Count Sketches and Bloom Filters

Recap from Last Time

Distribution Property:
Each element should have an equal probability of being placed in each slot.

For any $x \in \mathscr{U}$ and random $h \in \mathcal{H}$, the value of $h(x)$ is uniform over $[m]$.

Independence Property:
Where one element is placed shouldn't impact where a second goes.

For any distinct $x, y \in \mathscr{U}$ and random $h \in \mathcal{H}$, $h(x)$ and $h(y)$ are independent random variables.

A family of hash functions $\mathscr{H}$ is called ***2-independent*** (or ***pairwise independent***) if it satisfies the distribution and independence properties.

Suppose there are two tunable values

$$\varepsilon \in (0, 1]$$
$$\delta \in (0, 1]$$

where ε represents ***accuracy*** and δ represents ***confidence***.

Goal: Make an estimator $\hat{A}$ for some quantity A where

With probability at least $1 - \delta$, ← ***Probably***

$|\hat{A} - A| \leq \varepsilon \cdot size(input)$ ← ***Approximately Correct***

for some measure of the size of the input.

What does it mean for an approximation to be "good"?

How to Build an Estimator

	Count-Min Sketch
Step One: Build a Simple Estimator	Hash items to counters; add +1 when item seen.
Step Two: Compute Expected Value of Estimator	Sum of indicators; 2-independent hashes have low collision rate.
Step Three: Apply Concentration Inequality	One-sided error; use expected value and Markov's inequality.
Step Four: Replicate to Boost Confidence	Take min; only fails if all estimates are bad.

New Stuff!

The Count Sketch

Frequency Estimation

- ***Recall:*** A frequency estimator is a data structure that supports
  - ***increment***(x), which increments the number of times that we've seen x, and
  - ***estimate***(x), which returns an estimate of how many times we've seen x.
- ***Notation:*** Assume that the elements we're processing are $x_1, \ldots, x_n$, and that the true frequency of element x_i is $\boldsymbol{a}_i$.
- Remember that the frequencies are not random variables – we're assuming that they're not under our control. Any randomness comes from hash functions.

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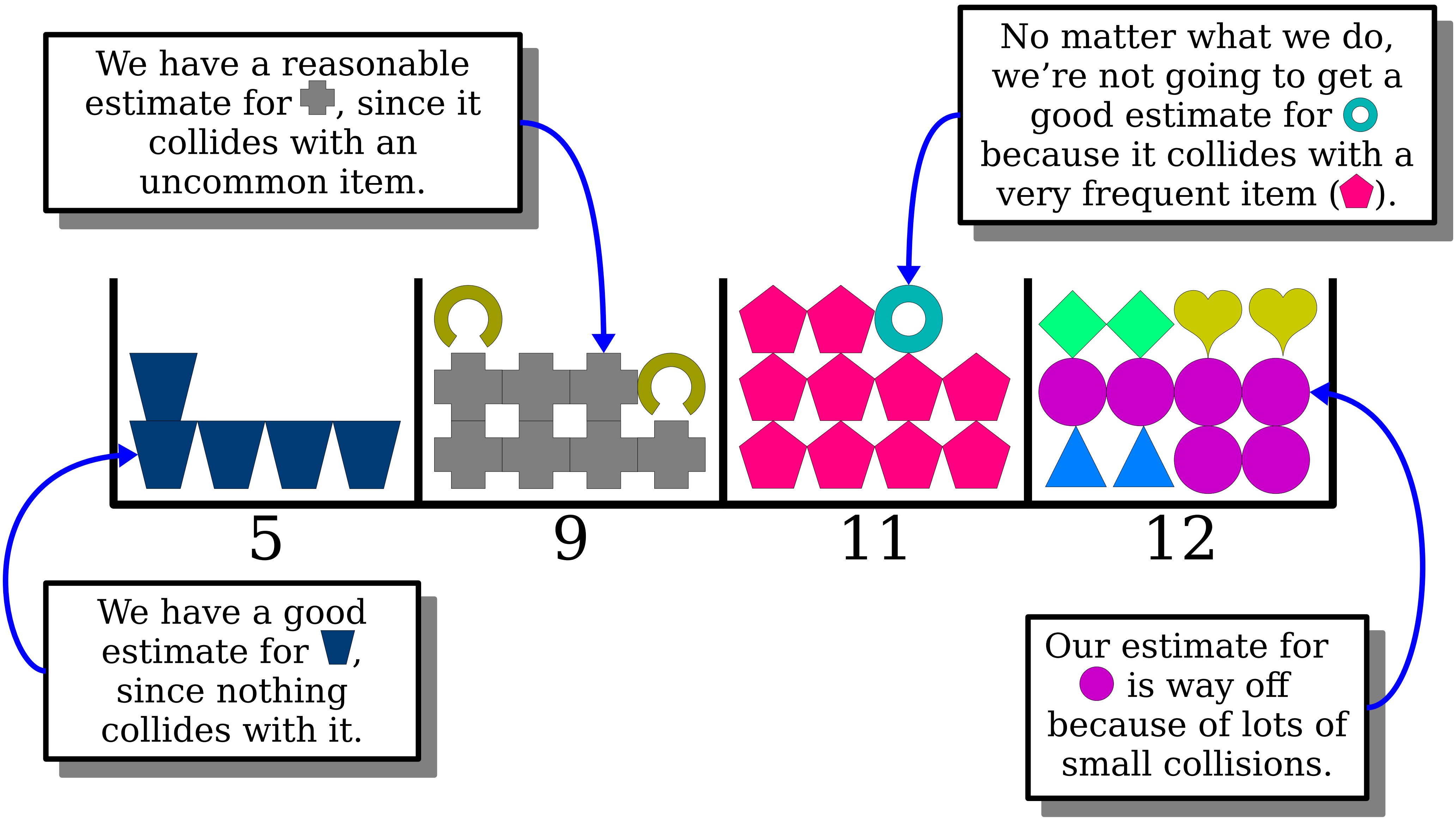
Revisiting Count-Min



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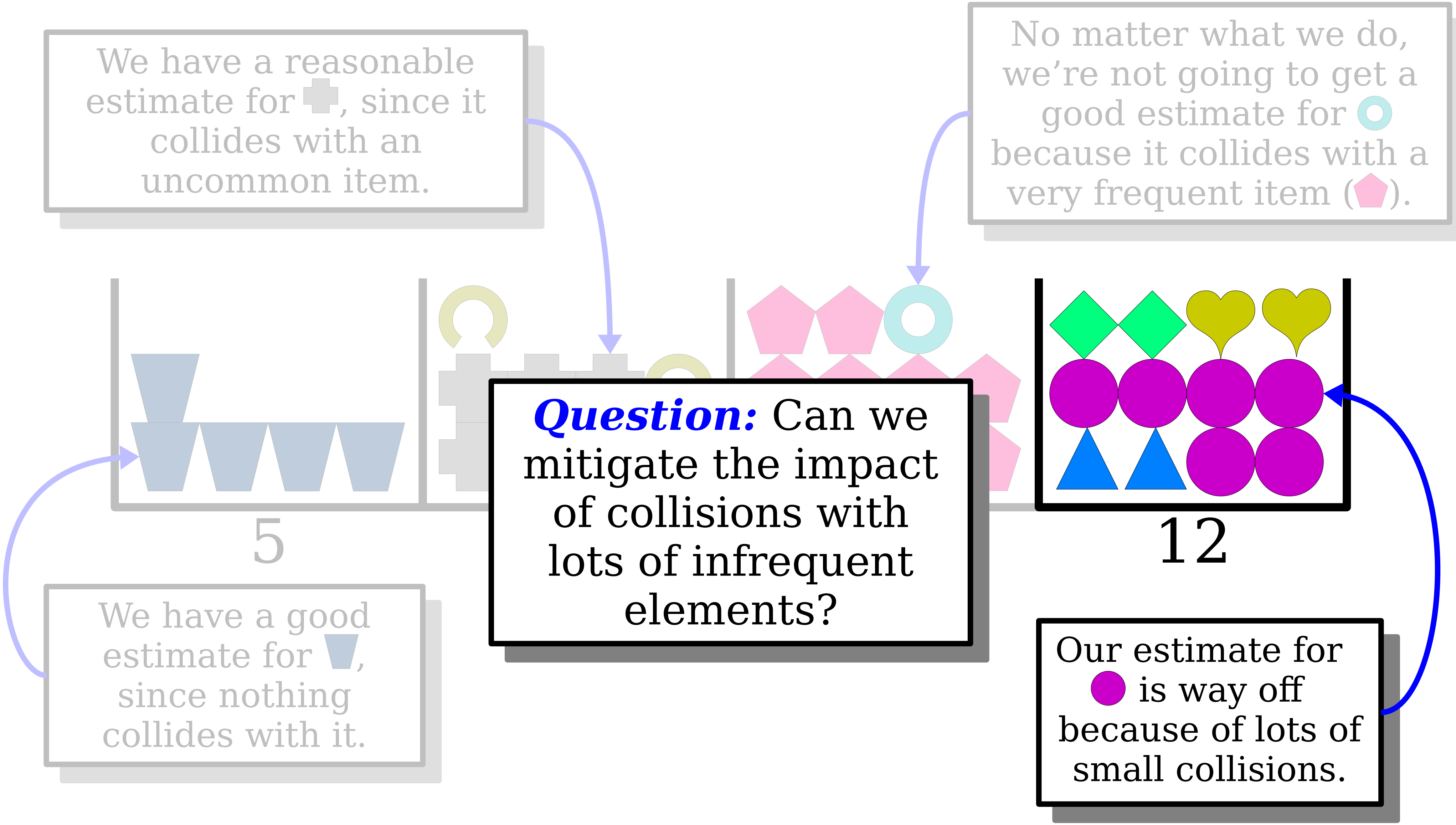
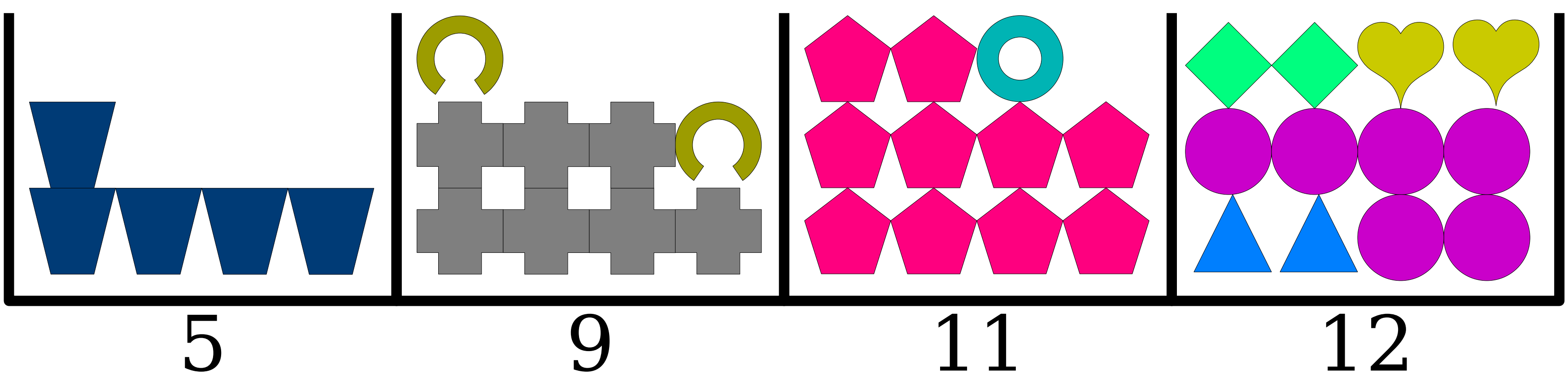


The Setup

- As before, create an array of counters and assign each item a counter.
- ***Key New Step:*** For each item x, assign x either +1 or -1.



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  - To ***increment***(x), go to **`count`**[$h(x)$] and add ±1 as appropriate.
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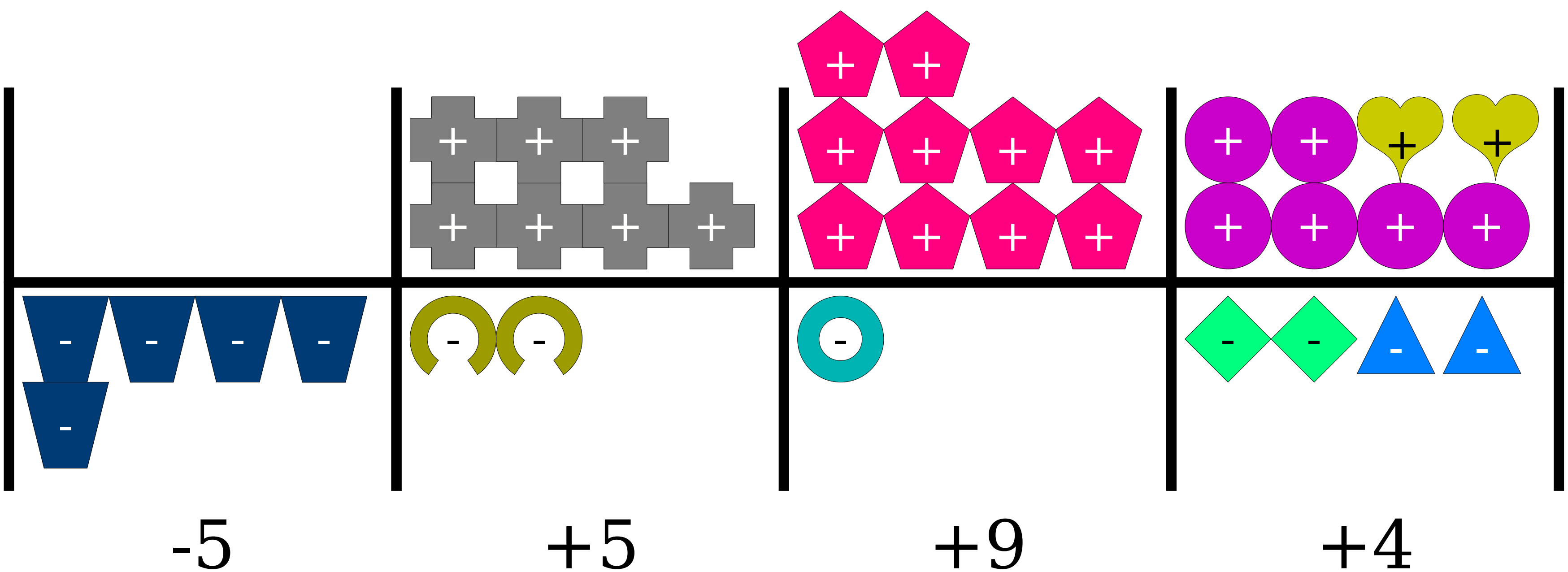

