LUND UNIVERSITY
MATHEMATICAL STATISTICS

Numerical methods and Statistical Inference for Linear SPDEs driven by Gaussian Noise.

Elmir Nahodovic
Erik Karlsson Strandh

April 8, 2024

Contents

Notes

Chapter 1

Introduction

1.1 What is an SPDE?

In the most general setting, a stochastic partial differential equation (SPDE) of order $k \in \mathbb{N}$ is a *formal* equation that involves the unknown multivariate function (stochastic process) $u : \Omega \times U \to \mathbb{R}$ and its Jacobian derivatives $\mathcal{D}^i u$, $i = 1, \dots, k$ such that, for all $y \in U$ and $\omega \in \Omega$:

$$\alpha(\omega, \mathcal{D}^k u(y), \dots, \mathcal{D}u(y), u(y), y) = \beta(\omega, \mathcal{D}^k u(y), \dots, \mathcal{D}u(y), u(y), y)\dot{\mathcal{F}}(\omega, y), \quad (1.1.1)$$

with initial- and boundary conditions if applicable. Here we have

$$\alpha, \beta : \Omega \times \mathbb{R}^{n^k} \times \dots \times \mathbb{R}^d \times \mathbb{R} \times U \to \mathbb{R}.$$

The factor $\dot{\mathcal{F}}$ is called the *noise* in the system, and the initial- and boundary conditions may also be stochastic. The space Ω is the sample space from the probability space $(\Omega, \Sigma, \mathcal{P})$ while U is usually some subspace of the Euclidean space $\mathbb{R}^d$.

The reason we are stating that equation 1.1.1 is a *formal* equation is because in general the noise and randomness of the system often doesn't admit a point-wise (let alone differentiable) solution to 1.1.1. It will become clear later what we mean by a rigorous solution to an SPDE and how it connects to the corresponding PDE without randomness.

1.1.1 Linear SPDE's

For this thesis, we consider SPDE's of the unknown process $u : \Omega \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ of the form

$$\mathcal{L}u(x,t) = a\Big(x,t,u(x,t)\Big) + b\Big(x,t,u(x,t)\Big)\dot{\mathcal{F}}(x,t),\ \ t > 0,\ x \in U \subseteq \mathbb{R}^d, \tag{1.1.2}$$

with deterministic initial conditions and if necessary also boundary condition. In the equations and calculations we usually omit the variable $\omega \in \Omega$. The operator $\mathcal{L}$ is a linear partial differential operator and $\dot{\mathcal{F}}$ is a Gaussian noise where we start with what is called white noise and later expand to other noise processes.

From PDE theory, if we assumed that eq. 1.1.2 did not contain any stochastic elements and with some further conditions[1] on the functions, we know that there exists a solution formula for the deterministic version of eq. 1.1.2 of the form

Add section on deterministic heat equation and show the weak formulation and mild solution in full.

$$u(x,t) = I_0(x,t) + \int_0^t \int_U \Psi(x,t;y,s)a(y,s,u(y,s))dsdy \tag{1.1.3}$$
$$+ \int_0^t \int_U \Psi(x,t;y,s)b(y,s,u(y,s))\mathcal{F}(dsdy). \tag{1.1.4}$$

Here $\Psi(x,t;y,s)$ is the fundamental solution to the determinstic PDE version of eq. 1.1.2 and I_0 comes from the initial- and boundary conditions. One big conundrum of this thesis is to give a rigorous meaning to the *stochastic* integral term above:

$$\int_0^t \int_U \Psi(x,t;y,s)b(y,s,u(y,s))\mathcal{F}(dsdy).$$

1.1.1.1 The Stochastic Heat Equation

The simplest example and the one we will devote the most time to in this thesis is the stochastic heat equation, formally given as

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \frac{1}{2}\Delta u(x,t) = \dot{\mathcal{W}}(x,t) & t > 0,\ x \in \mathbb{R}^d \\ u(x,0) = 0 & x \in \mathbb{R}^d. \end{cases} \tag{1.1.5}$$

Where $\dot{\mathcal{W}}(x,t)$ is supposed to represent a *white noise signal* at time t and point x to the system. What we would like is that $\dot{\mathcal{W}}$ should be a Gaussian stochastic process in the finite second moment space $L^2(\Omega)$ where the "white" of the noise means that $\dot{\mathcal{W}}(x_1,t_1)$ and $\dot{\mathcal{W}}(x_2,t_2)$ are independent for $(x_1,t_1) \neq (x_2,t_2)$ and that $E[\dot{\mathcal{W}}(x,t)] =$

[1] Need some good reference for this to get all the conditions correct and so on, currently this is found from Dalang and Sanz-Solé's new unpublished book [10] ch. 3. Maybe not so hard to derive on our own?

0. However, a process like $\dot{\mathcal{W}}(x,t)$ cannot exist. A short motivation is that if it did exist then we could construct uncountably many random variables in $L^2(\Omega)$ that are mutually orthogonal from these $\dot{\mathcal{W}}(x,t)$. But $L^2(\Omega)$ is separable, which implies that there are no such processes $\dot{\mathcal{W}}(x,t)$.

Not really certain about this discussion on existence of white noise.

If we simply close our eyes and pretend that $\dot{\mathcal{W}}(x,t)$ would be a regular integrable function it is not hard to derive the formula[2]

Expand discussion on what a white noise process is.

Again we instead add section on deterministic heat equation.

$$u(x,t) = \int_0^t \int_{\mathbb{R}^d} \frac{e^{-\frac{|x-y|^2}{2(t-s)}}}{(2\pi|t-s|)^{n/2}} \mathcal{W}(ds\,dy) \tag{1.1.6}$$

Just like the previous section stated we will come back to this strange integral and give a precise meaning to it.

1.1.1.2 The Stochastic Wave Equation

Additional example.

Add example on stochastic wave equation.

1.1.2 A primer on SDEs

Recall the basic stochastic differential equation on:

$$dX(t) = \mu X(t)dt + \sigma X(t)dB(t), \quad X(0) = x_0. \tag{1.1.7}$$

Which is the formal symbol for the integral equation

$$X(t) = x_0 + \mu \int_0^t X(s)ds + \sigma \int_0^t X(s)dB(s). \tag{1.1.8}$$

Idea of this section:

Add discussion on SDEs and white noise here?

1. The Riemann-Stiltjes integral (since the construction is so important for the idea of stochastic integration in several dimensions and how the one dimensional integral is just a special case)
2. Give simple SDE $dX_t = \mu X_t dt + \sigma dB_t$, $X_0 = x_0$ and look at the "equivalent" $\frac{dX_t}{dt} = \mu X_t + \sigma \frac{dB_t}{dt}$
3. What does $\frac{dB_t}{dt}$ mean?

[2] See appendix for a rough derivation.

   (a) Give the idea of a white-noise process as $\frac{dB_t}{dt}$. With the inner-product thingy from Dalang and Sanz-Solé [10].

4. We obtain the idea of what the white-noise means on $\mathbb{R}_+$.

Chapter 2

SPDEs with Gaussian Noise

2.1 Stochastic Calculus with White Noise

Here we go through the theory of SPDEs. The equations, definition of a solution, showing existence and properties of solution to SPDEs with white noise and white-coloured noise. This section aims to give the central definitions and properties needed to study stochastic partial differential equations. We will start with some basics on stochastic processes and Gaussian random fields. The presentation is largely inspired by [7]. We will be working on the complete (subsets of null sets are measurable) probability space $(\Omega, \Sigma, \mathcal{P})$. The space $L^2(\Omega)$ is the set of real-valued random variables with finite variance. It is a Hilbert space with norm $\|X\|_{L^2(\Omega)} := \sqrt{E(X^2)}$ for $X \in L^2(\Omega)$.

2.1.1 Gaussian processes

Definition 2.1.1 (Gaussian random vector)**.** *Let $g = (g_1, \ldots, g_n)$ be a random vector. We say that the distribution of g is Gaussian if $\alpha \cdot g := \sum_{j=1}^{n} \alpha_j g_j$ is Gaussian random variable for all $\alpha \in \mathbb{R}^d$.*

Recall that for a given probability space $(\Omega, \Sigma, \mathcal{P})$ and the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with some arbitrary index set $\mathcal{T}$, a *stochastic process* is given by the real-valued stochastic variables $X(t) := \{X(t, \omega)\ ,\ t \in \mathcal{T}\}$, where $\omega \in \Omega$ is any basic event for all t.

Definition 2.1.2 (Gaussian random field)**.** *The stochastic process $X = \{X(t, \omega)\ ,\ t \in \mathcal{T}\}$ is called a Gaussian random field or Gaussian stochastic process if for all integers $k > 0$ and $t_1, \ldots, t_k \in \mathcal{T}$, the random vector $(X(t_1), X(t_2), \ldots, X(t_k))$ is Gaussian.*

The finite dimensional distributions are the collection of probabilities obtained as follows:

$$p_{t_1,\dots,t_k}(A_1,\dots,A_k) := \mathcal{P}(X(t_1) \in A_1,\dots,X(t_k) \in A_k) \tag{2.1.1}$$

For $s,t \in \mathcal{T}$, the covariance function $C(s,t) = E(X(s)X(t)) - E(X(s))E(X(t))$ is a symmetric $C(s,t) = C(t,s)$- and non-negative definite kernel. There is a well known result that gives the existence of a Gaussian random field with a given covariance- and mean function.

We do not give proof of Kolmogorov Existence theorem. Find references instead and give them.

Lemma 2.1.3. *(1) Let X be a Gaussian random field. The probablity measures $p_{t_1,\dots,t_k}$ defined in 2.1.1 are determined by the mean function $m(t) = E(X(t))$ and the covariance function $C(s,t)$*
(2) Given functions $m : \mathcal{T} \to \mathbb{R}$ and a symmetric non-negative kernel $C : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$, then there exists a Gaussian random field $X(t)$ with mean function m and and covariance function C.

Some important Gaussian random fields are the Brownian motion $B(t)$, its continuous modification $W(t)$ (Wiener process), the Brownian Sheet $W(x,t)$, etc Our goal is to represent stochastic integrals with respect to Gaussian random fields, on functions h from a separable Hilbert space H with inner product $\langle \cdot,\cdot \rangle_H$ and norm $||\cdot||_H$.

We should give the definitions of some standard Gaussian random fields and so on.

Definition 2.1.4. *Let $(H, \langle \cdot,\cdot \rangle_H, \|\cdot\|_H)$ be a separable inner product space. A stochastic process $I(h)$ indexed by $\mathcal{T} = H$ on a complete probability space $(\Omega, \Sigma, \mathcal{P})$ is called an isonormal Gaussian process if for all $h, g \in H$, then $I(h) \in N(0, ||h||_H^2)$, and $E\Big(I(h)I(g)\Big) = \langle h, g\rangle_H$.*

We will see that the isonormal Gaussian processes $I(h)$ indexed by separable Hilbert spaces are a natural interpretation of the stochastic integral of deterministic processes with respect to some Gaussian noise process. Informally we want $I(h) = \int_E h d\mathcal{W}$ where E is some measure space with measure μ and we have a white noise $\mathcal{W}$ that is based on μ.

Proposition 2.1.5. *If $I(h)$ is an isonormal Gaussian process, then the map $I : H \to L^2(\Omega)$ such that $h \mapsto I(h)$, is a linear isometry.*

Proof. By definition 2.1.4, $Var(I(h)) = ||I(h)||^2_{L^2(\Omega)} = ||h||_H^2$. Now we show that it is linear in the sense that

$$I(ag + bh) \overset{d.}{=} aI(h) + bI(g)$$

for $a, b \in \mathbb{R}$ and $h, g \in H$. Both left- and right-hand side have zero mean. It suffices to show that the difference $I(ag + bg) - (aI(h) + bI(g))$ has zero variance.

$$E((I(ag+bg) - aI(h) - bI(g))^2) = ||ah+bg||_H^2 + a^2||h||_H^2 + b^2||g||_H^2$$
$$-2a\langle ah+bg, h\rangle_H - 2b\langle ah+bg, g\rangle_H + 2ab\langle h, g\rangle_H = 0$$

□

Proposition 2.1.5 guarantees that an isonormal Gaussian process is indeed a Gaussian random field, since the linear combination $aI(h) + bI(g) \in N(0, ||ag + bh||_H^2)$. This proposition will later also give us the natural property that stochastic integrals are linear in at least one sense. It is also important that equal in distribution could still mean that $I(ag+bh)$ and $aI(h)+bI(g)$ are equal with probability zero. For an example consider any $S \in N(0,1)$ and look at $-S \in N(0,1)$. Then

$$\mathcal{P}(\omega \in \Omega : \ S(\omega) = -S(\omega)\} = \mathcal{P}(\omega \in \Omega : \ S(\omega) = 0) = 0.$$

2.1.2 White Noise

Now it remains to define the Gaussian noise that we will integrate over. For our presentation we will use a σ-finite measurable subspace $(E, \mathcal{B}(E), \mu)$ of $\mathbb{R}^k$. Where $E \subseteq \mathbb{R}^k$. Let $\mathcal{B}_b(E)$ be the collection of Borel-measurable subsets of E with finite measure.

Definition 2.1.6 (White Noise). *A white noise based on μ is a Gaussian random field $\mathcal{W} = \{\mathcal{W}(A), \ A \in \mathcal{B}_b(E)\}$ defined on some probability space $(\Omega, \Sigma, \mathcal{P})$ with $E(\mathcal{W}(A)) = 0$ and covariance function*

$$C(A, B) = E\left(\mathcal{W}(A)\mathcal{W}(B)\right) = \mu(A \cap B).$$

The function C above is indeed a covariance function. It is obviously symmetric $C(A,B) = C(B,A)$. To see that it is non-negative definite, take $x_1, \ldots, x_n \in \mathbb{R}$ and $A_1, \ldots, A_n \in \mathcal{B}_b(E)$. Then

Walsh showed white noise covariance is covariance like this, not completely sure why last equality holds.

$$\sum_{k,l=1}^{n} x_k x_l C(A_k, A_l) = \sum_{k,l=1}^{n} x_k x_l \mu(A_k \cap A_l)$$

$$= \sum_{k,l=1}^{n} x_k x_l \left(\int_E \mathbb{1}_{A_k}(y) \mathbb{1}_{A_l}(y) \mu(dy) \right) = \int_E \left(\sum_{k=1}^{n} x_k \mathbb{1}_{A_k}(y) \right)^2 \mu(dy) \geq 0.$$

Thus the existence of the Gaussian random field $\mathcal{W}(A)$ with index set $\mathcal{T} = \mathcal{B}_b(E)$ follows from lemma 2.1.3.

Remark 2.1.7. *An important note is that the "white" in "white noise" refers to the covariance structure of the Gaussian random field $\mathcal{W}$. Notice how*

$$C(A, B) = \mu(A \cap B)$$

$$= \int_E \mathbb{1}_A(y) \mathbb{1}_B(y) \mu(dy) = \langle \mathbb{1}_A, \mathbb{1}_B \rangle_{L^2(E)}.$$

Therefore there exists a natural harmony between the finite second moment space $L^2(\Omega)$ and the Hilbert space $L^2(E)$. Later when we venture into coloured noise the fundamental difference is that we look at another covariance structure for the above Gaussian random field, which will give a correspondence to some other, hopefully larger, Hilbert space of functions to integrate.

Proposition 2.1.8. *Let $A, B \in \mathcal{B}_b(E)$ be two disjoint subsets of E. Then $\mathcal{W}(A)$ and $\mathcal{W}(B)$ are independent and*

$$\mathcal{W}(A \cup B) \overset{d}{=} \mathcal{W}(A) + \mathcal{W}(B).$$

Proof. Since A and B are disjoint, $C(A, B) = \mu(A \cap B) = \mu(\emptyset) = 0$. Therefore they are uncorrelated, and hence independent, since $\mathcal{W}(A)$, and $\mathcal{W}(B)$ are Gaussian.

