MA 351, HW 9

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Section 6.1: True/False: 6.1, 6.4, 6.5, 6.6, 6.7
Exercises: 6.1, 6.3, 6.9, 6.13, 6.14, 6.15

1 Section 6.1

1.1 True/False

Question 6.1. Let $\{P_1, P_2, P_3\}$ be an orthogonal subset of $\mathbb{R}^3$. Suppose that $X = P_1 - 2P_2 + 3P_3$ satisfies $X \cdot P_3 = 6$. Then $|P_3| = 2$

Answer: False,

$$X \cdot P_3 = (P_1 - 2P_2 + 3P_3)P_3 = P_1 \cdot P_3 - 2P_2 \cdot P_3 + 3P_3 \cdot P_3 = 3|P_3|^2 = 6$$

$$|P_3| = \sqrt{6/3} = \sqrt{2}$$

Question 6.4. The vectors $P_1 = [1, 1]^t$ and $P_2 = [1, 3]^t$ are perpendicular.

Answer: False,

$$1 * 1 + 1 * 3 = 1 + 3 \neq 0$$

Question 6.5. If $X \cdot Y = 0$, then either $X = \mathbf{0}$ or $Y = \mathbf{0}$

Answer: False, $Y = [1, 0]^t$, $X = [0, 1]^t$, $X \cdot Y = 0$

Question 6.6. $X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$

Answer: False, dot product is not associative, since dot product produces a scalar and is not defined for a scalar and a vector.

Question 6.7. $(X - Y) \cdot (X + Y) = |X|^2 - |Y|^2$

Answer: True,

$$\begin{aligned}(X - Y) \cdot (X + Y) &= (X - Y) \cdot X + (X - Y) \cdot Y \\ &= |X|^2 - Y \cdot X + X \cdot Y - |Y|^2 = |X|^2 - |Y|^2\end{aligned}$$

1.2 Exercises

Question 6.1. For each pair of vectors X and Y below, find (i) the distance between X and Y (ii) $|X|$ and $|Y|$, (iii) $X \cdot Y$, and (iv) the angle between X and Y.

Answer:

(a) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

- Distance $\sqrt{(3 - (-1))^2 + (4 - 2)^2} = 4.47213595499958$
- $|X| = \sqrt{(3)^2 + (4)^2} = 5.0$

- $|Y| = \sqrt{(-1)^2 + (2)^2} = 2.23606797749979$
- $X \cdot Y = 3 \cdot (-1) + 4 \cdot 2 = 5$
- Angle $\arccos \dfrac{5}{5.0 \cdot 2.23606797749979} = 1.1071487177940904$ rad

(b) $\begin{bmatrix} -3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 7 \end{bmatrix}$

- Distance $\sqrt{((-3) - (-4))^2 + (2 - 7)^2} = 5.0990195135927845$
- $|X| = \sqrt{(-3)^2 + (2)^2} = 3.605551275463989$
- $|Y| = \sqrt{(-4)^2 + (7)^2} = 8.06225774829855$
- $X \cdot Y = (-3) \cdot (-4) + 2 \cdot 7 = 26$
- Angle $\arccos \dfrac{26}{3.605551275463989 \cdot 8.06225774829855} = 0.4636476090008059$ rad

(c) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

- Distance $\sqrt{(1 - (-1))^2 + (2 - 1)^2 + (3 - 2)^2} = 2.449489742783178$
- $|X| = \sqrt{(1)^2 + (2)^2 + (3)^2} = 3.7416573867739413$
- $|Y| = \sqrt{(-1)^2 + (1)^2 + (2)^2} = 2.449489742783178$
- $X \cdot Y = 1 \cdot (-1) + 2 \cdot 1 + 3 \cdot 2 = 7$
- Angle $\arccos \dfrac{7}{3.7416573867739413 \cdot 2.449489742783178} = 0.7016741237876036$ rad

(d) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$

- Distance $\sqrt{(1 - 1)^2 + (1 - 1)^2 + (0 - 1)^2 + (2 - (-3))^2} = 5.0990195135927845$
- $|X| = \sqrt{(1)^2 + (1)^2 + (0)^2 + (2)^2} = 2.449489742783178$
- $|Y| = \sqrt{(1)^2 + (1)^2 + (1)^2 + (-3)^2} = 3.4641016151377544$
- $X \cdot Y = 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 + 2 \cdot (-3) = -4$
- Angle $\arccos \dfrac{-4}{2.449489742783178 \cdot 3.4641016151377544} = 2.061679005084208$ rad

