Problem Set 9

Exercises for the Lecture Fundamentals of Simulation Methods

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Submit the solution to your tutor in electronic form by **Wednesday December 18, 2019**.

1. Numerical hydrodynamics – part 2

In this exercise, we continue our exploration of numerical fluid dynamics as started in the last two homework assignments.

The first two equations of hydrodynamics in 1D are given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \,, \tag{1}$$
$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} = -\frac{\partial P}{\partial x} \,, \tag{2}$$

where ρ is the density, u the velocity, and P the pressure. We assume that the gas temperature is constant with value T_0, i. e. we adopt an isothermal equation of state of $P = c_s^2 \rho$. In equilibrium, we have the static solution with $\rho(x,t) = \rho_0$, $P(x,t) = P_0$, and $u(x,t) = 0$.

Consider a small linear perturbation of the density,

$$\rho(x,t) = \rho_0 + \delta\rho(x,t) \,. \tag{3}$$

We can Fourier decompose $\delta\rho(x,t)$ into modes of the type

$$\delta\rho(x,t) = A e^{i(kx - \omega t)} \,, \tag{4}$$

and we recall from exercise 7 that this gives rise to sound waves with a dispersion relation

$$\omega^2 - c_s^2 k^2 = 0 \,. \tag{5}$$

1.1. Isothermal 1D hydrodynamics solver (*12 points*)

1. Based on the general-purpose advection routine of the previous exercise construct a simple *isothermal* hydrodynamics solver in 1D. Implement periodic boundary conditions using the ghost cell technique.

2. Use your code to solve the following 1D isothermal hydrodynamics problem: The x-grid goes from $x = -100$ to $x = 100$, the boundary conditions are periodic, the isothermal sound speed is $c_s = 1$. The initial conditions are

$$\rho(x, t = 0) = 1 + \exp\left(-\frac{x^2}{200}\right) \tag{6}$$

$$u(x, t = 0) = 0 \tag{7}$$

   Plot the system at 10 different times: $t = 10$, $t = 20$, $\ldots$, $t = 90$, $t = 100$. Describe your results, and try to explain what you see.

   Choose a sensible cell size Δx. You may play with different values of Δx, to find out which choice leads to satisfying results. For simplicity, use also a fixed time step Δt, but make it small enough that the algorithm remains stable at all times (i.e. that the CFL condition is met at all times).

3. Now do the same, but with variable time step. Calculate the CFL condition Δt_{CFL} at each time step and choose $\Delta t = 0.4 * \Delta t_{\mathrm{CFL}}$ for safety.

4. Figure out how to produce a movie of your hydrodynamic waves. For this you must write intermediate results to a file after fixed time intervals $\Delta t_{\mathrm{write}}$. Since you have a variable time step you therefore must be clever to assure that the algorithm arrives exactly at those write-times, despite of the a-priori-unknown Δt. Once you have a file containing a sequence of snapshots, produce a sequence of images and use your favorite movie-making facility to make a movie.

1.2. HLL Riemann solver (*8 points*)

From the website you can download `codes_riemann1.zip`. It contains python code for a 1D Riemann solver of the HLL type.

1. Apply the Riemann solver to the above wave problem, and show that the results are better, i. e. have less spurious oscillations.

2. Redo the wave problem with 10x larger spatial resolution (careful: obey the CFL condition!), both with the classic solver (Section 1.1) and the Riemann solver. Explain the flow features and how they differ from the lower resolution case.