Problem Set 1

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Problem 1

(a). From the question, we know that

$$\pi_t dt = q_{t+dt}k_{t+dt} - q_t k_t + (a_i(k_t) - \iota_t k_t)\, dt$$
$$\frac{dk_t}{dt}dt = (\Phi(\iota_t) - \delta)k_t$$

Using the linear approximation for (q_{t+dt}, k_{t+dt}) around (q_t, k_t):

$$q_{t+dt} \approx q_t + \frac{dq_t}{dt}dt,\ k_{t+dt} \approx k_t + \frac{dk_t}{dt}dt$$

Substitute this approximation into the expression:

$$\pi_t dt = (q_t + \frac{dq_t}{dt}dt)(k_t + \frac{dk_t}{dt}dt) - q_t k_t + (a_i k_t - \iota_t k_t)dt$$

Expand and simplify the terms:

$$\pi_t dt = q_t k_t + q_t \frac{dk_t}{dt}dt + \frac{dq_t}{dt}k_t dt + \frac{dq_t}{dt}\frac{dk_t}{dt}dt^2 - q_t k_t + (a_i k_t - \iota_t k_t)dt$$

Ignoring higher-order terms (dt^2), and divide by dt:

$$\pi_t = q_t \frac{dk_t}{dt} + \frac{dq_t}{dt}k_t + a_i k_t - \iota_t k_t$$

Substitute $\frac{dk_t}{dt}$ with $(\Phi(\iota_t) - \delta)k_t$:

$$\pi_t = q_t(\Phi(\iota_t) - \delta)k_t + \frac{dq_t}{dt}k_t + a_i k_t - \iota_t k_t$$

Divide by $q_t k_t$:

$$r_t = \frac{\pi_t}{q_t k_t} = (\Phi(\iota_t) - \delta) + \frac{dq_t}{dt}\frac{1}{q_t} + \frac{a^i - \iota_t}{q_t}$$

Decomposition:

- Investment return: $\Phi(\iota_t) - \delta$
- Gain from holding and reselling: $\frac{dq_t}{dt}\frac{1}{q_t}$
- Dividend yield: $\frac{a^i}{q_t}$
- Investment cost: $\frac{\iota_t}{q_t}$

(b). Since the choice of investment rate is a static and time-separable problem. An agent chooses ι^i to maximize her return

$$r_t = \{\frac{\pi_t}{q_t k_t} = (\Phi(\iota_t) - \delta) + \frac{dq_t}{dt}\frac{1}{q_t} + \frac{a^i - \iota_t}{q_t}\}dt$$

Taking first order condition gives the Tobin q's condition: $\Phi'(\iota_t^i) = \frac{1}{q_t}$. Since q_t is the same for both agents, this imply $\iota_t = \iota_t^e = \iota_t^h$

(c). Using the function form $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$ and take first order condition we get:

$$\frac{1}{q_t} = \frac{1}{\phi\iota_t + 1} \Rightarrow \iota_t = \frac{q_t - 1}{\phi}$$

Problem 2

(a). We can derive the below equilibrium conditions:

$$q_t K_t \left(\rho^e \eta_t + \rho^h (1-\eta_t)\right) + K_t \left(\iota_t^e \kappa_t^e + \iota_t^h \kappa_t^h\right) = (a_t^e \kappa_t^e + a_t^h \kappa_t^h) K_t$$

$$\Rightarrow q_t \left((\rho^e - \rho^h)\eta_t + \rho^h\right) = \kappa_t^e (a_t^e - \iota_t^e) + (1-\kappa_t^e)(a_t^h - \iota_t^h)$$

$$\iota_t = \frac{q_t - 1}{\phi}$$

$$\kappa_t^e \leq \frac{\eta_t}{1-\ell}$$

$$\mu_t^\eta = (1-\eta_t)\left[-\left(\rho^e - \rho^h\right) + \frac{\kappa^*}{\eta_t}\left(\frac{a_t^e - a_t^h}{q_t}\right)\right]$$

(Above are global precise solutions to be taken to computer)

For steady state: let $\mu_t^\eta = 0$, also steady state $\kappa^* = \kappa^e = 1 - \kappa^h$ to obtain

$$\kappa^* \left(a_t^e - a_t^h\right) + q^* \rho^h = \kappa^* \left(a^e - \iota^*\right) + (1-\kappa^*)\left(a^h - \iota^*\right)$$

We get steady-state conditions:

$$q^* = \frac{a_h - \iota^*}{\rho^h}$$

$$\kappa^* = \frac{\eta^*}{1-\ell}$$

$$\frac{\rho^e - \rho^h}{\rho^h} = \frac{1}{1-\ell}\frac{a^e - a^h}{a^h - \iota^*}$$

Using $a^h = a^e \kappa^*$ and the Tobin's Q condition,

$$\kappa^* = \frac{\rho^h (1-\ell) + (\rho^e - \rho^h)\frac{q^*-1}{\phi a^e}}{(\rho^e - \rho^h) + \rho^h (1-\ell)}$$

At steady state for K, we know that the change in capital has to be zero, which requires:

$$\Phi(\iota^*) = \delta$$

$$\frac{1}{\phi}\log(\phi \iota^* + 1) = \delta$$

$$\iota^* = \frac{e^{\phi\delta} - 1}{\phi}$$

Therefore, when $\frac{q^*-1}{\phi a^e} = \frac{e^{\phi\delta}-1}{\phi}$, aggregate capital K reaches steady-state, thought its absolute level is not pinned down, since scaling it gives the same set of equilibrium conditions

Problem 3

Please see updated version.