MA 351, HW 12

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Section 6.5: Exercises: 6.91, 6.92, 6.95, 6.100
Section 6.6: True/False: 6.24, 6.26, 6.27, 6.28, 6.29, 6.33; Exercises: 6.106, 6.108, 6.109, 6.110, 6.115, 6.122

1 Section 6.5

1.1 Exercises

Question 6.91. Suppose that in the context of Example 6.16 it is proposed that our data would be better described by an equation of the form

$$T = a + bt + ct^2$$

1. Explain how you would use the techniques of this section to find approximate values for a, b, and c. Specifically, in the normal equation $A^tB = A^tAX$, what are appropriate choices for A and B?

   Answer:

$$A = [1, t, t^2] = \begin{bmatrix} 1.0 & 0.5 & 0.25 \\ 1.0 & 1.1 & 1.21 \\ 1.0 & 1.5 & 2.25 \\ 1.0 & 2.1 & 4.41 \\ 1.0 & 2.3 & 5.29 \end{bmatrix}$$

$$B = \begin{bmatrix} 32.0 \\ 33.0 \\ 34.2 \\ 35.1 \\ 35.7 \end{bmatrix}$$

2. If you have appropriate software available, compute the solution to the normal equation. Note the size of the constant c. What is your interpretation of this result?

   Answer:

$$A^tA = \begin{bmatrix} 5.0 & 7.5 & 13.41 \\ 7.5 & 13.41 & 26.259 \\ 13.41 & 26.259 & 54.0213 \end{bmatrix}$$

$$A^tB = \begin{bmatrix} 170.0 \\ 259.42 \\ 468.524 \end{bmatrix}$$

   Solution: Point, $\begin{bmatrix} 30.962159425626 \\ 1.98956799214518 \\ 0.0199435444281278 \end{bmatrix}$

Question 6.92. The data in the chart below is the estimated population of the United States (in millions), rounded to three digits, from 1980 to 2000.? Your goal in this exercise is to predict the U.S. population in the year 2010.

Year	1980	1985	1990	1995	2000
Population	227	238	249	263	273

For this, let t denote "years after 1980" and I represent the increase in population over the 1980 level (see the chart below). Use the method of least squares to find constants a and b such that I is approximately equal to $at + b$. Then use your formula to predict the 2010 population.

Years after 1980	0	5	10	15	20
Increase over 1980	0	11	22	36	46

Answer:

$$A = \begin{bmatrix} 1.0 & 0.0 \\ 1.0 & 5.0 \\ 1.0 & 10.0 \\ 1.0 & 15.0 \\ 1.0 & 20.0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 11 \\ 22 \\ 36 \\ 46 \end{bmatrix}$$

$$A^t A = \begin{bmatrix} 5 & 50 \\ 50 & 750 \end{bmatrix}, \quad A^t B = \begin{bmatrix} 115 \\ 1735 \end{bmatrix}$$

Solution: Point, $\begin{bmatrix} -\frac{2}{5} \\ \frac{117}{50} \end{bmatrix}$

Prediction: $349/5 + 227 = 69.8 + 227 = 296.8$

$$\begin{bmatrix} 1 & 30 \end{bmatrix} \begin{bmatrix} -\frac{2}{5} \\ \frac{117}{50} \end{bmatrix} = \begin{bmatrix} \frac{349}{5} \end{bmatrix}$$

Question 6.95. Let $\mathcal{W}$ be the subspace of $\mathbb{R}^4$ spanned by $A_1 = [1, 2, 0, 1, 0]^t$ and $A_2 = [1, 1, 1, 1, 1]^t$. Use Theorem 6.21 on page 376 to find the projection matrix P_w , $\mathrm{Proj}_{\mathcal{W}}(B)$ and $\mathrm{Orth}_{\mathcal{W}}(B)$, where $B = [1, 2, 3, 4, 5]^t$. Show by direct calculation that Orth $_{\mathcal{W}}(B)$ is orthogonal to A_1 and A_2

Answer:

$$P_{\mathcal{W}} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{14} & \frac{2}{7} & \frac{1}{7} & \frac{3}{14} & \frac{1}{7} \\ \frac{2}{7} & \frac{5}{7} & -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & \frac{3}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{3}{14} & \frac{2}{7} & \frac{1}{7} & \frac{3}{14} & \frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & \frac{3}{7} & \frac{1}{7} & \frac{3}{7} \end{bmatrix}$$

$$\mathrm{Proj}_{\mathcal{W}}(B) = \begin{bmatrix} \frac{39}{14} \\ \frac{12}{7} \\ \frac{27}{7} \\ \frac{39}{14} \\ \frac{27}{7} \end{bmatrix}$$

$$\mathrm{Orth}_{\mathcal{W}}(B) = \begin{bmatrix} -\frac{25}{14} \\ \frac{2}{7} \\ -\frac{6}{7} \\ \frac{17}{14} \\ \frac{8}{7} \end{bmatrix}$$

