MA 351

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1 Section 1.1

1.1 True/False

Question 1.1. A subset of a linearly independent set is linearly independent.
Answer: True, since none of the elements in the subset are linear combinations.

Question 1.2. A subset of a linearly dependent set is linearly dependent.
Answer: False, for example, lets take this linearly dependent set and remove the last element. The set will no longer be lineary dependant.

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 10 \\ 6 \end{bmatrix} \right\}$$

Question 1.3. A set that contains a linearly independent set is linearly independent.
Answer: False

Question 1.4. A set that contains a linearly dependent set is linearly dependent.
Answer: False

Question 1.5. If a set of elements of a vector space is linearly dependent, then each element of the set is a linear combination of the other elements of the set.
Answer: True

Question 1.8. If $\{X, A_1, A_2, A_3\}$ is linearly dependent then X is in the span of A_1, A_2, and A_3.
Answer: True

1.2 Exercises

Question 1.2. Each of the following sets of matrices is linearly dependent. Demonstrate this by explicitly exhibiting one of the elements of the set as a linear combination of the others. You should be able to find the constants by inspection (guessing).

a $\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} + 2 * \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + 2 * \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + 3 * \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

c $0 * \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2 * \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

d $-1 * \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 0 * \begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

e $2 * \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3 * \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - 4 * \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$

f $-3 * \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -9 & 3 & -6 \\ 0 & -3 & -12 \end{bmatrix}$

g $0 * \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$

Question 1.8. Verify the Remark following Example 1.2 on page 8, that is, show that A1 is not a linear combination of A2, A3, and A4.
Answer: It is not a linear combination since there is no way to get $A_{1_{21}} = 1$ by summing and scaling zeroes.

Question 1.10. Prove that the rows of the following matrix are linearly independent.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{bmatrix}$$

Answer:

$$\begin{aligned} A_3 &= xA_1 + yA_2 \\ \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} &= x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix} \\ &= 0 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix} && \text{Since it is the only way to get 0 in } A_{3_1} \\ &= 0 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix} && \text{Since it is the only way to get 0 in } A_{3_2} \end{aligned}$$

Same reasoning can be used to prove that $A_2 \neq xA_1 + yA_3$ and $A_1 \neq xA_2 + yA_3$.

Question 1.19. For each of the following sets of functions either find a function $f(x)$ in their span such that $f(x) > 0$ for all x or prove that no such function exists.

a $\{\sin x, 1\}$
Answer: Since $\forall x, -1 \leq \sin x \leq 1$, $f(x) = \sin x + 2 * 1 > 0$

b $\{\cos x, 1\}$
Answer: Since $\forall x, -1 \leq \cos x \leq 1$, $f(x) = \cos x + 2 * 1 > 0$

c $\{\sin x, \cos x\}$
Answer: It is not possible. $f(x) = c_1 \sin x + c_2 \cos x$, since $f(x) > 0$, $f(0) = c_2 > 0, f(\pi) = -c_2 > 0$ which is not possible.

Question 1.23. Let X, Y, and Z be as shown. Give four matrices (reader's choice) that belong to their span. Give a matrix that does not belong to their span.

$$X = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad Y = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Answer:
Matrices in their span:

$$\begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 8 & 0 \\ 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 5 & 5 \\ 0 & 9 \end{bmatrix}$$

Matrix not in their span:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Question 1.30. Construct an example of your own choice of a 4×4 matrix with linearly dependent columns having all of its entries nonzero.
Answer:

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 2 & 3 & 6 \\ 1 & 2 & 3 & 6 \\ 1 & 2 & 3 & 6 \end{bmatrix}$$

2 Section 1.2

2.1 True/False

Question 1.13. The solution set to a system of three equations in three unknowns cannot be a plane.
Answer: False

Question 1.14. A system of linear equations cannot have only two solutions.
Answer: True

Question 1.16. A system of four equations in four unknowns always has a solution.
Answer: False

2.2 Exercises

Question 1.49. One of these vectors is a solution to the system below and one is not. Which is which?

