Hashing and Sketching

Part One

Outline for Today

- ***Hash Functions***
  - Understanding our basic building blocks.
- ***Frequency Estimation***
  - Estimating how many times we've seen something.
- ***Concentration Inequalities***
  - "Correct on expectation" versus "correct with high probability."
- ***Probability Amplification***
  - Increasing our confidence in our answers.

Preliminaries: ***Hash Functions***

Hashing in Practice

- Hash functions are used extensively in programming and software engineering:
  - They make hash tables possible: think C++ `std::hash`, Python's `__hash__`, or Java's `Object.hashCode()`.
  - They're used in cryptography: SHA-256, HMAC, etc.
- ***Question:*** When we're in Theoryland, what do we mean when we say "hash function?"

Hashing in Theoryland

- In Theoryland, a hash function is a function from some domain called the ***universe*** (typically denoted $\mathscr{U}$) to some codomain.
- The codomain is usually a set of the form

$$\mathbf{[m]} = \{0, 1, 2, 3, \ldots, m - 1\}$$

$$h : \mathscr{U} \to [m]$$

Hashing in Theoryland

- ***Intuition:*** No matter how clever you are with designing a specific hash function, that hash function isn't random, and so there will be pathological inputs.
  - You can formalize this with the pigeonhole principle.
- ***Idea:*** Rather than finding the One True Hash Function, we'll assume we have a collection of hash functions to pick from, and we'll choose which one to use randomly.

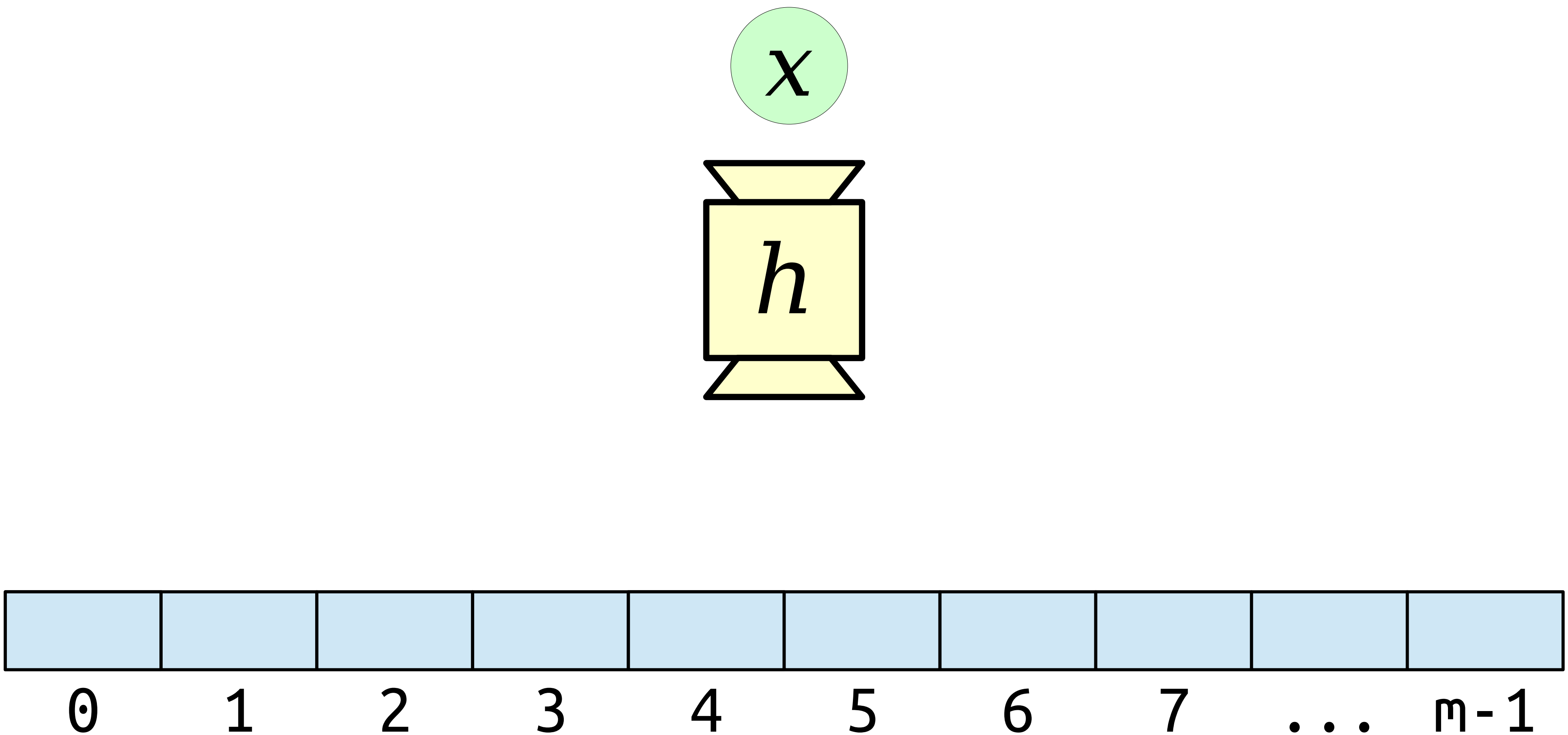
Families of Hash Functions

- A ***family*** of hash functions is a set $\mathscr{H}$ of hash functions with the same domain and codomain.
- We can then introduce randomness into our data structures by sampling a random hash function from $\mathscr{H}$.
- ***Key Point:*** The randomness in our data structures almost always derives from the random choice of hash functions, not from the data.

Data is adversarial.
Hash function selection is random.

- ***Question:*** What makes a family of hash functions $\mathscr{H}$ a "good" family of hash functions?

Goal: If we pick $h \in \mathscr{H}$ uniformly at random, then h should distribute elements uniformly randomly.



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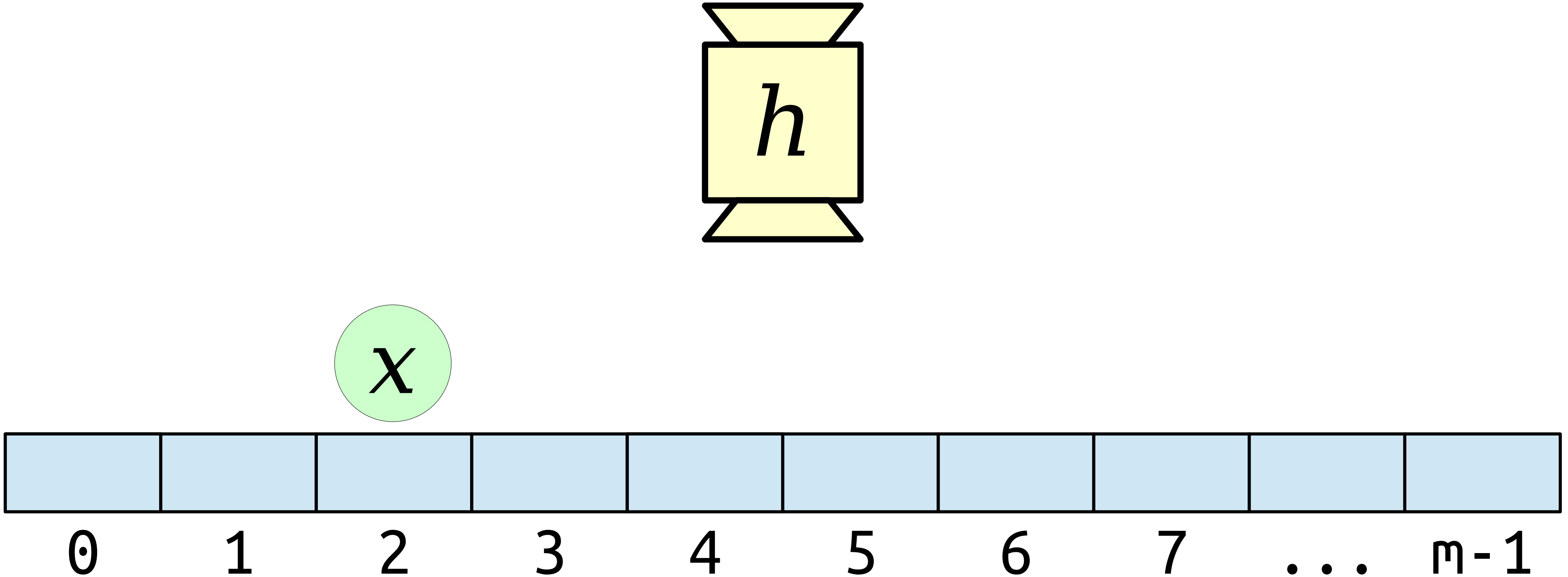

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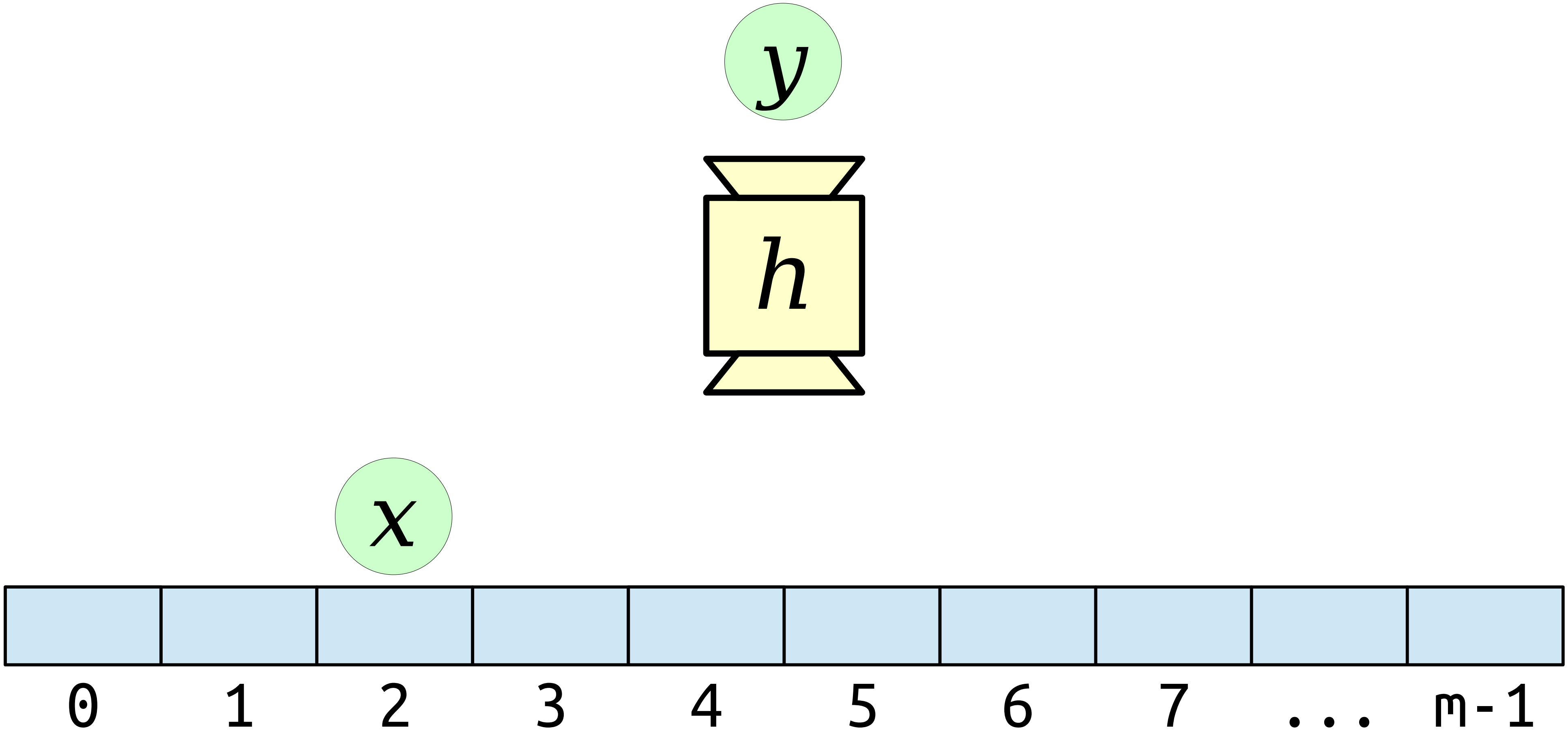

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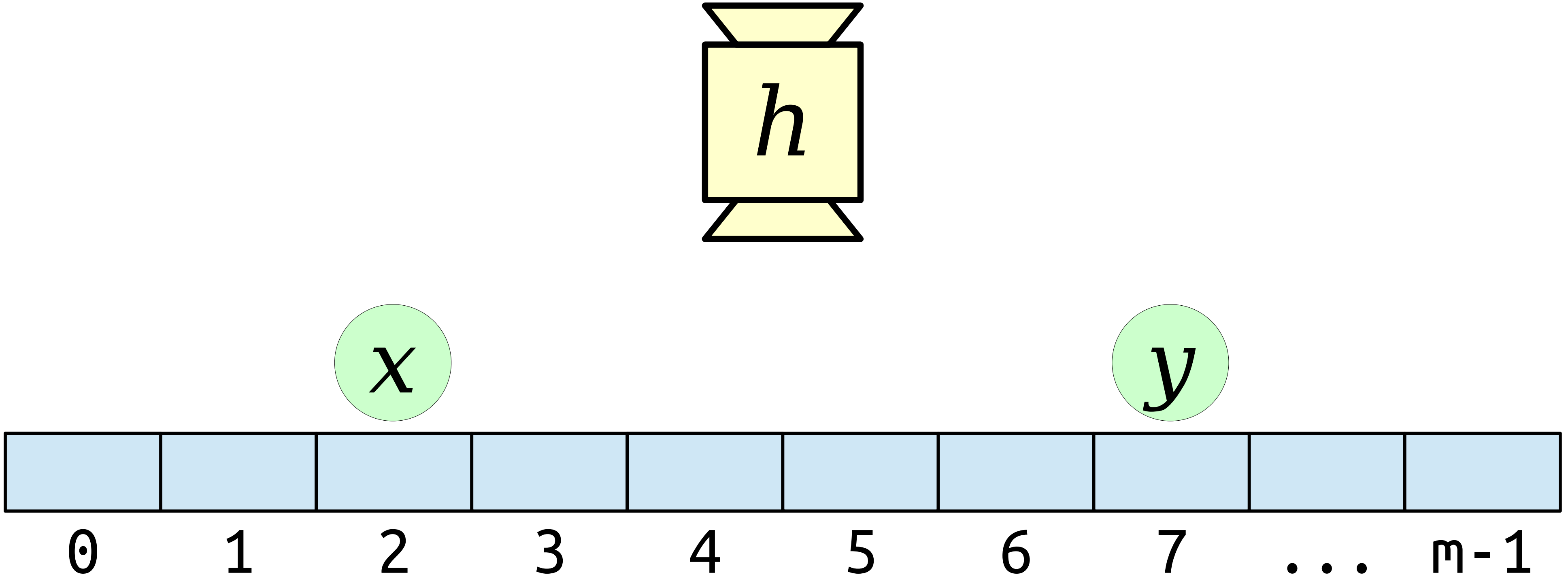

