MA 351, HW 8

Elnard Utiushev

March 21, 2019

Section 5.2: Exercises: 5.29 (a)(c)(e)(g), 5.32, 5.33, 5.34, 5.36, 5.37, 5.39
Section 5.3: Exercises: 5.44, 5.45, 5.46, 5.57

1 Section 5.2

1.1 Exercises

Question 5.29 (a)(c)(e)(g). For the following matrices, find (if possible) an invertible matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$.

Answer:

a.

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda-2 & -1 & 1 \\ -6 & -\lambda-2 & 0 \\ 13 & 7 & -\lambda-4 \end{vmatrix} = -\left(\lambda+8\right)\left(\lambda^2+1\right)$$

$\lambda = -8, -i, i$

Not diagonalizable over real numbers

c.

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda+1 & 3 & 0 & 0 \\ 3 & -\lambda+1 & 0 & 0 \\ 0 & 0 & -\lambda-1 & 2 \\ 0 & 0 & -1 & -\lambda-4 \end{vmatrix} = \left(\lambda-4\right)\left(\lambda+2\right)^2\left(\lambda+3\right)$$

$\lambda = -3, -2, 4$

1. $\lambda = -3$

$$A-(-3)I = \begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

2. $\lambda = -2$

$$A-(-2)I = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix}3 & 3 & 0 & 0 & 0\\3 & 3 & 0 & 0 & 0\\0 & 0 & 1 & 2 & 0\\0 & 0 & -1 & -2 & 0\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 2 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Eigenvectors: $\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\-2\\1\end{bmatrix}$

3. $\lambda = 4$

$$A - (4)I = \begin{bmatrix}-3 & 3 & 0 & 0\\3 & -3 & 0 & 0\\0 & 0 & -5 & 2\\0 & 0 & -1 & -8\end{bmatrix}$$

$$\begin{bmatrix}-3 & 3 & 0 & 0 & 0\\3 & -3 & 0 & 0 & 0\\0 & 0 & -5 & 2 & 0\\0 & 0 & -1 & -8 & 0\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & -1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Eigenvectors: $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$

$$Q = \begin{bmatrix}0 & -1 & 0 & 1\\0 & 1 & 0 & 1\\-1 & 0 & -2 & 0\\1 & 0 & 1 & 0\end{bmatrix}, \quad D = \begin{bmatrix}-3 & 0 & 0 & 0\\0 & -2 & 0 & 0\\0 & 0 & -2 & 0\\0 & 0 & 0 & 4\end{bmatrix}$$

e.

$$\det(A - \lambda I) = \begin{vmatrix}-\lambda + 7 & -1\\9 & -\lambda + 1\end{vmatrix} = (\lambda - 4)^2$$

$\lambda = 4$

1. $\lambda = 4$

$$A - (4)I = \begin{bmatrix}3 & -1\\9 & -3\end{bmatrix}$$

$$\begin{bmatrix}3 & -1 & 0\\9 & -3 & 0\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & -\frac{1}{3} & 0\\0 & 0 & 0\end{bmatrix}$$

Eigenvectors: $\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}$

Not diagonalizable

g.

$$\det(A - \lambda I) = \begin{vmatrix}-\lambda + 1 & -4 & 2\\-4 & -\lambda + 1 & -2\\2 & -2 & -\lambda - 2\end{vmatrix} = -(\lambda - 6)(\lambda + 3)^2$$

$\lambda = -3, 6$

1. $\lambda = -3$

$$A - (-3)I = \begin{bmatrix}4 & -4 & 2\\-4 & 4 & -2\\2 & -2 & 1\end{bmatrix}$$

$$\begin{bmatrix}4 & -4 & 2 & 0\\-4 & 4 & -2 & 0\\2 & -2 & 1 & 0\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & -1 & \frac{1}{2} & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

Eigenvectors: $\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}-\frac{1}{2}\\0\\1\end{bmatrix}$

2. $\lambda = 6$

$$A - (6)I = \begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix}$$

$$\begin{bmatrix} -5 & -4 & 2 & 0 \\ -4 & -5 & -2 & 0 \\ 2 & -2 & -8 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$$Q = \begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Question 5.32. In Example 5.6, on page 288, find a matrix B such that $B^2 = A$. Check your answer by direct computation.