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$
$$4x - 2y - z - w = 0$$
$$x + 3y - 2z - 2w = 0$$

Answer: Lets rewrite our system as a matrix.

$$\begin{bmatrix} 4 & -2 & -1 & -1 \\ 1 & 3 & -2 & -2 \end{bmatrix}$$

Now we can check if our vectors are solutions to this system:

$$\begin{bmatrix} 4 & -2 & -1 & -1 \\ 1 & 3 & -2 & -2 \end{bmatrix} X = \begin{bmatrix} 4 & -2 & -1 & -1 \\ 1 & 3 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad X \text{ is a solution}$$

$$\begin{bmatrix} 4 & -2 & -1 & -1 \\ 1 & 3 & -2 & -2 \end{bmatrix} Y = \begin{bmatrix} 4 & -2 & -1 & -1 \\ 1 & 3 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix} \qquad Y \text{ is not a solution}$$

Question 1.55. For each system: (i) Write the augmented matrix A. (ii) Find all solutions (if any exist). Express your answer in parametric form and give the translation vector and the spanning vectors. State whether the solution is a line or plane or neither. (iii) If one of the rows the solution of the augmented matrix becomes zero during process, explicitly exhibit one row of A as a linear combination of the other rows.

a

$$\left[\begin{array}{cc|c} 1 & -3 & 2 \\ -2 & 6 & -4 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cc|c} 1 & -3 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Solution: Line, $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3s+2 \\ s \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right] * s + \left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]$
Linear combination: $R_2 = -2R_1$

b

$$\left[\begin{array}{ccc|c}1&3&1&1\\2&4&7&2\\3&10&5&7\end{array}\right] \xrightarrow{R_2\to R_2-2R_1} \left[\begin{array}{ccc|c}1&3&1&1\\0&-2&5&0\\3&10&5&7\end{array}\right] \xrightarrow{R_3\to R_3-3R_1} \left[\begin{array}{ccc|c}1&3&1&1\\0&-2&5&0\\0&1&2&4\end{array}\right]$$

$$\xrightarrow{R_2\to \frac{1}{-2}R_2} \left[\begin{array}{ccc|c}1&3&1&1\\0&1&-\frac{5}{2}&0\\0&1&2&4\end{array}\right] \xrightarrow{R_1\to R_1-3R_2} \left[\begin{array}{ccc|c}1&0&\frac{17}{2}&1\\0&1&-\frac{5}{2}&0\\0&1&2&4\end{array}\right]$$

$$\xrightarrow{R_3\to R_3-1R_2} \left[\begin{array}{ccc|c}1&0&\frac{17}{2}&1\\0&1&-\frac{5}{2}&0\\0&0&\frac{9}{2}&4\end{array}\right] \xrightarrow{R_3\to \frac{1}{4}R_3} \left[\begin{array}{ccc|c}1&0&\frac{17}{2}&1\\0&1&-\frac{5}{2}&0\\0&0&1&\frac{8}{9}\end{array}\right]$$

$$\xrightarrow{R_1\to R_1-\frac{17}{2}R_3} \left[\begin{array}{ccc|c}1&0&0&-\frac{59}{9}\\0&1&-\frac{5}{2}&0\\0&0&1&\frac{8}{9}\end{array}\right] \xrightarrow{R_2\to R_2+\frac{5}{2}R_3} \left[\begin{array}{ccc|c}1&0&0&-\frac{59}{9}\\0&1&0&\frac{20}{9}\\0&0&1&\frac{8}{9}\end{array}\right]$$