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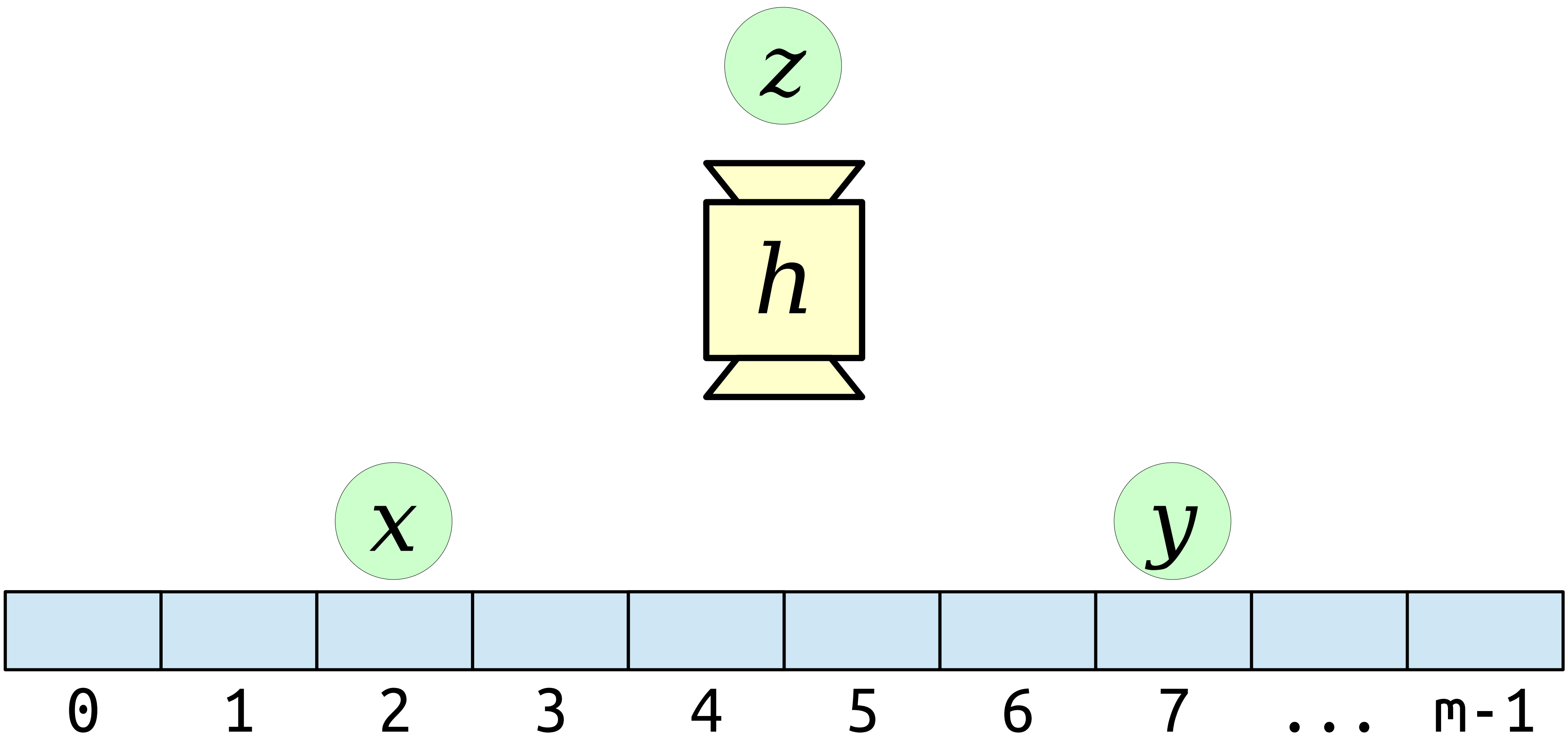

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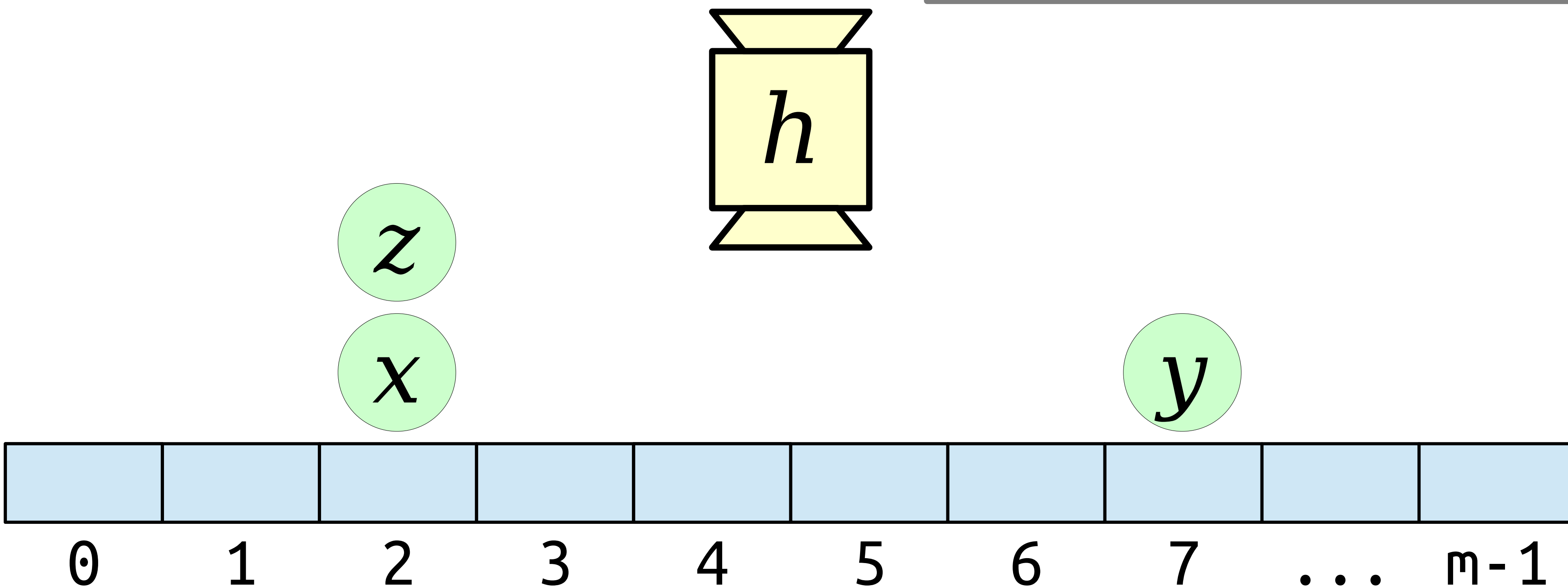
Problem: A hash function that distributes n elements uniformly at random over $[m]$ requires $\Omega(n \log m)$ space in the worst case.

Question: Do we actually need true randomness? Or can we get away with something weaker?



Distribution Property:
Each element should have an equal probability of being placed in each slot.

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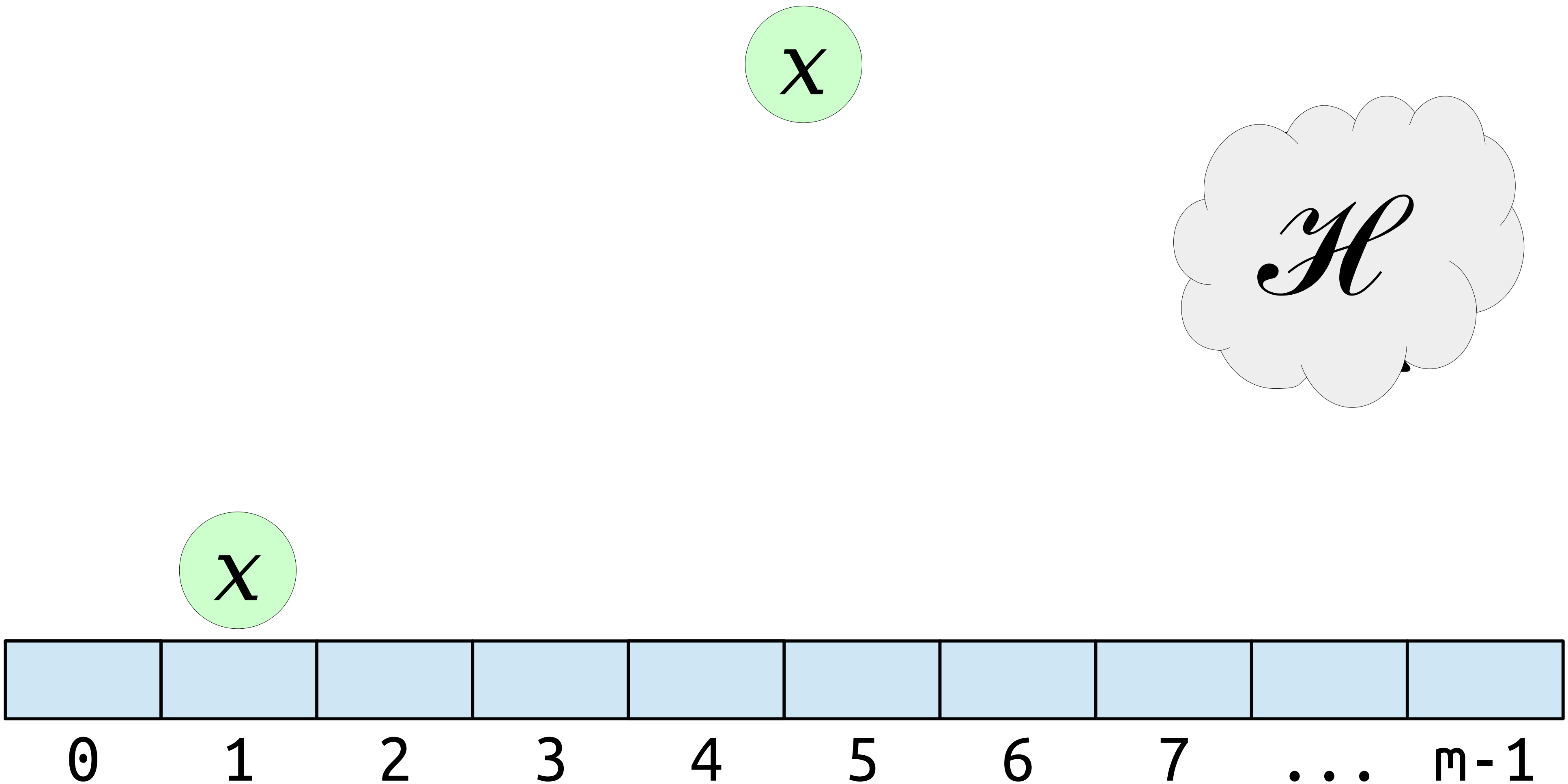

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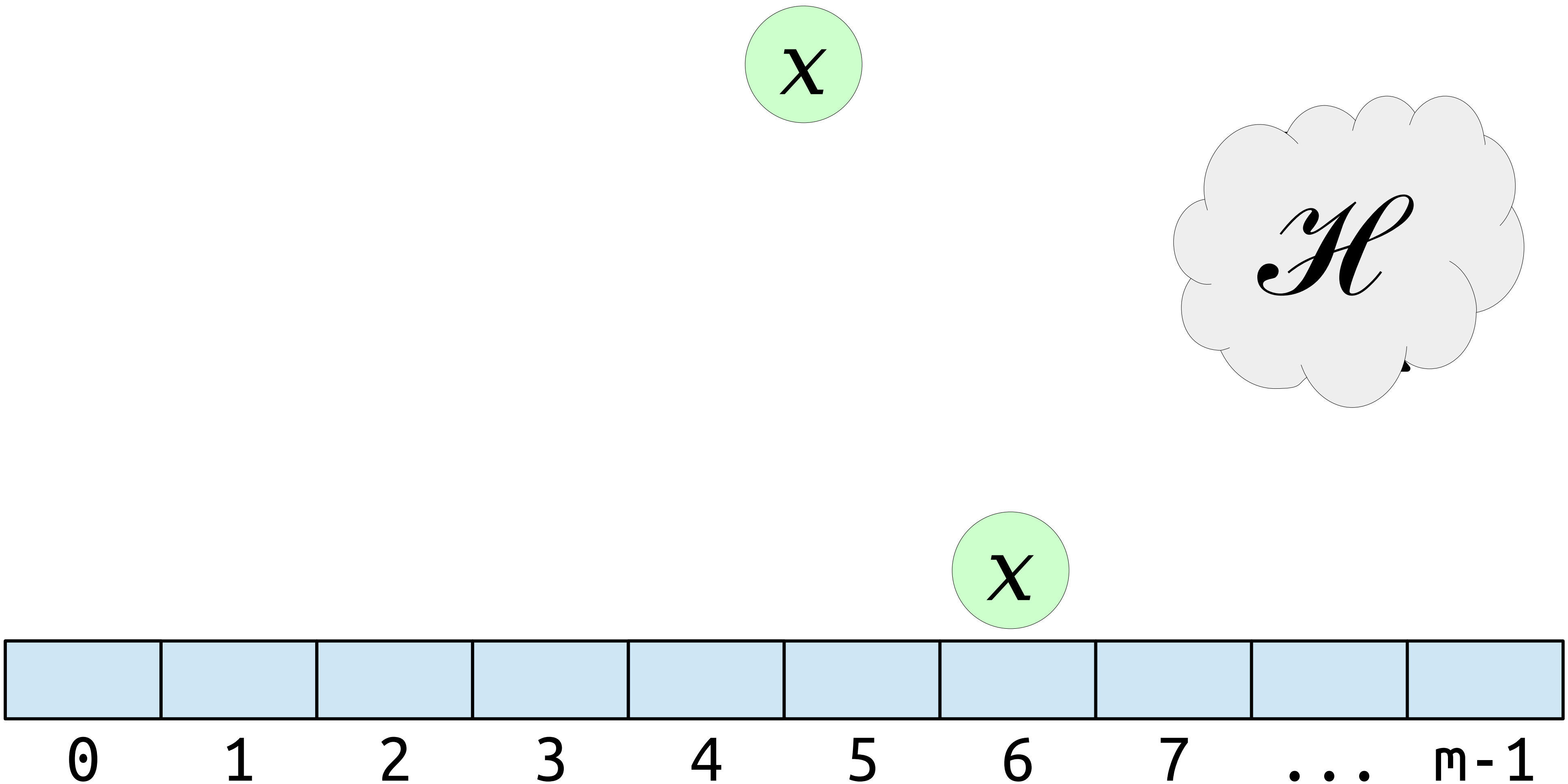

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Find an “obviously bad” family of hash functions that satisfies the distribution property.

Formulate a hypothesis, but ***don’t post anything in chat just yet***.

0	1	2	3	4	5	6	7	...	m-1

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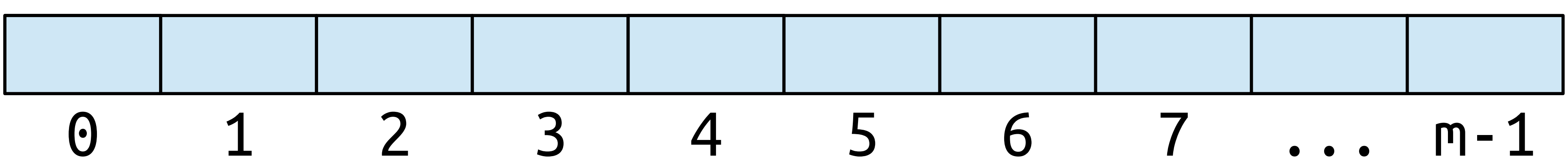
Now, ***private chat me your best guess***. Not sure? Just answer "??".



