MA 351, HW 7

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Section 4.2.1: Exercises: 4.31, 4.33
Section 4.3: Exercises: 4.35, 4.36, 4.38, 4.43
Section 5.1: True/False 5.1, 5.4, 5.5, 5.6, 5.11; Exercises: 5.5, 5.11, 5.12, 5.13

1 Section 4.2.1

1.1 Exercises

Question 4.31. Use volume to explain why it is expected that the determinant of a 3×3 matrix with linearly dependent columns has determinant zero.

Answer: If columns are linearly independent, the vectors that 3×3 matrix describes will be in the same plane, which makes volume equal to zero.

Question 4.33. Use volume to explain why it is expected that for 3×3 matrices B and C $|\det(BC)| = |\det B||\det C|$.[Hint: Use Theorem 4.13.]

Answer: Lets take an object with the volume of 1. By applying B on that object, we will get that the volume of that object has changed to $det(B)$. Similarly, with C. Applying both of those, the resulting volume will be $det(B) * det(C)$.

2 Section 4.3

2.1 Exercises

Question 4.35. Use Cramer's rule to express the value of y in system (a) and the value of z in system (b) as a ratio of two determinants. Do not evaluate the determinants.

(a)

$$
\begin{aligned}
2x + y + 3z + w &= 4 \\
x + 4y + 2z - 3w &= -1 \\
-x + y + z + w &= 0 \\
4x - y + z + 2w &= 0
\end{aligned}
$$

Answer:

$$
y = \frac{\begin{vmatrix} 2 & 4 & 3 & 1 \\ 1 & -1 & 2 & -3 \\ -1 & 0 & 1 & 1 \\ 4 & 0 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 4 & 2 & -3 \\ -1 & 1 & 1 & 1 \\ 4 & -1 & 1 & 2 \end{vmatrix}}
$$

(b)

$$
\begin{aligned}
x + 3y + z &= 1 \\
3x + 4y + 5z &= 7 \\
2x + 5y + 7z &= 2
\end{aligned}
$$

Answer:

$$z = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 3 & 4 & 7 \\ 2 & 5 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 1 \\ 3 & 4 & 5 \\ 2 & 5 & 7 \end{vmatrix}}$$

Question 4.36. Use Cramer's rule to solve the following system for z in terms of p_1, p_2, and p_3.(Note that we have not asked for x or y.)

$$\begin{aligned} x + 2y - 3z &= p_1 \\ 3x + y - z &= p_2 \\ 2x + 3y + 5z &= p_3 \end{aligned}$$

Answer:

$$\begin{vmatrix} 1 & 2 & p_1 \\ 3 & 1 & p_2 \\ 2 & 3 & p_3 \end{vmatrix} = +(1)((1)(p_3) - (p_2)(3)) - (2)((3)(p_3) - (p_2)(2)) + (p_1)((3)(3) - (1)(2)) = 7 * p_1 + p_2 - 5 * p_3$$

$$\begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix} = +(1)((1)(5) - (-1)(3)) - (2)((3)(5) - (-1)(2)) + (-3)((3)(3) - (1)(2)) = -47$$

$$\frac{\begin{vmatrix} 1 & 2 & p_1 \\ 3 & 1 & p_2 \\ 2 & 3 & p_3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix}} = \frac{7 * p_1 + p_2 - 5 * p_3}{-47}$$

Question 4.38. Use Theorem 4.15 to find the $(1, 2)$ entry for the inverse of each invertible matrix in Exercise 4.12 on page 258

Answer:

(a)

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 0, \quad \text{Matrix is not invertible}$$

(b)

$$\begin{bmatrix} -3 & 2 & 2 \\ 1 & 4 & 1 \\ 7 & 6 & -2 \end{bmatrix}, \quad (A^{-1})_{12} = \frac{-\begin{vmatrix} 2 & 2 \\ 6 & -2 \end{vmatrix}}{\begin{vmatrix} -3 & 2 & 2 \\ 1 & 4 & 1 \\ 7 & 6 & -2 \end{vmatrix}} = \frac{16}{16}$$

(c)

$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 0 & 5 \end{bmatrix}, \quad (A^{-1})_{12} = \frac{-\begin{vmatrix} 0 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 0 & 5 \end{vmatrix}} = \frac{0}{34}$$

