Spectral decomposition of self-adjoint compact operators

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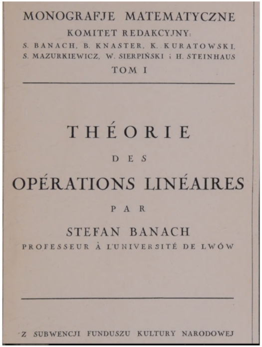
Historical perspective


MONOGRAFJE MATEMATYCZNE
KOMITET REDAKCYJNY:
S. BANACH, B. KNASTER, K. KURATOWSKI,
S. MAZURKIEWICZ, W. SIERPIŃSKI i H. STEINHAUS
TOM I

THÉORIE
DES
OPÉRATIONS LINÉAIRES
PAR
STEFAN BANACH
PROFESSEUR À L'UNIVERSITÉ DE LWÓW

Z SUBWENCJI FUNDUSZU KULTURY NARODOWEJ

Figure: Stephan Banach(1892-1945)

Frigyes Riesz(1880-1956)

Diagonalization of Self-adjoint compact operators

Lemma: Let $T \in B(H, H)$ be compact T is compact. Then

$$\text{either} -\|T\| \in \sigma_p(T) \text{ or } \|T\| \in \sigma_p(T).$$

Proof. If $T = 0$, then 0 is an eigenvalue and we have $\|T\| = 0$.

If $T \neq 0$, then (cf. section 2.6, theorem 2.74):

$$T \quad \text{self-adjoint} \implies \|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

There exists $\{x_n\}$, $\|x_n\| = 1$, such that $\|T\| = \lim_n |\langle Tx_n, x_n\rangle|$.
Wlog, $\langle Tx_n, x_n\rangle \to \lambda$ where $|\lambda| = \|T\|$. We have that

$$0 \leqslant \|(T - \lambda I)\, x_n\|^2 = \|Tx_n\|^2 - 2\lambda\langle Tx_n, x_n\rangle + \lambda^2 \to 0 \quad (n \to \infty)$$

By compactness, there exists $\{x_{n_k}\}$ such that
$\|Tx_{n_k} - x\| \to 0 \quad (k \to \infty)$. But $x_{n_k} = \frac{1}{\lambda}\,(\lambda - T)\, x_{n_k} + \frac{1}{\lambda}\, Tx_{n_k}$
converges then to $\frac{1}{\lambda}x$. Thus $1 = \|\lambda^{-1}x\| = |\lambda|\, \|x\|$ and $x \neq 0$.
Then $Tx_{n_k} \to \frac{1}{\lambda}\, Tx$ by continuity. By uniqueness of limit,
$x = \lambda^{-1}\, Tx$ and $Tx = \lambda x$. So $\lambda \in \sigma_p\,(T)$.

Proposition: Let $T \in B(H, H)$ be a compact self-adjoint linear operator. Then

$$H = \mathrm{Ker}\,(T) \oplus \widehat{\bigoplus}_{\lambda \in \sigma(T) \setminus \{0\}} \mathrm{Ker}\,(T_\lambda) \qquad (T_\lambda = T - \lambda I)\,.$$

Proof. Note that for any $\lambda \neq \mu$ in $\sigma(T)$, with T compact, we have $\mathrm{Ker}\,(T_\lambda) \perp \mathrm{Ker}\,(T_\mu)$. Then $F := \widehat{\bigoplus}_{\lambda \in \sigma(T) \setminus \{0\}} \mathrm{Ker}\,(T_\lambda)$ is closed and $T(F) \subset F$.[To see the stability, take any $x = \sum_{\lambda \in \sigma(T) \setminus \{0\}} x_\lambda$, then compute $Tx = \sum_\lambda T(x_\lambda)$. For each $\lambda \in \sigma(T) \setminus \{0\}$, $T_\lambda T(x_\lambda) = (T - \lambda I)\, T(x_\lambda) = \left(T^2 - \lambda T\right)(x_\lambda) = T(T_\lambda x_\lambda) = 0$.]

Consider $u = \sum_{\lambda \in \sigma(T) \setminus \{0\}} u_\lambda$ where $\sum_\lambda \|u_\lambda\|^2$ converges. We have that $Tu = \sum_{\lambda \in \sigma(T) \setminus \{0\}} \lambda u_\lambda \in F$. In addition, given that T is self-adjoint, $F^\perp$ is stable under taking T, too[1]. Let $T_0 : F^\perp \to F^\perp$ is the restriction of T on $F^\perp$. Then T_0 is self-adjoint and compact. Then (exercise), we have that $r_\sigma(T_0) = \|T_0\|$. Moreover, if $r_{\sigma(T_0)} > 0$ then T_0 has a **nonzero** eigenvalue λ since by **Riesz-Schauder theorem**, any nonzero element in $\sigma(T_0)$ is an eigenvalue due to the compactness of T_0. Since $\mathrm{Ker}(T_0 - \lambda_0 I) \subset \mathrm{Ker}(T_\lambda)$, we would have $\mathrm{Ker}(T_{\lambda_0}) \cap F^\perp \neq \{0\}$, which contradicts with $F^\perp \perp \mathrm{Ker}(T_\lambda)$ for any $\lambda \neq 0$. Thus $r_\sigma(T_0) = 0$, $\|T_0\| = 0$ and T_0 is the **null operator**. Furthermore, $\mathrm{Ker}(T) \subset (\mathrm{Ker}(T_\lambda))^\perp$ for all $\lambda \neq 0$ and $\mathrm{Ker}(T) \subset F^\perp$.

