The Mathematics of the Rubik's Cube

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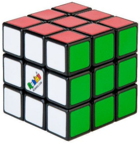
Composition of the Cube



- 27 cubes (cubies)
- 8 corner cubies
- 12 edge cubies
- 6 centre cubies
- One mysterious middle cubie (non visible)

Moves on the Cube

Figure 1: **Left (L), Up (U) and Right Moves (R)**

Figure 2: **Face (F), Back (B) and Down (D) Moves**

All moves M L,R,U,F,B,D are permuted by looking at each face and rotating the specific area 90 degress clockwise.

Inverse moves are defined as Lp,Rp,Up which is simply the move in the opposite direction.
Also moves L2,R2 denote the move applied twice.

The Rubik's Cube is a Group

To prove a group (G,*) is a group under it's operation we must follow the group definition :

- G is closed under * - So any move applied to the group will still leave the cube in a valid state.
- G has an identity e Do nothing
- G has an inverse R Rp
- G is Associative R (L D) = (R L) D

But what is this group ? What is the size ?

Size of the group

Considering the composition:

$$\begin{aligned} \textit{Each of } & \textit{the } 8 \textit{ corner cubies have } 3 \textit{ possible faces} \\ & \textit{the } 12 \textit{ edge cubies have } 2 \textit{ possible faces} \\ & \textit{the } 6 \textit{ center cubies have } 1 \textit{ possible face} \end{aligned}$$

$12! \cdot 8! \cdot 3^8 \cdot 2^{12} \cong 512$ quintillion different configurations
However not all these combinations are possible! This is called the Illegal Cube Group.

The Legal Rubik's Cube Group I

- To classify the legal group we must make it distinct from all non possible configurations.
- A non possible configuration is one which cant be reached from a set of moves.

The Legal Rubik's Cube Group II

Definition: Orbit

If G acts on a set A, then the orbit of $a \in A$ (under this action) is the set $\{a \cdot g : g \in G\}$ The stabiliser of an orbit is $\{g \in G | g \cdot x = x\}$ the set of all elements that fix x.

- The legal configurations are configurations which can be found from other possible moves. Moreover, when G acts on the set of cubies the configuration still exists within the orbit set.
- The solved configuration of the cube is fixed to allow reference to these such moves.

A Valid Configuration

Going back to invalid configurations, we construct a formal definition on what makes these invalid/valid.

$$(\tau, \delta, x, y)$$

The x,y values represent the number corresponding to the orientation. While τ and δ represent the number of transpositions on $C_{corners}$ and C_{edges}.
We know the group must be made out of some combination of the orientations and permutations. So we can informally classify this group as:

$$G_{RC} = C_{Orientations} \ x \ C_{Permutations}$$

Orientation Subgroups

- Let C be the set of cubies, let G act on C.
- Consider the two distinct orbits $C_{corners}$ and C_{edges}.
- Firstly consider the subgroup of orientations.

Corner Cubies

- Subgroup - $\mathbb{Z}_3$
- $\{0, 1, 2\}$ - 3 different faces
- Total $= \prod_{i=1}^{7} \mathbb{Z}_3$

Edge Cubies

- Subgroup - $\mathbb{Z}_2$
- $\{0, 1\}$ - 2 different faces
- Total $= \prod_{i=1}^{11} \mathbb{Z}_2$

The total size of this orientation is simply the direct product between them. $\mathbb{Z}_3^8$ X $\mathbb{Z}_2^{12}$, $\quad (3^7 \cdot 2^{11})$.

The Fundamental Theorem of Cube Theory

Theorem: The Fundamental Theorem of Cube Theory

Let $\tau \in \mathbb{Z}_3^8$, $\delta \in \mathbb{Z}_2^{12}$, x $\in S_8$, y $\in S_{12}$ make up a valid configuration (τ, δ, x, y) if and only if:

1. $sgn(\tau) = sgn(\delta)$ (Parity of Permutations)
2. $\Sigma x_i = 0 mod(3)$ (Orientation of Corners)
3. $\Sigma y_i = 0 mod(2)$ (Orientation of Edges)

Next consider the group of permutations, now this is a bit more involved. To do this introduce the idea of a semi-direct product.

Definition: Semi-Direct Product

Given group G, subgroup H, normal subgroup N/G and that the following hold.