(d)

$$\begin{bmatrix} 3 & 1 & 3 & 0 \\ 3 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ 6 & 3 & 4 & 5 \end{bmatrix}, \quad (A^{-1})_{12} = \frac{-\begin{vmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 3 & 4 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 3 & 0 \\ 3 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ 6 & 3 & 4 & 5 \end{vmatrix}} = \frac{-15}{-6}$$

(e)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} = 0, \quad \text{Matrix is not invertible}$$

(f)

$$\begin{bmatrix} 2 & 3 & 2 & 0 \\ 9 & 0 & 1 & 1 \\ 1 & 0 & 1 & 4 \\ 13 & 10 & 0 & 9 \end{bmatrix}, \quad (A^{-1})_{12} = \frac{-\begin{vmatrix} 3 & 2 & 0 \\ 0 & 1 & 4 \\ 10 & 0 & 9 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 2 & 0 \\ 9 & 0 & 1 & 1 \\ 1 & 0 & 1 & 4 \\ 13 & 10 & 0 & 9 \end{vmatrix}} = \frac{-107}{-973}$$

Question 4.43. Let A be an $n \times n$ matrix that has only integers as entries. State a necessary and sufficient condition on the determinant of such a matrix that guarantees that the inverse has only integers as entries. Prove your condition. [Hint: Consider the property $AA^{-1} = I$.]

Answer:

$$\det(AA^{-1}) = \det(I) = 1$$
$$\det(AA^{-1}) = \det(A)\det(A^{-1})$$
$$\det(A)\det(A^{-1}) = 1$$

Therefore, the inverse has only integers as entries if and only if $\det(A) = \pm 1$

3 Section 5.1

3.1 True/False

Question 5.1. If A is an $n \times n$ matrix that has zero for an eigenvalue, then A cannot be invertible.

Answer: True,

$$(A - 0I)X = 0$$
$$AX = 0$$
$$A^{-1}AX = A^{-1}0$$
$$X = 0$$

Question 5.4. The sum of two eigenvectors is an eigenvector.

Answer: False, for example, $\left[\begin{smallmatrix} 4 & 0 \\ 0 & 5 \end{smallmatrix}\right]$. Eigenvectors for that matrix are $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Their sum is $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ is not an eigenvector.

Question 5.5. If X is an eigenvector for A with eigenvalue 3, then $2X$ is an eigenvector for A with eigenvalue 6.

Answer: False, $2X$ will still have the eigenvalue of 3.

Question 5.6. If X is an eigenvector for an $n \times n$ matrix A, then X is also an eigenvector for $2A$.

Answer: True,

$$AX = \lambda X$$
$$2AX = 2\lambda X$$
$$(2A)X = (2\lambda)X$$

Question 5.11. There is a 3×3 matrix with eigenvalues $1, 2, 3$, and 4

Answer: False, since the cubic polynomial cannot have 4 roots.

3.2 Exercises

Question 5.5. For the following matrices, find all eigenvalues and a basis for each eigenspace. State whether or not the given matrix is diagonalizable over $\mathbb{R}$.

(a)

$$\begin{bmatrix} -18 & 30 \\ -10 & 17 \end{bmatrix}$$

Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda - 18 & 30 \\ -10 & -\lambda + 17 \end{vmatrix} = (\lambda - 2)(\lambda + 3) = 0$$

$\lambda = -3, 2$

1. $\lambda = -3$

$$A - (-3)I = \begin{bmatrix} -15 & 30 \\ -10 & 20 \end{bmatrix}$$

$$\begin{bmatrix} -15 & 30 & 0 \\ -10 & 20 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$

2. $\lambda = 2$

$$A - (2)I = \begin{bmatrix} -20 & 30 \\ -10 & 15 \end{bmatrix}$$

$$\begin{bmatrix} -20 & 30 & 0 \\ -10 & 15 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$

Matrix is diagonalizable

(b)

$$\begin{bmatrix} 10 & -17 \\ 6 & -10 \end{bmatrix}$$

Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda + 10 & -17 \\ 6 & -\lambda - 10 \end{vmatrix} = \lambda^2 + 2 = 0$$

$\lambda = -\sqrt{2}i, \sqrt{2}i$

Matrix is not diagonalizable over real numbers

(c)

$$\begin{bmatrix} -12 & 21 \\ -6 & 11 \end{bmatrix}$$

Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda - 12 & 21 \\ -6 & -\lambda + 11 \end{vmatrix} = (\lambda - 2)(\lambda + 3) = 0$$

