MA 351, HW 10

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Section 6.2: True/False: 6.11, 6.12, 6.14
Exercises: 6.19, 6.20, 6.22, 6.23, 6.24, 6.27, 6.30, 6.32, 6.37

1 Section 6.2

1.1 True/False

Question 6.11. $[1,1,1]^t$ is perpendicular to the span of $[1,-1,0]^t$ and $[2,2,-4]^t$ in $\mathbb{R}^3$

Answer: True, since $[1,1,1]^t \cdot [1,-1,0]^t = 0$, $[1,1,1]^t \cdot [2,2,-4]^t = 0$

Question 6.12. Let $\mathcal{W}$ be a two-dimensional subspace of $\mathbb{R}^5$. Suppose that $\{Q_1, Q_2\}$ and $\{P_1, P_2\}$ are two orthogonal bases for $\mathcal{W}$. Then for all $X \in \mathbb{R}^5$,

$$\frac{X \cdot Q_1}{Q_1 \cdot Q_1} Q_1 + \frac{X \cdot Q_2}{Q_2 \cdot Q_2} Q_2 = \frac{X \cdot P_1}{P_1 \cdot P_1} P_1 + \frac{X \cdot P_2}{P_2 \cdot P_2} P_2$$

Answer: True, according to Fourier Theorem

$$\frac{X \cdot Q_1}{Q_1 \cdot Q_1} Q_1 + \frac{X \cdot Q_2}{Q_2 \cdot Q_2} Q_2 = \text{Proj}_{\mathcal{W}}(X)$$
$$\frac{X \cdot P_1}{P_1 \cdot P_1} P_1 + \frac{X \cdot P_2}{P_2 \cdot P_2} P_2 = \text{Proj}_{\mathcal{W}}(X)$$

Question 6.14. Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ and let $X \in \mathbb{R}^n$. Then $\text{Proj}_{\mathcal{W}}(X - \text{Proj}_{\mathcal{W}}(X)) = \mathbf{0}$

Answer: True, $\text{Proj}_{\mathcal{W}}(X - \text{Proj}_{\mathcal{W}}(X)) = \text{Proj}_{\mathcal{W}}(\text{Orth}_{\mathcal{W}}(X)) = 0$

1.2 Exercises

Question 6.19. In each part let $\mathcal{W}$ be the subspace of $\mathbb{R}^4$ spanned by the set $\mathcal{B}$. Show that $\mathcal{B}$ is an orthogonal basis for $\mathcal{W}$ and find $\text{Proj}_{\mathcal{W}}([1,2,-1,-3]^t)$

(a) $\mathcal{B} = \{[2,-1,-1,-1]^t, [1,3,3,-4]^t, [1,1,0,1]^t\}$

Answer: $\mathcal{B}_1 \cdot \mathcal{B}_2 = 0$, $\mathcal{B}_2 \cdot \mathcal{B}_3 = 0$, $\mathcal{B}_1 \cdot \mathcal{B}_3 = 0$, therefore, the basis is orthogonal.

$$\text{Proj}_{\mathcal{W}}([1,2,-1,-3]^t) = \frac{\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix} \cdot \begin{bmatrix}2\\-1\\-1\\-1\end{bmatrix}}{\begin{bmatrix}2\\-1\\-1\\-1\end{bmatrix} \cdot \begin{bmatrix}2\\-1\\-1\\-1\end{bmatrix}} \begin{bmatrix}2\\-1\\-1\\-1\end{bmatrix} + \frac{\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix} \cdot \begin{bmatrix}1\\3\\3\\-4\end{bmatrix}}{\begin{bmatrix}1\\3\\3\\-4\end{bmatrix} \cdot \begin{bmatrix}1\\3\\3\\-4\end{bmatrix}} \begin{bmatrix}1\\3\\3\\-4\end{bmatrix} + \frac{\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\0\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\\1\end{bmatrix}} \begin{bmatrix}1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}1.6\\0.8\\0.8\\-2.4\end{bmatrix}$$

