Fokker-Planck Diffusion Approximation

Assumptions:

- Succesive moves are independent.
- Individual at location x moves at rate $\mu(x,t)$:

  Pr{move in $(t,t+\tau)$}=$\tau\mu(x,t)+O(\tau^2)$

  Pr{more than one move in $(t,t+\tau)$}= $O(\tau^2)$
- The step-length λ, for a move starting at x is random with density $M(\lambda,x,t)$.
- Movement is the only process affecting the population density $p(x,t)$.

Balance law:

Be here now = [been here, stayed put]

+ [made a move here]

+ [moved >1 time]

$$p(x,t+\tau)=p(x,t)-\tau\mu(x,t)p(x,t)+\int_{-\infty}^{\infty}\mu(x-\lambda,t)p(x-\lambda,t)M(\lambda,x-\lambda,t)d\lambda+O(\tau^2)$$

<u>Re-arrange to get</u>

$$\frac{p(x,t+\tau)-p(x,t)}{\tau}=-\mu(x,t)p(x,t)+\int_{-\infty}^{\infty}\mu(x-\lambda,t)p(x-\lambda,t)M(\lambda,x-\lambda,t)d\lambda+O(\tau)$$

<u>Let $\tau\to 0$</u>

$$\begin{aligned}\frac{\partial p}{\partial t}(x,t)&=-\mu(x,t)p(x,t)+\int_{-\infty}^{\infty}\mu(x-\lambda,t)p(x-\lambda,t)M(\lambda,x-\lambda,t)d\lambda\\&=-\mu(x,t)p(x,t)+\int_{-\infty}^{\infty}\Gamma(x-\lambda,\lambda,t)d\lambda\end{aligned}\tag{1}$$

$$\Gamma(x,\lambda,t)=\mu(x,t)p(x,t)M(\lambda,x,t)$$

(1) $$\frac{\partial p}{\partial t}(x,t) = -\mu(x,t)p(x,t) + \int_{-\infty}^{\infty} \Gamma(x-\lambda,\lambda,t)\,d\lambda$$
$$\Gamma(x,\lambda,t) = \mu(x,t)p(x,t)M(\lambda,x,t)$$

Diffusion Approximation to (1)

Taylor-series for $\Gamma(x-\lambda,\lambda,t) = \Gamma(x,\lambda,t) + \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{k!}\frac{\partial^k \Gamma}{\partial x^k}(x,\lambda,t)$

Do the integrals

$$\int_{-\infty}^{\infty} \Gamma(x,\lambda,t)\,d\lambda = \int_{-\infty}^{\infty} \mu(x,t)p(x,t)M(x,\lambda,t)\,d\lambda$$
$$= \mu(x,t)p(x,t)\underbrace{\int_{-\infty}^{\infty} M(x,\lambda,t)\,d\lambda}_{=1}$$

$$\int_{-\infty}^{\infty} \lambda^k \frac{\partial^k}{\partial x^k}\Gamma(x,\lambda,t)\,d\lambda = \frac{\partial^k}{\partial x^k}\{\int_{-\infty}^{\infty} p(x,t)\mu(x,t)\lambda^k M(x,\lambda,t)\,d\lambda\}$$
$$= \frac{\partial^k}{\partial x^k}\{p(x,t)\mu(x,t)\underbrace{\int_{-\infty}^{\infty} \lambda^k M(x,\lambda,t)\,d\lambda\}}_{=m_k(x,t)}$$

Plug back into (1)

(2) $$\frac{\partial p}{\partial t}(x,t) = -\frac{\partial}{\partial x}\{p(x,t)\mu(x,t)m_1(x,t)\} + \frac{\partial^2}{\partial x^2}\{p(x,t)\mu(x,t)m_2(x,t)/2\} + \cdots$$
$$= -\frac{\partial}{\partial x}\{p(x,t)v(x,t)\} + \frac{\partial^2}{\partial x^2}\{p(x,t)D(x,t)\} + \cdots$$