Solution: Point, $\begin{bmatrix}-\frac{59}{9}\\\frac{20}{9}\\\frac{8}{9}\end{bmatrix}$

c

$$\left[\begin{array}{ccc|c}1&3&1&1\\2&4&7&2\\4&10&9&4\end{array}\right] \xrightarrow{R_2\to R_2-2R_1} \left[\begin{array}{ccc|c}1&3&1&1\\0&-2&5&0\\4&10&9&4\end{array}\right] \xrightarrow{R_3\to R_3-4R_1} \left[\begin{array}{ccc|c}1&3&1&1\\0&-2&5&0\\0&-2&5&0\end{array}\right]$$

$$\xrightarrow{R_2\to \frac{1}{-2}R_2} \left[\begin{array}{ccc|c}1&3&1&1\\0&1&-\frac{5}{2}&0\\0&-2&5&0\end{array}\right] \xrightarrow{R_1\to R_1-3R_2} \left[\begin{array}{ccc|c}1&0&\frac{17}{2}&1\\0&1&-\frac{5}{2}&0\\0&-2&5&0\end{array}\right]$$

$$\xrightarrow{R_3\to R_3+2R_2} \left[\begin{array}{ccc|c}1&0&\frac{17}{2}&1\\0&1&-\frac{5}{2}&0\\0&0&0&0\end{array}\right]$$

Solution: Line, $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-\frac{17s}{2}+1\\\frac{5s}{2}\\s\end{bmatrix} = \begin{bmatrix}-\frac{17}{2}\\\frac{5}{2}\\1\end{bmatrix} * s + \begin{bmatrix}1\\0\\0\end{bmatrix}$

Linear combination: $R_3 = 2R_1 + R_2$

d

$$\left[\begin{array}{ccc|c}1&3&1&1\\2&4&7&2\\4&10&9&7\end{array}\right] \xrightarrow{R_2\to R_2-2R_1} \left[\begin{array}{ccc|c}1&3&1&1\\0&-2&5&0\\4&10&9&7\end{array}\right] \xrightarrow{R_3\to R_3-4R_1} \left[\begin{array}{ccc|c}1&3&1&1\\0&-2&5&0\\0&-2&5&3\end{array}\right]$$

$$\xrightarrow{R_2\to \frac{1}{-2}R_2} \left[\begin{array}{ccc|c}1&3&1&1\\0&1&-\frac{5}{2}&0\\0&-2&5&3\end{array}\right] \xrightarrow{R_1\to R_1-3R_2} \left[\begin{array}{ccc|c}1&0&\frac{17}{2}&1\\0&1&-\frac{5}{2}&0\\0&-2&5&3\end{array}\right]$$

$$\xrightarrow{R_3\to R_3+2R_2} \left[\begin{array}{ccc|c}1&0&\frac{17}{2}&1\\0&1&-\frac{5}{2}&0\\0&0&0&3\end{array}\right] \xrightarrow{R_3\to \frac{1}{3}R_3} \left[\begin{array}{ccc|c}1&0&\frac{17}{2}&1\\0&1&-\frac{5}{2}&0\\0&0&0&1\end{array}\right]$$

$$\xrightarrow{R_1\to R_1-1R_3} \left[\begin{array}{ccc|c}1&0&\frac{17}{2}&0\\0&1&-\frac{5}{2}&0\\0&0&0&1\end{array}\right]$$

Solution: Inconsistent

e

$$\left[\begin{array}{cccc|c}1&2&1&0&1\\0&1&4&3&2\\0&0&2&2&4\end{array}\right]\xrightarrow{R_1\to R_1-2R_2}\left[\begin{array}{cccc|c}1&0&-7&-6&-3\\0&1&4&3&2\\0&0&2&2&4\end{array}\right]\xrightarrow{R_3\to \frac{1}{2}R_3}\left[\begin{array}{cccc|c}1&0&-7&-6&-3\\0&1&4&3&2\\0&0&1&1&2\end{array}\right]$$

$$\xrightarrow{R_1\to R_1+7R_3}\left[\begin{array}{cccc|c}1&0&0&1&11\\0&1&4&3&2\\0&0&1&1&2\end{array}\right]\xrightarrow{R_2\to R_2-4R_3}\left[\begin{array}{cccc|c}1&0&0&1&11\\0&1&0&-1&-6\\0&0&1&1&2\end{array}\right]$$

