-1}R^{-1}DR \qquad h = U^{-1} \tag{13}$$

and applying the commutator algorithm $(ghg^{-1}h^{-1})$ gives the set of moves to orient two corner cubies again. The importance of these commutators is high as it allows the user to gain intuition into actually solving the cube rather than just following a set algorithm. This also becomes particularly useful when the cube is a few moves off the start configuration and say the user needs to permute two corner cubies, now with the knowledge of the correct group commutator this can be achieved.

Context on Cube

Generators and Orbits

Let G be a group , a subset $S \subseteq G$ is a generating set of G if $G = \langle S \rangle$. In other words every element of G can be written as a finite product (under group operation) of elements of S. These elements are called generators of G.Taking the group G to be composed of the 6 basic moves $G = \{F, L, R, U, D, B\}$, such that any cube permutation can be made, allows a comparison to be made of what a generator means to this specific example. For example taking $S = \{R\}$ gives the subgroup which obtains all possible cube permutations obtained by rotating the right face $\{R, R^2, R^3, R^4\}$. As the order of a group is defined as either the least $n \in \mathbb{N}$ such that $g^n = \epsilon$ (where g is a member of the group), or alternatively the size of the subgroup it generates, the order for S will be 4. Note it stops at R^4 as the group is cyclic* , so $R^5 = R$.*by cyclic it just means to say that there exists an element which creates a generating set.
In group theory many groups share the property of being Abelian, which means for a group G if $\forall x, y \in G$, $x * y = y * x$. One example could be $(\mathbb{Z}, +)$ the group of the integers on addition, for any integer x,y this will hold (3+4 = 4+3) as the integers are commutative and associative in definition. However applying this to the rubiks cube group this will not hold , showing us the order in which the moves are applied does in fact matter. This is shown by a simple case of the move M $= \{L, R2, D\}$ below



Where , from the starting configuration, the left hand cube has been applied the sequence $L, R2, D$ and the right hand cube $R2, D, L$. Visually one can see the are obviously not equal , thus breaking the condition of what it means to be abelian.This is because these moves share common cubies, a move consisting of opposite faces alone such as R,L would hold commutativity (RL=LR) however for the whole cube this can't be said. So clearly some elements of the group commute while others do not.
The set of elements to which the generators are made from is defined as the generating set. These generators are a useful way for defining groups , however for large groups such as the rubiks cube it can become difficult to gain information due to the sheer enormity of possible generators. When talking about these commuting element is often useful to mention orbits and their stabilisers.

Given the definition of an orbit from Group Theory ,

$$\begin{gathered} if\ G\ acts\ on\ a\ set\ A\ ,\ then\ the\ orbit\ of\ a \in A\ (under\ this\ action)\ is\ the\ set\ \{\ a * g\ :\ g \in G\} \\ The\ stabiliser\ of\ an\ orbit\ is\ \{g \in G | g * x = x\}\ the\ set\ of\ all\ elements\ that\ fix\ x \end{gathered} \tag{14}$$

From the definition the orbit is just the subset of A which can be moved by any move from G, with A being the total number of configurations of the cube. As said above the center cubies always stay in the 6 center positions hence will always be in the same orbit. This applies to the corner and edge cubies as well (relative to corner and edge positions). From properties of an orbit

, it must be either disjoint or the same (this can be shown from a single contrapositive argument). So one could consider 3 separate orbits within the Cube where,

$$A = A_x \cup A_e \cup A_c \cup A_e \tag{15}$$

A_x are the center pieces, A_e edge pieces and A_c the corner pieces. We know any of these respective cubies will stay under their respective positions from a move G, but what isn't so is trivial is when considering the whole set of the cubies. Moreover prove how any group action from the Rubiks Group $\mathcal{G}_R$ preserves any resulting configuration will be in the same orbit.(?, ?) An example of when the wouldn't apply is any illegal configuration of the cube (e.g. from pulling pieces out, switching stickers etc.) ,defining the difference into what is and isn't a valid configuration is key in proving the statement about same orbits above. This will be referenced later in the paper. Orbits aswell as describing how points, or in the case described, cubies move can also be used to compute stabilisers. These can be the subgroups of elements which fix one or more points, which can be very useful in solving the cube in a step by step solution.(Wolfram, 2018)

Suppose there was an algorithm which took a given scrambled state to the starting configuration. Where in this algorithm one piece has been permuted correctly, call it G_1, where the rest of the cube remains to be solved. These subsequent permutations will aim to put their respective pieces in the correct place (and orientation) without disturbing the previous permutation.

This algorithm is called a stabiliser chain where the first permutation fixing the first piece of the cube , G_1 shall be defined as the stabiliser of the chain. $G > G1 > G2 > .... > Gn = I$, where I is the identity (starting configuration).Each of these $G_i's$ should contain a set of moves which solves the i'th piece, therefore one of the move sequences for that G_i will be in G_{i+1}, a permuation in G_i will always lie in G_{i+1}. In other terms with a list of a move $a_{i1}, a_{i2}, a_{i3}, ...$ every $b \in G_i \in a_{ik}G_{i+1}$ for some k. So G_i is made up entirely of the set of all cosets $a_{i1}G_{i+1}, a_{i2}G_{i+1}, a_{i3}G_{i+1}, ...$ for every k. There now exists a way to find the generators for G_{i+1} a specific sequence solving the i+1'th piece, that given the generators of G_i and the list of move sequences a_{ik} (or coset representatives), the new stabiliser chain can be built.(Jaap, 1981)

The rubiks cube group $\mathbb{G}_{RC}$ if to be implemented by a computer using each individual configuration , which is $(8! * 3^8 * 12! * 2^12/3 * 2 * 2) \approx 4.3 * 10^{19}$, becomes computationally inefficient to try and store each sequence for each configuration.The aim of implementing this algorithm however is to identify the groups composition structure by using subgroups to factorise the problem down, to solve the cube from a given state. The algorithm in question the Schreier Sims Algorithm, is given motivation from the Schreier Lemma on subgroups.

Lemma 1 (Schreier Lemma on Subgroups)**.** *Let G be a group with a set of generators S. Let H be a subgroup of G, and let A be the set of coset representatives of $H \in G$. For any $g \in G$, let g denote the element of A that represents the coset gH.*
Then H is generated from the set $\{(sa)^{-1}(sa) | a \in R, s \in S\}$.

Proof. Take any $h \in H(\in G)$, h = $s_1 s_2 ... s_k$ for some sequence of generators $s_i \in S$.

$$Let \quad t_i = s_{i+1}...s_k \quad be\ the\ coset\ representative\ for\ the\ s\ sequence$$

, this is such that $t_0 = s_1...s_k$ = h and t_k defines the identity $t_k = e$. Rewriting h with new set of coset representatives t gives h = $(t_0^{-1} s_1 t_1)...(t_{k-1}^{-1} s_k t_k)$. We also find that $(s_i t_i)H = s_i(t_i H) = s_i(s_{i+1}..s_k H) = (s_i s_{i+1}..s_k)H = t_{i-1}H sos_i t_i = t_{i-1}$. Rewriting h with this new condition gives h = $((s_1 t_1)^{-1} s_1 t_1)((s_2 t_2)^{-1} s_2 t_2)...((s_k t_k)^{-1} s_k t_k)$. By definition t is the set of coset representatives, giving each i interval a factor of $s_i^{-1}s_i$. Any element $h \in H$ is a product of these factors , thus the set $\{(sa)^{-1}(sa)|a \in R, s \in S\}$ will generate H.

□

The method to factorise into smaller subgroups uses a combination of Lagranges Theorem with the Orbit Stabiliser Theorem,

Theorem 1.1 (Orbit Stabiliser Theorem)**.** .
$|G| = |Gx| * |G_x|$
applying Lagrange
$|Gx| = |G/G_x)| = |G : G_x|$

which simply states the size of the orbit of the group can be retrieved from the size of the group partitioned by its stabilisers. Any subgroup chain can also be completed in $log_2|G|$ as the stabiliser is a subgroup for the rubiks cube group (G $G > G_x$) meaning $[G : G_x] \geq 2$, so $|G| \geq 2|V|$ and $|G| \geq 2^k$. (Hulpke, 2018). So reducing the problem logarithmically can cause a major difference to understanding an algorithm to solve the cube in a certain state in the orbit, however the reduction is not that significant as $[G : G_x]$ still remains very large. In fact for each predecessing G_i the number of generators grows exponentially, G_1 = 6*24, G_2=6*24*22 ... For an efficient solver you would need to extract which of these generators are useful to the solving problem as a large number of these are not. The algorithm has been adapted to improve on these flaws , which along with other solvers shall be mentioned later.

Basic Moves and their Order

Below lists the generators with their corresponding orders. The combination of these in the set M allows derivation from the order of the whole rubiks cube group it generates.

Minimally Generating Set

The rubiks cube is generated by the set of basic moves M = $\{L, R, U, D, F, B\}$ where one can reach each configuration from a combination of these. However it actually stands that this set of moves M is not minimal and in fact the rubiks group can be generated by just 5 of these moves, $M' = R, L, F, B, U$ (Bandelow, 2012). With this set the move D and D' can be simulated using M' by doing the following two moves found by Roger Penrose:

$$D = R^2L^2U^{-1}B^2F^2U^{-1}B^2R^2B^2F^2L^2F^2U^{-1}$$
$$D^{-1} = R^2L^2UF^2B^2UF^2R^2F^2B^2U^2L^2U^2L^2R^2U^2R^2U^2R^2F^2U^{-1}R^2B^2R^2L^2F^2L^2UB^2F^2U$$

The downside to using this set is the efficiency when implementing the down moves using these replacements. However one cannot dispense another standard generator , 5 of the basic moves are needed.

