TAMS65 - Lecture 9
Random vector - Normal random vector

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Content

Syllabus

- Examiner, Teaching assistants, Textbook, Course Literature and Course plan are the same as VT1.
- 4 Distance lectures (on Lisam) - Two new topics.
- 2 Distance lessons - Microsoft Teams.
- 1 Distance seminar (on Lisam) - Same way as lectures.
- Distance Project (on Lisam : Lisam - Course documents - 7 Project) - **Instructions**.

Syllabus

- Distance quiz will be given on May 6, 2020.
  - which is based on the project and 4 lectures.
  - which contains five questions with multiple choice options.
  - Details will be given in the last Lecture.

Pass the course =
=Pass the written Exam (VT1) + Pass the project(VT2) + Pass the quiz(VT2)

Repetitions

If X, Y are independent(oberoende), then $E(XY) = E(X)E(Y)$.

Covariance(kovarians) of X and Y is defined as

$$\begin{aligned}\text{cov}(X, Y) = c(X, Y) = \sigma_{X,Y} &= E[(X - E(X))(Y - E(Y))] \\ &= E(XY) - E(X)E(Y)\end{aligned}$$

Remark 1:

1. Covariance measures how much two random variables change together.
2. $\text{cov}(X, Y) = \text{cov}(Y, X)$
3. $\text{cov}(X, X) = V(X)$
4. $\text{cov}(a, X) = 0$ for any constant a.

Repetitions

5. If X, Y are independent, then $\text{cov}(X, Y) = 0$
   - Independent $\Rightarrow$ covariance is 0, but covariance is $0 \not\Rightarrow$ independent.
   - Covariance is NOT 0 $\Rightarrow$ NOT independent.
6. $V(aX + bY) = a^2V(X) + 2ab\,\text{cov}(X, Y) + b^2V(Y)$.
   - If X, Y are independent, for ant constants a, b, then $V(aX + bY) = a^2V(X) + b^2V(Y)$.
7. If X, Y, W, V are random variables, a, b, c, d, e are constants, then

$$\begin{aligned}&\text{cov}(aX + bY + c, dW + eV) \\ &= ad\,\text{cov}(X, W) + ae\,\text{cov}(X, V) + bd\,\text{cov}(Y, W) + be\,\text{cov}(Y, V)\end{aligned}$$

Example 1

Example 1, If $X_1 \sim N(0, 1), X_2 \sim N(1, 2)$ and $X_3 \sim N(1, 3)$ are independent, calculate

$$\text{cov}(2X_1 - X_2 + 4X_3, 3X_1 + X_2).$$

$$\begin{aligned}&\text{cov}(2X_1 - X_2 + 4X_3, 3X_1 + X_2) \\ &= 6\,\text{cov}(X_1, X_1) + 2\,\text{cov}(X_1, X_2) - 3\,\text{cov}(X_2, X_1) - \text{cov}(X_2, X_2) \\ &+ 12\,\text{cov}(X_3, X_1) + 4\,\text{cov}(X_3, X_2) \\ &= 6V(X_1) - V(X_2) = 2.\end{aligned}$$

Repetitions

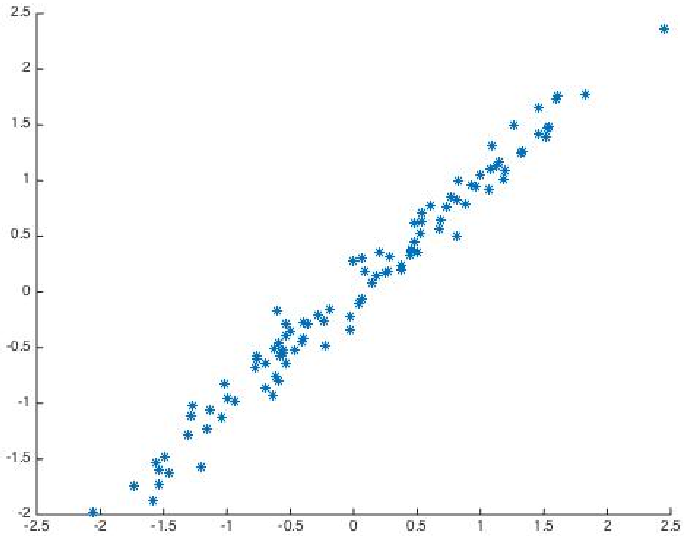
Correlation/Correlation coefficient(korrelations koefficient) of X and Y is defined as

$$\rho_{X,Y} = \frac{\text{cov}(X, Y)}{\sqrt{V(X)V(Y)}}$$

Remark 2:

