Cuckoo Hashing

Randomized Data Structures

- Randomization is a powerful tool for improving efficiency and solving problems under seemingly impossible constraints.
- Over the next three lectures, we'll explore a sampler of data structures that give a feel for the breadth of what's out there.
- You can easily spend an entire academic career just exploring this space; take CS265 for more on randomized algorithms!

Where We're Going

- ***Cuckoo Hashing (Today)***
  - Worst-case efficient hashing and deep properties of random graphs.
- ***Frequency Estimation (Next Week)***
  - Counting without counting, and how much randomness is needed to do it.

Outline for Today

- ***Cuckoo Hashing***
  - A simple, fast hashing system with worst-case efficient lookups.
- ***The Erdős–Rényi Model***
  - Randomly-generated graphs and their properties.
- ***Variants on Cuckoo Hashing***
  - Making a good idea even better.

Preliminaries: Hash Tables

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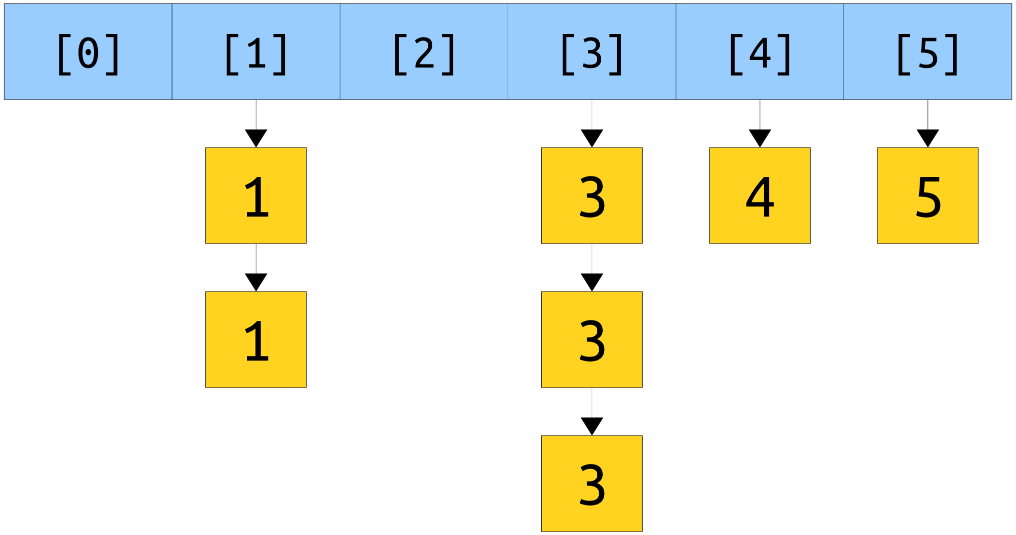

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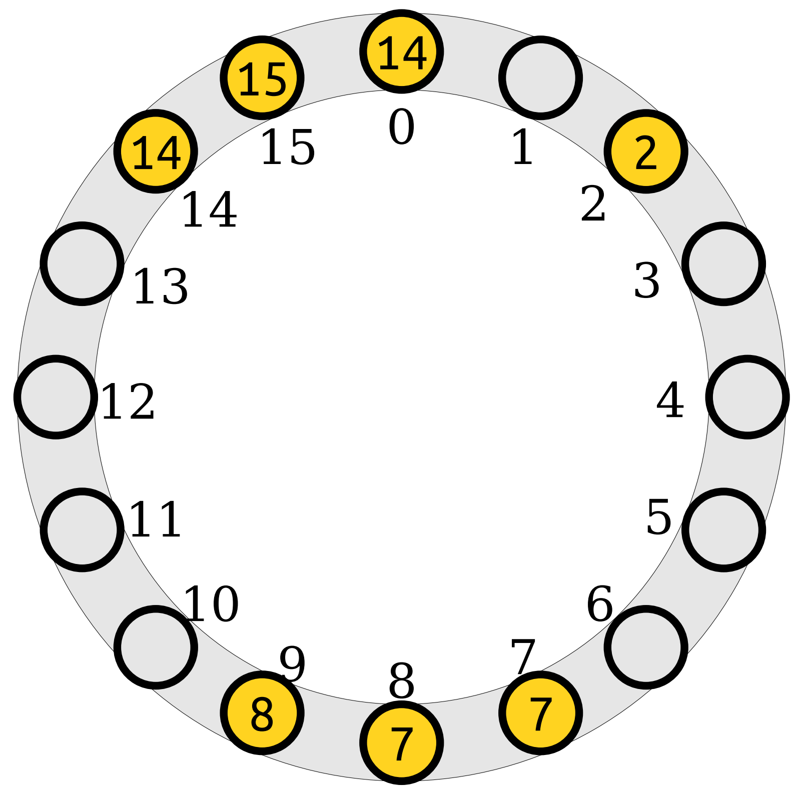

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  - ***Perfect hashing:*** Do something clever with multiple hash functions to eliminate collisions.
- What does that last option look like?

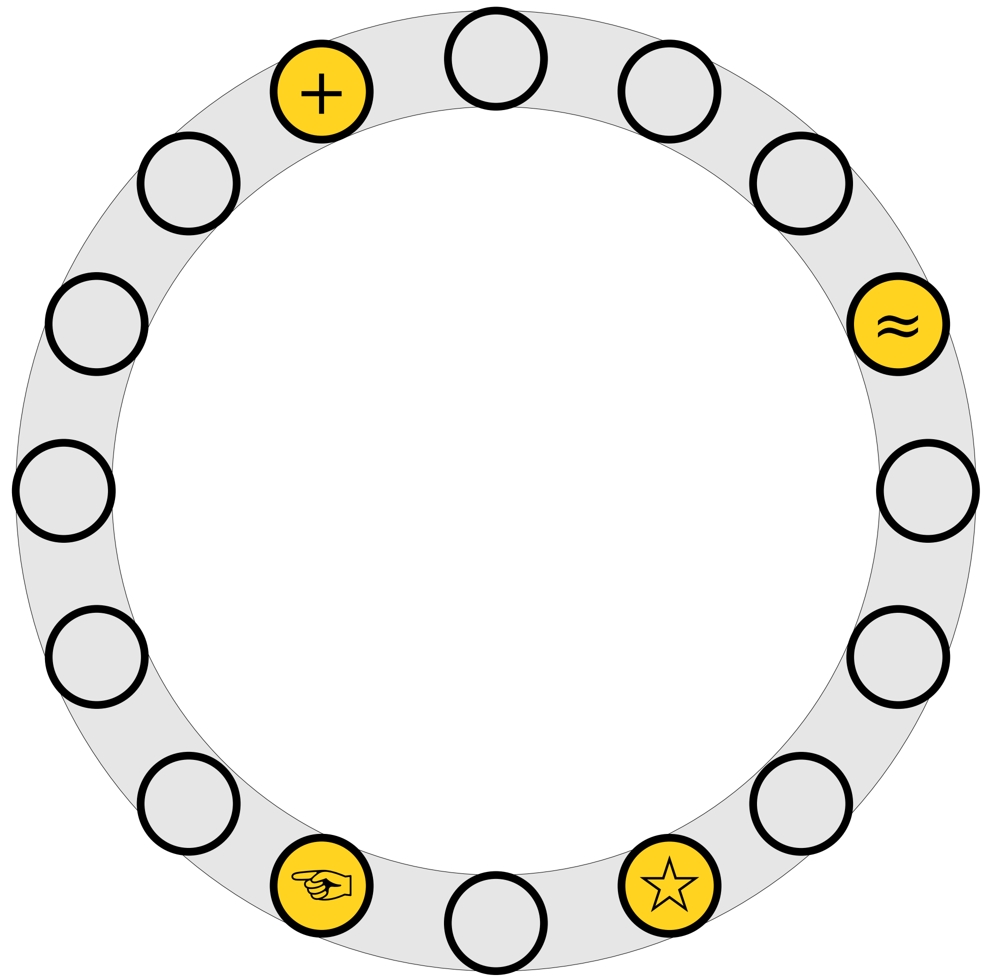
Cuckoo Hashing

Cuckoo Hashing

- Suppose we have a hash table with m slots.
- Unlike a normal hash table, we'll use *two* hash functions. We'll call them h_1 and h_2.
- Each hash function outputs a slot number in the set $\{ 0, 1, 2, \ldots, m - 1 \}$.
- We'll assume that these hash functions are truly random, with one constraint:

  $\boldsymbol{h_1(x) \neq h_2(x)}$ **for any key** $\boldsymbol{x}$**.**

- This is actually pretty easy to achieve both in theory and in practice – more on that later.



Cuckoo Hashing

- ***The Rule:*** Any item x must either be at position $h_1(x)$ or position $h_2(x)$ in the table.
- Lookups take *worst-case* O(1) time, since only two locations need to be checked.
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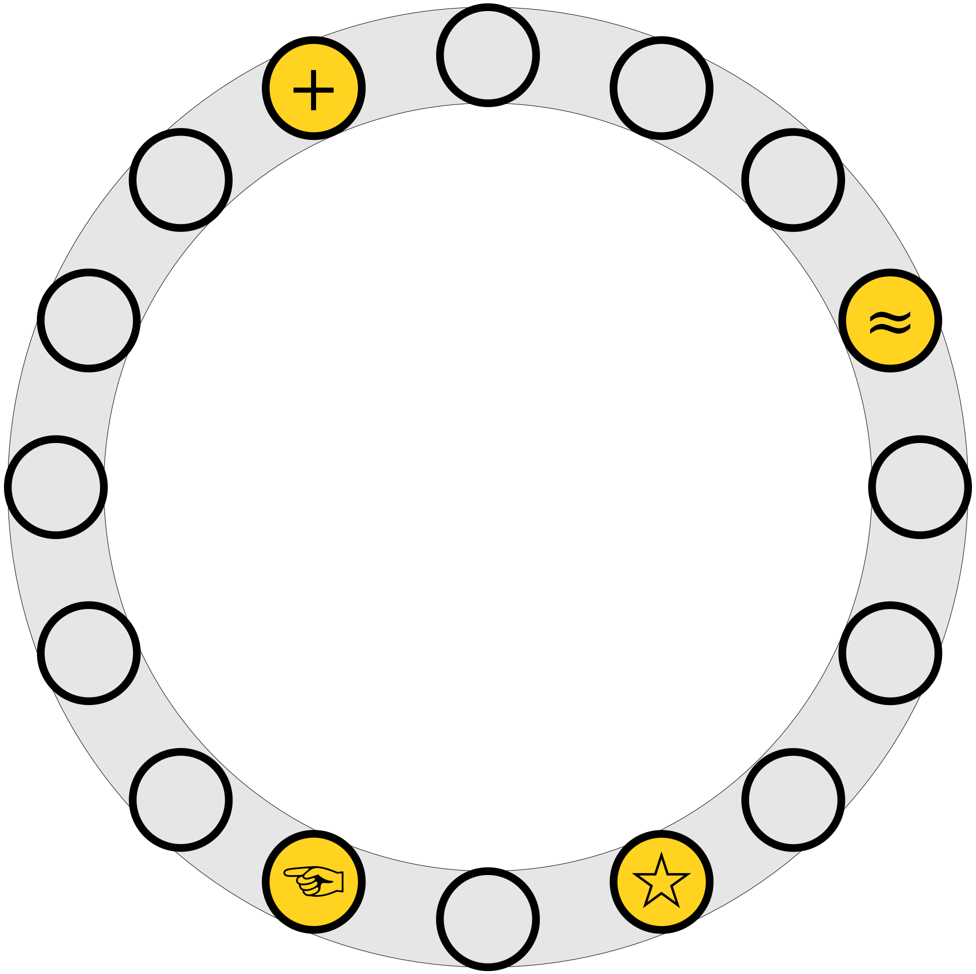

