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Simulation Methods - Problem Set 2

Ex 01 Order of an integration scheme

For a numerical solution y_{num} to be second-order accurate, it is necessary that $\|y - y_{num}\| = \mathcal{O}(h^2)$ where y is the exact solution and h is the step size.

Thus, we calculate $y_{n+1} - y(t_n + \Delta t)$ where Δt is the step size:

We begin with the Taylor expansion of $y(t_n + \Delta t)$ around t_n:

$$\Rightarrow y(t_n+\Delta t) = y(t_n) + \left.\frac{\partial y}{\partial t}\right|_{t=t_n} \Delta t + \mathcal{O}(\Delta t^2) = y_n + \left.\frac{\partial y}{\partial t}\right|_{t_n} + \mathcal{O}(\Delta t^2)$$

Plugging this in yields:

$$y_{n+1} - y_n - \left.\frac{\partial y}{\partial t}\right|_{t_n} + \mathcal{O}(\Delta t^2)$$

Now we use the definition of the Runge-Kutta scheme to rewrite y_{n+1} and simplify:

$$\left[\frac{1}{2}(k_1 + k_2) - \left.\frac{\partial y}{\partial t}\right|_{t_n}\right]\Delta t + \mathcal{O}(\Delta t^2) \quad ; \quad k_1 = f(t_n, y_n) = \left.\frac{\partial y}{\partial t}\right|_{t_n}$$

$$= \left(\frac{k_2}{2} - \frac{1}{2}\left.\frac{\partial y}{\partial t}\right|_{t_n}\right)\Delta t + \mathcal{O}(\Delta t^2)$$

We calculate another Taylor expansion:

$$k_2 = f(y_n + k_1 \Delta t, t_n + \Delta t) = f(y_n, t_n) + \left.\frac{\partial f}{\partial y}\right|_{y=y_n} k_1 \Delta t + \left.\frac{\partial f}{\partial t}\right|_{t=t_n} \Delta t + \mathcal{O}(\Delta t^2)$$

$$\Rightarrow \frac{1}{2}\left(k_2 - \left.\frac{\partial y}{\partial t}\right|_{t_n}\right)\Delta t = \frac{1}{2}\left(k_2 - f(y_n, t_n)\right)\Delta t$$

$$= \left[\left.\frac{\partial f}{\partial y}\right|_{y=y_n} \Delta t + \left.\frac{\partial f}{\partial t}\right|_{t=t_n} + \mathcal{O}(\Delta t^2)\right]\Delta t + \mathcal{O}(\Delta t^2)$$

$$= \mathcal{O}(\Delta t^2)$$

$\Rightarrow$ ~~[illegible]~~ Heuns method is of 2^{nd} order accuracy.

Ex 02 Double Pendulum

(a) Lagrangian of the double pendulum:

$$\mathcal{L} = \frac{m_1}{2}(l_1\dot\phi_1)^2 + \frac{m_2}{2}\left[(l_1\dot\phi_1)^2 + (l_2\dot\phi_2)^2 + 2 l_1 l_2 \dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)\right]$$
$$- m_1 g l_1(1-\cos\phi_1) - m_2 g\left[l_1(1-\cos\phi_1) + l_2(1-\cos\phi_2)\right]$$

Conjugate momenta:

$$q_1 = \frac{\partial \mathcal{L}}{\partial \dot\phi_1} \quad ; \quad \ddot q_2 = \frac{\partial \mathcal{L}}{\partial \dot\phi_2}$$

$$\Rightarrow \quad q_1 = m_1 l_1^2\dot\phi_1 + m_2 l_1^2\dot\phi_1 + m_2 l_1 l_2 \dot\phi_2\cos(\phi_1-\phi_2) \qquad (1)$$
$$q_2 = m_2 l_2^2\dot\phi_2 + m_2 l_1 l_2\dot\phi_1\cos(\phi_1-\phi_2) \qquad (2)$$

Derivatives after ϕ_1 and ϕ_2:

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = -m_2 l_1 l_2 \dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2) - (m_1+m_2) g l_1\sin(\phi_1)$$
$$\frac{\partial \mathcal{L}}{\partial \phi_2} = +m_2 l_1 l_2 \dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2) - m_2 g l_2\sin(\phi_2)$$

Thus we find the equations of motion to be:

$$\frac{d}{dt}q_1 = + m_2 l_1 l_2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2) - (m_1+m_2) g l_1\sin(\phi_1)$$
$$\frac{d}{dt}q_2 = m_2 l_1 l_2\dot\phi_1\dot\phi_2\sin(\phi_1-\phi_2) - m_2 g l_2\sin(\phi_2)$$

(b) From equations (1) and (2), we can obtain expressions for $\dot\phi_1$ and $\dot\phi_2$:

$$q_1 = (m_1+m_2) l_1^2\dot\phi_1 + m_2 l_1 l_2\dot\phi_2\cos(\phi_1-\phi_2)$$
$$q_2 = m_2 l_2^2\dot\phi_2 + m_2 l_1 l_2\dot\phi_1\cos(\phi_1-\phi_2)$$

This a 2×2 system of linear equations. The solution reads:

$$\dot\phi_1 = \frac{m_2 l_2^2 q_1 - m_2 l_1 l_2\cos(\phi_1-\phi_2) q_2}{m_2(m_1+m_2) l_1^2 l_2^2 - m_2^2 l_1^2 l_2^2\cos^2(\phi_1-\phi_2)} =: f_1(q_1, q_2, \phi_1, \phi_2)$$

$$\dot\phi_2 = \frac{(m_1+m_2) l_1^2 q_2 - m_2 l_1 l_2\cos(\phi_1-\phi_2) q_1}{m_2(m_1+m_2) l_1^2 l_2^2 - m_2^2 l_1^2 l_2^2\cos^2(\phi_1-\phi_2)} =: f_2(q_1, q_2, \phi_1, \phi_2)$$

We can then express $\dot q_1$ and $\dot q_2$:

$$\dot q_1 = -m_2 l_1 l_2 f_1 f_2\sin(\phi_1-\phi_2) - (m_1+m_2) g l_1\sin\phi_1 =: f_3(q_1, q_2, \phi_1, \phi_2)$$
$$\dot q_2 = m_2 l_1 l_2 f_1 f_2\sin(\phi_1-\phi_2) - m_2 g l_2\sin(\phi_2) =: f_4(q_1, q_2, \phi_1, \phi_2)$$

(c) + (d) see source code in .py file