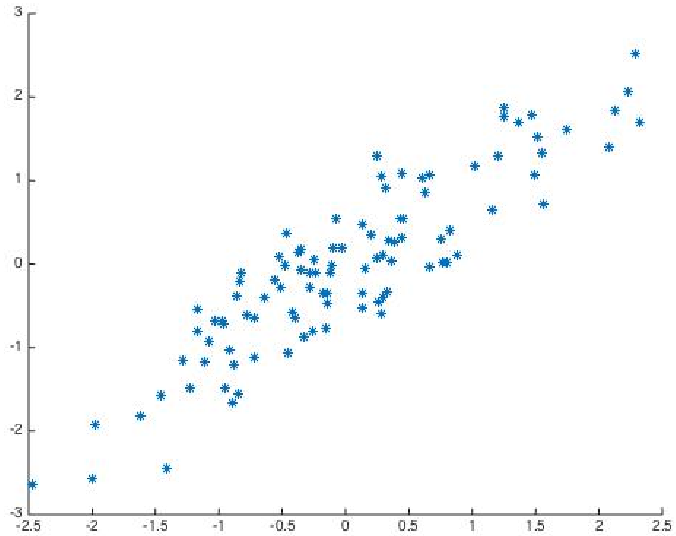
1. Correlation measures how much linear relation between X and Y.
   - If $|\rho_{X,Y}| \approx 1$, then X and Y are full linear.
   - If $|\rho_{X,Y}| \approx 0$, then X and Y are non-linear.

