MA 351, HW 2

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Section 1.3: True/False: 1.21, 1.23, 1.24; Exercises 1.65, 1.66, 1.69, 1.79, 1.80
Section 1.4: True/False: 1.25, 1.26, 1.31, 1.33, 1.34; Exercises: 1.93, 1.96, 1.102, 1.119

1 Section 1.3

1.1 True/False

Question 1.21. If the system has three unknowns and R has three nonzero rows, then the system can have an infinite number of solutions.

Answer: False, it will have either no solutions, or a single solution.

Question 1.23. The following matrix may be reduced to reduced echelon form using only one elementary row operation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Answer: False

Question 1.24. The matrix in question 1.23 is the coefficient matrix for a consistent system of equations

Answer: False, the system is inconsistent

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right] \xrightarrow{R_4 \to R_4 - (-1)R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{R_4 \to \frac{1}{2}R_4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - (3)R_4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - (1)R_4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

1.2 Exercises

Question 1.65. Find the reduced echelon form of each of the following matrices:

a.

$$\begin{bmatrix} 2 & 7 & -5 & -3 & 13 \\ 1 & 0 & 1 & 4 & 3 \\ 1 & 3 & -2 & -2 & 6 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2} R_1} \begin{bmatrix} 1 & \frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} & \frac{13}{2} \\ 1 & 0 & 1 & 4 & 3 \\ 1 & 3 & -2 & -2 & 6 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - (1) R_1} \begin{bmatrix} 1 & \frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} & \frac{13}{2} \\ 0 & -\frac{7}{2} & \frac{7}{2} & \frac{11}{2} & -\frac{7}{2} \\ 1 & 3 & -2 & -2 & 6 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - (1) R_1} \begin{bmatrix} 1 & \frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} & \frac{13}{2} \\ 0 & -\frac{7}{2} & \frac{7}{2} & \frac{11}{2} & -\frac{7}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{-7/2} R_2} \begin{bmatrix} 1 & \frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} & \frac{13}{2} \\ 0 & 1 & -1 & -\frac{11}{7} & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - \left(\frac{7}{2}\right) R_2} \begin{bmatrix} 1 & 0 & 1 & 4 & 3 \\ 0 & 1 & -1 & -\frac{11}{7} & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - \left(-\frac{1}{2}\right) R_2} \begin{bmatrix} 1 & 0 & 1 & 4 & 3 \\ 0 & 1 & -1 & -\frac{11}{7} & 1 \\ 0 & 0 & 0 & -\frac{9}{7} & 0 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{-9/7} R_3} \begin{bmatrix} 1 & 0 & 1 & 4 & 3 \\ 0 & 1 & -1 & -\frac{11}{7} & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - (4) R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & -\frac{11}{7} & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - \left(-\frac{11}{7}\right) R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

b.

$$\begin{bmatrix}1&1&1&1&1\\2&2&1&1&1\\1&0&2&3&2\\4&3&2&1&0\end{bmatrix}\xrightarrow{R_2\to R_2-(2)R_1}\begin{bmatrix}1&1&1&1&1\\0&0&-1&-1&-1\\1&0&2&3&2\\4&3&2&1&0\end{bmatrix}$$

$$\xrightarrow{R_3\to R_3-(1)R_1}\begin{bmatrix}1&1&1&1&1\\0&0&-1&-1&-1\\0&-1&1&2&1\\4&3&2&1&0\end{bmatrix}\xrightarrow{R_4\to R_4-(4)R_1}\begin{bmatrix}1&1&1&1&1\\0&0&-1&-1&-1\\0&-1&1&2&1\\0&-1&-2&-3&-4\end{bmatrix}$$