Solution: Line, $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-s+11\\s-6\\-s+2\\s\end{bmatrix}=\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}*s+\begin{bmatrix}11\\-6\\2\\0\end{bmatrix}$

f

$$\left[\begin{array}{cccc|c}1&-1&2&-2&1\\2&1&0&3&4\\2&3&2&0&6\end{array}\right]\xrightarrow{R_2\to R_2-2R_1}\left[\begin{array}{cccc|c}1&-1&2&-2&1\\0&3&-4&7&2\\2&3&2&0&6\end{array}\right]\xrightarrow{R_3\to R_3-2R_1}\left[\begin{array}{cccc|c}1&-1&2&-2&1\\0&3&-4&7&2\\0&5&-2&4&4\end{array}\right]$$

$$\xrightarrow{R_2\to \frac{1}{3}R_2}\left[\begin{array}{cccc|c}1&-1&2&-2&1\\0&1&-\frac{4}{3}&\frac{7}{3}&\frac{2}{3}\\0&5&-2&4&4\end{array}\right]\xrightarrow{R_1\to R_1+1R_2}\left[\begin{array}{cccc|c}1&0&\frac{2}{3}&\frac{1}{3}&\frac{5}{3}\\0&1&-\frac{4}{3}&\frac{7}{3}&\frac{2}{3}\\0&5&-2&4&4\end{array}\right]$$

$$\xrightarrow{R_3\to R_3-5R_2}\left[\begin{array}{cccc|c}1&0&\frac{2}{3}&\frac{1}{3}&\frac{5}{3}\\0&1&-\frac{4}{3}&\frac{7}{3}&\frac{2}{3}\\0&0&\frac{14}{3}&-\frac{23}{3}&\frac{2}{3}\end{array}\right]\xrightarrow{R_3\to \frac{1}{4}R_3}\left[\begin{array}{cccc|c}1&0&\frac{2}{3}&\frac{1}{3}&\frac{5}{3}\\0&1&-\frac{4}{3}&\frac{7}{3}&\frac{2}{3}\\0&0&1&-\frac{23}{14}&\frac{1}{7}\end{array}\right]$$

$$\xrightarrow{R_1\to R_1-\frac{2}{3}R_3}\left[\begin{array}{cccc|c}1&0&0&\frac{10}{7}&\frac{11}{7}\\0&1&-\frac{4}{3}&\frac{7}{3}&\frac{2}{3}\\0&0&1&-\frac{23}{14}&\frac{1}{7}\end{array}\right]\xrightarrow{R_2\to R_2+\frac{4}{3}R_3}\left[\begin{array}{cccc|c}1&0&0&\frac{10}{7}&\frac{11}{7}\\0&1&0&\frac{1}{7}&\frac{6}{7}\\0&0&1&-\frac{23}{14}&\frac{1}{7}\end{array}\right]$$

Solution: Line, $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-\frac{10s}{7}+\frac{11}{7}\\-\frac{s}{7}+\frac{6}{7}\\\frac{23s}{14}+\frac{1}{7}\\s\end{bmatrix}=\begin{bmatrix}-\frac{10}{7}\\-\frac{1}{7}\\\frac{23}{14}\\1\end{bmatrix}*s+\begin{bmatrix}\frac{11}{7}\\\frac{6}{7}\\\frac{1}{7}\\0\end{bmatrix}$

g

$$\left[\begin{array}{ccc|c}3&7&2&1\\1&-1&1&2\\5&5&4&5\end{array}\right]\xrightarrow{R_1\to \frac{1}{3}R_1}\left[\begin{array}{ccc|c}1&\frac{7}{3}&\frac{2}{3}&\frac{1}{3}\\1&-1&1&2\\5&5&4&5\end{array}\right]\xrightarrow{R_2\to R_2-1R_1}\left[\begin{array}{ccc|c}1&\frac{7}{3}&\frac{2}{3}&\frac{1}{3}\\0&-\frac{10}{3}&\frac{1}{3}&\frac{5}{3}\\5&5&4&5\end{array}\right]$$