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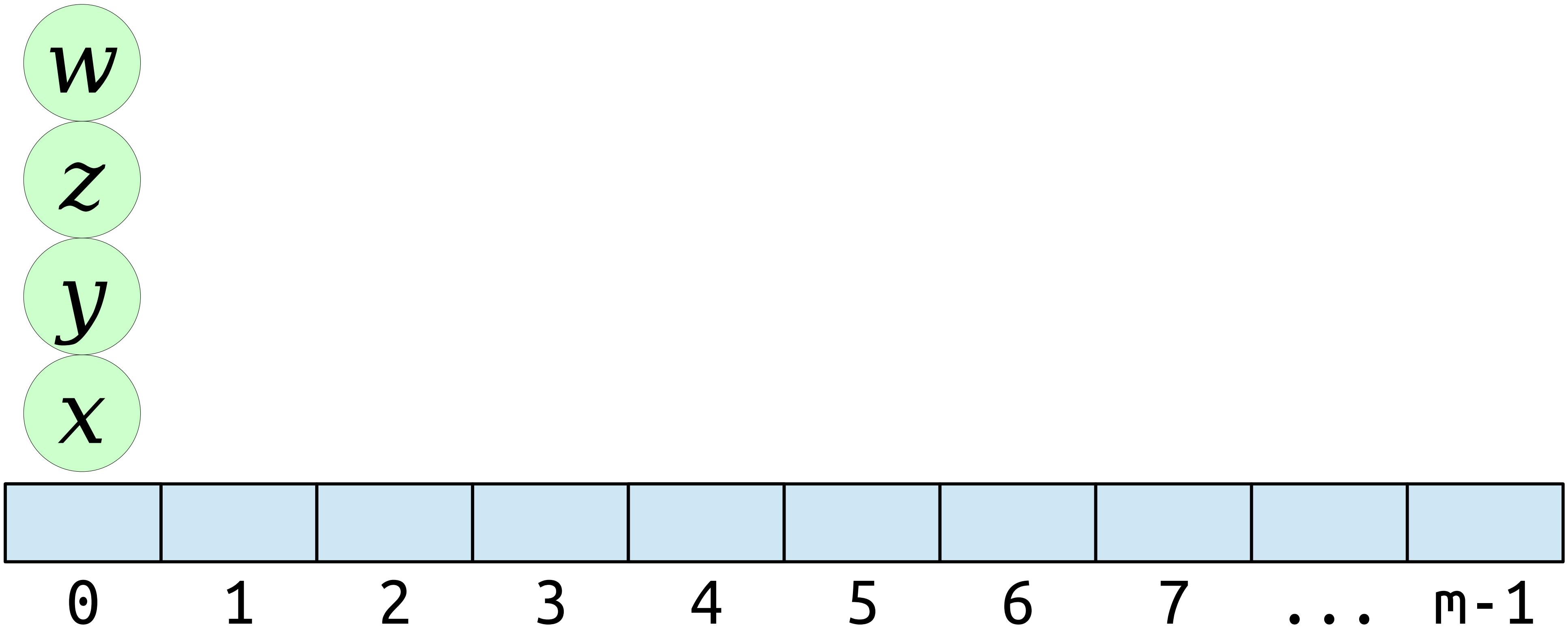

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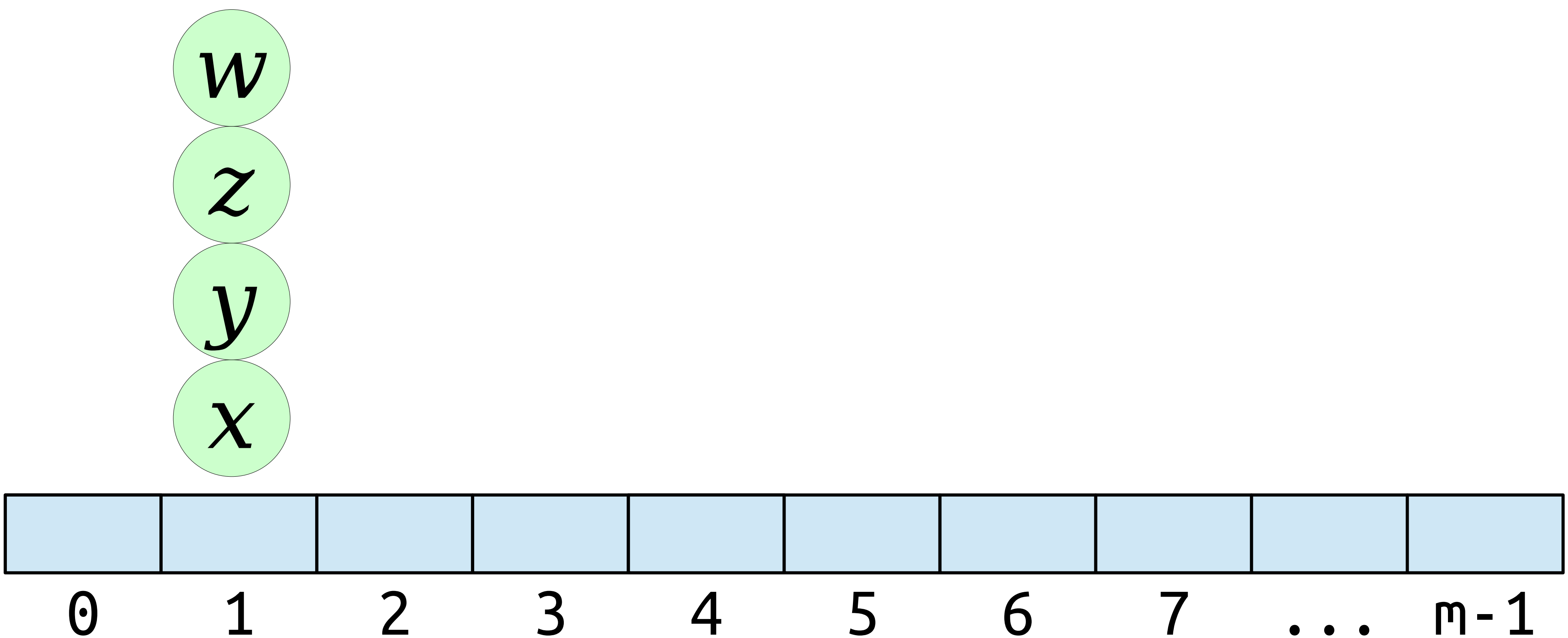

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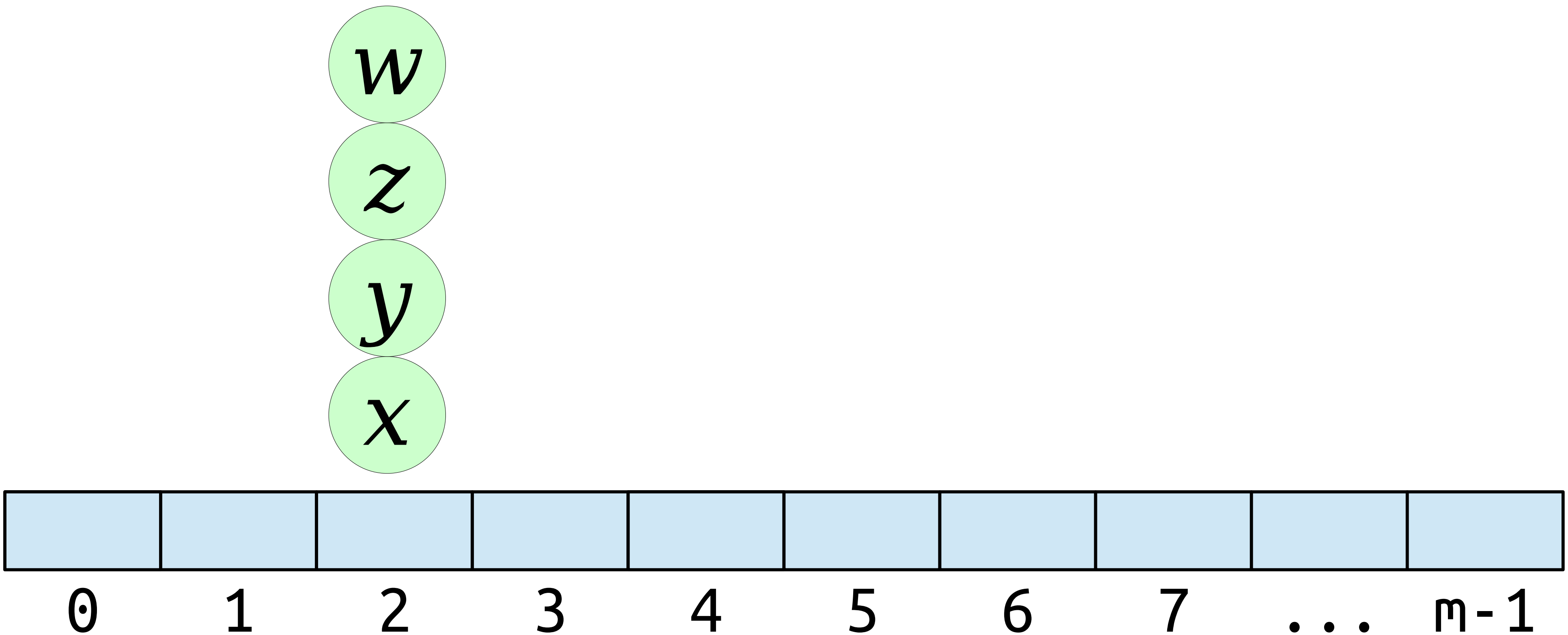

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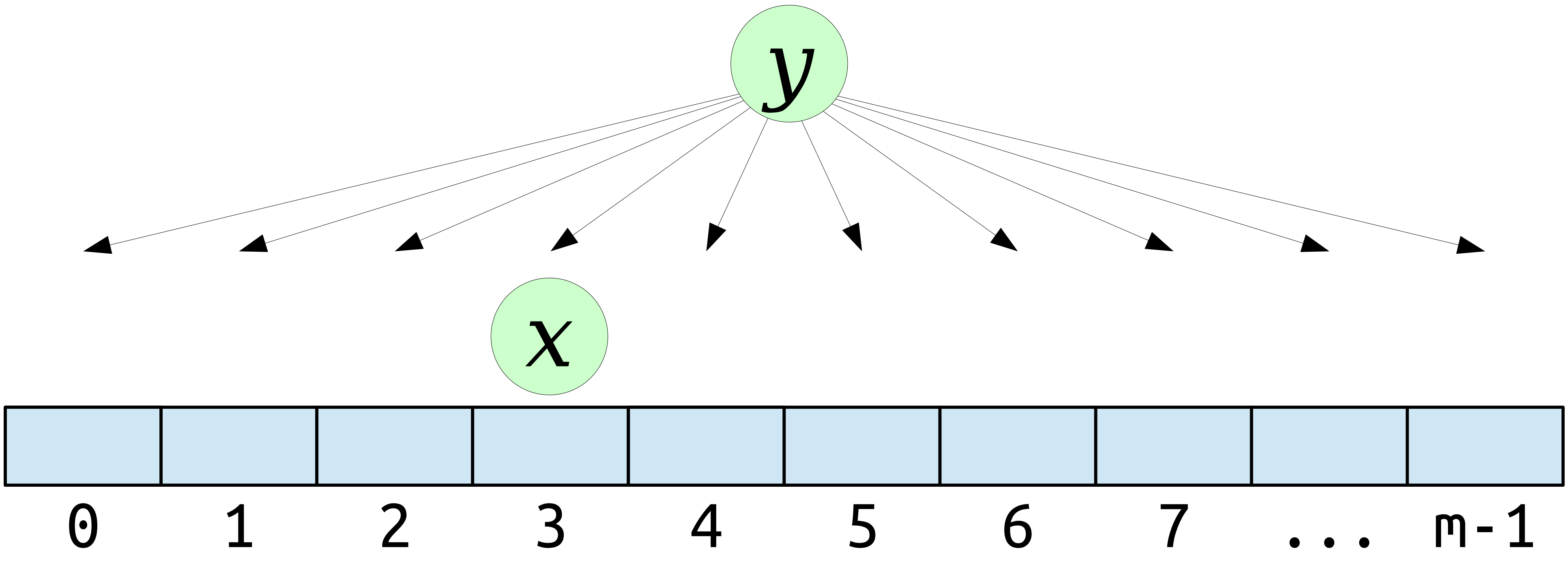

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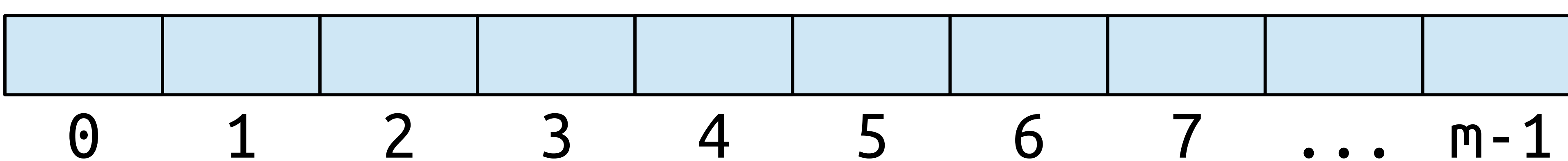
A family of hash functions $\mathscr{H}$ is called ***2-independent*** (or ***pairwise independent***) if it satisfies the distribution and independence properties.



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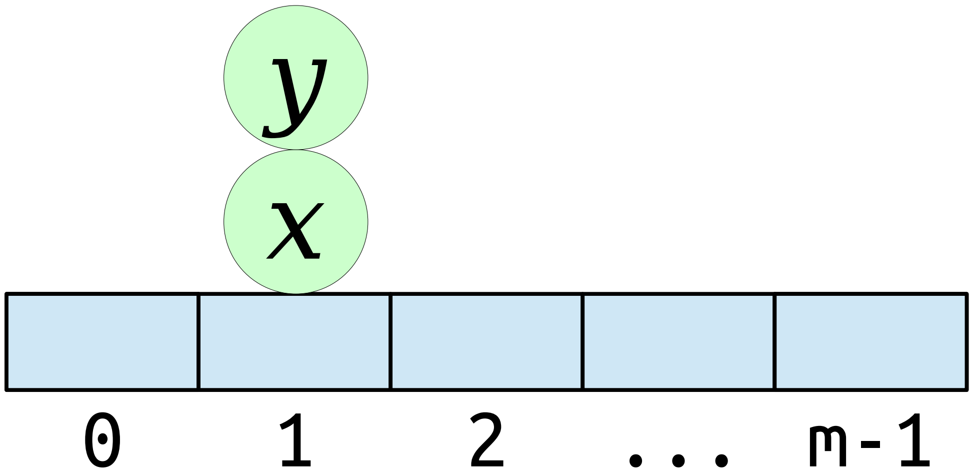

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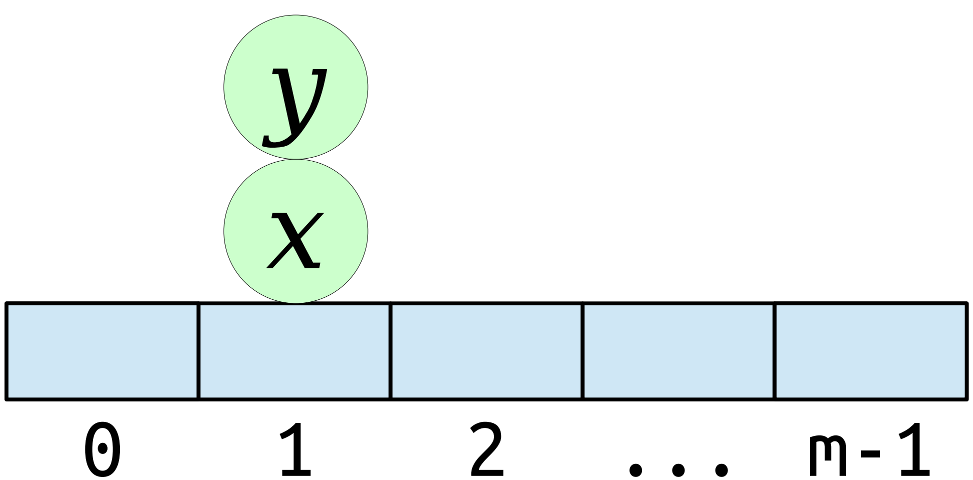

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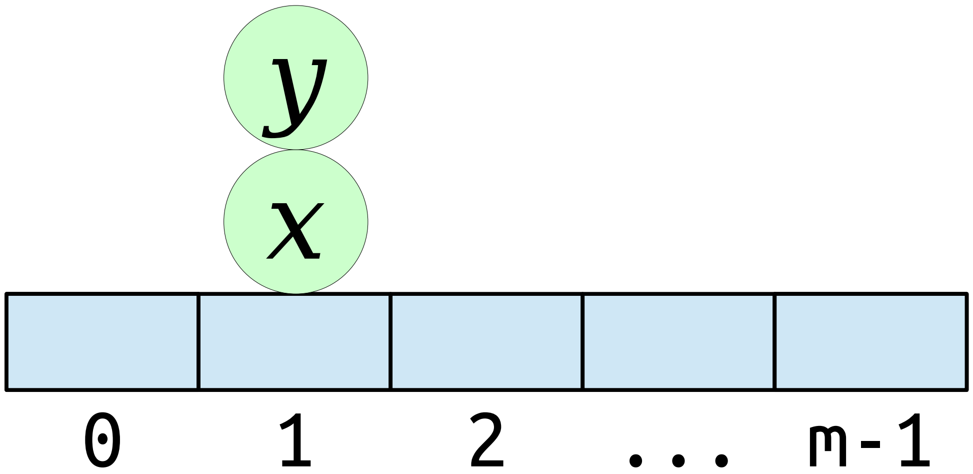

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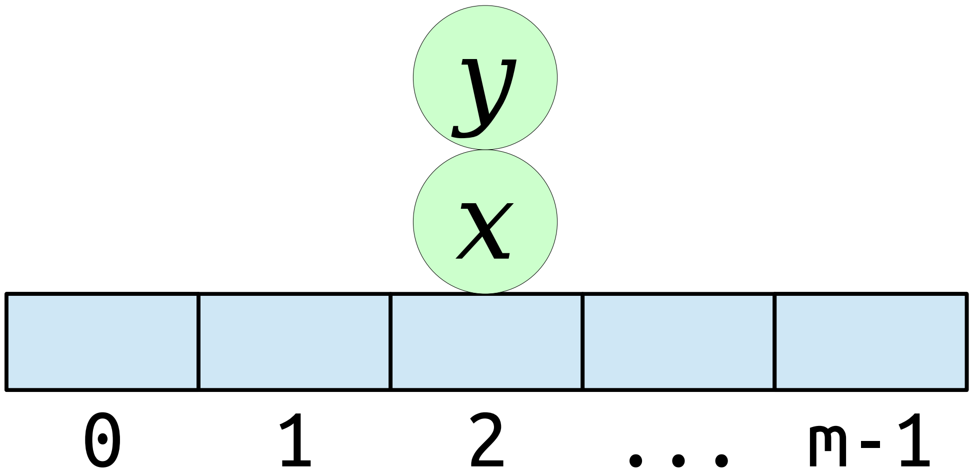