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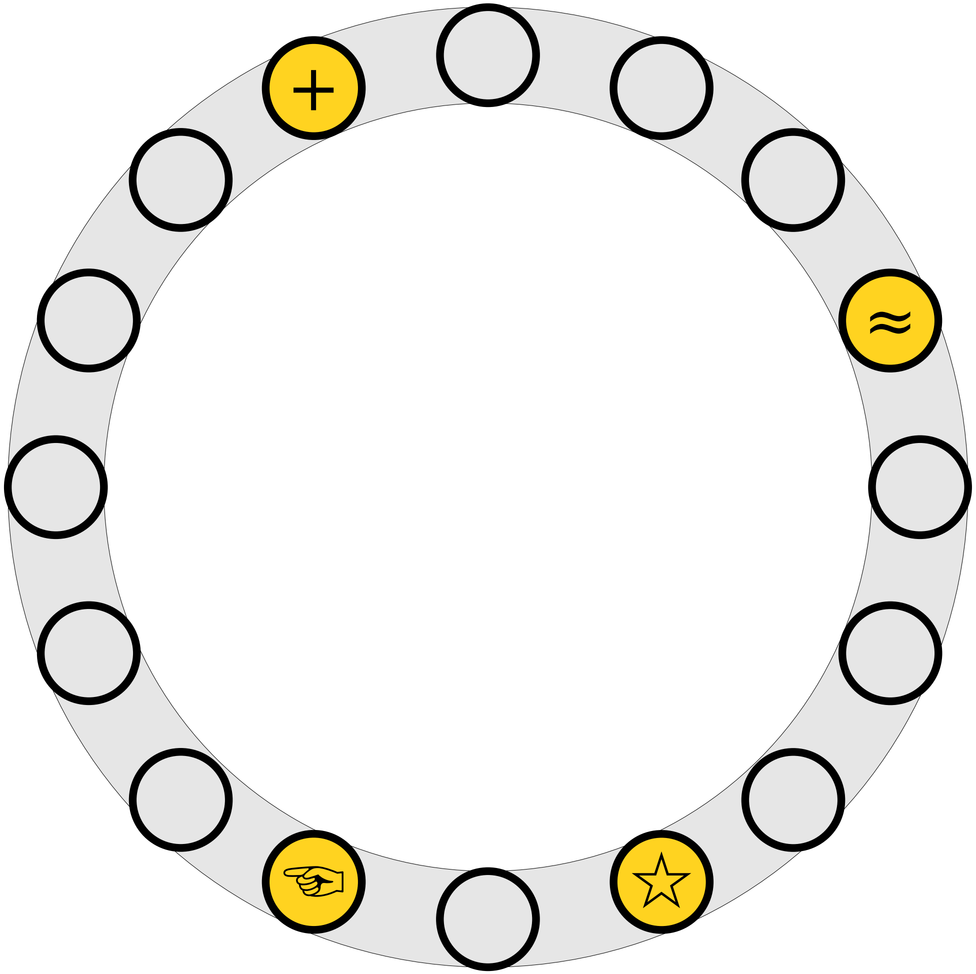

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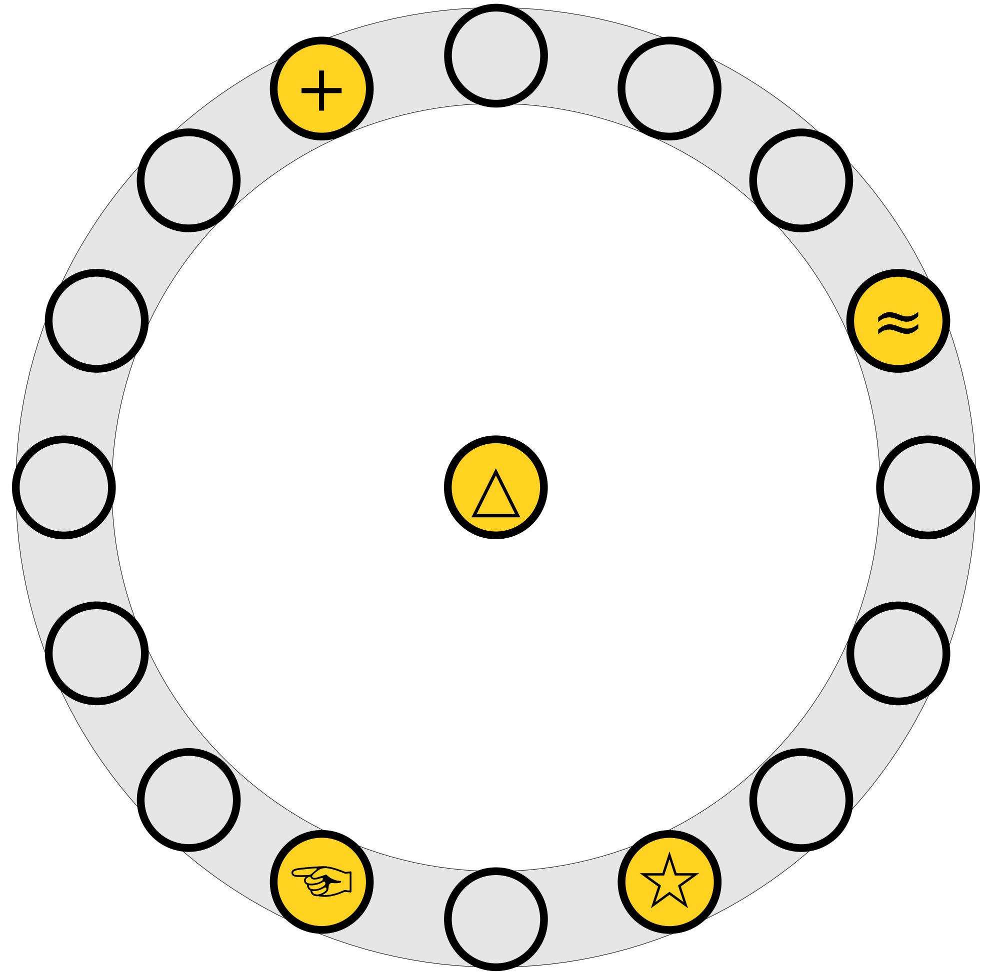

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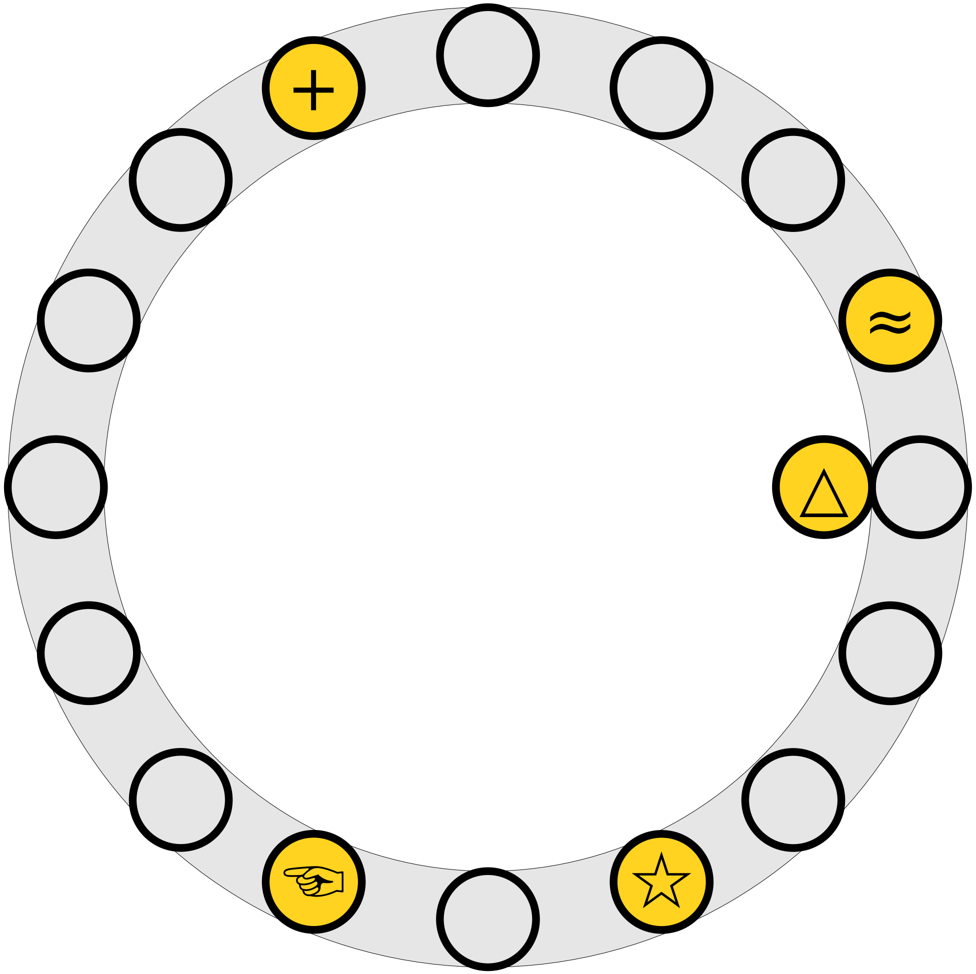