2-Generated Group

The Rubiks Cube group is a 2-generator group meaning that choosing any two elements of the cube will give a high chance of generating the whole group (p32) (Singmaster, 1981). Frank Barnes observed this using the two moves below.

$$\begin{aligned}
\alpha = L^2BRD^{-1}L^{-1} \quad &= (RF, RU, RB, UB, LD, LB, LU, BD, DF, FL, RD) \\
&\quad (FUR, UBR, LDB, LBU, DLF, BDR, DFR) \\
\beta = UFUR^{-1}U^{-1}F^{-1} \quad &= (UF, UL)_+(UR)_+(UBR, UFL)_-(URF)_+
\end{aligned}$$

Here the notation slightly differs from the usual disjoint-cycle so shallbe explained here. A move with the two character (LU,BD) is an edge cycle. This will move the LU edge piece to the BD edge cubie such that position of L will move to half end up in the B face, and U doing a similar action with D. Notation for a corner is similar with the corresponding 3 character cycles e.g. (URF URB). Finally $(UF, UL)_+$ is called a twisted cycle, taking the UF cubie to the UL position however the final edge cubie orientation flips when cycling back. Again for a corner this is similar where the cubie is rotated to the next face when cycling back.
The element α^7 is an 11-cycle of edges and α^{11} is an 11-cycle of corners, generating the orientation subgroups which wil be mentioned in the next section. β will act on the edges and corner cubies that α fixes. The fact that these 2 elements genrate the whole group can now be verified much easier with the implementation of cube related languages like GAP.

Possible and Non Possible Configurations

We have the orientations of a corner cubie defined as Z_3. This is because each of corner cubie has 3 coloured faces which can be oriented differently. This makes the set of 8 corner cubies the product of all these different permutations.

$$\mathcal{Z}_3^8 = \prod_{i=1}^{8} \mathcal{Z}_3$$

To think about the different orientations of a cubie the idea is to fix a 'starting orientation' .We know there must exist a homomorphism which takes the cubies to the correct orientation , as there exists a set of moves leading any valid configuration to the start configuration (solved). Thus using the group Z_3^8 , generators of such group would be basis elements, so fixing 7 cubies give us the homomorphism below:

$$\delta : G \Rightarrow G \qquad \delta(Z_3^7) = \epsilon \tag{16}$$

Also when considering the subgroups of G_{RC} it's necessary to define another subgroup along with the orientations of the cubies, this being S_8 the permutation group of 8 elements (the 8 corner cubies). This can be defined like such , as taking any corner cubie on the cube it has 3 faces as described above which can all be alternated between each other. This being equivalent to a 3-cyclic group , denoted C_3. Which leads again to a product of 8 groups

$$\mathcal{C}_3^8 = \prod_{i=1}^{8} \mathcal{C}_3$$

, where C_3^8 is denoted as S_8.These 2 definitons above analogously follow a similar argument when considering the 12 edge cubies along the cube Z_2^{12}.

Going back to orbits and considering the difference between correct configurations , we can start by giving notation on what defines a valid configuration.At any configuration all corner cubies , as well as edge cubies, should be on orientation number corresponding to that of their group. That is,

$$e.g. x_1 = 0, x_2 = 2..., x_8 = 1$$
$$y_1 = 1, y_2 = 1..., y_{12} = 0$$

One must also define a sign variable which states what the move has done to change where the arrangement of the corner cubies or that of the edge cubies. Define $\tau(\in S_8)$ as the sign of the corner cubies and $\delta(\in S_{12})$ as the sign of the edge cubies. This sign is 1 if there is an even number of transpositions, which are 2 cycle permutations, and -1 if there is an odd number.

$$\tau(x) = \begin{cases} 1, & \text{if x contains an even number of transpositions.} \\ -1, & \text{if x contains an odd number of transpositions.} \end{cases} \tag{17}$$

DEFINE K CYCLE PERM

Ignoring the center cubies as they always stay in the same position relative , a configuration is defined as (τ, δ, x, y) . For example an invalid configuration would be that below:

This is because there exists no set of moves to obtain the configuration shown on the right. The change of orientation of one singular cubie (from the start configuration) can only be achieved by breaking the cube ,so is invalid. Given our definitions of a configuration and the labelling previous REFERENCE LABEL ABOVE this can be shown mathematically.

To see if one configuration can be found from another is identical to saying $(\tau^{'}, \delta^{'}, x^{'}, y^{'}) = (\tau, \delta, x, y)$ * M e.g. applying a move M gives the different configuration. Using knowledge of orbits, define it to be the set of all possible configurations from the Rubiks Cube group G. It's known whenever a move M applied to a possible configuration is also valid then one can conclude any other valid configuration can be found from a move M. As this trivially holds for any one of the basic moves $M = \{F, L, R, U, D, B\}$ one can conclude using basic inductive principles, that for any intermediate set of moves this will still hold as the configuration will still be in the orbit. Giving reason to saying any valid configuration of the cube can be solves with a specific set of moves. However to prove one of the fundamental theorems on the cube we still need to look at the corresponding signs of two correct configurations, hence we define the two following maps.

$$\phi_{corner} : G \to S_8 \qquad \delta_{edge} : G \to S_{12} \tag{18}$$

This map all elements of the group to the subgroup S_8 (Symmetric group of the corner cubies). This map is a homomorphism as it is trivially well defined, and preserves the group operation of a move e.g. $\phi(L * L) = \phi(L) * \phi(L)$. With this homomorphism the above objective , $(\tau, \delta, x, y) \longrightarrow (\tau^{'}, \delta^{'}, x^{'}, y^{'})$ from $\tau^{'} = \tau\phi_{corner}(M)$, and similarly with edge pieces $\delta^{'} = \delta\phi_{edge}(M)$ works as from the following proof:

We know that the homomorphism is surjective, otherwise there would exist possibles moves in $M = \{F, L, R, U, D, B\}$ taking corner cubies out of subset of corner cubie positions (contradicting our definition of the orbit of the cube group). This trivially cannot happen , a cubie with 3 faces cannot live in an edge cubie consisting of 2 faces, meaning any application of $\phi_{corner} : G \to S_8$ will result in one of the 8 corner cubies. However to formally that this is a surjective homomorphism one must show the following:

Proof. It's known the symmetric group S_8 is genreated by the set of 2 cycles in S_8 aswell as that for any n. So to show surjectivity it's equivalent to show every 2 cycle (in S_8) is in the image subgroup $im_{\phi_{corner}}$. This just means the image is entirely those 2 cycles which make up the corner cubies, and hence would mean $sgn(\tau) = 1$. By taking any such commutator of the cube, and using trial and inspection on a said amount of cycles a move switching position of two corner cubies (while leaving the other corners fixed) can be found. One such move is M = $([D, R]F)^3$ with the commutator [D,R].This takes x_3 to x_8 making the cycle $(x_3\ x_8)$ while also switching edge cubies as there exists no move solely swithcing two corner cubies without affecting any of the other cubies. With this move applied to the map it gives, $\phi_{corner}(M) = (x_3\ x_8)$ thus $(x_3\ x_8)$ must lie in the image of ϕ_{corner}, it just remains to show the others.

Take x_i, x_j as any two coner cubies , any corner cubie can be placed in the position of another so there must exist a move p sending x_3 to x_i and x_j to x_8.Let $\sigma = \phi_{corner}$ and as ϕ_{corner} is a homomorphism the following applies:

$$\begin{aligned}
\phi_{corner}(p^{-1}Mp) &= \phi_{corner}(p)^{-1}\phi_{corner}(M)\phi_{corner}(p) \\
&= \sigma^{-1}(x_3\ x_8)\sigma \\
&= (\sigma(x_3)\ \sigma(x_8)) && \text{As } \sigma^{-1}(x_i...x_k)\sigma = (\sigma(x_i)..\sigma(x_k)) \\
&= (x_i\ x_j) && \square
\end{aligned}$$

So for any x_i and x_j distinct in $\{x_1, ..., x_8\}$ $(x_i\ x_j)$ is in the image of the map, moreover $(x_i\ x_j) \in im_{\phi_{corner}}$.

In having a surjective homomorphism , the Group Homomorphism Theorem can be applied once again (0.1) by inspection the kernel of this group is empty thus the image is equivalently S_8. Any $\tau \in S_8$ of this new image subgroup (S_8) can be described as an element which moves the corner cubies from their start positions to the new positions. From the way τ is defined it shall be either 1 or -1, using the homomorphic properties of S_8 must contain an inverse element (for every element in the group), meaning applying one of the basic moves M we can get cubie A to its original position from its new position. One of these basic moves, moves 4 corner cubies , making the permutation a 4 cycle thus being an even number of transpositions $\tau^{'} = 1$ and thus doing $(\tau^{'}, \delta^{'}, x^{'}, y^{'}) = (\tau, \delta, x, y)$ * M means $\tau^{'} = \tau\phi_{corner}(M)$. So if $\tau^{'} \neq \tau$ e.g. $\tau = -1$ applying M means τ will now alter from an even to an odd number of transpositions thus τ = -1 * -1 = 1 and sgn($\tau^{'}$) = sgn(τ).

There is an analogous argument with the edge cubies for $\delta^{'} = \delta\phi_{edge}(M)$. Combining the analogous argument and the fact that any basic move M moves 4 corner and 4 edge cubies the following can be concluded:

Proof.

$$\begin{aligned}
sgn(\tau^{'})sgn(\delta^{'}) &= sgn(\tau)sgn(\delta) \\
&= sgn(\tau)sgn(\phi_{corner}(M))sgn(\delta)sgn(\phi_{edge}(M)) && \text{Using homomorphisms above} \\
&= sgn(\tau)(-1)sgn(\delta)(-1) && \text{Both 4 cycles} \\
&= sgn(\tau)sgn(\delta) && \square
\end{aligned}$$