correlation $\rho = 0.99$



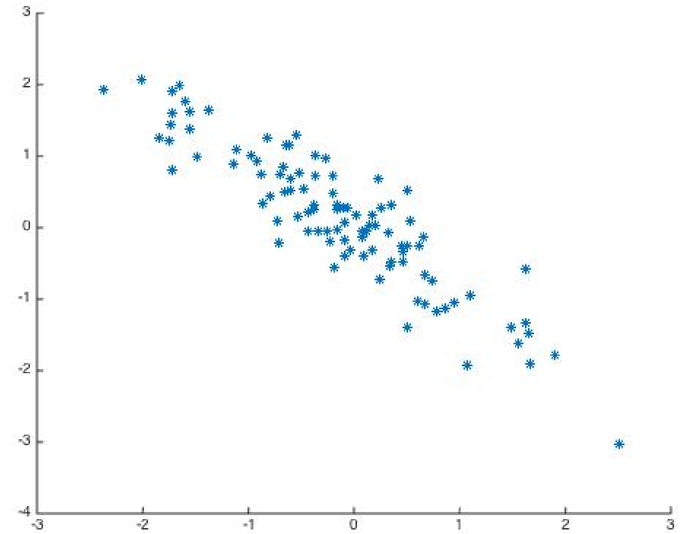
$\hat{\rho} = 0.9886$

correlation $\rho = 0.9$



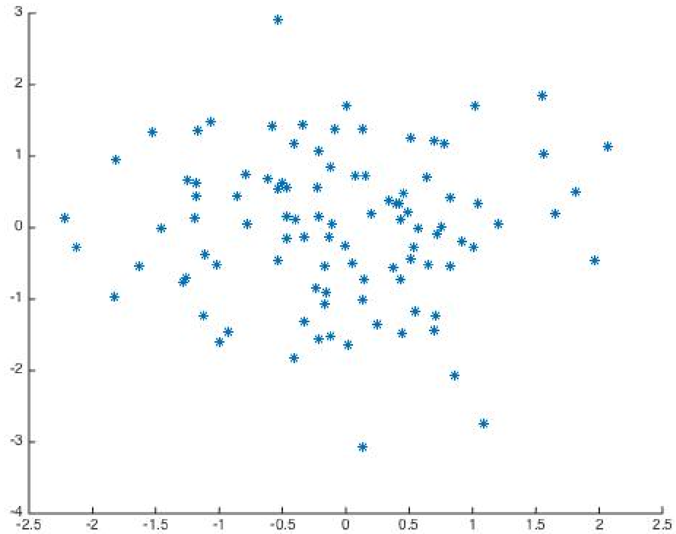
$\hat{\rho} = 0.8983$

correlation $\rho = -0.9$



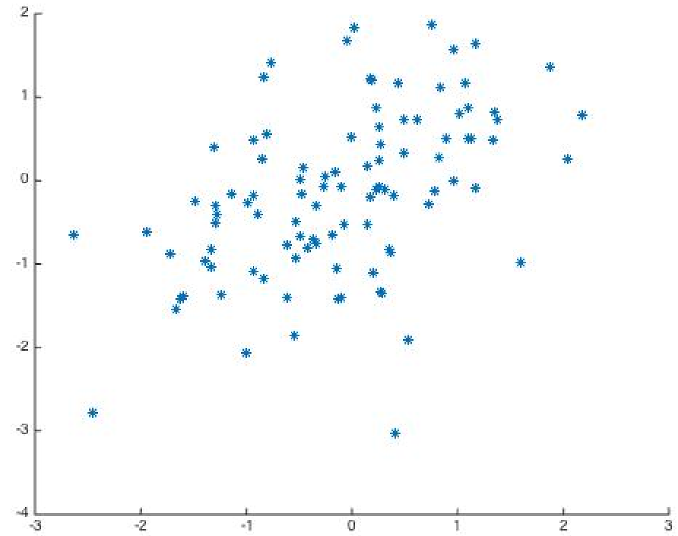
$\hat{\rho} = -0.9075$

correlation $\rho = 0$



$\hat{\rho} = -0.0041$

correlation $\rho = 0.5$



$\hat{\rho} = 0.4751$

Repetitions

Remark 2:

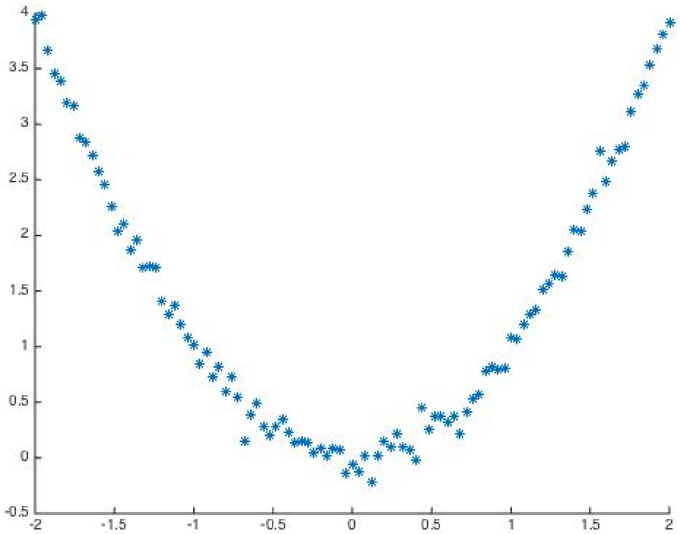
2 $-1 \leq \rho_{X,Y} \leq 1$
  - Proof: Cauchy-Schwarz inequality.
  - $|\rho_{X,Y}| = 1$ if and only if X and Y are full linear.
- If X and Y are independent, then $\rho_{X,Y} = 0$.

Definition X and Y is called **uncorrelated(okorrelerade)** if $\rho_{X,Y} = 0$.

Note:

- Independent $\Rightarrow$ uncorrelated, but uncorrelated $\not\Rightarrow$ independent.
- $|\rho_{X,Y}| \approx 0 \Rightarrow X$ and Y are non-linear, but it does NOT imply that they don't have other relation.

correlation $\rho = 0$, non linear but quadratic relation.



$\hat{\rho} = -0.0030$

Random vector

Definition: A **random vector(stokastisk vektor)** is defined as

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}_{n\times 1},$$

where the components(komponenterna) X_i, $i = 1, \ldots, n$ are one dimensional random variables.

Random vector

A random vector $\boldsymbol{X}$ has a **mean vector(väntevärdesvektor)** which is defined as

$$\boldsymbol{\mu_X} = \begin{pmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_n) \end{pmatrix}_{n\times 1}.$$

Random vector

A random vector $\boldsymbol{X}$ has also a **covariance matrix (kovariansmatris)** which is defined as

$$\boldsymbol{C_X} = \begin{pmatrix} \mathrm{cov}(X_1, X_1) & \mathrm{cov}(X_1, X_2) & \dots & \mathrm{cov}(X_1, X_n) \\ \mathrm{cov}(X_2, X_1) & \mathrm{cov}(X_2, X_2) & \dots & \mathrm{cov}(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}(X_n, X_1) & \mathrm{cov}(X_n, X_2) & \dots & \mathrm{cov}(X_n, X_n) \end{pmatrix}_{n\times n},$$

where the components $\mathrm{cov}(X_i, X_j) = E\left[(X_i - E(X_i))(X_j - E(X_j))\right]$ for $i, j = 1, 2, \dots, n$.