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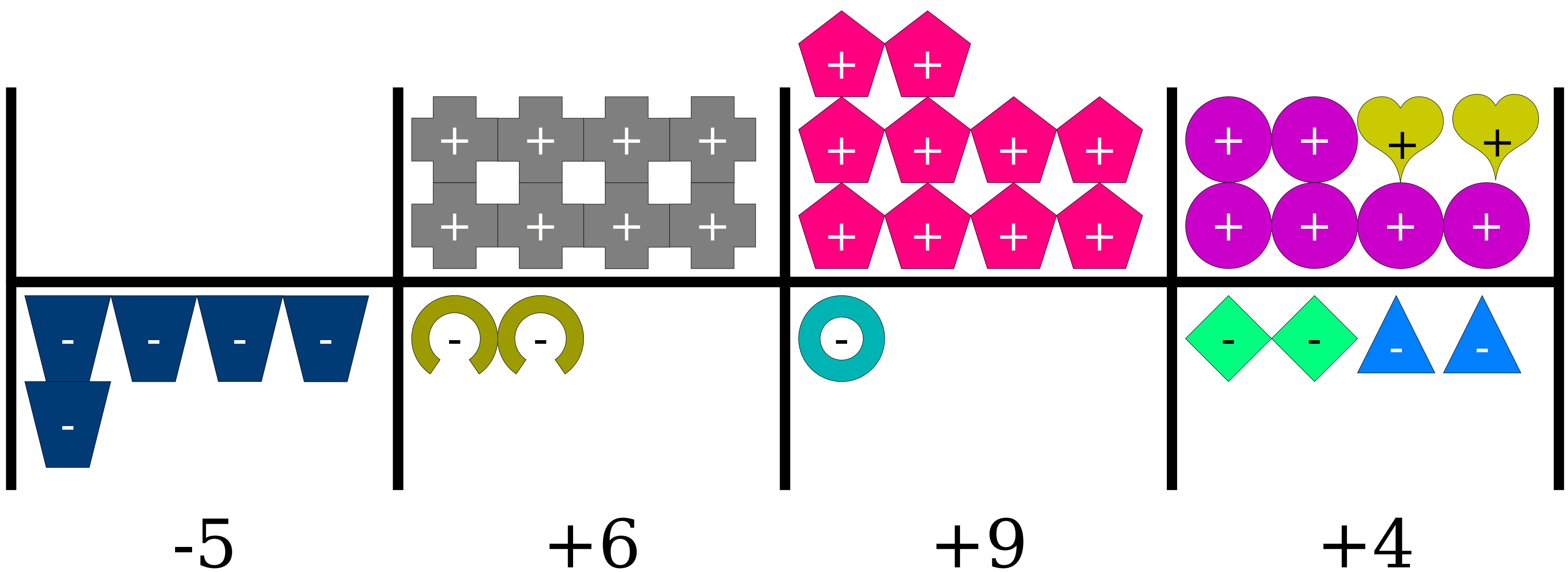

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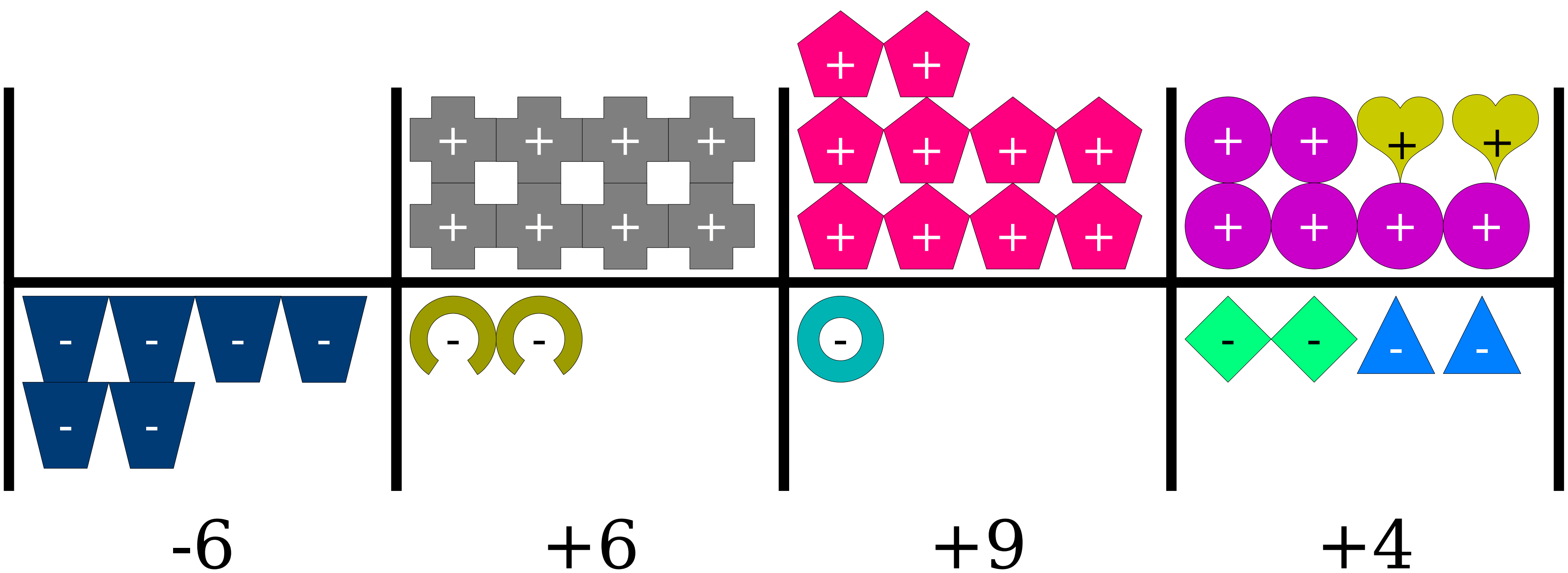

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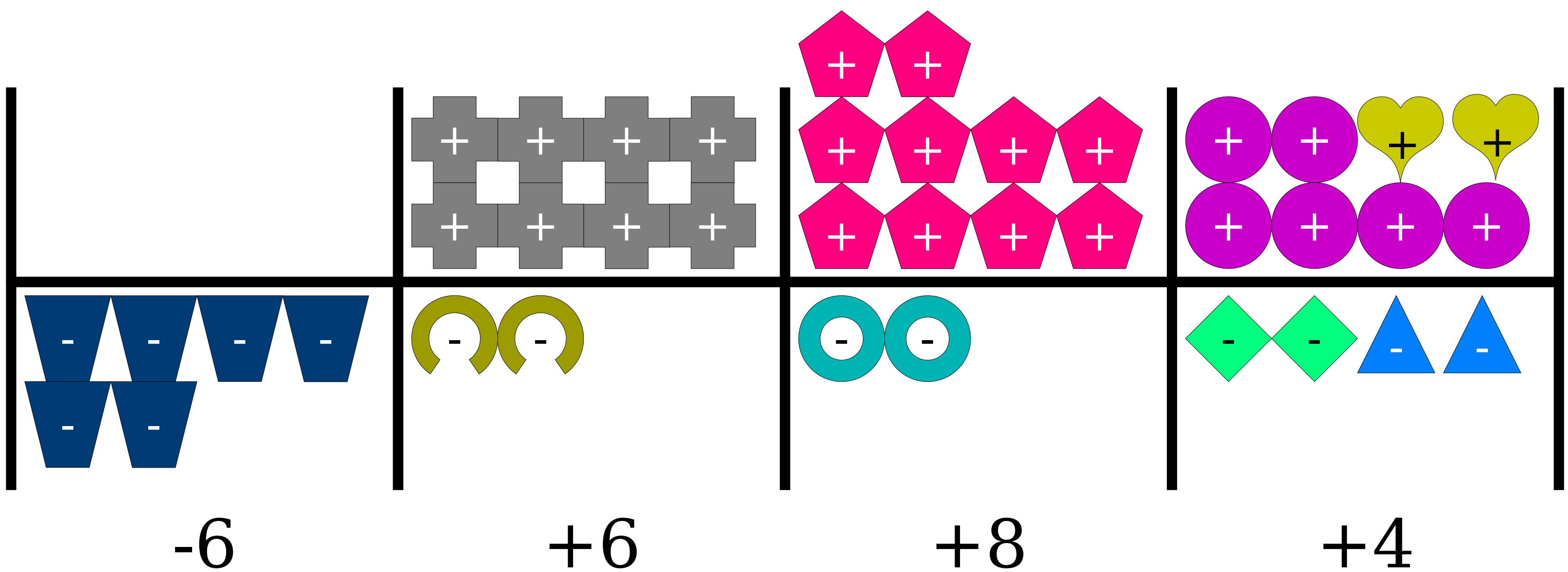

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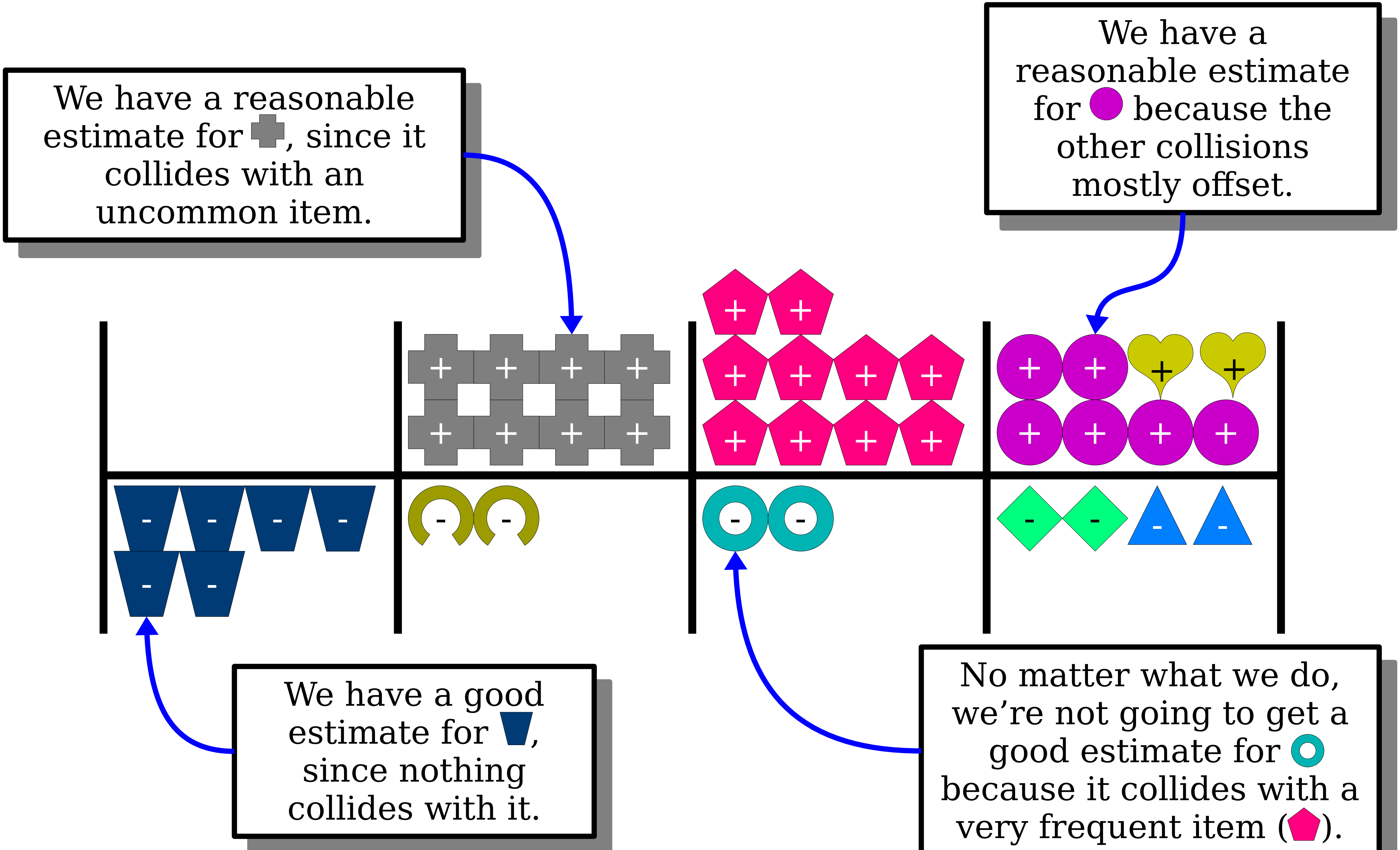
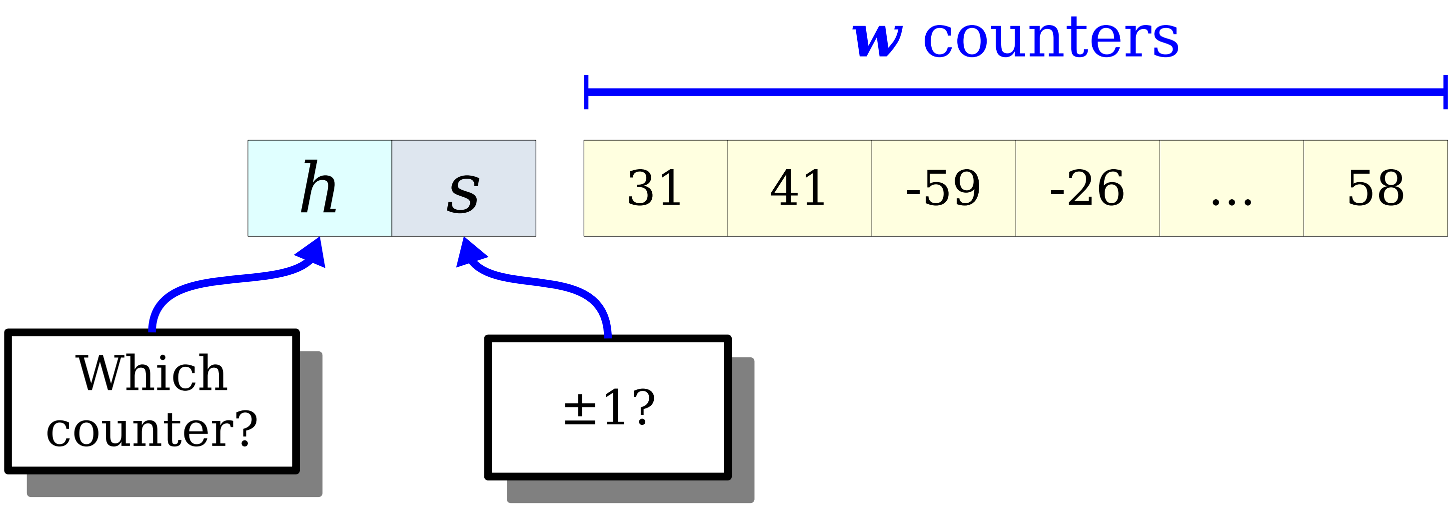

We have a reasonable estimate for , since it collides with an uncommon item.
We have a reasonable estimate for because the other collisions mostly offset.
We have a good estimate for , since nothing collides with it.
No matter what we do, we're not going to get a good estimate for because it collides with a very frequent item ( ).

Formalizing This

- Maintain an array of counters of length w.
- Pick $h \in \mathscr{H}$ chosen uniformly at random from a 2-independent family of hash functions from $\mathscr{U}$. to w.
- Pick $s \in \mathscr{U}$ uniformly randomly and independently of h from a 2-independent family from $\mathscr{U}$ to {-1, +1}.
- ***increment***(x): **`count`**$[h(x)]$ += $s(x)$.
- ***estimate***(x), return $s(x) \cdot$ **`count`**$[h(x)]$.



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Formalizing the Intuition

- Define $\hat{\boldsymbol{a}}_i$ to be our estimate of $\boldsymbol{a}_i$.
- As before, $\hat{\boldsymbol{a}}_i$ will depend on how the other elements are distributed. Unlike before, it now also depends on signs given to the elements by s.
- Specifically, for each other x_j that collides with x_i, the estimate $\hat{\boldsymbol{a}}_i$ includes an error term of

$$s(x_i) \cdot s(x_j) \cdot \boldsymbol{a}_j$$

- Why?

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- Why?
  - The counter for x_i will have $s(x_j)\ \boldsymbol{a}_j$ added in.
  - We multiply the counter by $s(x_i)$ before returning it.

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- Why?
  - If $s(x_i)$ and $s(x_j)$ point in the same direction, the terms add to the total.
  - If $s(x_i)$ and $s(x_j)$ point in different directions, the terms subtract from the total.