Question 6.100. From Exercises $3.80 - 3.82, (AB)^{-1} = B^{-1}A^{-1}$. What is wrong with the following calculation?

$$P_{\mathcal{W}} = A \left(A^t A\right)^{-1} A^t = AA^{-1} \left(A^t\right)^{-1} A^t = II = I$$

Answer: A may not be invertible

2 Section 6.6

2.1 True/False

Question 6.24. The matrix A is symmetric and has the characteristic polynomial $p(\lambda) = \lambda^3(\lambda - 1)^2(\lambda + 3)$. Then the nullspace of A might have dimension 2 .
Answer: True, since $\lambda = 0$ has multiplicity of 3

Question 6.26. The polynomial $p(\lambda) = (\lambda-1)(\lambda-2)^3\left(\lambda^2+1\right)$ could be the characteristic polynomial of a symmetric matrix.
Answer: True

Question 6.27. It is impossible for a symmetric 3×3 matrix A to have the vectors $[1,2,3]^t$ and $[1,1,1]^t$ as eigenvectors corresponding to the eigenvalues 3 and 5, respectively.
Answer: True, $[1,2,3]^t \cdot [1,1,1]^t \neq 0$

Question 6.28. It is impossible for a symmetric 3×3 matrix A to have the vectors $[1,2,3]^t$ and $[1,1,1]^t$ as eigenvectors corresponding to the eigenvalue 3.
Answer: False

Question 6.29. Given that the matrix A below has -5 and -10 as eigenvalues, it follows that the quadratic curve described by $-9x^2 + 4xy - 6y^2 = 1$ is an ellipse.

$$A = \begin{bmatrix} -9 & 2 \\ 2 & -6 \end{bmatrix}$$

Answer: False, $-5(x')^2 - 10(y')^2 = 1$

Question 6.33. Suppose that A is a 3×3 symmetric matrix with eigenvalues $1, 2$ and 3. Then there are exactly eight different orthogonal matrices Q such that $A = QDQ^t$ is a diagonal matrix.
Answer: True

2.2 Exercises

Question 6.106. Let

$$A = \begin{bmatrix} 10 & 2 & 2 \\ 2 & 13 & 4 \\ 2 & 4 & 13 \end{bmatrix}$$

Find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^t$ given that the eigenvalues of A are $\lambda = 9$ and $\lambda = 18$.
Answer:

$$D = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda+10 & 2 & 2 \\ 2 & -\lambda+13 & 4 \\ 2 & 4 & -\lambda+13 \end{vmatrix} = -(\lambda-18)(\lambda-9)^2$$

$\lambda = 9, 18$

1. $\lambda = 9$

$$A - (9)I = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 2 & 4 & 4 & 0 \\ 2 & 4 & 4 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\left[\begin{smallmatrix} -2 \\ 1 \\ 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} -2 \\ 0 \\ 1 \end{smallmatrix}\right]$

2. $\lambda = 18$

$$A - (18)I = \begin{bmatrix} -8 & 2 & 2 \\ 2 & -5 & 4 \\ 2 & 4 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -8 & 2 & 2 & 0 \\ 2 & -5 & 4 & 0 \\ 2 & 4 & -5 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$

$$Q = \begin{bmatrix} -\frac{2\sqrt{5}}{5} & -\frac{2\sqrt{5}}{15} & \frac{1}{3} \\ \frac{\sqrt{5}}{5} & -\frac{4\sqrt{5}}{15} & \frac{2}{3} \\ 0 & \frac{\sqrt{5}}{3} & \frac{2}{3} \end{bmatrix}, \quad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

Question 6.108. For each variety find a quadratic form in standard form that describes the variety relative to the B coordinates for some orthonormal basis $\mathcal{B}$. (You need not find the basis B.) What, geometrically, does each variety represent?

(a) $x^2 + xy + 2y^2 = 1$

Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda + 1 & \frac{1}{2} \\ \frac{1}{2} & -\lambda + 2 \end{vmatrix} = \frac{4\lambda^2 - 12\lambda + 7}{4}$$

$\lambda = -\frac{\sqrt{2}}{2} + \frac{3}{2}, \frac{\sqrt{2}}{2} + \frac{3}{2}$

Ellipse

$$(3 - \sqrt{2})(x')^2 + (3 + \sqrt{2})(y')^2 = 1$$

(b) $x^2 + 4xy + 2y^2 = 1$

Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda + 1 & 2 \\ 2 & -\lambda + 2 \end{vmatrix} = \lambda^2 - 3\lambda - 2$$

$\lambda = \frac{3}{2} + \frac{\sqrt{17}}{2}, -\frac{\sqrt{17}}{2} + \frac{3}{2}$