$\lambda = -3, 2$

1. $\lambda = -3$

$$A - (-3)I = \begin{bmatrix} -9 & 21 \\ -6 & 14 \end{bmatrix}$$

$$\begin{bmatrix} -9 & 21 & 0 \\ -6 & 14 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -\frac{7}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} \frac{7}{3} \\ 1 \end{bmatrix}$

2. $\lambda = 2$

$$A - (2)I = \begin{bmatrix} -14 & 21 \\ -6 & 9 \end{bmatrix}$$

$$\begin{bmatrix} -14 & 21 & 0 \\ -6 & 9 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$

Matrix is diagonalizable

(d)

$$\begin{bmatrix} 2 & 12 & -8 \\ 0 & -8 & 6 \\ 0 & -9 & 7 \end{bmatrix}$$

Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda + 2 & 12 & -8 \\ 0 & -\lambda - 8 & 6 \\ 0 & -9 & -\lambda + 7 \end{vmatrix} = -(\lambda - 2)(\lambda - 1)(\lambda + 2) = 0$$

$\lambda = -2, 1, 2$

1. $\lambda = -2$

$$A - (-2)I = \begin{bmatrix} 4 & 12 & -8 \\ 0 & -6 & 6 \\ 0 & -9 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 12 & -8 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & -9 & 9 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

2. $\lambda = 1$

$$A - (1)I = \begin{bmatrix} 1 & 12 & -8 \\ 0 & -9 & 6 \\ 0 & -9 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 12 & -8 & 0 \\ 0 & -9 & 6 & 0 \\ 0 & -9 & 6 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 0 \\ \frac{2}{3} \\ 1 \end{bmatrix}$

3. $\lambda = 2$

$$A - (2)I = \begin{bmatrix} 0 & 12 & -8 \\ 0 & -10 & 6 \\ 0 & -9 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 12 & -8 & 0 \\ 0 & -10 & 6 & 0 \\ 0 & -9 & 5 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Matrix is diagonalizable

(e)

$$\begin{bmatrix} 2 & 0 & 0 \\ -2 & -2 & 2 \\ -5 & -10 & 7 \end{bmatrix}$$

Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda + 2 & 0 & 0 \\ -2 & -\lambda - 2 & 2 \\ -5 & -10 & -\lambda + 7 \end{vmatrix} = -(\lambda - 3)(\lambda - 2)^2 = 0$$

$\lambda = 2, 3$

1. $\lambda = 2$

$$A - (2)I = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -4 & 2 \\ -5 & -10 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & -4 & 2 & 0 \\ -5 & -10 & 5 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

2. $\lambda = 3$

$$A - (3)I = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -5 & 2 \\ -5 & -10 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & -5 & 2 & 0 \\ -5 & -10 & 4 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 0 \\ \frac{2}{5} \\ 1 \end{bmatrix}$

Matrix is diagonalizable

(f)

$$\begin{bmatrix} 0 & -5 & 2 \\ -2 & -2 & 2 \\ -7 & -15 & 9 \end{bmatrix}$$

Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -5 & 2 \\ -2 & -\lambda - 2 & 2 \\ -7 & -15 & -\lambda + 9 \end{vmatrix} = -(\lambda - 3)(\lambda - 2)^2 = 0$$

$\lambda = 2, 3$

1. $\lambda = 2$

$$A-(2)I = \begin{bmatrix} -2 & -5 & 2 \\ -2 & -4 & 2 \\ -7 & -15 & 7 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -5 & 2 & 0 \\ -2 & -4 & 2 & 0 \\ -7 & -15 & 7 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

2. $\lambda = 3$

$$A-(3)I = \begin{bmatrix} -3 & -5 & 2 \\ -2 & -5 & 2 \\ -7 & -15 & 6 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -5 & 2 & 0 \\ -2 & -5 & 2 & 0 \\ -7 & -15 & 6 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 0 \\ \frac{2}{5} \\ 1 \end{bmatrix}$

Matrix is not diagonalizable

(g)

$$\begin{bmatrix} 10 & -24 & 7 \\ 6 & -14 & 4 \\ 6 & -15 & 5 \end{bmatrix}$$

Answer:

