1 Model Extensions: CHORN

We now describe a set of modifications to the original Harmonic Oscillating Recurrent Network (HORN), yielding the *Cayley Harmonic Oscillating Recurrent Network (CHORN)*. These extensions are designed to (i) enforce structured linear dynamics in the recurrent coupling, (ii) permit heterogeneous oscillator parameters, (iii) introduce a phase-equivariant readout aligned with the intrinsic symmetries of the system and (iv) define a hebbian inspired learning rule for the recurrent weights.

1.1 Heterogeneous Oscillator Parameters

Unlike the original HORN formulation, where oscillator parameters are shared across the reservoir, CHORN allows node-wise heterogeneity. Specifically, we define

$$\boldsymbol{\omega} = [\omega_1, \ldots, \omega_{n_{\text{res}}}]^\top, \quad \boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_{n_{\text{res}}}]^\top, \quad \boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_{n_{\text{res}}}]^\top,$$

with all quantities strictly positive. This enables frequency diversity and differential damping across the reservoir, improving expressivity while preserving oscillatory structure.

1.2 Cayley-Parametrised Recurrent Dynamics

In the original HORN model, the recurrent operator is implemented as an unconstrained linear mapping. In CHORN, we instead parameterise the recurrent dynamics via a skew-symmetric generator and the Cayley transform, ensuring norm-preserving linear flow in the absence of damping and forcing.

Let

$$S \in \mathbb{R}^{n_{\text{res}} \times n_{\text{res}}}, \quad A = S - S^\top,$$

so that $A^\top = -A$. The recurrent weight matrix is defined as

$$W = (I + A)^{-1}(I - A).$$

This construction guarantees that W is orthogonal whenever $I + A$ is invertible. The recurrent mapping is then given by

$$f^{\text{rec}}(\mathbf{v}) = W\mathbf{v},$$

where $\mathbf{v} \in \mathbb{R}^{n_{\text{res}}}$ denotes the reservoir velocity state.

1.3 Discrete-Time Reservoir Dynamics

Let $\mathbf{z}(t) \in \mathbb{R}^{n_{\text{res}}}$ denote the reservoir position state and $\dot{\mathbf{z}}(t) \in \mathbb{R}^{n_{\text{res}}}$ its velocity. Using a symplectic Euler discretisation with step size h, the CHORN dynamics are given by

$$\begin{aligned}
\dot{\mathbf{z}}(t+h) &= \dot{\mathbf{z}}(t) + h\Big(\boldsymbol{\alpha} \odot \tanh\left(f^{\text{in}}(\mathbf{x}(t)) + g_{\text{rec}} f^{\text{rec}}(\dot{\mathbf{z}}(t))\right) - \boldsymbol{\omega}^2 \odot \mathbf{z}(t) - 2\boldsymbol{\gamma} \odot \dot{\mathbf{z}}(t)\Big), \\
\mathbf{z}(t+h) &= \mathbf{z}(t) + h\,\dot{\mathbf{z}}(t+h),
\end{aligned}$$

where $\odot$ denotes elementwise multiplication and $g_{\text{rec}} = n_{\text{res}}^{-1/2}$ is a recurrent gain normalisation.

1.4 Phase–Amplitude Coordinates

To expose the intrinsic rotational structure of the dynamics, we introduce the complex-valued change of variables

$$\boldsymbol{\zeta}(t) = \mathbf{z}(t) + i\,\boldsymbol{\omega}^{-1} \odot \dot{\mathbf{z}}(t),$$

from which we define node-wise amplitude and phase variables

$$r_i(t)^2 = z_i(t)^2 + \frac{\dot{z}_i(t)^2}{\omega_i^2}, \quad \theta_i(t) = \tan^{-1}\left(\frac{\dot{z}_i(t)}{\omega_i z_i(t)}\right).$$