To show additivity we check that $\mathcal{W}(A \cup B) - (\mathcal{W}(A) + \mathcal{W}(B))$ has zero variance. By linearity of expectation and the definition of covariance for white noise:

$$\begin{aligned}
&E((\mathcal{W}(A \cup B) - \mathcal{W}(A) - \mathcal{W}(B))^2) \\
=&E((\mathcal{W}(A \cup B)^2) + E(\mathcal{W}(A)^2) + E(\mathcal{W}(B)^2) - 2E((\mathcal{W}(A \cup B)\mathcal{W}(A))^2) \\
-&2E((\mathcal{W}(A \cup B)\mathcal{W}(B))^2) + E(\mathcal{W}(A)\mathcal{W}(B)) \\
=&\mu(A \cup B) + \mu(A) + \mu(B) - 2\mu(A) - 2\mu(B) + 0 = 0.
\end{aligned}$$

□

Definition 2.1.9 (The Brownian Sheet). *Let $t = (t_1, \dots, t_n)$. The Brownian sheet $\{\mathcal{W}(t), t \in \mathbb{R}^d_+\}$ defined by $\mathcal{W}_t := \mathcal{W}\{(0, t]\} := \mathcal{W}\{(0, t_1] \times \ldots \times (0, t_n]\}$. This is a zero mean, Gaussian process with covariance function $E\{\mathcal{W}_s \mathcal{W}_t\} = \min(s_1, t_2) \cdot \ldots \cdot \min(s_n, t_n)$.*

For example if $n = 2$, the Brownian sheet over rectangles $R = (s, t] \times [x, y]$ (by proposition 2.1.8) is equal to $\mathcal{W}(R) = \mathcal{W}_{ty} - \mathcal{W}_{tx} - \mathcal{W}_{sy} - \mathcal{W}_{sx}$.

2.1.3 The stochastic integral

We take a general Lebesgue/Measure theory approach to the construction of a stochastic integral. Starting with integrals of simple functions, and extending with a density argument because of the linear isometry that will be created. Let our integrands come from the Hilbert space $H = L^2(E, \mu)$. We can now construct an isonormal Gaussian process $I(h)$ on H, given a white noise $\mathcal{W}$ based on μ.

Definition 2.1.10. *For simple functions $h = \sum_{k=1}^n a_k \mathbb{1}_{A_k} \in L^2(E,\mu)$ with $a_k \in \mathbb{R}$, and $A_k \in \mathcal{B}_b(E)$ pairwise disjoint. Then*

$$I(h) = I\left(\sum_{k=1}^n a_k \mathbb{1}_{A_k}\right) := \sum_{k=1}^n a_k \mathcal{W}(A_k). \tag{2.1.2}$$

Proposition 2.1.11. *The process $I(h)$ from the set of simple functions on $L^2(E)$ to random variables in $L^2(\Omega)$ is an isonormal Gaussian process.*

Proof. $I(h)$ is a finite sum of zero-mean normal variables, and hence it is also normal with zero mean. To see that $E(I(h)I(g)) = \langle I(h), I(g)\rangle_{L^2(\Omega)} = \langle h, g\rangle_{L^2(E)}$, we first observe that I is an isometry:

$$\begin{aligned}
\|I(h)\|^2_{L^2(\Omega)} &= \left\| I\left(\sum_{k=1}^n a_k \mathbb{1}_{A_k}\right)\right\|^2_{L^2(\Omega)} = E\left(\Big(\sum_{k=1}^n a_k \mathcal{W}(A_k)\Big)^2\right) \\
&= \sum_{k=1}^n a_k^2 E\Big(\mathcal{W}(A_k)^2\Big) = \sum_{k=1}^n a_k^2 \mu(A_k) = \sum_{k=1}^n \int_E a_k^2 \mathbb{1}_{A_k}(y) d\mu(y) \\
&= \int_E \sum_{k=1}^n \Big(a_k^2 \mathbb{1}_{A_k}(y)\Big) d\mu(y) = \int_E \left(\sum_{k=1}^n a_k^2 \mathbb{1}_{A_k}(y)\right)^2 d\mu(y) \\
&= \int_E h^2 d\mu = \|h\|^2_{L^2(E)}
\end{aligned}$$

Since the map $h \mapsto I(h)$ is an isometry between two inner products spaces, we know that the inner products are preserved in the mapping. Therefore $E(I(h)I(g)) = \langle I(h), I(g)\rangle_{L^2(\Omega)} = \langle h, g\rangle_{L^2(E)}$. □

By the preceeding proposition as well as proposition 2.1.5 we obtain the following important corollary.

Corollary 2.1.12. *The isonormal Gaussian process $I(h)$ is a linear isometry from the set of simple functions on $L^2(E)$ to $L^2(\Omega)$.*

This gives that stochastic integrals are linear in distribution, but I think it holds a.s as well. Maybe show that?

Remark 2.1.13. *The definition in 2.1.10 is well defined in the sense that if we have another representation of $\tilde{h} = \sum_{l=1}^m b_l \mathbb{1}_{B_l}$ with $b_l \in \mathbb{R}$ and pairwise disjoint $B_l \in \mathcal{B}_b(E)$, such that $h = \tilde{h}$, then $I(h) \overset{d.}{=} I(\tilde{h})$.*

Proof. We show that the difference $I(h) - I(\tilde{h})$ has zero variance.

$$
\begin{aligned}
&E\Bigg(\big(\sum_{k=1}^{n} a_k \mathcal{W}(A_k) - \sum_{l=1}^{m} b_k \mathcal{W}(A_l) \big)^2 \Bigg) = E\Bigg(\big(\sum_{k=1}^{n} a_k \mathcal{W}(A_k) \big)^2 \Bigg) \\
&+E\Bigg(\big(\sum_{l=1}^{m} b_l \mathcal{W}(B_l) \big)^2 \Bigg) - 2E\Bigg(\big(\sum_{k=1}^{n} \sum_{l=1}^{m} a_k b_l \mathcal{W}(A_k)\mathcal{W}(B_l) \big)^2 \Bigg) \\
&= \int_E \left(\sum_{k=1}^{n} a_k^2 \mathbb{1}_{A_k} + \sum_{l=1}^{m} b_l^2 \mathbb{1}_{B_l} - 2 \sum_{k=1}^{n} \sum_{l=1}^{m} a_k b_l \mathbb{1}_{A_k \cap B_l} \right) d\mu \\
&= \int_E \left(\sum_{k=1}^{n} a_k \mathbb{1}_{A_k} - \sum_{l=1}^{m} b_l \mathbb{1}_{B_l} \right)^2 d\mu = \int_E \left(h - \tilde{h} \right)^2 d\mu = 0.
\end{aligned}
$$

□

Since we have a linear isometry from the (dense) set of simple functions on $L^2(E)$ to the complete normed space $L^2(\Omega)$, the map $h \mapsto I(h)$ can be extended uniquely to $L^2(E)$. Take $h \in L^2(E)$ and a sequence of simple functions h_n such that $||h - h_n||_{L^2(E)} \to 0$. Then we define

$$
\int_E h d\mathcal{W} := I(h) := \lim_{n \to \infty} I(h_n). \tag{2.1.3}
$$

The above definition does not depend on the sequence of simple functions approximating h. To see this, let $h_n \to h$ and $g_n \to h$ be two sequences of simple functions that converge to h. Then by the linear isometry (corollary 2.1.12), we have

$$
||I(h_n) - I(g_n)||^2_{L^2(\Omega)} = ||I(h_n - g_n)||^2_{L^2(\Omega)} = ||h_n - g_n||^2_{L^2(E)} \to 0.
$$

That $I(h)$ for $h \in L^2(E)$ is an isonormal Gaussian process follows directly from the isometry $||I(h)||_{L^2(\Omega)} = ||h||_{L^2(E)}$ for $h \in L^2(E)$ in the same way that we proved it in proposition 2.1.11. It is important to realise that the stochastic integral is well defined if and only if $h \in L^2(E)$.

We will use both $\int_E h(x) d\mathcal{W}(x)$, and $\int_E h(x)\mathcal{W}(dx)$, to mean the stochastic integral $I(h)$ with respect to space-time white noise $\mathcal{W}$, of the $L^2(E)$ function h, such that $x \mapsto h(x)$. We obtain the following important formula, called Wiener's isometry, that shows symbolically how the inner product is preserved:

Theorem 2.1.14 (Wiener's isometry)**.** *For any $h, g \in L^2(E)$:*

$$
E\left(\int_E h(x) d\mathcal{W}(x) \int_E g(x) d\mathcal{W}(x) \right) = \int_E h(x) g(x) dx \tag{2.1.4}
$$

We will need the following corollary as well in the following chapters.

Corollary 2.1.15. *Assume $A_1, \ldots, A_n$ are disjoint sets in $\mathcal{B}(E)$, then for $f \in L^2(E)$*

$$E\left(\left[\sum_{i=1}^{n}\int_{A_i} f(x)d\mathcal{W}(x)\right]^2\right) = \sum_{i=1}^{n}\int_{A_i} f(x)^2 dx. \tag{2.1.5}$$

Proof. Expand the sum and note that the cross-term multiplications are independent Gaussian variables, then use Wiener's isometry on the square terms. □

TODO more rigorous on proof of Wiener's isometry on square of sum on disjoint subsets.

2.1.3.1 The case of white noise on $\mathbb{R}_+ \times \mathbb{R}^d$

In the coming presentation, $E = \mathbb{R}_+ \times \mathbb{R}^d$ for some integer $d > 0$, and for $g = h(s,y)\mathbb{1}_{(0,t)\times\mathbb{R}^d}(s,y)$ where $h \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$, we will write

$$\int_E g d\mathcal{W} = \int_{\mathbb{R}_+\times\mathbb{R}^d} g d\mathcal{W} =: \int_0^t \int_{\mathbb{R}} h(s,y)d\mathcal{W}(s,y).$$

The following identity is often useful.

Proposition 2.1.16. *If $0 < t_1 < t$ then*

$$\int_0^t \int_{\mathbb{R}} h(s,y)d\mathcal{W}(s,y) \overset{d}{=} \int_0^{t_1} \int_{\mathbb{R}} h(s,y)d\mathcal{W}(s,y) + \int_{t_1}^t \int_{\mathbb{R}} h(s,y)d\mathcal{W}(s,y)$$

Proof. Let $h(s,y) = \mathbb{1}_{[0,t_1]}(s)h(s,y) + \mathbb{1}_{(t_1,t]}(s)h(s,y)$ and use the linearity in distribution of the stochastic integral. □

2.1.4 Solution to an SPDE driven by white noise

We are now ready to give a definition of a solution to a simplified SPDE of the form presented in the introduction, eq. 1.1.2. We again consider the space $\mathbb{R}_+ \times U$ where $U \subseteq \mathbb{R}^d$ is bounded or unbounded. A linear SPDE with additive white noise $\mathcal{W}$ is the equation:

$$\mathcal{L}u(x,t) = \dot{\mathcal{W}}(x,t), \;\; t > 0, \; x \in U, \tag{2.1.6}$$

with deterministic initial conditions and if necessary also boundary condition.

Solution to and SPDE: Again this is taken from Dalang and Sanz-Solé's book [10], need to verify the condition and why they are needed. Want to give a show of this formula like Walsh did for the non-linear wave equation.

Definition 2.1.17 (Solution to 2.1.6)**.** *Suppose that there exists a fundamental solution to $\mathcal{L}$ that is a Borel function $\Psi(x,t;y,s)$ that lies in $L^2(\mathbb{R}_+ \times U)$. Then the SPDE 2.1.6 has a (unique) solution and it is the random field given by*

$$\boxed{u(x,t) = I_0(x,t) + \int_0^t \int_U \Psi(x,t;y,s)\mathcal{W}(dyds), \;\; t > 0, x \in U.}$$

I_0 is the solution to the homogeneous problem $\mathcal{L}u = 0$ with the same initial- and boundary conditions as 2.1.6.

2.1.5 The Stochastic Heat Equation with White Noise on $\mathbb{R}_+ \times \mathbb{R}^d$

We now know that the formula found in 1.1.6 is actually the solution to the equation by definition 2.1.17. In this section we will also give a proof of Hölder-continuity of the solution.

These calculations are taken primarily from Walsh's notes in [3].

2.1.5.1 Hölder continuity

Definition 2.1.18 (Hölder continuity)**.** *A function is Hölder continuous with Hölder exponent $\alpha \geq 0$ if there exists $C > 0$ such that*

$$|f(x) - f(y)| \leq C||x-y||^\alpha \tag{2.1.7}$$

for all x, y in the domain.

Since the solution to the stochastic heat $u(x,t) = u(\omega,x,t)$ is a stochastic process. We want the Hölder-continuity to hold hold for a.e. path i.e. for almost all $\omega \in \Omega$ the mapping $t \mapsto X_t(\omega)$ is Hölder continuous. Now this property will probably differ in time and space since ...

The proof of Hölder continuity consist of finding scalars such that Kolmogorov's theorem can be used.

Maybe we need some other variant on Kolmogorov's theorem that works jointly?

Theorem 2.1.19. *Let $\{X_t : t \in \mathbb{R}_+\}$ be a $\mathbb{R}$-valued stochastic process. If there exists $k > 1$, $K > 0$ and $\epsilon > 0$ such that for all $s,t \in \mathbb{R}_+$*

$$E\{|X_t - X_s|^k\} \leq K|t-s|^{n+\epsilon} \tag{2.1.8}$$

then there exists a version of X which is continuous and even locally Hölder α-continuous for every $\alpha \in (0, \frac{\epsilon}{k})$.

We have the solution for $d = 1$:

$$u(x,t) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)W(dyds)$$

where the Green kernel is denoted (for calculation purposes),

$$G_{t-s}(x-y) := G(x,t;y,s) = \frac{e^{-\frac{|x-y|^2}{2(t-s)}}}{(2\pi|t-s|)^{1/2}}.$$

Theorem 2.1.20. *The solution $(x,t) \mapsto u(x,t)$ is a locally Hölder continuous function with exponent $\frac{1}{4} - \epsilon$ in time and with $\frac{1}{2} - \epsilon$ in space.*

Proof. Given the solution and its Green kernel we have:

$$\begin{aligned} &E\{|u(x+h,t+k) - u(x,t)|^n\} \\ &\leq E\{|u(x+h,t+k) - u(x,t+k)|^n\} + E\{|u(x,t+k) - u(x,t)|^n\}. \end{aligned}$$

The inequality is motivated by the triangle inequality of the norm on $L^2(\Omega \times \mathbb{R} \times \mathbb{R}_+)$. We will estimate these two terms separately starting with the first term and for notations sake set $t+k$ to k.

$$\begin{aligned} &E\{|u(x+h,t) - u(x,t)|^n\} \\ &\leq \frac{2^{n/2}}{\sqrt{\pi}}\Gamma\Big(\frac{n+1}{2}\Big)E\{|\int_0^t \int_{\mathbb{R}} (G_{t-s}(x+h-y) - G_{t-s}(x-y))^2 dyds|^{n/2}\} \end{aligned}$$

Where the constant comes from the calculation of $E\{|Z|^n\}$ where $Z \sim N(0,\sigma^2)$. Then by a change of variable of $x-y$ to y does not change anything and $t-s$ to s

Need to show more clearly on the constants in Hölder-calculations.

$$= C\Big(\int_0^t \int_{\mathbb{R}} \frac{1}{2\pi|s|}(e^{-\frac{(y+h)^2}{4s}} - e^{-\frac{y^2}{4s}})^2 dyds\Big)^{n/2}$$

which by the change of variables $s = h^2 v$ and $y = hz$ gives

$$= C\Big(h\int_0^{t/h^2} \int_{\mathbb{R}} v^{-1}(e^{-\frac{(z+1)^2}{4v}} - e^{-\frac{z^2}{4v}})^2 dzdv\Big)^{n/2} \leq Ch^{n/2}. \tag{2.1.9}$$

The last inequality is motivated by the fact that h was arbitrarily small and the value of C changes since the finite value of the integral is multiplied. And hence the first term is bounded by $Ch^{n/2}$.

