MA 351, HW 4

Elnard Utiushev

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Section 2.3: True/False: 2.20, 2.25; Exercises: 2.66, 2.70, 2.80
Section 3.1: True/False: 3.1, 3.4, 3.5; Exercises: 3.9, 3.10, 3.17, 3.21, 3.24
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1 Section 2.3

1.1 True/False

Question 2.20. Suppose that A is a 3×5 matrix such that the vectors $X = [1,1,1,1,1]^t, Y = [0,1,1,1,1]^t$, and $Z = [0,0,1,1,1]^t$ belong to the nullspace of A. Classify the following statements as true or false.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(a) The rows of A are dependent.
Answer: False, since we need at least 3 independent row vectors, for X, Y, Z to belong to the nullspace.

(b) $AX = B$ has a solution for all $B \in \mathbb{R}^3$.
Answer: True, because A is a 3×5 matrix

(c) The solution to $AX = B$, when it exists, is unique.
Answer: False

Question 2.25. Suppose that A is a 3×7 matrix such that the equation $AX = B$ is solvable for all B in $\mathbb{R}^3$. Then A has rank 3.
Answer: True, according to the theorem 2.16

1.2 Exercises

Question 2.66. For each matrix $(a) - (d)$, find its rank and bases for its column and row spaces.

(a)

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 2 & 3 & 2 & 3 \\ 2 & 10 & 4 & 10 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 2 & 3 & 2 & 3 \\ 2 & 10 & 4 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{7}{2} \end{bmatrix}$$

Rank: 3

Row space basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{7}{2} \end{bmatrix}$

Column space basis: $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$

(b)

$$\begin{bmatrix} -1 & 4 & -2 \\ 4 & 4 & 2 \\ 3 & 0 & -3 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} -1 & 4 & -2 \\ 4 & 4 & 2 \\ 3 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank: 3

Row space basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Column space basis: $\begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}$

(c)

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 5 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

Rank: 2

Row space basis: $\begin{bmatrix} 1 \\ 0 \\ \frac{1}{5} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{3}{5} \end{bmatrix}$

Column space basis: $\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

(d)

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 3 & 6 & 6 \\ -2 & -4 & -4 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 3 & 6 & 6 \\ -2 & -4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank: 1

Row space basis: $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Column space basis: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix}$

Question 2.70. Let $\mathcal{W}$ be the span of the following vectors.

$$\begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 11 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

(a) Use Theorem 2.12 on page 134 to find a basis for $\mathcal{W}$

Answer: $\begin{bmatrix} 1 \\ 0 \\ \frac{1}{11} \\ \frac{2}{11} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ \frac{3}{11} \\ \frac{6}{11} \end{bmatrix}$

(b) Express each the given vectors as a linear combination of the basis elements.

Answer: $2 \begin{bmatrix} 1 \\ 0 \\ \frac{1}{11} \\ \frac{2}{11} \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ \frac{3}{11} \\ \frac{6}{11} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}$

$5 \begin{bmatrix} 1 \\ 0 \\ \frac{1}{11} \\ \frac{2}{11} \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ \frac{3}{11} \\ \frac{6}{11} \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

$1 \begin{bmatrix} 1 \\ 0 \\ \frac{1}{11} \\ \frac{2}{11} \end{bmatrix} - 4 \begin{bmatrix} 0 \\ 1 \\ \frac{3}{11} \\ \frac{6}{11} \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -1 \\ -2 \end{bmatrix}$

$11 \begin{bmatrix} 1 \\ 0 \\ \frac{1}{11} \\ \frac{2}{11} \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ \frac{3}{11} \\ \frac{6}{11} \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

(c) Use Theorem 2.3 on page 104 to find a basis for $\mathcal{W}$.

Answer: $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

Question 2.80. An $m \times n$ matrix A has a d -dimensional nullspace. What is the dimension of the nullspace of A^t?

Answer: According to Rank-Nullity Theorem, $m - (n - d)$

2 Section 3.1

2.1 True/False

Question 3.1. A linear transformation of $\mathbb{R}^2$ into $\mathbb{R}^2$ that transforms $[1, 2]^t$ to $[7, 3]^t$ and $[3, 4]^t$ to $[-1, 1]^t$ will also transform $[5, 8]^t$ to $[13, 7]^t$

Answer: True. We can find the transformation by solving this augmented matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 7 \\ 0 & 0 & 1 & 2 & 3 \\ 3 & 4 & 0 & 0 & -1 \\ 0 & 0 & 3 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -15 \\ 0 & 1 & 0 & 0 & 11 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

So the transformation is $\left[\begin{smallmatrix} -15 & 11 \\ -5 & 4 \end{smallmatrix}\right]$, we can verify that

$$\begin{bmatrix} -15 & 11 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -15 & 11 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -15 & 11 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}$$

Question 3.4. It is impossible for a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ to transform a parallelogram onto a line segment.