Formalizing the Intuition

- In our quest to learn more about $\boldsymbol{\hat{a}}_i$, let's have X_j be a random variable indicating whether x_i and x_j collided with one another:

$$X_j = \begin{cases} 1 & \text{if } h(x_i) = h(x_j) \\ 0 & \text{if } h(x_i) \neq h(x_j) \end{cases}$$

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- We can then express $\hat{\boldsymbol{a}}_i$ in terms of the signed contributions from the items x_i collides with:

$$\hat{\boldsymbol{a}}_i = \sum_j \boldsymbol{a}_j s(x_i) s(x_j) X_j$$

This is how much the collision impacts our estimate.

We only care about items we collided with.

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$$\hat{\boldsymbol{a}}_i = \sum_j \boldsymbol{a}_j s(x_i) s(x_j) X_j = \boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j$$

This is how much the collision impacts our estimate.

We only care about items we collided with.

$$\mathrm{E}[\hat{\boldsymbol{a}}_i] = \mathrm{E}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j]$$

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\mathrm{E}[\hat{\boldsymbol{a}}_i] &= \mathrm{E}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&= \mathrm{E}[\boldsymbol{a}_i] + \mathrm{E}[\sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j]
\end{aligned}
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Hey, it's linearity of expectation!

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Remember that $\boldsymbol{a}_i$ and the like aren't random variables.

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We chose the hash functions h and s independently of one another.

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Since s is drawn from a 2-independent family of hash functions, we know $s(x_i)$ and $s(x_j)$ are independent random variables.

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$\Pr[s(x_i) = -1] = ½$ $\Pr[s(x_i) = +1] = ½$

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$\mathrm{E}[s(x_i)] = ½ \cdot (-1) + ½ \cdot (+1)$

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\mathrm{E}[s(x_i)] &= \tfrac{1}{2} \cdot (-1) + \tfrac{1}{2} \cdot (+1) \\
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Step One: Build a Simple Estimator	Hash items to counters; add +1 when item seen.	Hash items to counters; add ±1 when item seen.
Step Two: Compute Expected Value of Estimator	Sum of indicators; 2-independent hashes have low collision rate.	2-independence breaks up products; ±1 variables have zero expected value.
Step Three: Apply Concentration Inequality	One-sided error; use expected value and Markov's inequality.	
Step Four: Replicate to Boost Confidence	Take min; only fails if all estimates are bad.	

How to Build an Estimator

	Count-Min Sketch	*Count Sketch*
Step One: Build a Simple Estimator	Hash items to counters; add +1 when item seen.	Hash items to counters; add ±1 when item seen.
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Step Four: Replicate to Boost Confidence	Take min; only fails if all estimates are bad.	

A Hitch

- In the count-min sketch, we used Markov's inequality to bound the probability that we get a bad estimate.
- This worked because we had a ***one-sided error***: the distance $\hat{a}_i - a_i$ from the true answer was nonnegative.
- With the count sketch, we have a ***two-sided error***: $\hat{a}_i - a_i$ can be negative in the count sketch because collisions can *decrease* the estimate $\hat{a}_i$ below the true value a_i.
- We'll need to use a different technique to bound the error.

Chebyshev to the Rescue

- ***Chebyshev's inequality*** states that for any random variable X with finite variance, given any $c > 0$, we have

$$\Pr\left[\ |X - \mathrm{E}[X]| \geq c\ \right] \leq \frac{\mathrm{Var}[X]}{c^2}.$$

- If we can get the variance of $\hat{\boldsymbol{a}}_i$, we can bound the probability that we get a bad estimate with our data structure.

$$\mathrm{Var}[\hat{\boldsymbol{a}}_i] \quad = \quad \mathrm{Var}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j]$$

$$\mathrm{Var}[\hat{\boldsymbol{a}}_i] = \mathrm{Var}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j]$$

$$\boxed{\mathrm{Var}[a + X] = \mathrm{Var}[X]}$$

$$
\begin{aligned}
\mathrm{Var}[\hat{\boldsymbol{a}}_i] &= \mathrm{Var}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
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$$\begin{aligned}\mathrm{Var}[\hat{\boldsymbol{a}}_i] &= \mathrm{Var}[\boldsymbol{a}_i + \sum_{j\neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\ &= \boxed{\mathrm{Var}[\sum_{j\neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j]}\end{aligned}$$

In general, Var is *not* a linear operator.

However, if the terms in the sum are ***pairwise uncorrelated***, then Var is linear.

Lemma: The terms in this sum are uncorrelated. *(Prove this!)*

$$\begin{aligned}
\mathrm{Var}[\hat{\boldsymbol{a}}_i] &= \mathrm{Var}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
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The “Sum-o’-Var” Samovar!

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Useful Fact: If X is an indicator, then $X^2 = X$.

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&= \sum_{j \neq i} \boldsymbol{a}_j^2 \mathrm{E}[X_j] \\
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I know this might look really dense, but many of these substeps end up being really useful techniques. These ideas generalize, I promise.

Think of [$\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, …] as a vector.

What does the following quantity represent?

$$\sum_j \boldsymbol{a}_j^2$$

$$\mathrm{Var}[\hat{\boldsymbol{a}}_i] \leq \frac{1}{w}\sum_{j \neq i} \boldsymbol{a}_j^2$$

Think of [$\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, …] as a vector.

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This is the square of the magnitude of the vector!

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This is the square of the magnitude of the vector!

The magnitude of a vector is called its ***L_2 norm*** and is denoted $\|\boldsymbol{a}\|_2$.

$$\|\boldsymbol{a}\|_2 = \sqrt{\sum_j \boldsymbol{a}_j^2}$$

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$$\|\boldsymbol{a}\|_2 \quad = \quad \sqrt{\sum_j \boldsymbol{a}_j^2}$$

Therefore, our above sum is $\|\boldsymbol{a}\|_2^2$.

$$\mathrm{Var}[\hat{\boldsymbol{a}}_i] \quad \leq \quad \frac{1}{w} \sum_{j \neq i} \boldsymbol{a}_j^2$$

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is denoted $\|$

Great exercise: Prove that the L_2 norm of a vector is never greater than the L_1 norm.

$$\|\boldsymbol{a}\|_2 = \sqrt{\sum_j \boldsymbol{a}_j^2}$$

Therefore, our above sum is $\|\boldsymbol{a}\|_2^2$.

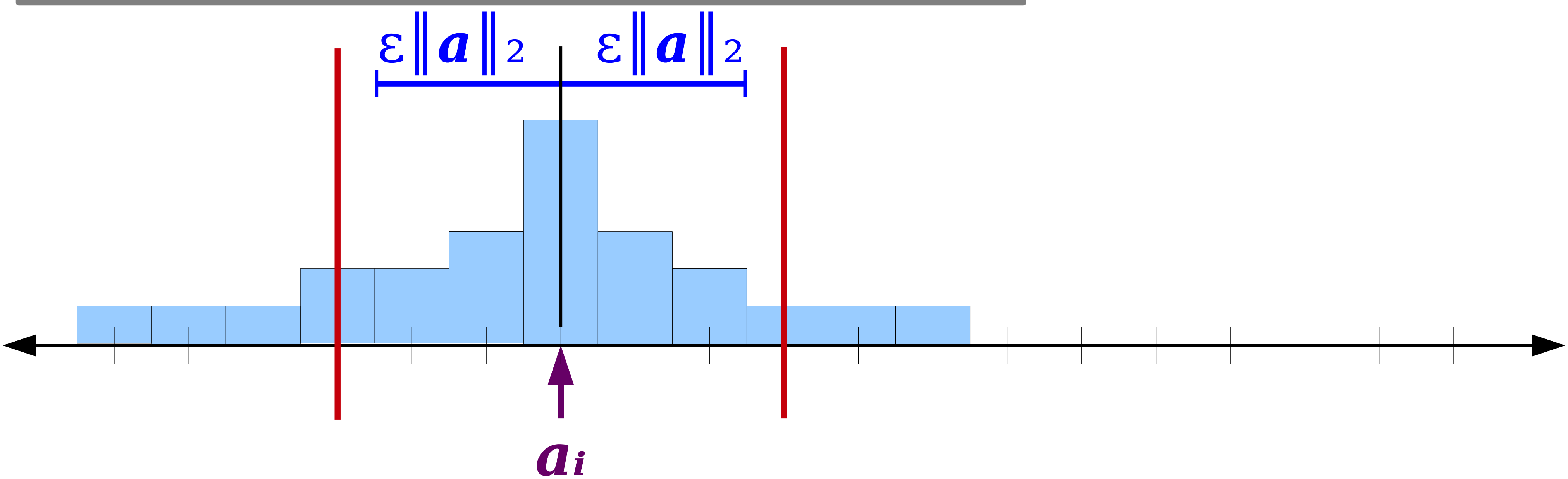
$$\mathrm{Var}[\hat{\boldsymbol{a}}_i] \le \frac{1}{w}\sum_{j \ne i} \boldsymbol{a}_j^2 \le \frac{\|\boldsymbol{a}\|_2^2}{w}$$

Goal: Make an estimator $\hat{\boldsymbol{a}}$ for some quantity $\boldsymbol{a}$ where

With probability at least $1 - \delta$, — ***Probably***

$|\hat{\boldsymbol{a}} - \boldsymbol{a}| \le \varepsilon \cdot \mathit{size}(\mathit{input})$ — ***Approximately Correct***

for some measure of the size of the input.



$$\mathrm{Var}[\hat{\boldsymbol{a}}_i] \le \frac{\|\boldsymbol{a}\|_2^2}{w}$$

$$\Pr\left[\left|\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i\right| > \varepsilon \|\boldsymbol{a}\|_2\right]$$

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Chebyshev's inequality says that

$$\Pr\left[\ |X - \mathrm{E}[X]| \geq c\ \right] \leq \frac{\mathrm{Var}[X]}{c^2}.$$

$$\Pr\left[\left| \hat{\boldsymbol{a}}_i - \boldsymbol{a}_i \right| > \varepsilon \|\boldsymbol{a}\|_2 \right]$$

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$$\leq \quad \frac{\mathrm{Var}[\hat{\boldsymbol{a}}_i]}{\left(\varepsilon \|\boldsymbol{a}\|_2\right)^2}$$

$$\leq \quad \frac{\|\boldsymbol{a}\|_2^2}{w} \cdot \frac{1}{\left(\varepsilon \|\boldsymbol{a}\|_2\right)^2}$$

$$\boxed{\mathrm{Var}[\hat{\boldsymbol{a}}_i] \quad \leq \quad \frac{\|\boldsymbol{a}\|_2^2}{w}}$$

$$
\begin{aligned}
& \Pr\left[\left|\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i\right| > \varepsilon \|\boldsymbol{a}\|_2\right] \\
\leq \quad & \frac{\mathrm{Var}[\hat{\boldsymbol{a}}_i]}{\left(\varepsilon \|\boldsymbol{a}\|_2\right)^2} \\
\leq \quad & \frac{\|\boldsymbol{a}\|_2^2}{w} \cdot \frac{1}{\left(\varepsilon \|\boldsymbol{a}\|_2\right)^2} \\
= \quad & \frac{1}{w\varepsilon^2}
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Goal: Make an estimator $\hat{\boldsymbol{a}}$ for some quantity $\boldsymbol{a}$ where

With probability at least $1 - \delta$, ***Probably***

$|\hat{\boldsymbol{a}} - \boldsymbol{a}| \le \varepsilon \cdot \mathit{size}(\mathit{input})$ ***Approximately Correct***

for some measure of input size.

$$\Pr\left[\left|\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i\right| > \varepsilon \left\|\boldsymbol{a}\right\|_2\right] \le \frac{1}{w\varepsilon^2}$$

Pick $w = 4 \cdot \varepsilon^{-2}$. Then

$$\Pr\left[\left|\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i\right| > \varepsilon \left\|\boldsymbol{a}\right\|_2\right] \le \frac{1}{4}.$$

We now have a single estimator with a not-so-great chance of giving a good estimate.

How do we fix this?

How to Build an Estimator

	Count-Min Sketch	***Count Sketch***
Step One: Build a Simple Estimator	Hash items to counters; add +1 when item seen.	Hash items to counters; add ±1 when item seen.
Step Two: Compute Expected Value of Estimator	Sum of indicators; 2-independent hashes have low collision rate.	2-independence breaks up products; ±1 variables have zero expected value.
Step Three: Apply Concentration Inequality	One-sided error; use expected value and Markov's inequality.	Two-sided error; compute variance and use Chebyshev's inequality.
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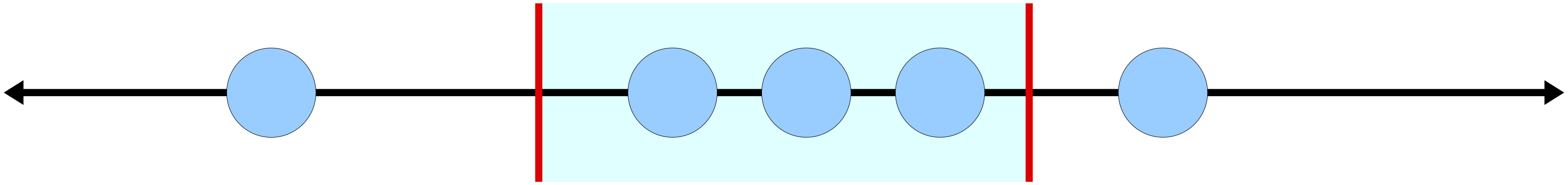
Running in Parallel

- Let's suppose that we run ***d*** independent copies of this data structure. Each has its own independently randomly chosen hash function.
- To ***increment***(x) in the overall structure, we call ***increment***(x) on each of the underlying data structures.
- The probability that at least one of them provides a good estimate is high.
- ***Question:*** How do you know which one?

Estimator 1:	*Estimator 2:*	*Estimator 3:*	*Estimator 4:*	*Estimator 5:*
137	271	166	103	261

Working with the Median

- ***Claim:*** If we output the median estimate given by the data structures, we have high probability of giving the right answer.
- ***Intuition:*** The only way that the median isn't in the "good" area is if ***at least half*** the estimates are in the "bad" area.
- Each individual data structure has a "reasonable" chance to be good, so this is very unlikely.