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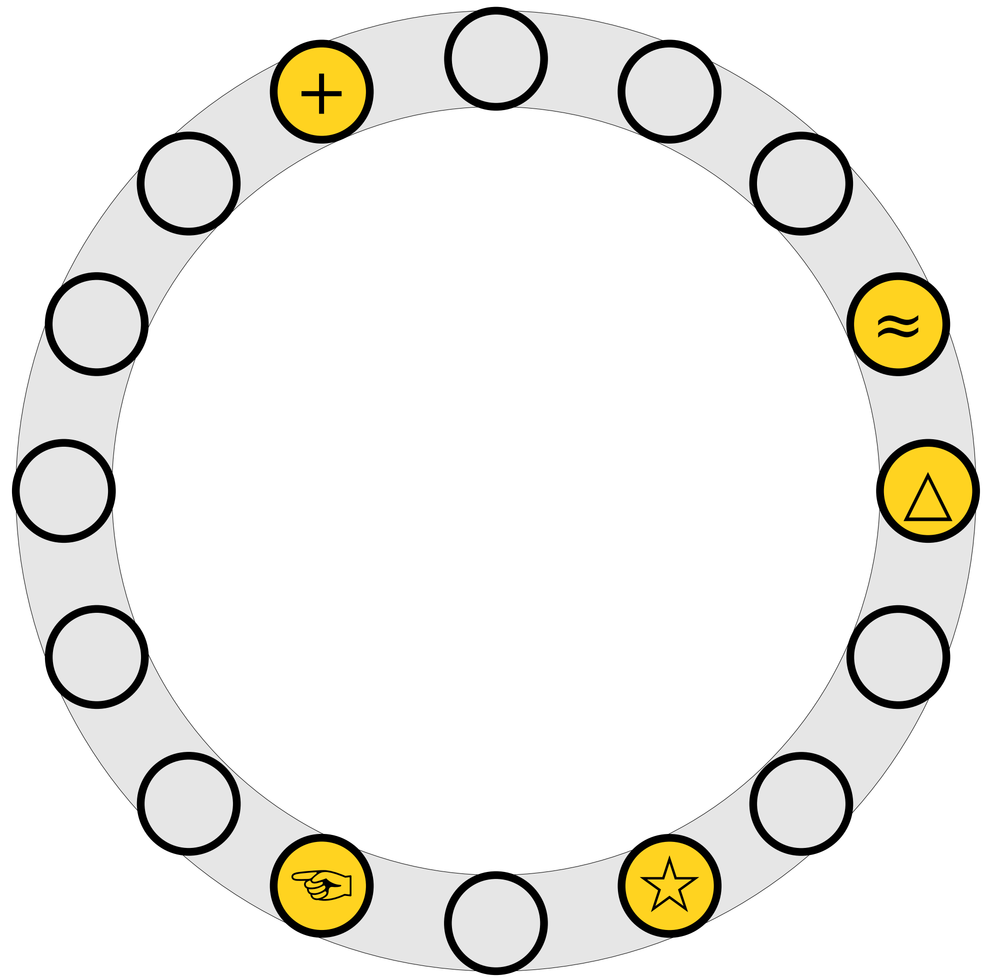

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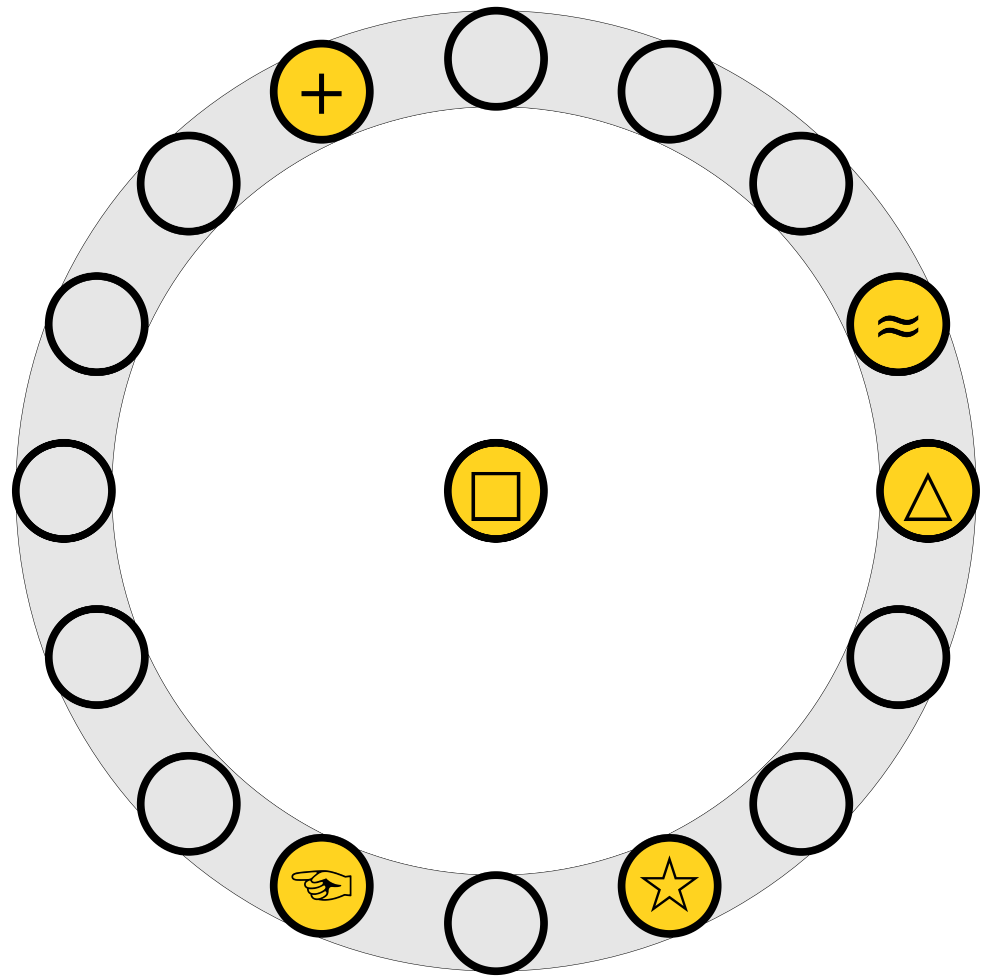

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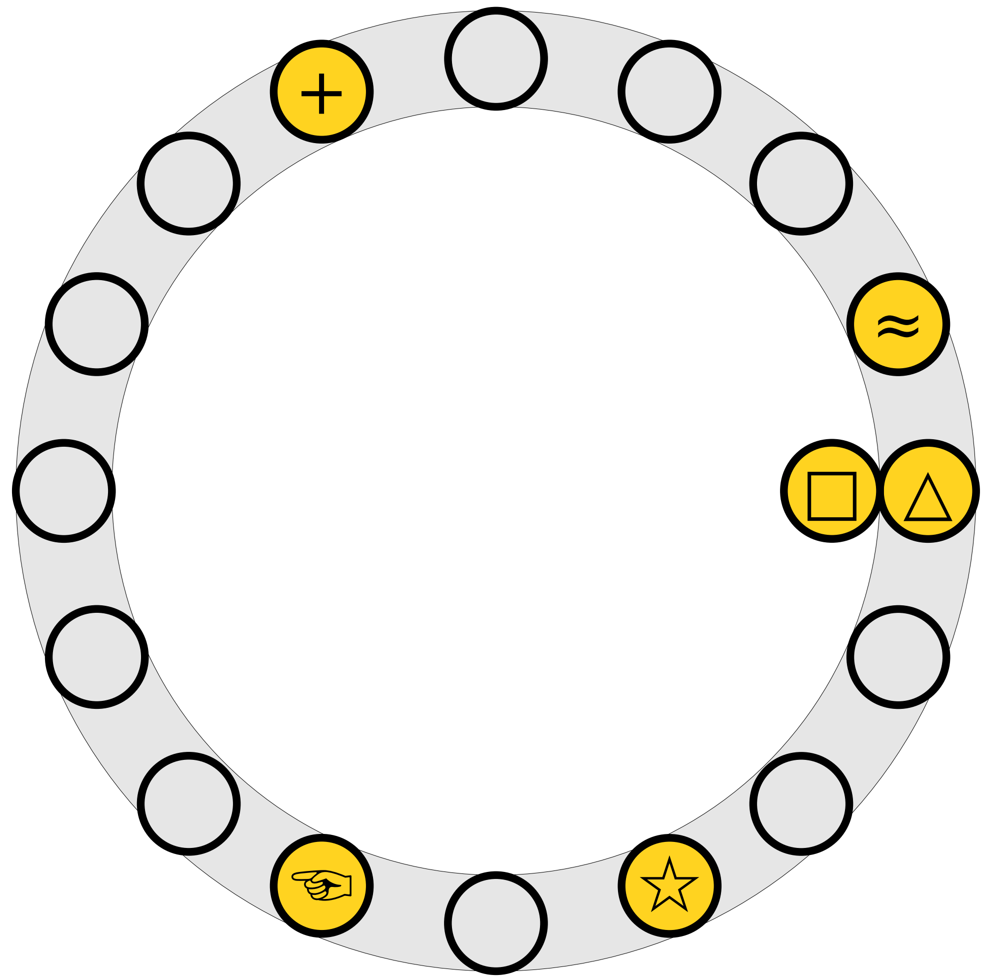