For the second term a similar argument is used. Splitting up the integral

$$\begin{aligned} &E\{|u(x,t+k) - u(x,t)|^n\} \\ &\leq \frac{2^{n/2}}{\sqrt{\pi}}\Gamma\Big(\frac{n+1}{2}\Big)E\{|\int_0^t \int_{\mathbb{R}} (G_{t+k-s}(x-y) - G_{t-s}(x-y))^2 dyds|^{n/2}\} \\ &+ \frac{2^{n/2}}{\sqrt{\pi}}\Gamma\Big(\frac{n+1}{2}\Big)E\{|\int_t^{t+k} \int_{\mathbb{R}} (G_{t+k-s}(x-y) - G_{t-s}(x-y))^2 dyds|^{n/2}\} \end{aligned}$$

and after shifting in x as before and changing $t-s$ to s the first term is equal to

$$= C(\int_0^t \int_{\mathbb{R}} (\frac{e^{-\frac{y^2}{4(s+k)}}}{\sqrt{2\pi|s+k|}} - \frac{e^{-\frac{y^2}{4s}}}{\sqrt{2\pi|s|}})^2 dyds)^{n/2}$$

which after the change of variable $s = ku$ and $y = \sqrt{k}z$ is

$$\leq C(k^{1/2} \int_0^\infty \int_{\mathbb{R}} (\frac{e^{-\frac{z^2}{4(u+1)}}}{\sqrt{u+1}} - \frac{e^{-\frac{z^2}{4u}}}{\sqrt{u}})^2 dzdu)^{n/2} \leq Ck^{n/4}$$

since the integral is finite. Then the second term is after shifting x and with change of variable from $s-t$ to s

$$= C(\int_0^k \int_{\mathbb{R}} (G_{s+k}(y) - G_s(y))^2 dyds)^{n/2} \leq C(\int_0^k \int_{\mathbb{R}} G_{s+k}(y)^2 dyds)^{n/2}$$

Which is equal to

$$= C(\int_0^k \int_{\mathbb{R}} s^{-1} e^{\frac{-y^2}{s}} dyds)^{n/2} = C(\int_0^k s^{-1} \frac{\sqrt{s}}{\pi} ds)^{n/2} = C(k^{1/2})^{n/2} = Ck^{n/4}.$$

This completes the proof since using Kolmogrov's theorem 2.1.19 with $k = n$ and ϵ equal to $n/2$ and $n/4$ in space and time respectively. This gives $\frac{n/2}{n} = 1/2$ and $\frac{n/4}{n} = 1/4$ locally Hölder continuity for the solution $u(x,t)$ of the stochastic heat equation. □

2.1.5.2 Covariance structure

We now look at the solution to the stochastic partial differential equation and its covariance structure.

Fix covariance calculations to drift-parameter = 1 (for the inference part to look better)

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \frac{e^{-\frac{|x-y|^2}{2(t-s)}}}{(2\pi|t-s|)^{n/2}} d\mathcal{W}(y,s). \tag{2.1.10}$$

It can be shown that a pointwise solution doesn't exist for $n \geq 2$. Although the expression for u in 2.1.10 can then be made sense of as a distribution. We will come back to this later.

TODO: Show that the Fundamental Solutions to the SHE do not lie in L^2

$$C(t_1,t_2,x_1,x_2) = E\{u(t_1,x_1)u(t_2,x_2)\}$$
$$= E\left\{ \int_0^{t_1} \int_{\mathbb{R}^d} \frac{e^{-\frac{|x_1-y|^2}{2(t_1-s)}}}{(2\pi(t_1-s))^{d/2}} d\mathcal{W}(s,y) \int_0^{t_2} \int_{\mathbb{R}^d} \frac{e^{-\frac{|x_2-y|^2}{2(t_2-s)}}}{(2\pi(t_2-s))^{d/2}} d\mathcal{W}(s,y) \right\}.$$

Because the stochastic integral also preserves the inner products by Wiener's isometry, for (deterministic) function α and β we have that

$$E\left\{\Big(\int_0^t\int_{\mathbb{R}^d}\alpha(s,y)\,d\mathcal{W}(s,y)\Big)\Big(\int_0^t\int_{\mathbb{R}^d}\beta(s,y)\,d\mathcal{W}(s,y)\Big)\right\}$$
$$=\int_0^t\int_{\mathbb{R}^d}\alpha(s,y)\beta(s,y)\,dsdy$$

Let $G_1(s,y)=\frac{e^{-\frac{|x_1-y|^2}{2(t_1-s)}}}{(2\pi(t_1-s))^{d/2}}$ and $G_2(s,y)=\frac{e^{-\frac{|x_2-y|^2}{2(t_2-s)}}}{(2\pi(t_2-s))^{d/2}}$. Assume that $t_1\le t_2$. Then the covariance can be written as:

$$E\left\{\int_0^{t_2}\int_{\mathbb{R}^d}G_1(s,y)\mathbb{1}(s\le t_1)d\mathcal{W}(s,y)\int_0^{t_2}\int_{\mathbb{R}^d}G_2(s,y)d\mathcal{W}(s,y)\right\}$$
$$=\int_0^{t_2}\int_{\mathbb{R}^d}\frac{e^{-\frac{|x_1-y|^2}{2(t_1-s)}}}{(2\pi(t_1-s))^{d/2}}\frac{e^{-\frac{|x_2-y|^2}{2(t_2-s)}}}{(2\pi(t_2-s))^{d/2}}\mathbb{1}(s\le t_1)dy\,ds$$
$$=\int_0^{t_1}\int_{\mathbb{R}^d}\frac{e^{-\frac{|x_1-y|^2}{2(t_1-s)}}}{(2\pi(t_1-s))^{d/2}}\frac{e^{-\frac{|x_2-y|^2}{2(t_2-s)}}}{(2\pi(t_2-s))^{d/2}}dy\,ds$$

If we instead assumed $t_2<t_1$, we would integrate to t_2. This gives us that the covariance can be simplified to

$$C(t_1,t_2,x_1,x_2)=\frac{1}{(2\pi)^d}\int_0^{t_1\wedge t_2}\int_{\mathbb{R}^d}\frac{e^{-\frac{|x_1-y|^2}{2(t_1-s)}}}{(t_1-s)^{n/2}}\frac{e^{-\frac{|x_2-y|^2}{2(t_2-s)}}}{(t_2-s)^{n/2}}dy\,ds \tag{2.1.11}$$

Looking at $\int_{\mathbb{R}^d}\frac{e^{-\frac{|x_1-y|^2}{2(t_1-s)}}}{(t_1-s)^{n/2}}\frac{e^{-\frac{|x_2-y|^2}{2(t_2-s)}}}{(t_2-s)^{n/2}}dy$. Set $y=y'+x_1$. The integral reduces to a convolution

$$\int_{\mathbb{R}^d}\frac{e^{-\frac{|y'|^2}{2(t_1-s)}}}{(t_1-s)^{n/2}}\frac{e^{-\frac{|x_2-x_1-y'|^2}{2(t_2-s)}}}{(t_2-s)^{n/2}}dy'=\int_{\mathbb{R}^d}\frac{e^{-\frac{|y|^2}{2(t_1-s)}}}{(t_1-s)^{n/2}}\frac{e^{-\frac{|x-y|^2}{2(t_2-s)}}}{(t_2-s)^{n/2}}dy$$
$$=\frac{1}{(t_1-s)^{n/2}(t_2-s)^{n/2}}\cdot\Big(e^{-\frac{|y|^2}{2(t_1-s)}}*e^{-\frac{|y|^2}{2(t_2-s)}}\Big)(x)$$

Where $x=x_2-x_1$. We have a convolution of two Gaussian kernels. If we have $f(y)=e^{-a|y|^2}$ and $g(y)=e^{-b|y|^2}$ then[1]

$$f(y)*g(y)(x)=\int_{\mathbb{R}^d}e^{-a|x-y|^2}e^{-b|y|^2}dy=\ldots=e^{\frac{-ab|x|^2}{a+b}}\left(\frac{\pi}{a+b}\right)^{n/2} \tag{2.1.12}$$

[1] is multi-dim convolution formula here correct?

In our case we have $a = \frac{1}{2(t_1-s)}$ and $b = \frac{1}{2(t_2-s)}$. This reduces the integral over $\mathbb{R}^d$ to

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{e^{-\frac{|x_1-y|^2}{2(t_1-s)}}}{(t_1-s)^{n/2}} \frac{e^{-\frac{|x_2-y|^2}{2(t_2-s)}}}{(t_2-s)^{n/2}} dy \\
=&\frac{\pi^{n/2}}{(\frac{1}{2(t_1-s)}+\frac{1}{2(t_2-s)})^{n/2}} \exp\left(\frac{-\frac{1}{2^2(t_1-s)(t_2-s)}|x|^2}{\frac{1}{2(t_1-s)}+\frac{1}{2(t_2-s)}} \right) \\
=&(t_1-s)^{n/2}(t_2-s)^{n/2} \frac{(2\pi)^{n/2}}{(t_1+t_2-2s)^{n/2}} \exp\left(\frac{-|x|^2}{2(t_1+t_2-2s)} \right)
\end{aligned}$$

Plugging the above expression into 2.1.11 we obtain the integral

$$\begin{aligned}
C(t_1,t_2,x_1,x_2) &= (2\pi)^{-n} \int_0^{t_1 \wedge t_2} \frac{(2\pi)^{n/2}}{(t_1+t_2-2s)^{n/2}} e^{\frac{-|x_2-x_1|^2}{2(t_1+t_2-2s)}} ds \\
&= (2\pi)^{-n/2} \int_{|t_1-t_2|}^{t_1+t_2} \frac{u^{-n/2}}{2} e^{\frac{-|x|^2}{2u}} du \qquad (2.1.13)
\end{aligned}$$

Where we made the substitution $u = t_1 + t_2 - 2s$. Looking at 2.1.13 we can make the "no solution in $n \geq 2$ argument". We proceed with $n = 1$ and calculate the covariance explicitly.

First consider the indefinite integral:

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \int \frac{1}{2\sqrt{u}} e^{-\frac{x^2}{2u}} du = \{\text{partial integration}\} \\
=&\frac{1}{\sqrt{2\pi}} \sqrt{u} e^{-\frac{x^2}{2u}} - \frac{1}{2\sqrt{2\pi}} \int u^{-3/2} e^{-\frac{x^2}{2u}} x^2 du
\end{aligned}$$

We have that

$$\begin{aligned}
&\int u^{-3/2} e^{-\frac{x^2}{2u}} x^2 du = \{v = \frac{x}{\sqrt{2u}}\} \\
=&x^2 \cdot \frac{-1}{x} \sqrt{2\pi} \int \frac{2}{\sqrt{\pi}} e^{-v^2} dv = -x\sqrt{2\pi}\Phi\Big(\frac{x}{\sqrt{2u}}\Big)
\end{aligned}$$

Therefore

$$\frac{1}{2\sqrt{2\pi}} \int_{|t_1-t_2|}^{t_1+t_2} \frac{1}{\sqrt{u}} e^{-\frac{x^2}{2u}} du = \left[\frac{1}{\sqrt{2\pi}} \sqrt{u} e^{\frac{-x^2}{2u}} + \frac{x}{2}\Phi\Big(\frac{x}{\sqrt{2u}}\Big) \right]_{|t_1-t_2|}^{t_1+t_2} \qquad (2.1.14)$$

Final true covariance

Need to get it right between Erf and Normal CDF in covariance calculations.

$$\begin{aligned}
C(t_1,t_2,x_1,x_2) = \frac{1}{\sqrt{2\pi}} &\left(\sqrt{t_1+t_2} e^{-\frac{|x_1-x_2|^2}{2(t_1+t_2)}} - \sqrt{|t_1-t_2|} e^{-\frac{|x_1-x_2|^2}{2|t_1-t_2|}} \right) \\
&+ (x_1-x_2) \left(\Phi\Big(\frac{x_1-x_2}{\sqrt{t_1+t_2}}\Big) - \Phi\Big(\frac{x_1-x_2}{\sqrt{|t_1-t_2|}}\Big) \right)
\end{aligned}$$

What properties does the solution have:

Add onto what the covariance tells us. Or maybe leave until later.

We can see that the covariance is space invariant, which means that $C(t_1, t_2, x_1, x_2) = C(t_1, t_2, x_1 - x_2, 0)$. Set $\tau = x_1 - x_2$ and fix some time points $t_1 = 1$, $t_2 = 2$ for example. The covariance plot is given by figure 2.1 below.

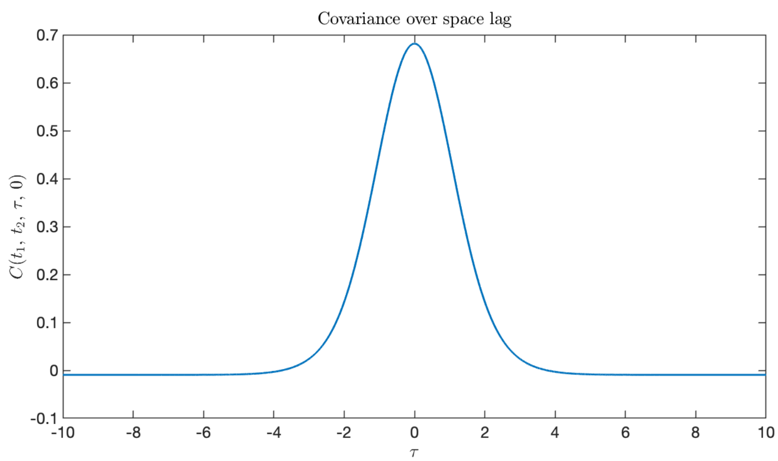

Figure 2.1: Covariance function plotted over the space lag. Here $t_1 = 1$ and $t_2 = 2$.

2.2 A splash of colour to the SPDE

We saw in the previous sections how the stochastic heat equation fails to admit a point-wise solution in $d \geq 2$. This pickle arises from the defintion of the stochastic integral, where we recall that $I(h) = \int_{\mathbb{R}\times\mathbb{R}^d} h d\mathcal{W}$ is well defined if and only if $h \in L^2(\mathbb{R}\times\mathbb{R}^d)$. In our case this is the green function $G(x,t;y,s)$ that we need to integrate over that does not lie in this L^2 space. Recall remark 2.1.7 of the white noise definition, where we saw how we acquire an isometry to some Hilbert space depending on the covariance structure of the noise. Before we venture into changing our noise we need some preliminaries on non-negative definite kernels.

Remove all unclear discussion on Kernels and inner products and so on. Do not think is needed.

Fix the structure of the coloured SPDE. And add existence of solution and covariance structure and so on.

2.2.1 Non-negative definite kernels

Definition 2.2.1 (Non-negative Definite Kernel)**.** *Let $\mathcal{T}$ be an arbitrary index set. A symmetric function $K : \mathcal{T}\times\mathcal{T} \to \mathbb{C}$ is called a non-negative definite kernel on the set $\mathcal{T}$ if given $n \in \mathbb{N}$ it holds that*

$$\sum_{k=1}^{n}\sum_{l=1}^{n} c_k \bar{c}_l K(t_k, t_l) \geq 0, \tag{2.2.1}$$

for each $c_1, \dots c_n \in \mathbb{R}$ and $t_1, \dots, t_n \in \mathcal{T}$.

Remark 2.2.2. *The above definition is equivalent to that every matrix $\mathbf{K} = (K(t_k, t_l))$ created for every $t_1, \dots, t_n \in \mathcal{T}$ and $n \in \mathbb{N}$ is a non-negative definite matrix, or equivalently that the eigenvalues to $\mathbf{K}$ are greater than or equal to zero.*

Remark 2.2.3. *Note on naming convention (a bit confusing): If the index set $\mathcal{T} = \mathbb{R}^d$ then the non-negative definite kernel is called a non-negative definite function. If we also require that 2.2.1 is equal to zero if and only if all $c_k = 0$ then we call the function/kernel positive definite.*

We will over and over again show that something is a non-negative definite kernel. It will be helpful to gather some propositions for later.