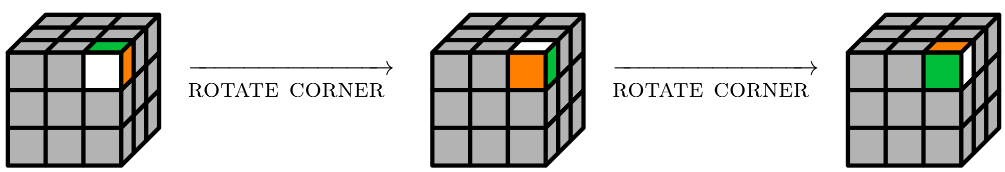
Furthermore as the start configuration is defined as (1,1,0,0) by looking at the definition of the sgns thus any other valid configuration is also in the start configuration as shown above giving that $sgn(\tau) = sgn(\delta)$. This fact is fundamental in cube theory, but a full definition of what makes a correct configuration is not fully complete. For this conservation of parity between edge and corner cubies must be verified. This touches on the point made earlier in the section discussing that each cubie , edge or corner, must be associated with a number to describe the orientation of the cubie in that specific cubicle, otherwise mathematically speaking , it cannot be told if the cubes are facing in the right directions. These x and y values must obey the following rules:

$$\Sigma x_i = 0mod(3) \qquad \Sigma y_i = 0mod(2) \tag{19}$$

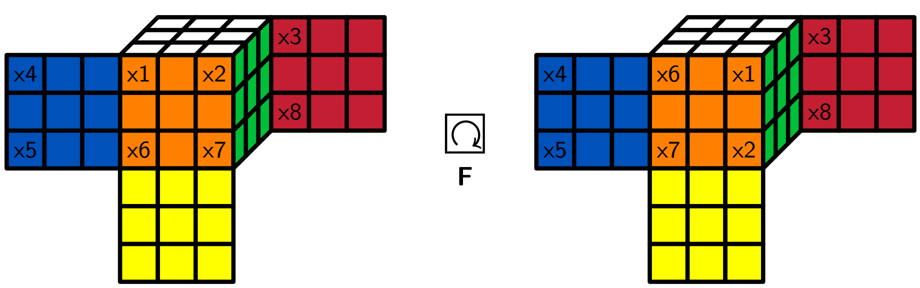
From the labelling defined previously for x_1 to x_8 and y_1 to y_{12} the basic move M will change these values on one configuration($(\tau^{'}, \delta^{'}, x^{'}, y^{'}) = (\tau, \delta, x, y)$ * M) in the table below(Chen, 2004):

Move	x' and y' value
D	$x_1, x_2, x_3, x_4, x_8, x_5, x_6, x_7$ $y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_{10}, y_{11}, y_{12}, y_9$
R	$x_1, x_7 + 1, x_2 + 2, x_4, x_5, x_6, x_8 + 2, x_3 + 1$ $y_1, y_7, y_3, y_4, y_5, y_2, y_{10}, y_8, y_9, y_6, y_{11}, y_{12}$
U	$x_2, x_3, x_4, x_1, x_5, x_6, x_7, x_8$ $y_4, y_1, y_2, y_3, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}$
L	$x_4 + 2, x_2, x_3, x_5 + 1, x_6 + 2, x_1 + 1, x_7, x_8$ $y_1, y_2, y_3, y_5, y_1 2, y_6, y_7, y_4, y_9, y_{10}, y_{11}, y_8$
F	$x_6 + 1, x_1 + 2, x_3, x_4, x_5, x_7 + 2, x_2 + 1, x_8$ $y_1, y_2, y_8 + 1, y_4, y_5, y_6, y_3 + 1, y_{11} + 1, y_9, y_{10}, y_7 + 1, y_{12}$
B	$x_1, x_2, x_8 + 1, x_3 + 2, x_4 + 1, x_6, x_7, x_5 + 2$ $y_6 + 1, y_2, y_3, y_4, y_1 + 1, y_9 + 1, y_7, y_8, y_5 + 1, y_{10}, y_{11}, y_{12}$

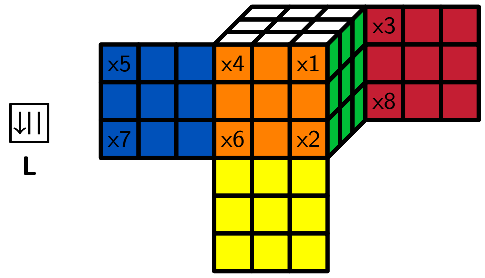
The method in which the corresponding x's and y's have been added by a number between mod 2 or mod 3 is defined below. Assume the below cube consists of a valid configuration , where there exists a move sequence which rotates the top front face corner cubie in its cubicle only.



Also assume the cubie fits in the cubicle correctly in the first cube , e.g. its in the correct position and orientation currently. Each time a move is applied and the cubie is rotated from it's correct orientation there will be an addition of 1 to preserve notational parity. For example let the cubie = x_2, when rotate corner sequence is applied the orange moves to the white facet clockwise 1 face, so the new $x_2^{'}$ becomes $x_2 + 1$. If the sequence was applied twice to go from the first to the third the new $x_2^{'}$ becomes $x_2 + 2$ as there are 2 clockwise movements. This is also similar for the edge pieces, but for only 2 changing faces there will only ever be a +1 or +0 (cycles back). As an example the front face turn is shown below.



Where in the solved position x1 has a white face up, blue face leftwards and orange face frontwards. After the permutation (1,2)(2,7)(7,6),(6,1), it now lives in the x2 cubicle where the blue



face is now upwards, the blue face is 2 clockwise positions from inital (white face), thus $x_1 + 2$. Same applies with other cubies affected giving after a F move $x_1 + 2$,$x_2 + 1$,$x_6 + 1$,$x_7 + 2$, complying with the table above. Applying an additional L move takes (5,4)(4,6)(6,7),(7,5) to their new positions. Giving $x_4 + 2$,$x_5 + 1$, $x_6 + 2$, x_7 +1 and the remaining corner cubies unchanged. Note these x's are taken as permutes from the start configuration not for the previous move.

So checking parity of all the corner cubies in this for the new configuration (τ', δ', x', y') under the move FL:

$$x' = x_1 + 2, x_2 + 1, x_3 + 0, x_4 + 2, x_5 + 1, x_6 + 2, x_7 + 1, x_8 + 0 \tag{20}$$

$$\Sigma x_i' = \Sigma x_i + 9 = \Sigma x_i (mod 3) \tag{21}$$

This will hold for any of the moves as shown in the parity table, and therefore any subsequent set of moves. In fact given the above information the conservation for the parity of corner and edge cubies can already be shown.

Theorem 1.2 (Parity of Edge and Corner Cubies)**.** *If* (τ, δ, x, y) *is a valid configuration then* $\Sigma x_i = 0 mod(3) \qquad \Sigma y_i = 0 mod(2)$

Because its shown any perumation of moves leaves the new configuration in the same orbit of the start configuration (1,1,0,0) and as $\Sigma x_i' = \Sigma x_i (mod 3)$ the theorem must hold.

Now the previous conditions have given the ability to characterise any valid configuration of a rubiks cube state. Let us define these conditions into the following theorem.

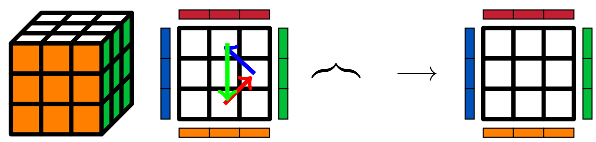
Theorem 1.3 (Fundamental Theorem of Cube Theory)**.** *Let* $\tau \in \mathbb{Z}_3^8$, $\delta \in \mathbb{Z}_2^{12}$, $x \in S_8$, $y \in S_{12}$ *make up a valid configuration* (τ, δ, x, y) *if and only if:*

1. $sgn(\tau) = sgn(\delta)$ *(Parity of Permutaions)*
2. $\Sigma x_i = 0 mod(3)$ *(Orientation of Corners)*
3. $\Sigma y_i = 0 mod(2)$ *(Orientation of Edges)*

Proof. In fact this foward direction has already been shown above in constructing what the definition of what makes a configuration above in the section. So the converse is left to show in this proof:

Assume there exits a configuration C = (τ, δ, x, y) which satisfies 1.,2.,3.
1. implies that the signs are equal meaning equal parity of permutations, meaning that they're either -1 (odd number of permutations) or 1 (even number of permutaions). In fact this proof just highlights what the surjectiveness of the homomorphism in 18, as this says any corner cubie can be

placed in the place of another one until the correct position is found). By following from the proof this means $\tau = 1$ in the configuration so to show parity, it must be now shown $\delta = 1$ for equality. Recalling the section on alternating and symmetric groups,3 the kernel is entirely A_n, which is generated by the set of 3-cycles in S_n in other words the odd transpositions (as the 1 cycle is trivial it is not considered in the explanation). Then for a map such as $\phi_{edge}|ker(\phi_{corner}) : ker(\phi_{corner}) \rightarrow S_{12}$ the image of it should just be the set of 3 cycles from the result of $\phi_{edge}(ker(\phi_{corner}))$ on an element from $ker(\phi_{corner})$ which is just the elements of the alternating group A_8. Note the function is restricted to $ker(\phi_{corner})$ to show if $\tau = 1$ the same applies with δ. So to show the parity is preserved is equivalent to showing the image (every 3 cycle) of this map is contained within A_{12}. Take a cycle permuting 3 edge pieces, $say(y_1, y_2, y_3)$ with the move M $= R^2UFB^{-1}R^2F^{-1}BUR^2$

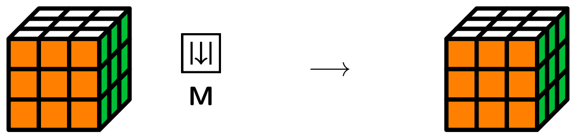


Where the red arrow is from x_1, blue from y_2 and green from y_3 complying with the usual labelling. With this cycle $(y_1, y_2)(y_2, y_3)$ there is an even number of even number of 2 cycle permutations (transpositions), thus will have $sgn(\delta) = 1$. Any of the remaining edge cubies for y_i (4 to 12) will be able to substitute for a_3 creating their respective cycles (y_1, y_2, y_4), (y_1, y_2, y_5) ..., (y_1, y_2, y_{12}). This is because there exists any move which will take y_i to y_3 without affecting the other two edge cubies. Call this move p, and substitute M for pMp^{-1} and we have the desired result. Note that this p will always be less than or equal to two of the basic moves just from the construction of the cube. This now gives $M \in (ker(\phi_{corner}))$ so $\phi_{edge}(M) = (y_1y_2y_3)$. Labelling the new move as $M^{'} = pMp^{-1}$, $M^{'}$ can be split into disjoint cycle decompisition $(y_iy_jy_k)$, aswell giving $M^{'} \in ker(\phi_{corner})$ so $\phi_{edge}(M^{'}) = (y_iy_jy_k)$ for any given i,j,k $\in \{1, ..., 12\}$ where i,j,k are distinct. So as for any edge cubie i,j,k , $(y_iy_jy_k)$ is in the image of the map, i.e.$\in im_{\phi_{edge|ker_{\phi_{corner}}}}$ thus we have reached condition above. So in correspondance with the previous proof of surjectiveness, again $sgn(\delta) = 1$. So now for configuration C , it must be such that $(1, \delta, x, y)$ δ =1, hence $\tau = \delta = 1$ giving us parity between permutations which concludes the proof of 1. For 2. and 3. it's just left to show that the orientation of the corners and edges always sums to 0 (mod3) and 0 (mod2) respectively. As in equation 11 a move was found to orient conrer cubies x_1 and x_2 without affecting any of the other corner cubies. This can be applied to any two such corner cubies (x_i x_j) rotating x_i clockwise one face and x_j anti-clockwise one face. So to first show 2. do an assumption by contradiction.