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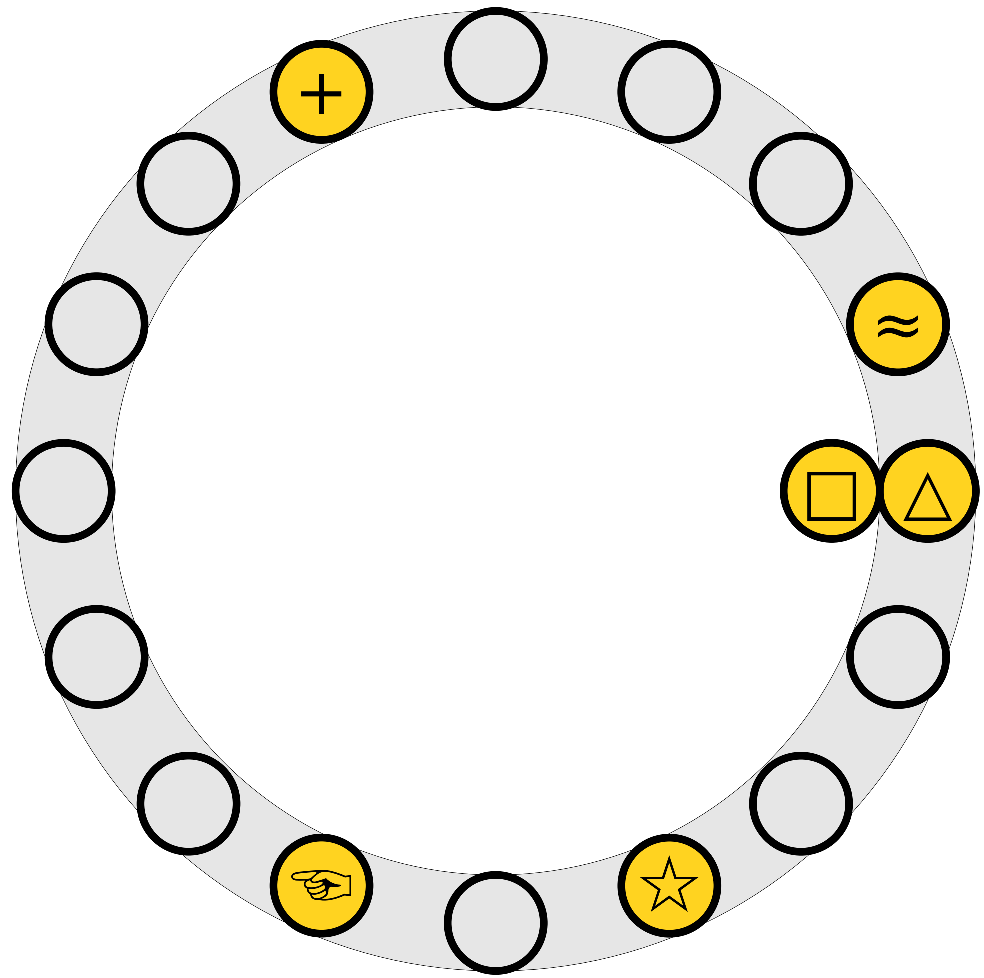

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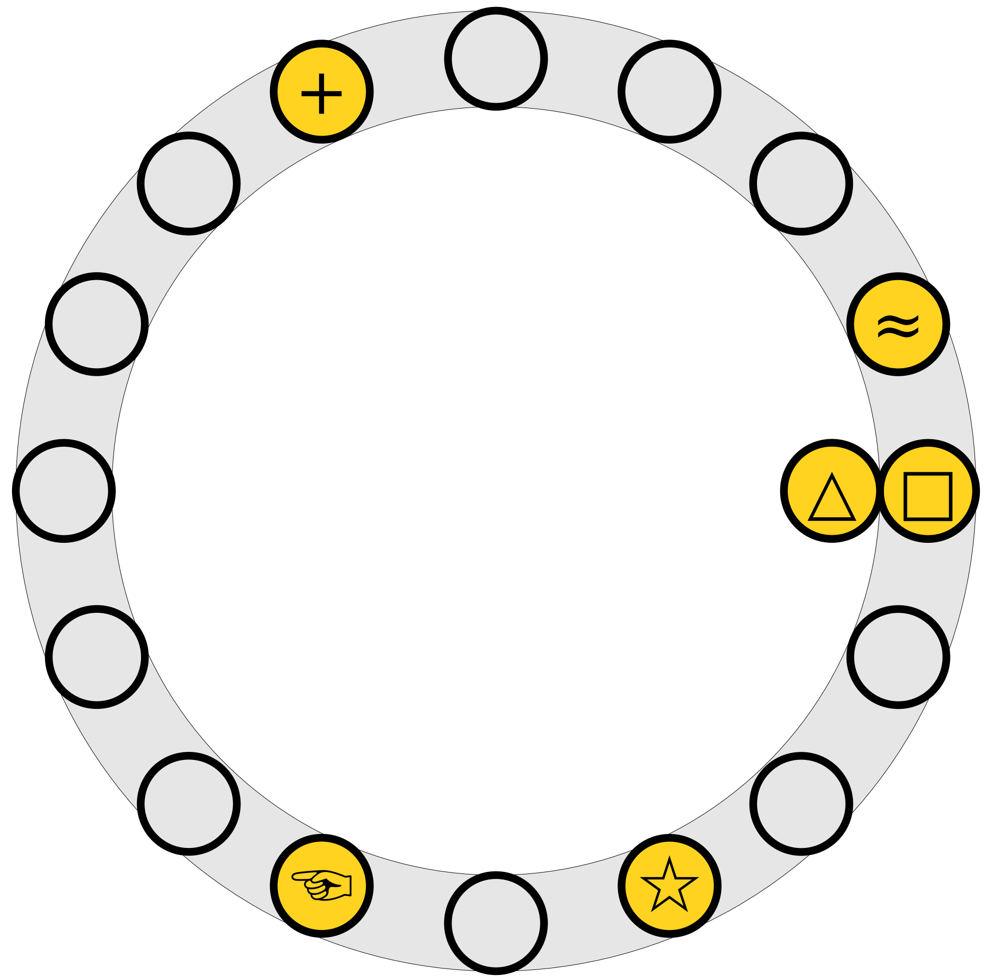

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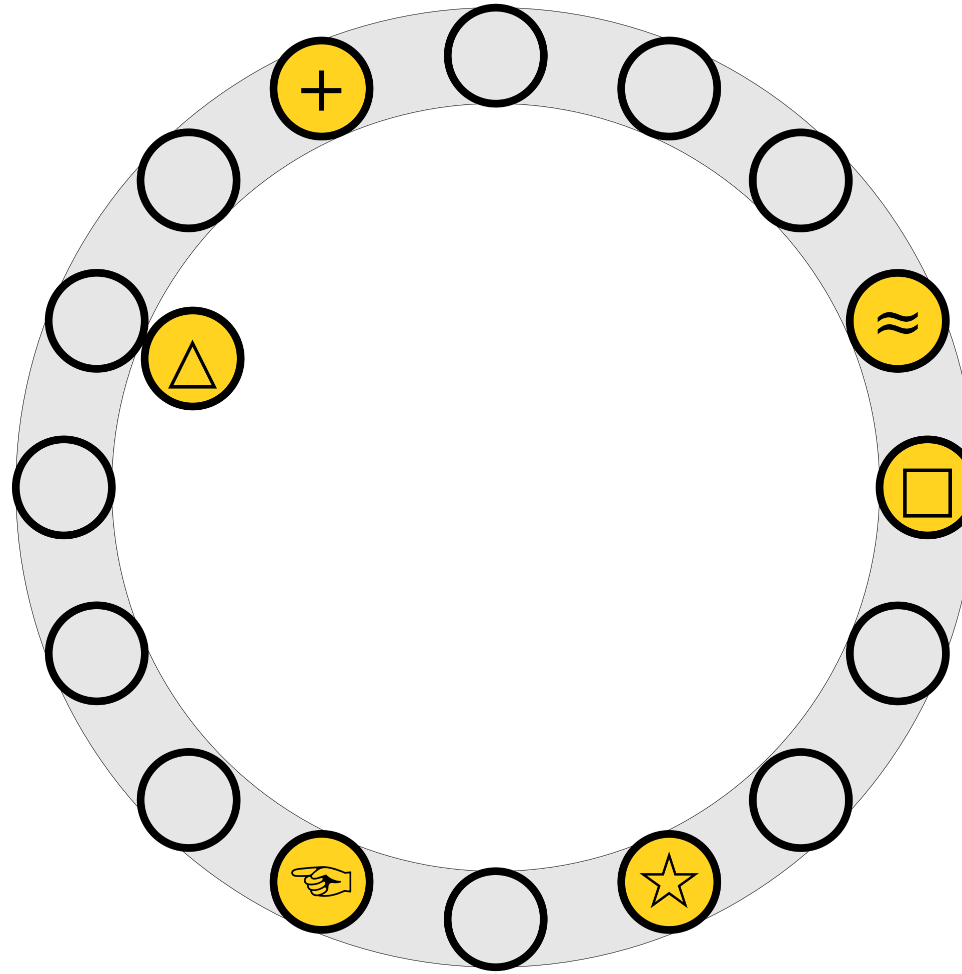

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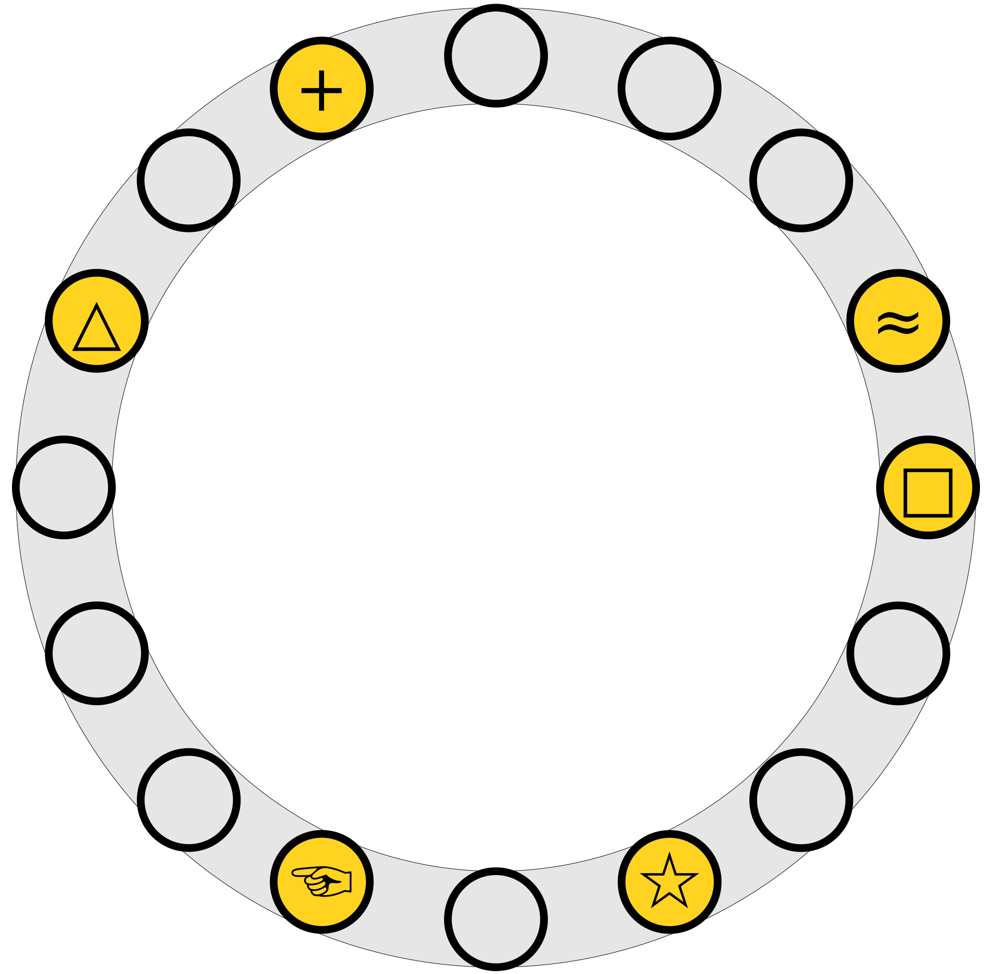

