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Simulation Methods – Sheet 6

Ex 01

1D advection equation $\partial_t u + v\,\partial_x u = 0$ (1)

Let $q \in C^1(\mathbb{R})$. Then we define $u(x,t) = q(x - vt)$ and find that

$$\frac{\partial u}{\partial t} = q'\,\partial_t(x - vt) = -vq'$$
$$\frac{\partial u}{\partial x} = q'\,\partial_x(x - vt = q'$$

$$\Rightarrow \quad \partial_t u + v\,\partial_x u = -vq' + vq' = 0$$

Thus $u(x,t) = q(x - vt)$ is a solution of (1).

Ex 02

Euler equations: $\partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} + \frac{\vec{\nabla}P}{\rho} = 0$ (2a)

$\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0$ (2b)

(for completeness:) $\partial_t \varepsilon + \vec{\nabla}\cdot(\varepsilon\vec{v}) + P\,\vec{\nabla}\cdot\vec{v} = 0$ (2c)

We know that the flow is steady and homogeneous, thus

$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial P}{\partial t} = 0, \quad \frac{\partial \vec{v}}{\partial t} = 0 \quad \text{and} \quad \frac{d\rho}{dt} = 0$$

$$\Rightarrow \quad \vec{v}\cdot\vec{\nabla}\rho = 0 \quad \Rightarrow \quad \rho = \text{const}$$

Before we consider small perturbations of the system, we transform our reference frame to the rest frame of the fluid. This is similar to letting ourselves drift with the flow. In this particular reference frame, we find $\vec{v} = 0$.

Now considering small perturbations, we find

$$\rho \rightarrow \rho + \delta\rho \longrightarrow \rho + \delta\rho$$
$$P \rightarrow P + \delta P \longrightarrow P + \delta P$$
$$\vec{v} \rightarrow \vec{v} + \delta\vec{v} \xrightarrow[\text{moving frame}]{} \delta\vec{v}$$

Plugging this into (2a) and (2b), we find

$$\partial_t (\rho+\delta\rho) + \vec{\nabla}\cdot\left((\rho+\delta\rho)(\vec{v}+\delta\vec{v})\right) = 0$$

$$(\rho+\delta\rho)\left[\partial_t(\vec{v}+\delta\vec{v}) + \left[(\vec{v}+\delta\vec{v})\cdot\vec{\nabla}\right](\vec{v}+\delta\vec{v})\right] + \vec{\nabla}(P+\delta P) = 0$$

completely dropped because everything is of 2^{nd} order at least or prop. to $\vec{v}$!

Dropping all terms of 2^{nd} order in the perturbations (mixed terms like $\delta\rho\,\delta\vec{v}$ are also dropped) and using the conditions for homogeneity, steadiness and the rest frame of the fluid, we see that

$$\partial_t(\delta\rho) + \rho\,\vec{\nabla}\cdot\delta\vec{v} = 0$$

$$\rho\,\partial_t\,\delta\vec{v} + \vec{\nabla}(P+\delta P) = 0$$

From (2b) we also see that

$$\rho\underbrace{\left(\partial_t\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right)}_{=0} + \vec{\nabla}P \overset{\vec{v}=0}{=} \vec{\nabla}P = 0$$

the pressure gradient vanishes in the new frame.

Applying $\delta P = c_s^2\,\delta\rho$, we reduce further to

$$\partial_t(\delta\rho) + \rho\,\vec{\nabla}\cdot(\delta\vec{v}) = 0 \qquad (3a)$$

$$\rho\,\partial_t(\delta\vec{v}) + c_s^2\vec{\nabla}(\delta\rho) = 0 \qquad (3b)$$

Throwing a gradient on eq. (3b) and a time derivative on (3a), we get (ρ is constant!):

$$\left(\partial_t^2(\delta\rho)\right) + \rho\,\vec{\nabla}\cdot(\partial_t\vec{v}) = 0$$

$$\rho\,\vec{\nabla}\cdot(\partial_t\vec{v}) + c_s^2\,\vec{\nabla}^2(\delta\rho) = 0$$

Plugging (3a) into (3b) yields

$$\left(-\partial_t^2 + c_s^2\vec{\nabla}^2\right)\delta\rho = 0$$

$\Rightarrow$ Pressure perturbations propagate like waves

This gives rise to sound waves in fluids.

Ex 03

From geography class, we remember that the fastest waves in earthquakes are P-waves, travelling at 5–7 km/s.

The speed of sound in air on the other hand reads 331.3 m/s

1) P-waves travel straight through the earth's interior. Thus the highest possible propagation time is

$$T_{P\text{wave}} = 2\cdot(6\,371\text{ km})\cdot(6\text{ km/s})^{-1} = 2.12\text{ s} \approx 2\text{ seconds}$$

2) The sound waves on the other hand cannot penetrate the earths crust and have to travel along the earths surface. Half or the circumference of the earth is given through: $U_{1/2} = 20037.5$ km

Thus $T_{sound} = \frac{20\,0\cancel{37},5 \text{ km}}{331,1 \text{ m/s}} = 60518 s \approx 16.8$ hours

The sound of the earthquake cannot be heard in Europe as the sound waves loose most of their energy due to distance and travel time.

Their amplitude becomes very small because energy is lost due to ~~the~~ amplitude $\propto \frac{1}{r^2}$ and friction (viscosity of air etc.)