Proof. Suppose the Rubiks Cube is in a configuration where at least two corner cubies have the wrong orientation. By the above we can implement a move rotating x_i clockwise and x_j counter-clockwise. Therefore we the application of this move x amount of times one can ensure x_i has the correct orientation. So now there is one fewer cubie with an incorrect orietation, as doing those moves would not have affected any of the other corner cubies. Repeating this action across the rest of the unoriented cubies leaves in a state with at most one corner cubie which is incorrectly oriented. But as it's already been shown in equation 21 and the following explananation aswell as theorem 1.2 that $\Sigma_{x_i^{'}} = \Sigma_{x_i} = 0$ (mod 3) for a move on a valid configuration. Thus as there are 7 $x_i^{'}$ with correct orientation as $\Sigma_{x_i^{'}} = \Sigma_{x_i} = 0$ the only possible value for the last $x_i^{'}$ must also be 0, therefore proving 2. □

Now to show 3. implement a similar argument as such:

Proof. Like for the corners there exists a move which orients two corner cubies. This move is M = $LR^{-1}FLR^{-1}DLR^{-1}BLR^{-1}ULR^{-1}F^{-1}LR^{-1}D^{-1}LR^{-1}B^{-1}LR^{-1}U^{-1}$
For notation purposes include the slice move M_R for an easier description of the move $(M_RU)^4(M_RU^{-1})^4$. This slice move rotates the middle slice clockwise as such: So this move M like above has disjoint



cycle decompositon $(y_2\ y_2^{-1})(y_4\ y_4^{-1})$. So for any two distinct edge cubies y_i, y_j there exists a move sending y_2^{-1} to y_i and a move sending y_4^{-1} to y_j.Let this move be denoted as p, and thus similar to the above argument pMp^{-1} changes the orientation of the two edge cubies y_i, y_j without affecting any other edge piece. Another proof by contradiction supposing the cube is in a state where at least two edge pieces have an incorrect orientation identical to the argument for corner pieces leads to the conclusion of 3. □

As all 3 arguments to the theorem have been proved there is now a conclusive proof on what makes and what is a valid configuration of the rubiks cube. □

Using facts and theorems above, one can now define some additional terms and conditions for operations on the cube **??**.

Theorem 1.4 (Second Fundamental Theorem of Cube Theory)**.** *An operation on the cube is possible if and only if:*

1. *Total number of cycles* of even length is even (*for corner and edge cycles)*
2. *Number of corner cycles twisted right equals number of corner cycles twisted left (mod 3)*
3. *The number of reorienting edge cycles is even*

Proof. Let M be an operation which takes the cube from its starting coniguration to a new configuration (τ, δ, x, y). 1. By 1.3(1) $sgn(\tau) = sgn(\delta)$, meaning that the permutation by the corner and egde set is even. As a single cycle is an even permutation if it's length is odd the two are equivalent.

2. For a move M the corners are either moved clockwise, anti-clockwise or remain the same. Thus the cycle alters the summation by 1,2 or 0 (mod3) respectively.$\Sigma X_i = 0$ by 1.3(2)(3) (summation over all cubies corner and edge). Thus the number of right twist must be equal to the number of left twists.

3. An edge cycle is reorienting (in the wrong orientation) if and only if the summation of the edge orientations changes by an odd number (+1). But by 1.3(3) as $\Sigma\delta = 0$, if one edge cycle is reorienting then another must be. Therefore this is an even number of reorienting edge cycles.

An edge cycle is reorienting, if and only if the sum oftheorientation coordinates Yi of the cubies in the cycle changes by an odd number. □

Applications

Mathematically to talk about these groups further and to understand the illegal rubiks cube group it's sensible to introduce normal subgroups, semi-direct products and wreath products.

Normal Subgroup

A subgroup H of G is normal if $gH = Hg$,$\forall\ g \in\ G$ this is denoted as $H \trianglelefteq G$ and $H \triangleleft G$ if $G \neq H$

Semi-Direct Product

Given group G , subgroup H , normal subgroup $N \triangleleft G$ and that the following hold ,

- G is the product of subgroups N,H such that G = NH where $N \cap H = e$ the identity of G
- $\exists$ *a homomorphism* $G \rightarrow H$ that is the identity on H whose kernel is N.

  Then G *=*$N \rtimes H$ *is a semi-direct product*

Wreath Products

The Wreath products of two groups G and H below:.

- G^H is the set of all functions defined on H with values in G, where G^H is an automorphism
- Let X be a finite set , where G^X is acted upon by H, this is just functions taking non identity values on a finite set of points X = $\{x_1, x_2, ..x_p\}$. So is trivially smaller than the set of all functions.

$$G^X \wr H = G^X \triangleleft H \tag{22}$$

is then the wreath product of G by H. So is just the semi direct product of p copies of G by H(of Mathematics, 2014). To further show these two properties the following examples firstly on the additive group $(\mathbb{Z}, +)$.
Let G = $\mathbb{Z}_5$ = $\{0, 1, 2, 3, 4, 5\}$ and H = $\{1, 2, 3\}$, and so $\forall\ g \in\ G$, g + H = H + g as addition in the group of integers is commutative, H is thus a normal subgroup of G.
Secondly considering the group of symmetric transpositions S_n, in semi-direct product notation it can be written as $S_n = A_n \rtimes \langle(12)\rangle$. This can be shown simply by following the conditions in the definition. Disjoint subgroups $A_n \cap \langle(12)\rangle = e$. For $a \in S_n$ and $b \in A_n$, as b are the even cycle permuations sgn(b) = 1 thus $aba^{-1} \in A_n$ as with homomorphic properties there is $sgn(aba^{-1}) = sgn(a)sgn(b)sgn(a^{-1}) = s^2 = 1$. Therefore $A_n \triangleleft S_n$ hence $S_n = A_n \rtimes \langle(12)\rangle$ as above.
Finally take the two groups G = $\mathbb{Z}_2$, H = S_3 and the set X = $\{1, 2, 3\}$. The wreath product of G by H is $\mathbb{Z}_2^3 \wr S_3$ with elements $\{(0,0,0)\sigma, (1,0,0)\sigma, (0,1,0)\sigma, (0,0,1)\sigma, (1,1,0)\sigma, (1,0,1)\sigma, (1,1,1)\sigma\}$ where $\sigma \in S_3$.

Illegal Rubiks Cube Group

This can be implemented into the groups already discussed in the project. As mentioned in the context of the cube, the group of corners and edges can be described as to being in seperate orbits of the cube group orbit. Where the group of corner cubies , let's say C_{corner} can be constructed from

the identities already defined. C_{corner} will act on the set of corner cubies S_8 to obtain the corner positions and to obtain the corner orientations $\mathbb{Z}_3^8$ it just performs the direct product between the two. As by following the definition the group of orientations is a normal subgroup for the corners it follows that the group of corner cubies is $C_{corners} = (S_8 \wr \mathbb{Z}_3^8)$. A similar argument can be made to find that the group of edges is $C_{edges} = (S_{12} \wr \mathbb{Z}_2^{12})$.
The rubiks cube group is made up of these separate orbits (the centre cubies can be disregarded as they do not change position) so the rubiks cube group is made out of their direct product $G_{RC} = C_{edges} \times C_{corners}$ thus $G_{RC} = (S_{12} \wr \mathbb{Z}_2^{12}) \times (S_8 \wr \mathbb{Z}_3^8)$.
Earlier the rubiks cube group was said to have $4.3 * 10^{19}$, these are actually just the valid configurations used computationally when algorithms such as using pre-made solver algorithms. So what was defined as G_{RC} is actually the illegal rubiks cube group I $= G_{RC}$ as the total number of configurations here are:

$$I = |\mathbb{Z}_2^{12}||\mathbb{Z}_3^8||S_{12}||S_8| = 12! * 8! * 3^7 * 2^{11} \tag{23}$$

Which is around $86 * 10^{19}$ configurations, what makes a portion of these configurations illegal is the double counting that takes places. This is because only one third of the permutations have the correct corner cubie configuration. Also half of these permutations have the correct edge cubie configurations and half of these such configurations have the correct parity's for their signs. This sum is basically achieved for all cases where the fundamental theorem is not met, thus the amount of valid configurations is actually near a 12th of the toal number, $4.3 * 10^{19}$ configurations.