$$\xrightarrow{\text{Swap } R_2 \text{ and } R_3}\begin{bmatrix}1&1&1&1&1\\0&-1&1&2&1\\0&0&-1&-1&-1\\0&-1&-2&-3&-4\end{bmatrix}\xrightarrow{R_2\to\frac{1}{-1}R_2}\begin{bmatrix}1&1&1&1&1\\0&1&-1&-2&-1\\0&0&-1&-1&-1\\0&-1&-2&-3&-4\end{bmatrix}$$

$$\xrightarrow{R_1\to R_1-(1)R_2}\begin{bmatrix}1&0&2&3&2\\0&1&-1&-2&-1\\0&0&-1&-1&-1\\0&-1&-2&-3&-4\end{bmatrix}\xrightarrow{R_4\to R_4-(-1)R_2}\begin{bmatrix}1&0&2&3&2\\0&1&-1&-2&-1\\0&0&-1&-1&-1\\0&0&-3&-5&-5\end{bmatrix}$$

$$\xrightarrow{R_3\to\frac{1}{-1}R_3}\begin{bmatrix}1&0&2&3&2\\0&1&-1&-2&-1\\0&0&1&1&1\\0&0&-3&-5&-5\end{bmatrix}\xrightarrow{R_1\to R_1-(2)R_3}\begin{bmatrix}1&0&0&1&0\\0&1&-1&-2&-1\\0&0&1&1&1\\0&0&-3&-5&-5\end{bmatrix}$$

$$\xrightarrow{R_2\to R_2-(-1)R_3}\begin{bmatrix}1&0&0&1&0\\0&1&0&-1&0\\0&0&1&1&1\\0&0&-3&-5&-5\end{bmatrix}\xrightarrow{R_4\to R_4-(-3)R_3}\begin{bmatrix}1&0&0&1&0\\0&1&0&-1&0\\0&0&1&1&1\\0&0&0&-2&-2\end{bmatrix}$$

$$\xrightarrow{R_4\to\frac{1}{-2}R_4}\begin{bmatrix}1&0&0&1&0\\0&1&0&-1&0\\0&0&1&1&1\\0&0&0&1&1\end{bmatrix}\xrightarrow{R_1\to R_1-(1)R_4}\begin{bmatrix}1&0&0&0&-1\\0&1&0&-1&0\\0&0&1&1&1\\0&0&0&1&1\end{bmatrix}$$

$$\xrightarrow{R_2\to R_2-(-1)R_4}\begin{bmatrix}1&0&0&0&-1\\0&1&0&0&1\\0&0&1&1&1\\0&0&0&1&1\end{bmatrix}\xrightarrow{R_3\to R_3-(1)R_4}\begin{bmatrix}1&0&0&0&-1\\0&1&0&0&1\\0&0&1&0&0\\0&0&0&1&1\end{bmatrix}$$

c.

$$\begin{bmatrix}3&9&13\\2&7&9\end{bmatrix}\xrightarrow{R_1\to\frac{1}{3}R_1}\begin{bmatrix}1&3&\frac{13}{3}\\2&7&9\end{bmatrix}$$

$$\xrightarrow{R_2\to R_2-(2)R_1}\begin{bmatrix}1&3&\frac{13}{3}\\0&1&\frac{1}{3}\end{bmatrix}\xrightarrow{R_1\to R_1-(3)R_2}\begin{bmatrix}1&0&\frac{10}{3}\\0&1&\frac{1}{3}\end{bmatrix}$$

d.

