Final Examination in Theory and Functions of Real Variable

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Theorem 1.1.8 *If a nonempty set S of real numbers is bounded below, then inf S is the unique real number α such that*

(a) $x \geq \alpha$ *for all x in S;*

(b) *if $\epsilon > 0$ (no matter how small), there is an x_0 in S such that $x_0 < \alpha + \epsilon$.*

Proof The set $T = \{x| - x \in S\}$ is bounded above if S is bounded below. If S is bounded below then T is bounded above, so T has a unique supremum, by Theorem 1.1.3. Denote sup $T = -\alpha$. Then (i) if $x \in S$ then $-x \leq -\alpha$, so $x \geq \alpha$; (ii) if $\epsilon > 0$ there is an $x_0 \in T$ such that $-x_0 < -\alpha - \epsilon$, so $x_0 > \alpha + \epsilon$. Therefore, there is an α with properties (i) and (ii). If (i) and (ii) hold with α replaced by α_1 then $-\alpha_1$ is a supremum of T, so $\alpha_1 = \alpha$ by the uniqueness assertion of Theorem 1.1.3. ■

Theorem 1.3.3

(a) *The union of open sets is open.*

(b) *The intersection of closed sets is closed.*

These statements apply to arbitrary collections, finite or infinite, of open and closed sets.

Additional Proof In **(b)**, showing that $T^c = \cup\{F^c|F \in \mathcal{F}\}$ where $\mathcal{F}$ is a collection of closed sets and $T = \cap\{F|F \in \mathcal{F}\}$ we have $x \in T^c \Leftrightarrow x \in F^c$ for some F in $F \Leftrightarrow x \in \cup\{F^c|F \in F\}$.■

Theorem 1.3.7 (Heine-Borel Theorem) *If $\mathcal{H}$ is an open covering of a closed and bounded subset S of the real line, then S has an open covering $\widetilde{\mathcal{H}}$ consisting of finitely many open sets belonging to $\mathcal{H}$.*

Additional Proof If $\epsilon > 0$, then $S \cap (\beta - \epsilon, \beta) \neq \emptyset$ and $S^c \cap (\beta, \beta + \epsilon) \neq \emptyset$. Hence, if S is bounded above and $\beta =$ sup S, then $\beta \in \partial S$. Analogously, If S is bounded below and $\alpha =$ inf S then $\alpha \in \partial S$. Therefore, the inf S and sup S are in ∂S.

(a) If $x_0 \in \partial S$ and U is a neighborhood of x_0 then, (A) $U \cap S \neq \emptyset$. If x_0 is not a limit point of S, then (B) $U \cap (S - \{x_0\}) = \emptyset$ for some U. Now (A) and (B) imply that $x_0 \in S$, and (B) implies that x_0 is an isolated point of S. Therefore, a boundary point of a set S is either a limit point or an isolated point of S. (b) If S is closed, Corollary 1.3.6 and (a) imply that $\partial S \subset S$; hence, $\overline{S} = S \cup \partial S = S$. If $\overline{S} = S$, then $\partial S \subset S$. Now, if x_0 is a limit point of S, then every neighborhood of x_0 contains points of S other than x_0. If every neighborhood of x_0 also contains a point in S^c, then $x_0 \in \partial S$. If there is a neighborhood of x_0 that does not contain a point in S^c, then $x_0 \in S^0$. These are the only possibilities. Therefore, a limit point of a set S is either an interior point or a boundary point of S. By this and Corollary 1.3.6 and since $S^0 \subset S$, S is closed. Hence, $\partial S \subset S$ if S is closed.

Therefore, If S is closed and bounded, then $\inf S$ and $\sup S$ are both in S. ■

Theorem 2.1.6 *A function f has a limit at x_0 if and only if it has left- and right-hand limits at x_0, and they are equal. More specifically,*

$$\lim_{x \to x_0} f(x) = L$$

if and only if

$$f(x_0+) = f(x_0-) = L.$$

Proof It follows immediately from Definition 2.1.2 and Definition 2.1.5 that the existence of the limit implies the existence of the left- and right-hand limits with the same value. Conversely, if both left- and right-hand limits exists and are equal to L, then given $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $x_0 - \delta_1 < x < x_0$ implies that $|f(x) - L| < \epsilon$, $x_0 < x < x_0 + \delta_2$ implies that $|f(x) - L| < \epsilon$. Choosing $\delta = \min(\delta_1, \delta_2) > 0$, we get that $|x - x_0| < \delta$ implies that $|f(x) - L| < \epsilon$, which show that the limit exist. ■

Theorem 2.1.9 *Suppose that f is monotonic on (a, b) and define*

$$\alpha = \inf_{a<x<b} f(x) \quad \textit{and} \quad \beta = \sup_{a<x<b} f(x).$$

(a) *If f is nondecreasing, then $f(a+) = \alpha$ and $f(b-) = \beta$.*

(b) *If f is nonincreasing, then $f(a+) = \beta$ and $f(b-) = \alpha$.*

(Here $a+ = -\infty$ if $a = -\infty$ and $b- = \infty$ if $b = \infty$.)

