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Sudden death of entanglement: Classical noise effects

T. Yu *, J.H. Eberly

Rochester Theory Center for Optical Science and Engineering and the Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627, United States

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Abstract

When a composite quantum state interacts with its surroundings, both quantum coherence of individual particles and quantum entanglement will decay. We have shown that under vacuum noise, i.e., during spontaneous emission, two-qubit entanglement may terminate abruptly in a finite time [T. Yu, J.H. Eberly, Phys. Rev. Lett. **93** (2004) 140404], a phenomenon termed entanglement sudden death (ESD). An open issue is the behavior of mixed-state entanglement under the influence of classical noise. In this paper we investigate entanglement sudden death as it arises from the influence of classical phase noise on two qubits that are initially entangled but have no further mutual interaction.

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1. Introduction

On the occasion of Bruce Shore's recent 70th birthday, we are pleased to join the Quantum Optics community in a celebration of his career, and specifically by this paper to extend into the domain of entanglement his contributions to understanding of the effects of classical noise on qubits and qutrits. His own and other works are clearly summarized in Chapter 23 of his well-known two-volume work [1].

Recently, quantum entanglement has become one theme of research connected with proposals for realization of many quantum information protocols, such as quantum cryptography [2], quantum teleportation and quantum computations [3,4]. Multi-partite entanglement is a key issue in quantum information processing (QIP), and has been under extensive research in the last few years [5–9]. So far static entanglement has been investigated extensively, but dynamic entanglement under the influence of environmental noises is what counts in realistic QIP protocols. Ideally, one hopes that entanglement needed for quantum information processing can be maintained for sufficiently long times to permit designed tasks to be fulfilled. However, as far as we know, noisy evolution of entanglement of a quantum system is largely unexplored [10–16].

In a previous publication [13], contrary to intuition based on experience about qubit decoherence, we showed that entanglement may decrease abruptly and non-smoothly to zero in a finite time due to the influence of quantum noise, specifically from vacuum fluctuations. This non-smooth finite-time decay is called entanglement sudden death (ESD), which is a new kind of nonlocal decoherence. In this paper, we investigate ESD caused by interaction with noisy classical environments. More precisely, we consider two qubits being affected by pure classical dephasing noises both collectively and individually. Based on this model, we show that classical noise can also cause ESD in a class of common mixed states.

The format of the paper is as follows. In Section 2, we introduce a model of open quantum systems interacting with their classical environments. In Section 3, Wootters concurrence for mixed states is briefly reviewed and explicit calculations of an important class of mixed states are given.

* Corresponding author.
E-mail addresses: ting@pas.rochester.edu (T. Yu), eberly@pas.rochester.edu (J.H. Eberly).

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In Section 4 we show that there exist a family of mixed states that have finite disentanglement times in the case of classical phase damping noises. In Section 5 we offer some conclusions.

2. Classical noisy environments

Consider two entangled qubits under the influence of two different noise models that have been well studied for single qubits [17] and also considered for entangled states [10] previously. Here we focus on mixed states. We consider the qubits to be spins, and we subject them to noise in these two different ways: (i) we impose a stochastic magnetic field $B(t)$ on both qubits together, and (ii) we impose stochastic magnetic fields $b_A(t)$ and $b_B(t)$ separately on qubit A and qubit B, respectively. The noises are assumed to be statistically independent. For simplicity, spatial inhomogeneity of the noises is not considered in this paper. Irreversible coherence decay and entanglement decay will be unavoidable because we assume that the qubits are members of an extended QIP network that is too large to rely on the intricate symmetries that would be necessary to guarantee protection by decoherence-free subspaces.

2.1. One global collective noise

First consider the two qubits to be affected collectively by a single stochastic field. The hamiltonian of the qubits plus the classical noisy field is given by

$$H(t) = -\frac{1}{2}\mu B(t)(\sigma_z^A + \sigma_z^B), \tag{1}$$

where μ is the gyromagnetic ratio, and $\sigma_z^{A,B}$ are the Pauli matrices in the standard basis

$$\begin{aligned} |1\rangle_{AB} &= |++\rangle_{AB}, \quad |2\rangle_{AB} = |+-\rangle_{AB}, \\ |3\rangle_{AB} &= |-+\rangle_{AB}, \quad |4\rangle_{AB} = |--\rangle_{AB}, \end{aligned} \tag{2}$$

where $|\pm\pm\rangle_{AB}$ denote the eigenstates of the product Pauli spin operator $\sigma_z^A \otimes \sigma_z^B$ with eigenvalues ± 1. For simplicity, we assume that $B(t)$ satisfies the Markov condition

$$\langle B(t)\rangle = 0, \tag{3}$$

$$\langle B(t)B(t')\rangle = \frac{\Gamma}{\mu^2}\delta(t-t'), \tag{4}$$

where $\langle\ldots\rangle$ stands for an ensemble average and Γ gives the dephasing damping rate due to the collective interaction with $B(t)$.

