MA 351, HW 6

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Section 4.1: True/False: 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7; Exercises: 4.1, 4.6
Section 4.2: True/False: 4.9, 4.10; Exercises: 4.12, 4.15, 4.16, 4.18, 4.25, 4.26

1 Section 4.1

1.1 True/False

Question 4.1. The following matrices have the same determinant.

$$\begin{bmatrix} 2 & 4 & 2 & 6 \\ 3 & 3 & 27 & 33 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 6 & 3 & 9 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \\ 2 & 2 & 18 & 22 \end{bmatrix}$$

Answer: True

$\left|\begin{smallmatrix} 2 & 4 & 2 & 6 \\ 3 & 3 & 27 & 33 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{smallmatrix}\right| = +(2)(+(3)((5)(3)-(2)(-3))-(27)((1)(3)-(2)(1))+(33)((1)(-3)-(5)(1)))-(4)(+(3)((5)(3)-(2)(-3))-(27)((2)(3)-(2)(6))+(33)((2)(-3)-(5)(6)))+(2)(+(3)((1)(3)-(2)(1))-(3)((2)(3)-(2)(6))+(33)((2)(1)-(1)(6)))-(6)(+(3)((1)(-3)-(5)(1))-(3)((2)(-3)-(5)(6))+(27)((2)(1)-(1)(6))) = 3318$

$\left|\begin{smallmatrix} 3 & 6 & 3 & 9 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \\ 2 & 2 & 18 & 22 \end{smallmatrix}\right| = +(3)(+(1)((-3)(22) - (3)(18)) - (5)((1)(22) - (3)(2)) + (2)((1)(18) - (-3)(2))) - (6)(+(2)((-3)(22) - (3)(18)) - (5)((6)(22) - (3)(2)) + (2)((6)(18) - (-3)(2))) + (3)(+(2)((1)(22) - (3)(2)) - (1)((6)(22) - (3)(2)) + (2)((6)(2) - (1)(2))) - (9)(+(2)((1)(18) - (-3)(2)) - (1)((6)(18) - (-3)(2)) + (5)((6)(2) - (1)(2))) = 3318$

Question 4.2. Let A be a 3×3 matrix. Then $\det\,(5A) = 5\det(A)$

Answer: False, for example

$$\begin{vmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 125$$

$$5 * \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 5$$

Question 4.3. Let A and B be 3×3 matrices. Then $\det(A + B) = \det(A) + \det(B)$

Answer: False, for example,

$$|A + B| = \begin{vmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 216$$

$$|A| + |B| = \begin{vmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 125 + 1 = 126$$

Question 4.4. The following statement is true:

$$\begin{vmatrix} 2 & 4 & 2 & 6 \\ 3 & 3 & 27 & 33 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{vmatrix} = 2 \begin{vmatrix} 2 & 4 & 2 & 6 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 4 & 2 & 6 \\ 1 & 1 & 23 & 33 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{vmatrix}$$

Answer: True

$$2 * \begin{vmatrix} 2 & 4 & 2 & 6 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 4 & 2 & 6 \\ 1 & 1 & 23 & 33 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 2 & 6 \\ 2 & 2 & 4 & 0 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 4 & 2 & 6 \\ 1 & 1 & 23 & 33 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 2 & 6 \\ 3 & 3 & 27 & 33 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{vmatrix}$$

Question 4.5. The following statement is true:

$$\begin{vmatrix} 2 & 4 & 2 & 6 \\ 3 & 3 & 7 & 8 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{vmatrix} = 4 \begin{vmatrix} 3 & 7 & 8 \\ 2 & 5 & 2 \\ 6 & -3 & 3 \end{vmatrix} - 3 \begin{vmatrix} 2 & 2 & 6 \\ 2 & 5 & 2 \\ 6 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 & 6 \\ 3 & 7 & 8 \\ 6 & -3 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 2 & 6 \\ 3 & 7 & 8 \\ 2 & 5 & 2 \end{vmatrix}$$

Answer: False, since the expansion started from an even column, signs should be inverted.

Question 4.6. The following matrices have the same determinant:

$$\begin{bmatrix} 1753 & 0 & 0 & 0 \\ 27 & 33 & 0 & 0 \\ 13 & 911 & 1411 & 0 \\ -15 & 44 & 32 & 1001 \end{bmatrix}, \begin{bmatrix} 1753 & 27 & 13 & -15 \\ 0 & 33 & 911 & 44 \\ 0 & 0 & 1411 & 32 \\ 0 & 0 & 0 & 1001 \end{bmatrix}$$

Answer: True, since the determinant of upper (or lower) triangular matrix is equal to the product of the diagonal elements.