Proposition 2.2.4. *If we have two non-negative definite kernels $K_1 : \mathcal{T}\times\mathcal{T} \to \mathbb{R}$ and $K_2 : \mathcal{S}\times\mathcal{S} \to \mathbb{R}$, then the product $K((t_1,t_2),(s_1,s_2)) := K_1(t_1,t_2)\cdot K_2(s_1,s_2)$ is also a non-negative definite kernel on $(\mathcal{T}\times\mathcal{T})\times(\mathcal{S}\times\mathcal{S})$.*

Proof. Will write out the proof later. But idea is to look at the matrices $\mathbf{K}_1$ and $\mathbf{K}_2$, and their corresponding symmetric decomposition or something to prove it. It checks out anyway. See Schur Product theorem [1] p. 14, Theorem VII (or the Wikipedia article `https://en.wikipedia.org/wiki/Schur_product_theorem`). $\square$

Proposition 2.2.5. *If H is an inner-product space, then its inner product $\langle\cdot,\cdot\rangle_H$ is a non-negative definite kernel on H.*

Proof. We go by the definition, let $c_1,\ldots,c_n \in \mathbb{C}$ and $h_1,\ldots,h_n \in H$. By the bi-linearity of the inner-product and its non-negative definiteness in norm we have

$$\sum_k \sum_l c_k \bar{c}_l \langle h_k, h_l\rangle_H = \left\langle \sum_{k=1}^n c_k h_k, \sum_{l=1}^n \bar{c}_l h_l, \right\rangle_H = \left\| \sum_{k=1}^n c_k h_k \right\|_H^2 \geq 0.$$

□

An additional proposition that will be useful when we venture into coloured noise (from [9] p.19)

Proposition 2.2.6. *If $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a continuous function then the condition*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x)\phi(y) f(x-y)dxdy \geq 0 \textit{ for all } \phi \in L^1(\mathbb{R}^d) \tag{2.2.2}$$

is equivalent to the condition that

$$\sum_{i,j=1}^n f(x_j - x_k)c_j c_k \geq 0 \textit{ for all } c_1,\ldots c_n \in \mathbb{R} \textit{ and } x_1,\ldots x_n \in \mathbb{R}^d. \tag{2.2.3}$$

Note how 2.2.3 is the condition that f is non-negative definite, since $f(x-y)$ is symmetric.

Proof. Would like to prove but don't know how... □

2.2.2 White-Coloured Noise

Let's restrict ourselves to the measure space $(E,\mu) = \Big(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d), \lambda\Big)$, with λ being the Lebesgue measure and where we use the notation $(t,A) := ([0,t]\times A) \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$. For the case of white noise based on λ, using standard results on set algebra and the Lebesgue measure we have that the covariance $C\Big((t,A),(s,B)\Big)$ can be factorised in a time and space component in the following manner:

$$\begin{aligned} C\Big((t,A),(s,B)\Big) &= \lambda\Big(([0,t]\times A)\cap([0,s]\times B)\Big) = \lambda\Big(([0,t]\cap[0,s])\times(A\cap B)\Big) \\ &= \lambda_{\mathbb{R}}([0,t]\cap[0,s])\cdot\lambda_{\mathbb{R}^d}(A\cap B) = t\wedge s\cdot\lambda_{\mathbb{R}^d}(A\cap B). \end{aligned} \tag{2.2.4}$$

Where $\lambda_{\mathbb{R}}$ and $\lambda_{\mathbb{R}^d}$ are the Lebesgue measures corresponding to $\mathbb{R}$ and $\mathbb{R}^d$ respectively. Given a smart choice of covariance for the noise, we can extend the stochastic integrals to a larger class of integrands.

One general idea is to establish a spatial parameter to the covariance structure, which is done by changing the spatial factor $\lambda_{\mathbb{R}^d}(A \cap B)$ from calculation 2.2.4 above. The following choice of covariance was introduced in Dalang's landmark paper [5] on the extension of martingale measures in the Walsh sense of SPDEs:

$$C\Big((t,A),(s,B)\Big) = t \wedge s \cdot \int_A \int_B f(x-y)dxdy. \tag{2.2.5}$$

We can therefore *colour* the spatial component by using an integrable function f as a spatial parameter to the noise. Note that the case of $f =$ gives us the white noise process.

What proceeds now is a *lengthy* discussion on what it means to have the above formula 2.2.5 for a covariance. Since we still need that the $C\Big((t,A),(s,B)\Big)$ defines a covariance function, i.e a symmetric- and non-negative definite function, we need some restrictions on this function f.

We know that the product of two non-negative functions is again a non-negative function, since the covariance $C_1(t,s) = t \wedge s$ of the Brownian motion is non-negative definite, we only need that the function $C_2 : \mathcal{B}_b(\mathbb{R}^d) \times \mathcal{B}_b(\mathbb{R}^d) \to \mathbb{R}$ such that $A \times B \mapsto \int_A \int_B f(x-y)dxdy$ is a non-negative definite function. We have that

$$C_2(A,B) = \int_A \int_B f(x-y)dxdy =$$
$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_A(x)\mathbb{1}_B(y) f(x-y)dxdy =: \langle \mathbb{1}_A, \mathbb{1}_B \rangle_{L^1(\mathbb{R}^d)}. \tag{2.2.6}$$

With some specific conditions on f (that we will sort out), the formula 2.2.6 above defines an inner product on the space of integrable functions in $L^1(\mathbb{R}^d)$ with weight function f by the formula

$$\langle \phi, \psi \rangle_{L^1(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x)\phi(y) f(x-y)dxdy. \tag{2.2.7}$$

As long as $\langle \cdot, \cdot \rangle_{L^1(\mathbb{R}^d)}$ actually is an inner product then we know from proposition 2.2.5 that $\langle \cdot, \cdot \rangle_{L^1(\mathbb{R}^d)}$ is a non-negative definite kernel on L^1, and since the indicator functions are a subset of L^1 it follows that $C_2 : \mathcal{B}_b(\mathbb{R}^d) \times \mathcal{B}_b(\mathbb{R}^d) \to \mathbb{R}$ is also non-negative definite kernel by the identifications $\mathbb{1}_A \mapsto A \in \mathcal{B}_b(\mathbb{R}^d)$.

It remains to decide which conditions we have on f in 2.2.7 to obtain a valid inner product. It is obvious that $\langle \cdot, \cdot \rangle_{L^1(\mathbb{R}^d)}$ is symmetric and bilinear. All that's left is to require that $\langle \phi, \phi \rangle_{L^1(\mathbb{R}^d)} \geq 0$, which by proposition 2.2.6 follows for continuous f that are themselves positive definite functions on $\mathbb{R}^d$. We are now ready to make the definition of white-coloured noise (that actually is well-defined).

Definition 2.2.7 (White-Coloured Noise)**.** *A spatial coloured noise that is white in time is a Gaussian random field $\mathcal{F} = \{\mathcal{F}(t, A),\ t > 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ defined on some probability space $(\Omega, \Sigma, \mathcal{P})$ with $E(\mathcal{F}(t, A)) = 0$, and covariance function*

$$C\Big((t, A), (s, B)\Big) = E\Big(\mathcal{F}(t, A)\mathcal{F}(s, B)\Big) = t \wedge s \cdot \int_A \int_B f(x - y)dxdy.$$

Where the function f (the spatial parameter of the noise) is continuous, symmetric and non-negative definite.

A note on the naming convention: A noise that is white in time and white in space will simply be referred to as *white noise*, if it is white in time but coloured in space it is called *white-coloured noise*, and later when we look at the time as a (bi-)fractional Brownian motion and the space as coloured we will call the noise *fractional-coloured*.

There is a way to characterise these spatial parameters f as Fourier transforms of tempered measures. This will give us some good formulas and tools to tackle the solutions to coloured SPDE's. Let's take a short detour into Fourier analysis:

2.2.2.1 Fourier Transforms of Tempered Measures

Note: The reason for this detour is (what seems to me) that we get some good formulas and tools for several proofs. Much of the math below is taken straight from Tudor's article [8] with some clarifying from us. Hard to show all the intuition for this stuff.

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space, which consists of the infinitely differentiable functions which are rapidly decreasing as $|x| \to \infty$ together with their derivatives of all orders and let $\mathcal{S}'(\mathbb{R}^d)$ be the corresponding dual containing the distributions on $\mathcal{S}(\mathbb{R}^d)$. (Recall that $\phi(x)$ is rapidly decreasing if $\lim_{|x|\to\infty} |x^k \phi(x)| = 0$ for all k.)

For any $\phi \in L^1(\mathbb{R}^d)$, we define $\hat{\phi}(\xi) := \int_{\mathbb{R}^d} e^{-i\xi\dot{y}}\phi(y)dy$ which is the Fourier transform of ϕ. Here comes a technical definition[2]:

Definition 2.2.8. *A non-negative measure ν on $\mathbb{R}^d$ is called a tempered measure if there exists some $k > 0$ such that*

$$\int_{\mathbb{R}^d} \left(\frac{1}{1 + |\xi|^2}\right)^k d\nu(\xi) < \infty$$

[2]Comes into proof of 2.2.12 in the "hard" direction of the equivalence I think. See [2] pp.145-147.

We will work Borel measures of the above form. Let f and g be two integrable functions. Then we have the relation

$$\int_{\mathbb{R}^d} \hat{f}(y)g(y)dy = \int_{\mathbb{R}^d} f(y)\hat{g}(y)dy \tag{2.2.8}$$

If we have $\phi \in \mathcal{S}$ then every integrable function f induces a distribution $A_f \in \mathcal{S}'$ in the sense that $A_f(\phi) = \langle f, \phi \rangle = \int_{\mathbb{R}^d} f(y)\phi(y)dy$. It is therefore natural to define the Fourier transform on any distribution $A \in \mathcal{S}'(\mathbb{R}^d)$ by $\hat{A}(\phi) := A(\hat{\phi})$ for every $\phi \in \mathcal{S}(\mathbb{R}^d)$. A (tempered) measure ν can naturally be seen as a distribution on test functions $\psi \in \mathcal{S}(\mathbb{R}^d)$ as $\psi \mapsto \langle \nu, \psi \rangle := \int_{\mathbb{R}^d} \psi(\xi)d\nu(\xi)$.

Definition 2.2.9 (Fourier transform of a tempered measure)**.** *If $\psi = \hat{\phi}$ is the Fourier transform of ϕ, then f is called the Fourier transform of the tempered measure ν if*

$$\int_{\mathbb{R}^d} f(x)\phi(x)dx = \int_{\mathbb{R}^d} \hat{\phi}(\xi)d\nu(\xi) \tag{2.2.9}$$

holds for all functions $\phi \in \mathcal{S}(\mathbb{R}^d)$.

Remark 2.2.10. *For the proof of one implication in the Bochner theorem we will actually need the equivalent definition of the Fourier transform of a measure in the sense of*

$$f(x) = \hat{\nu}(x) := \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} d\mu(\xi). \tag{2.2.10}$$

An important fact that will be used in several calculations is the following.

Proposition 2.2.11. *For any $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$ it holds that*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(x)\phi(y)f(x-y)dxdy = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{\psi}(\xi)\hat{\phi}^*(\xi)d\nu(\xi) \tag{2.2.11}$$

Where z^* denotes the complex conjugate of $z \in \mathbb{C}$.

Proof. Need to finish but some kind of Fubini $\to$ convolution $\to$ Fourier stuff gives (where $\tilde{g}(x) := g(-x)$).

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(x)\phi(y)f(x-y)dxdy &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \psi(x)\phi(y)f(x-y)dy\right)dx = \\
\int_{\mathbb{R}^d} \psi(x)\left(\int_{\mathbb{R}^d} \phi(y)f(x-y)dy\right)dx &= \int_{\mathbb{R}^d} \psi(x)\left(\int_{\mathbb{R}^d} \phi(x-y)f(y)dy\right)dx \\
= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(x)\phi(x-y)f(y)dxdy &= \int_{\mathbb{R}^d} f(y)\left(\int_{\mathbb{R}^d} \psi(x)\phi(-(y-x))dx\right)dy \\
&= \int_{\mathbb{R}^d} f(y)(\psi * \tilde{\phi})(y)dy = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{\psi}(\xi)\hat{\phi}^*(\xi)d\nu(\xi).
\end{aligned}$$

$\square$

Definition 2.2.9 ties into the discussion of the covariance definition 2.2.7 by the following theorem that is a special case of Bochner's theorem.

Theorem 2.2.12 (Bochner's theorem)**.** *A complex valued continuous function f defined on $\mathbb{R}^d$ is non-negative definite if and only if f is the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$ of a tempered measure ν on $\mathbb{R}^d$.*

Proof. For a full proof see [2] pp.145-147. We prove the simple implication that the Fourier transform of a tempered measure is non-negative definite. Let f be the Fourier transform of ν and recall that ν is a finite non-negative measure.

$$\begin{aligned}\sum_{k=1}^{n}\sum_{l=1}^{n} c_k \bar{c}_l f(x_k - x_l) &= \frac{1}{\sqrt{(2\pi)^d}} \sum_{k=1}^{n}\sum_{l=1}^{n}\left(c_k \bar{c}_l \int_{\mathbb{R}^d} e^{-i(x_k - x_l)\cdot\xi} d\nu(\xi)\right) \\ &= \frac{1}{\sqrt{(2\pi)^d}} \sum_{k=1}^{n}\sum_{l=1}^{n}\left(c_k \bar{c}_l \int_{\mathbb{R}^d} e^{-ix_k\cdot\xi} e^{ix_l\cdot\xi} d\nu(\xi)\right) \\ &= \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d}\left(\sum_{k=1}^{n} c_k e^{-ix_k\cdot\xi} \sum_{l=1}^{n} \bar{c}_l e^{ix_l\cdot\xi}\right) d\nu(\xi) \\ &= \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d}\left(\sum_{k=1}^{n} c_k e^{-ix_k\cdot\xi} \overline{\sum_{l=1}^{n} c_l e^{-ix_l\cdot\xi}}\right) d\nu(\xi) \\ &= \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d}\left|\sum_{k=1}^{n} c_k e^{-ix_k\cdot\xi}\right|^2 d\nu(\xi) \geq 0.\end{aligned}$$

□

It follows that the covariance structure defined in 2.2.5 is actually a covariance function if and only if the continuous spatial parameter function f is a Fourier transform of a tempered measure ν.

There are several examples of the spatial parameter f, as presented in C.A Tudor's paper [8]. Here is one that we will use throughout this paper.

Example 2.2.13. *The Riesz kernel of order α:*

$$f(x) = R_\alpha(x) := 2^{d-\alpha}\pi^{d/2}\frac{\Gamma\Big((d-\alpha)/2\Big)}{\Gamma(\alpha/2)}|x|^{-d+\alpha},\ \ 0<\alpha<d$$

with $\nu(d\xi) = |\xi|^{-\alpha} d\xi$.