[1] If a set W is invariant under taking an operator T, then its orthogonal complement must be stable under taking the adjoint operator, T^*.

Spectral theorem of self-adjoint compact operator

Then $\mathrm{Ker}\,(T) = F^{\perp}$.But F is closed, so $H = F \oplus F^{\perp}$ and we have $H = \mathrm{Ker}\,(T) \oplus \widehat{\bigoplus}_{\lambda \in \sigma(T)\setminus\{0\}} \mathrm{Ker}\,(T_\lambda)$.

Theorem: Let $T \in B(H,H)$ be a self-adjoint compact operator. Let $\sigma(T) \setminus \{0\} = \{\lambda_1, \lambda_2, \dots\} = \sigma_p(T) \setminus \{0\}$ and $P_n : H \to \mathrm{Ker}\,(T_{\lambda_n})$ be a projection, $n \in \mathbb{N}$. Then if $n \neq m$, we have $P_n P_m = 0 = P_m P_n$, $\mathrm{rk}\,(P_n) < \infty$ and $\lambda_n \to 0(n \to \infty)$ and:

$$T = \sum_{n=1}^{\infty} \lambda_n P_n$$

where the convergence is the convergence with respect to the operator norm.

It follows from the lemma that there exists $\lambda_1 \in \sigma_p(T)$ such that $|\lambda_1| = \|T\|$. Let $F_1 := \mathrm{Ker}\,(T_{\lambda_1})$ and $P_1 : H \to F_1$ be a projection of H onto F_1. Set $H_2 = F_1^{\perp}$.Since F_1 is invariant under T which is self-adjoint, so too H_2 is invariant under T. Let $T_2 = T|_{H_2}$. Then T_2 is a self-adjoint compact operator. The same lemma provides $\lambda_2 \in \sigma_p(T_2)$ such that $|\lambda_2| = \|T_2\|$. Let $F_2 = \mathrm{Ker}\,(T_{2\lambda_2})$. Then $F_2 = \mathrm{Ker}\,(T_{\lambda_2})$ and, since $F_2 \subset F_1^{\perp}$, $\lambda_1 \neq \lambda_2$.Let then $P_2 : H \to F_2$ be a projection; define $H_3 := (F_1 \oplus F_2)^{\perp}$. Then since $\|T_2\| \leqslant \|T_1\|$, we have $|\lambda_2| \leqslant |\lambda_1|$.

By induction, we construct a sequence $\{\lambda_1, \lambda_2, \dots\}$ of eigenvalues such that $|\lambda_1| \geqslant |\lambda_2| \geqslant \dots$ Furthermore, if, for any $n \geqslant 1$, we define $F_n := \mathrm{Ker}\,(T_{\lambda_n})$, then $|\lambda_{n+1}| = \left\| T|_{(F_1 \oplus \cdots \oplus F_n)^{\perp}} \right\|$ and $P_n : H \to F_n$ is a projection. The relationship $P_n P_m = 0 = P_m P_n$ whenever $n \neq m$ follows from the fact that the F_n's are pairwise mutually orthogonal. We know that for compact operators, the spectrum is always at most countably infinite. We have shown through previous constructions that

$$\{\lambda_1, \lambda_2, \dots\} = \sigma(T) \setminus \{0\}.$$

We now prove that the sequence $\{\lambda_n\}$ thus defined converge to 0. First $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots$ The sequence $\{|\lambda_n|\}$ converges to say $\alpha \geqslant 0$. For each $n \geqslant 1$, choose $x_n \in F_n$ with $\|x_n\| = 1$. Since T is compact, there exists $x \in H$ and a subsequence $\{x_{n_k}\}$ such that

$$\|T x_{n_k} - x\| \to 0.$$

So, for all $k, \ell \geqslant 1$, we have

$$\|T x_{n_k} - T x_{n_\ell}\|^2 = \lambda_{n_k}^2 + \lambda_{n_\ell}^2 \geqslant 2\alpha^2.$$

But since $\{T x_{n_k}\}$ is a Cauchy sequence, it must necessarily follow $\alpha = 0$.