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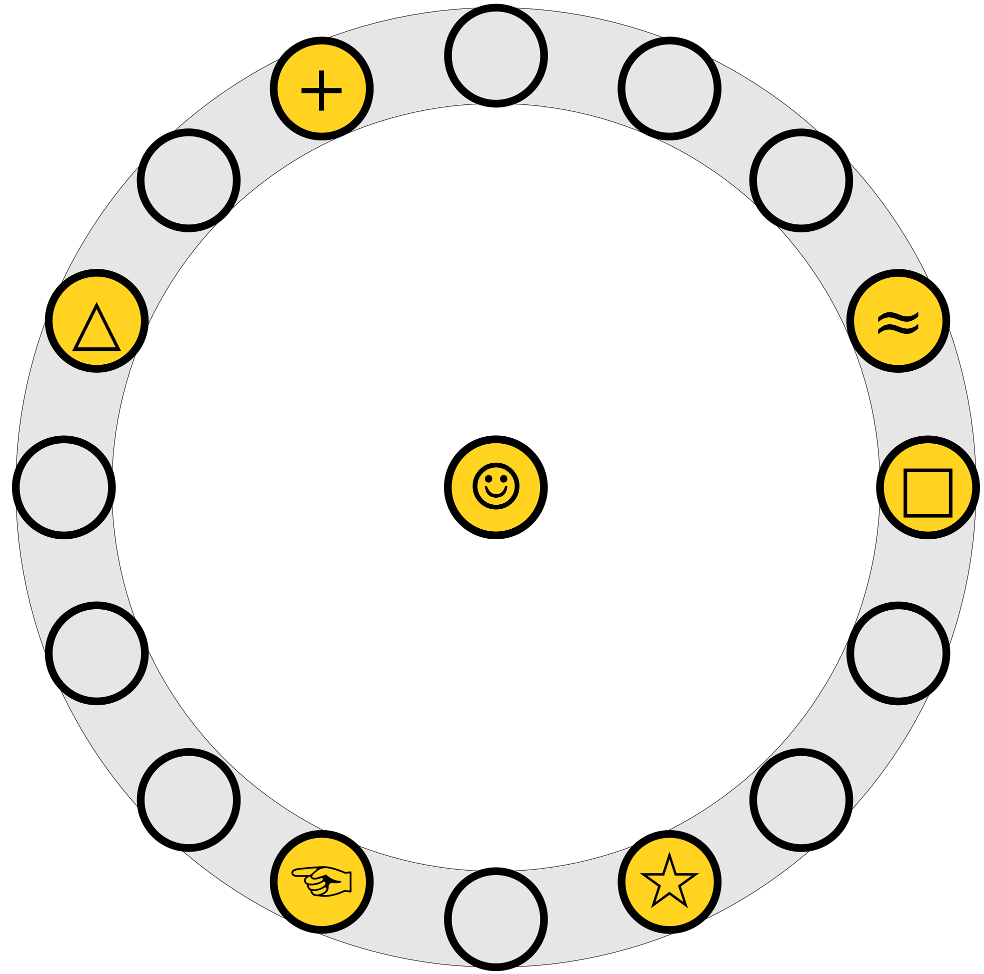

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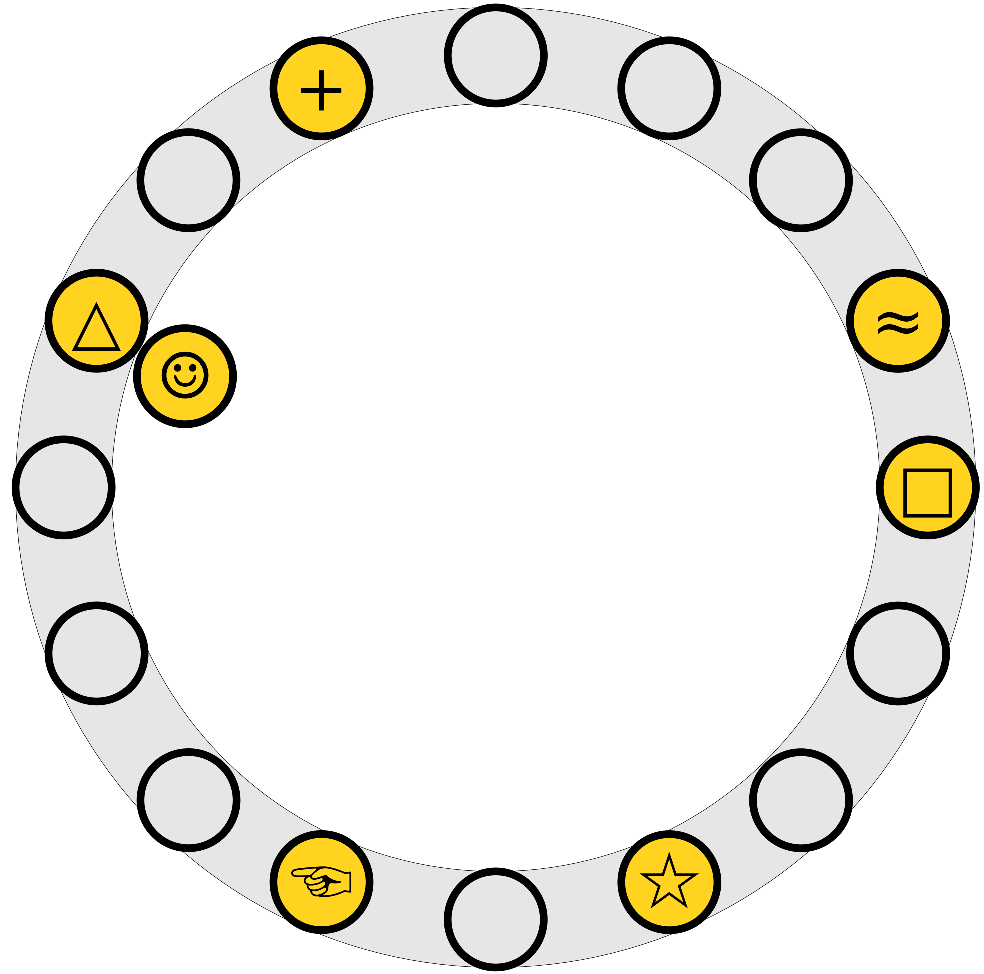

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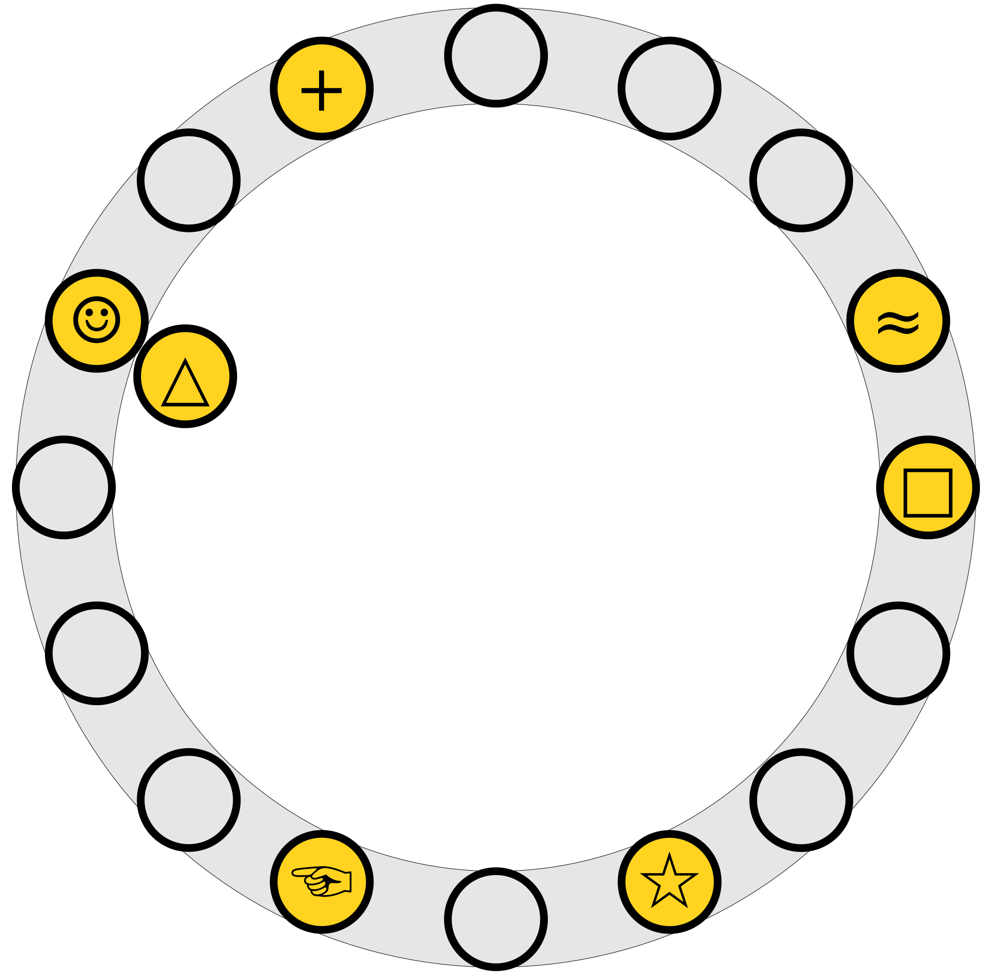