- G is the product of subgroups N, H such that G = NH where $N \cap H = e$ the identity of G
- There exists a homomorphism $G \to H$, that is the identity on H whose kernel is N. Then $G = N \rtimes H$ is a semi-direct product

Permutation subgroup

Again consider the two distinct orbits $C_{corners}$ and C_{edges}.

- By the fundamental theorem of cube theory parity of signs must be equal.
- By definition an alternating group only contains the even permutations.
- Thus the direct product contains half A_8 x A_{12} of the total permutations.

However this doesn't count any cases where both permutations are odd. It follows that we can take the semi-product of this with $\mathbb{Z}_2$ to obtain the whole subgroup.

The Legal Rubik's Cube Group III

Thus as the subgroup of permutations is found the legal group can now be defined:

$$G_{RC} = C_{orientations} \rtimes C_{permutaions} = (\mathbb{Z}_2^{12} x \mathbb{Z}_3^8 \rtimes ((A_8 x A_{12}) \rtimes \mathbb{Z}_2)$$

Where G_{RC} has a corresponding size:

$$\frac{1}{2} \cdot 8! \cdot 3^7 \cdot 12! \cdot 2^{11} \cong 43 \text{ quintillion}$$

One twelfth of the original size!

Perfect Efficiency?

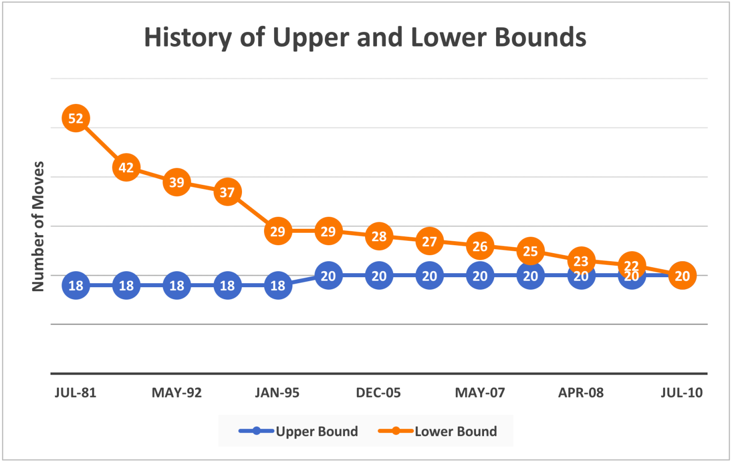
43 quintillion is still a huge amount of configurations. To achieve perfect efficiency at solving one would need to know the best way to solve the cube from any one of these 43 quintillion.

Definition: God's Algorithm

An algorithm which produces the most optimum solution in the fewest moves. An omniscient being (God) would know all of these optimal steps from any configuration.

Is this algorithm possible?

History of Efficiency



Technology and the Cube

- Optimisations were found by comparing related cosests.
- This reduced the 43 quintillion valid configurations to 2.2 billion.
- It took google over 2 weeks to verify this bound of 20 was correct.

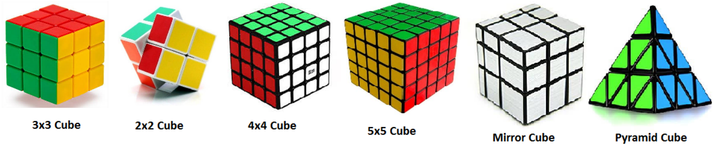
This algorithm though being stepwise efficient is no help when trying to solve the cube in a fast manner.
Computationally this in infeasible to implement, as storing all these different configurations becomes an issue.
There are many solvers out there where time/efficiency trade offs occur, to allow for the quickest near optimal case possible.

Many Cubes

The cube is just as popular 30 years from creation

Speedcube competitions still occur regularly in the world

Motivated the creation of many different variants.



Generators on the groups

2 by 2

- Legal Size - 3,674,160
- Minimal Generators - R,U
- God's Number - 20
- World Record = 0.59 seconds
- 2 generated

3 by 3

- Legal Size - 43,252,003,274,489,856,000
- Minimal Generators - R,L,F,B,U
- God's Number - 11
- World Record = 4.59 seconds
- 2 generated

$$D = R^2L^2U^{-1}B^2F^2U^{-1}B^2R^2B^2F^2L^2F^2U^{-1}$$

$$D^{-1} = R^2L^2UF^2B^2UF^2R^2F^2B^2U^2L^2U^2L^2R^2U^2$$
$$R^2U^2R^2F^2U^{-1}R^2B^2R^2L^2F^2L^2UB^2F^2U$$