MA 351, HW 5

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February 14, 2019

Section 3.2: True/False: 3.12, 3.20; Exercises: 3.41
Section 3.3: True/False: 3.21, 3.23, 3.30; Exercises 3.64 (a)(b)(f)(k), 3.71, 3.72, 3.78, 3.83, 3.88
Section 3.5: True/False: 3.32, 3.33 Exercises: 3.126 (a)(b)(c)(e), 3.130 (c)(d)(e)

1 Section 3.2

1.1 True/False

Question 3.12. If A and B are 2×2 matrices, $(AB)^2 = A^2B^2$.

Answer: False, since $AB \neq BA$ does not hold for all 2×2 matrices

Question 3.20. Suppose that matrices A and B satisfy $AB = 0$. Then either $A = 0$ or $B = 0$

Answer: False

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

1.2 Exercises

Question 3.41. For the given matrix A, find a 3×2 nonzero matrix B such that $AB = 0$. Prove that any such matrix B must have rank 1. [Hint: The columns of B belong to the nullspace of A.]

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ -1 & -2 \end{bmatrix}$$

Since the basis for all Bs only has one vector, any matrix B will have rank 1.

2 Section 3.3

2.1 True/False

Question 3.21. The following matrix is invertible:

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 0 \\ 4 & 6 & 2 & 6 \end{bmatrix}$$

Answer: False

$$\begin{bmatrix} 1 & 2 & -1 & 4 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 4 & 6 & 2 & 6 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 4 & 0 & \frac{1}{2} & -2 & \frac{1}{2} \\ 0 & 1 & 0 & -1 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix}$$

Question 3.23. Suppose that A is an invertible matrix and B is any matrix for which BA is defined. Then the matrices BA and B need not have the same rank.
Answer: False

Question 3.30. Suppose that A is an $n \times n$ invertible matrix and B is any $n \times n$ matrix. Then $ABA^{-1} = B$
Answer: False, matrix multiplication is generally not commutative

2.2 Exercises

Question 3.64. Use the method of Example 3.8 on page 185 to invert the following matrices (if possible).
Answer:

(a)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 4 & 4 & 2 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{13}{5} & \frac{3}{2} & -\frac{6}{5} \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & \frac{6}{5} & -\frac{1}{2} & \frac{2}{5} \end{array}\right]$$

(b)

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

(f)

$$\left[\begin{array}{cccc|cccc} 2 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 4 & -1 & 4 & -2 & 0 & 1 & 0 & 0 \\ 8 & -3 & 10 & 0 & 0 & 0 & 1 & 0 \\ 6 & -3 & 8 & 9 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{3}{14} & \frac{9}{7} & -\frac{11}{14} & \frac{2}{7} \\ 0 & 1 & 0 & 0 & -\frac{16}{7} & \frac{9}{7} & -\frac{2}{7} & \frac{2}{7} \\ 0 & 0 & 1 & 0 & -\frac{6}{7} & -\frac{9}{14} & \frac{9}{14} & -\frac{1}{7} \\ 0 & 0 & 0 & 1 & -\frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & \frac{1}{7} \end{array}\right]$$

(k)

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & 0 & 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{array}\right]$$

Question 3.71. Assume that $ad - bc \neq 0$. Find the inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Answer:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} + \frac{bc}{a^2\left[d - \frac{bc}{a}\right]} & -\frac{b}{a\left[d - \frac{bc}{a}\right]} \\ 0 & 1 & -\frac{c}{a\left[d - \frac{bc}{a}\right]} & \frac{1}{d - \frac{bc}{a}} \end{array}\right]$$

Question 3.72. Compute the inverse of the matrix A :

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

Answer:

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Question 3.78. Suppose that A is an $n \times n$ matrix such that $A^3 + 3A^2 + 2A + 5I = 0$. Show that A is invertible.

Answer:

$$\begin{aligned} A^3 + 3A^2 + 2A + 5I &= 0 \\ A^3 + 3A^2 + 2A &= -5I \\ (A^2 + 3A + 2)A &= -5I \\ -\frac{1}{5}(A^2 + 3A + 2)A &= I \end{aligned}$$

Therefore, matrix A is invertable

Question 3.83. Prove that if A is invertible, then so are A^2, A^3, and A^4. What are the inverses of these matrices? (Assume that you know A^{-1}.)

Answer:

$$AA_{-1} = I$$
$$A^2 = AA \Rightarrow A^2A^{-1}A^{-1} = AAA^{-1}A^{-1} \Rightarrow A^2A^{-2} = I$$
$$A^3 = AAA \Rightarrow A^3A^{-1}A^{-1}A^{-1} = AAAA^{-1}A^{-1}A^{-1} \Rightarrow A^3A^{-3} = I$$
$$A^4 = AAAA \Rightarrow A^4A^{-1}A^{-1}A^{-1}A^{-1} = AAAAA^{-1}A^{-1}A^{-1}A^{-1} \Rightarrow A^4A^{-4} = I$$

$$inv(A^2) = (A^{-1})^2, inv(A^3) = (A^{-1})^3, inv(A^4) = (A^{-1})^4$$

Question 3.88. We know that only square matrices can be invertible. We also know that if a square matrix has a right inverse, the right inverse is also a left inverse. It is possible, however, for a non square matrix to have either a right inverse or a left inverse (but not both). Parts (a)-(d) explore these possibilities.