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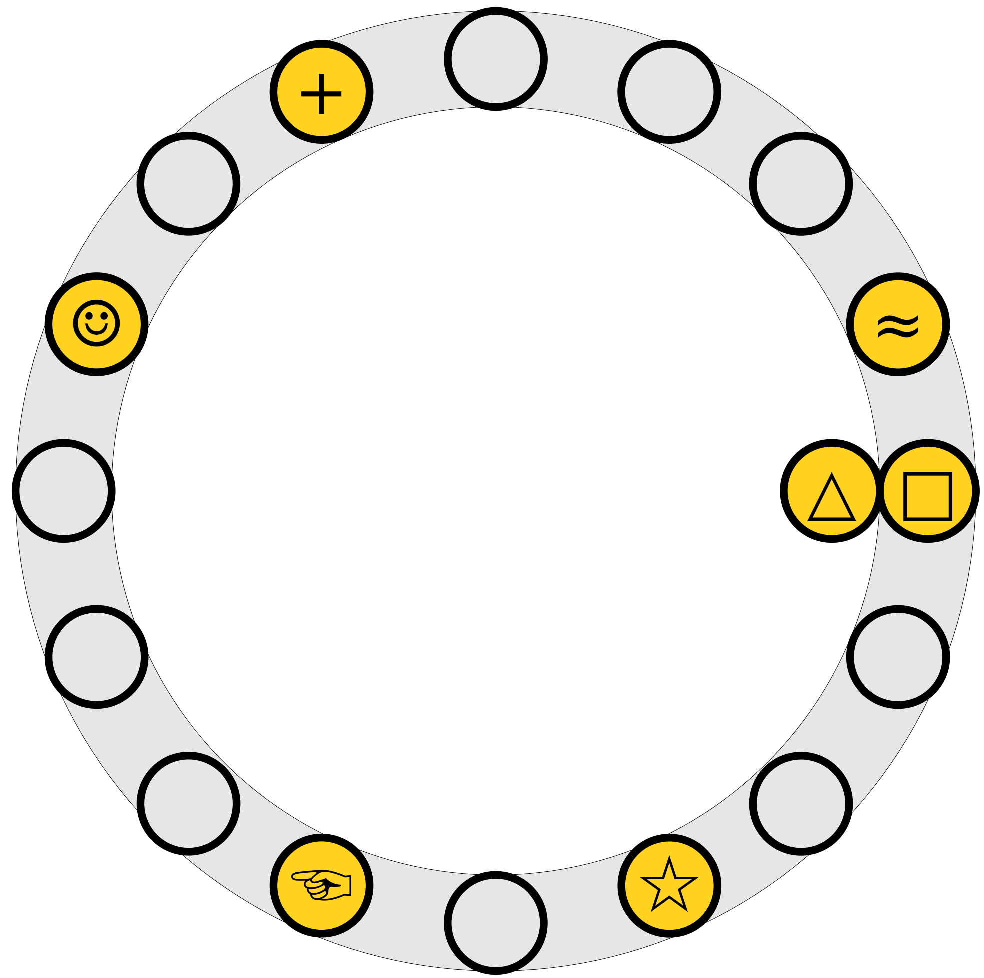

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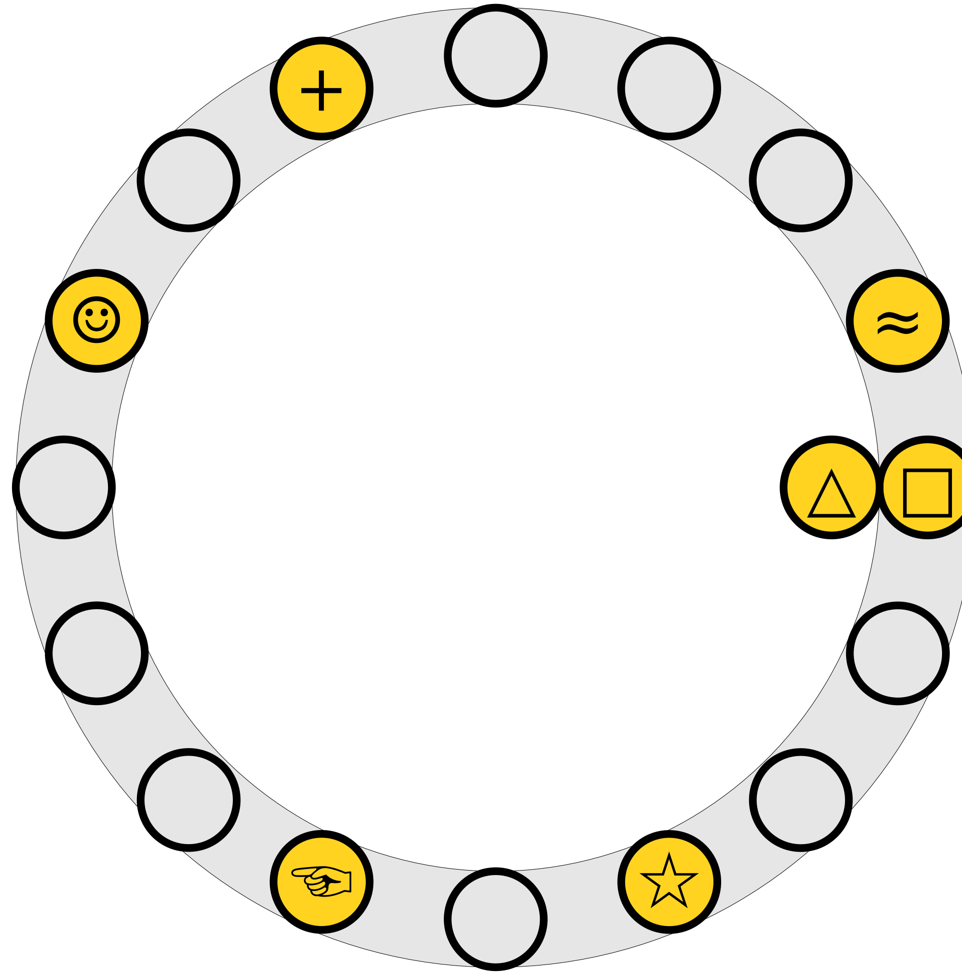

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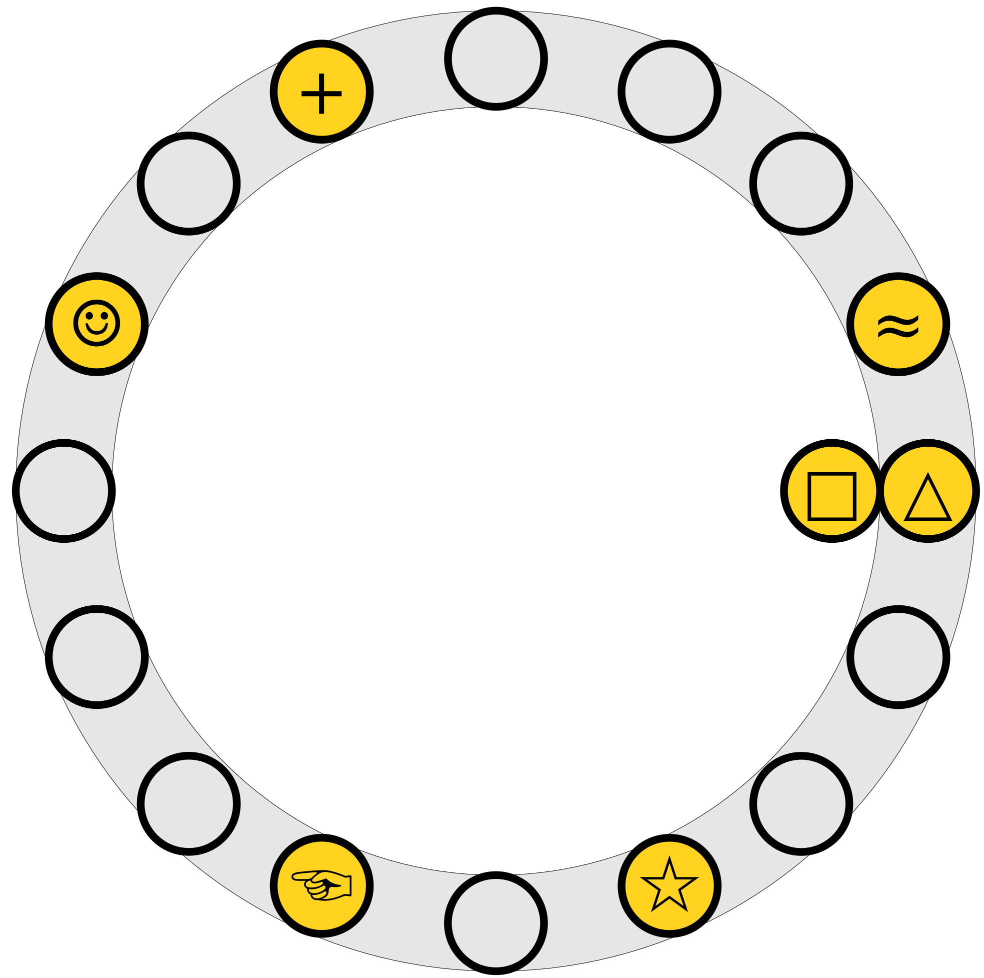