$$\xrightarrow{R_3\to R_3-5R_1}\left[\begin{array}{ccc|c}1&\frac{7}{3}&\frac{2}{3}&\frac{1}{3}\\0&-\frac{10}{3}&\frac{1}{3}&\frac{5}{3}\\0&-\frac{20}{3}&\frac{2}{3}&\frac{10}{3}\end{array}\right]\xrightarrow{R_2\to \frac{1}{-3}R_2}\left[\begin{array}{ccc|c}1&\frac{7}{3}&\frac{2}{3}&\frac{1}{3}\\0&1&-\frac{1}{10}&-\frac{1}{2}\\0&-\frac{20}{3}&\frac{2}{3}&\frac{10}{3}\end{array}\right]$$

$$\xrightarrow{R_1\to R_1-\frac{7}{3}R_2}\left[\begin{array}{ccc|c}1&0&\frac{9}{10}&\frac{3}{2}\\0&1&-\frac{1}{10}&-\frac{1}{2}\\0&-\frac{20}{3}&\frac{2}{3}&\frac{10}{3}\end{array}\right]\xrightarrow{R_3\to R_3+\frac{20}{3}R_2}\left[\begin{array}{ccc|c}1&0&\frac{9}{10}&\frac{3}{2}\\0&1&-\frac{1}{10}&-\frac{1}{2}\\0&0&0&0\end{array}\right]$$

Solution: Line, $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-\frac{9s}{10}+\frac{3}{2}\\\frac{s}{10}-\frac{1}{2}\\s\end{bmatrix}=\begin{bmatrix}-\frac{9}{10}\\\frac{1}{10}\\1\end{bmatrix}*s+\begin{bmatrix}\frac{3}{2}\\-\frac{1}{2}\\0\end{bmatrix}$

Linear combination: $R_3=R_1+2R_2$

h

$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 1 & -6 & 1 & 2 \\ -1 & -3 & -1 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2} R_1} \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 1 & \frac{1}{2} \\ 1 & -6 & 1 & 2 \\ -1 & -3 & -1 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 1 R_1} \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 1 & \frac{1}{2} \\ 0 & -\frac{9}{2} & 0 & \frac{3}{2} \\ -1 & -3 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 1 R_1} \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 1 & \frac{1}{2} \\ 0 & -\frac{9}{2} & 0 & \frac{3}{2} \\ 0 & -\frac{9}{2} & 0 & \frac{3}{2} \end{array}\right] \xrightarrow{R_2 \to \frac{1}{-4} R_2} \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & -\frac{9}{2} & 0 & \frac{3}{2} \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + \frac{3}{2} R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & -\frac{9}{2} & 0 & \frac{3}{2} \end{array}\right] \xrightarrow{R_3 \to R_3 + \frac{9}{2} R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: Plane, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ -\frac{1}{3} \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} * s + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} * k + \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 0 \end{bmatrix}$

Linear combination: $R_3 = R_2 - R_1$

i

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -2 \\ 1 & -1 & 1 & 2 \\ 2 & 3 & 4 & 5 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2} R_1} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & -1 \\ 1 & -1 & 1 & 2 \\ 2 & 3 & 4 & 5 \end{array}\right] \xrightarrow{R_2 \to R_2 - 1 R_1} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & -1 \\ 0 & -\frac{5}{2} & \frac{3}{2} & 3 \\ 2 & 3 & 4 & 5 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2 R_1} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & -1 \\ 0 & -\frac{5}{2} & \frac{3}{2} & 3 \\ 0 & 0 & 5 & 7 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{-2} R_2} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -\frac{3}{5} & -\frac{6}{5} \\ 0 & 0 & 5 & 7 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - \frac{3}{2} R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{5} & \frac{4}{5} \\ 0 & 1 & -\frac{3}{5} & -\frac{6}{5} \\ 0 & 0 & 5 & 7 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{5} R_3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{5} & \frac{4}{5} \\ 0 & 1 & -\frac{3}{5} & -\frac{6}{5} \\ 0 & 0 & 1 & \frac{7}{5} \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - \frac{2}{5} R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{6}{25} \\ 0 & 1 & -\frac{3}{5} & -\frac{6}{5} \\ 0 & 0 & 1 & \frac{7}{5} \end{array}\right] \xrightarrow{R_2 \to R_2 + \frac{3}{5} R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{6}{25} \\ 0 & 1 & 0 & -\frac{9}{25} \\ 0 & 0 & 1 & \frac{7}{5} \end{array}\right]$$