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda+10 & -24 & 7 \\ 6 & -\lambda-14 & 4 \\ 6 & -15 & -\lambda+5 \end{vmatrix} = -(\lambda-1)\left(\lambda^2+2\right) = 0$$

$\lambda = 1, -\sqrt{2}i, \sqrt{2}i$

1. $\lambda = 1$

$$A-(1)I = \begin{bmatrix} 9 & -24 & 7 \\ 6 & -15 & 4 \\ 6 & -15 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -24 & 7 & 0 \\ 6 & -15 & 4 & 0 \\ 6 & -15 & 4 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 1 \\ \frac{2}{3} \\ 1 \end{bmatrix}$

Matrix is not diagonalizable over real numbers

(h)

$$\begin{bmatrix} -2 & -1 & 1 \\ -6 & -2 & 0 \\ 13 & 7 & -4 \end{bmatrix}$$

Answer:

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda-2 & -1 & 1 \\ -6 & -\lambda-2 & 0 \\ 13 & 7 & -\lambda-4 \end{vmatrix} = -(\lambda+8)\left(\lambda^2+1\right) = 0$$

$\lambda = -8, -i, i$

1. $\lambda = -8$

$$A - (-8)I = \begin{bmatrix} 6 & -1 & 1 \\ -6 & 6 & 0 \\ 13 & 7 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -1 & 1 & 0 \\ -6 & 6 & 0 & 0 \\ 13 & 7 & 4 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} -\frac{1}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix}$

Matrix is not diagonalizable over real numbers

(i)

$$\begin{bmatrix} 1 & 2 & 0 \\ -3 & 2 & 3 \\ -1 & 2 & 2 \end{bmatrix}$$

Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda + 1 & 2 & 0 \\ -3 & -\lambda + 2 & 3 \\ -1 & 2 & -\lambda + 2 \end{vmatrix} = -(\lambda - 2)^2 (\lambda - 1) = 0$$

$\lambda = 1, 2$

1. $\lambda = 1$

$$A - (1)I = \begin{bmatrix} 0 & 2 & 0 \\ -3 & 1 & 3 \\ -1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ -3 & 1 & 3 & 0 \\ -1 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

2. $\lambda = 2$

$$A - (2)I = \begin{bmatrix} -1 & 2 & 0 \\ -3 & 0 & 3 \\ -1 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 0 & 0 \\ -3 & 0 & 3 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$

Matrix is not diagonalizable

(j)

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

Answer:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda+1 & 3 & 0 & 0 \\ 3 & -\lambda+1 & 0 & 0 \\ 0 & 0 & -\lambda-1 & 2 \\ 0 & 0 & -1 & -\lambda-4 \end{vmatrix} = (\lambda-4)(\lambda+2)^2(\lambda+3) = 0$$

$\lambda = -3, -2, 4$

1. $\lambda = -3$

$$A - (-3)I = \begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

2. $\lambda = -2$

$$A - (-2)I = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

3. $\lambda = 4$

$$A - (4)I = \begin{bmatrix} -3 & 3 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 0 & -5 & 2 \\ 0 & 0 & -1 & -8 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 3 & 0 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 \\ 0 & 0 & -5 & 2 & 0 \\ 0 & 0 & -1 & -8 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Matrix is diagonalizable

Question 5.11. Suppose that A is an $n \times n$ matrix such that $A^2 = I$ and that λ is an eigenvalue for A. Prove that $\lambda = \pm 1$

Answer:

$$AX = \lambda X$$
$$AAX = A\lambda X$$
$$IX = \lambda AX$$

$$X = \lambda\lambda X$$
$$X = \lambda^2 X$$

Since X is a nonzero matrix, $\lambda = \pm 1$

Question 5.12. Let A be an $n \times n$ matrix and let λ be an eigenvalue for A. Prove that λ^2 is an eigenvalue for A^2.

Answer:

$$AX = \lambda X$$
$$AAX = A\lambda X$$
$$A^2X = \lambda AX$$
$$A^2X = \lambda\lambda X$$
$$A^2X = \lambda^2 X$$

Question 5.13. Suppose that A in Exercise 5.12 is invertible. Prove that λ^{-1} is an eigenvalue for A^{-1}

Answer:

$$AX = \lambda X$$
$$A^{-1}AX = A^{-1}\lambda X$$
$$IX = \lambda A^{-1}X$$
$$\lambda^{-1}X = A^{-1}X$$