(b) $\mathcal{B} = \{[1,1,1,1]^t, [1,-2,1,0]^t, [1,1,1,-3]^t\}$

Answer: $\mathcal{B}_1 \cdot \mathcal{B}_2 = 0$, $\mathcal{B}_2 \cdot \mathcal{B}_3 = 0$, $\mathcal{B}_1 \cdot \mathcal{B}_3 = 0$, therefore, the basis is orthogonal.

$$\text{Proj}_{\mathcal{W}}([1,2,-1,-3]^t) = \frac{\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\\1\end{bmatrix}} \begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix} \cdot \begin{bmatrix}1\\-2\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\-2\\1\\0\end{bmatrix}} \begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + \frac{\begin{bmatrix}1\\2\\-1\\-3\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\\-3\end{bmatrix}}{\begin{bmatrix}1\\1\\1\\-3\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\\-3\end{bmatrix}} \begin{bmatrix}1\\1\\1\\-3\end{bmatrix} = \begin{bmatrix}0.0\\2.0\\0.0\\-3.0\end{bmatrix}$$

Question 6.20. Below, you are given two sets of vectors $\mathcal{B}_1$ and $\mathcal{B}_2$ in $\mathbb{R}^2$.

$$\mathcal{B}_1 = \{[-1,1,-1]^t, [1,3,2]^t\}$$
$$B_2 = \{[3,1,4]^t, [-4,16,-1]^t\}$$

(a) Show that both $\mathcal{B}_1$ and $\mathcal{B}_2$ are orthogonal sets.

(a) $\mathcal{B}_{1_1} \cdot \mathcal{B}_{1_2} = 0$, therefore, basis 1 is orthogonal

(b) $\mathcal{B}_{2_1} \cdot \mathcal{B}_{2_2} = 0$, therefore, basis 2 is orthogonal

(b) Show that B_1 and B_2 both span the same subspace $\mathcal{W}$ of $\mathbb{R}^3$.

Answer: $-2\begin{bmatrix}-1\\1\\-1\end{bmatrix} + 1\begin{bmatrix}1\\3\\2\end{bmatrix} = \begin{bmatrix}3\\1\\4\end{bmatrix}$, $7\begin{bmatrix}-1\\1\\-1\end{bmatrix} + 3\begin{bmatrix}1\\3\\2\end{bmatrix} = \begin{bmatrix}-4\\16\\-1\end{bmatrix}$

(c) Find $\text{Proj}_w([1,2,2]^t)$ using formula (6.20) on page 321 and $\mathcal{B}_1$

Answer:

$$\frac{\begin{bmatrix}1\\2\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\-1\end{bmatrix}}{\begin{bmatrix}-1\\1\\-1\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\-1\end{bmatrix}}\begin{bmatrix}-1\\1\\-1\end{bmatrix} + \frac{\begin{bmatrix}1\\2\\2\end{bmatrix}\cdot\begin{bmatrix}1\\3\\2\end{bmatrix}}{\begin{bmatrix}1\\3\\2\end{bmatrix}\cdot\begin{bmatrix}1\\3\\2\end{bmatrix}}\begin{bmatrix}1\\3\\2\end{bmatrix} = \begin{bmatrix}1.11904761904762\\2.02380952380952\\1.9047619047619\end{bmatrix}$$

(d) Find $\text{Proj}_w([1,2,2]^t)$ using formula (6.20) on page 321 and B_2. You should get the same answer as in part (c). Why?

Answer:

$$\frac{\begin{bmatrix}1\\2\\2\end{bmatrix}\cdot\begin{bmatrix}3\\1\\4\end{bmatrix}}{\begin{bmatrix}3\\1\\4\end{bmatrix}\cdot\begin{bmatrix}3\\1\\4\end{bmatrix}}\begin{bmatrix}3\\1\\4\end{bmatrix} + \frac{\begin{bmatrix}1\\2\\2\end{bmatrix}\cdot\begin{bmatrix}-4\\16\\-1\end{bmatrix}}{\begin{bmatrix}-4\\16\\-1\end{bmatrix}\cdot\begin{bmatrix}-4\\16\\-1\end{bmatrix}}\begin{bmatrix}-4\\16\\-1\end{bmatrix} = \begin{bmatrix}1.11904761904762\\2.02380952380952\\1.9047619047619\end{bmatrix}$$

The answer is the same since $\mathcal{B}_1$ and $\mathcal{B}_2$ describe the same subspace according to (b)

(e) Let $X = [x,y,z]^t$. Use formula (6.20) on page 321 to find $\text{Proj}_w(X)$ using the basis B_1. Then find a matrix R such that $\text{Proj}_w(X) = RX$.