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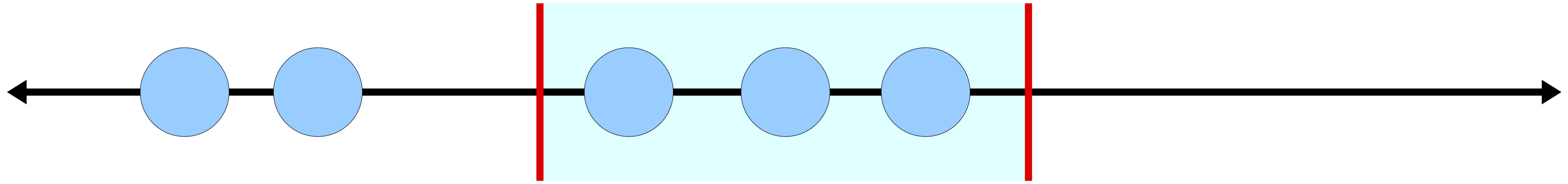

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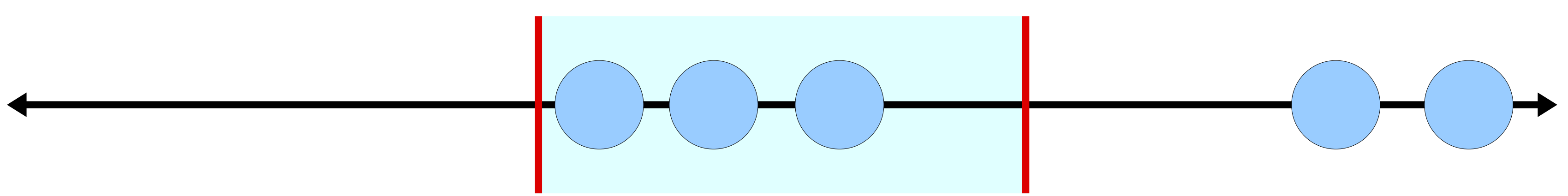

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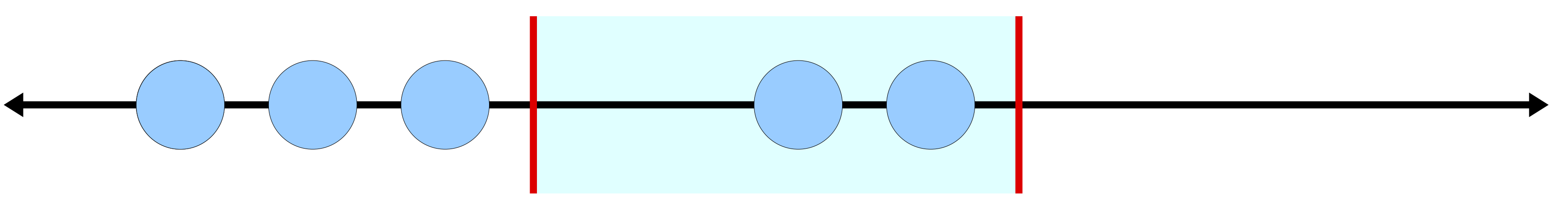
- Each individual data structure has a "reasonable" chance to be good, so this is very unlikely.



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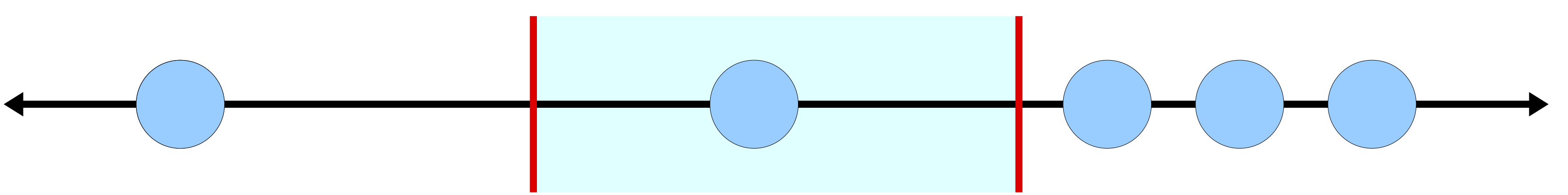

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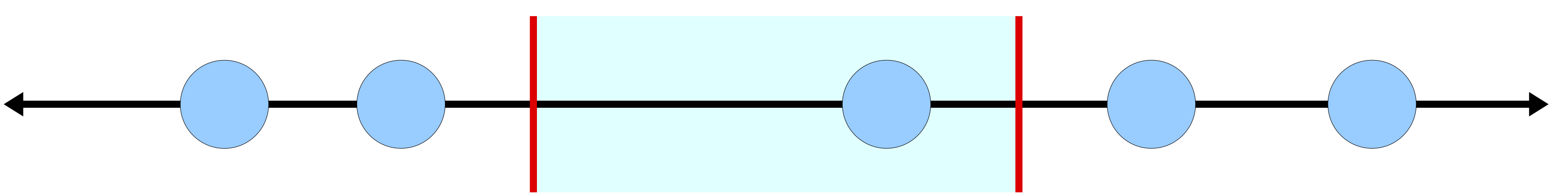

The Setup

- Let D denote a random variable equal to the number of data structures that produce an answer *not* within $\varepsilon||\boldsymbol{a}||_2$ of the true answer.
- Since each independent data structure has failure probability at most ¼, we can upper-bound D with a Binom(d, ¼) variable.
- We want to know $\Pr[D > d / 2]$.
- How can we determine this?

Chernoff Bounds

- The ***Chernoff bound*** says that if $X \sim \text{Binom}(n, p)$ and $p < 1/2$, then

$$\Pr[X \geq n/2] < \boxed{e^{-n \cdot z(p)}}$$

where $z(p) = (½ – p)^2 / 2p$.

Intuition: For any fixed value of p, this quantity decays exponentially quickly as a function of n. It's extremely unlikely that more than half our estimates will be bad.

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- In our case, $D \sim \text{Binom}(d, ¼)$, so we know that

$$\Pr[D \geq \frac{d}{2}] \leq e^{-n \cdot z(1/4)} = e^{-d/8}$$

- Therefore, choosing $\boldsymbol{d = 8 \log \delta^{-1}}$ ensures that

$$\Pr[|\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i| > \varepsilon \|\boldsymbol{a}\|_2] \leq \Pr[D \geq \frac{d}{2}] \leq \delta$$

How to Build an Estimator

	Count-Min Sketch	***Count Sketch***
Step One: Build a Simple Estimator	Hash items to counters; add +1 when item seen.	Hash items to counters; add ±1 when item seen.
Step Two: Compute Expected Value of Estimator	Sum of indicators; 2-independent hashes have low collision rate.	2-independence breaks up products; ±1 variables have zero expected value.
Step Three: Apply Concentration Inequality	One-sided error; use expected value and Markov's inequality.	Two-sided error; compute variance and use Chebyshev's inequality.
Step Four: Replicate to Boost Confidence	Take min; only fails if all estimates are bad.	Take median; only can fail if half of estimates are wrong; use Chernoff.

The Count Sketch

$w = \lceil 4 \cdot \varepsilon^{-1} \rceil$

h_1	s_1
h_2	s_2
h_3	s_3
…	…
h_d	s_d

31	41	-59	-26	…	58
27	-18	28	-18	…	-45
16	-18	-3	39	…	-75
…					
69	-31	47	-18	…	59

$d = \lceil 8 \ln \delta^{-1} \rceil$

Sampled uniformly and independently from 2-independent families of hash functions

The Count Sketch

$w = \lceil 4 \cdot \varepsilon^{-1} \rceil$

h_1	s_1
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$d = \lceil 8 \ln \delta^{-1} \rceil$

```
increment(x):
   for i = 1 … d:
      count[i][hi(x)] += si(x)
```

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$w = \lceil 4 \cdot \varepsilon^{-1} \rceil$

h_1	s_1
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h_2	s_2
h_3	s_3
…	…
h_d	s_d

31	**40**	-59	-26	…	58
27	-18	28	**-19**	…	-45
16	-18	-3	**40**	…	-75
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```
estimate(x):
   options = []
   for i = 1 … d:
      options += count[i][hᵢ(x)] * sᵢ(x)
   return medianOf(options)
```

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h_1	s_1	31	40	-59	-26	…	58
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increment(x):
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   return medianOf(options)
```

The Final Analysis

- Here's a comparison of these two structures.
- ***Question to ponder:*** When is a count-min sketch better than a count sketch, and vice-versa?

Count-Min Sketch

Space: $\Theta(\varepsilon^{-1} \log \delta^{-1})$

increment: $\Theta(\log \delta^{-1})$

estimate: $\Theta(\log \delta^{-1})$

Accuracy: within $\varepsilon||\boldsymbol{a}||_1$.

Count Sketch

Space: $\Theta(\varepsilon^{-2} \log \delta^{-1})$

increment: $\Theta(\log \delta^{-1})$

estimate: $\Theta(\log \delta^{-1})$

Accuracy: within $\varepsilon||\boldsymbol{a}||_2$

Bloom Filters

Exact Membership Queries

- Suppose you're in a memory-constrained environment where every bit of memory counts, and you want to store a set of n items drawn from some universe U.
- Examples:
  - You're working on an embedded device with some maximum amount of working RAM.
  - You're working with large n (say, $n = 10^9$) on a modern machine.
  - You're building a consumer application like a web browser and don't want to hog all system resources.
- ***Question:*** How many bits of memory are needed to do this?

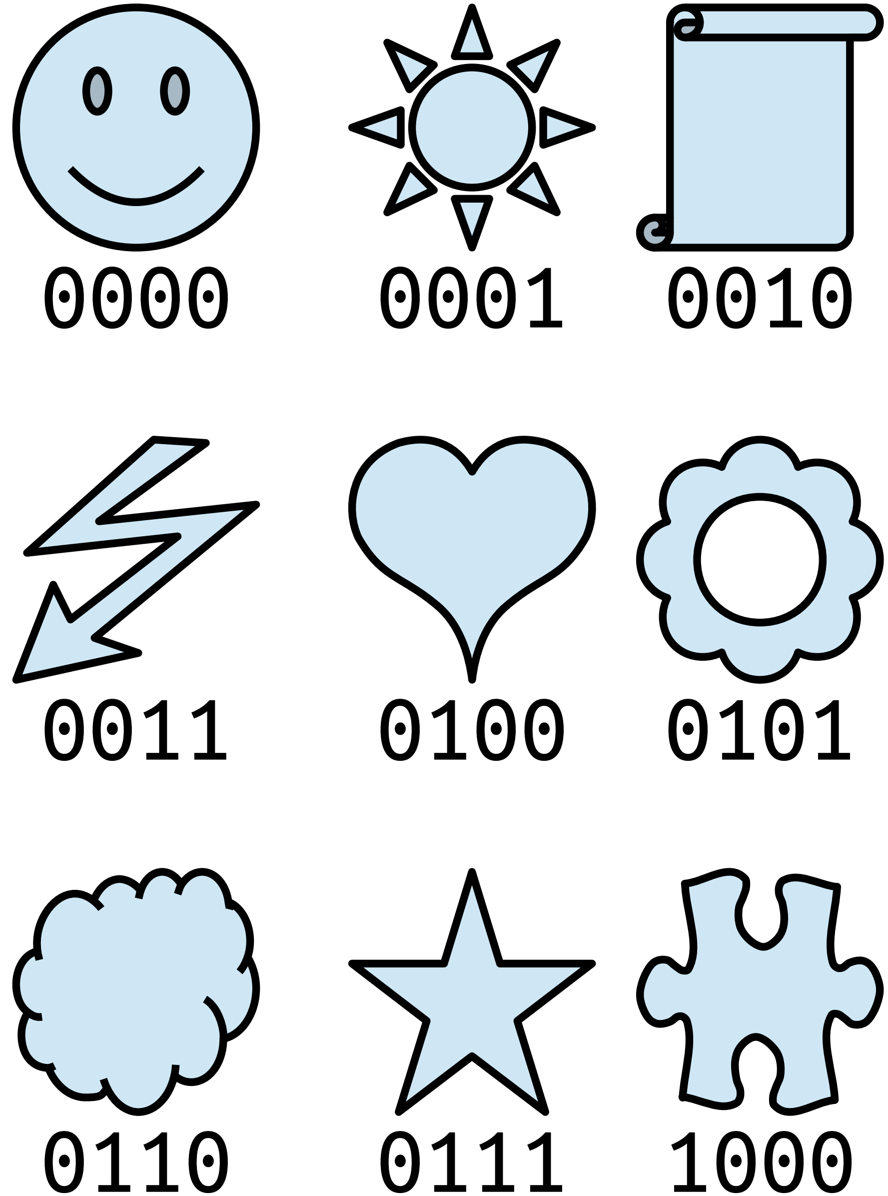
A Quick Detour

Goal: Design a simple data structure that can hold a single one of the objects shown to the right.

What is the minimum number of *bits* (not *words*) required for this data structure in the worst case?

We can get away with four bits by numbering each item and just storing the number.

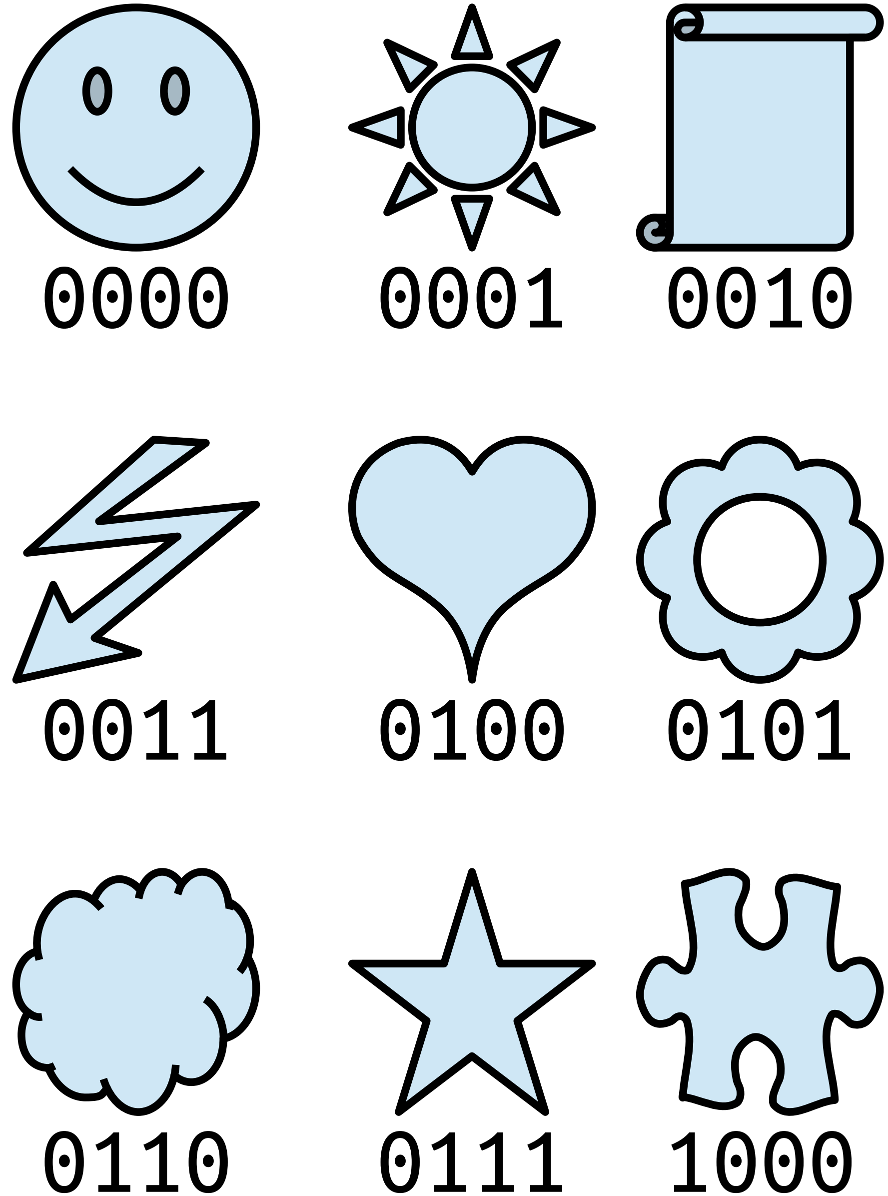
Question: Can we do better?



Goal: Design a simple data structure that can hold a single one of the objects shown to the right.