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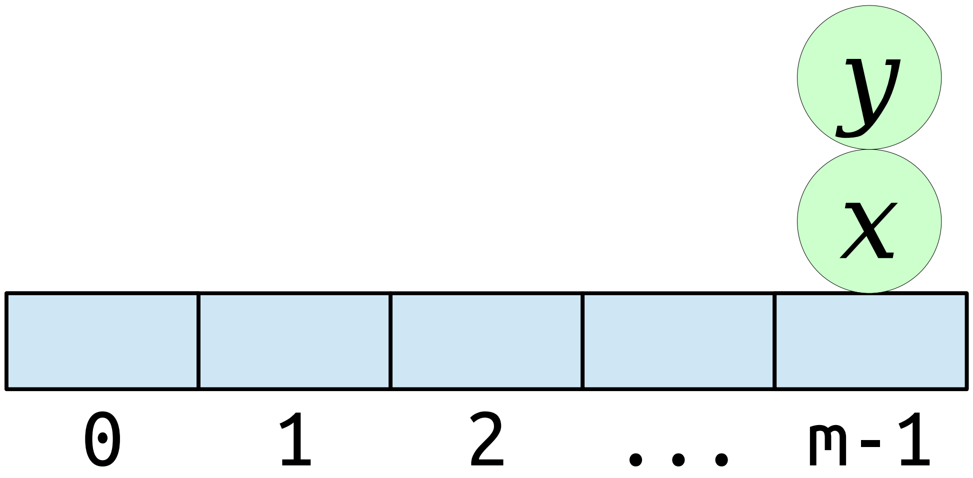

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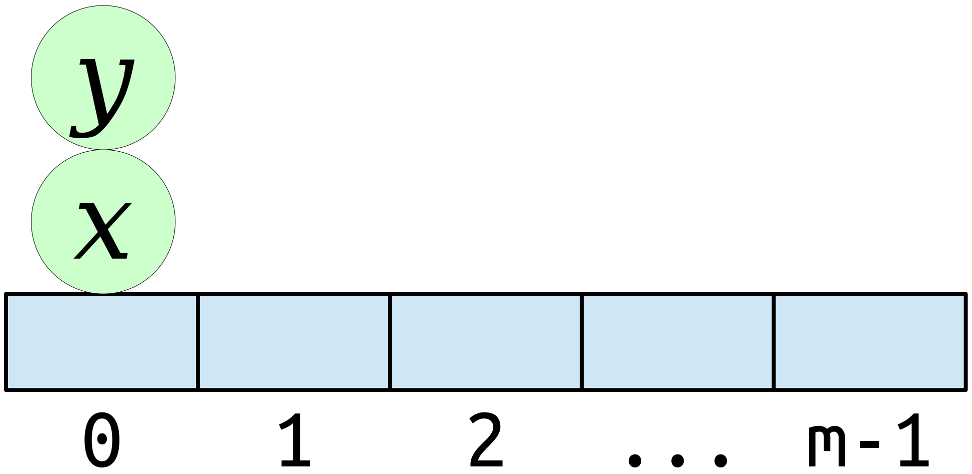

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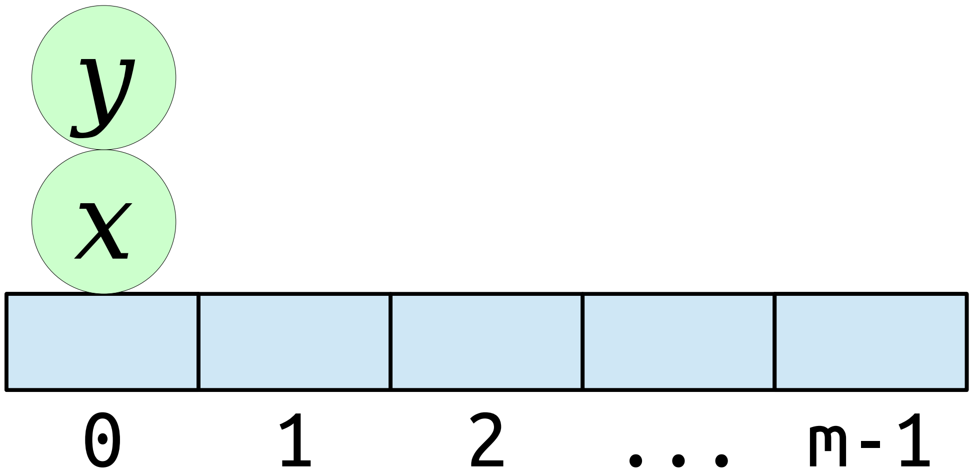

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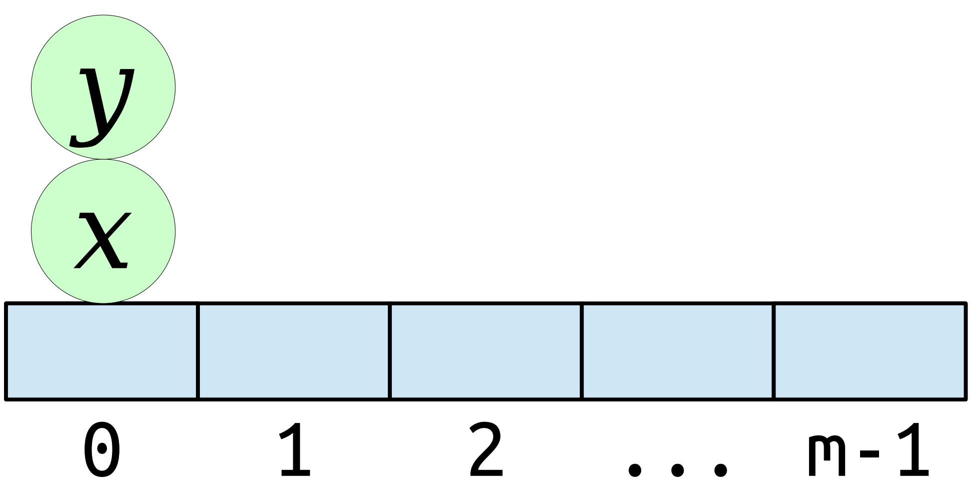

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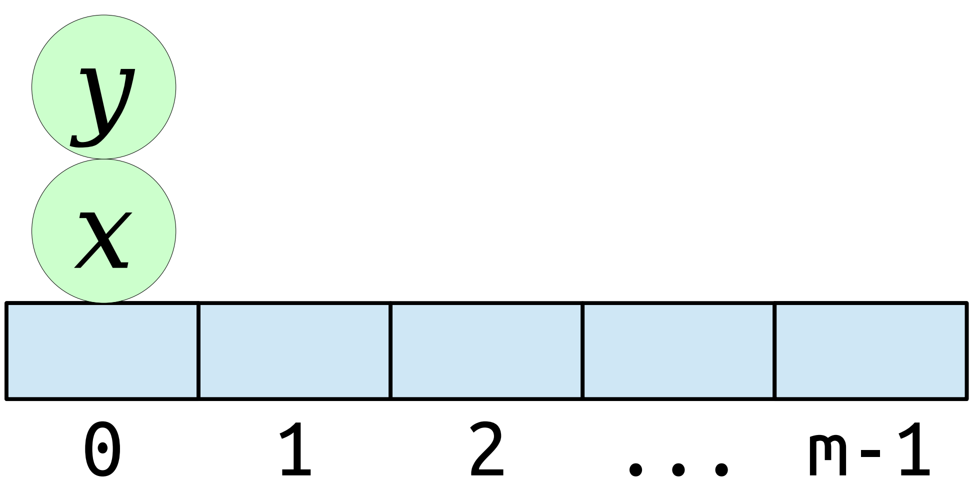

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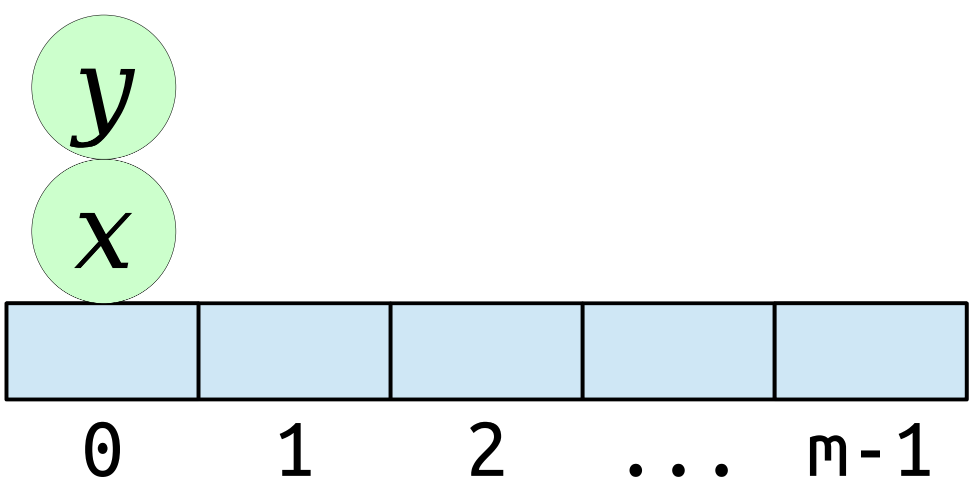

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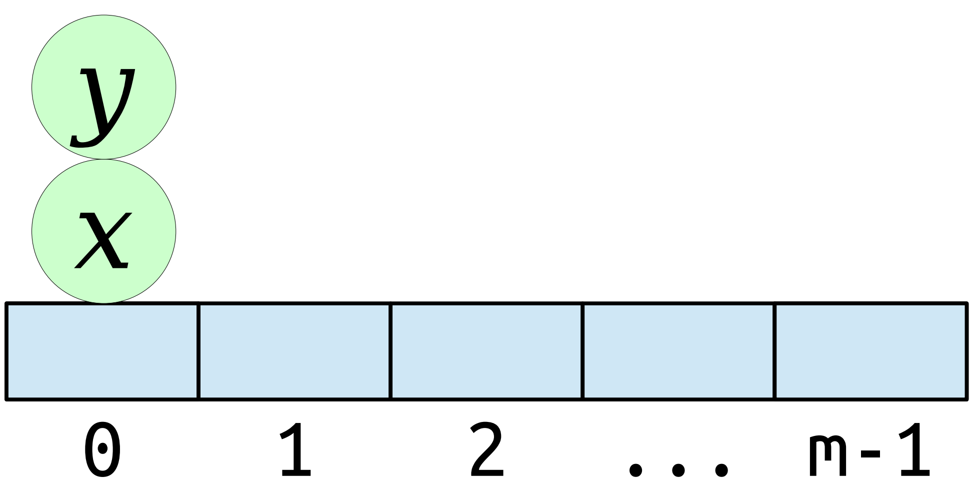

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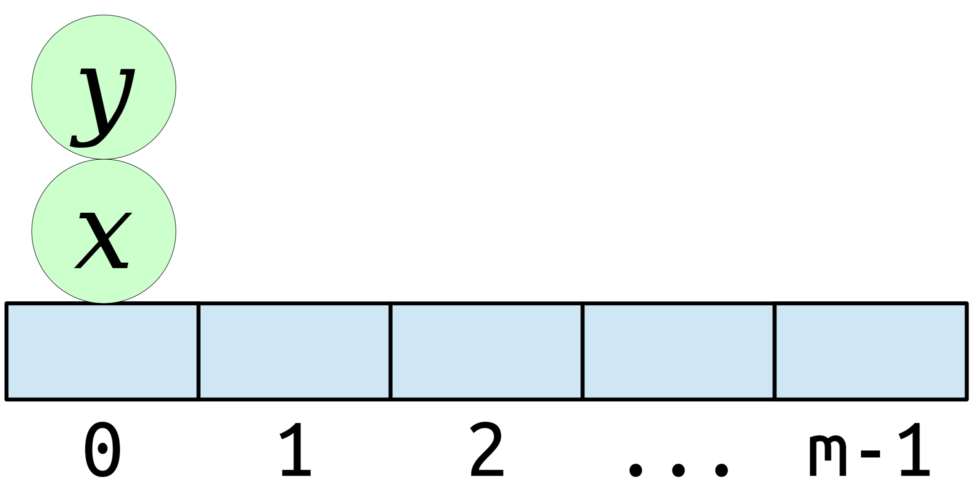

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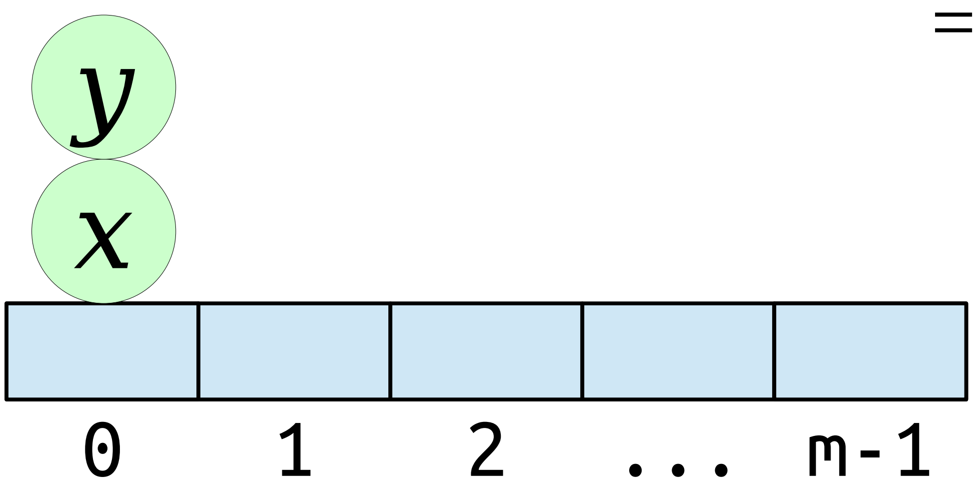

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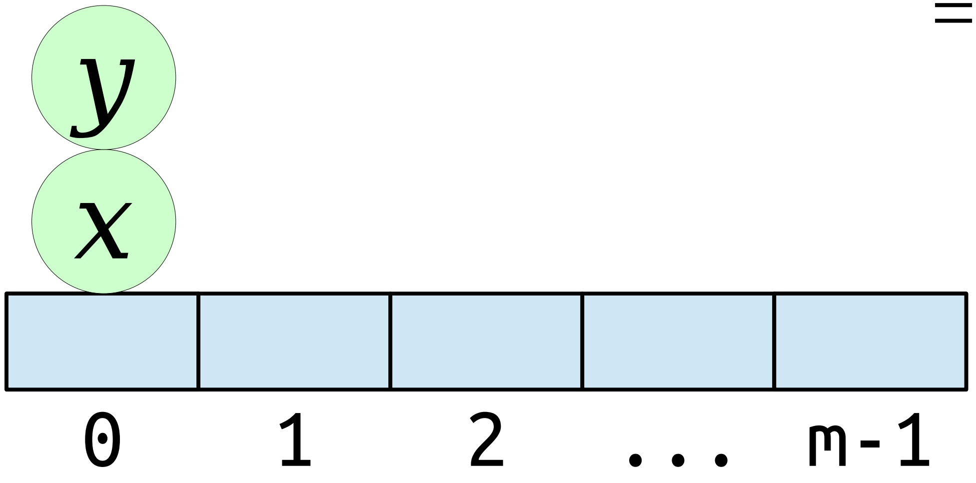