Answer: Using the formula obtained in Example 5.6, we get

$$A^k = QD^kQ^{-1} = \frac{1}{2} \begin{bmatrix} 2\left(3^k\right) & -1 + 3^k & 0 \\ 0 & 2 & 0 \\ -4 + 4\left(3^k\right) & -2 + 2\left(3^k\right) & 2 \end{bmatrix}$$

$$B = A^{0.5} = QD^{0.5}Q^{-1} = \frac{1}{2} \begin{bmatrix} 2\left(3^{0.5}\right) & -1 + 3^{0.5} & 0 \\ 0 & 2 & 0 \\ -4 + 4\left(3^{0.5}\right) & -2 + 2\left(3^{0.5}\right) & 2 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 3.0 & 1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 4.0 & 2.0 & 1.0 \end{bmatrix} = A$$

Question 5.33. Suppose that A is diagonalizable over $\mathbb{R}$ and A has only ± 1 as eigenvalues. Show that $A^2 = I$

Answer: Since A is diagonalizable over $\mathbb{R}$, we can write $A^k = QD^kQ^{-1}$,where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore,

$$A^2 = QD^2Q^{-1} = Q\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)^2 Q^{-1} = QIQ^{-1} = I$$

Question 5.34. Suppose that A is diagonalizable over $\mathbb{R}$ and A has only 0 and 1 as eigenvalues. Show that $A^2 = A$

Answer: Since A is diagonalizable over $\mathbb{R}$, we can write $A^k = QD^kQ^{-1}$,where

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore,

$$A^2 = QD^2Q^{-1} = Q\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)^2 Q^{-1} = QDQ^{-1} = A$$

Question 5.36. Suppose that A is diagonalizable over $\mathbb{R}$ with eigenvalues λ_i. Suppose also that

$$q(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$$

is some polynomial such that $q(\lambda_i) = 0$ for all i. Prove that

$$a_nA^n + a_{n-1}A^{n-1} + \cdots + a_0I = \mathbf{0}$$

Answer: Since A is diagonalizable over $\mathbb{R}$

$$\begin{aligned}
a_nA^n + a_{n-1}A^{n-1} + \cdots + a_0I &= a_nQD^nQ^{-1} + a_{n-1}QD^{n-1}Q^{-1} + \cdots + a_0I \\
&= a_nQ\begin{bmatrix} \lambda_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_i \end{bmatrix}^n Q^{-1} + a_{n-1}Q\begin{bmatrix} \lambda_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_i \end{bmatrix}^{n-1} Q^{-1} + \cdots + a_0I \\
&= a_nQ\begin{bmatrix} \lambda_0^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_i^n \end{bmatrix} Q^{-1} + a_{n-1}Q\begin{bmatrix} \lambda_0^{n-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_i^{n-1} \end{bmatrix} Q^{-1} + \cdots + a_0I \\
&= Q(a_n\begin{bmatrix} \lambda_0^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_i^n \end{bmatrix} + a_{n-1}\begin{bmatrix} \lambda_0^{n-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_i^{n-1} \end{bmatrix} + \cdots)Q^{-1} + a_0I \\
&= Q\begin{bmatrix} a_n\lambda_0^n + a_{n-1}\lambda_0^{n-1} + \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n\lambda_i^n + a_{n-1}\lambda_i^{n-1} + \cdots \end{bmatrix} Q^{-1} + a_0IQQ^{-1} \\
&= Q\begin{bmatrix} a_n\lambda_0^n + a_{n-1}\lambda_0^{n-1} + \cdots + a_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n\lambda_i^n + a_{n-1}\lambda_i^{n-1} + \cdots + a_0 \end{bmatrix} Q^{-1} \\
&= Q\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} Q^{-1} \\
&= 0
\end{aligned}$$

Question 5.37. Find values of a, b, and c, all nonzero, such that the matrix A below is diagonalizable over $\mathbb{R}$:

$$A = \begin{bmatrix} 2 & a & b \\ 0 & -5 & c \\ 0 & 0 & 2 \end{bmatrix}$$

Answer: The eigenvalues are 2 with multiplicity 2 and -5 with multiplicity 1. The dimension of an eigenspace $\lambda = 2$ must be 2.

$$A - 2I = \begin{bmatrix} 0 & a & b \\ 0 & -7 & c \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, $[a, b] = t[-7, c]$, where t is a scalar

Question 5.39. Suppose that A and B are $n \times n$ matrices such that $A = QBQ^{-1}$ for some invertible matrix Q. Prove that A and B have the same characteristic polynomials. Suppose that X is an eigenvector for B. Show that QX is an eigenvector for A