(c) *If $a < x_0 < b$, then $f(x_0+)$ and $f(x_0-)$ exist and are finite; moreover,*

$$f(x_0-) \le f(x_0) \le f(x_0+)$$

if f is nondecreasing, and

$$f(x_0-) \ge f(x_0) \ge f(x_0+)$$

if f is nonincreasing.

Additional Proof (b) We first prove that $f(a+) = \beta$. If $M < \beta$ there is an x_0 in (a, b) such that $f(x_0) > M$. Since f is nonincreasing, $f(x) > M$ if $a < x < x_0$. Therefore, if

$\beta = \infty$ then $f(a+) = \infty$. If $\beta < \infty$ let $M = \beta - \epsilon$ where $\epsilon > 0$. Then $\beta - \epsilon < f(x) \leq \beta + \epsilon$, so

$$|f(x) - \beta| < \epsilon \quad \text{if} \quad a < x < x_0. \tag{A}$$

If $a = -\infty$ this implies that $f(-\infty) = \beta$. If $a > -\infty$ let $\delta = x_0 - a$. Then (A) is equivalent to

$$|f(x) - \beta| < \epsilon \quad \text{if} \quad a < x < a + \delta,$$

which implies that $f(a+) = \beta$.

Now we prove that $f(b-) = \alpha$. If $M > \alpha$ there is an x_0 in (a, b) such that $f(x_0) < M$. Since f is nonincreasing, $f(x) < M$ if $x_0 < x < b$. Therefore, if $\alpha = -\infty$ then $f(b-) = -\infty$. If $\alpha > -\infty$ let $M = \alpha + \epsilon$ where $\epsilon > 0$. Then $\alpha \leq f(x) < \alpha + \epsilon$, so

$$|f(x) - \alpha| < \epsilon \quad \text{if} \quad x_0 < x < b. \tag{B}$$

If $b = \infty$ this implies that $f(\infty) = \alpha$. If $b < \infty$ let $\delta = b - x_0$. Then (B) is equivalent to

$$|f(x) - \alpha| < \epsilon \quad \text{if} \quad b - \delta < x < b,$$

which implies that $f(b-) = \alpha$.

(c) Applying **(b)** to f on (a, x_0) and (x_0, b) separately shows that

$$f(x_0-) = \inf_{a<x_1<x_0} f(x_1) \quad \text{and} \quad f(x_0+) = \sup_{x_0<x_2<b} f(x_2).$$

However, if $x_1 < x_0 < x_2$ then $f(x_1) \geq f(x_0) \geq f(x_2)$; hence $f(x_0-) \geq f(x_0) \geq f(x_0+)$. ■

Theorem 2.2.9 *Suppose that f is continuous on a finite closed interval $[a, b]$. Let*

$$\alpha = \inf_{a\leq x\leq b} f(x) \quad \textit{and} \quad \beta = \sup_{a\leq x\leq b} f(x).$$

Then α and β are respectively the minimum and maximum of f on $[a, b]$; that is, there are points x_1 and x_2 in $[a, b]$ such that

$$f(x_1) = \alpha \quad \textit{and} \quad f(x_2) = \beta.$$

Additional Proof Showing that there is an x_2 such that $f(x_2) = \beta$, suppose there is no x_2 in $[a, b]$ such that $f(x_2) = \beta$. Then $f(x) < \beta$ for all $x \in [a, b]$. We will show that this leads to a contradiction. Suppose $t \in [a, b]$. Then $f(t) < \beta$, so $f(t) < (f(t) + \beta)/2 < \beta$. Since f is continuous at t, there is an open interval I_t about t such that (A) $f(x) < (f(t) + \beta)/2$ if $x \in I_t \cap [a, b]$. The collection $H = \{I_t | a \leq t \leq b\}$ is an open covering of $[a, b]$. Since $[a, b]$ is compact, the Heine-Borel theorem implies that there are finitely many points $t_1, t_2, \dots, t_n$ such that the intervals $I_{t1}, I_{t2}, \dots, I_{tn}$ cover $[a, b]$. Define $\beta_1 = \max\{(f(t_i) + \beta)/2 | 1 \leq i \leq n\}$. Then, since $[a, b] \subset \bigcup_{i=1}^{n}(I_{ti} \cap [a, b])$, (A) implies that $f(x) < \beta_1 (a \leq t \leq b)$. But $\beta_1 < \beta$, so this contradicts the definition of β. Therefore $f(x_2) = \beta$ for some x_2 in $[a, b]$. ■