Question 4.7. The following matrices have the same determinant:

$$\begin{bmatrix} 1753 & 0 & 0 & 0 \\ 27 & 33 & 0 & 0 \\ 13 & 911 & 1411 & 0 \\ -15 & 44 & 32 & 1001 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1753 \\ 0 & 0 & 33 & 27 \\ 0 & 1411 & 911 & 13 \\ 1001 & 32 & 44 & -15 \end{bmatrix}$$

Answer: True, we can transform the second matrix to upper triangular matrix by switching the rows (1 and 4, 2 and 3). Since we switched rows twice, the determinant for the second matrix did not change. By applying the same reasoning as in 4.6, we prove that the matrices have the same determinants.

1.2 Exercises

Question 4.1. Compute the following determinants:

(a) $\begin{vmatrix} 1 & 4 \\ -3 & 2 \end{vmatrix} = (1)(2) - (4)(-3) = 14$

(b) $\begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix} = (1)(8) - (4)(2) = 0$

(c) $\begin{vmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ -2 & 2 & 2 \end{vmatrix} = +(2)((1)(2) - (1)(2)) + (1)((-1)(2) - (1)(-2)) = 0$

(d) $\begin{vmatrix} 7 & 1 & 1 \\ 0 & a & b \\ 0 & d & c \end{vmatrix} = +(7)((a)(c) - (b)(d)) - (1)((0)(c) - (b)(0)) + (1)((0)(d) - (a)(0)) = 7 * a * c - 7 * b * d$

(e) $\begin{vmatrix} 0 & 5 & 1 \\ -1 & 1 & 3 \\ -2 & -2 & 2 \end{vmatrix} = -(5)((-1)(2) - (3)(-2)) + (1)((-1)(-2) - (1)(-2)) = -16$

(f) $\begin{vmatrix} 2 & 1 & 1 \\ 5 & 4 & 3 \\ 7 & 5 & 4 \end{vmatrix} = +(2)((4)(4)-(3)(5)) - (1)((5)(4)-(3)(7)) + (1)((5)(5)-(4)(7)) = 0$

(g) $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = +(1)((1)(1)-(1)(2)) + (1)((2)(2)-(1)(3)) = 0$

(h) $\begin{vmatrix} -3 & 2 & 2 \\ 1 & 4 & 1 \\ 7 & 6 & -2 \end{vmatrix} = +(-3)((4)(-2)-(1)(6)) - (2)((1)(-2)-(1)(7)) + (2)((1)(6)-(4)(7)) = 16$

(i) $\begin{vmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 0 & 5 \end{vmatrix} = +(2)(+(1)((3)(5)-(2)(0))-(1)((0)(5)-(2)(0))+(1)((0)(0)-(3)(0)))+(2)(+(1)((0)(5)-(2)(0)) - (1)((0)(5)-(2)(1)) + (1)((0)(0)-(0)(1))) = 34$

(j) $\begin{vmatrix} 3 & 1 & 3 & 0 \\ 3 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ 6 & 3 & 4 & 5 \end{vmatrix} = +(3)(+(1)((2)(5)-(1)(4))-(3)((0)(5)-(1)(3))+(1)((0)(4)-(2)(3)))-(1)(+(3)((2)(5)-(1)(4)) - (3)((0)(5)-(1)(6)) + (1)((0)(4)-(2)(6))) + (3)(+(3)((0)(5)-(1)(3)) - (1)((0)(5)-(1)(6)) + (1)((0)(3)-(0)(6))) = -6$

Question 4.6. This exercise discusses the proof of the statement that a matrix with integral entries has integral determinant.

(a) Prove that if all the entries of a 2×2 matrix are integers, then its determinant must be an integer.

Answer: Let a, b, c, d to be integers. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ Since integer set is closed under addition and multiplication, $ad - bc$ is an integer

(b) Use part (a) and formula (4.2) on page 239 to prove the statement in part (a) for 3×3 matrices.

Answer: Since integer set is closed under addition and multiplication, using the same reasoning as in part (a), the determinant of 3 by 3 matrix is an integer if all elements of the matrix are integers.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(c) Use part (b) to prove the statement in part (a) for 4×4 matrices. If you are familiar with mathematical induction, prove the statement in part (a) for all $n \times n$ matrices.