Legal Rubiks Cube Group

The official legal rubiks cube group G_{RC} is made up of the cube orientations and the cube permutations, where the intersection of these two is the identity, it follows by definition the rubiks cube group is the semi-direct product of the two.

$$G_{RC} = C_{orientations} \rtimes C_{permutations} \tag{24}$$

Where $C_{orientations}$ is just $\mathbb{Z}_2^{12} \times \mathbb{Z}_3^8$ as rotations groups under integers are abeliean and such following rotations determine the orientation of previous rotations. The $C_{permutations}$ is more complex , so firstly think of this permuation group on the cube as a product of two subgroups. The first being the group of permutations on the 8 corner cubies and the second being on the 12 edge cubies. Due to 1.3(1) there parity must be equal. So as the alternating group contains the even permutations the direct product $A_8 \times A_{12}$ contains half the permutations in the permutation group. As any combination of even permutations is even it follows from the definition that this identity is a normal subgroup of the permutation group. For the second half of this group consider the group $\{e, (ur, uf)(urf, urb)\}$. Now the permutation group can be expressed as a semidirect product between these two groups. The identity element e is chosen is both permutations are already even and (ur,uf)(urf,urb) is chosen if both permuations are odd (Beeler, 2016). Trivially the group $\{e, (ur, uf)(urf, urb)\}$ is isomorphic to $\mathbb{Z}_2$ with two elements one being the idenity e. Combining all these facts to the legal rubiks cube group gives us:

$$G_{RC} = C_{orientations} \rtimes C_{permutations} = (\mathbb{Z}_2^{12} \times \mathbb{Z}_3^8) \rtimes ((A_8 \times A_{12}) \rtimes \mathbb{Z}_2) \tag{25}$$

Possible Extension- CRAN cubing package

For the section relating gods number and efficiency to actually solving the cube, it will be implementing the CRAN cubing package (seesource). From this it will aid in the explanation about ways to solve the cube as well as showing how the mathematics behind the cube ties in with the programming. After installing the CRAN package and using the R command to further access specific packages within CRAN , the cube functions can finally be accessed. The notation of defining the cube in terms of an array is as such:

```
ellisp97@ellisp97: ~/Desktop/rubikscube/cubing
File  Edit  View  Search  Terminal  Help
> hello.cube
$cp
URF UFL ULB UBR DFR DLF DBL DRB
  1   2   3   4   5   6   7   8

$ep
FR FL BL BR UR UF UL UB DR DF DL DB
 1  2  3  4  5  6  7  8  9 10 11 12

$co
URF UFL ULB UBR DFR DLF DBL DRB
  0   0   0   0   0   0   0   0

$eo
FR FL BL BR UR UF UL UB DR DF DL DB
 0  0  0  0  0  0  0  0  0  0  0  0

$spor
U R F D L B
1 2 3 4 5 6
```

Entering the command hello.cube displays how the cube is stored with a similar notation to how it's labelled earlier in the project. Where **cp,ep,co,eo** stand for corner/edge permutation and orientation (where the cubicles live in the cubie).

Visual aid however is important as without this it becomes hard to see what's actually happening to the cube. This comes in 2d and 3d plots, with the commands plot(yourCube) and plot3d(yourCube) once defined(To enable the use of a 3D plot with RGL, start the RGL device in the R command)**??**.

Using a construction like so allows the user to input any configuration of a cube into the system, whether it's valid or invalid (in terms of solvability defined previously) so long as it obeys methods of input e.g.

```
>yourCube = cubieCube("UUUUUUUU RLLRRLLR BBFFFFBB DDDDDDDD LRRLLRRL FFBBBBFF")
>plot(yourCube, numbers = TRUE)
```

and then can plot in usual way (including numbers = TRUE displays the cubie positions on the cube like so)**??**. A function which is imperative for the input of these cubes is **is.solvable**. When on input of a cube previously defined will return a boolean value depending on whether the configuration entered is correct or not. Furthermore when a cube which has an invalid configuration has been input,

```
>is.solvable(yourCube, split = TRUE)

  parity   co      eo
  TRUE    TRUE   FALSE
```

is solvable will tell you the condition of how yourCube fails to be solvable. The items displayed above simply correspond to the funademntal theorem described earlier 1.3, where (1) describes if *parity* is met, (2) if *co* (corner orientation) is met and (3) if *eo* (edge orientation) is met.
The package allows for an input of a random cube using the function randCube(). Using a combination of these functions (including is.solvable), one can construct a comparison between the ratio of solvable cubes and that statement following the equation 23.

```
> sum(sapply(randCube(100, solvable = FALSE), is.solvable))
```

The graph below has been constructed using ten different samples of 1000 randomly generated rubiks cubes, by 23 the amount of cubes which should be solvable would be $1000/12 \approx 83.3$, which is the expected average below.

The actual results deviate around this mean value, with an actual mean value of 84.2 solvable cubes this is only 0.9 away from the expected means, and with more trials should converge closer to this value.
Using the following command a user can create their own move by separating the standard moves with spaces (or even separate lines). This has uses and can be used in conjunction with the is.solvable to see if a certain set of moves can take one configuration to the start configuration (solved).

```
> yourCube <- getMovesCube(scan(what = character()))

1: D2 F2 U F2 D R2 D B L' B R U L R U L2 F L' U
20:
19 moves read

> mv <- scan(what = character())
1: D2 F2 U F2 D R2 D B L' B R U L R U L2 F L' U
20:
19 moves read

> result <- move(yourCube, mv)
> is.solved(result)

[1] TRUE
```

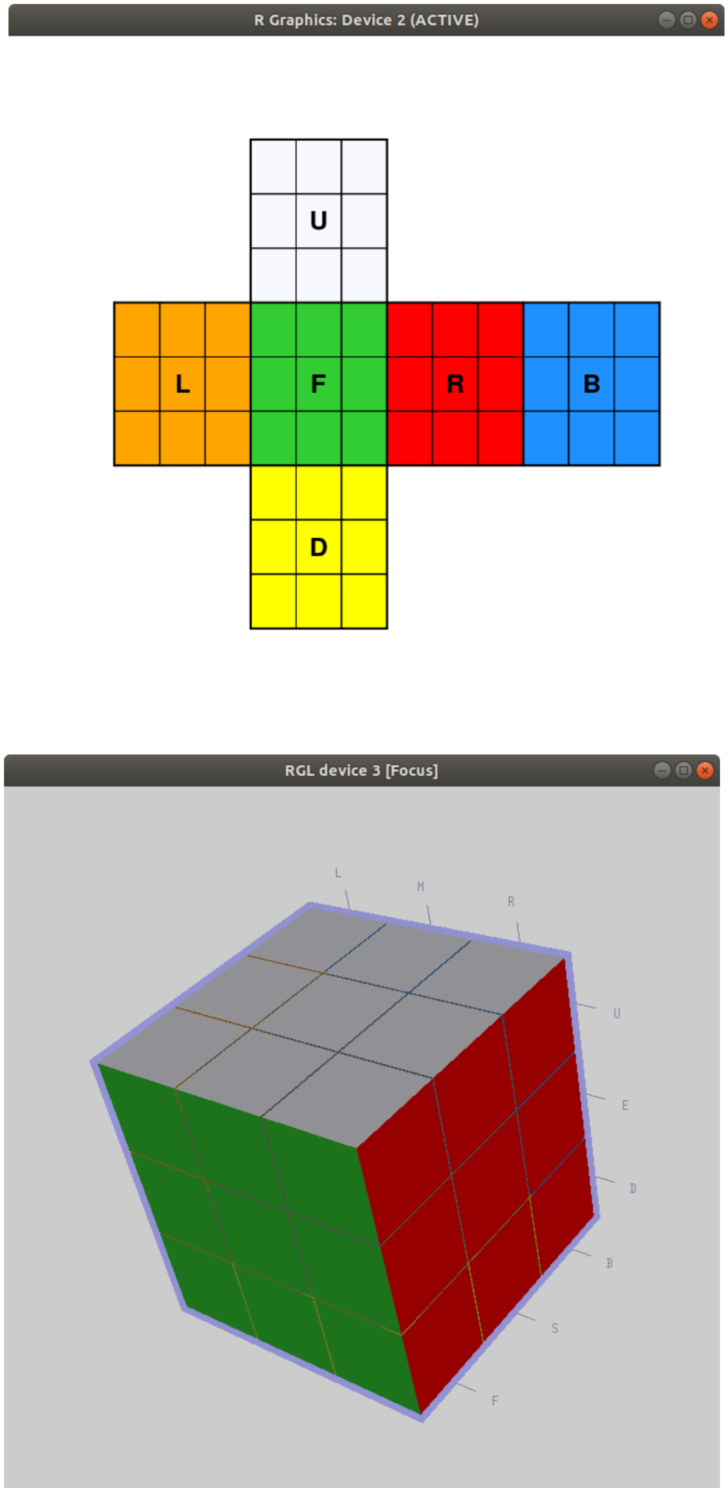


Figure 4: 2D and 3D plot of the solved cube in R

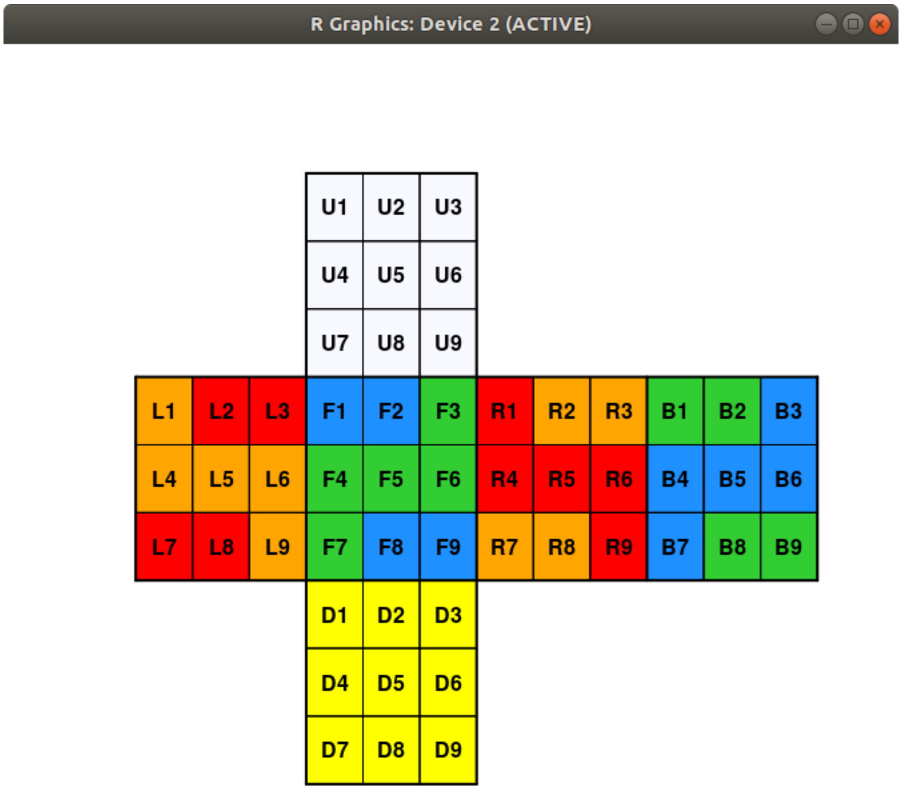


Figure 5: 2D cube with R labelling

INSERT STUFF RELATING ON ANIMATION E.G. WR MOVE ANIMATE(ACUBE,MV)

A question must be asked on the efficiency of this 'solver', how it works and how fast it can be done. The solver used in the is.solvable function is called the *Kociemba Solver*. This algorithm was created by Herbert Kociemba in 1992 which is an adapted version of the Thistlethwaite algorithm often referred to as the breakthrough for cube solvers(Thistlethwaite, 1981) . So to understand how efficiency works between solvers this algorithm must be considered first. Note all algorithms below use the Half-Turn Metric, this just means any one of the basic moves (quarter turn) counts as one move alongwith half turns such as R2 , U2 etc.