Answer:

$$\frac{\begin{bmatrix}x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\-1\end{bmatrix}}{\begin{bmatrix}-1\\1\\-1\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\-1\end{bmatrix}}\begin{bmatrix}-1\\1\\-1\end{bmatrix} + \frac{\begin{bmatrix}x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}1\\3\\2\end{bmatrix}}{\begin{bmatrix}1\\3\\2\end{bmatrix}\cdot\begin{bmatrix}1\\3\\2\end{bmatrix}}\begin{bmatrix}1\\3\\2\end{bmatrix} = \begin{bmatrix}\frac{17x}{42} - \frac{5y}{42} + \frac{10z}{21}\\-\frac{5x}{42} + \frac{41y}{42} + \frac{2z}{21}\\\frac{10x}{21} + \frac{2y}{21} + \frac{13z}{21}\end{bmatrix}$$

$$R = \begin{bmatrix}\frac{17}{42} & -\frac{5}{42} & \frac{10}{21}\\-\frac{5}{42} & \frac{41}{42} & \frac{2}{21}\\\frac{10}{21} & \frac{2}{21} & \frac{13}{21}\end{bmatrix}$$

(f) Repeat part (e) using basis $\mathcal{B}_2$. You should get the same matrix R

Answer:

$$\frac{\begin{bmatrix}x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}3\\1\\4\end{bmatrix}}{\begin{bmatrix}3\\1\\4\end{bmatrix}\cdot\begin{bmatrix}3\\1\\4\end{bmatrix}}\begin{bmatrix}3\\1\\4\end{bmatrix} + \frac{\begin{bmatrix}x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}-4\\16\\-1\end{bmatrix}}{\begin{bmatrix}-4\\16\\-1\end{bmatrix}\cdot\begin{bmatrix}-4\\16\\-1\end{bmatrix}}\begin{bmatrix}-4\\16\\-1\end{bmatrix} = \begin{bmatrix}\frac{17x}{42} - \frac{5y}{42} + \frac{10z}{21}\\-\frac{5x}{42} + \frac{41y}{42} + \frac{2z}{21}\\\frac{10x}{21} + \frac{2y}{21} + \frac{13z}{21}\end{bmatrix}$$

$$R = \begin{bmatrix}\frac{17}{42} & -\frac{5}{42} & \frac{10}{21}\\-\frac{5}{42} & \frac{41}{42} & \frac{2}{21}\\\frac{10}{21} & \frac{2}{21} & \frac{13}{21}\end{bmatrix}$$

(g) Show that the matrix R from part (e) satisfies $R^2 = R$. Explain the geometric meaning of this equality.

Answer:

$$R^2 = \begin{bmatrix}\frac{17}{42} & -\frac{5}{42} & \frac{10}{21}\\-\frac{5}{42} & \frac{41}{42} & \frac{2}{21}\\\frac{10}{21} & \frac{2}{21} & \frac{13}{21}\end{bmatrix}\begin{bmatrix}\frac{17}{42} & -\frac{5}{42} & \frac{10}{21}\\-\frac{5}{42} & \frac{41}{42} & \frac{2}{21}\\\frac{10}{21} & \frac{2}{21} & \frac{13}{21}\end{bmatrix} = \begin{bmatrix}\frac{17}{42} & -\frac{5}{42} & \frac{10}{21}\\-\frac{5}{42} & \frac{41}{42} & \frac{2}{21}\\\frac{10}{21} & \frac{2}{21} & \frac{13}{21}\end{bmatrix} = R$$

Question 6.22. Use the Gram-Schmidt process to find an orthogonal basis for the subspaces of $\mathbb{R}^n$ spanned by the following ordered sets of vectors for the appropriate n :

Answer:

(a)