Answer: False

Question 3.5. All transformations of $\mathbb{R}^2$ into $\mathbb{R}^2$ transform line segments onto line segments.

Answer: False, it is possible to transform (example $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$) line segment to a point.

2.2 Exercises

Question 3.9. Describe geometrically the effect of the transformation of $\mathbb{R}^3$ into $\mathbb{R}^3$ defined by multiplication by the following matrices.

(a)

$$R_\psi^x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

Answer: Rotation counterclockwise around x axis for ψ

(b)

$$R_\psi^y = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix}$$

Answer: Rotation counterclockwise around y axis for ψ

(c)

$$R_\psi^z = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer: Rotation counterclockwise around z axis for ψ

Question 3.10. Consider the points

$$X_1 = [1,1]^t, \quad X_2 = [2,2]^t$$
$$Y_1 = [4,5]^t, \quad Y_2 = [5,6]^t$$

Is it possible to find a 2×2 matrix A for which multiplication by A transforms X_1 into Y_1 and X_2 into Y_2?

Answer: No

$$X_2 = 2X_1 Y_2 = AX_2 = A(2X_1) = 2AX_1 = 2Y_1$$

Question 3.17. What matrix describes rotation in $\mathbb{R}^2$ clockwise by θ radians?

Answer:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Question 3.21. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation. Prove that $T(0) = 0$. (Note: You should not assume that T is a matrix transformation. Instead, think about the property that in any vector space $0, X = 0$.)

Answer: $T(0+0) = T(0) + T(0)$

Question 3.24. Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation between two vector spaces. We define the image $T(\mathcal{V})$ by $T(\mathcal{V}) = \{T(X) | X \in \mathcal{V}\}$. The image is the set of $Y \in \mathcal{W}$ such that the equation $T(X) = Y$ is solvable for $X \in \mathcal{V}$. Show that the image of T is a subspace of $\mathcal{V}$.

Answer: Let $T(X_1) = Y_1, T(X_2) = Y_2$ to be in $T(\mathcal{V})$. We can show that $T(aY_1 + bY_2) = aT(Y_1) + bT(Y_2)$, where a and b are scalars, meaning that image is closed under linear combinations.

3 Section 3.2

3.1 True/False

Question 3.14. Let $A = R_{\pi/2}$ be the matrix that describes rotation by $\pi/2$ radians [formula (3.1) on page 150]. Then $A^4 = I$, where I is the 2×2 identity matrix.

Answer: True, since application of the rotation by $\pi/2$ radians will result in the full circle of rotation.

Question 3.15. Assume that A and B are matrices such that AB is defined and B has a column that has all its entries equal to zero. Then one of the columns of AB also has all its entries equal to zero.

Answer: True

Question 3.18. Assume that A and B are matrices such that AB is defined and the columns of B are linearly dependent. Then the columns of AB are also linearly dependent.

Answer: True

3.2 Exercises

Question 3.34. Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the following rule: $T(X)$ is the result of first rotating X counterclockwise by $\pi/6$ radians and then multiplying by

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$B = AR_{\pi/6} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ \frac{3}{2} & \frac{3\sqrt{3}}{2} \end{bmatrix}$$

(a) What is the image of the circle $x^2 + y^2 = 1$ under T?
Answer:

$$\begin{bmatrix} \sqrt{3} & -1 & u \\ \frac{3}{2} & \frac{3\sqrt{3}}{2} & v \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{\sqrt{3}u}{4} + \frac{v}{6} \\ 0 & 1 & \frac{\sqrt{3}}{6}\left[-\frac{\sqrt{3}u}{2} + v\right] \end{bmatrix}$$

Our image of the circle:

$$\frac{1}{12}\left(-\frac{\sqrt{3}u}{2} + v\right)^2 + \left(\frac{\sqrt{3}u}{4} + \frac{v}{6}\right)^2 = 1$$

(b) What is the image of the unit square under T?
Answer:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \sqrt{3} \\ \frac{3}{2} \end{bmatrix}, \quad \begin{bmatrix} -1 \\ \frac{3\sqrt{3}}{2} \end{bmatrix}, \quad \begin{bmatrix} -1 + \sqrt{3} \\ \frac{3}{2} + \frac{3\sqrt{3}}{2} \end{bmatrix}$$

(c) Find a matrix B such that $T(X) = BX$ for all $X \in \mathbb{R}^2$
Answer:

$$B = \begin{bmatrix} \sqrt{3} & -1 \\ \frac{3}{2} & \frac{3\sqrt{3}}{2} \end{bmatrix}$$

Question 3.49. Find a 3×3 matrix A such that $A^3 = 0$ but $A^2 \neq 0$.

Answer:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$