Random vector

Remark 3:

1. $\mathrm{cov}(X_i, X_i) = V(X_i), i = 1, \dots, n$. Note: On main diagonal, there are $V(X_i)$.
2. If $n = 1$, then $\boldsymbol{C_X} = V(X_1)$.
3. $\boldsymbol{C_X}$ is symmetric since $\mathrm{cov}(X_i, X_j) = \mathrm{cov}(X_j, X_i)$. Thus, we can see

$$\boldsymbol{C_X} = \boldsymbol{C'_X} = \boldsymbol{C_X^T} \quad \text{transpose(transponat) of } \boldsymbol{C_X}$$

Question: Can a random vector $\boldsymbol{X}$ have a variance, if $n > 1$? NO!

Random vector

Theorem 1: Let $\boldsymbol{X}$ be a n dimensional random vector with mean vector $\boldsymbol{\mu_X}$ and covariance matrix $\boldsymbol{C_X}$. Define a new random vector

$$\boldsymbol{Y} = \boldsymbol{AX} + \boldsymbol{b},$$

where $\boldsymbol{A}$ is a constant matrix and $\boldsymbol{b}$ is a constant vector.

Then $\boldsymbol{Y}$ has the following mean vector and covariance matrix

$$\boldsymbol{\mu_Y} = \boldsymbol{A\mu_X} + \boldsymbol{b},$$

$$\boldsymbol{C_Y} = \boldsymbol{AC_XA'}.$$

Recall: $E(aX + b) = aE(X) + b, \quad V(aX + b) = a^2V(X)$.

Example 2

Example 2, A random vector $\begin{pmatrix} X \\ Y \end{pmatrix}$ has covariance matrix $\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$.

(1) Find $V(X)$ and $V(Y)$.
(2) Are X and Y independent?

(1) $V(X) = 2$, $V(Y) = 6$.
(2) X and Y are not independent since $\text{cov}(X_1, X_2) = 3 \neq 0$.

Standard normal random vector

Definition: A random vector $\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}_{n\times 1}$ is called **standard normal random vector** if its components satisfy

$$X_i \sim N(0,1), i = 1, \dots, n \text{ and } X_1, \dots, X_n \text{ are independent .}$$

Remark 4:

1 $\boldsymbol{\mu_X} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1} = 0 = \vec{0}$, $\boldsymbol{C_X} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}_{n\times n} = \boldsymbol{I}_{n\times n}$.

Standard normal random vector

2 The standard normal random vector is denoted by

$$\boldsymbol{X} \sim N(\boldsymbol{\mu_X}, \boldsymbol{C_X}) \text{ or } \boldsymbol{X} \sim N(\vec{0}, \boldsymbol{I}_{n\times n}).$$

3 The density function(täthesfunktion) is

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{\boldsymbol{X}}(x_1, \dots, x_n) = f_{X_1}(x_1) \dots f_{X_n}(x_n) = \frac{1}{\left(\sqrt{2\pi}\right)^n} e^{-\frac{1}{2}\boldsymbol{x}'\boldsymbol{x}},$$

where $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

General Normal random vector

Recall: If $X \sim N(0,1)$, then $Y = \sigma X + \mu \sim N(\mu, \sigma)$.

Definition:
If there exist constant matrix $\boldsymbol{A}$ and constant vector $\boldsymbol{b}$, such that $\boldsymbol{Y} = \boldsymbol{AX} + \boldsymbol{b}$ where $\boldsymbol{X}$ is a standard normal random vector, then $\boldsymbol{Y}$ is called (general) **normal random vector**.

Note: Normal random vector is also called **Multivariate random vector**.

Remark 5:

1 $\boldsymbol{\mu_Y} = \boldsymbol{A\mu_X} + \boldsymbol{b} = \boldsymbol{b}$, and $\boldsymbol{C_Y} = \boldsymbol{AC_XA'} = \boldsymbol{AA'}$ (by **Theorem 1**.)
2 The normal random vector is denoted by

$$\boldsymbol{Y} \sim N(\boldsymbol{\mu_Y}, \boldsymbol{C_Y}) \text{ or } \boldsymbol{Y} \sim N(\boldsymbol{b}, \boldsymbol{AA'}).$$

General Normal random vector

3 The density function(täthesfunktion) is

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = f_{\boldsymbol{Y}}(y_1, \ldots, y_n) = \frac{1}{\left(\sqrt{2\pi}\right)^n \sqrt{\det \boldsymbol{C_Y}}} e^{-\frac{1}{2}[(\boldsymbol{y}-\boldsymbol{\mu_Y})' \boldsymbol{C_Y}^{-1}(\boldsymbol{y}-\boldsymbol{\mu_Y})]},$$

where $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$,

$\det \boldsymbol{C_Y}$ is determinant of covariance matrix $\boldsymbol{C_Y}$ and $\boldsymbol{C}_{\boldsymbol{Y}}^{-1}$ is the inverse matrix of $\boldsymbol{C_Y}$.