Theorem 2.2.14 *If f is monotonic and nonconstant on $[a,b]$, then f is continuous on $[a,b]$ if and only if its range $R_f = \{f(x)|x \in [a,b]\}$ is the closed interval with endpoints $f(a)$ and $f(b)$.*

Proof In the case where f is nonincreasing, Theorem 2.1.9(b) implies that the set $\widetilde{R_f} = \{f(x)|x \in (a,b)\}$ is a subset of the open interval $(f(b-), f(a+))$. Therefore

$$R_f = \{f(b)\} \cup \widetilde{R_f} \cup \{f(a)\} \subset \{f(b)\} \cup (f(b-), f(a+)) \cup \{f(a)\}. \tag{A}$$

Now suppose f is continuous on $[a,b]$. Then $f(a) = f(a+)$, $f(b-) = f(b)$, so (A) implies that $R_f \subset [f(b), f(a)]$. If $f(b) < \mu < f(a)$, then Theorem 2.2.10 implies that $\mu = f(x)$ for some x in (a,b). Hence, $R_f = [f(b), f(a)]$.

For the converse, suppose $R_f = [f(b), f(a)]$. Since $f(a) \geq f(a+)$ and $f(b-) \geq f(b)$, (A) implies that $f(a) = f(a+)$ and $f(b-) = f(b)$. We know from Theorem 2.1.9(c) that if f is nonincreasing and $a < x_0 < b$, then $f(x_0-) \geq f(x_0) \geq f(x_0+)$. If either of these inequalities is strict, then R_f cannot be an interval. Since this contradicts our assumption, $f(x_0-) = f(x_0) = f(x_0+)$. Therefore f is continuous at x_0. We can now conclude that f is continuous on $[a,b]$. ■

Theorem 2.3.4 *If f and g are differentiable at x_0, then so are $f+g$, $f-g$, and fg, with*

(a) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$;

(b) $(f-g)'(x_0) = f'(x_0) - g'(x_0)$;

(c) $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

The quotient f/g is differentiable at x_0 if $g(x_0) \neq 0$, with

(d) $\displaystyle\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$.

Additional Proof

(a)

$$\begin{aligned}\lim_{x\to x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= f'(x_0) + g'(x_0).\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to x_0} \frac{(f-g)(x) - (f-g)(x_0)}{x - x_0} &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} - \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= f'(x_0) - g'(x_0).\end{aligned}$$

(d)

$$\begin{aligned}
\lim_{x\to x_0}\frac{(f/g)(x)-(f/g)(x_0)}{(x-x_0)} &= \lim_{x\to x_0}\frac{f(x)g(x_0)-f(x_0)g(x)}{(x-x_0)g(x)g(x_0)}\\
&= \lim_{x\to x_0}\frac{[f(x)g(x_0)-f(x_0)g(x_0)+f(x_0)g(x_0)-f(x_0)g(x)]}{(x-x_0)g(x)g(x_0)}\\
&= \lim_{x\to x_0}\frac{1}{g(x)}\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\\
&\quad+\frac{f(x_0)}{g(x_0)}\lim_{x\to x_0}\frac{1}{g(x)}\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}\\
&= \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{(g(x_0))^2}.
\end{aligned}$$

■

Theorem 2.4.1 (L'Hospital's Rule) *Suppose that f and g are differentiable and g' has no zeros on (a,b). Let*

$$\lim_{x\to b-} f(x)=\lim_{x\to b-} g(x)=0$$

or

$$\lim_{x\to b-} f(x)\pm\infty \quad \textit{and} \quad \lim_{x\to b-} g(x)=\pm\infty,$$