The solution for the dynamic evolution under the Hamiltonian (1) can be obtained in several different ways (master equation, stochastic Schrödinger equation, etc.), and we use the operator-sum (Kraus) representation [13]. The reduced density matrix for the two qubits together can be obtained from the statistical density operator $\rho_{\rm st}(t)$ for both qubits and a classical Gaussian field by taking the ensemble average over the noise field $B(t)$

$$\rho(t) = \langle\rho_{\rm st}(t)\rangle, \tag{5}$$

where the statistical density operator $\rho_{\rm st}(t)$ is given by

$$\rho_{\rm st}(t) = U(t)\rho(0)U^\dagger(t), \tag{6}$$

with the unitary operator $U(t) = \exp\left[-i\int_0^t dt' H(t')\right]$. Clearly, the unitary operator $U(t)$ is dependent on the noise $B(t)$.

In our case the statistical unitary operator $U(t)$ can be explicitly written as

$$U(t) = \exp\left[i\frac{\mu}{2}\int_0^t dt' B(t')(\sigma_z^A + \sigma_z^B)\right]. \tag{7}$$

By taking the statistical mean of (6) over the noise $B(t)$, it can be shown that the most general solution (5) can be expressed in a very compact way in terms of Kraus operators [18]

$$\rho(t) = \mathscr{E}_D(\rho(0)) = \sum_{\mu=1}^{3} D_\mu^\dagger(t)\rho(0)D_\mu(t), \tag{8}$$

where the Kraus operators describing the collective interaction are given by

$$D_1 = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}, \tag{9}$$

$$D_2 = \begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \end{pmatrix}, \tag{10}$$

$$D_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_3 \end{pmatrix}, \tag{11}$$

$$\gamma = e^{-t/2T_2}, \quad \omega_1 = \sqrt{1 - e^{-t/T_2}}, \tag{12}$$

$$\omega_2 = -\omega_1 e^{-t/T_2}, \quad \omega_3 = \omega_1^2\sqrt{1 + e^{-t/T_2}}, \tag{13}$$

where $T_2 = 1/\Gamma$ is the phase relaxation time due to the collective interaction with $B(t)$.

2.2. Two local noises

For the local dephasing model in which two qubits interact with their own environments represented by two independent classical noises, the Hamiltonian of the two-qubit system plus the classical noises is given by

$$H(t) = -\frac{1}{2}\mu(b_A(t)\sigma_z^A + b_B(t)\sigma_z^B), \tag{14}$$

where the noises $b_A(t)$ and $b_B(t)$ are assumed to be statistically independent and satisfy

$$\langle b_i(t)\rangle = 0, \tag{15}$$

$$\langle b_i(t)b_i(t')\rangle = \frac{\Gamma_i}{\mu^2}\delta(t-t'), \quad i = A, B. \tag{16}$$

where $\Gamma_i\,(i = A, B)$ are the phase damping rates of qubits A and B due to the coupling to the stochastic magnetic fields $b_1(t)$, $b_2(t)$, respectively.

Similar to the collective noise case, the general solution of density matrix $\rho(t)$ of the two qubits can be expressed in terms of four Kraus operators

$$\rho(t) = \mathscr{E}_{AB}(\rho(0)) = \sum_{\mu,\nu=1}^{2} E_\mu^\dagger(t) F_\nu^\dagger(t)\rho(0)F_\nu(t)E_\mu(t), \tag{17}$$

where the Kraus operators describing the interaction with the local environments are given by

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_A \end{pmatrix} \otimes I, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & \omega_A \end{pmatrix} \otimes I, \tag{18}$$

$$F_1 = I \otimes \begin{pmatrix} 1 & 0 \\ 0 & \gamma_B \end{pmatrix}, \quad F_2 = I \otimes \begin{pmatrix} 0 & 0 \\ 0 & \omega_B \end{pmatrix}. \tag{19}$$

The parameters appearing in (18) and (19) are given by

$$\gamma_A = \mathrm{e}^{-t/2T_2^A}, \quad \gamma_B = \mathrm{e}^{-t/2T_2^B}, \tag{20}$$

$$\omega_A = \sqrt{1 - \mathrm{e}^{-t/T_2^A}}, \quad \omega_B = \sqrt{1 - \mathrm{e}^{-t/T_2^B}}, \tag{21}$$

where $T_2^A = 1/\Gamma_A$ and $T_2^B = 1/\Gamma_B$ are the phase relaxation times for qubit A and qubit B due to the interaction with their own environments $b_A(t)$, $b_B(t)$, respectively.