$$\begin{bmatrix} 2 & 1 & 3 & 4 & 0 & -1 \\ -2 & -1 & -3 & -4 & 5 & 6 \\ 4 & 2 & 7 & 6 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2} R_1} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 2 & 0 & -\frac{1}{2} \\ -2 & -1 & -3 & -4 & 5 & 6 \\ 4 & 2 & 7 & 6 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - (-2) R_1} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 2 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 5 & 5 \\ 4 & 2 & 7 & 6 & 1 & -1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - (4) R_1} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 2 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{Swap } R_2 \text{ and } R_3} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 2 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 5 & 5 \end{bmatrix} \xrightarrow{R_1 \to R_1 - \left(\frac{3}{2}\right) R_2} \begin{bmatrix} 1 & \frac{1}{2} & 0 & 5 & -\frac{3}{2} & -2 \\ 0 & 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 5 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{1}{5} R_3} \begin{bmatrix} 1 & \frac{1}{2} & 0 & 5 & -\frac{3}{2} & -2 \\ 0 & 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - \left(-\frac{3}{2}\right) R_3} \begin{bmatrix} 1 & \frac{1}{2} & 0 & 5 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - (1) R_3} \begin{bmatrix} 1 & \frac{1}{2} & 0 & 5 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

e.

$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{5} R_1} \begin{bmatrix} 1 & \frac{4}{5} \\ 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - (1) R_1} \begin{bmatrix} 1 & \frac{4}{5} \\ 0 & \frac{6}{5} \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{6/5} R_2} \begin{bmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - \left(\frac{4}{5}\right) R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

f.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{a} R_1} \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - (c) R_1} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{bc}{a} \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{d - b * c / a} R_2} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - \left(\frac{b}{a}\right) R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

g.

$$\begin{bmatrix} 2 & 4 & 3 & 0 & 6 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2} R_1} \begin{bmatrix} 1 & 2 & \frac{3}{2} & 0 & 3 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - (2) R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & -2 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{2} R_3} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & -2 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - (-2) R_3} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & 5 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - (1) R_3} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & 5 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

h.

$$\begin{bmatrix}1&2&3&4\\0&5&6&7\\0&0&9&10\\0&0&0&13\end{bmatrix}\xrightarrow{R_2\to\frac{1}{5}R_2}\begin{bmatrix}1&2&3&4\\0&1&\frac{6}{5}&\frac{7}{5}\\0&0&9&10\\0&0&0&13\end{bmatrix}$$

$$\xrightarrow{R_1\to R_1-(2)R_2}\begin{bmatrix}1&0&\frac{3}{5}&\frac{6}{5}\\0&1&\frac{6}{5}&\frac{7}{5}\\0&0&9&10\\0&0&0&13\end{bmatrix}\xrightarrow{R_3\to\frac{1}{9}R_3}\begin{bmatrix}1&0&\frac{3}{5}&\frac{6}{5}\\0&1&\frac{6}{5}&\frac{7}{5}\\0&0&1&\frac{10}{9}\\0&0&0&13\end{bmatrix}$$

$$\xrightarrow{R_1\to R_1-\left(\frac{3}{5}\right)R_3}\begin{bmatrix}1&0&0&\frac{8}{15}\\0&1&\frac{6}{5}&\frac{7}{5}\\0&0&1&\frac{10}{9}\\0&0&0&13\end{bmatrix}\xrightarrow{R_2\to R_2-\left(\frac{6}{5}\right)R_3}\begin{bmatrix}1&0&0&\frac{8}{15}\\0&1&0&\frac{1}{15}\\0&0&1&\frac{10}{9}\\0&0&0&13\end{bmatrix}$$

$$\xrightarrow{R_4\to\frac{1}{13}R_4}\begin{bmatrix}1&0&0&\frac{8}{15}\\0&1&0&\frac{1}{15}\\0&0&1&\frac{10}{9}\\0&0&0&1\end{bmatrix}\xrightarrow{R_1\to R_1-\left(\frac{8}{15}\right)R_4}\begin{bmatrix}1&0&0&0\\0&1&0&\frac{1}{15}\\0&0&1&\frac{10}{9}\\0&0&0&1\end{bmatrix}$$

$$\xrightarrow{R_2\to R_2-\left(\frac{1}{15}\right)R_4}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&\frac{10}{9}\\0&0&0&1\end{bmatrix}\xrightarrow{R_3\to R_3-\left(\frac{10}{9}\right)R_4}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

i.