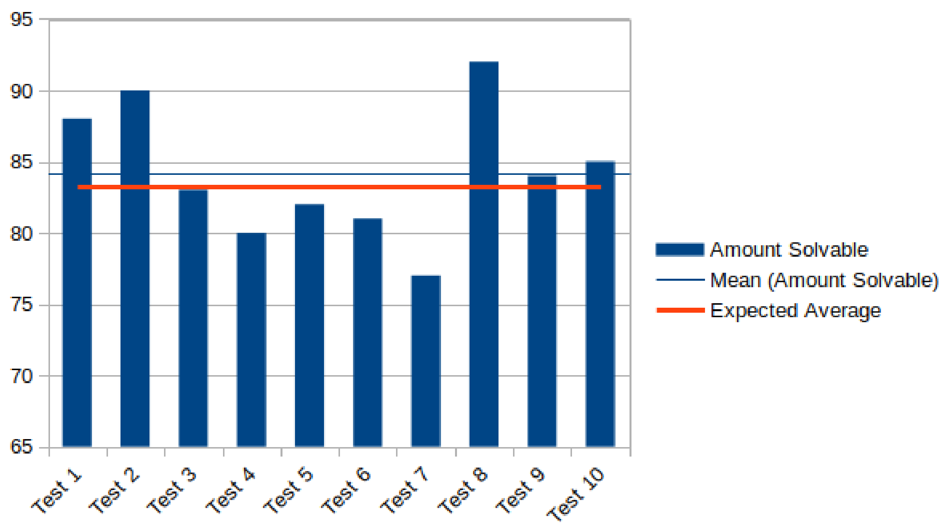


Figure 6: sample of random cubes against theorised result

Thistlethwaite's Algorithm

The idea of this was to create a *Divide and Conquer* style implementation, which essentially means to divide one big problem (solving the cube) into many different smaller subproblems. It uses the main concepts of group theory explained above, while utilising complex computer searches on very large lookup tables. The cube group $\mathcal{G}_{RC}$is split up into the following sequence of subgroups.

$$\mathcal{G}_{RC} = \begin{cases} G_0 = \langle L, R, F, B, U, D \rangle. \\ G_1 = \langle L, R, F, B, U^2, D^2 \rangle. \\ G_2 = \langle L, R, F^2, B^2, U^2, D^2 \rangle. \\ G_3 = \langle L^2, R^2, F^2, B^2, U^2, D^2 \rangle. \\ G_4 = 1. \end{cases} \tag{26}$$

This technique allows the cube to be solved in a maximum of 52 moves. Using the smaller subgroups he restricts the possible space of moves which can be made, this works by moving down the chain of G_0 to G_4 where the solved position has been found. Due to the nature of the groups G they can be described as a sequence of nested subgroups. Nested subgroups allow an evaluation of the worst case scenario in the amount of moves it takes to solve using a set method. Given nested subgroups G_1 to G_6 an example can be made which explains how these subgroups help in efficiency of solving, where the below process is called *The Restoration Sequence*. This moves through subgroups fixing more and more cubies each time until all cubies are in their cubicles with the correct orientation. Thus this restoration process is chracterised by the sequence of nested subgroups $G \supset G_2 \supset G_3 \supset G_4 \supset G_5 \supset G_6$ (Frey & Singmaster, 1982). However *The Restoration Sequence* works by fixing cubies at each stage, thus this can be applied to a general case. For any set of locations on the cube S, there is a subgroup G(S) which leaves the elements of S fixed. Therefore alongwith nested groups one can define a sequence of nested sets $S \supset S_2 \supset S_3 ... \supset S_n$ until the set where all cubies are fixed has been reached (S_n). This means for any sequence of solving the cube (not just *The Restoration Sequence*) thus after step i $G(S_i) \supset G_(S_{i+1})$ characterises that particular method. So a solving technique can be characterised by a sequence of nested subgroups but what remains to see is the converse is which is the motivation behind Thistlethwaite's Algorithm. For the algorithm in question each succesive group G_{i+1} up until G_4 is a subgroup of the predecessor as such $G_{RC} = G_0 = G_1 = G_2 = G_3 = G_4 = I$. At each stage of the algorithm a look-up table is used which shows a solution for each element in the coset space, this just means how to fix the selected cubie in the corresponding space. The table below shows how the subgroups reduce the configuration space at each stage in the Thistlewaite algorithm **??**.

Group	Configuration Number	Factor
$G_0 = \langle L, R, F, B, U, D \rangle$	$4.3 * 10^{19}$	2048
$G_1 = \langle L, R, F, B, U^2, D^2 \rangle$	$2.1 * 10^{16}$	1,082,565
$G_2 = \langle L, R, F^2, B^2, U^2, D^2 \rangle$	$1.95 * 10^{10}$	29,400
$G_3 = \langle L^2, R^2, F^2, B^2, U^2, D^2 \rangle$	$6.63 * 10^5$	663,552
$G_4 = I$	1	.

The factor corresponds to the action of the algorithm in that stage. In stage 1 the orientation of the edges are fixed which as explained previously is the group $\mathbb{Z}_2^{11}$, fixing this group is equivalent to factoring out the size of this group $2^{11} = 2048$. Similarly with stage 2 the algorithm fixes the orientations of the corners, as well as placing middle edges correctly. This corresponds to factoring out the group $\mathbb{Z}_3^7$, 3^7 and a combinatorial argument gives the result of fixing the edge cubies in the middle to give the result $3^7 * 12!(8! * 4!) = 1,082,655$. Using the lookup tables Thistlewaite was able to efficiently make use of a computer to reduce the permutations needed in a given subgroup. Thistlewait theorised these results in a table to describe the best and worst case scenarios below.

.	1	2	3	4	Total
Proven Algorithm	7	13	15	17	52
Improved Proof	7	13	15	15	50
Best Possible Case	7	10	13	15	45

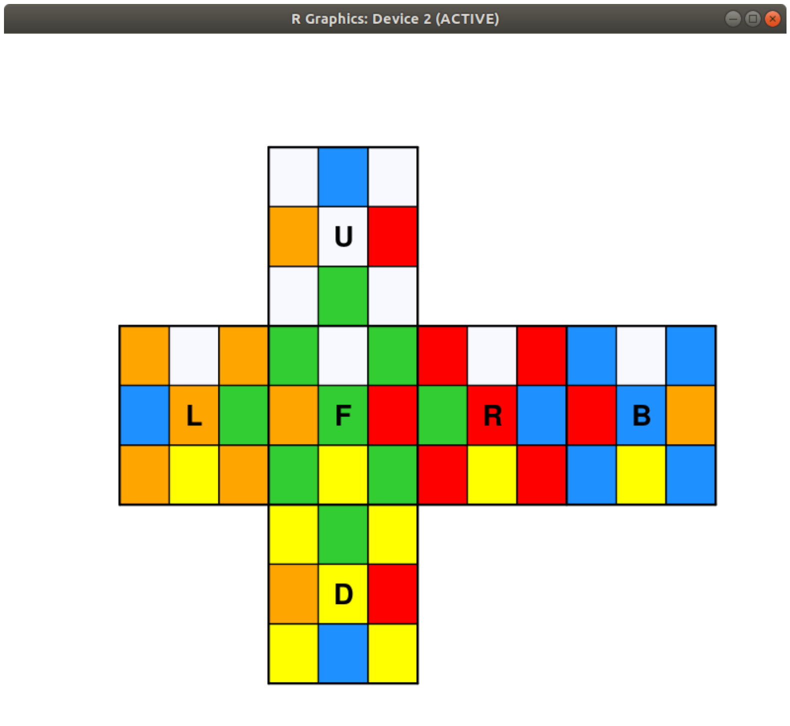
When published the full proof of the algorithm to reduce stage 3 to stage 4 (G_3 to G_4) was made in 17 moves by Thistlewaite. Whereas he predicted the improved case would reduce this by 2 moves, later this was correctly identified by his students. This started a somewhat trend in opening up the field of cube theory to reduce this bound. This section now relates to mentioning God's number.

God's Number

The number of moves it would take to solve the most difficult configuration of the cube is referred to as God's Number. If one had perfect knowledge on the rubiks cube, would be able to solve that cube in the most efficent way using the most efficient algorithm. This is essentially just an upper bound or a worst case scenario on the maximum least number of moves from a given configuration. From the publication of Thistlewait's proof of 52 in 1981, God's number has now been discovered from over 30 years of mathematic uncertainty to be 20**??**.

Previously a lower bound for God's number was thought to be 18, this is just a position which requires a minimum 18 moves to solve. However 15 years after the introduction of the cube, with the discovery of the *Superflip* position this lower bound was deemed to be false. This configuration is in the solved position expect that all edge pieces have incorrect orientations. A proof was made by Michael Reid in January 1995 proving that this configuration requires at least 20 moves, updating the lower bound from the previous. In fact more and more of these 20 distance configurations are being found recently where in March 2014 there were over 93 million known 20 distance positions with a total 490 million positions thought to be possible. These positions are found by analysing related cosets to positions such as the *Superflip***??**.

What was left now was to show that this lower bound complies for all configurations , or that God's number is 20**??**. However to prove this one needed to implement a proof by exhaustion, which as previously mentioned for that time and with a sample size of $4.3 * 10^{19}$ this was just computationally inefficient. This is why factoring methods such as the one in Thistlewaite's algorithm were constructed, therefore reducing the number of cases you had to check. Also a efficient way of checking if these cases could be solved was needed, motivating the explosion of these algorithms in the 30 year span. These two optimisations were made to the full extent of the knowledge behind them, supercomputers at Google Headquarters ran this subset size of the 2.2 billion positions (reduced from the $4.3 * 10^{19}$) computing the result in a few weeks. Concluding that any possible configuration can be solved in a minimum of 20 moves.