3. Measuring entanglement

To describe the dynamic evolution of quantum entanglement we need a concrete measure of the degree of entanglement contained in a quantum state. For any two-qubit case Wootters concurrence [19] is particularly convenient. Any reliable measure of entanglement will yield the same conclusions. The concurrence varies from $C = 0$ for a separable state to $C = 1$ for a maximally entangled state. For two qubits, the concurrence may be calculated explicitly from the density matrix ρ for qubits A and B

$$C(\rho) = \max\left(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right), \tag{22}$$

where the quantities λ_i are the eigenvalues in decreasing order of the matrix

$$\zeta = \rho(\sigma_y^A \otimes \sigma_y^B)\rho^*(\sigma_y^A \otimes \sigma_y^B), \tag{23}$$

where ρ^* denotes the complex conjugation of ρ in the standard basis (2) and σ_y is the Pauli matrix expressed in the same basis as:

$$\sigma_y^{A,B} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{24}$$

In the following we will examine the evolution of entanglement under noise-induced relaxation of a class of bipartite density matrices having the "standard" form

$$\rho^{AB} = \begin{pmatrix} a & 0 & 0 & w \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ w^* & 0 & 0 & d \end{pmatrix}. \tag{25}$$

where $a + b + c + d = 1$. This class of mixed state arises naturally in a wide variety of physical situations (see [20]). Particularly, it includes pure Bell states as well as the well-known Werner mixed state [21] as special cases.

It may be surprising that this class is invariant under dephasing evolution. Of course the diagonal elements retain their initial values, but all other elements only pick up a time-dependent factor multiplying the initial value, so all the initially zero elements remain zero. A direct calculation shows that the concurrence of any mixed state of this type is thus given by

$$C(\rho^{AB}) = 2\max\left\{0, |w(t)| - \sqrt{bc}, \quad |z(t)| - \sqrt{ad}\right\}. \tag{26}$$

In order for the entanglement to be zero, both of the following inequalities must be satisfied:

$$|w(t)| - \sqrt{bc} \leqslant 0, \quad |z(t)| - \sqrt{ad} \leqslant 0. \tag{27}$$

In the following sections we show that they can both be satisfied for finite times t, i.e., that classical noise on entangled spins is a case where sudden death can occur.

4. Entanglement sudden death under classical noise

4.1. Global dephasing noise

For this case we return to (8) and use the three Kraus operators to obtain:

$$\rho(t) = \sum_{\mu=1}^{3} D_\mu^\dagger(t)\rho(0)D_\mu(t), \tag{28}$$

where D_μ are given in (9)–(11).

For the initial mixed states (25), the explicit solution of (28) in the standard basis (2) can be then expressed as

$$\rho^{AB}(t) = \mathscr{E}_D(\rho^{AB}(0)) = \begin{pmatrix} a & 0 & 0 & \gamma^4 w \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ \gamma^4 w^* & 0 & 0 & d \end{pmatrix}, \tag{29}$$

where $\gamma = \mathrm{e}^{-t/2T_2}$ is defined in (12).

From (29), we see that the collective noise only affects the off-diagonal elements ρ_{14} and ρ_{41} and leaves the off-diagonal elements z, z^* intact. As we noted previously in discussing pure state decoherence [10], the collective global field allows certain phase combinations to cancel out, generating what is effectively a decoherence-free subspace spanned by $|+-\rangle$, $|-+\rangle$. For our present mixed-state purpose, we'll avoid this coincidence by assuming that $z(0) = 0$, so the mixed state is not protected in this way. The concurrence of the mixed state at t can be easily computed as:

$$C(\rho^{AB}(t)) = 2\max\left\{0, |w(t)| - \sqrt{bc}\right\}. \tag{30}$$

Therefore, the state $\rho^{AB}(t)$ (29) is separable if and only if $|w(t)| - \sqrt{bc} \leqslant 0$. From this we see that there is a critical time t_c for the end of entanglement, such that

$$t_c = \frac{1}{2\Gamma} \ln \frac{|w|}{\sqrt{bc}}, \tag{31}$$

at which $C(\rho^{AB}(t_c)) = 0$. From (31), we see that for the entangled density matrix with $b \neq 0$ and $c \neq 0$, the entanglement sudden death will occur at t_c. These features are illustrated in Fig. 1, where the sudden death time is $\Gamma t_c = \frac{1}{2}\ln 2$.