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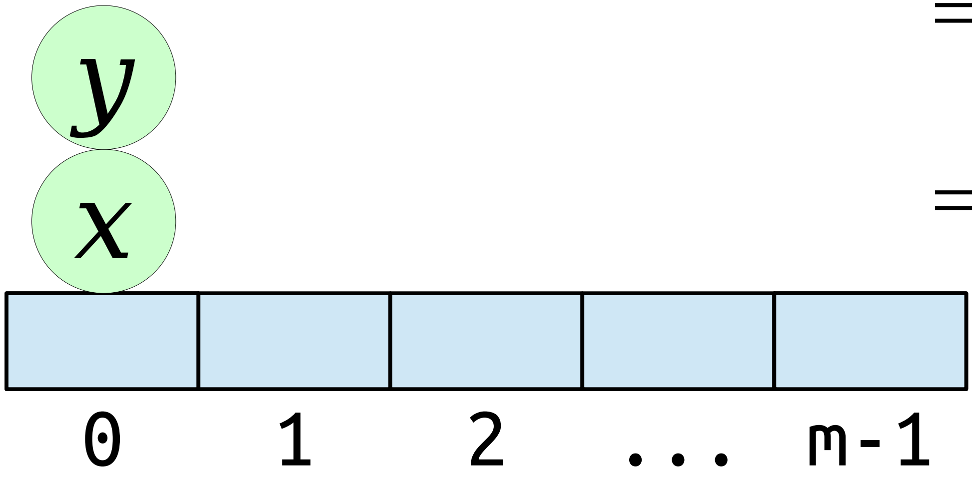

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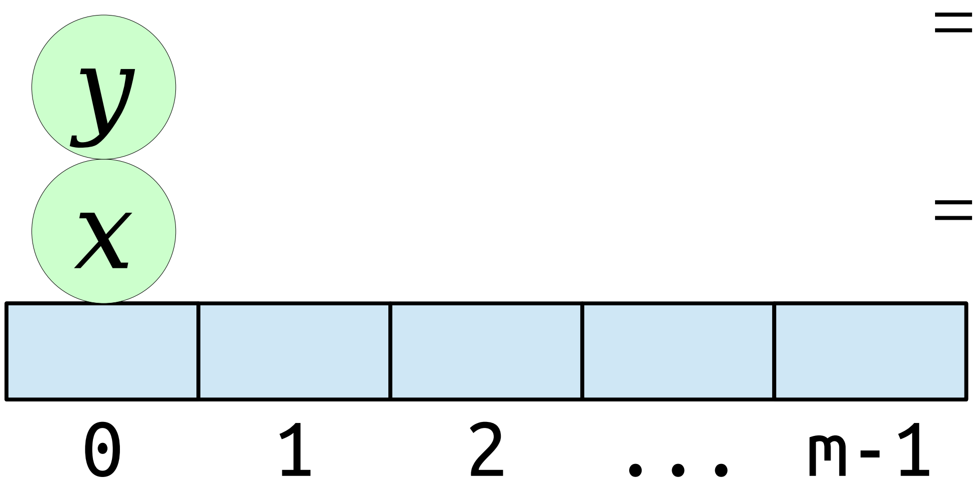
This is the same as if h were a truly random function.



For more on hashing outside of Theoryland, check out ***this Stack Exchange post***.

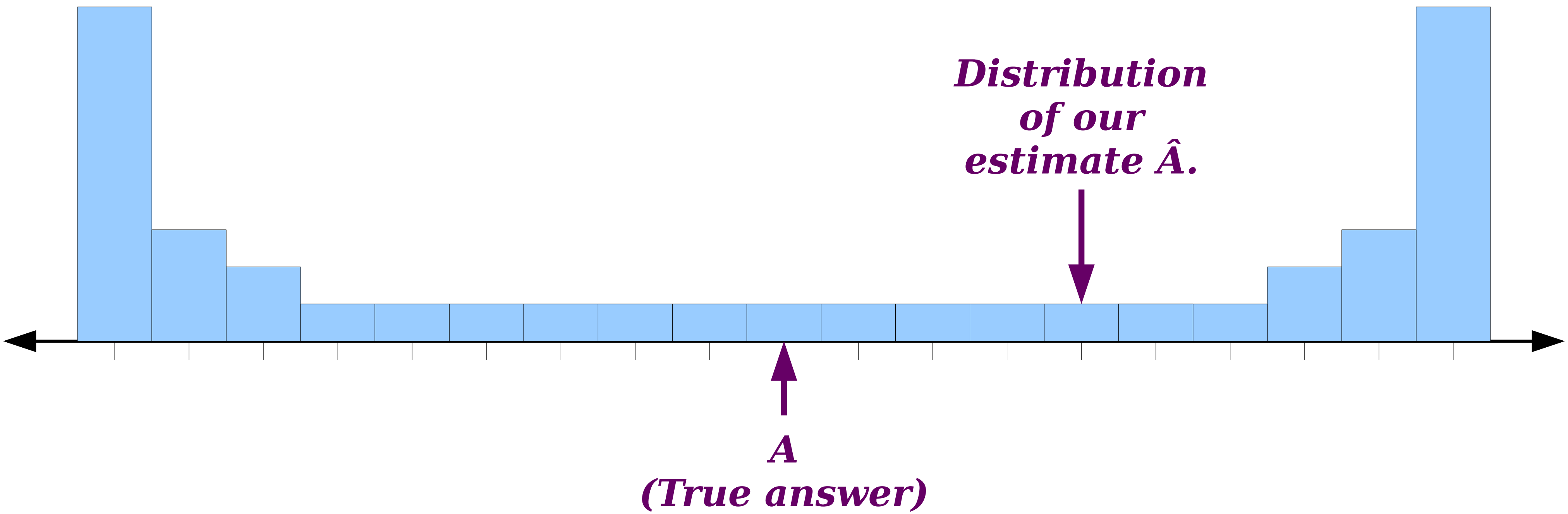
Approximating Quantities

What makes for a good “approximate” solution?

Let $\boldsymbol{A}$ be the true answer. Let $\boldsymbol{\hat{A}}$ be a random variable denoting our estimate.

This would not make for a good estimate. However, we have $E[\hat{A}] = A$.

Observation 1: Being correct in expectation isn't sufficient.



What does it mean for an approximation to be "good"?

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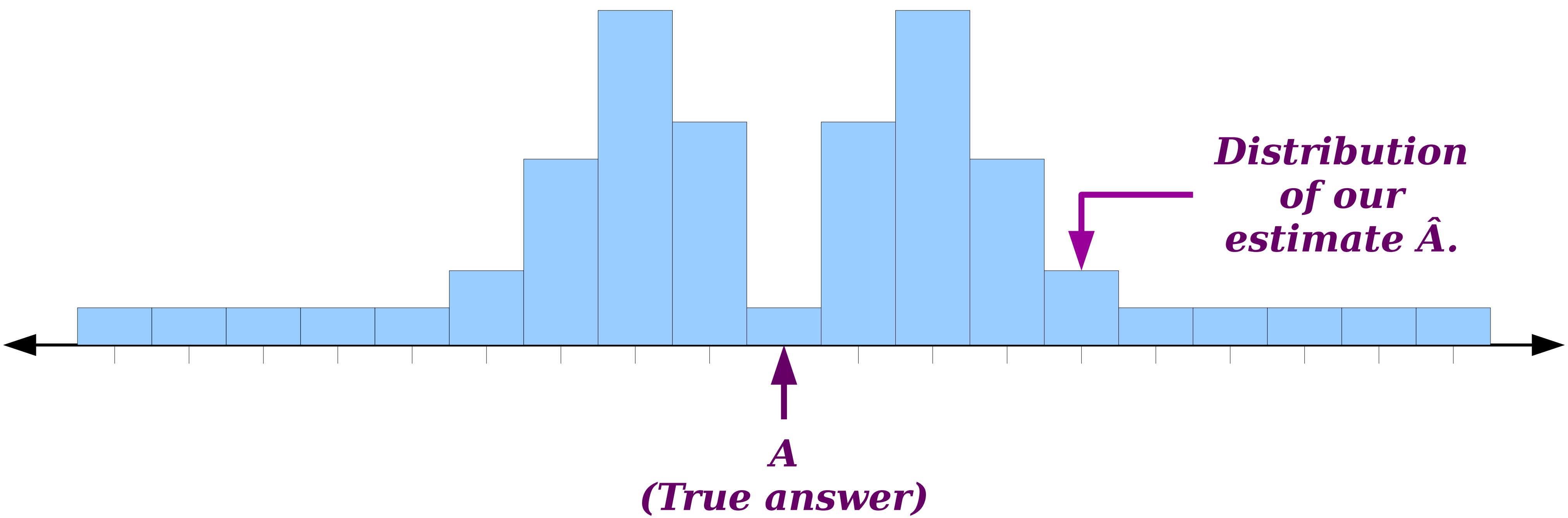
It's unlikely that we'll get the right answer, but we're probably going to be close.

Observation 2: The difference $|\hat{A} - A|$ between our estimate and the truth should ideally be small.



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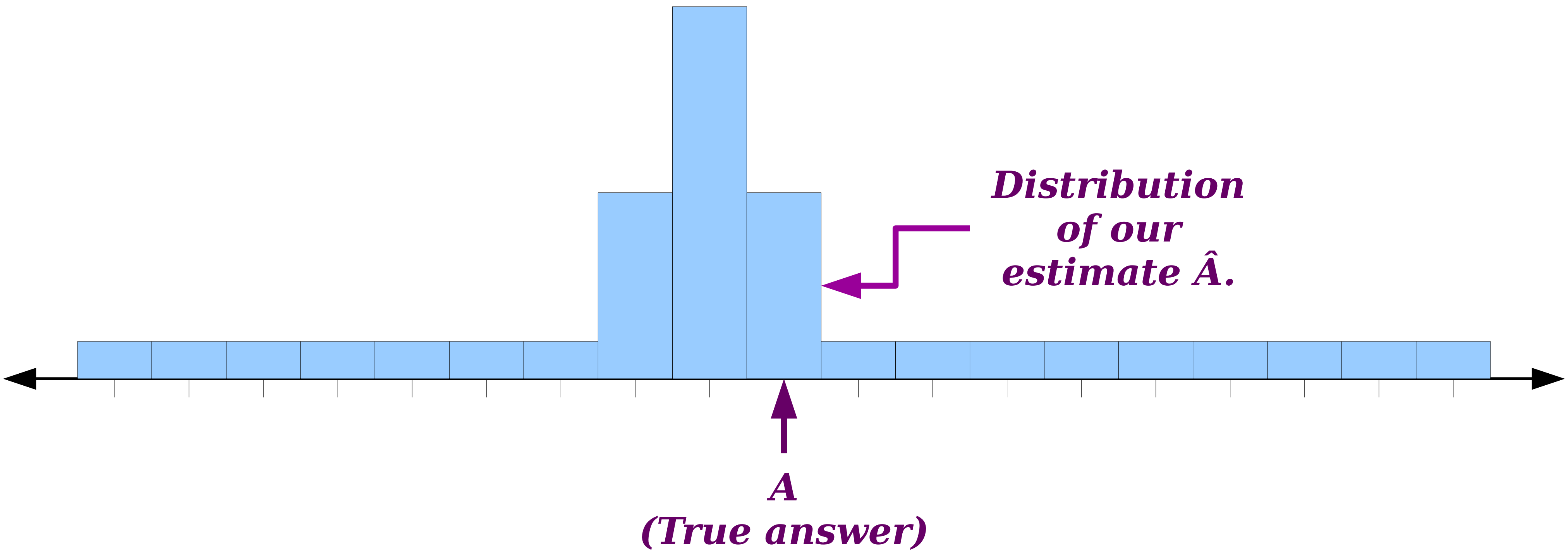
This estimate skews low, but it's very close to the true value.

Observation 3: An estimate doesn't have to be unbiased to be useful.



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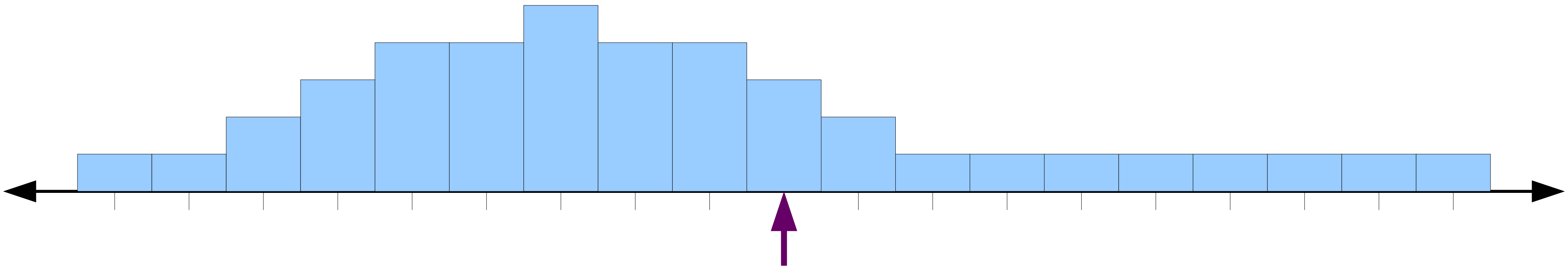

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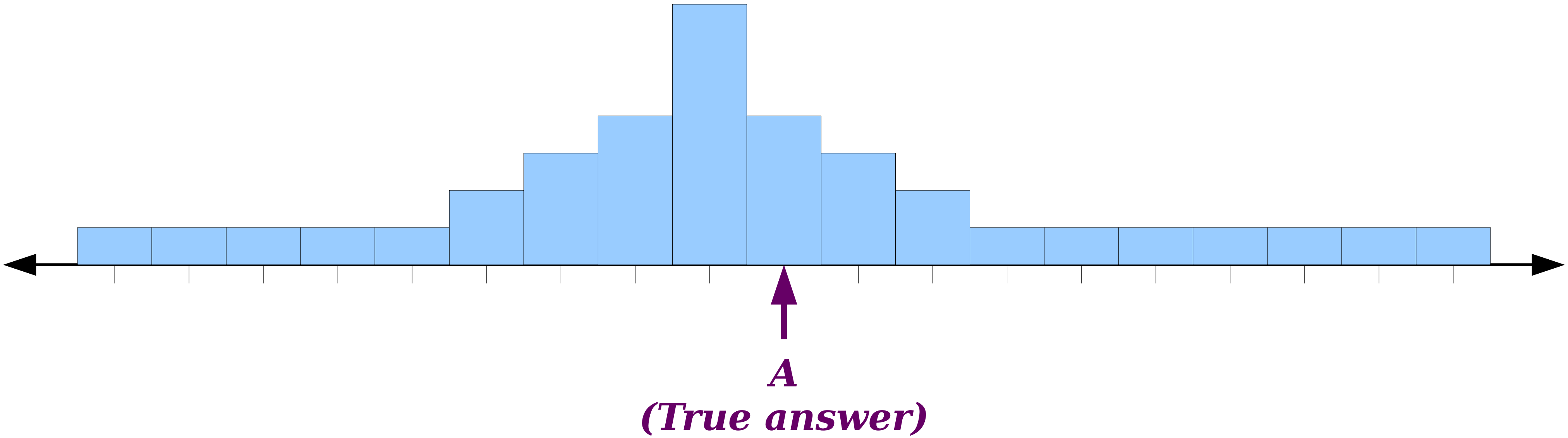
Let $\boldsymbol{A}$ be the true answer. Let $\boldsymbol{\hat{A}}$ be a random variable denoting our estimate.