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{a} R_1} \begin{bmatrix} 1 & \frac{b}{a} & \frac{c}{a} & \frac{d}{a} \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{e} R_2} \begin{bmatrix} 1 & \frac{b}{a} & \frac{c}{a} & \frac{d}{a} \\ 0 & 1 & \frac{f}{e} & \frac{g}{e} \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - \left(\frac{b}{a}\right) R_2} \begin{bmatrix} 1 & 0 & -\frac{bf}{ae} + \frac{c}{a} & -\frac{bg}{ae} + \frac{d}{a} \\ 0 & 1 & \frac{f}{e} & \frac{g}{e} \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{1}{h} R_3} \begin{bmatrix} 1 & 0 & -\frac{bf}{ae} + \frac{c}{a} & -\frac{bg}{ae} + \frac{d}{a} \\ 0 & 1 & \frac{f}{e} & \frac{g}{e} \\ 0 & 0 & 1 & \frac{i}{h} \\ 0 & 0 & 0 & j \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - \left(-\frac{bf}{ae} + \frac{c}{a}\right) R_3} \begin{bmatrix} 1 & 0 & 0 & -\frac{i}{h}\left[-\frac{bf}{ae} + \frac{c}{a}\right] - \frac{bg}{ae} + \frac{d}{a} \\ 0 & 1 & \frac{f}{e} & \frac{g}{e} \\ 0 & 0 & 1 & \frac{i}{h} \\ 0 & 0 & 0 & j \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - \left(\frac{f}{e}\right) R_3} \begin{bmatrix} 1 & 0 & 0 & -\frac{i}{h}\left[-\frac{bf}{ae} + \frac{c}{a}\right] - \frac{bg}{ae} + \frac{d}{a} \\ 0 & 1 & 0 & -\frac{fi}{eh} + \frac{g}{e} \\ 0 & 0 & 1 & \frac{i}{h} \\ 0 & 0 & 0 & j \end{bmatrix}$$

$$\xrightarrow{R_4 \to \frac{1}{j} R_4} \begin{bmatrix} 1 & 0 & 0 & -\frac{i}{h}\left[-\frac{bf}{ae} + \frac{c}{a}\right] - \frac{bg}{ae} + \frac{d}{a} \\ 0 & 1 & 0 & -\frac{fi}{eh} + \frac{g}{e} \\ 0 & 0 & 1 & \frac{i}{h} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - \left(-\frac{i}{h}\left(-\frac{bf}{ae} + \frac{c}{a}\right) - \frac{bg}{ae} + \frac{d}{a}\right) R_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{fi}{eh} + \frac{g}{e} \\ 0 & 0 & 1 & \frac{i}{h} \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - \left(-\frac{fi}{eh} + \frac{g}{e}\right) R_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{i}{h} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - \left(\frac{i}{h}\right) R_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

j.

$$\begin{bmatrix} 2 & 5 & 11 & 6 \\ 1 & 4 & 9 & 5 \\ -1 & 2 & 5 & 3 \\ 2 & -1 & -3 & -2 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{5}{2} & \frac{11}{2} & 3 \\ 1 & 4 & 9 & 5 \\ -1 & 2 & 5 & 3 \\ 2 & -1 & -3 & -2 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - (1)R_1} \begin{bmatrix} 1 & \frac{5}{2} & \frac{11}{2} & 3 \\ 0 & \frac{3}{2} & \frac{7}{2} & 2 \\ -1 & 2 & 5 & 3 \\ 2 & -1 & -3 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - (-1)R_1} \begin{bmatrix} 1 & \frac{5}{2} & \frac{11}{2} & 3 \\ 0 & \frac{3}{2} & \frac{7}{2} & 2 \\ 0 & \frac{9}{2} & \frac{21}{2} & 6 \\ 2 & -1 & -3 & -2 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - (2)R_1} \begin{bmatrix} 1 & \frac{5}{2} & \frac{11}{2} & 3 \\ 0 & \frac{3}{2} & \frac{7}{2} & 2 \\ 0 & \frac{9}{2} & \frac{21}{2} & 6 \\ 0 & -6 & -14 & -8 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{3/2}R_2} \begin{bmatrix} 1 & \frac{5}{2} & \frac{11}{2} & 3 \\ 0 & 1 & \frac{7}{3} & \frac{4}{3} \\ 0 & \frac{9}{2} & \frac{21}{2} & 6 \\ 0 & -6 & -14 & -8 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - \left(\frac{5}{2}\right)R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & \frac{7}{3} & \frac{4}{3} \\ 0 & \frac{9}{2} & \frac{21}{2} & 6 \\ 0 & -6 & -14 & -8 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \left(\frac{9}{2}\right)R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & \frac{7}{3} & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & -6 & -14 & -8 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - (-6)R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & \frac{7}{3} & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Question 1.66. Suppose that the matrices in Exercise 1.65 are the augmented matrices for a system of equations. In each case, write the system down and find all solutions (if any) to the system. [(a), (c), (e), (g)]