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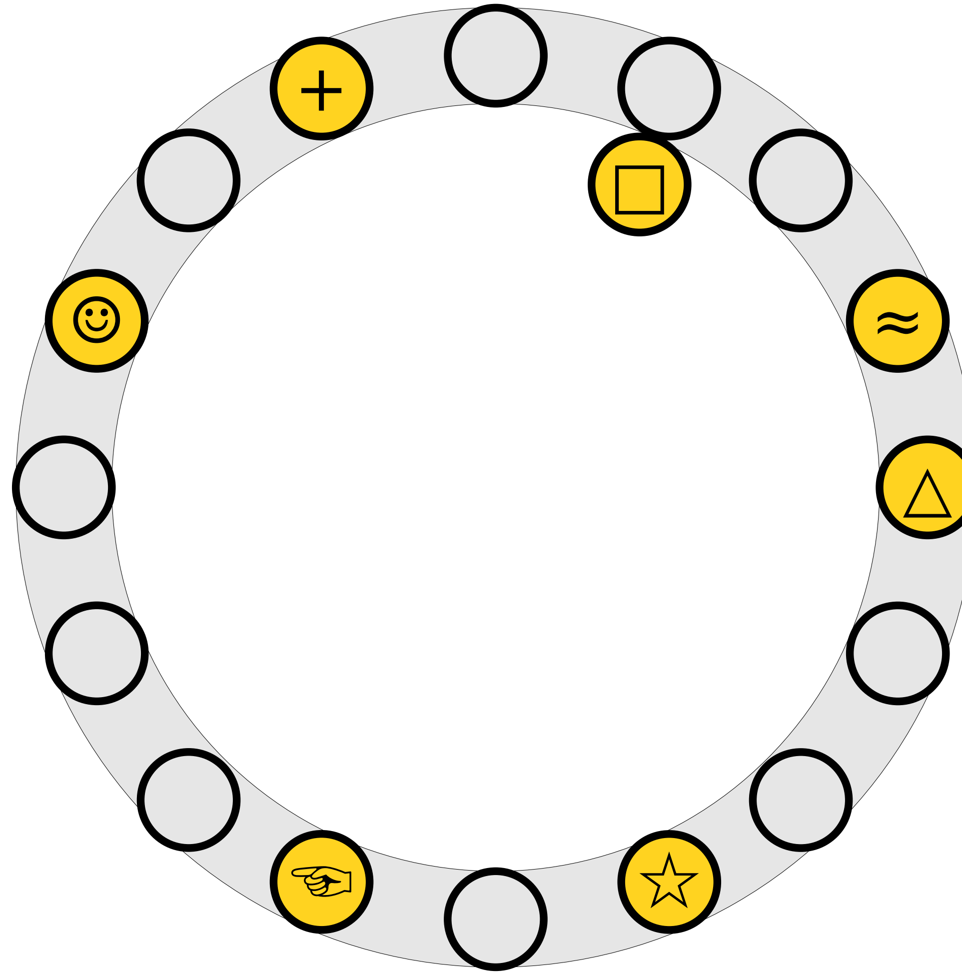

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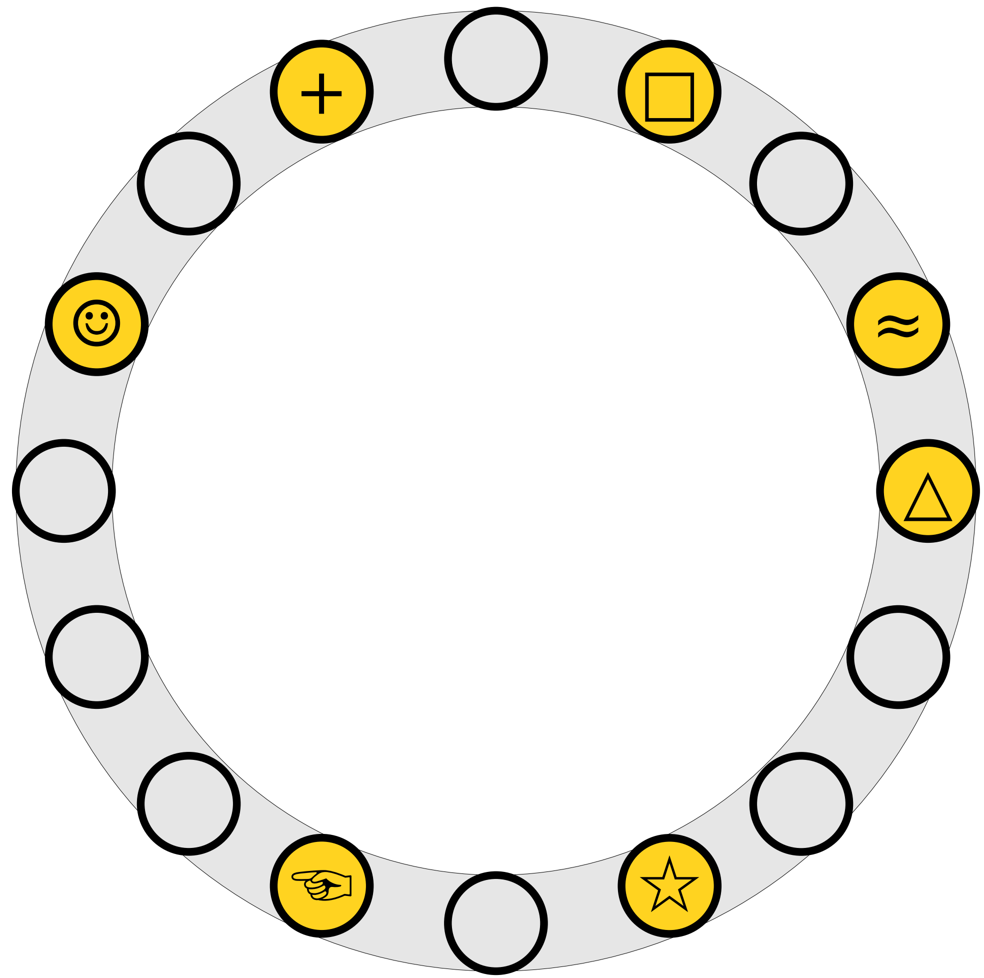

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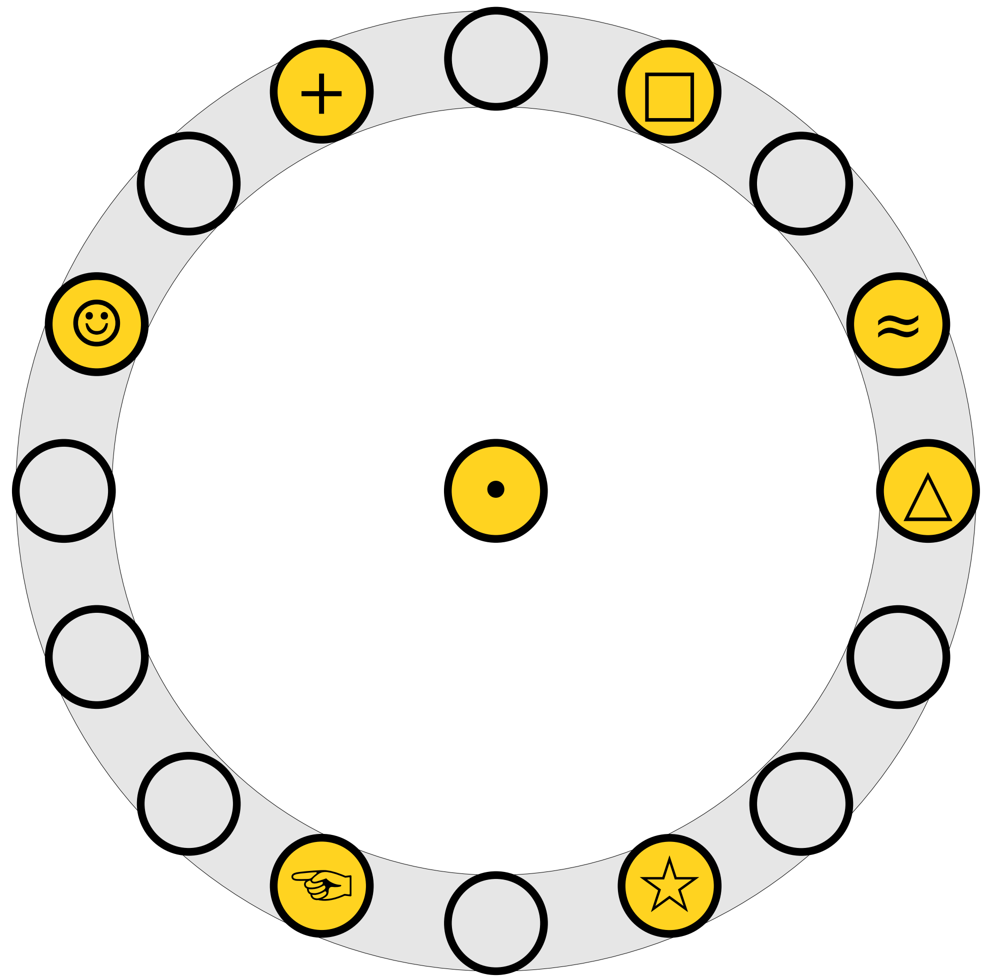

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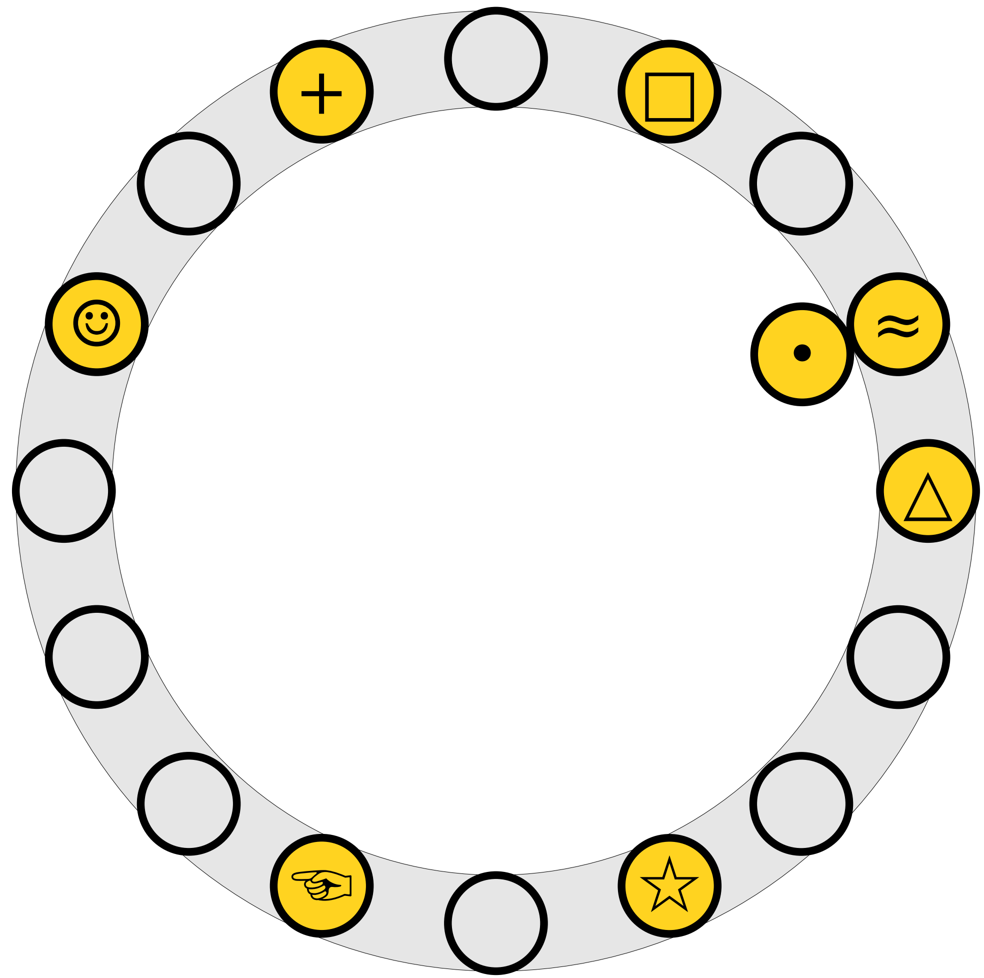