To remove trivial phase rotation induced by intrinsic frequencies, we define the demodulated phase

$$\tilde{\theta}_i(t) = \theta_i(t) - \omega_i t.$$

1.5 Phase-Equivariant Readout

The CHORN readout operates on a phase-equivariant feature vector constructed from amplitudes and pairwise phase differences. Specifically, we define

$$\boldsymbol{\phi}(t) = \Big[\mathbf{r}(t), \{\cos(\tilde{\theta}_i(t) - \tilde{\theta}_j(t))\}_{i<j}, \{\sin(\tilde{\theta}_i(t) - \tilde{\theta}_j(t))\}_{i<j}\Big]^\top \in \mathbb{R}^{n_{\text{phase}}},$$

with

$$n_{\text{phase}} = n_{\text{res}} + n_{\text{res}}(n_{\text{res}} - 1).$$

The predicted output is then given by

$$\hat{\mathbf{y}}(t) = f^{\text{out}}(\boldsymbol{\phi}(t)),$$

where f^{out} is a linear mapping. By construction, this readout is invariant under global phase shifts $\theta_i(t) \mapsto \theta_i(t) + \varphi$, ensuring equivariance to the natural symmetry of the oscillator ensemble.

1.6 Initial Conditions

Rather than zero initial conditions, CHORN initialises the reservoir on the unit circle with evenly spaced phases:

$$\theta_i(0) = \frac{2\pi i}{n_{\text{res}}}, \quad z_i(0) = \cos\theta_i(0), \quad \dot{z}_i(0) = \sin\theta_i(0),$$

optionally perturbed by small stochastic deviations in phase and amplitude. This induces maximal phase diversity at initialisation and accelerates symmetry breaking during transient dynamics.

1.7 Phase-Synchrony Hebbian Learning Rule

In addition to gradient-based optimisation of the readout, CHORN allows for a local Hebbian learning rule acting directly on the recurrent generator S. This rule uses phase relationships between oscillators to modify the skew-symmetric coupling structure. Recall the instantaneous phase

$$\theta_i(t) = \tan^{-1}\left(\frac{\dot{z}_i(t)}{\omega_i z_i(t)}\right).$$

For each ordered pair (i, j), define the phase difference

$$\Delta\theta_{ij}(t) = \theta_i(t) - \theta_j(t).$$

The Hebbian interaction signal is

$$H_{ij}(t) = \sin\big(\Delta\theta_{ij}(t)\big).$$

Noticing that

$$H_{ij}(t) = -H_{ji}(t),$$

so $H(t)$ is skew-symmetric for every t therefore the learning signal crucially preserves the constraint required by the Cayley parametrisation.

During a forward pass over a sequence of length T, the accumulated Hebbian signal is

$$\bar{H}_{ij} = \frac{1}{T}\sum_{t=1}^{T} \sin\big(\theta_i(t) - \theta_j(t)\big),$$

followed by averaging over the batch dimension.

Thus,

$$\bar{H} \in \mathbb{R}^{n_{\mathrm{res}} \times n_{\mathrm{res}}}, \quad \bar{H}^\top = -\bar{H}.$$

The recurrent operator is defined via the skew-symmetric generator

$$A = S - S^\top, \qquad W = (I + A)^{-1}(I - A).$$

Rather than updating W directly, we update the underlying parameter S according to

$$S \leftarrow S + \eta\,\bar{H},$$

where $\eta > 0$ is a Hebbian learning rate.

Because $\bar{H}$ is skew-symmetric, the induced update on

$$A = S - S^\top$$

remains skew-symmetric. Consequently, the Cayley transform continues to produce an orthogonal recurrent matrix W, preserving norm-controlled linear dynamics.

The learning rule strengthens directed coupling when oscillator i systematically leads oscillator j in phase:

$$\sin(\theta_i - \theta_j) > 0 \quad \Longrightarrow \quad S_{ij} \text{ increases}.$$

Conversely, if j leads i, the coupling is adjusted in the opposite direction. When oscillators are phase-locked ($\Delta\theta_{ij} \approx 0$ or π), the update vanishes. It can be interpreted as a continuous-time analogue of spike-timing-dependent plasticity (STDP).