Question 6.3. Find $c, d, e,$ and f such that $[c, d, e, f]^t$ is perpendicular to $[a, b, a, b]^t$.
Answer: These 2 vectors will be perpendicular if their dot product is zero

$$\begin{aligned} ca + db + ea + fb &= 0 \\ (c + e)a + (d + f)b &= 0 \\ c &= -e \\ d &= -f \end{aligned}$$

Question 6.9. Show that the following set $\mathcal{B}$ is an orthogonal basis for $\mathbb{R}^4$. Find the $\mathcal{B}$ coordinate vector for $X = [1, 2, -1, -3]^t$

$$\mathcal{B} = \left\{ [2, -1, -1, -1]^t, [1, 3, 3, -4]^t, [1, 1, 0, 1]^t, [1, -2, 3, 1]^t \right\}$$

Answer:

$$P_1 \cdot P_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 3 \\ -4 \end{bmatrix} = 0$$

$$P_1 \cdot P_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0$$
$$P_1 \cdot P_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix} = 0$$
$$P_2 \cdot P_3 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0$$
$$P_2 \cdot P_4 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix} = 0$$
$$P_3 \cdot P_5 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix} = 0$$

Therefore, $\mathcal{B}$ is an orthogonal basis.

$$\begin{aligned} x_1' &= \frac{X \cdot P_1}{|P_1|^2} = \frac{4}{7} \\ x_2' &= \frac{X \cdot P_1}{|P_1|^2} = \frac{16}{35} \\ x_3' &= \frac{X \cdot P_1}{|P_1|^2} = \frac{0}{3} = 0 \\ x_4' &= \frac{X \cdot P_1}{|P_1|^2} = \frac{-9}{15} \end{aligned}$$

$$X' = [4/7, 16/35, 0, -9/15]^t$$

Question 6.13. Let X and Y be vectors in $\mathbb{R}^n$. Prove that $|X|^2 + |Y|^2 = |X+Y|^2$ holds if and only if $X \cdot Y = 0$. Using a diagram, explain why this identity is referred to as the Pythagorean theorem.

Answer:

$$\begin{aligned} |X+Y|^2 &= (X+Y)(X+Y) = X(X+Y) + Y(X+Y) \\ &= |X|^2 + X \cdot Y + Y \cdot X + |Y|^2 \\ &= |X|^2 + 2X \cdot Y + |Y|^2 \end{aligned}$$

$|X|^2 + 2X \cdot Y + |Y|^2 = |X|^2 + |Y|^2$ if and only if $X \cdot Y = 0$

If we draw a right triangle, where X and Y will be legs of the right triangle. We can see that if we apply Pythagorean theorem on the X and Y, we will get the equation above.

Question 6.14. Prove Theorem 6.1 on page 310− that is, prove that if X and Y are vectors in $\mathbb{R}^n$, then

$$-|X||Y| \leq X \cdot Y \leq |X||Y|$$

Answer: If $Y = X = 0$ the proof is trivial. Assuming that X and Y are not 0 vectors. This inequality is true, since $X \cdot Y = |X||Y| \cos\theta$

Question 6.15. Prove that if $X \cdot Y = \pm|X||Y|$ where X and Y are nonzero vectors in $\mathbb{R}^n$, then $X/|X| = \pm Y/|Y|$; hence, X and Y are scalar multiples of each other.

Answer: Let $U = X/|X| \mp Y/|Y|$

$$\begin{aligned} U \cdot U = (X/|X| \mp Y/|Y|)^2 &= (\frac{X}{|X|})^2 \mp 2\frac{X}{|X|} \cdot \frac{Y}{|Y|} + (\frac{Y}{|Y|})^2 \\ &= 1 \mp 2\frac{\pm|X||Y|}{|X||Y|} + 1 = 2 - 2 = 0 \end{aligned}$$

$$\begin{aligned} (X/|X| \mp Y/|Y|)^2 &= 0 \\ X/|X| \mp Y/|Y| &= 0 \\ X/|X| &= \pm Y/|Y| \end{aligned}$$