Answer: Since integer set is closed under addition and multiplication, using the same reasoning as in part (b), the determinant of 4 by 4 matrix is an integer if all elements of the matrix are integers.

2 Section 4.2

2.1 True/False

Question 4.9. For all $n \times n$ matrices A and $B, \det(AB) = \det(BA)$

Answer: True

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

Question 4.10. Suppose that $\det(A + I) = 3$, and $\det(A - I) = 5$. Then

$$\det\left(A^2 - I\right) = 20$$

Answer: False

$$\det(A^2 - I) = \det(A^2 - I^2) = \det((A - I)(A + I)) = \det(A - I)\det(A + I) = 3 * 5 = 15$$

2.2 Exercises

Question 4.12. Use row reduction to compute the following determinants:

(a)

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} \overset{R_2 \to R_2 - (2)R_1}{=} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$\overset{R_3 \to R_3 - (3)R_1}{=} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{vmatrix} \overset{R_3 \to R_3 - (2)R_2}{=} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

(b)

$$\begin{vmatrix} -3 & 2 & 2 \\ 1 & 4 & 1 \\ 7 & 6 & -2 \end{vmatrix} \overset{R_1 \to \frac{1}{-3} R_1}{=} -3 \begin{vmatrix} 1 & -\frac{2}{3} & -\frac{2}{3} \\ 1 & 4 & 1 \\ 7 & 6 & -2 \end{vmatrix}$$

$$\overset{R_2 \to R_2 - (1)R_1}{=} -3 \begin{vmatrix} 1 & -\frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{14}{3} & \frac{5}{3} \\ 7 & 6 & -2 \end{vmatrix} \overset{R_3 \to R_3 - (7)R_1}{=} -3 \begin{vmatrix} 1 & -\frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{14}{3} & \frac{5}{3} \\ 0 & \frac{32}{3} & \frac{8}{3} \end{vmatrix}$$

$$\overset{R_2 \to \frac{1}{14/3} R_2}{=} -14 \begin{vmatrix} 1 & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 1 & \frac{5}{14} \\ 0 & \frac{32}{3} & \frac{8}{3} \end{vmatrix} \overset{R_1 \to R_1 - \left(-\frac{2}{3}\right) R_2}{=} -14 \begin{vmatrix} 1 & 0 & -\frac{3}{7} \\ 0 & 1 & \frac{5}{14} \\ 0 & \frac{32}{3} & \frac{8}{3} \end{vmatrix}$$

$$\overset{R_3 \to R_3 - \left(\frac{32}{3}\right) R_2}{=} -14 \begin{vmatrix} 1 & 0 & -\frac{3}{7} \\ 0 & 1 & \frac{5}{14} \\ 0 & 0 & -\frac{8}{7} \end{vmatrix} \overset{R_3 \to \frac{1}{-8/7} R_3}{=} 16 \begin{vmatrix} 1 & 0 & -\frac{3}{7} \\ 0 & 1 & \frac{5}{14} \\ 0 & 0 & 1 \end{vmatrix}$$

$$\overset{R_1 \to R_1 - \left(-\frac{3}{7}\right) R_3}{=} 16 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{5}{14} \\ 0 & 0 & 1 \end{vmatrix} \overset{R_2 \to R_2 - \left(\frac{5}{14}\right) R_3}{=} 16 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 16$$

(c)

$$\begin{vmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 0 & 5 \end{vmatrix} \overset{R_1 \to \frac{1}{2} R_1}{=} 2 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 0 & 5 \end{vmatrix}$$

$$\overset{R_2 \to R_2 - (1)R_1}{=} 2 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 0 & 5 \end{vmatrix} \overset{R_4 \to R_4 - (1)R_1}{=} 2 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 5 \end{vmatrix}$$

$$\overset{R_3 \to \frac{1}{3} R_3}{=} 6 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & -1 & 5 \end{vmatrix} \overset{R_1 \to R_1 - (1)R_3}{=} 6 \begin{vmatrix} 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & -1 & 5 \end{vmatrix}$$

$$\overset{R_4 \to R_4 - (-1)R_3}{=} 6 \begin{vmatrix} 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & \frac{17}{3} \end{vmatrix} \overset{R_4 \to \frac{1}{17/3} R_4}{=} 34 \begin{vmatrix} 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\overset{R_1\to R_1-\left(-\frac{2}{3}\right)R_4}{=} 34\begin{vmatrix}1&0&0&0\\0&1&0&1\\0&0&1&\frac{2}{3}\\0&0&0&1\end{vmatrix} \overset{R_2\to R_2-(1)R_4}{=} 34\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&\frac{2}{3}\\0&0&0&1\end{vmatrix}$$

$$\overset{R_3\to R_3-\left(\frac{2}{3}\right)R_4}{=} 34\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix} = 34$$