2.2.3 Stochastic Integral with White-Coloured Noise

Proceeding with the white-coloured noise $\mathcal{F}$ we construct a stochastic integral with respect to $\mathcal{F}$. The procedure is the same as for white noise, we define the stochastic process $I(h)$ on deterministic functions $h \in H$ where the Hilbert space H is the completion of the set of indicator functions $\mathbb{1}_{(t,A)}$ on $\mathbb{R}_+ \times \mathbb{R}^d$ with inner product

$$\langle \phi, \psi \rangle_H = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x,t) f(x-y) \psi(y,t) dy dx dt. \tag{2.2.12}$$

Call the space of simple functions $\mathscr{E}$. Define the isometry mapping from $\mathscr{E}$ to finite second moment space $L^2(\Omega)$:

$$I : \mathscr{E} \to L^2(\Omega), \qquad g = \sum a_k \mathbb{1}_{(t_k, A_k)} \mapsto \sum a_k \mathcal{F}(t_k, A_k) := \int g d\mathcal{F}. \tag{2.2.13}$$

Proposition 2.2.14. *The mapping $I : \mathscr{E} \to L^2(\Omega)$ is an isonormal Gaussian process*

Proof. It suffices to show that I is an isometry. Taking a simple $g = \sum_{k=1}^n a_k \mathbb{1}_{(t_k,A_k)}(x,t) = \sum a_k \mathbb{1}_{(0,t_k]}(t) \mathbb{1}_{A_k}(x)$ we see that

$$\begin{aligned}
\|I(g)\|^2_{L^2(\Omega)} &= \left\| \sum a_k \mathcal{F}(t_k, A_k) \right\|^2_{L^2(\Omega)} = E\left(\Big(\sum a_k \mathcal{F}(t_k, A_k) \Big)^2 \right) \\
&= \sum_{k=1}^n a_k^2 E \mathcal{F}(t_k, A_k)^2 + 2 \sum_{k<l}^n a_k a_l E\left(\mathcal{F}(t_k, A_k) \mathcal{F}(t_l, A_l) \right)
\end{aligned}$$

Proceeding with the norm of g we find

$$\begin{aligned}
\|g\|^2_{\mathscr{E}} &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) \Big(\sum_{k=1}^n a_k \mathbb{1}_{(t_k,A_k)}(x,t) \Big) \Big(\sum_{l=1}^n a_l \mathbb{1}_{(t_l,A_l)}(y,t) \Big) dt dx dy \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) \left(\sum_{k=1}^n \left(a_k^2 \mathbb{1}_{(t_k,A_k)}(x,t) \mathbb{1}_{(t_k,A_k)}(y,t) \right) \right. \\
&\quad \left. + 2 \sum_{k<l}^n \left(a_k a_l \mathbb{1}_{(t_k,A_k)}(x,t) \mathbb{1}_{(t_l,A_l)}(y,t) \right) \right) dx dy dt
\end{aligned}$$

Splitting up the integral over the sum of the two sums we obtain:

$$
\begin{aligned}
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) (\sum_{k=1}^{n} \left(a_k^2 \mathbb{1}_{(t_k, A_k)}(x,t) \mathbb{1}_{(t_k, A_k)}(y,t) \right) dxdydt \\
&+ \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) 2 \sum_{k<l}^{n} \left(a_k a_l \mathbb{1}_{(t_k, A_k)}(x,t) \mathbb{1}_{(t_l, A_l)}(y,t) \right) dxdydt \\
&= \sum_{k=1}^{n} a_k^2 \left(\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) \mathbb{1}_{(t_k, A_k)}(x,t) \mathbb{1}_{(t_k, A_k)}(y,t) dxdydt \right) \\
&+ 2 \sum_{k<l}^{n} a_k a_l \left(\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) \mathbb{1}_{(t_k, A_k)}(x,t) \mathbb{1}_{(t_l, A_l)}(y,t) dxdydt \right) \\
&= \sum_{k=1}^{n} a_k^2 E \mathcal{F}(t_k, A_k)^2 + 2 \sum_{k<l}^{n} a_k a_l E \left(\mathcal{F}(t_k, A_k) \mathcal{F}(t_l, A_l) \right),
\end{aligned}
$$

which gives that the norms are equal. □

The rest of the construction follows the white-noise case. We take the completion of $\mathscr{E}$ with respect to the inner-product induced from 2.2.12 which is the Hilbert space H. Since we have a linear isometry from the set of simple functions, the map $h \mapsto I(h)$ can be extended uniquely to H. Take $h \in H$ and a sequence of simple functions h_n such that $||h - h_n||_H \to 0$. Then we define

$$
\int_E h d\mathcal{F} := I(h) := \lim_{n \to \infty} I(h_n).
$$

The above definition does not depend on the sequence of simple functions approximating h.

2.2.3.1 The stochastic Heat Equation w.r.t white-coloured noise: TODO

Consider the stochastic heat equation with white-coloured noise (coloured by the Riesz-kernel in example 2.2.13).

$$
\begin{cases} \frac{\partial}{\partial t} u(x,t) - \frac{1}{2} \Delta u(x,t) = \dot{\mathcal{F}}(x,t) & t > 0,\ x \in \mathbb{R}^d \\ u(x,0) = 0 & x \in \mathbb{R}^d. \end{cases} \tag{2.2.14}
$$

As before we define the solution to this equation to be

$$
u(x,t) = \int_0^t \int_{\mathbf{R}^d} \frac{e^{-\frac{|x-y|^2}{2(t-s)}}}{(2\pi|t-s|)^{d/2}} \mathcal{F}(ds\, dy), \tag{2.2.15}
$$

as long as the integral above is well defined which is answered by the following proposition, where we denote the Green-kernel $G(x,t;y,s) := \frac{e^{-\frac{|x-y|^2}{2(t-s)}}}{(2\pi|t-s|)^{d/2}}$.

Proposition 2.2.15. *The equation 2.2.14 with noise coloured by an $f = \hat{\nu}$ admits a unique solution if and only if*

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} d\nu(\xi) < \infty.$$

Specifically for f which is equal to the Riesz-kernel with order α the solution exists if and only if

$$d < 2 + \alpha.$$

Proof. : But we show the variance (like in [8])

TODO show solution of coloured stochastic heat

$$E(u(x,t)^2) = \|u(x,t)\|^2_{L^2(\Omega)} < \infty.$$

Where we note that $E(u(x,t)^2) = \|G(x,t;y,s)\|_H$ by the isometry which defines the integrals of G with respect to $\mathcal{F}$. □

Further TODO: We can probably compute the complete covariance $C(x_1,t_1,x_2,t_2)$ of the solution to the coloured stochastic heat equation with the Riesz-kernel and not just $C(x,t_1,x,t_2)$ like Tudor did.

2.2.4 Fractional-Coloured Noise TODO

incorporates long-range dependence while preserving the self-similarity and Gaussianity [3]

I think we will remove all this on fractional coloured? Or at least not make it into own section and instead move what's relevant other places

A specific type of noise

A typical noise to be studied is fractional Brownian motion which is a zero mean Gaussian process with the covariance function $R_H(t,s) = (t^{2H} + s^{2H} - |t-s|^{2H})/2$ where $H \in (0,1)$. The parameter H is called the Hurst index since he was the first to use it while modelling the Nile. The parameter determines the regularity of the processes sample paths. If $H = 1/2$ the process is the standard Brownian motion and when $H < 1/2$ and $H > 1/2$ it is more irregular and less irregular than Brownian motion respectively. This is because fBm is almost surely δ-Hölder continuous with $\delta \in (0,H)$. It has stationary increments

$$E\{(W^H(t) - W^H(s))^2\} = |t-s|^{2H} \tag{2.2.16}$$

and is self similar

$$(W^H(ct), t \geq 0) \stackrel{d}{=} (c^H W^H(t), t \geq 0). \tag{2.2.17}$$

[3]https://arxiv.org/pdf/1903.02364.pdf

Now using the machinery above above we can create the stochastic integral with respect tot fractional Brownian motion over time. The covariance is given and hence an inner product which creates an isometry between the two is

$$< \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} >_{\mathcal{H}(0,T)} = R_H(t,s). \tag{2.2.18}$$

This because this can be expanded by simple integration to the inner product

$$< \phi, \psi >_{\mathcal{H}(0,T)} = H(2H-1) \int_0^T \int_0^T \phi(t,x)|t-s|^{2H-2}\psi(t,y) ds dt. \tag{2.2.19}$$

Now taking the completion of the space of simple functions with the metric induced by the inner product gives the function space for which this integral is defined. But now the question is, what is this space? It can be seen that it is a subset of a fractional Sobolov(I DO NOT CARE) space and larger than L^2. But one way of describing this space is with a transfer operator.

Transfer operator
Now, instead of checking all the properties for this new stochastic integral, we can instead create an operator for which the end goal is the following property

$$\int K^*(f) dW = \int f dF. \tag{2.2.20}$$

This is a sought after property mainly since now all the previous results can be applied but also since now we do not need to study the Hilbert spaces spanned by the different noises. Instead, one needs to check if $K^*(f)$ is square intolerable.

Transfer operator for fBm in time
Now in this section a operator which transfers the integrand from this new stochastic integral to the white noise stochastic integral is defined. Hence from the space $\mathcal{H}(0,T)$ to $L^2(0,T)$ where $\mathcal{H}(0,T)$ is the closure of simple functions with respect to the covariance inner product. Now this transfer operator is defined by the set of simple functions and then the domain is extended to the completion, hence $\mathcal{H}(0,T)$.

The kernel for the covariance for fBm is

$$K_H(t,s) = \Big(\frac{H(2H-1)\Gamma(3/2-H)}{\Gamma(2-2H)\Gamma(H-1/2)}\Big)^{1/2} s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du. \tag{2.2.21}$$

Hence $R_H(t,s) = \int_0^{t\wedge s} K(t,u)K(s,u)du$ which gives

$$< K_H^* \mathbb{1}_{[0,t]}, K_H^* \mathbb{1}_{[0,s]} >_{L^2} = \int \int_0^{t\wedge s} K(t,u)K(s,u)du = R_H(t,s) = < \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} >_{\mathcal{H}(0,T)} . \tag{2.2.22}$$

Hence the operator K_H^* is an isomorphism from the set of $\mathscr{E}(0,T)$ to $L^2(0,T)$, since $\bar{\mathscr{E}}^{<>_H} = \mathcal{H}(0,T)$ the operator can be extended as usual. Now the white noise stochastic integral can be used in the following way

$$\int K^*(f)dW = \int f dW^H \tag{2.2.23}$$

where $f \in \mathcal{H}(0,T)$ and hence, $K^*(f) \in L^2(0,T)$ and hence the integral $\int K^*(f)dW$ is well defined and $\int K^*(f)dW \in L^2(\Omega)$.

fBm in time and white in space Now to create the noise which is fractional in time and white in space we let the covariance be fractional in time and white in space by

$$E\{F(t,A)F(s,B)\} = \int_0^T \int_0^T \int_{\mathbb{R}^d} \mathbb{1}_A(x)\mathbb{1}_B(x)|u-v|^{2H-2} dudvdx. \tag{2.2.24}$$

(((Snear så vet ej om det är rätt)))

Chapter 3

Statistical Inference for the Drift Parameter of the Stochastic Heat Equation

3.1 Inference

Inference on the stochastic heat equation will focus on the estimation of the drift of the solution. The drift term is θ in

Erik, gå igenom inference delen och lägg till såna här notes så vi har tydligt vad vi ska lägga till osv.

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \theta\Delta u(x,t) = \sigma\dot{\mathcal{W}}(x,t) & t > 0,\ x \in \mathbb{R}^d \\ u(x,0) = 0 & x \in \mathbb{R}^d. \end{cases} \tag{3.1.1}$$

The diffusion term σ will also be studied possibly via the ergodic properties. However I do not know if both can be identified?

3.1.1 Distribution to solution

3.1.1.1 In time

Firstly the distributions for the solution are needed since the aim is to be able to discretely sample the solution in either time or space. The distributions of the solutions will be decomposed into the sum of a smooth process plus a fractional Brownian motion.

The mild solution to the white heat equation with diffusion $\theta = 1$(Here is a big issue, it is really $1/2$!!!) have the following property in time:

Theorem 3.1.1. *The mild solution is well defined for every $x \in \mathbb{R}$, the process $\{u(x,t), t \in \mathbb{R}^+\}$ have the same distribution, (modolo a constant) to bi fractional Brownian motion*

$$\{u(x,t), t \in \mathbb{R}^+\} \stackrel{d}{=} \{ \underbrace{(2\pi)^{-\frac{1}{4}}}_{Thisiswrong} B_t^{\frac{1}{2},\frac{1}{2}}, t \in \mathbb{R}^+\}. \tag{3.1.2}$$

Fixa det här så att det blir rätt, tror vi ska lita på hans bok

Proof. The covariance for the solution have been calculated above and letting $x = x_1 = x_2$ the covariance of the solution is

$$C(t_1, t_2, x, x) = \frac{1}{\sqrt{2\pi}}((t_1 + t_2)^{\frac{1}{2}} - |t_1 - t_2|^{\frac{1}{2}}). \tag{3.1.3}$$

Since the covariance of the solution above is the same as the one for the bi fractional Brownian motion with parameters $H = \frac{1}{2}$, $K = \frac{1}{2}$ they have the same law. $\square$

3.1.1.2 In space

Theorem 3.1.2. *The mild solution is well defined for every $t \in \mathbb{R}^+$, and*

$$\{u(x,t), x \in \mathbb{R}\} \stackrel{d}{=} \{m_\alpha B^{\frac{1}{2}}(x) + S_t(x), x \in \mathbb{R}\}, \tag{3.1.4}$$

where $B^{\frac{1}{2}}$ is a fractional Brownian motion with Hurst parameter $1/2$. $\{S_t(x), x \in \mathbb{R}\}$ is a centred Gaussian process with sample paths in C^∞ and m_α is a numerical constant.

Proof. The process $S_t(x)$ can be created as

$$S_t(x) := \int_{(t,\infty)\times\mathbb{R}} G_s(y) - G_s(x-y) W(dyds). \tag{3.1.5}$$

This is a mean zero Gaussian random field since the kernel is in $L^2(\mathbb{R}^+ \times \mathbb{R})$. The desired property that

$$E\{|S_t(y) - S_t(x)|^2\} \leq C|x-y|^2 \tag{3.1.6}$$

is derived from the fact that for $x_2 > x_1$,

$$E\{|S_t(x_2) - S_t(x_1)|^2\} = E\{|\int_{(t,\infty)\times\mathbb{R}} G_s(y) - G_s(x_2-y) - G_s(y) + G_s(x_1-y)W(dyds)|^2\} \tag{3.1.7}$$

because the stochastic integral is linear in distribution. Then by using the isometry, and Plancherel's formula with a shift

$$\leq \int_t^\infty \int_{\mathbb{R}} |G_s(y-x_1) - G_s(y-x_2)|^2 dyds \tag{3.1.8}$$

$$= (2\pi)^{-1} \int_t^\infty \int_{\mathbb{R}} |\mathscr{F}(G_s(y))(\xi)\ (e^{-ix_1\xi} - e^{-ix_2\xi})|^2 d\xi ds \tag{3.1.9}$$

$$= (2\pi)^{-1} \int_t^\infty \int_{\mathbb{R}} e^{-s|\xi|^2}\ |e^{-ix_1\xi}(1 - e^{-i\xi\epsilon})|^2 d\xi ds \tag{3.1.10}$$

where $x_2 - x_1 = \epsilon$. Continuouing with the fact that $|1 - exp(\theta i)|^2 = 2(1 - cos(\theta))$ and Fubinis theorem,

$$\frac{1}{\pi} \int_{\mathbb{R}} \int_t^\infty (e^{-s|\xi|^2}(1 - cos(\epsilon\xi))dsd\xi = \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{-2t|\xi|^2}}{2\xi^2}(1 - cos(\epsilon\xi))d\xi. \tag{3.1.11}$$

The fact that $1 - cos(\theta) \leq \theta^2$ and symmetry is used,

$$\leq \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{-2t|\xi|^2}}{2\xi^2}(\epsilon\xi)^2 d\xi = \epsilon^2 \frac{1}{\pi} \int_0^\infty e^{-2t|\xi|^2} d\xi = C|x_2 - x_1|^2. \tag{3.1.12}$$

Now the process F

Proof of the fact that it can be decomposed into the sum is Prop 3.1 in `https://arxiv.org/pdf/1406.5246.pdf` Then look at the covariance. □ skriv in bevis.

3.1.2 Distribution of θ SHE

The fundamental solution to the non-parameterised stochastic heat equation is $G(t, x)$ and the mild solution to the parameterised stochastic heat equation 3.1.1 with $\sigma = 1$ is

$$u_\theta(t, x) = \int_0^t \int_{\mathbb{R}} G_{\theta t, x}(\theta s, y) W(dsdy).^{1} \tag{3.1.13}$$

To be able to estimate the parameters the new distributions must be derived. A general result for both time and space is derived below.

Lemma 3.1.3. *Suppose that the process $\{u_\theta(t, x), t \in \mathbb{R}^+, x \in \mathbb{R}\}$ is a solution to 3.1.1. Define*

$$v_\theta(t, x) := u_\theta(\frac{t}{\theta}, x), t \in \mathbb{R}^+, x \in \mathbb{R} \tag{3.1.14}$$

then the process $\{v_\theta(t, x), t \in \mathbb{R}^+, x \in \mathbb{R}\}$ satisfies the stochastic partial differential equation

$$\frac{\partial}{\partial t} v_\theta(x, t) - \Delta v_\theta(x, t) = \theta^{-\frac{1}{2}} \dot{\dot{W}}(x, t) \tag{3.1.15}$$

where $\dot{\dot{W}}$ is a space-time white noise.

[1] Prove this

Proof. For all $t \in \mathbb{R}^+, x \in \mathbb{R}$

$$v_\theta(t,x) = u_\theta(\frac{t}{\theta}, x) = \int_0^{\frac{t}{\theta}} \int_{\mathbb{R}} G_{t,x}(\theta s, y) W(dsdy) = \Big\{ s = \frac{s'}{\theta} \Big\} \tag{3.1.16}$$

$$\int_0^t \int_{\mathbb{R}} G_{t,x}(s,y) W(d\frac{s}{\theta} dy) = \theta^{-\frac{1}{2}} \int_0^t \int_{\mathbb{R}} G_{t,x}(s,y) \tilde{W}(dsdy) \tag{3.1.17}$$

where the last step is motivated by the self similarly property of Brownian motion. □

When looked at over time the distribution changes with a constant as can be seen below.