Let $k \in \{1, \dots, n\}$ and $x \in F_k$. Then

$$\left(T - \sum_{j=1}^{n} \lambda_j P_j \right)(x) = Tx - \lambda_k x = 0.$$

Then $F_1 \oplus \cdots \oplus F_n \subset \text{Ker}\left(T - \sum_{j=1}^{n} \lambda_j P_j\right)$. If now $x \in (F_1 \oplus \cdots \oplus F_n)^\perp$, then $P_j x = 0$ for any $j \in \{1, \cdots, n\}$ and $\left(T - \sum_j \lambda_j P_j\right) x = Tx$. Since moreover T leaves $(F_1 \oplus \cdots \oplus F_n)^\perp$ invariant, we have that

$$\left\| T - \sum_{j=1}^{n} \lambda_j P_j \right\| = \left\| T^{\perp}_{(F_1 \oplus \cdots \oplus F_n)} \right\| = \|\lambda_{n+1}\| \to 0 \qquad (n \to \infty)$$

Therefore the series $\sum_n \lambda_n P_n$ converges to T with respect to the operator norm.

Example 1:

$T : \ell^2 \to \ell^2,\ Tx = \{\frac{x_i}{i}\}$ where $i \geqslant 1$.We know T is compact and

$$\langle Tx, y\rangle = \sum_{i=1}^{\infty} \frac{x_i}{i}\overline{y_i} = \sum_{i=1}^{\infty} x_i \overline{\frac{y_i}{i}} = \langle x, Ty\rangle.$$

Thus T is a compact self-adjoint linear operator. Then $T = \sum_{n=1}^{\infty} \lambda_n P_n$. In this particular example, we have $\sigma_p(T) = \sigma(T) \setminus \{0\} = \{\frac{1}{i} : i = 1, 2, \dots\}$. Therefore $T = \sum_{i=1}^{\infty} \frac{1}{i} P_i$ where $P_i x = \langle x, e_i\rangle e_i$ with $\{e_i\}_{i=1}^{\infty}$ being the canonical ONB for ℓ^2.

Example 2:

$T : \ell^2 \to \ell^2,\ Tx = \{\xi_k\}$ where

$$\xi_1 = \frac{x_1 + x_2}{\sqrt{2}}, \xi_2 = \frac{x_1 - x_2}{\sqrt{2}}, \xi_n = \frac{x_n}{n}, \quad n \neq 3.$$

Define $T_N : \ell^2 \to \ell^2$ by $T_N(x) = \sum_{k=1}^{N} \xi_k e_k$. It's easy to show $\| T - T_N\| \to 0\ (n \to \infty)$. Thus T is compact.

T self-adjoint? We have $\langle Te_1, e_2\rangle = \frac{1}{\sqrt{2}} = \langle e_1, Te_2\rangle$. If $i, j \geqslant 3$, we have $Te_i = \frac{1}{i} e_i$ and $Te_j = \frac{1}{j} e_j$ so that $\langle Te_i, e_j\rangle = \frac{1}{ij}\delta_{ij} = \frac{1}{ji}\delta_{ji} = \langle e_i, Te_j\rangle$. If $i \leqslant 2$ and $j \geqslant 3$, we have $\langle Te_i, e_j\rangle = 0 = \langle e_i, Te_j\rangle$.

Therefore T is self-adjoint.

For $n \geqslant 3$: eignevalues are $\lambda_n = \frac{1}{n}$. For $n = 1, 2$, consider $H = \mathrm{span}\,(e_1, e_2)$. Then $T|_H$ is given by

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Consider $\widetilde{Q} = \sqrt{2}\, Q$.

Solve $\det\left(\widetilde{Q}-\alpha I\right)=0$, that is $(1-\alpha)(-1-\alpha)-1=0$. We have $2-\alpha^2=0$, thus $\alpha=\pm\sqrt{2}$. Then the enigenvalues corresponding to Q are $\lambda=\frac{\alpha}{\sqrt{2}}$. So $\lambda_1=1$ and $\lambda_2=-1$. we check that the eigenvectors corresponding to these eigenvalues are

$$u_1=\frac{1}{\sqrt{4-2\sqrt{2}}}\left(e_1+\left(\sqrt{2}-1\right)e_2\right) \quad \text{and}$$

$$u_2=\frac{1}{\sqrt{4+2\sqrt{2}}}\left(e_1-\left(\sqrt{2}+1\right)e_2\right).$$

Thus $\{u_1, u_2, e_3, e_4, \cdots\}$ forms an ONB of eigenvectors on which T becomes

$$T = P_{u_1} - P_{u_2} + \sum_{i=3}^{\infty} \frac{1}{i} P_{e_i}.$$

Symmetric finite dimensional matrix	$T = \sum_{i=1}^{N} \lambda_i P_i$
Compact self-adjoint linear operator	$T = \sum_{i=1}^{\infty} \lambda_i P_i$
Bounded self-adjoint linear operator	$T = \int_{m-0}^{M} \lambda dE_\lambda$

Figure: Comparison with finite and continuous situations

Thank you!