

Problem Set 1

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Problem 1

(a). From the question, we know that

$$\begin{aligned}\pi_t dt &= q_{t+dt} k_{t+dt} - q_t k_t + (a_i(k_t) - \iota_t k_t) dt \\ \frac{dk_t}{dt} dt &= (\Phi(\iota_t) - \delta) k_t\end{aligned}$$

Using the linear approximation for (q_{t+dt}, k_{t+dt}) around (q_t, k_t) :

$$q_{t+dt} \approx q_t + \frac{dq_t}{dt} dt, \quad k_{t+dt} \approx k_t + \frac{dk_t}{dt} dt$$

Substitute this approximation into the expression:

$$\pi_t dt = (q_t + \frac{dq_t}{dt} dt)(k_t + \frac{dk_t}{dt} dt) - q_t k_t + (a_i k_t - \iota_t k_t) dt$$

Expand and simplify the terms:

$$\pi_t dt = q_t k_t + q_t \frac{dk_t}{dt} dt + \frac{dq_t}{dt} k_t dt + \frac{dq_t}{dt} \frac{dk_t}{dt} dt^2 - q_t k_t + (a_i k_t - \iota_t k_t) dt$$

Ignoring higher-order terms (dt^2), and divide by dt :

$$\pi_t = q_t \frac{dk_t}{dt} + \frac{dq_t}{dt} k_t + a_i k_t - \iota_t k_t$$

Substitute $\frac{dk_t}{dt}$ with $(\Phi(\iota_t) - \delta) k_t$:

$$\pi_t = q_t (\Phi(\iota_t) - \delta) k_t + \frac{dq_t}{dt} k_t + a_i k_t - \iota_t k_t$$

Divide by $q_t k_t$:

$$r_t = \frac{\pi_t}{q_t k_t} = (\Phi(\iota_t) - \delta) + \frac{dq_t}{dt} \frac{1}{q_t} + \frac{a^i - \iota_t}{q_t}$$

Decomposition:

- Investment return: $\Phi(\iota_t) - \delta$
- Gain from holding and reselling: $\frac{dq_t}{dt} \frac{1}{q_t}$
- Dividend yield: $\frac{a^i}{q_t}$
- Investment cost: $\frac{\iota_t}{q_t}$

(b). Since the choice of investment rate is a static and time-separable problem. An agent chooses ι^i to maximize her return

$$r_t = \left\{ \frac{\pi_t}{q_t k_t} = (\Phi(\iota_t) - \delta) + \frac{dq_t}{dt} \frac{1}{q_t} + \frac{a^i - \iota_t}{q_t} \right\} dt$$

Taking first order condition gives the Tobin q's condition: $\Phi'(\iota_t^i) = \frac{1}{q_t^i}$. Since q_t is the same for both agents, this imply $\iota_t = \iota_t^e = \iota_t^h$

(c). Using the function form $\Phi(\iota) = \frac{1}{\phi} \log(\phi \iota + 1)$ and take first order condition we get:

$$\frac{1}{q_t} = \frac{1}{\phi \iota_t + 1} \Rightarrow \iota_t = \frac{q_t - 1}{\phi}$$

Problem 2

(a). We can derive the below equilibrium conditions:

$$q_t K_t (\rho^e \eta_t + \rho^h (1 - \eta_t)) + K_t (\iota_t^e \kappa_t^e + \iota_t^h \kappa_t^h) = (a_t^e \kappa_t^e + a_t^h \kappa_t^h) K_t$$

$$\Rightarrow q_t ((\rho^e - \rho^h) \eta_t + \rho^h) = \kappa_t^e (a_t^e - \iota_t^e) + (1 - \kappa_t^e) (a_t^h - \iota_t^h)$$

$$\iota_t = \frac{q_t - 1}{\phi}$$

$$\kappa_t^e \leq \frac{\eta_t}{1 - \ell}$$

$$\mu_t^\eta = (1 - \eta_t) \left[-(\rho^e - \rho^h) + \frac{\kappa^*}{\eta_t} \left(\frac{a_t^e - a_t^h}{q_t} \right) \right]$$

(Above are global precise solutions to be taken to computer)

For steady state: let $\mu_t^\eta = 0$, also steady state $\kappa^* = \kappa^e = 1 - \kappa^h$

From drift condition $\mu_t^\eta = 0$ alone we get

$$q^* = \frac{1}{1 - \ell} \left(\frac{a^e - a^h}{\rho^e - \rho^h} \right)$$

Combine goods market and drift condition to obtain

$$\kappa^* (a_t^e - a_t^h) + q^* \rho^h = \kappa^* (a^e - \iota^*) + (1 - \kappa^*) (a^h - \iota^*)$$

$$\Rightarrow q^* = \frac{a_h - \iota^*}{\rho^h}$$

Further:

$$\iota^* = \frac{q^* - 1}{\phi} \Rightarrow q^* = \frac{a_h \phi + 1}{\rho^h \phi + 1}$$

$$\kappa^* = \frac{\eta^*}{1 - \ell}$$

Using $a^h = a^e \kappa^*$,

$$\kappa^* = 1 - \frac{(1 - \ell)(\rho^e - \rho^h)(a_h \phi + 1)}{a^e(\rho^h \phi + 1)}$$

At steady state for K, we know that the change in capital has to be zero, which requires:

$$\Phi(\iota^*) = \delta$$

$$\frac{1}{\phi} \log(\phi \iota^* + 1) = \delta$$

$$\iota^* = \frac{e^{\phi \delta} - 1}{\phi}$$

Therefore, when $\frac{q^* - 1}{\phi a^e} = \frac{e^{\phi \delta} - 1}{\phi}$, aggregate capital K reaches steady-state, though its absolute level is not pinned down, since scaling it gives the same set of equilibrium conditions

Problem 3

Please see [updated version](#).