Proposition 3.1.4. *For every $x \in \mathbb{R}$ and $\theta > 0$, we have*

$$\{u_\theta(t,x), t \in \mathbb{R}^+\} \stackrel{d}{=} \{(\theta \underbrace{2\pi}_{\text{THis is wrong}})^{-\frac{1}{4}} B_t^{\frac{1}{2},\frac{1}{2}}, t \in \mathbb{R}^+\}. \tag{3.1.18}$$

Proof. Fix $x \in \mathbb{R}$ and $\theta > 0$, we have the following equality in distribution

$$E\{u_\theta(t,x)u_\theta(s,x)\} = E\{v_\theta(\theta t, x)v_\theta(\theta s, x)\} = \theta^{-1} E\{u_1(\theta t, x)u_1(\theta s, x)\} \tag{3.1.19}$$

where the second equality is motivated by same change of variable as in the proof of lemma 3.1.3 and seen below

$$v_\theta(\theta t, x) = \int_0^t \int_{\mathbb{R}} G_{\theta t,x}(\theta s, y) \tilde{W}(dsdy) \tag{3.1.20}$$

$$= \int_0^{\theta t} \int_{\mathbb{R}} G_{\theta t,x}(s,y) \tilde{W}(d\frac{s}{\theta} dy) = \theta^{-\frac{1}{2}} u_1(\theta t, x). \tag{3.1.21}$$

Continuing on equation 3.1.19 firstly with the use of theorem 3.1.1 and secondly the self-similarity property of bi fractional Brownian motion,

$$\theta^{-1} E\{u_1(\theta t, x) 1_1(\theta s, x)\} = \theta^{-1}(2\pi)^{-\frac{1}{2}} E\{B_{\theta t}^{\frac{1}{2},\frac{1}{2}} B_{\theta s}^{\frac{1}{2},\frac{1}{2}}\} = (\theta 2\pi)^{-\frac{1}{2}} E\{B_t^{\frac{1}{2},\frac{1}{2}} B_s^{\frac{1}{2},\frac{1}{2}}\}. \tag{3.1.22}$$

This completes the proof since as for all these profs, zero mean Gaussian processes are govern by first and second moments. □

When looked at over space the distribution changes with a constant as can be seen below.

Theorem 3.1.5. *For every $t \geq 0$ and, $\theta > 0$, we have the following equality in distribution*

$$\{u_\theta(t,x), x \in \mathbb{R}\} \stackrel{d}{=} \{\theta^{-\frac{1}{2}} m B^{\frac{1}{2}}(x) + S_{\theta t}(x), x \in \mathbb{R}\} \tag{3.1.23}$$

where $B^{\frac{1}{2}}(x)$ is fractional Brownian motion with Hurst parameter $H = \frac{1}{2}$ and, $\{S_{\theta t}(x)\}_{x\in\mathbb{R}}$ is a centred Gaussian process with C^∞ sample paths.

Proof. For every $t > 0$, $\theta > 0$

$$\{u_\theta(t,x), x \in \mathbb{R}\} = \{v_\theta(\theta t,x), x \in \mathbb{R}\} \stackrel{d}{=} \{\theta^{-\frac{1}{2}} u_1(\theta t,x), x \in \mathbb{R}\} \tag{3.1.24}$$

where the derivations in equations 3.1.21 are used once again. Continuing the calculations using the distribution given in lemma 3.1.2,

$$\{\theta^{-\frac{1}{2}} u_1(\theta t,x), x \in \mathbb{R}\} = \{\theta^{-\frac{1}{2}} m B^{\frac{1}{2}}(x) + S_{\theta t}(x), x \in \mathbb{R}\}. \tag{3.1.25}$$

□

3.1.3 Distribution of the variation

In this section the distribution of the variation is derived via a decomposition of the bi fractional Brownian motion and have a deep connection with the Hölder continuity of the process. A new way of deriving the Hölder continuity is studied, again via the Kolmogorov continuity theorem though.

This distribution can be decomposed into a sum of a smooth function and a fractional Brownian motion as the theorem below proposes such that the variation in time can be studied can be analysed. This is done because the variance of the fractional Brownian motion is known combined with the fact that the variation of the smooth function vanishes which will be shown afterwards.

Hela detta och satsen under måste skrivas om

Theorem 3.1.6. [2] *The bi fractional Brownian motion $\{B^{H,K}\}_{t\geq 0}$ have the same distribution as a perturbed fractional Brownian motion with Hurst parameter HK, i.e.*

$$\{B^{H,K}(t)\}_{t\geq 0} \stackrel{d.}{=} \{\sqrt{2^{1-K}} B^{HK}(t) - \sqrt{\frac{2^{-K}K}{\Gamma(1-K)}} X_t^{HK}\}_{t\geq 0}{}^{3} \tag{3.1.26}$$

where X_t^{HK} is a centred Gaussian process (with C^∞ sample paths), more precisely it satisfies $E\{|X_t^{HK} - X_s^{HK}|^2\} \leq C|t-s|^2$.[4]

[2]https://arxiv.org/pdf/0803.2227.pdf

[3]The soruse says variant is an implication of the other one?

[4]Here some things needs to be discussed, **1**. The C^∞ property is not needed right? **2**. Need to prove the more precisely statement.

Proof. Rewriting the covariance function of the bi-fractional Brownian motion by adding and subtracting $t^{2HK} - s^{2HK}$ results in,

$$R^{H,K}(t,s) = 2^{-K}((t^{2H} + s^{2H})^K - |t-s|^{2HK}) = \tag{3.1.27}$$

$$2^{-K}[((t^{2H} + s^{2H})^K - t^{2HK} - s^{2HK}) + (t^{2HK} + s^{2HK} - |t-s|^{2HK})]. \tag{3.1.28}$$

Where the second term is the covariance of a fractional Brownian motion with Hurst parameter HK and a constant. The first term is !!!"The first summand turns out to be non-positive definite and with a change of sign it will be the covariance of a Gaussian process" Måste lista ut vad de har gjort här.!!!! With a change of sign of the first term, which is no issue distribution wise since it is a Gaussian zero mean process. This leads to a positive definite covariance and is the Gaussian process X_t^K defined below.

$$X_t^K := \int_0^\infty (1 - e^{-t\alpha})\alpha^{\frac{-1-K}{2}} dW_\alpha. \tag{3.1.29}$$

The definition comes from the fact that the covariance matches the sought after covariance structure in the first term of the decomposition which can be seen by,

$$E\{X_t^K X_s^K\} = \int_0^\infty (1 - e^{-t\alpha})(1 - e^{-s\alpha})\alpha^{-1-K} d\alpha \tag{3.1.30}$$

$$= \frac{\Gamma(1-K)}{K}(t^K + s^K - (t+s)^K). \tag{3.1.31}$$

The decomposition of the bi fractional Brownian motion covariance now consist of the sum of the process X_t^K and a fractional Brownian with Hurst parameter HK. (This completes the proof since the expected value is equal to zero, the covariance is the same and hence the variance is the same. Since it is a Gaussian variable which in turn is completely govern by it first and second moment this completes the proof.) □

The distribution for the solution in both time and space have the same form derived in two different ways, the distribution in space is split into a sum fractional Brownian motion directly while the distribution in time took some more work as seen above. The variation of the distributions have to be analysed.

Definition 3.1.7. *The exact q-variation of a stochastic process over an interval $[A,B]$. For all $n \geq 1$, let $t_i = A + \frac{i}{n}(B-A)$ for $i = 0, ..., n$. A continuous stochastic process $\{X_t\}_{t \in \mathbb{R}^+}$ admits the q-variation*

$$S^{n,q}_{[A,B]}(X) = \sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}|^q \tag{3.1.32}$$

if the sequence converges in probability as $n \to \infty$.

The fact that variation on the interval (a, b) is $||f||_{\frac{1}{\alpha}} \leq ||f||_{\alpha}(b-a)$ where $||\cdot||_{\alpha}$ is the α-Hölder norm. And since we know the Hölder continuity we can use this. The Hölder coefficient for fBm is the Hurst parameter H and hence we know that the $1/H$-variation is bounded which gives reason to the following lemma.

This need to be written better!!!!!!!!!!!!!!

Lemma 3.1.8. *Let $\{B_t^H\}_{t\geq 0}$ be an fBm with $H \in (0, \frac{1}{2}]$ and consider a centred Gaussian process $\{X_t\}_{t\geq 0}$ such that*

$$E\{(X_t - X_s)^2\} \leq C|t-s|^2, \text{ for every } s, t \geq 0. \tag{3.1.33}$$

Then the process $Y_t^H = aB_t^H + X_t$ has $1/H$-variation over the interval $[A, B]$ which is equal to

Denna egenskapen måste visas

$$a^{\frac{1}{H}} E\{Z^{1/H}\}(B-A). \tag{3.1.34}$$

Proof. We use the Minkowski inequality to write

$$(\sum_{i=0}^{n-1} |aB_{t_{i+1}}^H - aB_t^H|^{1/H})^H - (\sum_{i=0}^{n-1} |X_{t_{i+1}} - X_t|^{1/H})^H \leq (\sum_{i=0}^{n-1} |Y_{t_{i+1}}^H - Y_t^H|^{1/H})^H \tag{3.1.35}$$

$$\leq (\sum_{i=0}^{n-1} |aB_{t_{i+1}}^H - aB_t^H|^{1/H})^H + (\sum_{i=0}^{n-1} |X_{t_{i+1}} - X_t|^{1/H})^H. \tag{3.1.36}$$

This fact coupled with the fact that

$$E\{\sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}|^{1/H}\} \leq E\{\sum_{i=0}^{n-1} (|X_{t_{i+1}} - X_{t_i}|^2)^{1/(2H)}\} \tag{3.1.37}$$

motivated by (which theorem, i forgot) and then

$$E\{\sum_{i=0}^{n-1} (|X_{t_{i+1}} - X_{t_i}|^2)^{1/(2H)}\} \leq CE\{\sum_{i=0}^{n-1} (|t_{i+1} - t_i|^2)^{1/(2H)}\} \tag{3.1.38}$$

$$= C(B-A)\sum_{i=0}^{n-1} (\frac{1}{n^2})^{1/(2H)} = C(B-A)n(\frac{1}{n^2})^{1/(2H)} \leq n^{1-1/H} \tag{3.1.39}$$

which converges to 0 as $n \to \infty$ since $1 - 1/H < 0$. Hence, by and enclosing argument we need only prove that

$$\sum_{i=0}^{n-1} |aB_{t_{i+1}}^H - aB_t^H|^{1/H} = \sum_{i=0}^{n-1} a^{1/H} |B_{t_{i+1}}^H - B_t^H|^{1/H} \xrightarrow{p} a^{\frac{1}{H}}{}^5 E\{Z^{1/H}\}(B-A). \tag{3.1.40}$$

This is almost a whole paper long proof which can be found in [6] I do not know if this is that important, it is at least a guaranteed appendix proof!

However, the proof for only $q = 2, 4$ is a bit easier and can be written. □

Det här beviset kan skrivas in

3.1.4 Estimators based on the temporal variation

Now since we have the distribution of the solution to the SHE we can use lemma 3.1.8 to construct an estimate for θ.

Given everything above an estimator of θ can be created vi the following lemma

Lemma 3.1.9. *Let u_θ be the solution to 3.1.1 with $\sigma = 1$. Then for every $x \in \mathbb{R}$,*

$$S^{n,4}_{[A,B]}(u_\theta) := \sum_{i=0}^{n-1} |u_\theta(t_{i+1}, x) - u_\theta(t_i, x))|^4 \xrightarrow{p} \theta^{-1} C_4^4 2^{-1} E\{Z^4\}(B - A). \tag{3.1.41}$$

Proof. The solution can be written as

$$u_\theta(t, x) = \theta^{-1/4} C_4 B^{\frac{1}{2},\frac{1}{2}}(t) \stackrel{d}{=} \Big(\frac{2}{\theta}\Big)^{\frac{1}{4}} C_4 B^{\frac{1}{4}}(t) + \frac{\theta^{-1/4} C_4}{2\sqrt{2}\Gamma(1-K)} X_t^{\frac{1}{4}} \tag{3.1.42}$$

firstly using proposition 3.1.4 and secondly using theorem 3.1.6. where $C_4 = (2\pi)^{-1/4}$, but this is WRONG!! Then using lemma 3.1.8 the exact 4-variation converges in probability to

$$\Big(\Big(\frac{2}{\theta}\Big)^{\frac{1}{4}} C_4\Big)^4 E\{Z^4\}(B - A) = 2\theta^{-1} C_4^4 E\{Z^4\}(B - A). \tag{3.1.43}$$

□

From this a construction of the diffusion term θ can be constructed as seen below

$$\hat{\theta} = \frac{2C_4^4 E\{Z^4\}(B - A)}{\sum_{i=0}^{n-1} |u_\theta(t_{i+1}, x) - u_\theta(t_i, x))|^4}. \tag{3.1.44}$$

[6] The 1/H-variation of the divergence integral with respect to the fractional Brownian motion for H41=2 and fractional Bessel processes Joa˜o M.E. Guerraa,b,1, David Nualartb

3.1.5 Estimators based on the spacial variation

Lemma 3.1.10. *Let u_θ be the solution to 3.1.1 with $\sigma = 1$. Then for every $t \in \mathbb{R}^+$,*

$$S^{n,4}_{[A,B]}(u_\theta) := \sum_{i=0}^{n-1} |u_\theta(t, x_{i+1}) - u_\theta(t, x_i))|^2 \xrightarrow{p} \theta^{-1} m^2 E\{Z^2\}(B - A). \tag{3.1.45}$$

Proof. The solution can be written as

$$u_\theta(t, x) = \theta^{-\frac{1}{2}} m B^{\frac{1}{2}}(x) + S_{\theta t}(x) \tag{3.1.46}$$

shown in lemma 3.1.5. Then using lemma 3.1.8 the exact 2-variance converges in probability to

$$(\theta^{-\frac{1}{2}} m)^2 E\{Z^2\}(B - A) = \theta^{-1} m^2 E\{Z^2\}(B - A). \tag{3.1.47}$$

□

From this a construction of the diffusion term θ can be constructed as seen below

$$\hat{\theta} = \frac{m^2 E\{Z^2\}(B - A)}{\sum_{i=0}^{n-1} |u_\theta(t, x_{i+1}) - u_\theta(t, x_i))|^2}. \tag{3.1.48}$$

3.1.6 Properties of the estimators

Theorem 3.1.11. *Let $\{B^H_t\}_{t\geq 0}$ be an fBm with $H \in (0, \frac{1}{2}]$ and consider a centred Gaussian process $\{X_t\}_{t\geq 0}$ such that*

$$E\{|X_t - X_s|^2\} \leq C|t - s|^2 \qquad \textit{for every } s, t \geq 0. \tag{3.1.49}$$

Define

$$Y^H_t = B^H(t) + X_t \qquad \textit{for every } t \geq 0. \tag{3.1.50}$$

Then

$$V_{q,n}(Y^H) := \sum_{i=0}^{n-1} \left(\frac{n^{Hq}}{(B - A)^{Hq}} (Y^H_{t_{i+1}} - Y^H_{t_i})^q - E\{Z^q\}\right) \tag{3.1.51}$$

A motivation for the coefficient is needed, it is because we want the renormalised q-variation which is

$$V_{q,n}(X) := \sum_{i=0}^{n-1} \Big(\frac{(X_{t_{i+1}} - X_{t_i})^q}{E\{(X_{t_{i+1}} - X_{t_i})^q\}} - E\{Z^q\}\Big). \tag{3.1.52}$$

For the process Y_t^H the denominator is

$$E\{(Y^H_{t_{i+1}} - Y^H_{t_i})^q\} = E\{(B^H(t_{i+1}) + X_{t_{i+1}} - B^H(t_i) - X_{t_i})^q\} \tag{3.1.53}$$

$$E\{(B^H(t_{i+1} - t_i) + X_{t_{i+1}} - X_{t_i})^q\} = E\{(B^H(\frac{B-A}{n}) + X_{t_{i+1}} - X_{t_i})^q\} \tag{3.1.54}$$

$$= E\{\sum_{j=0}^{q} \binom{q}{j} (B^H(\frac{B-A}{n}))^{q-j}(X_{t_{i+1}} - X_{t_i})^j\} \tag{3.1.55}$$

$$= E\{B^H(\frac{B-A}{n}))^q\} + E\{\sum_{j=1}^{q} \binom{q}{j} (B^H(\frac{B-A}{n}))^{q-j}(X_{t_{i+1}} - X_{t_i})^j\} \tag{3.1.56}$$

$$= \left(\frac{B-A}{n}\right)^{Hq} E\{Z^q\} + \underbrace{E\{\sum_{j=1}^{q} \binom{q}{j} (B^H(\frac{B-A}{n}))^{q-j}(X_{t_{i+1}} - X_{t_i})^j\}}_{\to 0 \text{ as } n \to \infty} \tag{3.1.57}$$

where the binomial expansion is motivated by the fact that q is an integer which is sufficient for out purposes and the convergence is motivated by 3.1.49. The constant is motivated and the proof of the theorem can commence.