Memory used: 32MB



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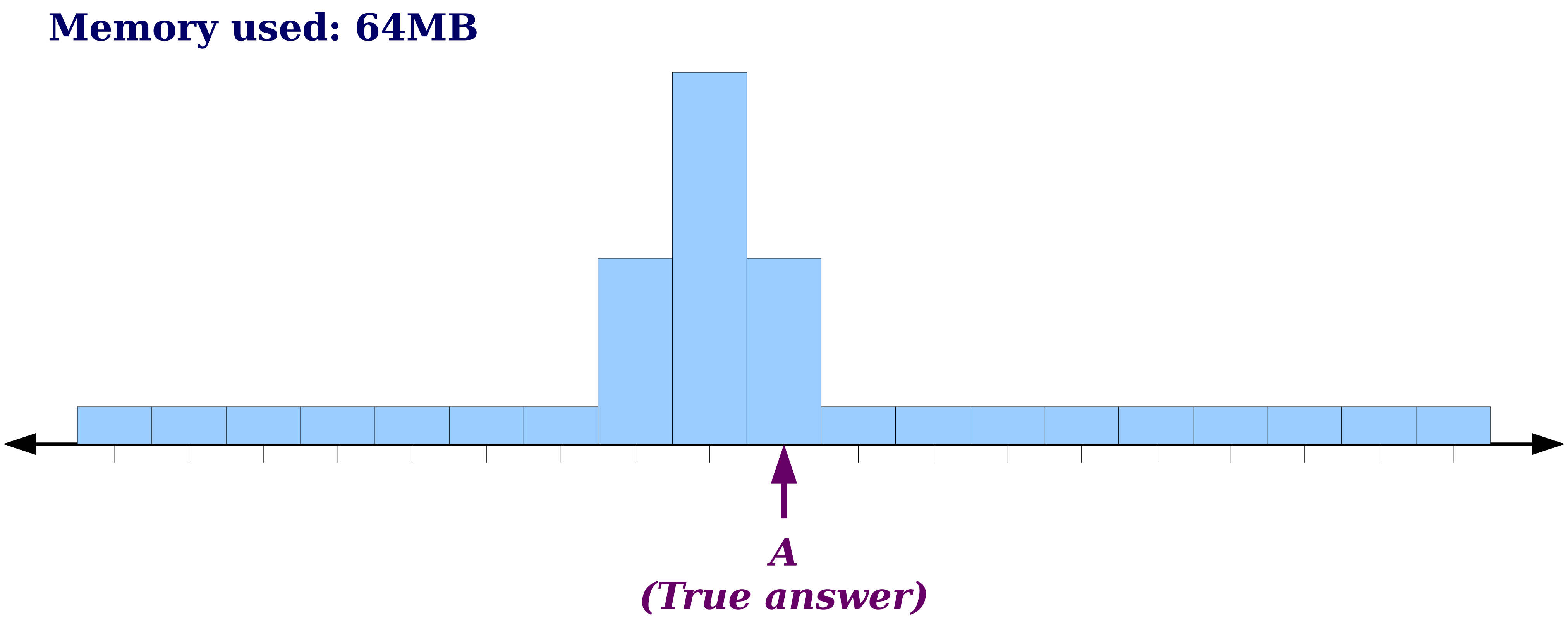

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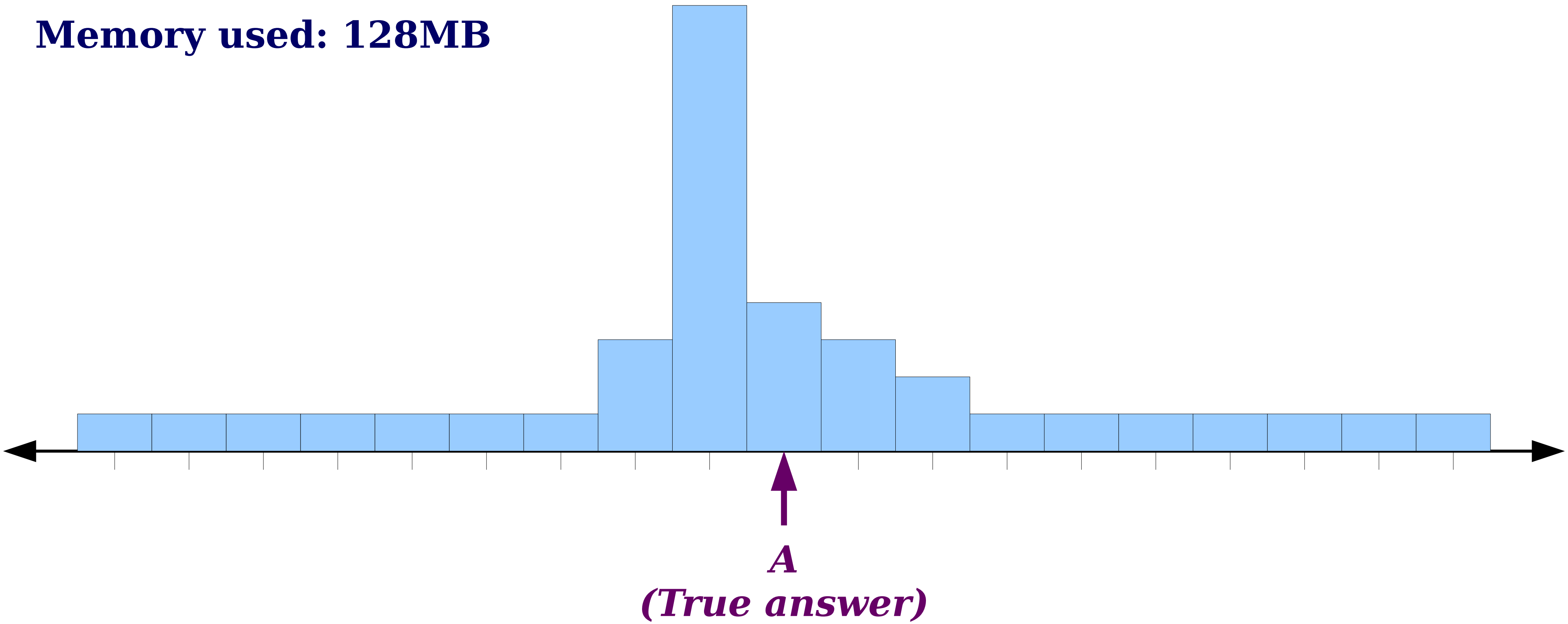
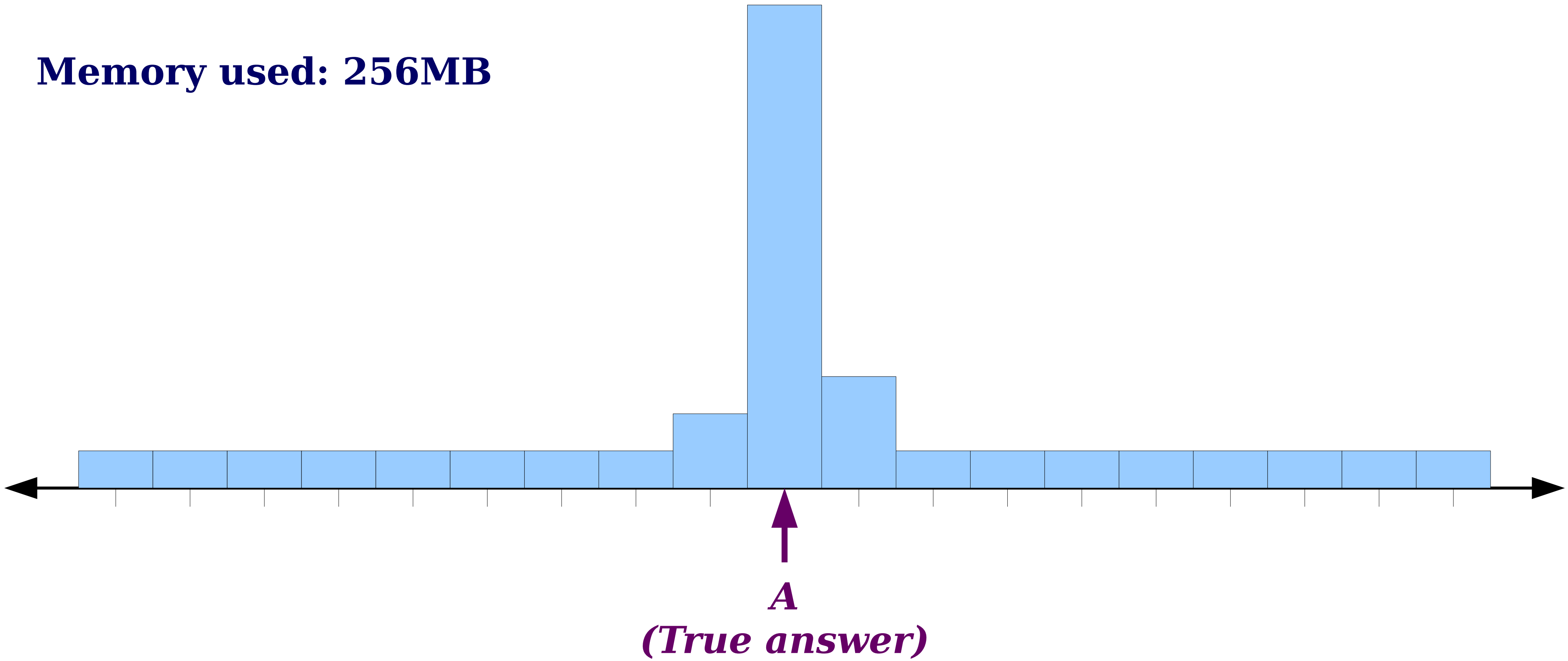


What does it mean for an approximation to be “good”?

Let A be the true answer. Let $\hat{A}$ be a random variable denoting our estimate.

The more resources we allocate, the better our estimate should be.

Observation 4: A good approximation should be tunable.



What does it mean for an approximation to be "good"?

We have two user-provided values

$$\varepsilon \in (0, 1]$$
$$\delta \in (0, 1]$$

where ε represents ***accuracy*** and δ represents ***confidence***.

Goal: Make an estimator $\hat{A}$ for some quantity A where

With probability at least $1 - \delta$,

$$|\hat{A} - A| \leq \varepsilon \cdot size(input)$$

for some measure of the size of the input.

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$\delta = \frac{1}{2}$

ε small

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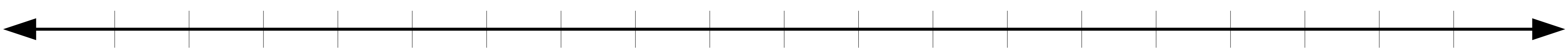
$\delta = ½$

ε small



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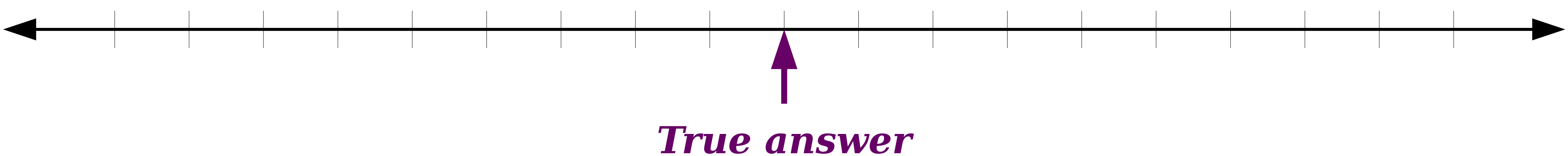

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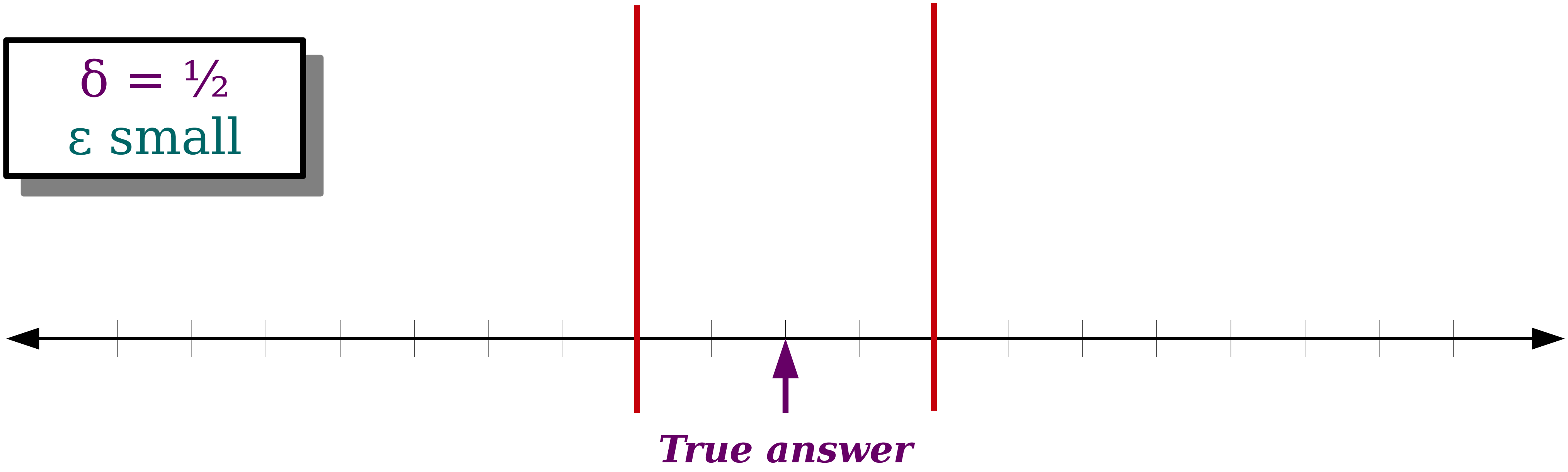

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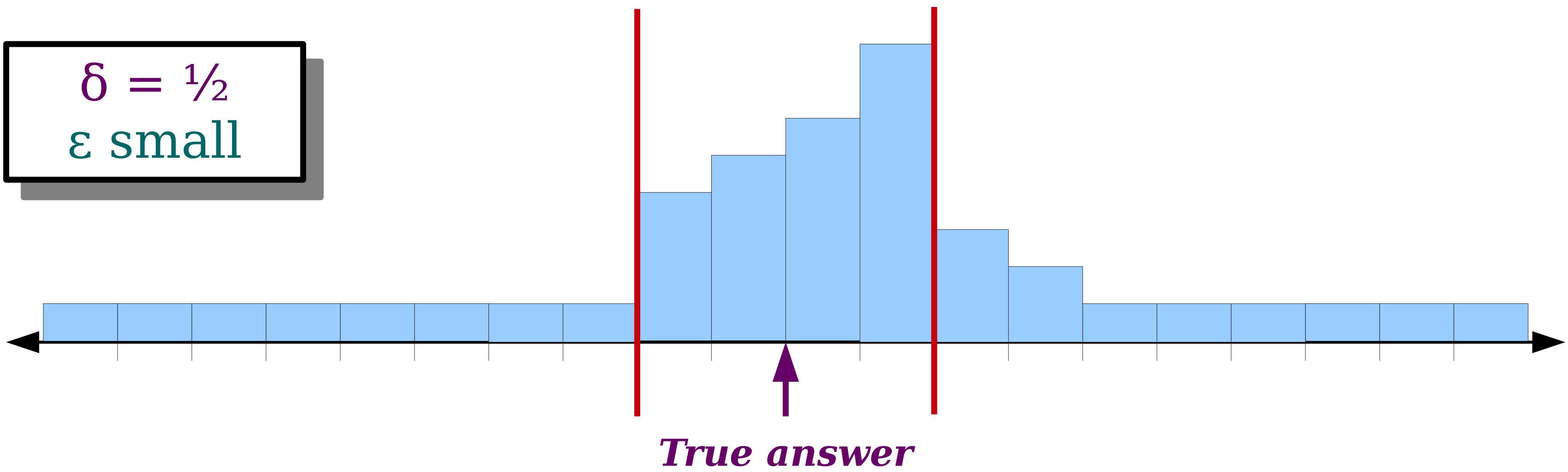

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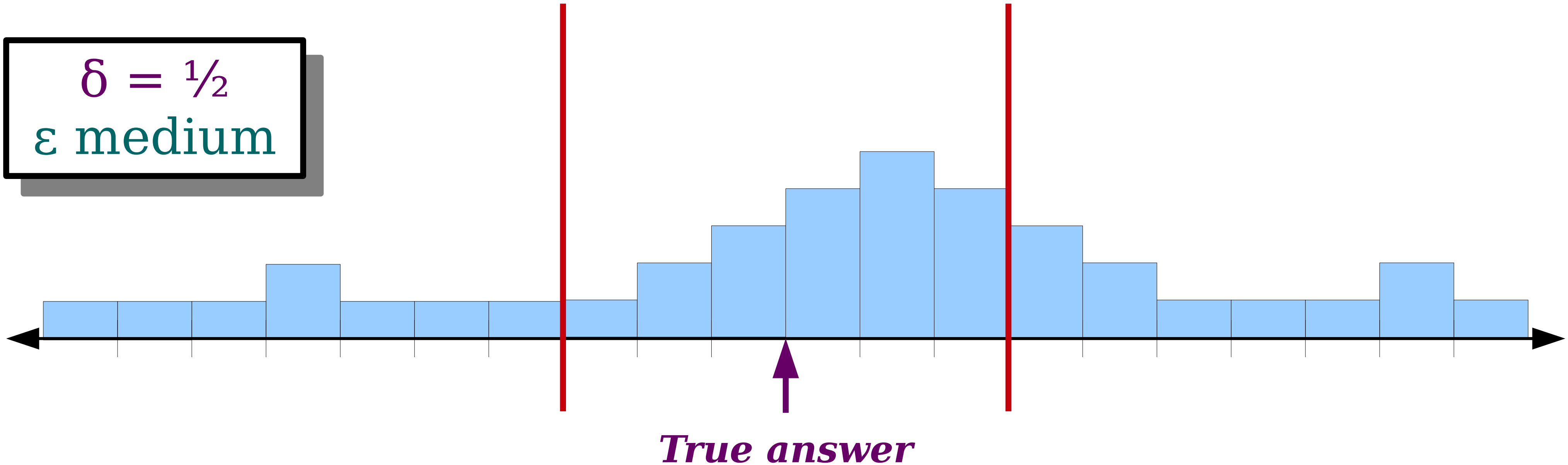
$\delta = ½$