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- $P_1 = X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
- $Y_2 = \dfrac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $P_2 = X_2 - Y_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(b)

$$\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

- $P_1 = X_1 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$
- $Y_2 = \dfrac{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}}{\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} \\ \frac{3}{14} \\ \frac{9}{14} \end{bmatrix}$, $P_2 = X_2 - Y_2 = \begin{bmatrix} \frac{17}{7} \\ \frac{11}{14} \\ \frac{19}{14} \end{bmatrix}$

(c)

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

- $P_1 = X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$
- $Y_2 = \dfrac{\begin{bmatrix} -2 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $P_2 = X_2 - Y_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$
- $Y_3 = \dfrac{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \dfrac{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} -2 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} -2 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{13}{7} \\ \frac{16}{7} \\ 1 \\ \frac{11}{7} \end{bmatrix}$, $P_3 = X_3 - Y_3 = \begin{bmatrix} -\frac{6}{7} \\ -\frac{2}{7} \\ 0 \\ \frac{10}{7} \end{bmatrix}$

(d)

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

- $P_1 = X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$
- $Y_2 = \dfrac{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, $P_2 = X_2 - Y_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

- $$Y_3 = \frac{\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}} \begin{bmatrix}1\\1\\-1\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}}{\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}} \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} = \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}, P_3 = X_3 - Y_3 = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Question 6.23. Compute Proj $_{\mathcal{W}}\left([1,2,3]^t\right)$, where $\mathcal{W}$ is the subspace of $\mathbb{R}^3$ spanned by the vectors in Exercise 6.22. a. Repeat for the subspace spanned by the vectors in Exercise 6.22.b.

Answer:

(a) 6.22a

$$\frac{\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}0\\1\\1\end{bmatrix}}{\begin{bmatrix}0\\1\\1\end{bmatrix} \cdot \begin{bmatrix}0\\1\\1\end{bmatrix}} \begin{bmatrix}0\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix}} \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\\frac{5}{2}\\\frac{5}{2}\end{bmatrix}$$

(b) 6.22b

$$\frac{\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\3\end{bmatrix}}{\begin{bmatrix}-2\\1\\3\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\3\end{bmatrix}} \begin{bmatrix}-2\\1\\3\end{bmatrix} + \frac{\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}\frac{17}{7}\\\frac{11}{14}\\\frac{19}{14}\end{bmatrix}}{\begin{bmatrix}\frac{17}{7}\\\frac{11}{14}\\\frac{19}{14}\end{bmatrix} \cdot \begin{bmatrix}\frac{17}{7}\\\frac{11}{14}\\\frac{19}{14}\end{bmatrix}} \begin{bmatrix}\frac{17}{7}\\\frac{11}{14}\\\frac{19}{14}\end{bmatrix} = \begin{bmatrix}\frac{124}{117}\\\frac{164}{117}\\\frac{379}{117}\end{bmatrix}$$

Question 6.24. Compute Proj $w\left([1,2,3,4]^t\right)$, where $\mathcal{W}$ is the subspace of $\mathbb{R}^4$ spanned by the vectors in Exercise 6.22.c. Repeat for the subspace spanned by the vectors in Exercise 6.22.d.

Answer:

(a) 6.22c

$$\frac{\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \cdot \begin{bmatrix}1\\2\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\2\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\1\\1\end{bmatrix}} \begin{bmatrix}1\\2\\1\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\1\\-1\end{bmatrix}}{\begin{bmatrix}-2\\1\\1\\-1\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\1\\-1\end{bmatrix}} \begin{bmatrix}-2\\1\\1\\-1\end{bmatrix} + \frac{\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \cdot \begin{bmatrix}-\frac{6}{7}\\-\frac{2}{7}\\0\\\frac{10}{7}\end{bmatrix}}{\begin{bmatrix}-\frac{6}{7}\\-\frac{2}{7}\\0\\\frac{10}{7}\end{bmatrix} \cdot \begin{bmatrix}-\frac{6}{7}\\-\frac{2}{7}\\0\\\frac{10}{7}\end{bmatrix}} \begin{bmatrix}-\frac{6}{7}\\-\frac{2}{7}\\0\\\frac{10}{7}\end{bmatrix} = \begin{bmatrix}\frac{5}{7}\\\frac{20}{7}\\\frac{11}{7}\\4\end{bmatrix}$$