a.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Solution: Line, $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} a + \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

c.

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{10}{3} \\ 0 & 1 & \frac{1}{3} \end{array}\right]$$

Solution: Point, $\begin{bmatrix} \frac{10}{3} \\ \frac{1}{3} \end{bmatrix}$

e.

$$\left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right]$$

Solution: Inconsistent

g.

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{2} & 0 & 5 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: Line, $\begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \\ 0 \end{bmatrix} a + \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix}$

Question 1.69. Find conditions on a, b, c, and d for which the following system has solutions:
Answer:

$$\begin{bmatrix} 2 & 4 & 1 & 3 & a \\ -3 & 1 & 2 & -2 & b \\ 13 & 5 & -4 & 12 & c \\ 12 & 10 & -1 & 13 & d \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{11}{14} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{5}{14} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: Inconsistent

Question 1.79.

$$X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$xX + yY = \begin{bmatrix} a \\ b \end{bmatrix}$$

Answer:

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 2 & -2 & b \end{array}\right] \xrightarrow{R_2 \to R_2 - (2)R_1} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -4 & -2a + b \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{-4} R_2} \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & \frac{a}{2} - \frac{b}{4} \end{array}\right] \xrightarrow{R_1 \to R_1 - (1)R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{a}{2} + \frac{b}{4} \\ 0 & 1 & \frac{a}{2} - \frac{b}{4} \end{array}\right]$$

Therefore, it is always possible to solve this equation regardless of a and b

Question 1.80. Show that the vectors do not span $\mathbb{R}^3$ by finding a vector that cannot be expressed as a linear combination of them

$$X1 = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \quad X2 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \quad X3 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 4 \\ 4 \\ 13 \end{bmatrix}$$

2 Section 1.4

2.1 True/False

Question 1.25. The nullspace of a 3 × 4 matrix cannot consist of only the zero vector.
Answer: True, because there will always be a free variable.

Question 1.26. The nullspace of a 4×3 matrix cannot consist of only the zero vector.

Answer: False

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

Question 1.31. The set of 3×1 column vectors for which the system below is solvable is a plane in $\mathbb{R}^3$.

Answer:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 3 & 6 & 9 & b_3 \end{array}\right] \xrightarrow{R_2 \to R_2 - (2)R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & -2b_1 + b_2 \\ 3 & 6 & 9 & b_3 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - (3)R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & -2b_1 + b_2 \\ 0 & 0 & 0 & -3b_1 + b_3 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{-2*b_1+b_2} R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3b_1 + b_3 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - (b_1)R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3b_1 + b_3 \end{array}\right] \xrightarrow{R_3 \to R_3 - (-3b_1 + b_3)R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: Inconsistent

Question 1.33. The set of all vectors of the form $\left[\begin{smallmatrix} 1 \\ x \\ y \end{smallmatrix}\right]$, where x and y range over all real numbers, is a subspace of $\mathbb{R}^3$.

