On the Realizability of Neural Codes

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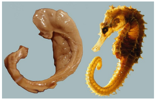
Neuroanatomy of the limbic system (Human)

- Limbic system, hippocampus Vs seahorse
- Short-term and long-term memories, **spatial memory**
- Disorders: alzheimer ...



- Firing, action potential
- Hodgking - Huxley model (H-H model)

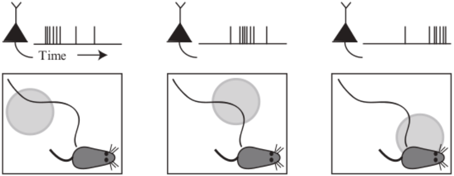
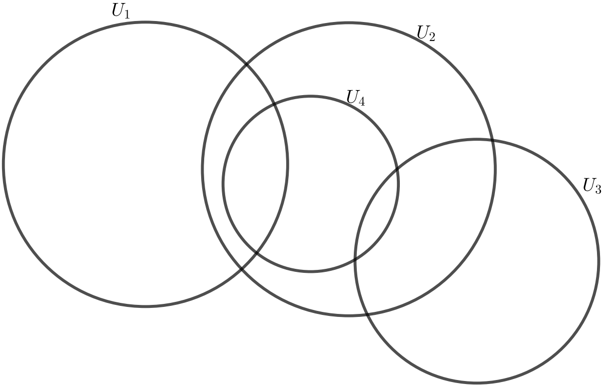


Figure: Neurons in the hippocampus (place cells) fire when the animal passes through place fields

source: article of Carina Curto, *What can topology tell us about the neural code ?*

Place fields in neuroscience



Codewords

In our example, the **neural code** is

$$\mathcal{C} = \{0000, 1100, 1101, 0100, 0001, 0111, 0110, 0010\}$$

An element $c \in \mathcal{C}$ is called a **codeword**.

Its **support** is $\text{supp}(c) = \{i \in [n] : c_i = 1\}$;

$\text{supp}(\mathcal{C}) := \{\sigma \subset [n]; \sigma = \text{supp}(c) \text{ for some } c \in \mathcal{C}\}$;

Notations: $x_\sigma = \prod_{i \in \sigma} x_i$ and $U_\sigma = \bigcap_{i \in \sigma} U_i$ where $\sigma \subset [n]$

The reverse engineering problem

- **Easy**: given $\mathcal{U} = \{U_1, \ldots, U_n\}$, construct the neural code;
- **Hard**: given a neural code $\mathcal{C}$, reconstruct $\mathcal{U}$ (realization);

Question

Can we always do this? in $2D$? $3D$? with connected sets? convex sets?

Some of these can be NP-hard

Key technique

Encode the structure algebraically, with *pseudomonomial ideals*.

- *Monomials*: products of variables, like $x_1x_2x_4x_5$.
- *Pseudomonomials*: allow "negations," like $x_1\overline{x_2}\,\overline{x_4}x_5$.

Some useful ideals

Consider the following in $\mathbb{F}_2[x_1, \ldots, x_n]$, where $\overline{x_j} = 1 + x_j$.

Characteristic polynomial: $\rho_{\mathbf{v}}(c) = 1$ *iff* $c = v$, *and* 0 *otherwise*

$$\rho_{\mathbf{v}} = \prod_{i \in \mathsf{supp}(\mathbf{v})} x_i \prod_{j \in [n] \setminus \mathsf{supp}(\mathbf{v})} \overline{x_j}$$

Vanishing ideal: "*polynomials that vanish on all codewords*"

$$I_{\mathcal{C}} = \{f \in \mathbb{F}_2[x_1, \ldots, x_n] \mid f(c) = 0 \text{ for all } c \in \mathcal{C}\}$$

Neural ideal: "*ideal generated by non-codewords*"

$$J_{\mathcal{C}} = \langle \rho_{\mathbf{v}} \mid \mathbf{v} \in \mathbb{F}_2^n \setminus \mathcal{C} \rangle$$

Basic properties

- $J_{\mathcal{C}} \subset I_{\mathcal{C}}$;
- $I_{\mathcal{C}} = J_{\mathcal{C}} + \mathcal{B}$, where $\mathcal{B} = \langle x_i \overline{x_i} |\ i \in [n] \rangle$, the "*Boolean ideal.*"
- $V(I_{\mathcal{C}}) = \mathcal{C} = V(J_{\mathcal{C}})$

An example

Consider the neural code $\mathcal{C} = \{000, 010, 110, 011\}$.