(d)

$$\begin{vmatrix}3&1&3&0\\3&1&3&1\\0&0&2&1\\6&3&4&5\end{vmatrix} \overset{R_1\to\frac{1}{3}R_1}{=} 3\begin{vmatrix}1&\frac{1}{3}&1&0\\3&1&3&1\\0&0&2&1\\6&3&4&5\end{vmatrix}$$

$$\overset{R_2\to R_2-(3)R_1}{=} 3\begin{vmatrix}1&\frac{1}{3}&1&0\\0&0&0&1\\0&0&2&1\\6&3&4&5\end{vmatrix} \overset{R_4\to R_4-(6)R_1}{=} 3\begin{vmatrix}1&\frac{1}{3}&1&0\\0&0&0&1\\0&0&2&1\\0&1&-2&5\end{vmatrix}$$

$$\overset{\text{Swap } R_2 \text{ and } R_4}{=} -3\begin{vmatrix}1&\frac{1}{3}&1&0\\0&1&-2&5\\0&0&2&1\\0&0&0&1\end{vmatrix} \overset{R_1\to R_1-\left(\frac{1}{3}\right)R_2}{=} -3\begin{vmatrix}1&0&\frac{5}{3}&-\frac{5}{3}\\0&1&-2&5\\0&0&2&1\\0&0&0&1\end{vmatrix}$$

$$\overset{R_3\to\frac{1}{2}R_3}{=} -6\begin{vmatrix}1&0&\frac{5}{3}&-\frac{5}{3}\\0&1&-2&5\\0&0&1&\frac{1}{2}\\0&0&0&1\end{vmatrix} \overset{R_1\to R_1-\left(\frac{5}{3}\right)R_3}{=} -6\begin{vmatrix}1&0&0&-\frac{5}{2}\\0&1&-2&5\\0&0&1&\frac{1}{2}\\0&0&0&1\end{vmatrix}$$

$$\overset{R_2\to R_2-(-2)R_3}{=} -6\begin{vmatrix}1&0&0&-\frac{5}{2}\\0&1&0&6\\0&0&1&\frac{1}{2}\\0&0&0&1\end{vmatrix} \overset{R_1\to R_1-\left(-\frac{5}{2}\right)R_4}{=} -6\begin{vmatrix}1&0&0&0\\0&1&0&6\\0&0&1&\frac{1}{2}\\0&0&0&1\end{vmatrix}$$

$$\overset{R_2\to R_2-(6)R_4}{=} -6\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&\frac{1}{2}\\0&0&0&1\end{vmatrix} \overset{R_3\to R_3-\left(\frac{1}{2}\right)R_4}{=} -6\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix} = -6$$

(e)

$$\begin{vmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\\13&14&15&16\end{vmatrix} \overset{R_2\to R_2-(5)R_1}{=} \begin{vmatrix}1&2&3&4\\0&-4&-8&-12\\9&10&11&12\\13&14&15&16\end{vmatrix}$$

$$\overset{R_3\to R_3-(9)R_1}{=} \begin{vmatrix}1&2&3&4\\0&-4&-8&-12\\0&-8&-16&-24\\13&14&15&16\end{vmatrix} \overset{R_4\to R_4-(13)R_1}{=} \begin{vmatrix}1&2&3&4\\0&-4&-8&-12\\0&-8&-16&-24\\0&-12&-24&-36\end{vmatrix}$$

$$\overset{R_2\to\frac{1}{-4}R_2}{=} -4\begin{vmatrix}1 & 2 & 3 & 4\\ 0 & 1 & 2 & 3\\ 0 & -8 & -16 & -24\\ 0 & -12 & -24 & -36\end{vmatrix} \overset{R_1\to R_1-(2)R_2}{=} -4\begin{vmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 3\\ 0 & -8 & -16 & -24\\ 0 & -12 & -24 & -36\end{vmatrix}$$

$$\overset{R_3\to R_3-(-8)R_2}{=} -4\begin{vmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\\ 0 & -12 & -24 & -36\end{vmatrix} \overset{R_4\to R_4-(-12)R_2}{=} -4\begin{vmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{vmatrix} = 0$$