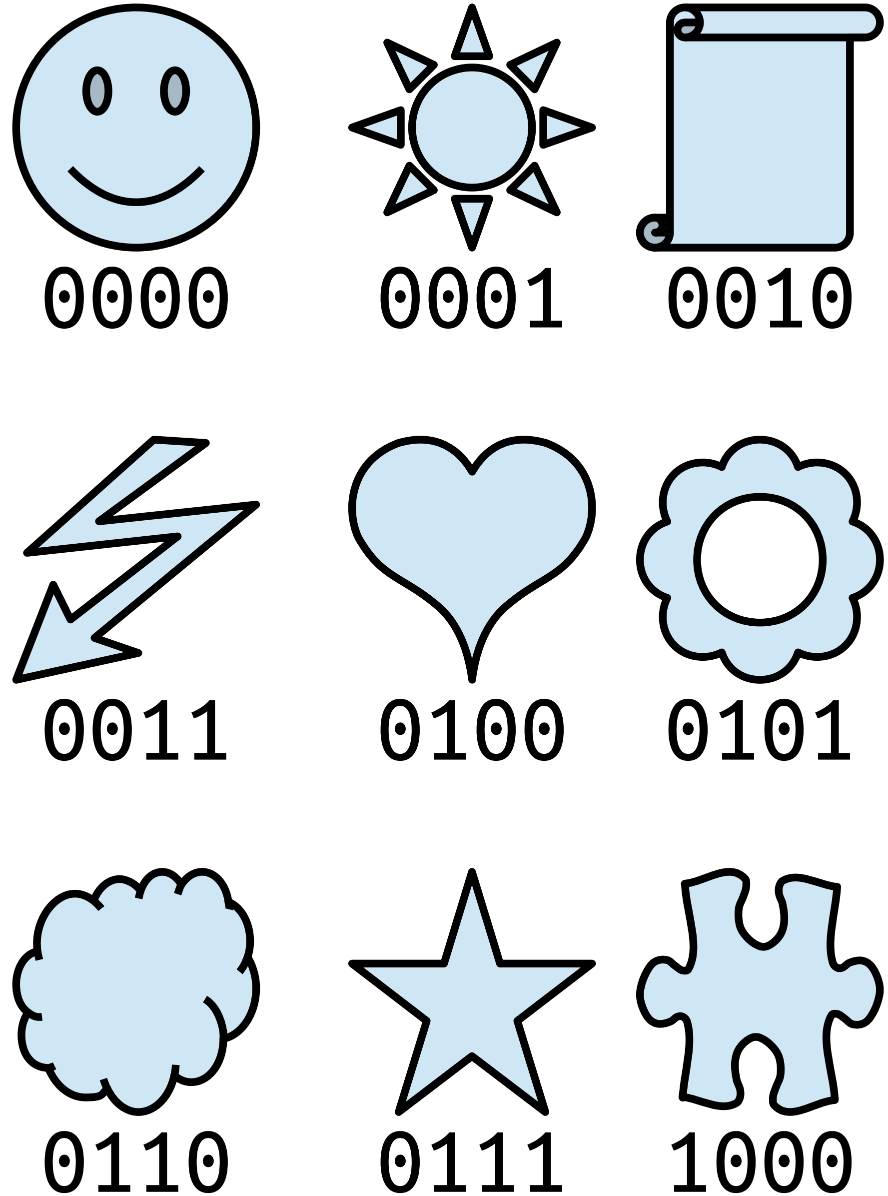
Claim: Every data structure for this problem must use at least four bits of memory in the worst case.

Proof: If we always use three or fewer bits, there are at most $2^3 = 8$ combinations of those bits, not enough to uniquely identify one of the nine different items.



Theorem: A data structure that stores one object out of a set of k possibilities must use at least $\lg k$ bits in the worst case.

Proof: Using fewer than $\lg k$ bits means there are fewer than $2^{\lg k} = k$ possible combinations of those bits, not enough to uniquely identify each item out of the set.



Question: How much memory is needed to solve the exact membership query problem?

Suppose we want to store a set $S \subseteq U$ of size n. How many bits of memory do we need?

Number of n-element subsets of universe U:

$$\binom{|U|}{n}$$

Bits needed:

$\Omega(n \lg |U| - n \lg n)$

$$
\begin{aligned}
& \lg \binom{|U|}{n} \\
= \; & \lg \left(\frac{|U|\,!}{n\,!\,(|U|-n)\,!} \right) \\
\geq \; & \lg \left(\frac{(|U|-n)^n}{n^n} \right) \\
= \; & n \lg \left(\frac{|U|-n}{n} \right) \\
= \; & n \lg \left(\frac{|U|}{n} - 1 \right) \\
\geq \; & n \lg \left(\frac{|U|}{n} \right) - n \\
\geq \; & n \lg |U| - n \lg n - n \\
= \; & \boldsymbol{\Omega\left(n \lg |U| - n \lg n \right)}
\end{aligned}
$$

Bitten by Bits

- Assuming $|U| \gg n$, we need $\boldsymbol{\Omega(n \log |U|)}$ bits to encode a solution to the exact membership query problem.
- If we're resource-constrained, this might be way too many bits for us to fit things in memory.
  - Think $n = 10^8$ and U is the set of all possible URLs or human genomes.
- Can we do better?

Approximate Membership Queries

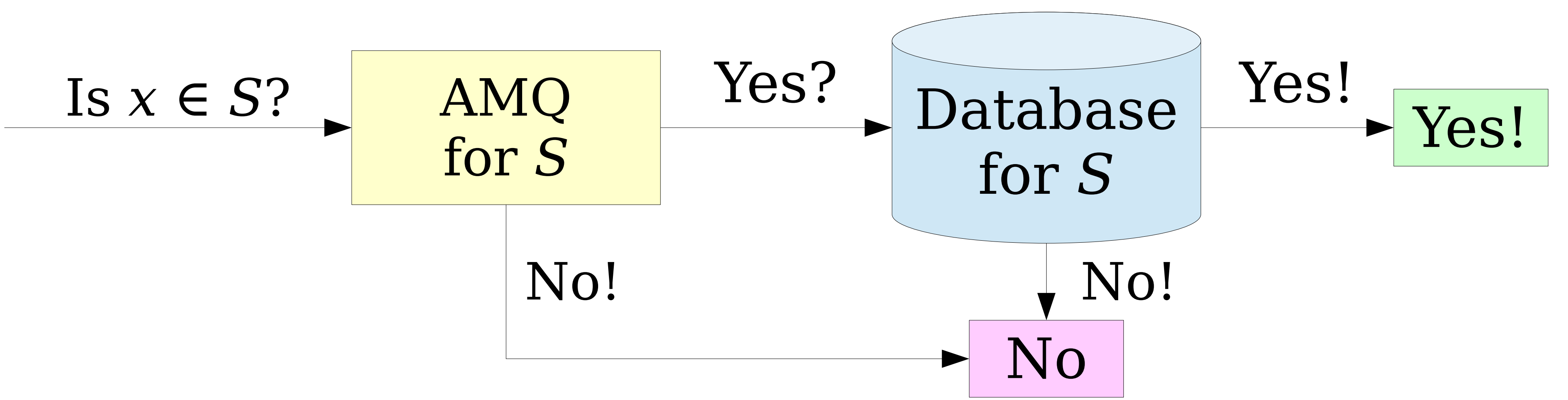
- The ***approximate membership query*** problem is the following:

  Maintain a set S in a way that gives approximate answers to queries of the form "is x ∈ S?"

- Questions we need to answer:
  - How do you give an "approximate" answer to the question "is $x \in S$?"
  - Does this relaxation let us save memory?
- We'll address each of these in turn.

Our Model

- ***Goal:*** Design our data structures to allow for false positives but not false negatives.
- That is:
  - if $x \in S$, we always return true, but
  - if $x \notin S$, we have a small probability of returning true.
- This is often a good idea in practice.



Our Model

- Let’s assume we have a tunable accuracy parameter $\varepsilon \in (0, 1)$.
- ***Goal:*** Design our data structure so that
  - if $x \in S$, we always return true;
  - if $x \notin S$, we return false with probability at least $1 - \varepsilon$; and
  - the amount of space we need depends only on n and ε, not on the size of the universe.
- Is this even possible?

Suppose we're storing an n-element set S with error rate ε.

How much memory is needed to solve the approximate membership query problem?

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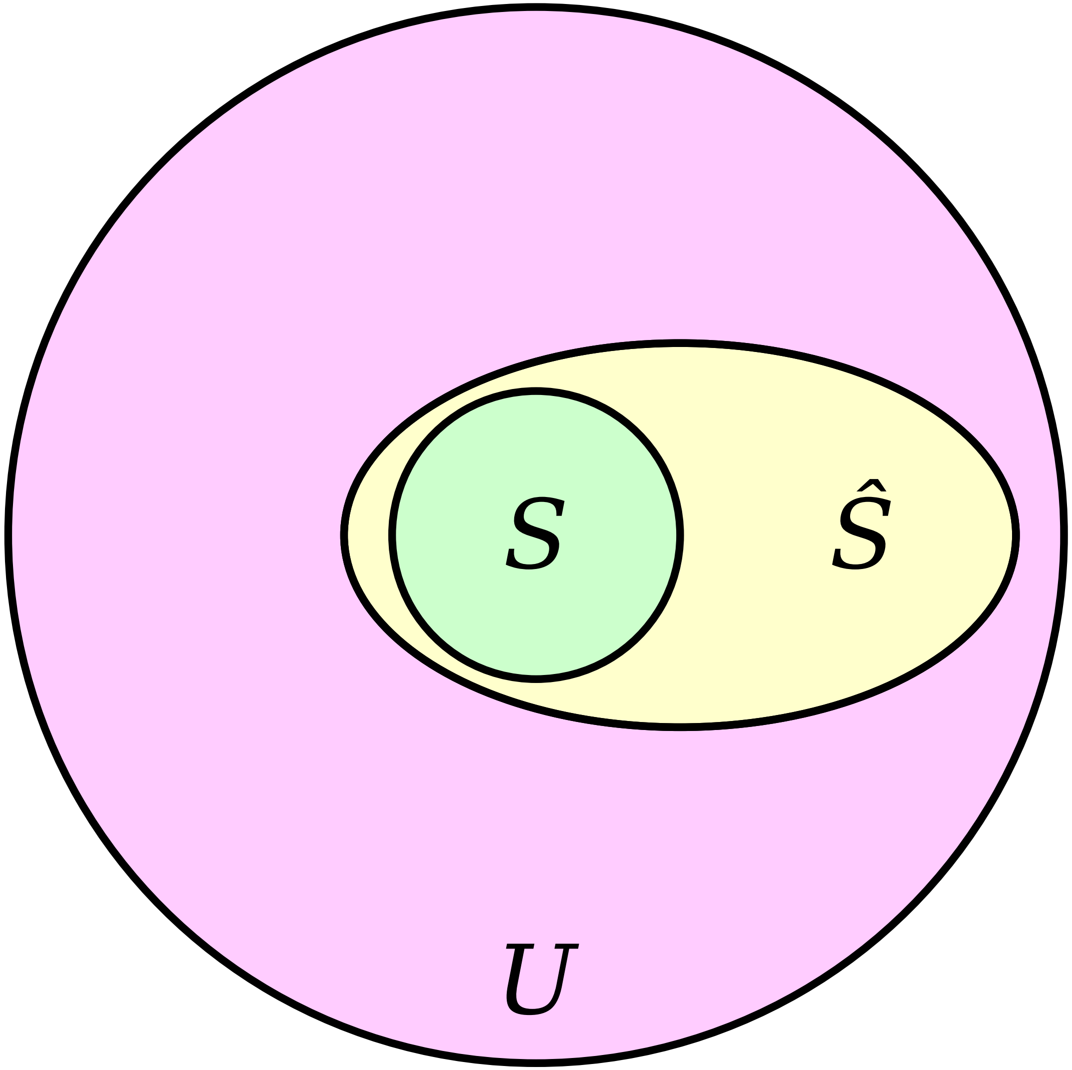
Intuition: An AMQ structure stores a set $\hat{S}$: S plus approximately $\varepsilon|U|$ extra elements due to the error rate.



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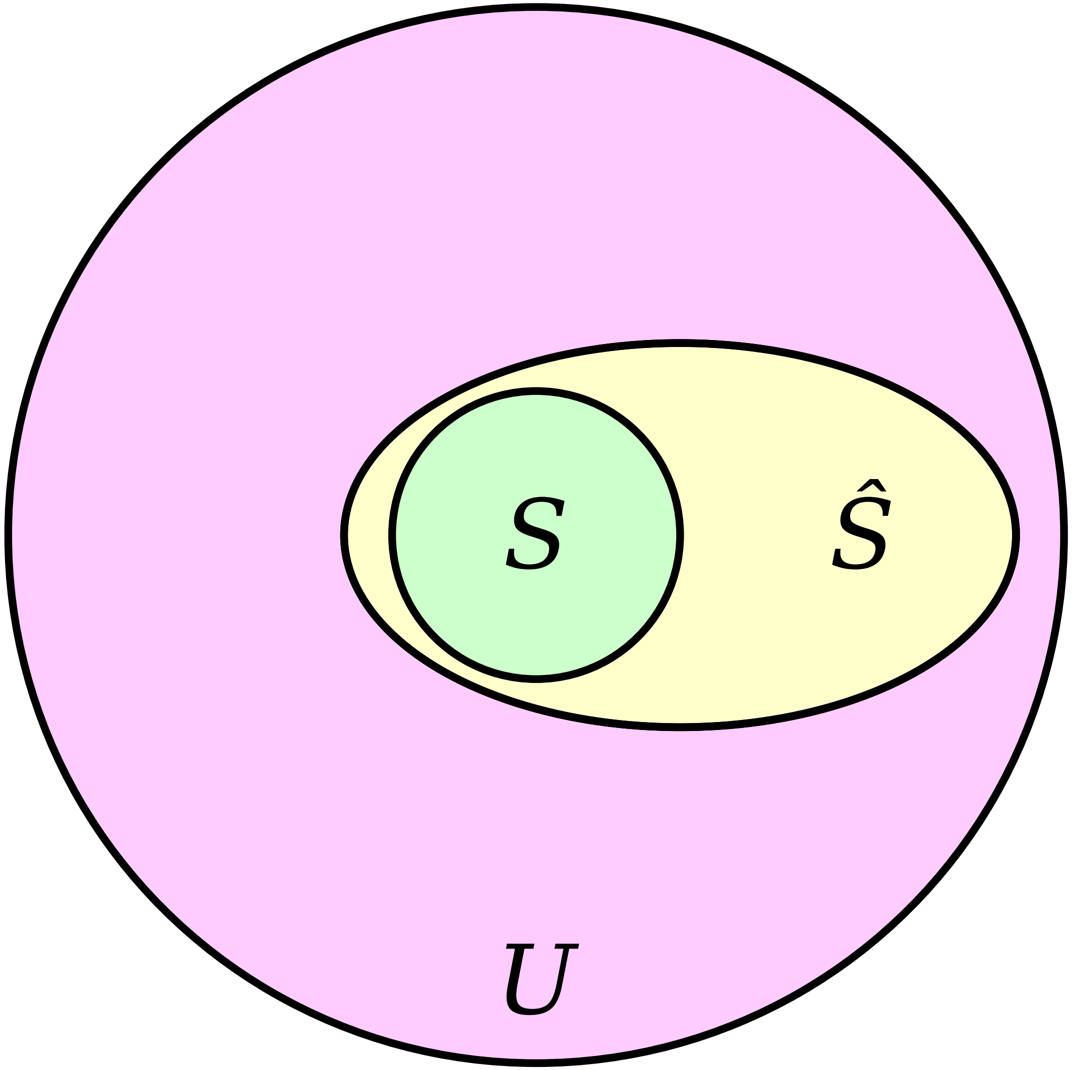