Ellipse

$$(3 + \sqrt{17})(x')^2 + (3 - \sqrt{17})(y')^2 = 1$$

(c) $x^2 + 4xy + 4y^2 = 1$

Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda + 1 & 2 \\ 2 & -\lambda + 4 \end{vmatrix} = \lambda\,(\lambda - 5)$$

$\lambda = 0, 5$

Lines

$$5(y')^2 = 1$$

(d) $14x^2 + 4xy + 11y^2 = 1$

Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda + 14 & 2 \\ 2 & -\lambda + 11 \end{vmatrix} = (\lambda - 15)\,(\lambda - 10)$$

$\lambda = 10, 15$

Ellipse

$$10(x')^2 + 15(y')^2 = 1$$

Question 6.109. For each variety in parts (a) – (c) in Exercise 6.108, find an orthonormal basis B for $\mathbb{R}^2$ for which the variety is described by a quadratic form in standard form in the $\mathcal{B}$ coordinates.

(a) $x^2 + xy + 2y^2 = 1$

Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda + 1 & \frac{1}{2} \\ \frac{1}{2} & -\lambda + 2 \end{vmatrix} = \frac{4\lambda^2 - 12\lambda + 7}{4}$$

$\lambda = -\frac{\sqrt{2}}{2} + \frac{3}{2}, \frac{\sqrt{2}}{2} + \frac{3}{2}$

Eigenvectors: $\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -\sqrt{2}-1 \\ 1 \end{bmatrix}$

$$Q = \begin{bmatrix} \frac{-1+\sqrt{2}}{\sqrt{\left[-1+\sqrt{2}\right]^2+1}} & \frac{-\sqrt{2}-1}{\sqrt{1+\left[1+\sqrt{2}\right]^2}} \\ \frac{1}{\sqrt{\left[-1+\sqrt{2}\right]^2+1}} & \frac{1}{\sqrt{1+\left[1+\sqrt{2}\right]^2}} \end{bmatrix}$$

(b) $x^2 + 4xy + 2y^2 = 1$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda + 1 & 2 \\ 2 & -\lambda + 2 \end{vmatrix} = \lambda^2 - 3\lambda - 2$$

$\lambda = \frac{3}{2} + \frac{\sqrt{17}}{2}, -\frac{\sqrt{17}}{2} + \frac{3}{2}$

Eigenvectors: $\begin{bmatrix} -\frac{1}{4}+\frac{\sqrt{17}}{4} \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4}+\frac{\sqrt{2}}{4} \\ 1 \end{bmatrix}$

$$Q = \begin{bmatrix} \frac{-\frac{1}{4}+\frac{\sqrt{17}}{4}}{\sqrt{\left[-\frac{1}{4}+\frac{\sqrt{17}}{4}\right]^2+1}} & \frac{-\frac{1}{4}+\frac{\sqrt{2}}{4}}{\sqrt{\left[-\frac{1}{4}+\frac{\sqrt{2}}{4}\right]^2+1}} \\ \frac{1}{\sqrt{\left[-\frac{1}{4}+\frac{\sqrt{17}}{4}\right]^2+1}} & \frac{1}{\sqrt{\left[-\frac{1}{4}+\frac{\sqrt{2}}{4}\right]^2+1}} \end{bmatrix}$$

(c) $x^2 + 4xy + 4y^2 = 1$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda + 1 & 2 \\ 2 & -\lambda + 4 \end{vmatrix} = \lambda(\lambda - 5)$$

$\lambda = 0, 5$

Eigenvectors: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

$$Q = \begin{bmatrix} -\frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{bmatrix}$$

Question 6.110. Give an equation in the standard coordinates for $\mathbb{R}^2$ that describes an ellipse centered at the origin with a length 4 major cord parallel to the vector $[3, 4]^t$ and a length 2 minor axis. (The major cord is the longest line segment that can be inscribed in the ellipse.)

Question 6.115. Let A be as shown. Find an orthogonal matrix Q and a diagonal mattrix D such that $Q'AQ = D$. To save you time, we have provided the eigenvalues (there only two).

$$A = \begin{bmatrix} 3 & 1 & 1 & 2 \\ 1 & 3 & 1 & 2 \\ 1 & 1 & 3 & 2 \\ 2 & 2 & 2 & 6 \end{bmatrix}, \quad \lambda_1 = 9, \quad \lambda_2 = 2$$

Question 6.122. Let A be an $n \times n$ matrix. Suppose that there is an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$. Prove that A must be symmetric. Thus, symmetric matrices are only matrices that may be diagonalized using an orthogonal diagonalizing basis.

Answer: $A = QDQ^{-1} = QDQ^t = Q^tD^t(Q^t)^t = (QDQ^t)^t = (QDQ^{-1})^t = A^t$