ε medium



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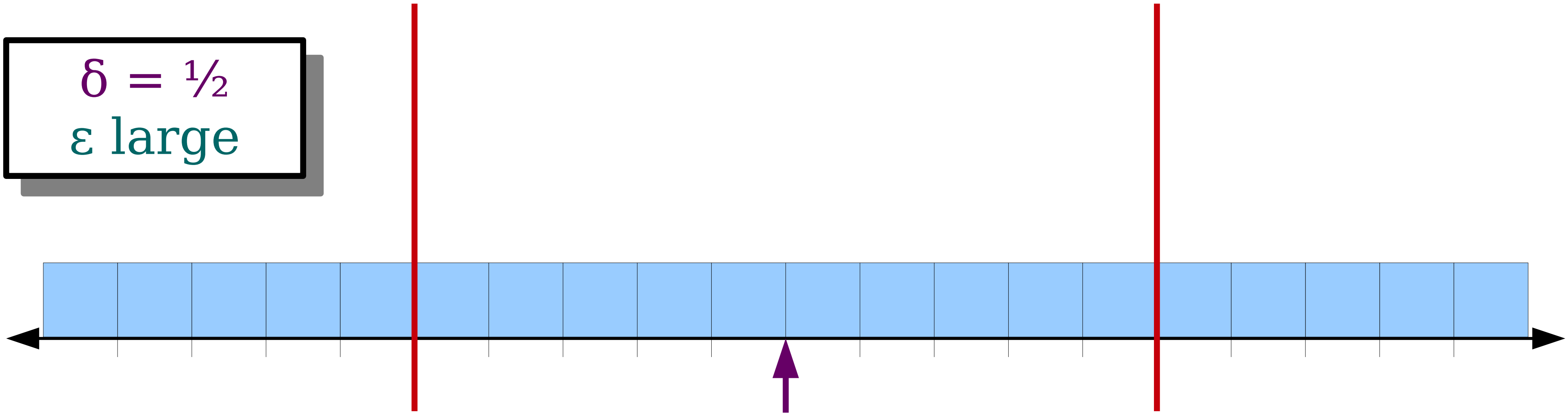
δ = ½
ε large



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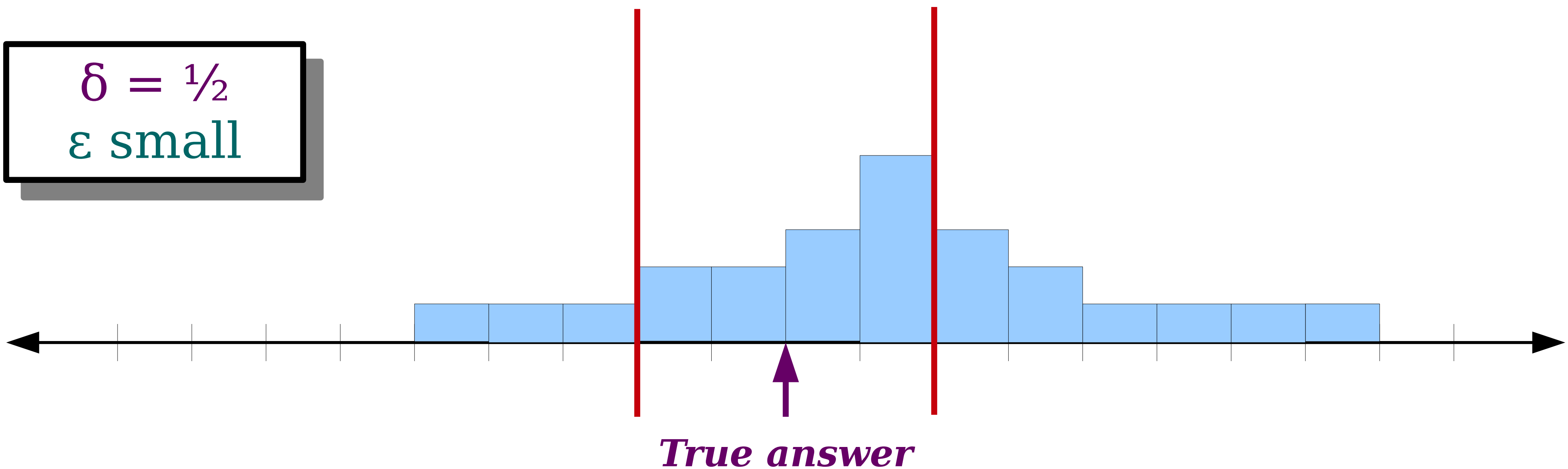

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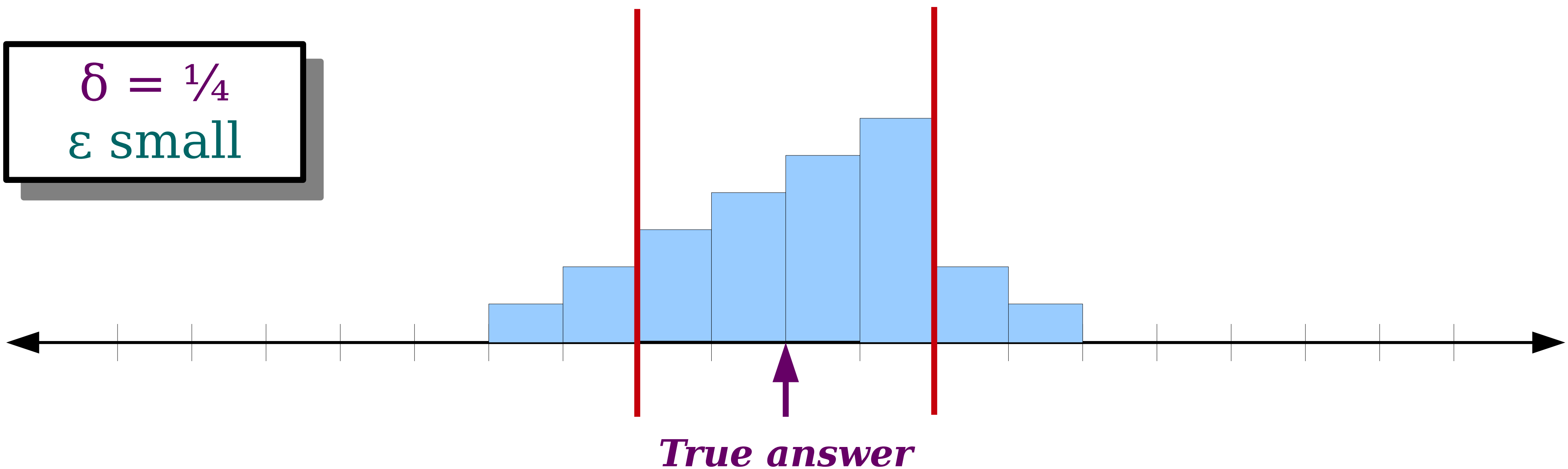
With probability at least $1 - \delta$, **← *Probably***

$|A - \hat{A}| \leq \varepsilon \cdot size(input)$ **← *Approximately Correct***

for some measure of the size of the input.

$\delta = ¼$
ε small



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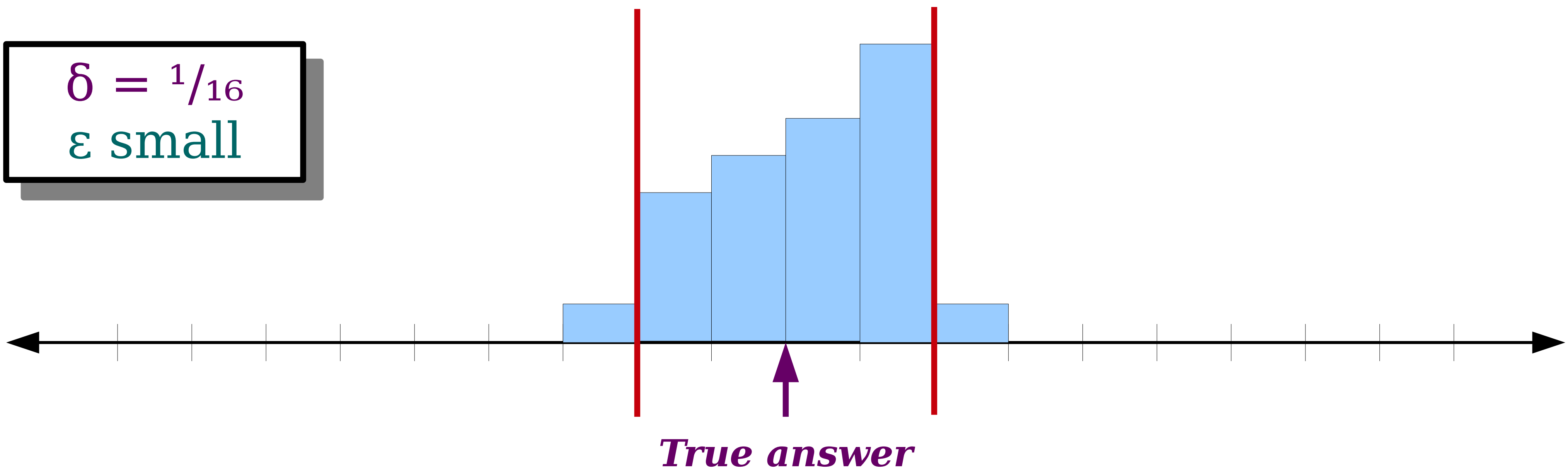
$\delta = 1/16$

ε small



What does it mean for an approximation to be "good"?

Frequency Estimation

Frequency Estimators

- A ***frequency estimator*** is a data structure supporting the following operations:
  - ***increment***(x), which increments the number of times that x has been seen, and
  - ***estimate***(x), which returns an estimate of the frequency of x.
- Using BSTs, we can solve this in space $\Theta(n)$ with worst-case $O(\log n)$ costs on the operations.
- Using hash tables, we can solve this in space $\Theta(n)$ with expected $O(1)$ costs on the operations.

Frequency Estimators

- Frequency estimation has many applications:
  - Search engines: Finding frequent search queries.
  - Network routing: Finding common source and destination addresses.
- In these applications, $\Theta(n)$ memory can be impractical.
- ***Goal:*** Get *approximate* answers to these queries in sublinear space.

The Count-Min Sketch

How to Build an Estimator

1. Design a simple data structure that, intuitively, gives you a good estimate.

2. Use a ***sum of indicator variables*** and ***linearity of expectation*** to prove that, on expectation, the data structure is pretty close to correct.

3. Use a ***concentration inequality*** to show that, with decent probability, the data structure's output is close to its expectation.

4. Run multiple copies of the data structure in parallel to amplify the success probability.

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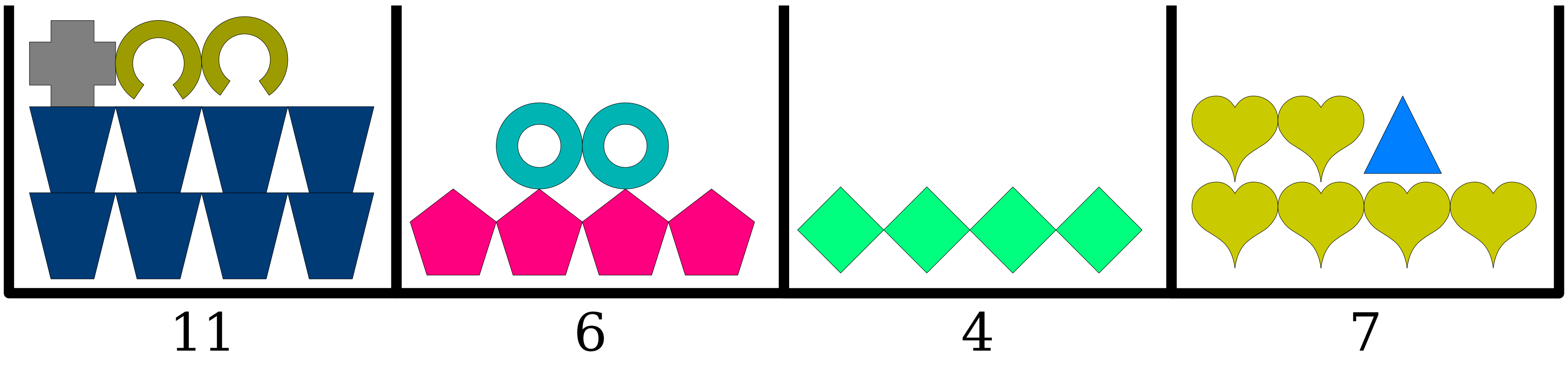
Revisiting the Exact Solution

- In the exact solution to the frequency estimation problem, we maintained a single counter for each distinct element. This is too space-inefficient.
- ***Idea:*** Store a fixed number of counters and assign a counter to each $x_i \in \mathscr{U}$. Multiple x_i's might be assigned to the same counter.
- To ***increment***(x), increment the counter for x.
- To ***estimate***(x), read the value of the counter for x.



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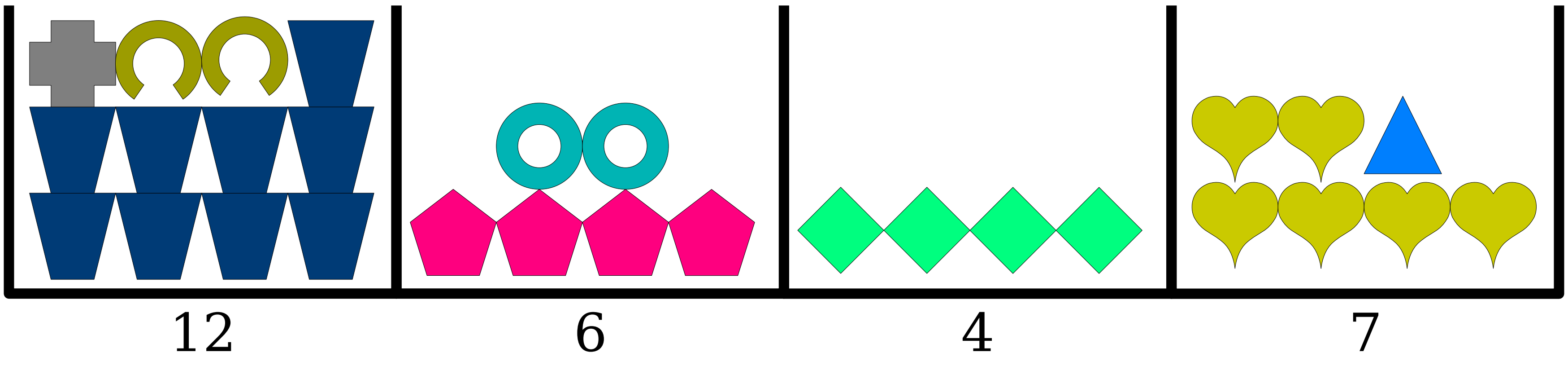