(f)

$$\begin{vmatrix}2 & 3 & 2 & 0\\ 9 & 0 & 1 & 1\\ 1 & 0 & 1 & 4\\ 13 & 10 & 0 & 9\end{vmatrix} \overset{R_1\to\frac{1}{2}R_1}{=} 2\begin{vmatrix}1 & \frac{3}{2} & 1 & 0\\ 9 & 0 & 1 & 1\\ 1 & 0 & 1 & 4\\ 13 & 10 & 0 & 9\end{vmatrix}$$

$$\overset{R_2\to R_2-(9)R_1}{=} 2\begin{vmatrix}1 & \frac{3}{2} & 1 & 0\\ 0 & -\frac{27}{2} & -8 & 1\\ 1 & 0 & 1 & 4\\ 13 & 10 & 0 & 9\end{vmatrix} \overset{R_3\to R_3-(1)R_1}{=} 2\begin{vmatrix}1 & \frac{3}{2} & 1 & 0\\ 0 & -\frac{27}{2} & -8 & 1\\ 0 & -\frac{3}{2} & 0 & 4\\ 13 & 10 & 0 & 9\end{vmatrix}$$

$$\overset{R_4\to R_4-(13)R_1}{=} 2\begin{vmatrix}1 & \frac{3}{2} & 1 & 0\\ 0 & -\frac{27}{2} & -8 & 1\\ 0 & -\frac{3}{2} & 0 & 4\\ 0 & -\frac{19}{2} & -13 & 9\end{vmatrix}$$

$$\overset{R_2\to\frac{1}{-27/2}R_2}{=} -27\begin{vmatrix}1 & \frac{3}{2} & 1 & 0\\ 0 & 1 & \frac{16}{27} & -\frac{2}{27}\\ 0 & -\frac{3}{2} & 0 & 4\\ 0 & -\frac{19}{2} & -13 & 9\end{vmatrix}$$

$$\overset{R_1\to R_1-\left(\frac{3}{2}\right)R_2}{=} -27\begin{vmatrix}1 & 0 & \frac{1}{9} & \frac{1}{9}\\ 0 & 1 & \frac{16}{27} & -\frac{2}{27}\\ 0 & -\frac{3}{2} & 0 & 4\\ 0 & -\frac{19}{2} & -13 & 9\end{vmatrix}$$

$$\overset{R_3\to R_3-\left(-\frac{3}{2}\right)R_2}{=} -27\begin{vmatrix}1 & 0 & \frac{1}{9} & \frac{1}{9}\\ 0 & 1 & \frac{16}{27} & -\frac{2}{27}\\ 0 & 0 & \frac{8}{9} & \frac{35}{9}\\ 0 & -\frac{19}{2} & -13 & 9\end{vmatrix}$$

$$\overset{R_4\to R_4-\left(-\frac{19}{2}\right)R_2}{=} -27\begin{vmatrix}1 & 0 & \frac{1}{9} & \frac{1}{9}\\ 0 & 1 & \frac{16}{27} & -\frac{2}{27}\\ 0 & 0 & \frac{8}{9} & \frac{35}{9}\\ 0 & 0 & -\frac{199}{27} & \frac{224}{27}\end{vmatrix}$$

$$\overset{R_3\to\frac{1}{8/9}R_3}{=} -24\begin{vmatrix}1 & 0 & \frac{1}{9} & \frac{1}{9}\\ 0 & 1 & \frac{16}{27} & -\frac{2}{27}\\ 0 & 0 & 1 & \frac{35}{8}\\ 0 & 0 & -\frac{199}{27} & \frac{224}{27}\end{vmatrix}$$

$$\overset{R_1\to R_1-\left(\frac{1}{9}\right)R_3}{=} -24\begin{vmatrix}1 & 0 & 0 & -\frac{3}{8}\\ 0 & 1 & \frac{16}{27} & -\frac{2}{27}\\ 0 & 0 & 1 & \frac{35}{8}\\ 0 & 0 & -\frac{199}{27} & \frac{224}{27}\end{vmatrix}$$