Detta beviset måste skrivas klart

Proof. □

(Now he introduces the Wasserstein distance metric, I THINK this is only for the proof of the fact that the estimators are normal, so I will wait with this fact.) In fact this is to prove the Berry-Esséen bounds

`https://arxiv.org/pdf/1912.07917.pdf`

Chapter 4

Numerical Methods for the Stochastic Heat Equation

There are a multitude of ways to simulate solutions to stochastic partial differential equations. Aim is to explore the one-step Θ finite difference schemes.

perhaps look at some some finite element as well?

This chapter will explore the numerical methods to simulate the paths of the following stochastic heat equation

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \alpha\Delta u(x,t) = \dot{\mathcal{F}}(x,t) & t > 0,\ x \in U \\ u(x,0) = u_0(x) & x \in U. \end{cases} \tag{4.0.1}$$

With $U \subset \mathbb{R}^d$, and for our purposes we will consider the compact interval $U = [a,b]$. The noise term $\dot{\mathcal{F}}$ will first be the simple white-noise and later we will explore white-coloured noise. Note that we now denote the drift term as α instead of θ, this is to not confuse it with the Θ-schemes presented below.

If I find a good way of simulating

4.1 Note on heat equation

In this thesis we have primarily studied the heat equation on $\mathbb{R} \times [0,T]$, but to simulate solutions effectively we will restrict ourselves to the heat equation on $[a,b] \times [0,T]$ instead.

4.2 The case of white noise

We know that the equation defined in 4.0.1 is only formal, since the solutions $u(x,t)$ are nowhere differentiable. By approximating the white noise process $\mathcal{W}$ to something smoother, we can actually apply numerical methods such as difference schemes to equation 4.0.1.

Let the space domain be $U = [a,b]$ and time $t \in [0,T]$. We first consider the stochastic heat equation driven by space-time white noise

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \alpha\Delta u(x,t) = \dot{\mathcal{W}}(x,t) & 0 < t < T,\ x \in [a,b] \\ u(x,0) = u_0(x) & x \in [a,b]. \end{cases} \tag{4.2.1}$$

Where $u_0(x)$ is some integrable function. First our goal is to find a suitable discretisation of the white noise process $\mathcal{W}(x,t)$. We follow the approach of Allen et al. [4]. We discretize the time $[0,T]$ and space $[0,1]$ into rectangles $[x_j, x_{j+1}] \times [t_m, t_{m+1}]$, with $j = 0,1,2,\ldots,N+1$, and $m = 0,1,2,\ldots,M$. The desired number of points of the solution in space and time being $N+2$ and M respectively, and $\Delta t = T/M$, $\Delta x = \frac{b-a}{N+1}$, such that $x_j = a + j\Delta x$ and $t_m = m\Delta t$.

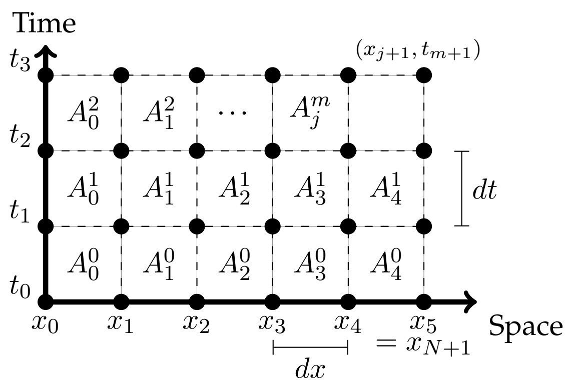

Figure 4.1: Illustration of the grid scheme

Employing the following approximation of the mixed derivative $\mathcal{W}(x,t)$. We let $A_k^l = [x_k, x_{k+1}) \times [t_m, t_{m+1})$ be the half open rectangles over the discritization of the domain. The white noise $\mathcal{W}(x,t)$ can be approximated as

Something is a bit off here with the approximation of white noise, at least with rectangles. Check to make sure it's correct.

$$\hat{\mathcal{W}}(x,t) = \sum_{k=0}^{N}\sum_{l=0}^{M-1} \mathcal{W}(R_k^l)\mathbb{1}_{A_k^l}(x,t). \tag{4.2.2}$$

Such that

$$\mathcal{W}_j^m := \hat{\mathcal{W}}(x_j, t_m) \in N(0, \Delta t\Delta x). \tag{4.2.3}$$

For the proof of the coming theorem 4.2.1 it is useful to have the following representation of the white noise in 4.2.2:

$$\dot{\hat{\mathcal{W}}}(x,t) = \frac{\partial^2 \hat{\mathcal{W}}}{\partial x \partial t}(x,t) := \frac{\sum_{k=0}^{N}\sum_{l=1}^{M-1} \eta_k^l \sqrt{\Delta t \Delta x}\, \mathbb{1}_{[x_k,x_{k+1})}(x)\mathbb{1}_{[t_l,t_{l+1})}(t)}{\Delta t \Delta x}. \tag{4.2.4}$$

Where we set $\eta_k^l = \frac{1}{\sqrt{\Delta t \Delta x}} \int_{t_l}^{t_{l+1}} \int_{x_j}^{x_{j+1}} \mathcal{W}(dxdt) \in N(0,1)$ i.i.d. for every k and l. We have especially in every discrete point (x_j, t_m) that

$$\begin{aligned} \hat{\mathcal{W}}(x_j,t_m) = &\sum_{k=0}^{N}\sum_{l=1}^{M-1} \eta_k^l \sqrt{\Delta t \Delta x}\, \mathbb{1}_{[x_k,x_{k+1})}(x_j)\mathbb{1}_{[t_l,t_{l+1})}(t_m) \\ =&\eta_j^m \sqrt{\Delta t \Delta x} = \mathcal{W}_j^m \end{aligned} \tag{4.2.5}$$

The following theorem gives how well $\hat{\mathcal{W}}$ approximates the white noise in the sense of the integrals of a Hölder continuous deterministic function f. Note that the integrals with respect to $\hat{\mathcal{W}}$ are to be considered in the Riemann-Stiltjes sense.

Lemma 4.2.1. *Assume that f is a deterministic function defined on $[0,T] \times [a,b]$ with constants $\alpha, \beta \in [0,1]$ and $\gamma \geq 0$ such that*

$$|f(x,t) - f(y,s)| \leq \gamma \left(|t-s|^\beta + |x-y|^\alpha \right), \tag{4.2.6}$$

for any (x,t) and $(y,s) \in [0,T] \times [a,b]$ (i.e. f is Hölder-continuous). Then it holds that

$$\begin{aligned} &E\left(\left[\int_0^T \int_a^b f(x,t)\mathcal{W}(dxdt) - \int_0^T \int_a^b f(x,t)\hat{\mathcal{W}}(dxdt) \right]^2 \right) \\ \leq &2T(b-a)\gamma^2 \left((\Delta t)^{2\beta} + (\Delta x)^{2\alpha} \right). \end{aligned} \tag{4.2.7}$$

Proof. We note first that:

$$\int_0^T \int_a^b f(x,t)\hat{\mathcal{W}}(dxdt) = \sum_{k=0}^{N}\sum_{l=0}^{M-1} \int_{t_l}^{t_{l+1}} \int_{x_k}^{x_{k+1}} f(x,t)\hat{\mathcal{W}}(dxdt).$$

And the same holds for the integrals with respect to $\mathcal{W}$. Since $\hat{\mathcal{W}}(x,t)$ is a step function on the rectangles A_k^l for every sample of the η_k^l we have that

Rewriting Riemann-Stiltjes integral w.r.t step function. Don't see why this holds completely. Something with rules on Riemann-Stiltjes integrals (see `https://personal.math.ubc.ca/~feldman/m321/step.pdf`) I guess and using the expression for η_k^l but I'll go with it for now.

$$\begin{aligned} &\int_{t_l}^{t_{l+1}} \int_{x_k}^{x_{k+1}} f(x,t)\hat{\mathcal{W}}(dxdt) \\ =&\int_{t_l}^{t_{l+1}} \int_{x_k}^{x_{k+1}} \left(\frac{1}{\Delta x \Delta t} \int_{t_l}^{t_{l+1}} \int_{x_k}^{x_{k+1}} f(y,s)dyds \right) \mathcal{W}(dxdt). \end{aligned}$$

With the above expression we obtain

$$
\begin{aligned}
&E\left(\left[\int_0^T\int_a^b f(x,t)\mathcal{W}(dxdt)-\int_0^T\int_a^b f(x,t)\hat{\mathcal{W}}(dxdt)\right]^2\right)\\
=&E\left(\left[\sum_{k=0}^{N}\sum_{l=0}^{M-1}\left(\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}} f(x,t)\mathcal{W}(dxdt)-\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}} f(x,t)\hat{\mathcal{W}}(dxdt)\right)\right]^2\right)\\
=&E\left(\left[\sum_{k=0}^{N}\sum_{l=0}^{M-1}\left(\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}}\left(f(x,t)-\frac{1}{\Delta x\Delta t}\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}} f(y,s)dyds\right)\mathcal{W}(dxdt)\right)\right]^2\right)
\end{aligned}
$$

Now we apply corollary 2.1.15 from Wiener's isometry and then the Hölder-inequality of f. Note also that $|t-s|\leq\Delta t$ and $|x-y|\leq\Delta x$ for $t,s\in[t_l,t_{l+1}]$ and $x,y\in[x_k,x_{k+1}]$.

$$
\begin{aligned}
&=\sum_{k=0}^{N}\sum_{l=0}^{M-1}\left(\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}}\left(\frac{1}{\Delta x\Delta t}\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}}\big(f(x,t)-f(y,s)\big)\,dyds\right)^2 dxdt\right)\\
&\leq\frac{\gamma^2}{(\Delta x\Delta t)^2}\sum_{k=0}^{N}\sum_{l=0}^{M-1}\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}}\left(\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}}\left(|t-s|^\beta+|x-y|^\alpha\right)dyds\right)^2 dxdt\\
&\leq\frac{\gamma^2}{(\Delta x\Delta t)^2}\sum_{k=0}^{N}\sum_{l=0}^{M-1}\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}}\left(\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}}\left((\Delta t)^\beta+(\Delta x)^\alpha\right)dyds\right)^2 dxdt\\
&=\frac{\gamma^2}{(\Delta x\Delta t)^2}\sum_{k=0}^{N}\sum_{l=0}^{M-1}\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}}\left((\Delta t)^\beta+(\Delta x)^\alpha\right)^2\left(\int_{t_l}^{t_{l+1}}\int_{x_k}^{x_{k+1}}1dyds\right)^2 dxdt\\
&=\gamma^2\left((\Delta t)^\beta+(\Delta x)^\alpha\right)^2(N+1)\Delta xM\Delta t\\
&=T(b-a)\gamma^2\left((\Delta t)^\beta+(\Delta x)^\alpha\right)^2\\
&\leq 2T(b-a)\gamma^2\left((\Delta t)^{2\beta}+(\Delta x)^{2\alpha}\right).
\end{aligned}
$$

Last inequality follows because $(x+y)^2\leq 2(x^2+y^2)$ for $x,y\in\mathbb{R}$. □

4.3 One-Step Θ Schemes

We discretise equation 4.0.1 in the following way:

$$
\begin{cases}
\frac{u_j^{m+1}-u_j^m}{\Delta t}=\alpha\left(\Theta\frac{u_{j+1}^{m+1}-2u_j^{m+1}+u_{j-1}^{m+1}}{\Delta x^2}+(1-\Theta)\frac{u_{j+1}^{m+1}-2u_j^{m+1}+u_{j-1}^{m+1}}{\Delta x^2}\right)+\frac{\mathcal{W}_j^m}{\Delta t\Delta x}\\
u_0^m=u_{N+1}^m=0,\ m=0,1,\ldots,M.\\
u_j^0=u_0(x_j),\qquad j=1,2,\ldots,N.
\end{cases}
\tag{4.3.1}
$$

With N and M being the desired number of points in space and time respectively, and $\Delta t = T/M$, $\Delta x = \frac{b-a}{N+1}$, the white noise $\mathcal{W}_j^m := \mathcal{W}(A_j^m) \in N(0, \Delta x \Delta t)$ are i.i.d normal variables, where A_j^m is the rectangle $[x_j, x_{j+1}] \times [t_m, t_{m+1}]$.

We can rewrite the equations, with $r_1 = \frac{\Theta \alpha \Delta t}{\Delta x^2}$ and $r_2 = \frac{(1-\Theta)\alpha \Delta t}{\Delta x^2}$ we have for every $j = 1, 2, \ldots, N$ and $m = 0, 1, \ldots, M$.

$$-r_1 u_{j-1}^{m+1} + (1 + 2r_1) u_j^{m+1} - r_1 u_{j+1}^{m+1} = -r_2 u_{j-1}^m + (1 - 2r_2) u_j^m - r_2 u_{j+1}^m. \tag{4.3.2}$$

In figure below we can see some visual ways of remembering the schemes. :

Discuss the different schemes more. Fix with references and so on

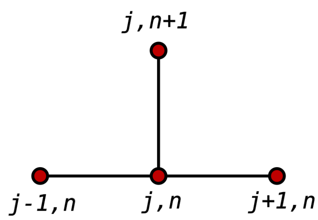

Figure 4.2: Explicit Euler Stencil, $\Theta = 0$.

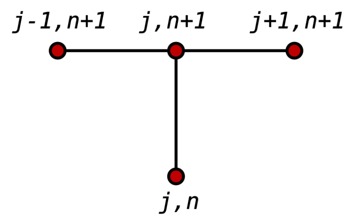

Figure 4.3: Implicit Euler Stencil, $\Theta = 1$.

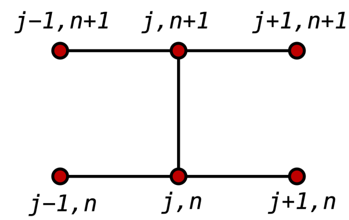

Figure 4.4: Crank-Nicolson Stencil, $\Theta = 1/2$.

$\alpha \Delta t / \Delta x^2 = c \iff \Delta t = (c/\alpha) \Delta x^2$

Chapter 5

Further Work and Applications

"Modern climate models pose an ever-increasing storage burden to computational facilities, and the upcoming generation of global simulations from the next Intergovernmental Panel on Climate Change will require a substantial share of the budget of research centers worldwide to be allocated just for this task. A statistical model can be used as a means to mitigate the storage burden by providing a stochastic approximation of the climate simulations. Indeed, if a suitably validated statistical model can be formulated to draw realizations whose spatiotemporal structure is similar to that of the original computer simulations, then the estimated parameters are effectively all the information that needs to be stored. In this work we propose a new statistical model defined via a stochastic partial differential equation (SPDE) on the sphere and in evolving time"[1]

[1]COMPRESSION OF CLIMATE SIMULATIONS WITH A NONSTATIONARY GLOBAL SPATIOTEMPORAL SPDE MODEL, GEIR-ARNE FUGLSTAD1, STEFANO CASTRUCCIO

Chapter 6

Appendix

6.1 Appendix: Probability theory

6.1.1 Karhunen-Loeve expansions

There are several ways of representing Gaussian processes. An analog of Fourier series expansion is the so called Karhunen-Loève expansion. The definitions and proof of the theorems that are presented can be found in [6].

