Factoring Polynomials

Introduction

A polynomial with integer coefficients is said to be **prime** if it has no monomial or polynomial factors with integer coefficients other than itself and one. Furthermore, a polynomial with integer coefficients is said to be in **completely factored form** when each of its polynomial factors is prime.

Removing a Common Monomial Factor

$$ax + ay + az = a(x + y + z)$$

- Note that the factor $(x + y + z)$ is obtained by dividing the given polynomial by the common monomial factor a.

Difference of Two Squares

$$x^2 - y^2 = (x + y)(x - y)$$

Factoring Trinomials

From special product 1 we have the formula

$$x^2 + (a + b)x + ab = (x + a)(x + b)$$

From special product 2 we have the formula

$$x^2 + 2xy + y^2 = (x + y)^2$$

- Note that the trinomial is called a **perfect-square trinomial**.

From special product 4 we have the formula

$$acx^2 + (ad + bc)xy + bdy^2 = (ax + by)(cx + dy)$$

Sum and Difference of Two Cubes

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Factoring by Grouping

Illustration 1

To factor the polynomial

$$3x^2 + 7x - 6xy - 14y$$

we can group the first and third terms and second and fourth terms to get

$$\begin{aligned}(3x^2 - 6xy) + (7x - 14y) &= 3x(x - 2y) + 7(x - 2y) \\ &= (x - 2y)(3x + 7)\end{aligned}$$

Illustration 2

If we have the polynomial $x^2 + 2xy + y^2 - 1$, we can group the first three terms to form a perfect-square trinomial, which we factor as the square of a binomial. Then we have the difference of two squares. The computation follows:

$$\begin{aligned}x^2 + 2xy + y^2 - 1 &= (x^2 + 2xy + y^2) - 1 \\ &= (x + y)^2 - 1^2 \\ &= [(x + y) + 1][(x + y) - 1] \\ &= (x + y + 1)(x + y - 1)\end{aligned}$$

If we have the polynomial $x^2 - y^2 + x - y$, we can group the first two terms to form the difference of two squares, which we factor; then we have a common binomial factor of $x - y$ and we get

$$\begin{aligned}x^2 - y^2 + x - y &= (x^2 - y^2) + (x - y) \\ &= (x + y)(x - y) + (x - y) \\ &= (x - y)[(x + y) + 1] \\ &= (x - y)(x + y + 1)\end{aligned}$$

Illustration 3

In the trinomial $2st^4 - 8st^2 - 90s$ there is a common monomial factor of $2s$. Hence we can write the trinomial as $2s(t^4 - 4t^2 - 45)$. The new trinomial can be factored and written as the product of two binomials, one of which is the difference of two squares. The computation follows:

$$\begin{aligned}2st^4 - 8st^2 - 90s &= 2s(t^4 - 4t^2 - 45) \\ &= 2s(t^2 + 5)(t^2 - 9) \\ &= 2s(t^2 + 5)(t + 3)(t - 3)\end{aligned}$$

Practice Items

Factor the polynomials. In items 3, 6, 15 and 34, n is a positive integer.

1. $8x^2 + 4x$

2. $a^5 - 3a^4 + a^3$

3. $a^{2n+1} + a^{n+2} + a^{n+1}$

4. $16 - y^2$

5. $36x^2 - 81y^2$

6. $b^{2n} - c^{8n}$

7. $x^2 - 9x + 18$

8. $a^2 + 4ab - 21b^2$

9. $x^2 + 5x - 24$

10. $y^2 - 10y + 25$

11. $9x^2 - 30xy + 25y^2$

12. $25y^6 - 10y^3 + 1$

13. $10y^2 - 11y - 6$

14. $32a^2 + 12ab - 9b^2$

15. $x^{6n} - 14x^{3n} + 49$

16. $27 - x^3$

17. $t^3 + 8$

18. $a^6b^3 - 27c^3$

19. $x^3 + 3x^2 + x + 3$

20. $10a^3 - 4a^2 + 25a - 10$

21. $3xy - yz + 3xw - zw$

22. $6st^2 - 9s^2t - 2t^3 + 27s^3$

23. $4 - (3a + 2b)^2$

24. $(2x + y)^2 - (5z - 3w)^2$

25. $r^2 + 10rs + 25s^2 - 9$

26. $x^2 - 8xy + 16y^2 - 36a^2 + 12ab - b^2$

27. $9a^2 - 16b^2 - 3a - 4b$

28. $(x + 2y)^3 - 1$

29. $81c^4 - d^4$

30. $a^6 + 2a^3 + 1$

31. $t^6 + t^4 + t^2 + 1$

32. $64x^6 - y^6$

33. $abx + acs - bcy - aby + bcx - acy$

34. $x^{8n} - 16y^{4n}$