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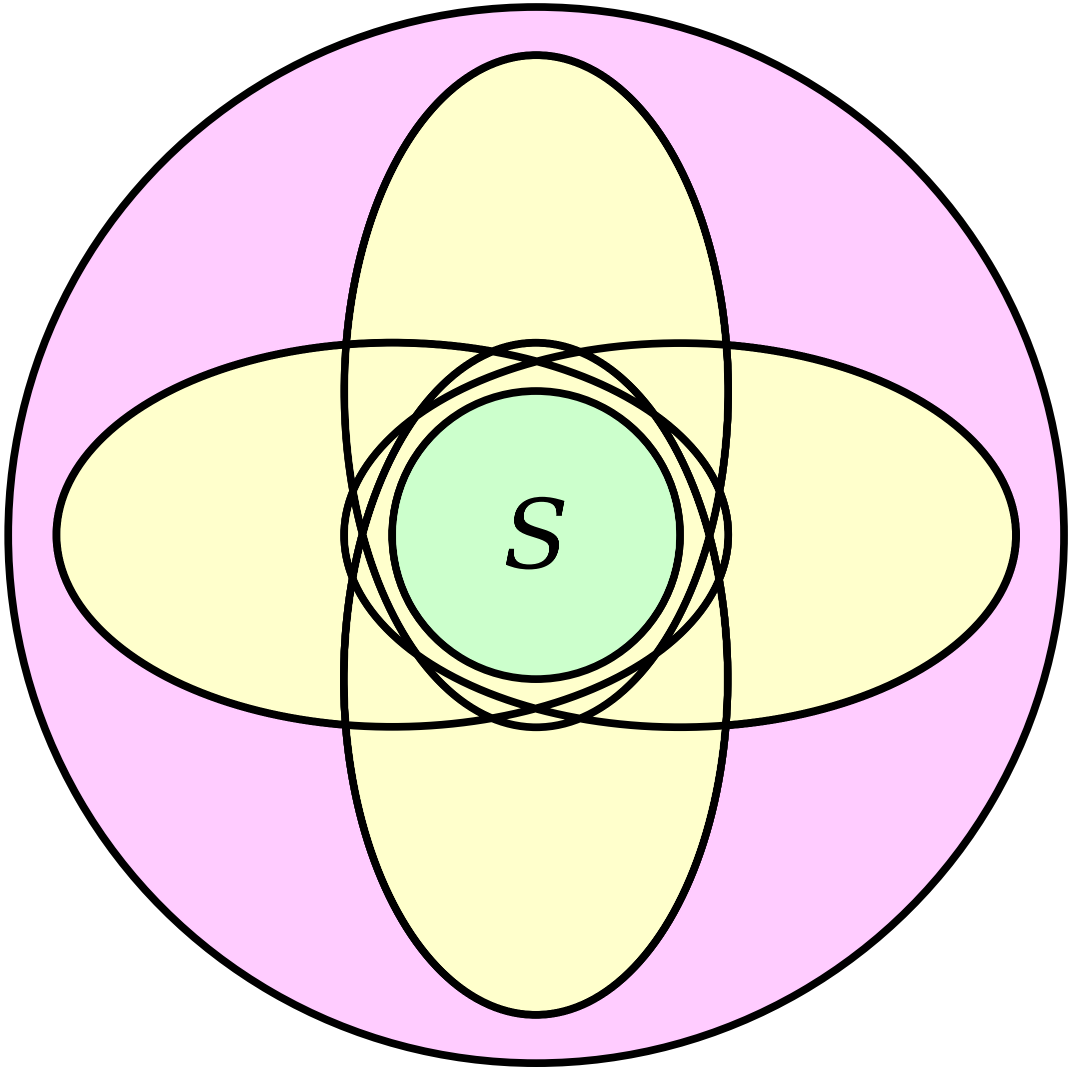

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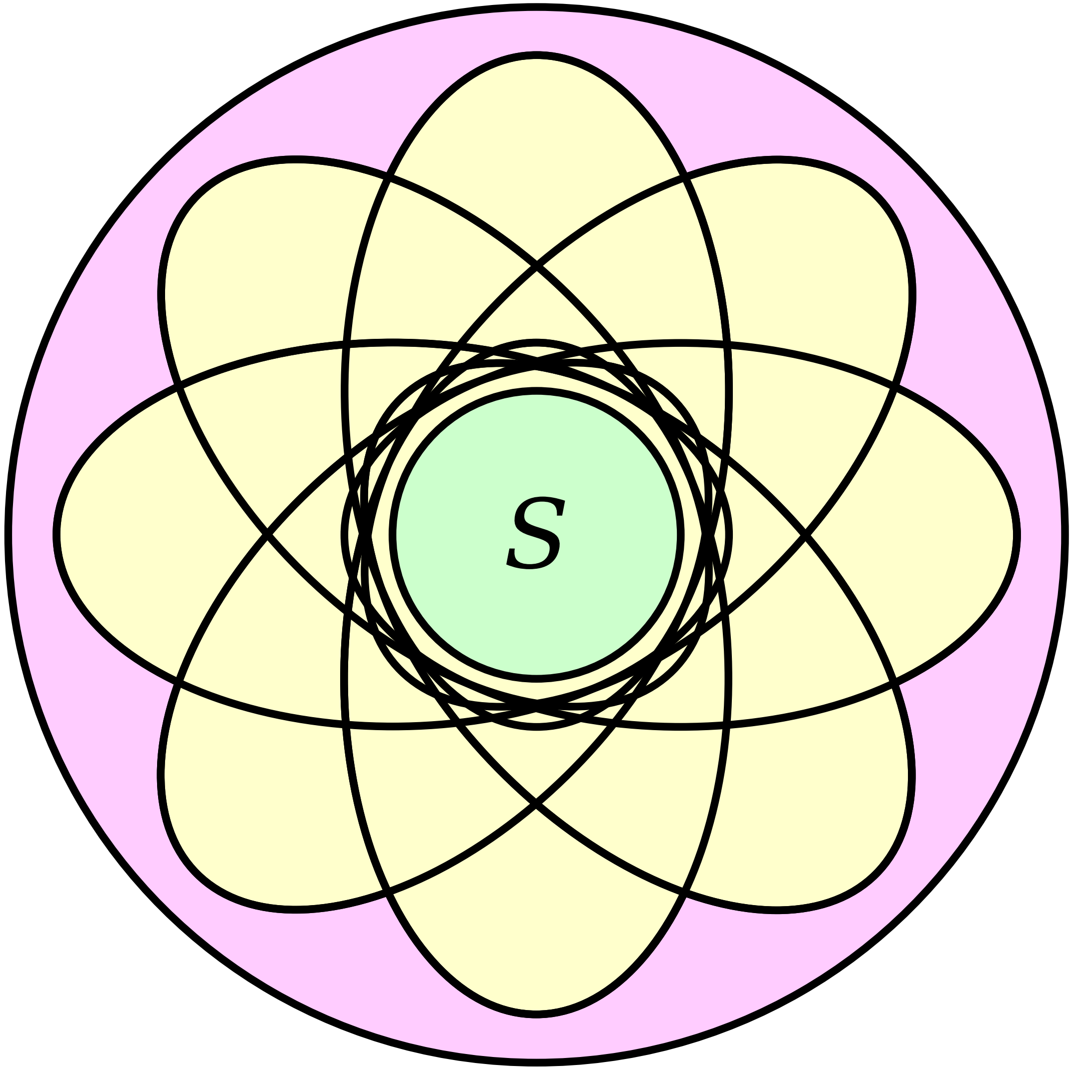

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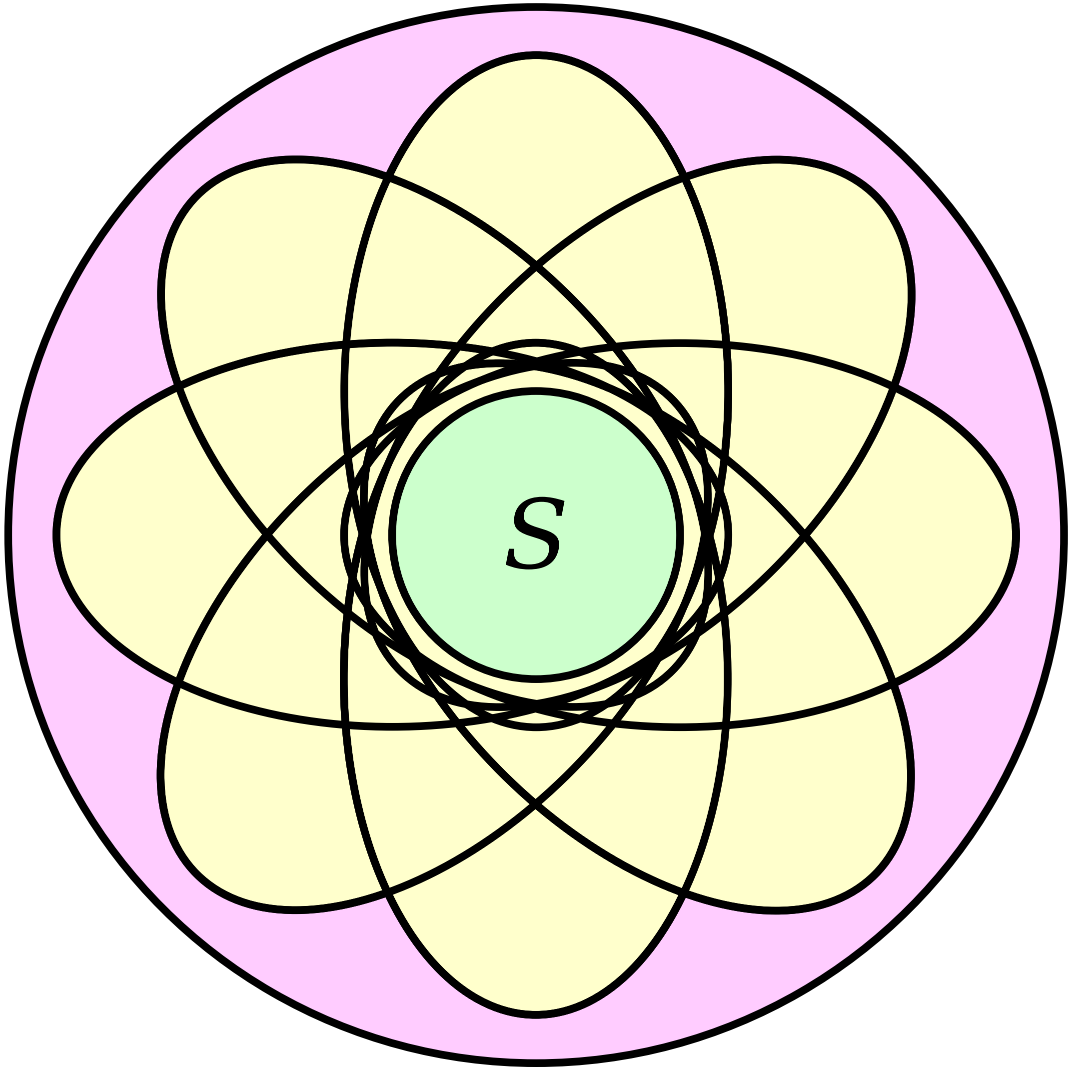
How does that affect our lower bound?



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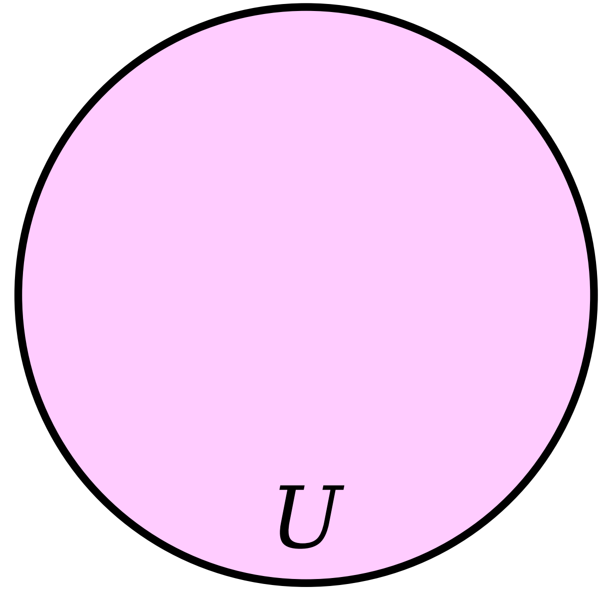

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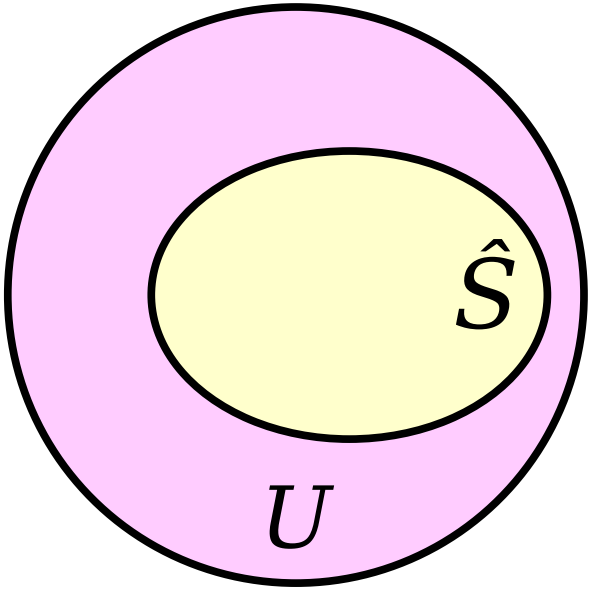

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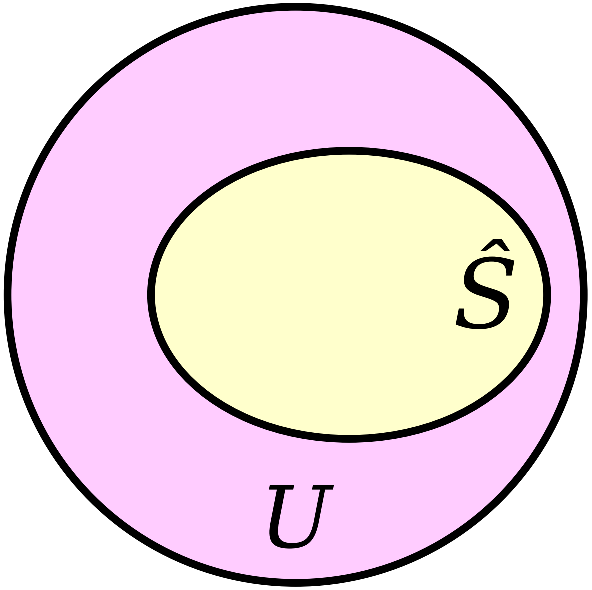

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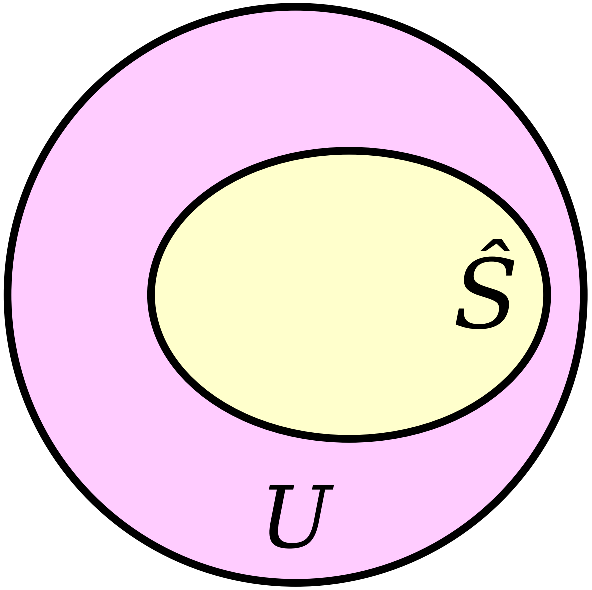
$$\lg \binom{\varepsilon |U|}{n}$$