Answer: False, for example, vectors $\left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right]$ belong to the subspace, but their difference does not.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Question 1.34. The set of all vectors of the form $\left[\begin{smallmatrix} 0 \\ x \\ y \end{smallmatrix}\right]$, where x and y range over all real numbers, is a subspace of $\mathbb{R}^3$.

Answer: True. Let $u = \left[\begin{smallmatrix} 0 \\ a \\ b \end{smallmatrix}\right], v = \left[\begin{smallmatrix} 0 \\ c \\ d \end{smallmatrix}\right]$ to be vectors in this subspace, then $u + v = \left[\begin{smallmatrix} 0 \\ a+c \\ b+d \end{smallmatrix}\right]$ and $s * u = \left[\begin{smallmatrix} 0 \\ as \\ bs \end{smallmatrix}\right]$ belong to the subspace.

2.2 Exercises

Question 1.93. For each matrix A and each vector X, compute AX.

a.

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 2 & -2 & 1 & 1 \\ 3 & 2 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -11 \end{bmatrix}$$

b.

$$\begin{bmatrix} -5 & 17 \\ 4 & 2 \\ 3 & 1 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 7 \\ 5 \end{bmatrix}$$

c.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{bmatrix}$$

Question 1.96. Find the nullspace for each of the matrices A in Exercise 1.93. Express each answer as a span.

a.

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 2 & -2 & 1 & 1 \\ 3 & 2 & -2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{4}{7} \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{6}{7} \end{bmatrix}$$

Solution: Line, $\begin{bmatrix} \frac{4}{7} \\ \frac{3}{2} \\ \frac{6}{7} \\ 1 \end{bmatrix} a + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

b.

$$\begin{bmatrix} -5 & 17 \\ 4 & 2 \\ 3 & 1 \\ 5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution: Point, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

c.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution: Line, $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} a + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Question 1.102. Create a system of four equations in five unknowns (reader's choice) such that the solution space is a plane in $\mathbb{R}^5$ (Definition 1.10 on page 38). Do not make any coefficients equal 0. Explain why your example works.

Answer:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 12 & 11 & 10 & 9 & 8 & 7 \\ 71 & 90 & 61 & 91 & 83 & 99 \\ 24 & 22 & 20 & 18 & 16 & 14 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{59}{48} & -\frac{5}{3} & -\frac{125}{48} \\ 0 & 1 & 0 & \frac{35}{24} & \frac{4}{3} & \frac{53}{24} \\ 0 & 0 & 1 & \frac{37}{48} & \frac{4}{3} & \frac{67}{48} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

I made one of the rows of the matrix to be a linear combination of another row, so the number of free variables at the end will be equal to 2.

Question 1.119. A square matrix A is upper triangular if all the entries below the main diagonal are zero. Prove that the set $\mathcal{T}$ of all 3 × 3 upper triangular matrices is a subspace of M(3, 3).

Answer: Let A and B to be in the set $\mathcal{T}$. Let s and t to be scalars

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{bmatrix}$$

$$sA + tB = \begin{bmatrix} a_1 s & a_2 s & a_3 s \\ 0 & a_4 s & a_5 s \\ 0 & 0 & a_6 s \end{bmatrix} + \begin{bmatrix} b_1 t & b_2 t & b_3 t \\ 0 & b_4 t & b_5 t \\ 0 & 0 & b_6 t \end{bmatrix} = \begin{bmatrix} a_1 s + b_1 t & a_2 s + b_2 t & a_3 s + b_3 t \\ 0 & a_4 s + b_4 t & a_5 s + b_5 t \\ 0 & 0 & a_6 s + b_6 t \end{bmatrix}$$

$sA + tB$ is an upper triangular, therefore $\mathcal{T}$ closed under linear combination and a subspace.