and suppose that

$$\lim_{x\to b-}\frac{f'(x)}{g'(x)}=L \quad (\textit{finite or } \pm\infty).$$

Then

$$\lim_{x\to b-}\frac{f(x)}{g(x)}=L.$$

Additional Proof For the case where $\lim_{x\to b-}\frac{f'(x)}{g'(x)}=\pm\infty$, If $\lim_{x\to b-}\frac{f'(x)}{g'(x)}=\infty$ and M is an arbitrary real number, there is an x_0 in (a,b) such that $\frac{f'(c)}{g'(c)}>M$ if $x_0<c<b$. By the argument given in the text, we can assume also that g has no zeros in $[x_0,b)$ and (A) $\frac{f(x)-f(t)}{g(x)-g(t)}>M$ if $x,t\in[x_0,b)$. If $\lim_{t\to b-} f(t)=\lim_{t\to b-} g(t)=0$ then letting $t\to b-$ in (A) shows that $\frac{f(x)}{g(x)}\ge M$ if $x,t\in[x_0,b)$, so $\lim_{x\to b-}\frac{f(x)}{g(x)}=\infty$ in this case. If $\lim_{t\to b-} f(t)=\lim_{t\to b-} g(t)=\infty$, let u and x_1 be as in the proof given in the text. Then (B) $\frac{f(x)}{g(x)u(x)}>M$ if $x_1<x<b$. Since $\lim_{x\to b-} u(x)=1$, there is an $x_2\ge x_1$ such that $u(x)\ge\frac{1}{2}$ if $x_2<x<b$. Therefore, (B) implies that $\frac{f(x)}{g(x)}>\frac{M}{2}$ if $x_2<x<b$, so $\lim_{x\to b-}\frac{f(x)}{g(x)}=\infty$ in this case also. ■

Theorem 3.3.2 *If f is integrable on $[a,b]$ and c is a constant, then cf is integrable on $[a,b]$ and*

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx.$$

Proof Trivial if $c=0$. Suppose $c \neq 0$ and $\epsilon > 0$. If $\widehat{\sigma}$ is a Riemann sum of cf, then $\widehat{\sigma} = \sum_{j=1}^{n} cf(c_j)(x_j - x_{j-1}) = c\sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) = c\sigma$, where σ is a Riemann sum for f. Since f is integrable on $[a,b]$, Definition 3.1.1 implies that there is a $\delta > 0$ such that $\left|\sigma - \int_a^b f(x)dx\right| < \frac{\epsilon}{|c|}$ if σ is a Riemann sum of f over any partition P of $[a,b]$ such that $\|P\| < \delta$. Therefore, $\left|\widehat{\sigma} - \int_a^b cf(x)dx\right| < \epsilon$ if $\widehat{\sigma}$ is a Riemann sum of cf over any partition P of $[a,b]$ such that $\|P\| < \delta$, so cf is integrable over $[a,b]$, again by Definition 3.1.1. ∎

Theorem 3.3.3 *If $f_1, f_2, \dots, f_n$ are integrable on $[a,b]$ and $c_1, c_2, \dots, c_n$ are constants, then $c_1f_1 + c_2f_2 + \cdots + c_nf_n$ is integrable on $[a,b]$ and*

$$\int_a^b (c_1f_1 + c_2f_2 + \cdots + c_nf_n)(x)dx = c_1\int_a^b f_1(x)dx + c_2\int_a^b f_2(x)dx + \cdots + c_n\int_a^b f_n(x)dx.$$

Proof If f_1 and f_2 are integrable on $[a,b]$ and c_1 and c_2 are constants, then Theorem 3.3.2 implies that c_1f_1 and c_2f_2 are integrable on $[a,b]$ and $\int_a^b c_if_i(x)dx = c_i\int_a^b f_i(x)dx$, $i = 1,2$. Therefore, Theorem 3.3.1 implies P_2. Now suppose $n \geq 2$ and P_n is true. Let $f_1, f_2, \dots, f_{n+1}$ be integrable on $[a,b]$ and $c_1, c_2, \dots, c_{n+1}$ be constants. By Theorem 3.3.1, $c_1f_1, c_2f_2, \dots, c_{n+1}f_{n+1}$ are integrable on $[a,b]$, and $\int_a^b c_if_i(x)dx = c_i\int_a^b f_i(x)dx$, $i = 1, 2, \dots, n+1$. Now $\int_a^b (c_1f_1 + c_2f_2 + \cdots + c_{n+1}f_{n+1})(x)dx = \int_a^b [(c_1f_1 + c_2f_2 + \cdots + c_nf_n)(x) + c_{n+1}f_{n+1}(x)]dx = \int_a^b (c_1f_1 + c_2f_2 + \cdots + c_nf_n)(x)dx + \int_a^b c_{n+1}f_{n+1}(x)dx$ (by P_2) $= c_1\int_a^b f_1(x)dx + c_2\int_a^b f_2(x)dx + \cdots + c_n\int_a^b f_n(x)dx + c_{n+1}\int_a^b f_{n+1}(x)dx$ by P_n and Theorem 3.3.2. Therefore, P_n implies P_{n+1}. ∎