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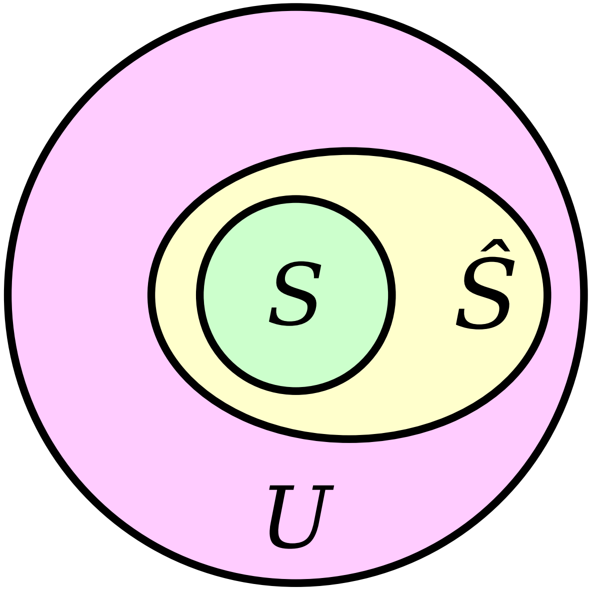

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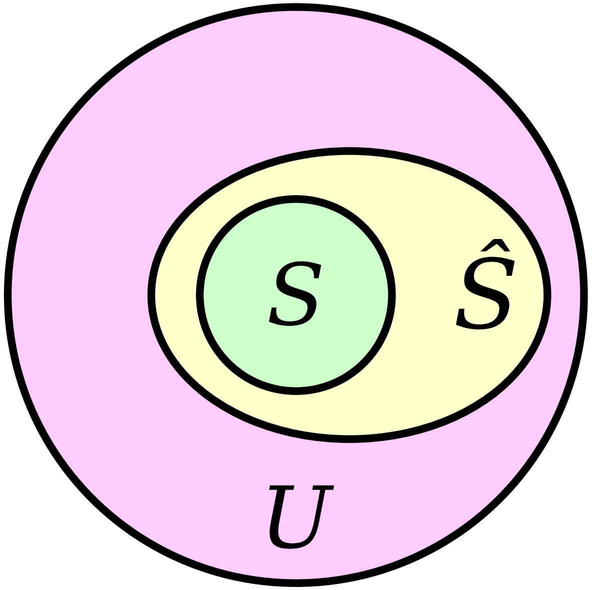

$$b + \lg \binom{\varepsilon|U|}{n}$$

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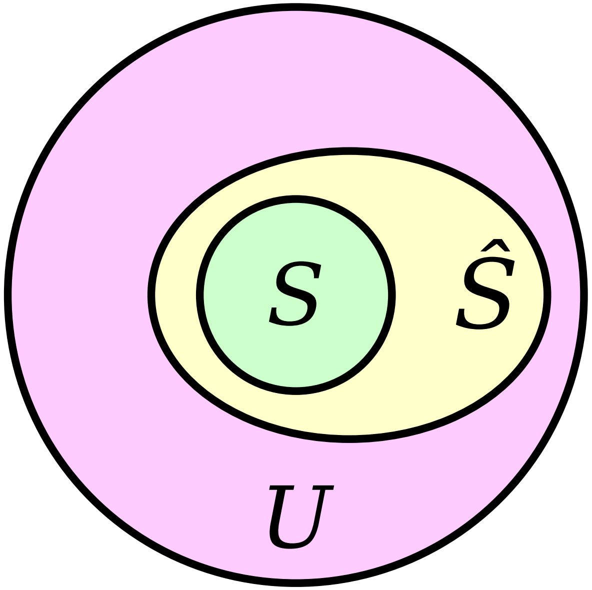

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(Number of bits needed to describe S.)

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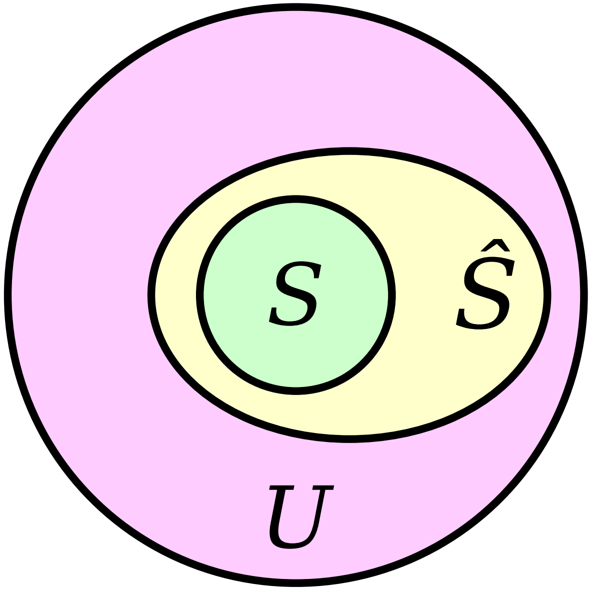

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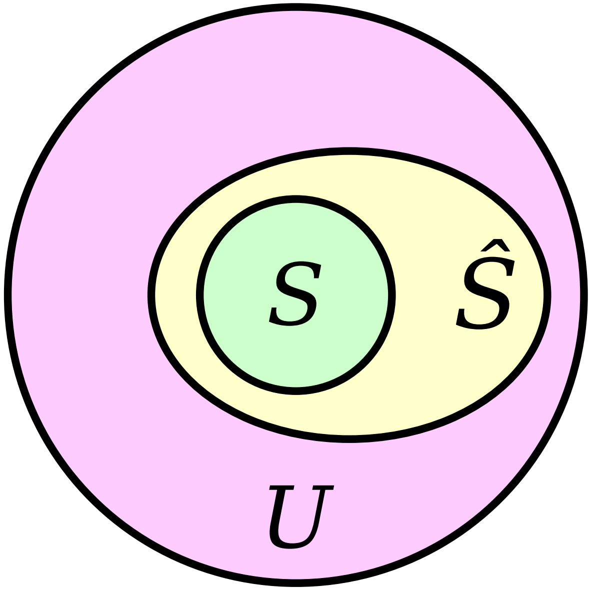

$$b + \lg \binom{\varepsilon|U|}{n} \geq \lg \binom{|U|}{n}$$

(Number of bits needed to describe S.)

(Lower bound on number of bits needed.)

How much memory is needed to solve the approximate membership query problem?

Theorem: Assuming $\varepsilon|U| \gg n$, any AMQ structure needs at least approximately $\boldsymbol{n \lg \varepsilon^{-1}}$ bits in the worst case.

This lower bound suggests we should aim for $\Theta(n \lg \varepsilon^{-1})$ bits of storage space.

How might we use them?

The math, if you're curious:

$$
\begin{aligned}
b &\geq \lg \left(\frac{\binom{|U|}{n}}{\binom{\varepsilon |U|}{n}} \right) \\
&\approx \lg \left(\frac{|U|^n / n!}{(\varepsilon |U|)^n / n!} \right) \\
&= \lg \varepsilon^{-n} \\
&= n \lg \varepsilon^{-1}
\end{aligned}
$$

How much memory is needed to solve the approximate membership query problem?

Bloom Filters

Idea 1: Adapt the "hash to a bucket" idea of the count-min and count sketches.

As an example, let's have

S = {103, 137, 166, 271, 314 }

`0000000000000000000000000000000000000000`

Number of bits: $\boldsymbol{cn \lg \varepsilon^{-1}}$

(We'll pick c later.)

How can we approximate a set in a small number of bits and with a low error rate?

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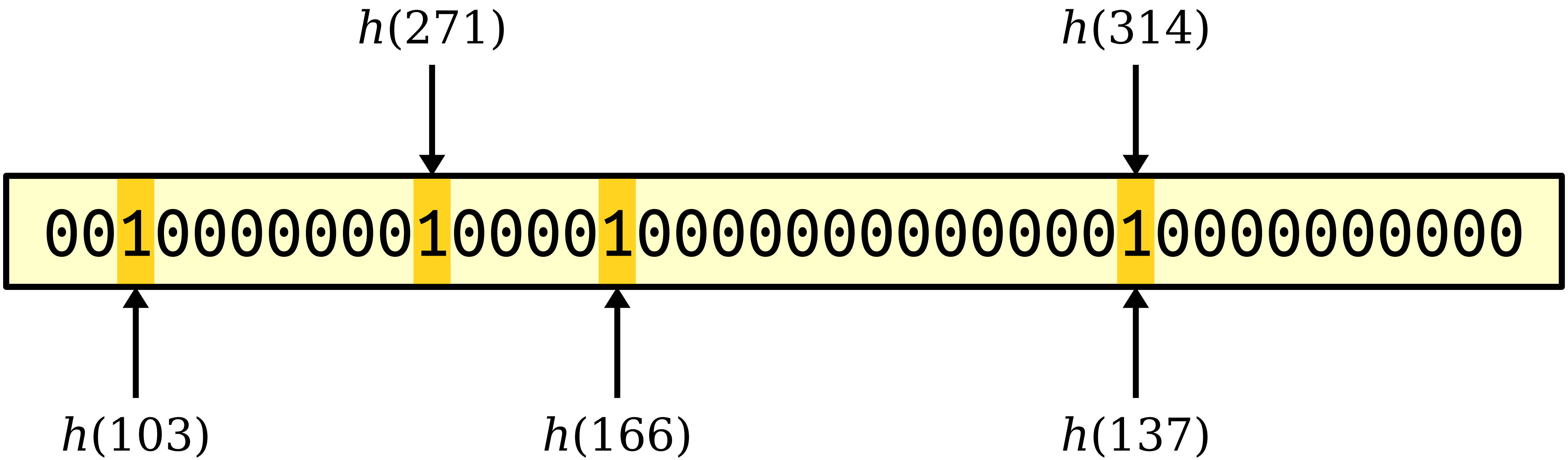

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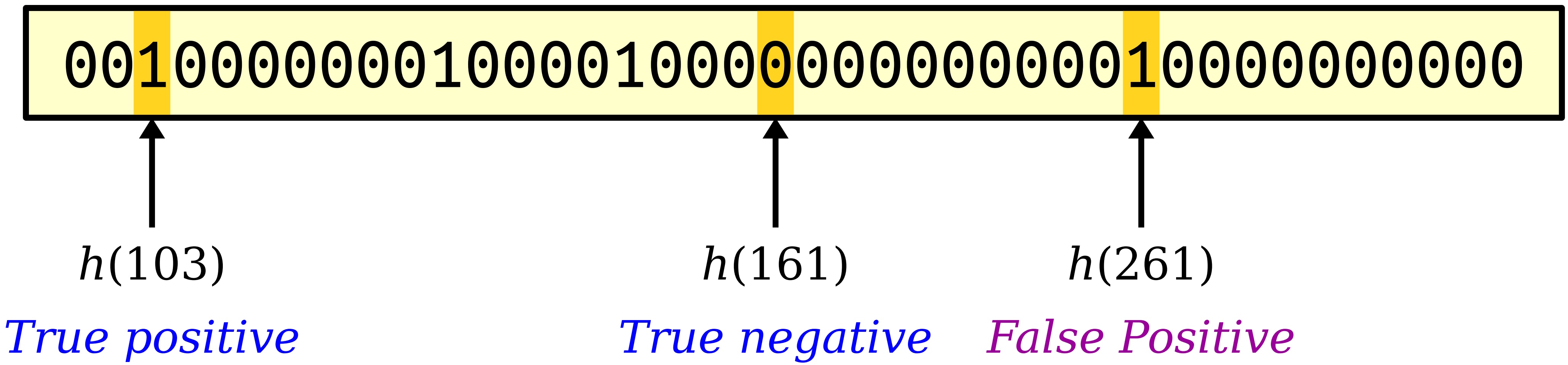

How can we approximate a set in a small number of bits and with a low error rate?

Idea 1: Adapt the "hash to a bucket" idea of the count-min and count sketches.

Suppose we store a set of n elements in collection of $cn \lg \varepsilon^{-1}$ bits.

Question: What is our false positive rate?

`0010000000100001000000000000010000000000`

Intuition: At most n of our $cn \lg \varepsilon^{-1}$ bits will be 1. We only have false positives if we see a 1. So our false positive rate will be about $1 / c \lg \varepsilon^{-1}$. That's not great.

How can we approximate a set in a small number of bits and with a low error rate?

Idea 2: Adapt the "run in parallel" approach of the count-min sketch.

Make several copies of the previous data structure, each with a random hash function.

`0001000000000000000011000000010000000001`

`0010000000100001000000000000010000000000`

`0000010000000000010000001000001000000100`

Can we get the same accuracy while using fewer bits?

Idea 2: Adapt the "run in parallel" approach of the count-min sketch.

Question: Each copy provides its own estimate. Which one should we pick?

```
00010000000000000000 … 0001
00100000001000010000 … 0000
0000010000000000010000001000001000000100
```

This entry is 0, so it's not possible for this element to be in our set *S*.

Can we get the same accuracy while using fewer bits?

Idea 2: Adapt the "run in parallel" approach of the count-min sketch.

Question: Each copy provides its own estimate. Which one should we pick?

```
0001000000000000000011000000010000000001
```

```
0010000000100001000000000000010000000000
```

```
0000010000000000010000001000001000000100
```

Can we get the same accuracy while using fewer bits?

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```
0001000000000000000011000000010000000001
```

```
0010000000100001000000000000010000000000
```

```
0000010000000000010000001000001000000100
```

Can we get the same accuracy while using fewer bits?

Idea 2: Adapt the “run in parallel” approach of the count-min sketch.

Question: Each copy provides its own estimate. Which one should we pick?

`0001000000000000000011000000010000000001`

`0010000000100001000000000000010000000000`

`00000100000000000100000010000`

We only say “yes” if all bits are 1’s.

Can we get the same accuracy while using fewer bits?

Idea 2: Adapt the "run in parallel" approach of the count-min sketch.

We have some fixed number of bits to use. How should we split them across these copies?

`0010000000100001000000000000010000000000`

Can we get the same accuracy while using fewer bits?

Idea 2: Adapt the "run in parallel" approach of the count-min sketch.

We have some fixed number of bits to use. How should we split them across these copies?

`0010010000100000100010`

`1001100001000000001000`

Can we get the same accuracy while using fewer bits?

Idea 2: Adapt the "run in parallel" approach of the count-min sketch.

We have some fixed number of bits to use. How should we split them across these copies?

```
0010001100100100
0101000010001010
0000011100000010
```

Can we get the same accuracy while using fewer bits?

Idea 2: Adapt the "run in parallel" approach of the count-min sketch.

We have some fixed number of bits to use. How should we split them across these copies?