The non-codewords are $\mathbb{F}_2^3 \setminus \mathcal{C} = \{001, 100, 101, 111\}$.

They generate the neural ideal

$$\begin{aligned} J_{\mathcal{C}} &= \{\rho_\nu \mid \nu \in \{001, 100, 101, 111\}\} \\ &= \langle \overline{x_1}\,\overline{x_2}x_3, x_1\overline{x_2}\,\overline{x_3}, x_1\overline{x_2}x_3, x_1x_2x_3 \rangle \\ &= \langle x_1x_3, \overline{x_2}x_3, x_1\overline{x_2} \rangle = \langle x_1x_3, \overline{x_2}x_3, x_1\overline{x_2} \rangle. \end{aligned}$$

The neural ideal encodes combinatorial information:

$$\begin{aligned} x_1x_3 \in J_{\mathcal{C}} \subset I_{\mathcal{C}} \quad &\Rightarrow \quad c_1c_3 = 0 \text{ for all } \mathbf{c} \in \mathcal{C} \\ &\Rightarrow \quad U_1 \cap U_3 = \emptyset \\ &\Rightarrow \quad \text{neuons 1 \& 3 never fire at the same time.} \end{aligned}$$

More combinatorial structure encoded by the neural ideal

A **RF relationship** is a relation of the type

$$\bigcap_{i\in\sigma} U_i \subset \bigcup_{i\in\tau} U_i$$

Special cases:

- $i, j \in \mathsf{supp}(c) \implies c_i = 1 = c_j$ (both firing);
- Overlapping: $U_i \cap U_j \neq \emptyset$;
- $U_i \subset U_j$ (i doesn't fire without j firing);

Many of these can be characterized algebraically!

Proposition

If $\sigma \cap \tau = \emptyset$, then

$$\bigcap_{i\in\sigma} U_i \subset \bigcup_{j\in\tau} U_j \iff \prod_{i\in\sigma} x_i \prod_{j\in\tau} \overline{x_j} \in J_{\mathcal{C}}.$$

Simplicial complex & Neural Code

Simplicial complex of a code.
$\Delta\left(\mathcal{C}\right) := \{\sigma|\sigma \subset \text{supp}(c) \text{ for some } c \in \mathcal{C}\}$

[Nerve] The nerve of a cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ is

$$N(\mathcal{U}) := \left\{\sigma \subset [n]; \bigcap_{j\in\sigma} U_j \neq 0\right\}$$

For any code $\mathcal{C} \subset \{0,1\}^n$ and any realization $\mathcal{U}$ of $\mathcal{C}$, we have

$$\Delta(\mathcal{C}) = N(\mathcal{U})$$

A nice generating set of a neural ideal

The Canonical form of a neural ideal is

$$CF(J_{\mathcal{C}}) := \{f \in J_{\mathcal{C}} \mid f \text{ is a minimal pseudo-monomial}\}$$

Proposition

The canonical form generates the neural ideal: $J_{\mathcal{C}} = \langle CF(J_{\mathcal{C}}) \rangle$

For example: if $J_{\mathcal{C}} = \langle x_1\overline{x_2}x_3, x_1x_2x_3 \rangle$, then $CF(J_{\mathcal{C}}) = \{x_1x_3\}$.

There is an algorithm to compute it in SageMath (available in github.com/nebneuron/neural-ideal);

One can also use the Primary Decomposition to compute it.

$$J_{\mathcal{C}} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_\mathfrak{n}$$

Convex Realizability

Nerve Lemma: $\mathcal{U} = \{U_1, \ldots, U_n\}$ are convex.
Then, $\pi_k\left(\bigcup_{1 \le i \le n} U_i\right) = \pi_k\left(N(\mathcal{U})\right)$ (homotopy type)

In particular, $\bigcup_{1 \le i \le n} U_i$ and $N(\mathcal{U})$ have exactly the same homology groups.

Theorem If $\mathcal{C}$ is a simpicial complex, then $\mathcal{C}$ has a convex realization.

Proposition. A neural code $\mathcal{C}$ is a simplicial complex if and only if $CF(J_{\mathcal{C}})$ consists **only** of monomials (i.e., no $\overline{x_i}$).

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Thank you!