Solution: Point, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{6}{25} \\ -\frac{9}{25} \\ \frac{7}{5} \end{bmatrix}$

j

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & -2 & 1 & 2 & 3 \\ 2 & 6 & 3 & 2 & 1 \\ 5 & -3 & 3 & 5 & 7 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -4 & -1 & 0 & 1 \\ 2 & 6 & 3 & 2 & 1 \\ 5 & -3 & 3 & 5 & 7 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -4 & -1 & 0 & 1 \\ 0 & 4 & 1 & 0 & -1 \\ 5 & -3 & 3 & 5 & 7 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 - 5R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -4 & -1 & 0 & 1 \\ 0 & 4 & 1 & 0 & -1 \\ 0 & -8 & -2 & 0 & 2 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{-4} R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 4 & 1 & 0 & -1 \\ 0 & -8 & -2 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 1R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{3}{4} & 1 & \frac{5}{4} \\ 0 & 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 4 & 1 & 0 & -1 \\ 0 & -8 & -2 & 0 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 - 4R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{3}{4} & 1 & \frac{5}{4} \\ 0 & 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -8 & -2 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 + 8R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{3}{4} & 1 & \frac{5}{4} \\ 0 & 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: Plane, $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -\frac{3k}{4} - s + \frac{5}{4} \\ -\frac{k}{4} - \frac{1}{4} \\ k \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} * s + \begin{bmatrix} -\frac{3}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix} * k + \begin{bmatrix} \frac{5}{4} \\ -\frac{1}{4} \\ 0 \\ 0 \end{bmatrix}$

Linear combination: $R_3 = 4R_1 - R_2$, $R_4 = R_1 + 2R_2$

k

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & -2 & 1 & 2 & 3 \\ 2 & 6 & 3 & 2 & 1 \\ 5 & -3 & 3 & 5 & 8 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -4 & -1 & 0 & 1 \\ 2 & 6 & 3 & 2 & 1 \\ 5 & -3 & 3 & 5 & 8 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -4 & -1 & 0 & 1 \\ 0 & 4 & 1 & 0 & -1 \\ 5 & -3 & 3 & 5 & 8 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 - 5R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -4 & -1 & 0 & 1 \\ 0 & 4 & 1 & 0 & -1 \\ 0 & -8 & -2 & 0 & 3 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{-4} R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 4 & 1 & 0 & -1 \\ 0 & -8 & -2 & 0 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 1R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{3}{4} & 1 & \frac{5}{4} \\ 0 & 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 4 & 1 & 0 & -1 \\ 0 & -8 & -2 & 0 & 3 \end{array}\right] \xrightarrow{R_3 \to R_3 - 4R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{3}{4} & 1 & \frac{5}{4} \\ 0 & 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -8 & -2 & 0 & 3 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 + 8R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{3}{4} & 1 & \frac{5}{4} \\ 0 & 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Swap } R_3 \text{ and } R_4} \left[\begin{array}{cccc|c} 1 & 0 & \frac{3}{4} & 1 & \frac{5}{4} \\ 0 & 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - \frac{5}{4} R_3} \left[\begin{array}{cccc|c} 1 & 0 & \frac{3}{4} & 1 & 0 \\ 0 & 1 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + \frac{1}{4} R_3} \left[\begin{array}{cccc|c} 1 & 0 & \frac{3}{4} & 1 & 0 \\ 0 & 1 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: Inconsistent