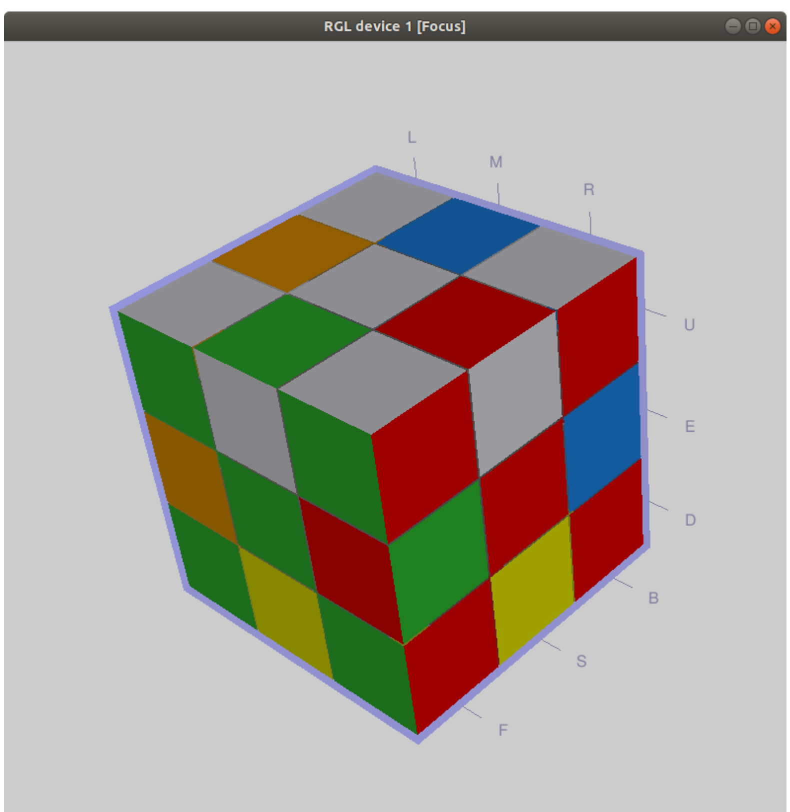


Figure 7: Superflip configuration in 2d and 3d

Computer vs God

With optimisations and a seemingly unlimited computing power at google this still takes an infeasible amount of time when trying to solve the cube at a fast rate. So one would need to compare their efficiency with a 'God like power' who knows where every configuration can be returned to solve in a maximum number of 20 moves. As said before the algorithm used in the CRAN package uses the Kociemba Solver. This algorithm is a two phase approach. It first gets the cube in $G_1 = \langle U, D, R2, L2, F2, B2 \rangle$ from G_0 and the second phase from G_1 to $G_2 = \mathrm{I}$. Combining both phases of Thistlewaite's algorithm to compute an efficient alternative for sub-optimal solutions on Gods algorithm. Phase 1 has a maximum length of 12, while Phase 2 has a length of 16 (previously 18 but last 2 can be avoided), thus the whole bound of Kociemba's Algorithm is 28. This stood for a number of years as the global optimum before the proof for God's number as 20 in 2010(?, ?). The algorithm uses backtracking in both phases to lower this bound to attempt to retrieve an optimal solution. It does this by implementing the following algorithms:

```
function KociembaAlg(position)
  loop depth from 0 to maxLength:
    Phase1Search(position,depth)
  end loop
end function
```

```
function Phase1Search(position,depth)
  if depth = 0 then:
    if last move was a quarter turn of [R,L,F,B] and subgoal reached then:
      Phase2Search(position)
    end if
  else if depth > 0 then:
    if prune1[position] <= depth then:
      loop through all moves M:
        Phase1Search(M applied to position, depth -1)
      end loop
    end if
  end if
end function
```

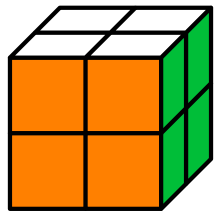
```
function Phase2Search(position,depth)
  loop depth from 0 to maxLength - currentDepth
    if depth = 0 then:
      if cube solved:
        "solution found"
        maxLength = currentDepth - 1
      end if
    else if depth > 0 then:
      if prune2[position] <= depth then:
        loop through all moves M:
          Phase2Search(M applied to position, depth -1)
        end loop
      end if
    end if
  end loop
end function
```

This format of the Kociemba Algorithm is also in pseudo code to make ease of understanding for the reader, thus line by line it just be fairly trivial to understand what it's doing. Although it also

uses new aspects uncovered previously , for instance in Phase1Search it looks in the pruning table and makes a comparison to the depth. The pruning table will provide a lower bound for a given position on the number of moves needed to solve. This may seem more powerful at first, as this implies full efficiency, hence Gods number. However this is not the case as a pruning table does not change for each configuration of the cube, and instead represents a set of similar configurations as an index. So God's algorithm is essentially identical however the full position is stored, as opposed to part of the position to index the pruning tables like in Kociemba.
A comparison could be made to test the efficiency between the two by graphing out some results. However as computing power of google's computers is hard to come by, also alongwith a sensible time constraint this makes a comparison between the two difficult. From Tomas Rokiki paper the optimised version of Kociemba's Algorithm can calculate 3900 20-length (max-length) solutions. The time to solve all positions using this method is around 3.7 million CPU years **??**. However slow it is still much faster than several billion years for a perfect solution for each configuration. The below table shows this efficency argument, additionally combining the rate for solving a coset at a time.

.	Optimal Solution	Near Optimal
Individual Position	2	3900
Cosets of the Cube Group	$2 * 10^6$	10^9

Pocket Cube



Therehave been alot of variants made since the production of the original 3x3x3 rubik cube in 1974. These include 4 by 4, 5 by 5's, tetrahedron and triangular based cubes and so on. The majority of these cubes aim to highten the difficulty of the original puzzle, where more complex moves and algorithms would be needed. However describing the original 3 by 3 cube already implements a high brute force cost on computational efficiency. Here the 2by2 cube is introduced to show the comparison between the two, and how the group size grows exponentially. The following will explore analysing Daniel Bump and Daniel Auerbachs paper on the minature rubik cube and it's two generator group.

Again the 2by2 cube has the exact same set of basic moves M = $\{L, R, U, D, F, B\}$. However this time the group structure will differ alot because there are no such edge cubies. It is made up of just 8 cubes which can act as corner cubies using the group already defined ($\mathbb{Z}_3^8$). The proof of the pocket cube being a group is identical to that of the 3 by 3 cube. For the basic moves of the pocket cube, similar to the 3 by 3, there is disjoint cycle notation which describe the action of performing the respective move on the cube.

$$
\begin{aligned}
U &= (5, 13, 21, 17)(6, 14, 22, 18)(1, 2, 4, 3) \\
R &= (6, 22, 3, 10)(8, 4, 21, 12)(17, 18, 20, 19) \\
F &= (5, 6, 7, 8)(3, 17, 10, 16)(14, 4, 19, 9) \\
D &= (7, 19, 23, 15)(8, 20, 24, 16)(9, 10, 11, 12) \\
L &= (13, 14, 16, 15)(5, 9, 24, 1)(7, 11, 22, 3) \\
B &= (21, 22, 24, 23)(2, 13, 11, 20)(1, 15, 12, 18)
\end{aligned}
$$

These cycles are based on the following labelling of the cube, similar to the 3 by 3 but there are now only 4 cubies to label not 9.

However for this cube though the minimal generating subgroup can be generated by just two moves as opposed to 5. So the two generating subgroup G is denoted as $G = \langle R, U \rangle$. Also the non valid configurations must be considered, so one will need to consider the valid group of the pocket cube. When applying a move to the pocket cube, one can see similar to that of the ($\mathbb{Z}_3^7$) subgroup thus the last cubie will be determined automatically (as in the 3 by 3). While in this version of the cube any the orientation group can be defined as the subgroup of G, say H. This which can also be thought of as the basic move operations that dont't change the position of any other 6 cubies **??**. As the orientation subgroup H only describes the change of orientations of cubies, it does not depend on order operations are performed. Thus the subgroup is abelian, but trivially by inspecting moves the pocket cubes group must not be. It follows from the definition that K is a normal subgroup

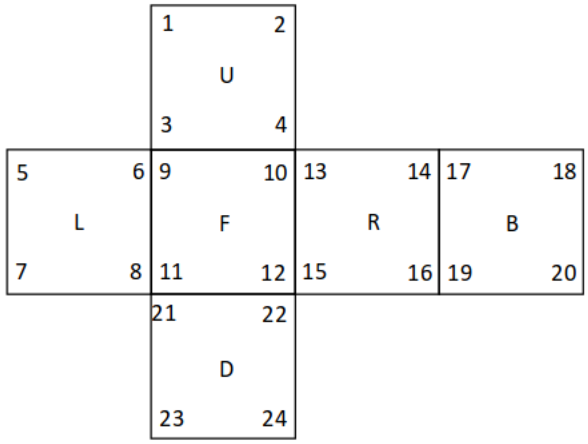
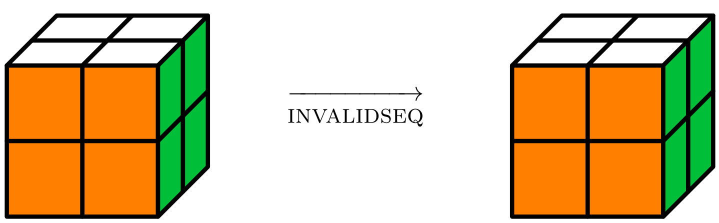


Figure 8: Labelling of the Pocket Cube

of G as gH = Hg,or equivalently for $g \in G$ an operation, $gHg^{-1} = H$. To get a bound on the pocket cube group, one must first get a bound on the orientation subgroup to which it's defined by. There are 3 possible orientations for each of the 6 cubies, and following from the fundamental theorem for cube theory (note this still applies for the pocket cube as all individual cubies here are essentially acting as corner cubies) 1.3(2) these orientations must equal 0 (mod 3). Again note that 6 cubies are considered as the original group is a 2-generated group by the two moves R,U which by inspection only effect 6 cubies on the poecket cube. Thus there is a free choice for 5 of the 6 corners which are movable, while the remaining one is determined automically by construction. This means that the bound for the subgroup H must be $|H| \leq 3^5$. To show $|H| \leq 3^5$ there just needs an example given to show there is a move which generates the group from the elements of the two generator such that it equals 0 (mod 3). Such a move is:

$$RUR^{-1}URU^2R^{-1}U^2 \in H \tag{27}$$



Thus the bound on the subgroup is $K = 3^5$.
H described these orientations of the 6 cubies in their cubicles. There must be a group such that it describes permutations which affect the locations of the 6 pieces. It's useful here to introduce the notion of a projective general linear group.