$$\overset{R_2\to R_2-\left(\frac{16}{27}\right)R_3}{=} -24\begin{vmatrix}1 & 0 & 0 & -\frac{3}{8}\\ 0 & 1 & 0 & -\frac{8}{3}\\ 0 & 0 & 1 & \frac{35}{8}\\ 0 & 0 & -\frac{199}{27} & \frac{224}{27}\end{vmatrix}$$

$$\overset{R_4\to R_4-\left(-\frac{199}{27}\right)R_3}{=} -24\begin{vmatrix}1 & 0 & 0 & -\frac{3}{8}\\ 0 & 1 & 0 & -\frac{8}{3}\\ 0 & 0 & 1 & \frac{35}{8}\\ 0 & 0 & 0 & \frac{973}{24}\end{vmatrix} \overset{R_4\to \frac{1}{973/24}R_4}{=} -973\begin{vmatrix}1 & 0 & 0 & -\frac{3}{8}\\ 0 & 1 & 0 & -\frac{8}{3}\\ 0 & 0 & 1 & \frac{35}{8}\\ 0 & 0 & 0 & 1\end{vmatrix}$$

$$\overset{R_1\to R_1-\left(-\frac{3}{8}\right)R_4}{=} -973\begin{vmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & -\frac{8}{3}\\ 0 & 0 & 1 & \frac{35}{8}\\ 0 & 0 & 0 & 1\end{vmatrix} \overset{R_2\to R_2-\left(-\frac{8}{3}\right)R_4}{=} -973\begin{vmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & \frac{35}{8}\\ 0 & 0 & 0 & 1\end{vmatrix}$$

$$\overset{R_3\to R_3-\left(\frac{35}{8}\right)R_4}{=} -973\begin{vmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{vmatrix} = -973$$

Question 4.15. Suppose that

$$\begin{vmatrix} a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = 5$$

Compute the following determinants:

(a) $\begin{vmatrix} 2a & 2b & 2c\\ 3d-a & 3e-b & 3f-c\\ 4g+3a & 4h+3b & 4i+3c\end{vmatrix}$

Answer:

$$\begin{vmatrix} 2a & 2b & 2c\\ 3d-a & 3e-b & 3f-c\\ 4g+3a & 4h+3b & 4i+3c\end{vmatrix} = 2*3*4\begin{vmatrix} a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = 2*3*4*5 = 120$$

(b) $\begin{vmatrix} a+2d & b+2e & c+2f\\ g & h & i\\ d & e & f\end{vmatrix}$

Answer:

$$\begin{vmatrix} a+2d & b+2e & c+2f\\ g & h & i\\ d & e & f\end{vmatrix} = \begin{vmatrix} a & b & c\\ g & h & i\\ d & e & f\end{vmatrix} = -\begin{vmatrix} a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = -5$$

Question 4.16. Use row reduction to prove that for all numbers x, y, and z

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (y-x)(z-x)(z-y)$$

reduction.
Answer:

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & y-x & z-x \\ 0 & y^2-x^2 & z^2-x^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & y-x & z-x \\ 0 & 0 & z^2-x^2-(y+x)(z-x) \end{vmatrix} = (y-x)(z-x)(z-y)$$

Question 4.18. Use the row interchange, scalar, and additive properties to prove that any 3×3 matrix with linearly dependent rows has a zero determinant.
Answer:

$$\begin{vmatrix} A_1 \\ A_2 \\ A_1 n + A_2 k \end{vmatrix} = -\begin{vmatrix} A_1 n + A_2 k \\ A_1 \\ A_2 \end{vmatrix} = \begin{vmatrix} nA_1 + kA_2 - nA_1 \\ A_1 \\ A_2 \end{vmatrix} = \begin{vmatrix} kA_2 - kA_2 \\ A_1 \\ A_2 \end{vmatrix} = \begin{vmatrix} 0 \\ A_1 \\ A_2 \end{vmatrix} = 0$$

Question 4.25. Two $n \times n$ matrices A and B are said to be similar if there is an invertible matrix Q such that $A = QBQ^{-1}$. Prove that similar matrices have the same determinant.
Answer:

$$\det(QBQ^{-1}) = \det(Q)\det(B)\det(Q^{-1}) = \det(QQ^{-1})\det(B) = \det(B)$$

Question 4.26. Later we shall study $n \times n$ matrices with the property that $AA^t = I$. What are the possible values of the determinant of such a matrix?
Answer:

$$\det(I) = 1 = \det(AA^t) = \det(A)\det(A^t)$$

Since this must hold even if A consists of integers, $\det(A) = 1$