Answer:

$$\det(A - \lambda I) = \det(QBQ^{-1} - Q\lambda IQ^{-1}) = \det(Q(B - \lambda I)Q^{-1}) = \det(Q)\det(B - \lambda I)\det(Q^{-1})$$

$$= \det(QQ^{-1})\det(B-\lambda I) = \det(B-\lambda I)$$

$$\begin{aligned} BX &= \lambda X \\ BQ^{-1}QX &= \lambda X \\ QBQ^{-1}QX &= Q\lambda X \\ AQX &= \lambda QX \end{aligned}$$

2 Section 5.3

2.1 Exercises

Question 5.44. Compute AB and BA for the matrices A and B .

$$A = \begin{bmatrix} 1+i & 2i \\ 2 & 3i \end{bmatrix}, \quad B = \begin{bmatrix} -i & 3 \\ 2+i & 4i \end{bmatrix}$$

Answer:

$$AB = \begin{bmatrix} -1+3i & -5+3i \\ -3+4i & -6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 7-i & 2+9i \\ 1+11i & -14+4i \end{bmatrix}$$

Question 5.45. For the matrix A below, find complex matrices Q and D where D is diagonal such that $A = QDQ^{-1}$. Use your answer to find A^{20}. (Use a calculator to approximate the answer.)

$$A = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$$

Answer:

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda+1 & -4 \\ 1 & -\lambda+1 \end{vmatrix} = \lambda^2 - 2\lambda + 5$$

$\lambda = 1-2i, 1+2i$

1. $\lambda = 1-2i$

$$A - (1-2i)I = \begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix}$$

$$\begin{bmatrix} 2i & -4 & 0 \\ 1 & 2i & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 2i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} -2i \\ 1 \end{bmatrix}$

2. $\lambda = 1+2i$

$$A - (1+2i)I = \begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix}$$

$$\begin{bmatrix} -2i & -4 & 0 \\ 1 & -2i & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -2i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 2i \\ 1 \end{bmatrix}$

$$Q = \begin{bmatrix} -2i & 2i \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1-2i & 0 \\ 0 & 1+2i \end{bmatrix}$$

$$A^{20} = QD^{20}Q^{-1} = \begin{bmatrix} -2i & 2i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (1-2i)^{20} & 0 \\ 0 & (1+2i)^{20} \end{bmatrix} \begin{bmatrix} \frac{i}{4} & \frac{1}{2} \\ -\frac{i}{4} & \frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} -9653287.0 + 7.0 \cdot 10^{-15}i & 2953968.0 + 1.0 \cdot 10^{-13}i \\ -738492.0 + 3.0 \cdot 10^{-14}i & -9653287.0 + 7.0 \cdot 10^{-15}i \end{bmatrix}$$

Question 5.46. Find all eigenvalues for the following matrix A :

$$A = \begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix}$$

It might help to know that -7 is one eigenvalue.
Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda + 2 & 3 & 6 \\ 6 & -\lambda + 2 & -3 \\ 3 & -6 & -\lambda + 2 \end{vmatrix} = -(\lambda + 7)\left(\lambda^2 - 13\lambda + 49\right)$$

$\lambda = -7, \frac{13}{2} - \frac{3\sqrt{3}i}{2}, \frac{13}{2} + \frac{3\sqrt{3}i}{2}$

Question 5.47. Consider the transformation $T : \mathbb{C} \to \mathbb{C}$ given by $T(z) = (2+3i)z$. We commented that we may interpret the complex numbers as being $\mathbb{R}^2$. Thus, we may think of T as transforming $\mathbb{R}^2$ into $\mathbb{R}^2$. Explicitly, for example, T transforms the point $[1, 2]^t$ into $[-4, 7]^t$ since $(2+3i)(1+2i) = -4+7i$

Show that as a transformation of $\mathbb{R}^2$ into $\mathbb{R}^2, T$ is linear. Find a real 2×2 matrix A such that $T = T_A$. Find all eigenvalues of this matrix.
Answer:

$$T(z_1 + z_2) = (2+3i)(z_1 + z_2) = (2+3i)z_1 + (2+3i)z_2 = T(z_1) + T(z_2)$$
$$T(az) = (2+3i)(az) = a(2+3i)z = aT(z)$$

Therefore, the transformation T is linear.

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda + 2 & -3 \\ 3 & -\lambda + 2 \end{vmatrix} = \lambda^2 - 4\lambda + 13$$

$\lambda = 2 - 3i, 2 + 3i$