Say that we have a random field $X(t)$ indexed by the *normed* set $\mathcal{T}$[1] with a mean process, $\mu : \mathcal{T} \to \mathbf{R}$ that is identically zero ($\mu(t) \equiv 0$) and a continuous covariance function $C : \mathcal{T} \times \mathcal{T} \to \mathbf{R}$. Let also $E(X(t)^2) < \infty$ for all $t \in \mathcal{T}$. Such a random field $X(t)$ is called a zero-mean second order process with continuous covariance function.

Theorem 6.1.1. *Let $X(t)$ be a zero-mean second-order process over the probability space $(\Omega, \Sigma, \mathcal{P})$, with t in the normed measure space $\mathcal{T}$ and with a continuous covariance function C. Let e_k be a set of orthogonal functions of $L^2(\mathcal{T})$ and assume that e_k and λ_k satisfy the Fredholm equation:*

$$\int_{\mathcal{T}} C(t,s)e_k(t)dt = \lambda_k e_k(s) \tag{6.1.1}$$

where the e_k are called the eigenfunctions of C with corresponding eigenvalues λ_k. Then

$$X(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\xi_k e_k(t)$$

[1]I think $\mathcal{T}$ needs to additionaly be a measure space for all of this to work but I am not sure... Since we are integrating with respect to $t \in \mathcal{T}$ later and so on...

where the convergence is in $L^2(\Omega)$ and uniform in t.

The random variables ξ_k are zero-mean, uncorrelated, have variance 1 and are given by

$$\xi_k = \frac{1}{\sqrt{\lambda_k}} \int_{\mathcal{T}} X(t) e_k(t) dt.$$

The ξ_k above are actually the inner product of the zero-mean process $X(t)$ and the ON-basis functions $e_k(t)$ on the Hilbert space $L^2(\mathcal{T})$.

Remark 6.1.2. *Note that in general ξ_k are only uncorrelated, but for Gaussian processes they are even i.i.d.*

Important example

Example 6.1.3. ***Brownian Motion***

When the Gaussian process $X(t)$ has $C(t,s) = \min(t,s)$ (Wiener Process), where $t, s \in \mathcal{T} = [0,T]$, we have[2]

$$X(t) = \sum_{k=1}^{\infty} \frac{2\sqrt{2T}}{(2k-1)\pi} \xi_k \sin\left((2k-1)\frac{\pi t}{2T}\right)$$

Where ξ_k are i.i.d[3] *standard Gaussian variables. The eigenfunction are*

$$e_k(t) = \sqrt{2T} \sin\left((2k-1)\frac{\pi t}{2T}\right)$$

with eigenvalues

$$\lambda_k = \frac{1}{(2k-1)^2 \pi^2}.$$

When $\mathcal{T} = \mathcal{T}_1 \times \cdots \times \mathcal{T}_d$ is a product of one-dimensional spaces the Fredholm integral equations is almost always horrible to solve analytically since it involves d-times integration. The following theorem gives a way to reduce this problem to the one-dimensional case.

Theorem 6.1.4. *Let $X(t)$ satisfy the same conditions as in theorem 6.1.1 where $t = (t_1, \ldots, t_d) \in \mathcal{T} = \mathcal{T}_1 \times \cdots \times \mathcal{T}_d$. Assume that the covariance function C is separable, i.e that*

$$C(t,s) = \prod_{k=1}^{d} C(t_i, s_i)$$

[2] We should show this!
[3] Note remark 6.1.2

and the univariate Karhunen-Loeve expansion for $X(t_i)$ with its covariance function $C(t_i, s_i)$, $t_i, s_i \in \mathcal{T}_i$ is

$$X(t_i) = \sum_{k\geq 1} \sqrt{\lambda_k^{(i)}} e_k^{(i)}(t_i)\xi_k^{(i)}.$$

Then the Karhunen-Loeve expansion for $X(t)$ is

$$X(t) = X(t_1, \ldots, t_d) = \sum_{k\geq 1} \sqrt{\lambda_k} e_k(t_1, \ldots, t_d)\xi_k \tag{6.1.2}$$

where

$$\lambda_k = \prod_{i=1}^{d} \lambda_k^{(i)} \qquad\qquad e_k(t_1, \ldots, t_d) = \prod_{i=1}^{d} e_k^{(i)}(t_i)$$

Note that the sum in 6.1.2 is a multisum running over $i = 1, \ldots, d$ for all k.

Since we will be looking at stochastic *partial* differential equations, it will be useful to have an expansion of the Brownian sheet. This expansion follows from the expansion for Brownian motion.

Example 6.1.5. ***Brownian Sheet*** *The Brownian sheet $W(x, t)$, where $x \in [0, D]$*[4] *and $t \in [0, T]$ is a biparameter Gaussian random field with a zero mean and covariance function*

$$Cov(W(x_1, t_1), W(x_2, t_2)) = C(x_1, t_1; x_2, t_2) = \min(x_1, x_2) \cdot \min(t_1, t_2).$$

Since the covariance function is separable this satisfies the conditions for theorem 6.1.4 and we have that

$$W(x, t) = \sum_{i\geq 1} \sum_{j\geq 1} \frac{8\sqrt{TD}\xi_k}{\pi^2(2k-1)(2j-1)} \sin\left((2k-1)\frac{\pi t}{2T}\right) \sin\left((2j-1)\frac{\pi x}{2D}\right).$$

Where $\xi_k \in N(0, 1)$ are i.i.d.

[4] How when x negative again??

6.2 Appendix

6.2.1 Distributions and Fourier analysis

6.2.2 Functional analysis

Proposition 6.2.1. *Let $A : X \to Y$ be a map between two inner product spaces X and Y. If A is an isometry then A preserves inner products.*

Proof. Suppose that A is an isometry. Then for any $x_1, x_2 \in X$ we have that

$$\begin{aligned}
\|Ax_1 - Ax_2\|_Y^2 = \|x_1 - x_2\|_X^2 = \\
\langle Ax_1 - Ax_2, Ax_1 - Ax_2\rangle_Y = \langle x_1 - x_2, x_1 - x_2\rangle_X = \\
\|Ax_1\|_Y^2 + \|Ax_2\|_Y^2 - 2\langle Ax_1, Ax_2\rangle_Y = \|x_1\|_X^2 + \|x_2\|_X^2 - 2\langle x_1, x_2\rangle_X \iff \\
\langle Ax_1, Ax_2\rangle_Y = \langle x_1, x_2\rangle_X
\end{aligned}$$

□

Proposition 6.2.2. *A linear isometry from a dense subset of a normed space to a complete normed space admits a unique extension.*

6.2.3 Solution formula of linear heat equation

Aim of this section is to give an explicit solution u over $\mathbb{R}^d$ and $t > 0$. Assuming that all the following manipulations are well defined for u (like the Fourier transforms, inverse transform, all integrals, etc.). We have

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \alpha^2 \Delta u(x,t) = f(x,t) & t > 0,\ x \in \mathbb{R}^d \\ u(x,0) = g(x) & x \in \mathbb{R}^d. \end{cases} \tag{6.2.1}$$

To solve 6.2.1 we solve first the homogeneous problem

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \alpha^2 \Delta u(x,t) = 0 & t > 0,\ x \in \mathbb{R}^d \\ u(x,0) = g(x) & x \in \mathbb{R}^d. \end{cases} \tag{6.2.2}$$

Recall the Fourier transform

$$\mathcal{F}\Big(u(x,t)\Big)(\omega,t) = \hat{u}(\omega,t) := \int_{\mathbb{R}^d} u(x,t) e^{-2\pi\omega\cdot x} dx.$$

Where $\omega \cdot x$ is the inner product on $\mathbb{R}^d$. Using that $\mathcal{F}\Big(\Delta u\Big) = -|\omega|^2 \hat{u}$ and the convolution identity $\mathcal{F}(f * g) = \hat{f}\hat{g}$. We will do a Fourier transform on 6.2.2 to get the corresponding ODE

$$\begin{cases} \frac{d}{dt}\hat{u}(\omega,t) + |\omega|^2 \alpha^2 \hat{u}(\omega,t) = 0 & t > 0,\ \omega \in \mathbb{R}^d \\ \hat{u}(\omega,0) = \hat{g}(\omega) & \omega \in \mathbb{R}^d. \end{cases} \tag{6.2.3}$$

Solving 6.2.3 and doing an inverse transform gives us the homogeneous solution

$$u_h(x,t) = \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha^2 t}}}{(2\alpha)^d (\pi t)^{n/2}} g(y) dy,\ \ x \in \mathbb{R}^d\ , t > 0. \tag{6.2.4}$$

We proceed to the inhomogenuous problem with zero initial value to complete the superpositition:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \alpha^2 \Delta u(x,t) = f(x,t) & t > 0,\ x \in \mathbb{R}^d \\ u(x,0) = 0 & x \in \mathbb{R}^d. \end{cases} \tag{6.2.5}$$

To solve 6.2.5 we will invoke **Duhamel's principle**. Consider once again a homogenuous equation, for $0 < s < t$, of the form:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \alpha^2 \Delta u(x,t) = 0 & t > s,\ x \in \mathbb{R}^d \\ u(x,s) = f(x,s) & x \in \mathbb{R}^d. \end{cases} \tag{6.2.6}$$

By a translation $t' = t - s$ we obtain a PDE of the form in 6.2.2, which admits the solution:

$$u_s(x,t) = \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha^2(t-s)}}}{(2\alpha)^d(\pi(t-s))^{d/2}} f(y,s)dy, \;\; x \in \mathbb{R}^d\,, t > s. \tag{6.2.7}$$

Duhamel's principle gives us that the solution to 6.2.6 is simply to integrate 6.2.7 with respect to s for $0 < s < t$, which gives the particular solution

$$u_p(x,t) = \int_0^t \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha^2(t-s)}}}{(2\alpha)^d(\pi(t-s))^{d/2}} f(y,s)dyds, \;\; x \in \mathbb{R}^d\,, t > 0. \tag{6.2.8}$$

We can now state the solution.

Proposition 6.2.3. *The solution to 6.2.1 is by the superposition principle:*

$$u(x,t) = \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha^2 t}}}{(2\alpha)^d(\pi t)^{n/2}} g(y)dy + \int_0^t \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha^2(t-s)}}}{(2\alpha)^d(\pi(t-s))^{d/2}} f(y,s)dyds$$

We where $G(x,t;s,y) = \frac{e^{\frac{-|x-y|^2}{4\alpha^2(t-s)}}}{(2\alpha)^d(\pi(t-s))^{d/2}}$ is the fundamental solution to 6.2.1. Compare this to the solution stated in section 1 1.1.4.

Remark 6.2.4 (Regarding the integral $\int_0^t \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha^2(t-s)}}}{(2\alpha)^d(\pi(t-s))^{d/2}} f(y,s)dyds$)**.** *If* $\frac{\partial^2 F}{\partial t \partial x} = f$, *other ways of writing the integral are*

$$\int_0^t \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha^2(t-s)}}}{(2\alpha)^d(\pi(t-s))^{d/2}} F(dy,ds) = \int_0^t \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha^2(t-s)}}}{(2\alpha)^d(\pi(t-s))^{d/2}} dF(y,s)$$

6.2.4 MATLAB Code

Below is the code for simulating the solution to the SHE with white noise.

```
%% Walsh Finite Difference Crank Nicolson and Estimation of
   Drift
clear
alpha = 10;
theta = 0.5;

M = 1000; % Number of time points. Including t = 0
N = 999; % Inner space points. N + 2 with boundary
x_end = 1;
% Time,  m
% Space, j
% *: u_jm,
% Below ----- is initial cond.

%   Bound         Bound
%     *    * * *    *
%     *    * * *    *
%     *    * * *    *
%          -----
%     *    * * *    *
% j = 0    1 2 3    4 = N + 1
%              N
dx = x_end/(N + 1);

U = zeros(N + 2,M);
c = 1/(pi - 2); %  c = dt / dx^2 = 1/(pi -2) by the paper
dt = dx^2 * c / alpha;
T = M * dt; % Stopping time is not decided by us, still sol.
   is self similar
r_1 = alpha * dt * theta /(dx^2);
r_2 = alpha * dt * (1 - theta) / (dx^2);

A = diag((1+2*r_1)*ones(1,N)) + diag(-r_1*ones(1,N-1),1) + ...
    diag(-r_1*ones(1,N-1),-1);

A2 = diag((1-2*r_2)*ones(1,N)) + diag(r_2*ones(1,N-1),1) + ...
    diag(r_2*ones(1,N-1),-1);
```

```
A_tilde = diag((1-2*r_2)*ones(1,N+2)) + diag(r_2*ones(1,N+1)
   ,1) + ...
    diag(r_2*ones(1,N+1),-1);
A_tilde(1,:) = [];
A_tilde(end,:) = [];
x_points = linspace(0, x_end, N + 2);
u0 = @(x) x.*(1-x)/2;
U(:, 1) = u0(x_points);

W = sqrt(dt*dx)*normrnd(zeros(N, M), 1);
BC = zeros(N, M);
BC(1, :) = U(1, :);
BC(end, :) = U(end, :);

time_points = linspace(0, T, M);

%%
for m = 1:M-1
    m
    b = A2*U(2:end-1, m) + W(:, m)/(dx) + r_1*BC(:, m + 1) +
       r_2*BC(:, m);
    U(2:end-1, m + 1) = A\b;
end

%%
u_time = U(round(N/2), :);
sum = 0;
for m = 1:M-1
    sum = sum + (u_time(m + 1) - u_time(m))^4;
end
drift_param_real = alpha
drift_param_est = 3/(pi*sum)*T


%%
close all
figure
h = surf(time_points, x_points, U)
set(h,'LineStyle','none')

figure
plot(time_points, u_time);
```

Bibliography

[1] J. Schur. "Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen." In: *Journal für die reine und angewandte Mathematik* 140 (1911), pp. 1–28. URL: http://eudml.org/doc/149352.

[2] Gel'fand I.M and Ya Vilenkin N. *Generalized functions. Vol. 4, Applications of harmonic analysis*. New York : Academic Press, 1964. ISBN: 9781483252995. URL: https://archive.org/details/gelfand-vilenkin-generalized-functions-vol-4-applications-of-harmonic-analysis/page/147/mode/2up.

[3] John B. Walsh. "An introduction to stochastic partial differential equations". In: *École d'Été de Probabilités de Saint Flour XIV - 1984*. Ed. by P. L. Hennequin. Berlin, Heidelberg: Springer Berlin Heidelberg, 1986, pp. 265–439. ISBN: 978-3-540-39781-6.

[4] S. J. Novosel E. J. Allen and Z. Zhang. "Finite element and difference approximation of some linear stochastic partial differential equations". In: *Stochastics and Stochastic Reports* 64.1-2 (1998), pp. 117–142. DOI: 10.1080/17442509808834159. eprint: https://doi.org/10.1080/17442509808834159. URL: https://doi.org/10.1080/17442509808834159.

[5] Robert Dalang. "Extending the Martingale Measure Stochastic Integral With Applications to Spatially Homogeneous S.P.D.E.'s". In: *Electronic Journal of Probability* 4.none (1999), pp. 1–29. DOI: 10.1214/EJP.v4-43. URL: https://doi.org/10.1214/EJP.v4-43.

[6] Limin Wang. "Karhunen-Loeve Expansions and their Applications". Available at https://etheses.lse.ac.uk/2950/1/U615901.pdf. PhD thesis. The London School of Economics and Political Science, Mar. 2008.

[7] R Dalang et al. *A Minicourse on Stochastic Partial Differential Equations*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009. ISBN: 978-3-540-85993-2.

[8] Ciprian Tudor. In: *Fractional Calculus and Applied Analysis* 17.1 (2014), pp. 224–246. DOI: doi:10.2478/s13540-014-0164-6. URL: https://doi.org/10.2478/s13540-014-0164-6.

[9] Saulius Norvidas. "On positive definite distributions with compact support". In: *Banach Journal of Mathematical Analysis* 9.3 (2015), pp. 14–23. DOI: 10.15352/bjma/09-3-2. URL: https://doi.org/10.15352/bjma/09-3-2.

[10] Robert C. Dalang and Marta Sanz-Solé. *An Introduction to Stochastic Partial Differential Equations: Part I*. 2024. arXiv: 2402.02119 [math.PR].