MA 351, HW 11

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April 18, 2019

Section 6.4: True/False: 6.15, 6.16, 6.17, 6.18, 6.19, 6.20, 6.22
Exercises: 6.70, 6.71, 6.74, 6.75, 6.76, 6.81, 6.82, 6.83, 6.84

1 Section 6.4

1.1 True/False

Question 6.15. There exist 4×4 orthogonal matrices with rank 3.
Answer: It is not possible since $\det(AA^t) = \det(I) = 1$, so A cannot have a rank less than 4.

Question 6.16. If A is an $n \times n$ orthogonal matrix and B is an $n \times 1$ matrix, then the equation $AX = B$ has a unique solution.
Answer: Since A is an orthogonal matrix, $\det(A) \neq 0$, therefore equation $AX = B$ has a unique solution.

Question 6.17. Multiplication by an orthogonal matrix transforms congruent triangles into congruent triangles.
Answer: True, since an orthogonal matrix transform does not change the length of triangles' sides

Question 6.18. If multiplication by an orthogonal matrix transforms a given parallelogram into a square, then the parallelogram was a square to begin with.
Answer: True, since orthogonal matrix transformation preserves both angles and lengths

Question 6.19. The following matrix is orthogonal:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Answer: False

$$AA^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \neq I$$

Question 6.20. Suppose that $A = [A_1, A_2, A_3]$ is an orthogonal matrix where A_i are the columns of A. Then $|A_1 + A_2 + A_3| = \sqrt{3}$.
Answer: True, since we can form an orthonormal basis from the columns of an orthogonal matrix

Question 6.22. The following matrix is orthogonal:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [1, 1, 1]$$

Answer: True

$$A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$AA^t = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} = I$$

1.2 Exercises

Question 6.70. Change just one column of each of the following matrices to make them orthogonal:

(a)

$$\frac{1}{5}\begin{bmatrix} 3 & 4 & 3 \\ -4 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Answer:

$$\frac{1}{5}\begin{bmatrix} 3 & 4 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(b)

$$\frac{1}{25}\begin{bmatrix} 16 & -12 & -15 \\ -12 & 9 & -20 \\ -15 & -20 & 1 \end{bmatrix}$$

Answer:

$$\frac{1}{25}\begin{bmatrix} 16 & -12 & -15 \\ -12 & 9 & -20 \\ -15 & -20 & 0 \end{bmatrix}$$

(c)

$$\frac{1}{9}\begin{bmatrix} 8 & 1 & -4 \\ -4 & 1 & -7 \\ 1 & 1 & 4 \end{bmatrix}$$

Answer:

$$\frac{1}{9}\begin{bmatrix} 8 & 1 & -4 \\ -4 & 4 & -7 \\ 1 & 8 & 4 \end{bmatrix}$$

Question 6.71. In each part, find numbers C or a, b, c, and d such that the given matrix A is orthogonal:

1.

$$A = C\begin{bmatrix} 9 & -12 & -8 \\ -12 & -1 & -12 \\ -8 & -12 & 9 \end{bmatrix}$$

Answer: $C = \pm\frac{1}{17}$

2.

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & a \\ 1 & -1 & 0 & b \\ 1 & 1 & -\sqrt{2} & c \\ 1 & -1 & 0 & d \end{bmatrix}$$

Answer: $a = 0, b = \pm\sqrt{2}, c = 0, d = \mp\sqrt{2}$

Question 6.74. Let A and B be $n \times n$ orthogonal matrices. Prove that AB is orthogonal by showing that for all $X \in \mathbb{R}^n, |(AB)X| = |X|$

Answer: $|(AB)X| = |A(BX)| = |BX| = |X|$

Question 6.75. Redo Exercise 6.74 by showing that $(AB)^t(AB) = I$

Answer: $(AB)^t(AB) = B^tA^tAB = B^tIB = B^tB = I$

Question 6.76. Prove that the inverse of an orthogonal matrix is orthogonal.

Answer: Let A to be an orthogonal matrix. A^t is an orthogonal matrix according to $A^tA = I$. Since $A^{-1} = A^t$, A^{-1} is an orthogonal matrix

Question 6.81. Is it possible to find a 3×2 matrix with orthonormal rows? Explain.

Answer: It is not possible, since we are going to have 3 vectors, with size 2×1 spanning $\mathbb{R}^2$, so one of them will be a linear combination of others.