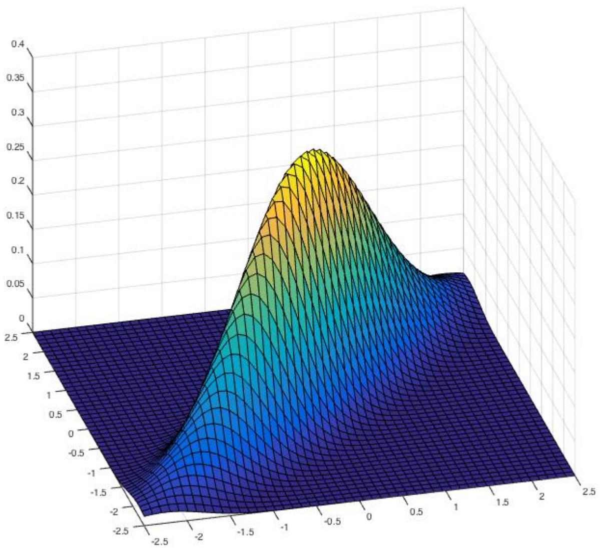
General Normal random vector

Theorem 2 $\boldsymbol{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}_{n\times 1}$ is a normal random vector if and only if $c_1X_1 + \ldots + c_nX_n$ is normally distributed for any **not all zero** constants $c_1, \ldots, c_n$.

Theorem 3 If $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a normal random vector, then we get

- X, Y are independent $\Leftrightarrow$ $\text{cov}(X, Y) = 0$.
- X, Y are independent $\Leftrightarrow$ X, Y are uncorrelated ($\rho_{X,Y} = 0$).

For example, $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N(\vec{0}, \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix})$



Example 3

Example 3, Suppose that $\boldsymbol{X}$ is a normal random vector with

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 2 & 3 & 4 \end{pmatrix}\right).$$

Define $Y = (X_1 + X_2 + 2X_3)/2$.

(1) Find the distributions of X_1, X_2. Are they independent?

(2) Find $P(Y \leq 6)$.

(1), By Theorem 1, $X_1 \sim N(1, 1)$ and $X_2 \sim N(1, 1)$. $\text{cov}(X_1, X_2) = 0$ and X_1, X_2 are components of a normal random vector which imply that they are independent. (By Theorem 3)
(2) By Theorem 2, Y is normally distributed since it is a linear combination of components of a normal random vector.

Example 3

$$Y = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = A\boldsymbol{X}$$

By Theorem 1,

$$\boldsymbol{\mu_Y} = \boldsymbol{A\mu_X} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 3,$$

$$\boldsymbol{C_Y} = \boldsymbol{AC_XA'} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}' = \frac{19}{2}.$$

Thus $Y \sim N(3, \sqrt{\frac{19}{2}})$.

$$P(Y \leq 6) = \Phi(0.97) = 0.834.$$

Example 4

Random variables X_1, X_2 are independent and $X_i \sim N(0,1), i{=}1,2$.

$$Y_1 = X_1 - X_2,$$
$$Y_2 = X_1 + X_2.$$

Find the density function for $\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$.

$$Y = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = A\boldsymbol{X}, \quad \boldsymbol{X} \sim N(\vec{0}, \boldsymbol{I}_{2\times 2})$$

Example 4

By Theorem 1,

$$\boldsymbol{\mu_Y} = \boldsymbol{A\mu_X} = \vec{0},$$

$$\boldsymbol{C_Y} = \boldsymbol{AC_XA'} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\boldsymbol{I}_{2\times 2}.$$

det $\boldsymbol{C_Y} = 4$, $\boldsymbol{C_Y}^{-1} = \frac{1}{2}\boldsymbol{I}_{2\times 2}$ and $\boldsymbol{y} - \boldsymbol{\mu_Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

Thus

$$f_{\boldsymbol{Y}}(y_1, y_2) = \frac{1}{\left(\sqrt{2\pi}\right)^2 \sqrt{\det \boldsymbol{C_Y}}}\, e^{-\frac{1}{2}[(\boldsymbol{y}-\boldsymbol{\mu_Y})'\boldsymbol{C_Y}^{-1}(\boldsymbol{y}-\boldsymbol{\mu_Y})]}$$

$$= \frac{1}{4\pi} e^{-\frac{y_1^2+y_2^2}{4}}.$$

Practice after the lecture:

Exercises:

(I) PS-15, PS-16, PS-20, PS-21.

(II) PS-23, PS-17, PS-18, PS-22.

Thank you!

http://courses.mai.liu.se/GU/TAMS65/

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