Linear combination: $R_3 = 4R_1 - R_2$

l

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 1 \\ -3 & -3 & 1 & 10 & -6 \\ -5 & -4 & 1 & 18 & -11 \\ -2 & 5 & -4 & 16 & -11 \end{array}\right] \xrightarrow{R_2 \to R_2 + 3R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 1 \\ 0 & 3 & -2 & 7 & -3 \\ -5 & -4 & 1 & 18 & -11 \\ -2 & 5 & -4 & 16 & -11 \end{array}\right] \xrightarrow{R_3 \to R_3 + 5R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 1 \\ 0 & 3 & -2 & 7 & -3 \\ 0 & 6 & -4 & 13 & -6 \\ -2 & 5 & -4 & 16 & -11 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 + 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 1 \\ 0 & 3 & -2 & 7 & -3 \\ 0 & 6 & -4 & 13 & -6 \\ 0 & 9 & -6 & 14 & -9 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{3} R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 1 \\ 0 & 1 & -\frac{2}{3} & \frac{7}{3} & -1 \\ 0 & 6 & -4 & 13 & -6 \\ 0 & 9 & -6 & 14 & -9 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{3} & -\frac{17}{3} & 3 \\ 0 & 1 & -\frac{2}{3} & \frac{7}{3} & -1 \\ 0 & 6 & -4 & 13 & -6 \\ 0 & 9 & -6 & 14 & -9 \end{array}\right] \xrightarrow{R_3 \to R_3 - 6R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{3} & -\frac{17}{3} & 3 \\ 0 & 1 & -\frac{2}{3} & \frac{7}{3} & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 9 & -6 & 14 & -9 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 - 9R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{3} & -\frac{17}{3} & 3 \\ 0 & 1 & -\frac{2}{3} & \frac{7}{3} & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -7 & 0 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{-1} R_3} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{3} & -\frac{17}{3} & 3 \\ 0 & 1 & -\frac{2}{3} & \frac{7}{3} & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -7 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + \frac{17}{3} R_3} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{3} & 0 & 3 \\ 0 & 1 & -\frac{2}{3} & \frac{7}{3} & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -7 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - \frac{7}{3} R_3} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{3} & 0 & 3 \\ 0 & 1 & -\frac{2}{3} & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -7 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 + 7R_3} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{3} & 0 & 3 \\ 0 & 1 & -\frac{2}{3} & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: Line, $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -\frac{s}{3}+3 \\ \frac{2s}{3}-1 \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} * s + \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

Question 1.57. Create an example of a system of five equations in five unknowns that has rank 2. How about one with rank 3? Rank 1?

Answer:

$$\left[\begin{array}{ccccc|c} 4603 & 3039 & 3385 & 5681 & 6319 & 3208 \\ 46030 & 30390 & 33850 & 56810 & 63190 & 32080 \\ 32221 & 21273 & 23695 & 39767 & 44233 & 22456 \\ 9206 & 6078 & 6770 & 11362 & 12638 & 6416 \\ 18412 & 12156 & 13540 & 22724 & 25276 & 12832 \end{array}\right] \qquad \text{Rank 1}$$

$$\left[\begin{array}{ccccc|c} 4603 & 3039 & 3385 & 5681 & 6319 & 3208 \\ 5399 & 6263 & 3947 & 3762 & 235 & 2332 \\ -796 & -3224 & -562 & 1919 & 6084 & 876 \\ 9206 & 6078 & 6770 & 11362 & 12638 & 6416 \\ 21596 & 25052 & 15788 & 15048 & 940 & 9328 \end{array}\right] \qquad \text{Rank 2}$$

$$\left[\begin{array}{ccccc|c} 4603 & 3039 & 3385 & 5681 & 6319 & 3208 \\ 5399 & 6263 & 3947 & 3762 & 235 & 2332 \\ 2355 & 499 & 4654 & 2770 & 7647 & 6554 \\ 9206 & 6078 & 6770 & 11362 & 12638 & 6416 \\ 21596 & 25052 & 15788 & 15048 & 940 & 9328 \end{array}\right] \qquad \text{Rank 3}$$