(b) 6.22d

$$\frac{\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}} \begin{bmatrix}1\\1\\-1\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}}{\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}} \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\\frac{3}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$$

Question 6.27. Let A be an $m \times n$ matrix and let $\mathcal{W} \subset \mathbb{R}^n$ be the nullspace of A. Prove that for all $X \in \mathbb{R}^n, AX = A\left(\operatorname{Orth}_{\mathcal{W}} X\right)$

Answer: $A\left(\operatorname{Orth}_{\mathcal{W}} X\right) = A(X - \operatorname{Proj}_{\mathcal{W}} X) = AX - A(\operatorname{Proj}_{\mathcal{W}} X)$, since $\mathcal{W}$ is the nullspace of A and $\operatorname{Proj}_{\mathcal{W}} X$ is in $\mathcal{W}$, $A(\operatorname{Proj}_{\mathcal{W}} X) = 0$, therefore $AX - A(\operatorname{Proj}_{\mathcal{W}} X) = AX$

Question 6.30. Find an orthogonal basis for $S^{\perp}$ for the following sets of vectors.

(a)

$$\begin{bmatrix}1\\3\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\6\\0\\1\end{bmatrix}, \begin{bmatrix}4\\12\\2\\-1\end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & -1 \\ 2 & 6 & 0 & 1 \\ 4 & 12 & 2 & -1 \end{bmatrix} X = 0$$

Solution: Plane, $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} a + \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{3}{2} \\ 1 \end{bmatrix} b + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Basis: $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{3}{2} \\ 1 \end{bmatrix}$

(b)

$$\begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & -1 \\ 2 & 6 & 0 & 1 \end{bmatrix} X = 0$$

Solution: Plane, $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} a + \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{3}{2} \\ 1 \end{bmatrix} b + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Basis: $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{3}{2} \\ 1 \end{bmatrix}$

(c)

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} X = 0$$

Solution: Plane, $\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} a + \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} b + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Basis: $\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

Question 6.32. Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Let $\{P_1, P_2, \ldots, P_k\}$ be an orthogonal basis for $\mathcal{W}$ and let $\{X_1, X_2, \ldots, X_m\}$ be an orthogonal basis for $\mathcal{W}^\perp$. Show that the set

$$\mathcal{B} = \{P_1, P_2, \ldots, P_k, X_1, X_2, \ldots, X_m\}$$

is an orthogonal set. Prove that this set is a basis for $\mathbb{R}^n$ and, hence, $\dim(\mathcal{W}) + \dim\left(\mathcal{W}^\perp\right) = n$

Answer: This set is orthogonal since any two vectors P_i, P_k and X_i, X_j are orthogonal and any X_i is orthogonal to any P_j since P_j is in $\mathcal{W}$ and X_i is in $\mathcal{W}^\perp$. This set is a basis for $\mathbb{R}^n$, since any vector v can be written as $v = \text{Proj}_\mathcal{W} + \text{Orth}_\mathcal{W}$, where $\text{Proj}_\mathcal{W} \in \mathcal{W}$ and $\text{Orth}_\mathcal{W} \in \mathcal{W}^\perp$. The set is linearly independent, so $\dim(\mathcal{W}) + \dim\left(\mathcal{W}^\perp\right) = k + m = n$

Question 6.37. Let $\{P_1, P_2\}$ be an ordered orthogonal (but not orthonormal) basis for some subspace $\mathcal{W}$ of $\mathbb{R}^n$. Let X and Y be elements of $\mathcal{W}$ whose coordinate vectors with respect to these bases are $X' = [x_1, x_2]^t$ and $Y' = [y_1, y_2]^t$. Prove that

$$X \cdot Y = x_1 y_1 \left|P_1\right|^2 + x_2 y_2 \left|P_2\right|^2$$

Answer:

$$X \cdot Y = x_1 P_1 y_1 P_1 + x_2 P_2 y_2 P_2 = x_1 y_1 \left|P_1\right|^2 + x_2 y_2 \left|P_2\right|^2$$

What is the corresponding formula for $|X|^2$?
Answer:

$$|X|^2 = x_1^2 \left|P_1\right|^2 + x_2^2 \left|P_2\right|^2$$