Projective General Linear Group (PGL)

The projective general linear group denoted PGL(n,k) of order n over k where $n \in \mathbb{N}$, k a field is defined such that:

- It's the group of automorphisms of projective space of dimension n-1, that arise from linear automorphisms of the vector space of dimension n
- It's the quotient of GL(n,k) by it's center C, the scalar matrices of the identity

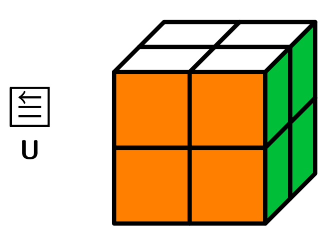
The elements can be projected onto $\mathbb{P}^1(\mathbb{F}_5)$ this just describes labelling of points of the line $\mathbb{P}$ over a field with 5 elements. One can individually reference cubie locations here, considering the 3 by 3 rubiks cube there would be far too many elements to talk about projective line mapping. The quotient group G/H fits this definition of the group of permutations on $\mathbb{P}^1(\mathbb{F}_5)$ which exclude orientation. Using this definition one can accurately reference the group e.g. instead of calling it a subgroup of S_6 and deal with non valid permuatations. So from the definition above, the group of permutations is PGL(2,$\mathbb{F}_5$) acting on elements $\mathbb{P}^1(\mathbb{F}_5)$ by fractional linear transformations. From there one can tell the generalised linear group GL(2,$\mathbb{F}_5$) below which describes all the invertible matrices.

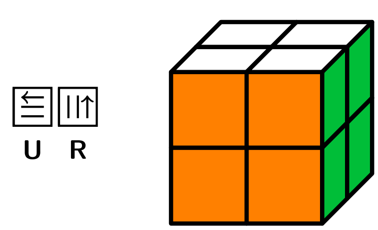
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : x \mapsto \frac{ax+b}{cx+d}, \qquad x \in \mathbb{P}^1(\mathbb{F}_5) \tag{28}$$

The projective line here is actually the projective extended real line due to working in real numbers. It extends the number line to which the elements are mapped onto by a infinity point ∞. This infinite point describes the point to which both ends of the real number line (sequence) meet, the limit of the sequence. One can also think about any such parallel lines on a visual plane that despite no intersection ∞ can be used to categorise such lines. Taking this information into the equation above, $x \in \mathbb{P}^1(\mathbb{F}_5)$ simply means $x \in \mathbb{F}_5 \cup \infty$. Thus for $x = \infty$, $\frac{ax+b}{cx+d} = \frac{a}{c}$ and for $x = 0$, $\frac{ax+b}{cx+d} = \infty$. As from the definition the center C of the generalised group GL(2,$\mathbb{F}_5$) consists of the scalar matrices on the identity, which is just the kernel of the generlised group. An action of the group G is actually just an action on GL(2,$\mathbb{F}_5$)/C = PGL(2,$\mathbb{F}_5$), giving the desired action on the group of cubie locations. This can be shown formally below:

Proposition 2 (Singmaster)**.** *The permutation group acting on* $\mathbb{P}^1(\mathbb{F}_5)$ *is* $G/H = PGL(2,\mathbb{F}_5)$

Proof. To see this one needs to check that the generators of the quotient G/H (and thus the moves) are contained in PGL(2,$\mathbb{F}_5$). As the group is two generated from U,R , one just needs to show U,UR (or a combination of such) $\in$ PGL(2,$\mathbb{F}_5$). With the labelling in **??**, U generates the cycle (10,9,1,∞) and UR generates (10,9,1,16,12). Now look at their fractional transformations by comparing the signs of the respective cubies.





$$U = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \in PGL(2, \mathbb{F}_5) \qquad UR = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in PGL(2, \mathbb{F}_5) \tag{29}$$

In particular this linear transform for both of these can be examined. For U there is

$$x \mapsto \frac{1}{2x+1}$$

, and for UR there is

$$x \mapsto \frac{x+1}{1}$$

such that using these transforms one can show that these moves are in fact $\in$ PGL(2,$\mathbb{F}_5$). This is done by checking each element in the field $\mathbb{F}_5$ (congruent mod 5) under the remains in the permutation group.

$$\begin{aligned}
0 &\mapsto \frac{1}{2*0+1} = \frac{1}{1} = 1 \\
1 &\mapsto \frac{1}{2*1+1} = \frac{1}{3} = 3^{-1} \equiv 2 \\
2 &\mapsto \frac{1}{2*2+1} = \frac{1}{5} \equiv \frac{1}{0} = \infty \\
3 &\mapsto \frac{1}{2*3+1} = \frac{1}{7} \equiv \frac{1}{2} = 2^{-1} = 3 \\
4 &\mapsto \frac{1}{2*4+1} = \frac{1}{9} \equiv \frac{1}{4} = 4
\end{aligned}
\qquad
\begin{aligned}
0 &\mapsto \frac{0+1}{1} = \frac{1}{1} = 1 \\
1 &\mapsto \frac{1+1}{1} = \frac{2}{1} = 2 \\
2 &\mapsto \frac{2+1}{1} = \frac{3}{1} = 3 \\
3 &\mapsto \frac{3+1}{1} = \frac{4}{1} = 4 \\
4 &\mapsto \frac{4+1}{1} = \frac{5}{1} \equiv \frac{0}{1} = 0
\end{aligned}$$

Where on the left there's the transformation for U, and the UR for the right. Note $\infty \mapsto 0$ in U and $\infty \mapsto \infty$ in UR. Thus all elements have been checked and $G/H \subset PGL(2, \mathbb{F}_5)$. Combined with the fact these two elements generate $PGL(2, \mathbb{F}_5)$ gives the desired result. □

The moves which generate the group U,R only change the positions of the 6 cubies, but the last one is determined. Following this it can be shown there is an isomorphism between this and the symmetric group S_5 (Bandelow, 2012).

Proposition 3. *$G/K \cong S_5$ $|G/K|$ = 5!*

Proof. To show G/K or $PGL(2, \mathbb{F}_5)$ is isomorphic to the symmetric group of 5 elements one needs to consider the 5 Sylow groups. For a p prime , the highest power p^k which divides the order of the group means there exists a subgroup of G, of order p^k called a p-Sylow subgroup. Taking the group G as the symmetric group S_5 the 5-Sylow subgroup can be labelled in the following way:

$$\begin{aligned}
\infty &= \langle(1,2,3,4,5)\rangle & 3 &= \langle(1,2,5,4,3)\rangle \\
1 &= \langle(1,2,3,5,4)\rangle & 4 &= \langle(1,2,5,3,4)\rangle \\
2 &= \langle(1,2,4,5,3)\rangle & 5 &= \langle(1,2,4,3,5)\rangle
\end{aligned}$$

Conjugating these above subgroups will describe how S_5 acts on the permutation group $\mathbb{P}^1(\mathbb{F}_5)$. The claim to show the isomorphism is that the group of permuations as a result of conjugacy is in fact $PGL(2, \mathbb{F}_5)$. The action of conjugacy is described below:

Conjugate Subgroup

For a subgroup H with elements h_i and a fixed element x $\in$ G,where x $\notin$ H. The transformation xh_ix^{-1} for (i=1,2,...) generates the conjugate subgroup.

Where in this case taking H as a sylow subgroup $\in \{\infty, 0, 1, 2, 3, 4\}$ and x a fixed element of an alternate cycle the operation can proceed. To see the group obtained is $PGL(2, \mathbb{F}_5)$ one must check the generators of S_5 induces linear fractional transformation as in the proof of the previous proposition 2. Taking conjugation on the subgroup $\infty = \langle(12345)\rangle$ gives the following:

$$(12345)(12345)(12345)^{-1} = (12345) : \infty \mapsto \infty$$
$$(12345)(12354)(12345)^{-1} = (15234) : 0 \mapsto 1$$
$$(12345)(12453)(12345)^{-1} = (14235) : 1 \mapsto 2$$
$$(12345)(12543)(12345)^{-1} = (12534) : 2 \mapsto 3$$
$$(12345)(12534)(12345)^{-1} = (14523) : 3 \mapsto 4$$
$$(12345)(12435)(12345)^{-1} = (13425) : 4 \mapsto 0$$

Where the fixed element x has been chosen from the group S_5's Sylow subgroups. This conjugation creates the transformation which fixes ∞ and permutes the cycle (01234). Thus this linear transformation is just $x \mapsto x + 1$ (where $x \neq \infty$). This transformation corresponds to the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{F}_5)$. Secondly consider another conjugation which produces a similar result namely the conjugation on the transposition (45). This is shown similarly below:

$$(45)(12345)(45) = (12354) : \infty \mapsto 0$$
$$(45)(12354)(45) = (12345) : 0 \mapsto \infty$$
$$(45)(12453)(45) = (12543) : 1 \mapsto 2$$
$$(45)(12543)(45) = (12453) : 2 \mapsto 1$$
$$(45)(12534)(45) = (12435) : 3 \mapsto 4$$
$$(45)(12435)(45) = (12534) : 4 \mapsto 3$$

$\square$

As a transpotion, cycle of 2 elements, is conjugated by here it's inverse is identical $(45)^{-1} = (45)$. To see it's linear transform inspect the maps on the field (mod 5), similarly as in the proof of 2 this can be shown like so:

$$0 \mapsto \frac{2}{0} = \infty \qquad 1 \mapsto \frac{2}{1} = \frac{2}{2} = 2$$
$$2 \mapsto \frac{2}{2} = 1 \qquad 3 \mapsto \frac{2}{3} = 2 * 3^{-1} \equiv 2 * 2 = 4$$
$$4 \mapsto \frac{2}{4} = 2 * 4^{-1} \equiv 2 * 4 \equiv 3$$

By inspection the map is $x \mapsto \frac{2}{x}$ which is verified above, permuting like so $0 \leftrightarrow \infty, 1 \leftrightarrow 2, 3 \leftrightarrow 4$.

Where earlier saw the 3 by 3 cube had a subgroup for the orientation group , one which changed corner pieces, it seems isomorphic ORNOT

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