Question 6.82. Give an example of a 3×2 matrix A with all entries nonzero that has orthonormal columns. Compute AA^t and A^tA. Which is the identity? Prove that the similar product equals I for any A that has orthonormal columns.

Answer:

$$A = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} \frac{5}{9} & -\frac{4}{9} & \frac{2}{9} \\ -\frac{4}{9} & \frac{5}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{8}{9} \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A^tA results in an identity matrix.

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

$$A^tA = \begin{bmatrix} a^2 + c^2 + e^2 & ab + cd + ef \\ ab + cd + ef & b^2 + d^2 + f^2 \end{bmatrix}$$

Since the columns are orthogonal, $ab+cd+ef = 0$. Since the columns are orthonormal, $a^2+c^2+e^2 = b^2 + d^2 + f^2 = 1$

Question 6.83. In parts (a) – (c), find the matrix M such that multiplication by M describes reflection about the given line, plane, or hyperplane. (Use formula (6.62) on page 362. .

1. $2x - 5y = 0$

   Answer:

$$P = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$M_P = I - \frac{2}{|P|^2}PP^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{29}\begin{bmatrix} 4 & -10 \\ -10 & 25 \end{bmatrix} = \begin{bmatrix} \frac{21}{29} & \frac{20}{29} \\ \frac{20}{29} & -\frac{21}{29} \end{bmatrix}$$

2. $x + y - 3z = 0$

   Answer:

$$P = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

$$M_P = I - \frac{2}{|P|^2}PP^t = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \frac{2}{11}\begin{bmatrix}1 & 1 & -3\\1 & 1 & -3\\-3 & -3 & 9\end{bmatrix} = \begin{bmatrix}\frac{9}{11} & -\frac{2}{11} & \frac{6}{11}\\-\frac{2}{11} & \frac{9}{11} & \frac{6}{11}\\\frac{6}{11} & \frac{6}{11} & -\frac{7}{11}\end{bmatrix}$$

3. $2x + y - z - 3w = 0$

   Answer:

$$P = \begin{bmatrix}2\\1\\1\\-3\end{bmatrix}$$

$$M_P = I - \frac{2}{|P|^2}PP^t = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} - \frac{2}{15}\begin{bmatrix}4 & 2 & 2 & -6\\2 & 1 & 1 & -3\\2 & 1 & 1 & -3\\-6 & -3 & -3 & 9\end{bmatrix} = \begin{bmatrix}\frac{7}{15} & -\frac{4}{15} & -\frac{4}{15} & \frac{4}{5}\\-\frac{4}{15} & \frac{13}{15} & -\frac{2}{15} & \frac{2}{5}\\-\frac{4}{15} & -\frac{2}{15} & \frac{13}{15} & \frac{2}{5}\\\frac{4}{5} & \frac{2}{5} & \frac{2}{5} & -\frac{1}{5}\end{bmatrix}$$

Question 6.84. For each given vector $X \in \mathbb{R}^n$, find a scalar k, a vector $P \in \mathbb{R}^n$, and a Householder matrix M_P[formula (6.62) on page 362] such that $M_P X = kI_1$ where I_1 is the first standard basis element of $\mathbb{R}^n$[formula (6.62) on page 362] .

1. $X = [3, 4]^t$

   Answer:

$$k = |X| = 5$$

$$P = X - |X|I_1 = \begin{bmatrix}-2\\4\end{bmatrix}$$

$$M_P = I - \frac{2}{|P|^2}PP^t = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \frac{2}{20}\begin{bmatrix}4 & -8\\-8 & 16\end{bmatrix} = \begin{bmatrix}\frac{3}{5} & \frac{4}{5}\\\frac{4}{5} & -\frac{3}{5}\end{bmatrix}$$

2. $X = [1, 1]^t$

   Answer:

$$k = |X| = 1$$

$$P = X - |X|I_1 = \begin{bmatrix}0\\1\end{bmatrix}$$

$$M_P = I - \frac{2}{|P|^2}PP^t = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \frac{2}{1}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

3. $X = [1, 2, 2]^t$

   Answer:

$$k = |X| = 3$$

$$P = X - |X|I_1 = \begin{bmatrix}-2\\2\\2\end{bmatrix}$$

$$M_P = I - \frac{2}{|P|^2}PP^t = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \frac{2}{12}\begin{bmatrix}4 & -4 & -4\\-4 & 4 & 4\\-4 & 4 & 4\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\\frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\\frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix}$$