4.2. Two-qubit local dephasing noises

In the case of local dephasing the independence of the two local noises prevents appearance of a decoherence-free subspace, and the general solution is given by (17), which here takes form

$$\rho(t) = \begin{pmatrix} \rho_{11} & \gamma_B\rho_{12} & \gamma_A\rho_{13} & \gamma_A\gamma_B\rho_{14} \\ \gamma_B\rho_{21} & \rho_{22} & \gamma_A\gamma_B\rho_{23} & \gamma_A\rho_{24} \\ \gamma_A\rho_{31} & \gamma_A\gamma_B\rho_{32} & \rho_{33} & \gamma_B\rho_{34} \\ \gamma_A\gamma_B\rho_{41} & \gamma_A\rho_{42} & \gamma_B\rho_{43} & \rho_{44} \end{pmatrix}. \tag{32}$$

The general solution (26) easily shows that under independent phase noises one will always find entanglement sudden death for the initial mixed density matrix (25).

An interesting sub-category is worth attention. This is the case when one of the two qubits experiences very weak noise or no noise. One such limiting case is, for example, $\Gamma_B = 0$. The one-qubit dephasing model $\mathscr{E}_A$ can be described by the Kraus operators E_1, E_2 and one can write the explicit solution for any initial state $\rho(0)$, but the result is easy to anticipate.We find the result (32) again, but have to put $\Gamma_B = 0$, and otherwise the critical time is the same. Naturally it is a longer time, but still finite. Again we find ESD. As a most striking example, we have shown that the effect of the dephasing noises on entanglement and quantum coherence of a single qubit is indeed very different. Particularly, we have shown here that for some mixed states, entanglement may experience a sudden death process even if the local coherence of one participating particle can be well preserved and the other only decays to zero asymptotically. This may be one of the best examples to show the difference between entanglement decay and local coherence decay.

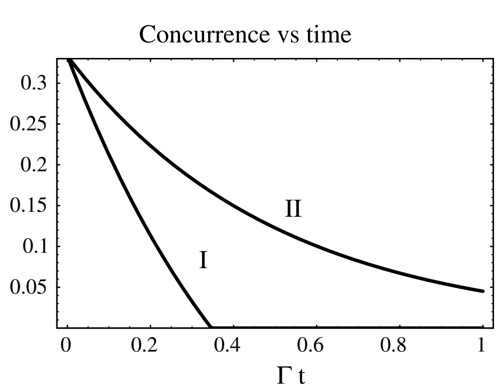

Fig. 1. In the presence of global dephasing noise there is long-lived concurrence of mixed entangled states for $b = 0$ or $c = 0$. Otherwise, entanglement sudden death is unavoidable. The graph shows the sudden death process (I) with initial values $z = 0$, $w = 1/3$, $a = d = 1/3$, $b = c = 1/6$ and the exponential decay in (II) with $z = 0$, $w = 1/6$, $a = c = d = 1/3$, $b = 0$. Both start from $C = 1/3$.

5. Conclusion

In this paper for a standard set of initial mixed two-qubit states we have investigated quantum entanglement decay due to interaction with classical noises. We have shown that noisy classical environments may cause entanglement to vanish completely in a finite time, while they allow the coherence of either one or both of the engaged qubits to remain non-zero for an infinitely long time. A deeper understanding of entanglement decay processes is of interest in quantum computation and in any branch of quantum information science where preservation of entanglement is essential for the action of desired operations and devices.

A few closing comments are in order: (1) The degree of entanglement in this paper is measured by Wootters' concurrence, which is defined for both pure and mixed states. However, we emphasize that entanglement sudden death (ESD) is independent of the choice of measures of entanglement. This can be easily seen as follows: Entanglement sudden death occurs when a quantum entangled state becomes separable beginning at a certain time, but separability of a state is independent of entanglement measure. Concurrence is a conveniently normalized measure. (2) We note that because we have found unexpected evolution of standard mixed states in the presence of white noise sources, it will be very interesting to extend these results to the case of colored noises (e.g., see [9]). (3) We note that entanglement sudden death is a generic feature for a still larger class of mixed states than we have specified in this paper. (4) It will of great interest if analogies of entanglement sudden death can be identified in multi-partite systems.

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