```
001000110010
001000110010
010100001000
000001110000
```

More copies means fewer bits per copy. The error rate per copy is higher, but there's more copies.

Can we get the same accuracy while using fewer bits?

Idea 2: Adapt the "run in parallel" approach of the count-min sketch.

Approach: Use one giant array. Have all hash functions edit and read that array.

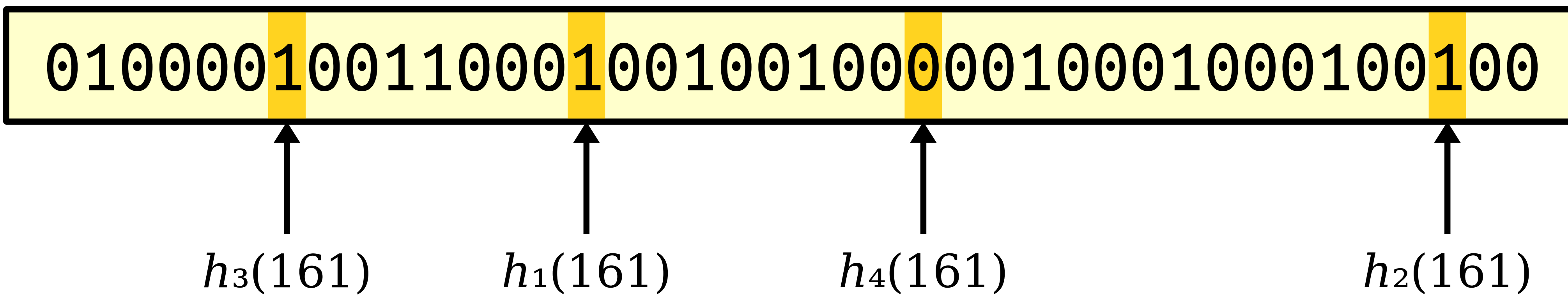
This is called a ***Bloom filter***, named after its inventor.

`01000010011000100100100000010001000100100`

Can we get the same accuracy while using fewer bits?

Assume we use k hash functions, each of which is chosen independently of the others. We'll pick k later on.

(In this example, $k = 4$.)

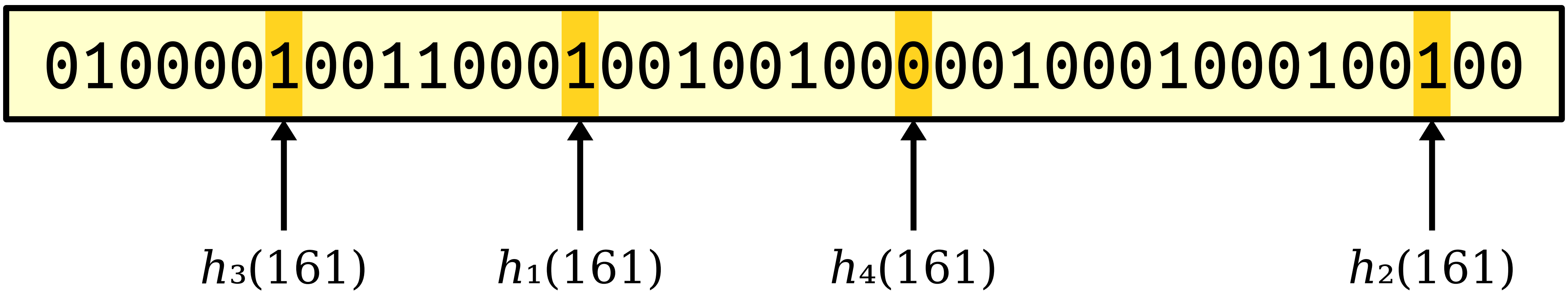


Can we get the same accuracy while using fewer bits?

create(S): Select k hash functions. Hash each element with all hash functions and set the indicated bits to 1.

query(x): Hash x with all k hash functions.

Return whether all the indicated bits are 1.



Can we get the same accuracy while using fewer bits?

We have two knobs to turn: the coefficient c in the number of bits $cn \lg \varepsilon^{-1}$, and the number of hash functions k.

Intuition: If c is too low, we'll get too many false positives. If c is too large, we'll use too much memory.

`01000010011000100100100000010001000100100`

Can we get the same accuracy while using fewer bits?

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`1000100010001100101010001000101`

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`110011001111000111000101`

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111111

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`01000010011000100100100000010001000100100`

Can we get the same accuracy while using fewer bits?

We have two knobs to turn: the coefficient c in the number of bits $cn \lg \varepsilon^{-1}$, and the number of hash functions k.

Intuition: If k is either too low *or* too high, we'll get too many false positives.

`0100001001100010010010000001000100010010`

Can we get the same accuracy while using fewer bits?

We have two knobs to turn: the coefficient c in the number of bits $cn \lg \varepsilon^{-1}$, and the number of hash functions k.

Intuition: If k is either too low *or* too high, we'll get too many false positives.

`010100100110101001101000101000100011010 0`

Can we get the same accuracy while using fewer bits?

We have two knobs to turn: the coefficient c in the number of bits $cn \lg \varepsilon^{-1}$, and the number of hash functions k.

Intuition: If k is either too low *or* too high, we'll get too many false positives.

`0101001111101010011010001011001000110111`

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1111111111111111111111111111111111111111

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`0100001001100010010010000010001000100100`

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```
0000001001000010000000000000001000000100
```

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We have two knobs to turn: the coefficient c in the number of bits $cn \lg \varepsilon^{-1}$, and the number of hash functions k.

Intuition: If k is either too low *or* too high, we'll get too many false positives.

Question: How do we tune k, the number of hash functions?

`0100001001100010010010000001000100010100`

Can we get the same accuracy while using fewer bits?

In what follows, let's have

$$\boldsymbol{m = cn \lg \varepsilon^{-1}}$$

be the number of bits in our array.

How do we quantify our error rate?

Question: In what circumstance do we get a false positive?

Answer: Each of the element's bits are set, but the element isn't in the set S.

Question: What is the probability that this happens?

`0100001001100010010010000001000100010 0100`

How do we quantify our error rate?

Question 1: What is the probability that any particular bit is set?

`0011010100101000`

Focus on a bit at index i.

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Focus on a bit at index i.

Pick some $x \in S$ and hash function h.

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Focus on a bit at index i.

Pick some $x \in S$ and hash function h.

What's the probability that $h(x) \neq i$? (Assume truly random hash functions.)

How do we quantify our error rate?

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Focus on a bit at index i.

Pick some $x \in S$ and hash function h.

What's the probability that $h(x) \neq i$? (Assume truly random hash functions.)

Answer: $1 - 1/m$.

How do we quantify our error rate?

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`0011010100 0101000`

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Answer: $1 - {}^{1}/_{m}$.

What's the probability that, across all n elements and k hash functions, bit i isn't set?

How do we quantify our error rate?

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What's the probability that $h(x) \neq i$? (Assume truly random hash functions.)

Answer: $1 - {}^1/_m$.

What's the probability that, across all n elements and k hash functions, bit i isn't set?

Answer: $(1 - {}^1/_m)^{kn}$.

How do we quantify our error rate?

Question 1: What is the probability that any particular bit is set?

Useful fact: $(1 - {}^{1}/_{p})^{p} \approx e^{-1}$.

```
0011010100101000
```

How do we quantify our error rate?

Question 1: What is the probability that any particular bit is set?

```
0011010100010100
```

Useful fact: $(1 - {}^{1}/_{p})^{p} \approx e^{-1}$.

Probability that bit i is unset after inserting n elements:

$$\left(1 - \frac{1}{m}\right)^{kn}$$

How do we quantify our error rate?

Question 1: What is the probability that any particular bit is set?

```
0011010100101000
```

Useful fact: $(1 - {}^{1}/_{p})^{p} \approx e^{-1}$.

Probability that bit *i* is unset after inserting *n* elements:

$$\left(1 - \frac{1}{m}\right)^{kn}$$

$$= \left(\left(1 - \frac{1}{m}\right)^{m}\right)^{\frac{kn}{m}}$$

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$$\approx e^{-\frac{kn}{m}}$$

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```
00110101000101000
```

Useful fact: $(1 - {}^{1}/_{p})^{p} \approx e^{-1}$.

Probability that bit i is unset after inserting n elements:

$$
\begin{aligned}
& \left(1 - \frac{1}{m}\right)^{kn} \\
= \quad & \left(\left(1 - \frac{1}{m}\right)^{m}\right)^{\frac{kn}{m}} \\
\approx \quad & e^{-\frac{kn}{m}} \\
= \quad & e^{-k/c \lg \varepsilon^{-1}}
\end{aligned}
$$

How do we quantify our error rate?

Question 2: What is the probability of a false positive?

`0011010100010100 0`

Probability that a fixed bit is 1 after n elements have been added:

$$\approx \; 1 - e^{-k / c \lg \varepsilon^{-1}}$$

How do we quantify our error rate?

Question 2: What is the probability of a false positive?

`00110101000101000`

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False positive probability is approximately

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False positive probability is approximately

$$(1 - e^{-k / c \lg \varepsilon^{-1}})^k$$

How do we quantify our error rate?

Question 2: What is the probability of a false positive?

`0011010100001010000`

This value isn't exactly correct because certain bits being 1 decrease the probability that other bits are 1. With a more advanced analysis we can show that this is very close to the true value.

Probability that a fixed bit is 1 after n elements have been added:

$$\approx \ 1 - e^{-k / c \lg \varepsilon^{-1}}$$

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$$\approx \; 1 - e^{-k / c \lg \varepsilon^{-1}}$$

False positive probability is approximately

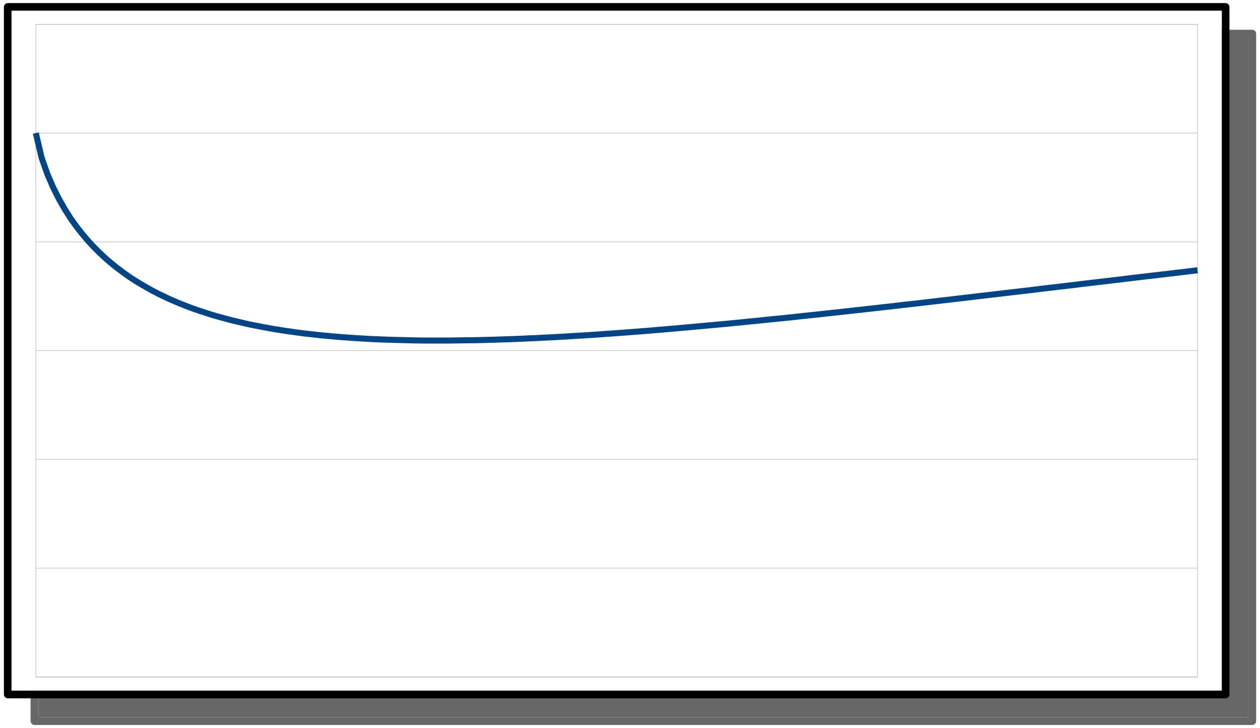
$$(1 - e^{-k / c \lg \varepsilon^{-1}})^k$$

Question: What choice of k minimizes this expression?

How do we quantify our error rate?

Goal: Pick k to minimize

$$(1 - e^{-k / c \lg \varepsilon^{-1}})^k.$$



How many hash functions should we use?

Goal: Pick k to minimize

$$(1 - e^{-k / c \lg \varepsilon^{-1}})^k.$$

Claim: This expression is minimized when

$$\boldsymbol{k = c \lg \varepsilon^{-1} \ln 2}.$$

You can show this using some symmetry arguments or calculus.

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Good exercise: This claim is often repeated and seldom proved. Confirm I am not perpetuating lies.

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$$(1 - e^{-k / c \lg \varepsilon^{-1}})^k.$$

Claim: This expression is minimized when

$$\boldsymbol{k = c \lg \varepsilon^{-1} \ln 2}.$$

You can show this using some symmetry arguments or calculus.

Good exercise: This claim is often repeated and seldom proved. Confirm I am not perpetuating lies.

Challenge: Give an explanation for this result that is “immediately obvious” from the original expression.

How many hash functions should we use?

The false positive rate is

$$(1 - e^{-k / c \lg \varepsilon^{-1}})^k.$$

and we know to pick

$$k = c \lg \varepsilon^{-1} \ln 2.$$

Plugging this value into the expression gives a false positive rate of

$$\varepsilon^{c \ln 2}$$

(The derivation, for those of you who are curious.)

$$\begin{aligned}
& \left(1 - e^{-k/c \lg \varepsilon^{-1}}\right)^k \\
= \; & \left(1 - e^{-c \lg \varepsilon^{-1} \ln 2 / c \lg \varepsilon^{-1}}\right)^k \\
= \; & \left(1 - e^{-\ln 2}\right)^k \\
= \; & \left(1 - \frac{1}{2}\right)^k \\
= \; & 2^{-k} \\
= \; & 2^{-c \lg \varepsilon^{-1} \ln 2} \\
= \; & 2^{\lg \varepsilon^{c \ln 2}} \\
= \; & \varepsilon^{c \ln 2}
\end{aligned}$$

Knowing what we know now, how many bits do we need to get a false positive rate of ε?

Our false positive rate, as a function of c, is

$$\varepsilon^{c \ln 2}.$$

Our goal is to get a false positive rate of ε.

To do so, pick

$$\boldsymbol{c = 1 / \ln 2}$$
$$\boldsymbol{= \lg e}$$

Which is about 1.44.

Knowing what we know now, how many bits do we need to get a false positive rate of ε?

Given a number of elements n and an error rate ε, pick

$$m = cn \lg \varepsilon^{-1}$$

$$k = c \lg \varepsilon^{-1} \ln 2.$$

Optimal c:

$$c = \lg e$$

How did we do overall?

Given a number of elements n and an error rate ε, pick

$$m = n \lg \varepsilon^{-1} \lg e$$

$$k = \lg \varepsilon^{-1}$$

Optimal c:

$$c = \lg e$$

How did we do overall?

Given a number of elements n and an error rate ε, pick

$$m = n \lg \varepsilon^{-1} \lg e$$

$$k = \lg \varepsilon^{-1}$$

Space usage: **$\Theta(n \log \varepsilon^{-1})$**. *(Within a factor of 1.44 of optimal!)*

Query time: **$O(\log \varepsilon^{-1})$**.

More recent data structures, like the ***quotient filter*** and ***cuckoo filter***, push this coefficient down and are still viable in practice.

We know that it's possible to reduce this coefficient to $1 + o(1)$ (***Pagh, Pagh, and Rao's optimal AMQ structure***), though this is currently only of theoretical interest.

How did we do overall?

To Summarize

- ***Chebyshev's inequality*** is a useful concentration inequality for two-sided errors. It involves bounding the variance of the underlying variable.
- To simplify variance expressions, look for pairwise independence.
- Some random variables (± 1 variables, indicator variables, etc.) simplify nicely when squared.
- Taking the median of many estimators with two-sided error is a good way to amplify success probabilities. The ***Chernoff bound*** is helpful for showing how effective this strategy is.
- Information-theoretic lower bounds can be helpful in determining how much progress can be made with a data structure.

Next Time

- ***Cuckoo Hashing***
  - Hashing with worst-case O(1) lookups.
- ***Random Graph Theory***
  - Properties of random bipartite graphs.