Theorem 3.3.11 *If f is integrable on $[a,b]$ and $a \leq c \leq b$, then $F(x) = \int_c^x f(t)dt$ is differentiable at any point x_0 in (a,b) where f is continuous, with $F'(x_0) = f(x_0)$. If f is continuous from the right at a, then $F_+^{'}(a) = f(a)$. If f is continuous from the left at b, then $F_-^{'}(b) = f(b)$.*

Additional Proof Since $\frac{1}{x-a}\int_a^x f(a)dt = f(a)$, we can write

$$\frac{F(x) - F(a)}{x-a} - f(a) = \frac{1}{x-a}\int_a^x [f(t) - f(a)]dt.$$

From this and Theorem 3.3.5, (A) $\left|\dfrac{F(x)-F(a)}{x-a}-f(a)\right| \le \dfrac{1}{|x-a|}\left|\int_a^x |f(t)-f(a)|dt\right|$. Since f is continuous from the right at a, there is for each $\epsilon > 0$ a $\delta > 0$ such that $|f(t)-f(a)| < \epsilon$ if $a \le x < a+\delta$ and t is between x and a. Therefore, from (A), $\left|\dfrac{F(x)-F(a)}{x-a}-f(a)\right| < \epsilon\dfrac{|x-a|}{|x-a|} = \epsilon$ if $a < x < a+\delta$. This proves that $F_+'(a) = f(a)$.

Since $\dfrac{1}{b-x}\int_x^b f(b)dt = f(b)$, we can write

$$\frac{F(x)-F(b)}{b-x} - f(b) = \frac{1}{b-x}\int_x^b [f(b)-f(t)]dt.$$

From this and Theorem 3.3.5, (B) $\left|\dfrac{F(x)-F(b)}{b-x}-f(b)\right| \le \dfrac{1}{|b-x|}\left|\int_x^b |f(b)-f(t)|dt\right|$. Since f is continuous from the left at b, there is for each $\epsilon > 0$ a $\delta > 0$ such that $|f(b)-f(t)| < \epsilon$ if $b-\delta \le x < b$ and t is between x and b. Therefore, from (B), $\left|\dfrac{F(x)-F(b)}{b-x}-f(b)\right| < \epsilon\dfrac{|b-x|}{|b-x|} = \epsilon$ if $b-\delta < x < b$. This proves that $F_-'(b) = f(b)$. ■

Theorem 4.1.7 *Let* $\lim_{x\to\infty} f(x) = L$*, where* L *is in the extended reals, and suppose that* $s_n = f(n)$ *for large* n*. Then*

$$\lim_{n\to\infty} s_n = L.$$

Proof Suppose that $s_n = f(n)$ for $n \ge N_1$. Let $\epsilon > 0$. Since $\lim_{x\to\infty} f(x) = L$ there is an integer $N > N_1$ such that $|f(x)-L| < \epsilon$ if $x > N$, so $|s_n - L| = |f(n)-L| < \epsilon$ if $n \ge N$. Therefore, $\lim_{n\to\infty} s_n = L$. ■

Theorem 4.2.3 *If* $\{s_n\}$ *is monotonic and has a subsequence* $\{s_{n_k}\}$ *such that*

$$\lim_{k\to\infty} s_{n_k} = s \quad (-\infty \le s \le \infty),$$

then

$$\lim_{n\to\infty} s_n = s.$$

Additional Proof If $\{s_n\}$ is nonincreasing, then $\{s_{n_k}\}$ is also, so it suffices to show that (A) $\inf\{s_{n_k}\} = \inf\{s_n\}$ and apply Theorem 4.1.6(b). Since the set of terms of $\{s_{n_k}\}$ is contained in the set of terms of $\{s_n\}$, (B) $\inf\{s_n\} \le \inf\{s_{n_k}\}$. Since $\{s_n\}$ is nonincreasing, there is for every n an integer $n_k > n$ such that $s_n \ge s_{n_k}$. This implies that $\inf\{s_n\} \ge \inf\{s_{n_k}\}$. This and (B) imply the conclusion. ■

Theorem 4.3.19 *If* $\sum a_n$ *converges absolutely, then* $\sum a_n$ *converges.*

Proof Suppose that $\sum_{n=m}^{\infty} |a_n| < \infty$. Let $b_n = |a_n| - a_n$; then $0 \le b_n \le 2|a_n|$, so $\sum_{n=m}^{\infty} b_n$

converges absolutely, by the comparison test. Since $a_n = |a_n| - b_n$, $\sum_{n=m}^{\infty} a_n$ converges, by Theorem 4.3.3. ■

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