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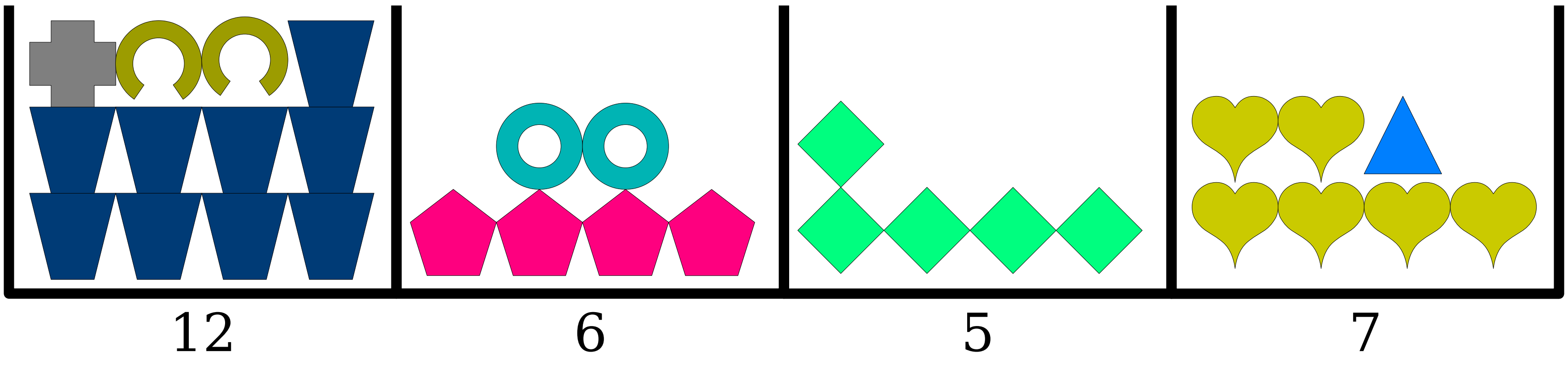
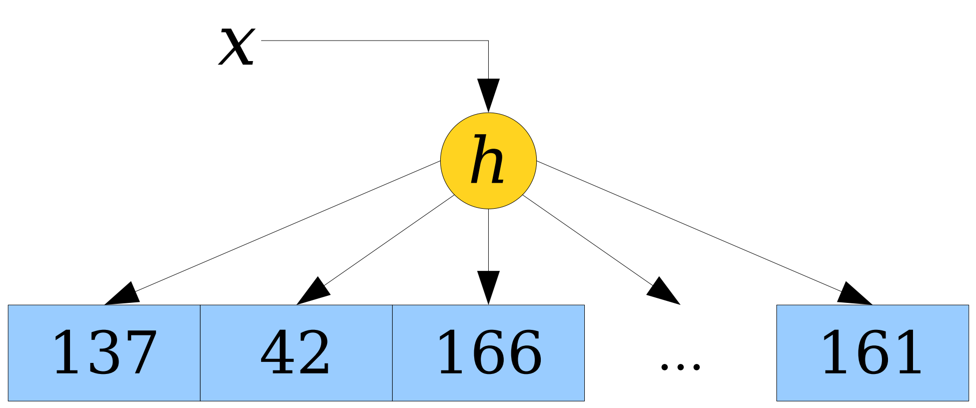


Our Initial Structure

- We can model "assigning each x_i to a counter" by using hash functions.
- Choose, from a family of 2-independent hash functions $\mathscr{H}$, a uniformly-random hash function $h : \mathscr{U} \to [w]$.
- Create an array **count** of w counters, each initially zero.
  - We'll choose w later on.
- To ***increment***(x), increment **count**$[h(x)]$.
- To ***estimate***(x), return **count**$[h(x)]$.



Analyzing our Structure

For each $x_i \in \mathscr{U}$, let $\boldsymbol{a}_i$ denote the number of times we've seen x_i.

Similarly, let $\hat{\boldsymbol{a}}_i$ denote our estimated value of the frequency of x_i.

Goal: Bound the probability that the error $(\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i)$ is too high.

Idea: Think of our element frequencies $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, … as a vector

$$\boldsymbol{a} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \ldots].$$

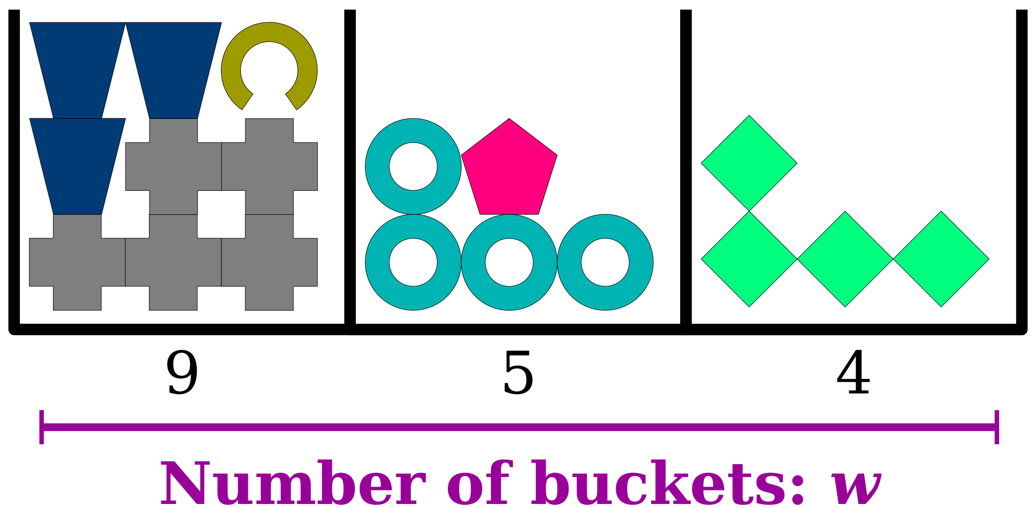
The total number of objects is the sum of the vector entries.

This is called the ***L_1 norm*** of $\boldsymbol{a}$, and is denoted $\|\boldsymbol{a}\|_1$:

$$\|\boldsymbol{a}\|_1 = \sum_i |\boldsymbol{a}_i|$$

There are $\|\boldsymbol{a}\|_1$ total elements distributed across w buckets. We're using a 2-independent hash family.

Reasonable guess: each bin has $\|\boldsymbol{a}\|_1 / w$ elements in it, so $\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i \le \|\boldsymbol{a}\|_1 / w$



Question: Intuitively, what should we expect our approximation error to be?

How to Build an Estimator

1. Design a simple data structure that, intuitively, gives you a good estimate.
2. Use a ***sum of indicator variables*** and ***linearity of expectation*** to prove that, on expectation, the data structure is pretty close to correct.
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Analyzing this Structure

- Let's look at $\hat{\boldsymbol{a}}_i$ = `count`[$h(x_i)$] for some choice of x_i.
- For each element x_j:
  - If $h(x_i) = h(x_j)$, then x_j contributes $\boldsymbol{a}_j$ to `count`[$h(x_i)$].
  - If $h(x_i) \neq h(x_j)$, then x_j contributes 0 to `count`[$h(x_i)$].

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- To pin this down precisely, let's define a set of random variables $X_1, X_2, \ldots,$ as follows:

$$X_j = \begin{cases} 1 & \text{if } h(x_i) = h(x_j) \\ 0 & \text{otherwise} \end{cases}$$

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$$X_j = \begin{cases} 1 & \text{if } h(x_i) = h(x_j) \\ 0 & \text{otherwise} \end{cases}$$

Each of these variables is called an ***indicator random variable***, since it "indicates" whether some event occurs.

Analyzing this Structure

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$$X_j = \begin{cases} 1 & \text{if } h(x_i) = h(x_j) \\ 0 & \text{otherwise} \end{cases}$$

- The value of $\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i$ is then given by

$$\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i = \sum_{j \neq i} \boldsymbol{a}_j X_j$$

$$\mathrm{E}[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i] = \mathrm{E}[\sum_{j \neq i} \boldsymbol{a}_j X_j]$$

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i] &= \mathrm{E}[\sum_{j \neq i} \boldsymbol{a}_j X_j] \\
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\end{aligned}
$$

$$
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\end{aligned}
$$

This follows from ***linearity of expectation***. We'll use this property extensively over the next few days.

$$
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The values of $\boldsymbol{a}_j$ are not random. ***The randomness comes from our choice of hash function.***

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&= \sum_{j \neq i} \boldsymbol{a}_j E[X_j]
\end{aligned}
$$

$$E[X_j] = 1 \cdot \Pr[h(x_i) = h(x_j)] + 0 \cdot \Pr[h(x_i) \neq h(x_j)]$$

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If X is an indicator variable for some event $\mathcal{E}$, then **E[X] = Pr[$\mathcal{E}$]**. This is really useful when using linearity of expectation!

$$
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&= \Pr[h(x_i) = h(x_j)]
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$$

Hey, we saw this earlier!

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\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i] &= \mathrm{E}\Big[\sum_{j \neq i} \boldsymbol{a}_j X_j\Big] \\
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How to Build an Estimator

1. Design a simple data structure that, intuitively, gives you a good estimate.
2. Use a ***sum of indicator variables*** and ***linearity of expectation*** to prove that, on expectation, the data structure is pretty close to correct.
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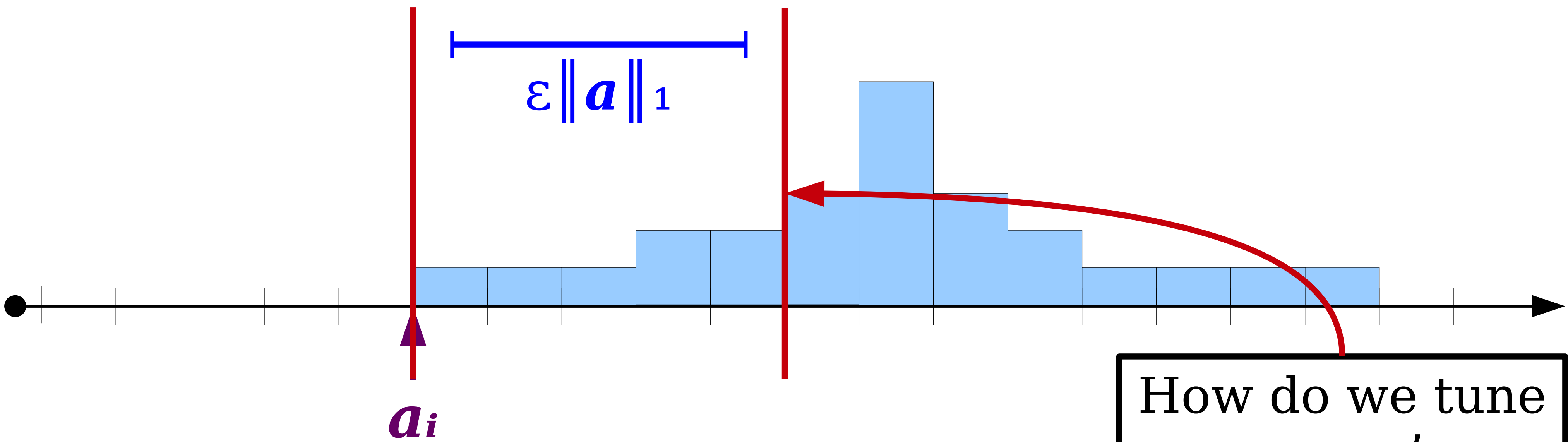
Goal: Make an estimator $\hat{\boldsymbol{a}}$ for some quantity $\boldsymbol{a}$ where

With probability at least $1 - \delta$, **Probably**

$|\hat{\boldsymbol{a}} - \boldsymbol{a}| \le \varepsilon \cdot size(input)$ **Approximately Correct**

for some measure of the size of the input.



How do we tune w so we're likely to fall in this range?

$$\mathrm{E}[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i] \le \frac{\|\boldsymbol{a}\|_1}{w}$$

$$\Pr\left[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i > \varepsilon \left\|\boldsymbol{a}\right\|_1\right]$$

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We don't know the exact distribution of this random variable.

However, we have a ***one-sided error***: our estimate can never be lower than the true value. This means that $\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i \geq 0$.

Markov's inequality says that if X is a nonnegative random variable, then

$$\Pr[X \geq c] \leq \frac{\mathrm{E}[X]}{c}.$$

$$\Pr\left[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1\right]$$

$$\leq \frac{\mathrm{E}\left[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i\right]}{\varepsilon \|\boldsymbol{a}\|_1}$$

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With probability at least $1 - \delta$, — ***Probably***

$|\hat{\boldsymbol{a}} - \boldsymbol{a}| \leq \varepsilon \cdot \mathit{size}(\mathit{input})$ — ***Approximately Correct***

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Pick $w = \varepsilon^{-1} \cdot \delta^{-1}$. Then

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Answer: **$1{,}000 \cdot \varepsilon^{-1}$**.

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$$\left.\text{With probability at least } 1 - \delta,\right\} \leftarrow \textbf{\textit{Probably}}$$

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Can we do better?

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$$\Pr\left[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1\right] \leq \frac{1}{\varepsilon w}$$

We could choose $w = k \cdot \varepsilon^{-1}$ for any constant k to get a failure probability of at most k^{-1}. The choice of e is (mostly) arbitrary.

Revised Idea: Pick $w = e \cdot \varepsilon^{-1}$. Then

$$\Pr\left[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1\right] < e^{-1}$$

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This simple data structure, by itself, is likely to be wrong.

What happens if we run a bunch of copies of this approach in parallel?

Running in Parallel

- Let's run ***d*** copies of our data structure in parallel with one another.
- Each row has its hash function sampled uniformly at random from our hash family.
- Each time we ***increment*** an item, we perform the corresponding ***increment*** operation on each row.

$w = \lceil e \cdot \varepsilon^{-1} \rceil$

h_1	31	41	59	26	53	…	58
h_2	27	18	28	18	28	…	45
h_3	16	18	3	39	88	…	75
…				…			
h_d	69	31	47	18	5	…	59

$d = ??$

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Running in Parallel

- Imagine we call ***estimate***(x) on each of our estimators and get back these estimates.
- We need to give back a single number.
- ***Question:*** How should we aggregate these numbers into a single estimate?

Formulate a hypothesis, but ***don't post anything in chat just yet***.

Estimator 1: 137	*Estimator 2:* 271	*Estimator 3:* 166	*Estimator 4:* 103	*Estimator 5:* 261

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Now, ***private chat me your best guess***. Not sure? Just answer “??”.

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Intuition: The smallest estimate returned has the least "noise," and that's the best guess for the frequency.

Estimator 1: 137	*Estimator 2:* 271	*Estimator 3:* 166	*Estimator 4:* 103	*Estimator 5:* 261

Let $\hat{\boldsymbol{a}}_{ij}$ be the estimate from the jth copy of the data structure.

Our final estimate is $\min \{\hat{\boldsymbol{a}}_{ij}\}$

$$\Pr\left[\min\{\hat{\boldsymbol{a}}_{ij}\} - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1\right]$$

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The only way the minimum estimate is inaccurate is if *every* estimate is inaccurate.

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Each copy of the data structure is independent of the others.

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$$= \prod_{j=1}^{d} \Pr\left[\hat{\boldsymbol{a}}_{ij} - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1\right]$$

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$$\Pr[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i \geq \varepsilon \|\boldsymbol{a}\|_1] \leq e^{-1}$$

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$$\le \prod_{j=1}^{d} e^{-1}$$

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Idea: Choose d = -ln δ.
(Equivalently: $d = \ln \delta^{-1}$.) Then

$$\Pr[\min\{\hat{\boldsymbol{a}}_{ij}\} - \boldsymbol{a}_i > \varepsilon \|\boldsymbol{a}\|_1] \leq \delta$$

The Count-Min Sketch

$w = \lceil e \cdot \varepsilon^{-1} \rceil$

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$d = \lceil \ln \delta^{-1} \rceil$

Sampled uniformly and independently from a 2-independent family of hash functions

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```
increment(x):
   for i = 1 … d:
      count[i][hᵢ(x)]++
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The Count-Min Sketch

h_1	31	41	59	26	53	...	58
h_2	27	18	28	18	28	...	45
h_3	16	18	3	39	88	...	75
...				...			
h_d	69	31	47	18	5	...	59

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The Count-Min Sketch

- Update and query times are $\Theta(d)$, which is $\Theta(\log \delta^{-1})$.
- Space usage: $\Theta(\varepsilon^{-1} \cdot \log \delta^{-1})$ counters.
  - This is a *major* improvement over our earlier approach that used $\Theta(\varepsilon^{-1} \cdot \delta^{-1})$ counters.
  - This can be *significantly* better than just storing a raw frequency count!
- Provides an estimate to within $\varepsilon\|\boldsymbol{a}\|_1$ with probability at least $1 - \delta$.

Major Ideas From Today

- ***2-independent hash families*** are useful when we want to keep collisions low.
- A “good” approximation of some quantity should have tunable ***confidence*** and ***accuracy*** parameters.
- ***Sums of indicator variables*** are useful for deriving expected values of estimators.
- ***Concentration inequalities*** like ***Markov’s inequality*** are useful for showing estimators don’t stay too much from their expected values.
- Good estimators can be built from multiple parallel copies of weaker estimators.

Next Time

- ***Count Sketches***
  - An alternative frequency estimator with different time/space bounds.
- ***Cardinality Estimation***
  - Estimating how many different items you’ve seen in a data stream.