(a) For the given matrix A find a 3×2 matrix B such that $AB = I$, where I is the 2×2 identity matrix. [Hint: If B_1 and B_2 are the columns of B , then $AB_j = I_j$.]

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Answer:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

$$B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$$

(b) Suppose that A is any 2×3 matrix with rank 2. Prove that A has a right inverse. [Hint: Is the equation $AX = B$ solvable for all $B \in \mathbb{R}^2$]?

Answer: $AB = I$, $rank(I) = 2$. Since $3 > 2$, there always will be a solution to $AB = I$

(c) Show conversely that if A is a 2×3 matrix that has a right inverse, then A has rank 2

Answer: Let B to be the right inverse of A

$$rank(AB) = rank(I) = 2$$
$$rank(AB) \leq rank(B) \leq 2$$

(d) Under what circumstances does an $m \times n$ matrix have a right inverse? State your condition in terms of rank and prove your answer.

Answer: $m \leq rank(A)$ and rows are linearly independent. This way we ensure that the matrix will be either square and invertable, or will not be square and will have at least 1 free variable.

3 Section 3.5

3.1 True/False

Question 3.32. Let $\mathcal{B}$ and $\overline{\mathcal{B}}$ be ordered bases for $\mathbb{R}^n$. Then the matrix of the identity transformation of $\mathbb{R}^n$ into itself with respect to $\mathcal{B}$ and $\overline{\mathcal{B}}$ is the $n \times n$ identity matrix I.

Answer: False

Question 3.33. Let $\mathcal{B}$ and $\overline{\mathcal{B}}$ be ordered bases for $\mathbb{R}^n$ where $\mathcal{B} = \overline{\mathcal{B}}$. Then the matrix of the identity transformation of $\mathbb{R}^n$ into itself with respect to $\mathcal{B}$ and $\overline{\mathcal{B}}$ is the $n \times n$ identity matrix I.

Answer: True

3.2 Exercises

Question 3.126. Compute the matrix M with respect to the standard ordered basis of $M(2,2)$ for the linear transformation $L : M(2,2) \to M(2,2)$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(a) $L(X) = AX$

Answer:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix}$$
$$\begin{bmatrix} a+2c \\ b+2d \\ 3a+4c \\ 3b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

(b) $L(X) = XA$

Answer:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix}$$
$$\begin{bmatrix} a+3b \\ 2a+4b \\ c+3d \\ 2c+4d \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

(c) $L(X) = AXA^t$

Answer:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} a+2b+2c+4d & 3a+4b+6c+8d \\ 3a+6b+4c+8d & 9a+12b+12c+16d \end{bmatrix}$$

$$\begin{bmatrix} a+2b+2c+4d \\ 3a+4b+6c+8d \\ 3a+6b+4c+8d \\ 9a+12b+12c+16d \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 4 & 6 & 8 \\ 3 & 6 & 4 & 8 \\ 9 & 12 & 12 & 16 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

(e) $L(X) = X + X^t$

Answer:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$$

$$\begin{bmatrix} 2a \\ b+c \\ b+c \\ 2d \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Question 3.130. Compute the matrix M with respect to the standard ordered basis of $\mathcal{P}_2$ for the linear transformation $L : \mathcal{P}_2 \to \mathcal{P}_2$, where:

(c) $L(y) = y' - y$

Answer:

$$\begin{aligned} L(a+bx+cx^2) &= (a+bx+cx^2)' - (a+bx+cx^2) \\ &= b + 2cx - a - bx - cx^2 \\ &= b - a + (2c-b)x - cx^2 \end{aligned}$$

$$\begin{bmatrix} -a+b \\ -b+2c \\ -c \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

(d) $L(y) = y' + 3y$

Answer:

$$\begin{aligned} L(a+bx+cx^2) &= (a+bx+cx^2)' + 3(a+bx+cx^2) \\ &= b + 2cx + 3a + 3bx + 3cx^2 \\ &= (3a+b) + (2c+3b)x + 3cx^2 \end{aligned}$$

$$\begin{bmatrix} 3a+b \\ 3b+2c \\ 3c \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

(e) $L(y) = x^2y'' + 3xy' - 7y$

Answer:

$$\begin{aligned} L(a+bx+cx^2) &= x^2(a+bx+cx^2)'' + 3x(a+bx+cx^2)' - 7(a+bx+cx^2) \\ &= x^2(2c) + 3x(b+2cx) - 7(a+bx+cx^2) \\ &= 2cx^2 + 3bx + 6cx^2 - 7a - 7bx - 7cx^2 \\ &= -7a - 4bx + cx^2 \end{aligned}$$

$$\begin{bmatrix} -7a \\ -4b \\ c \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$