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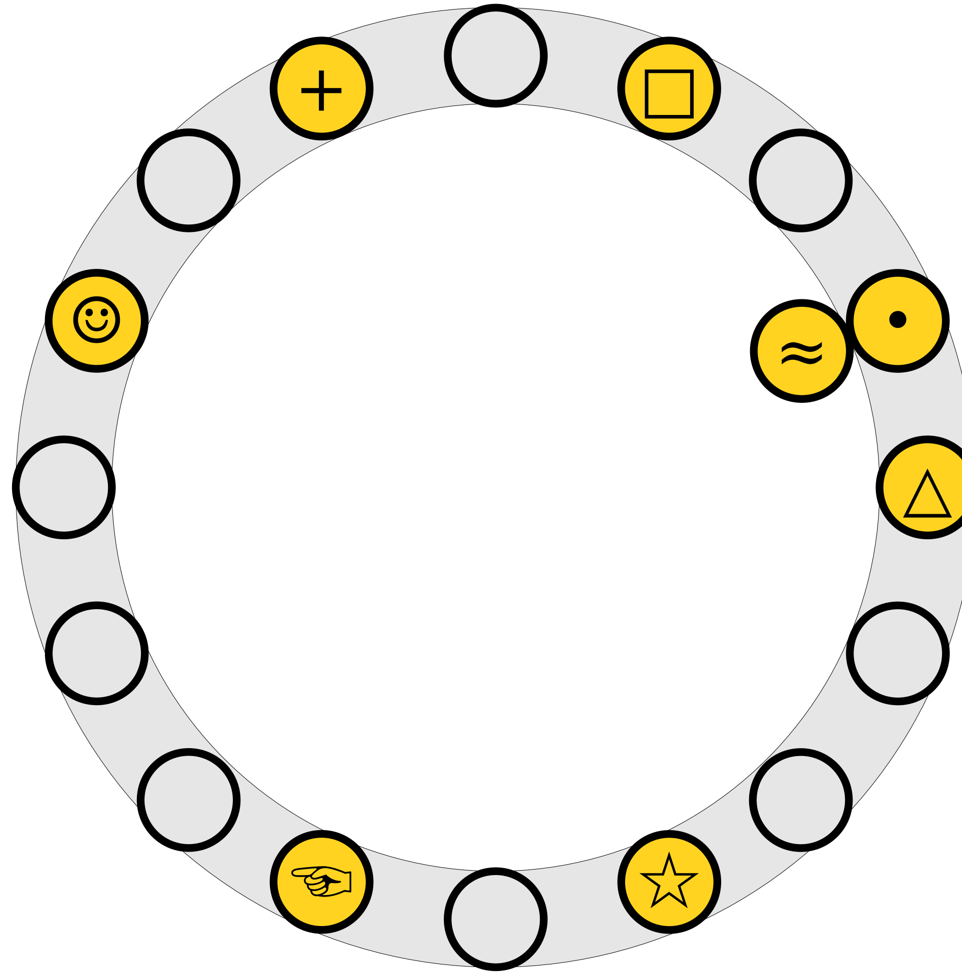

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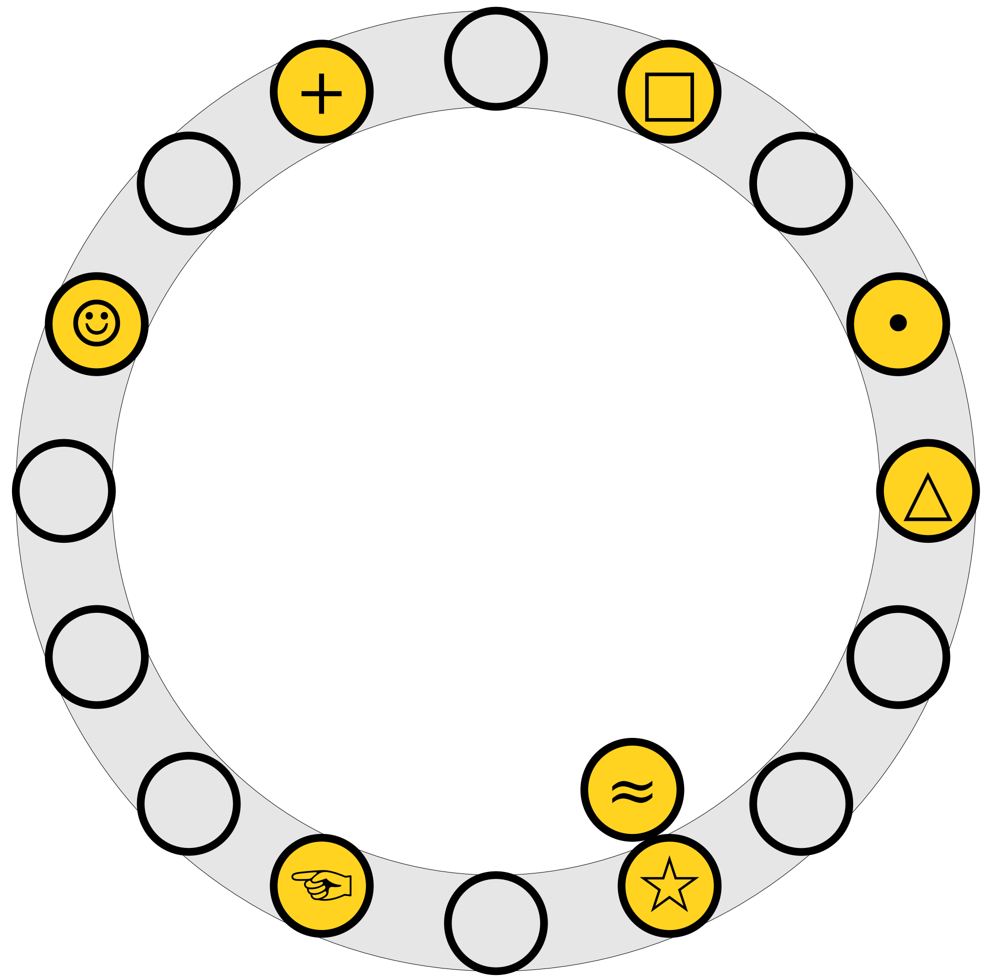

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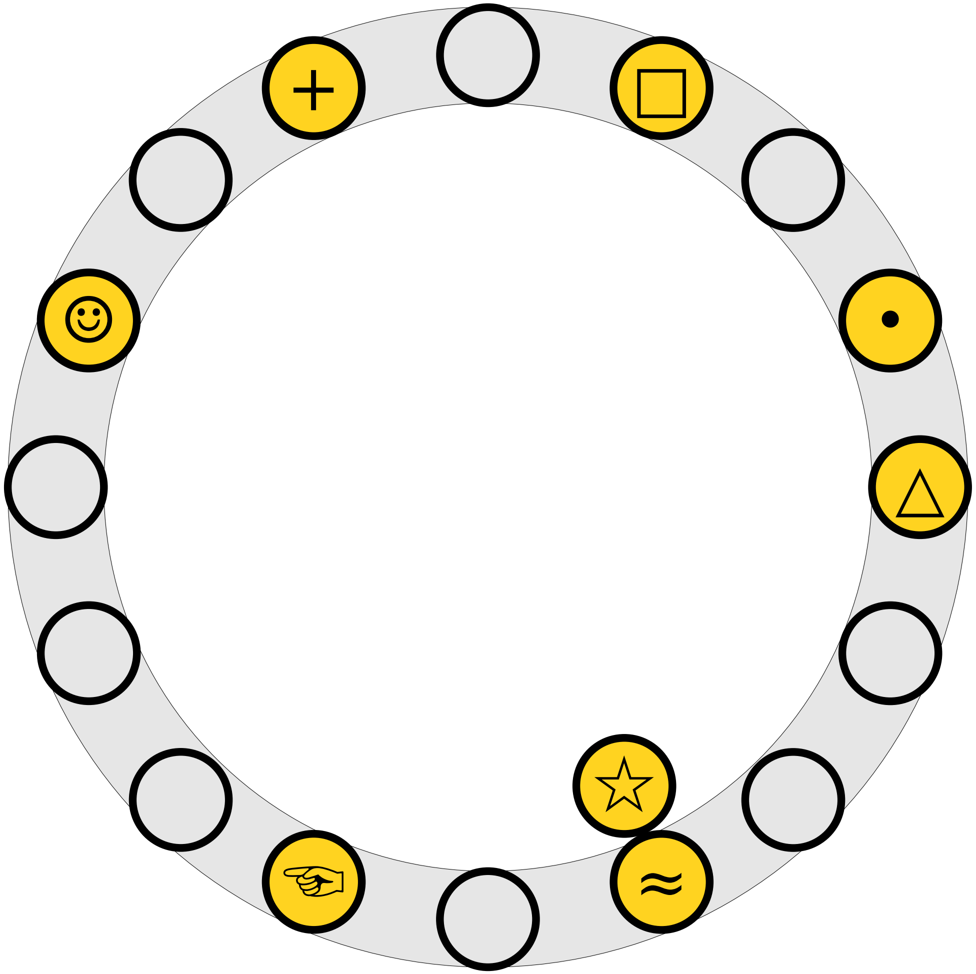

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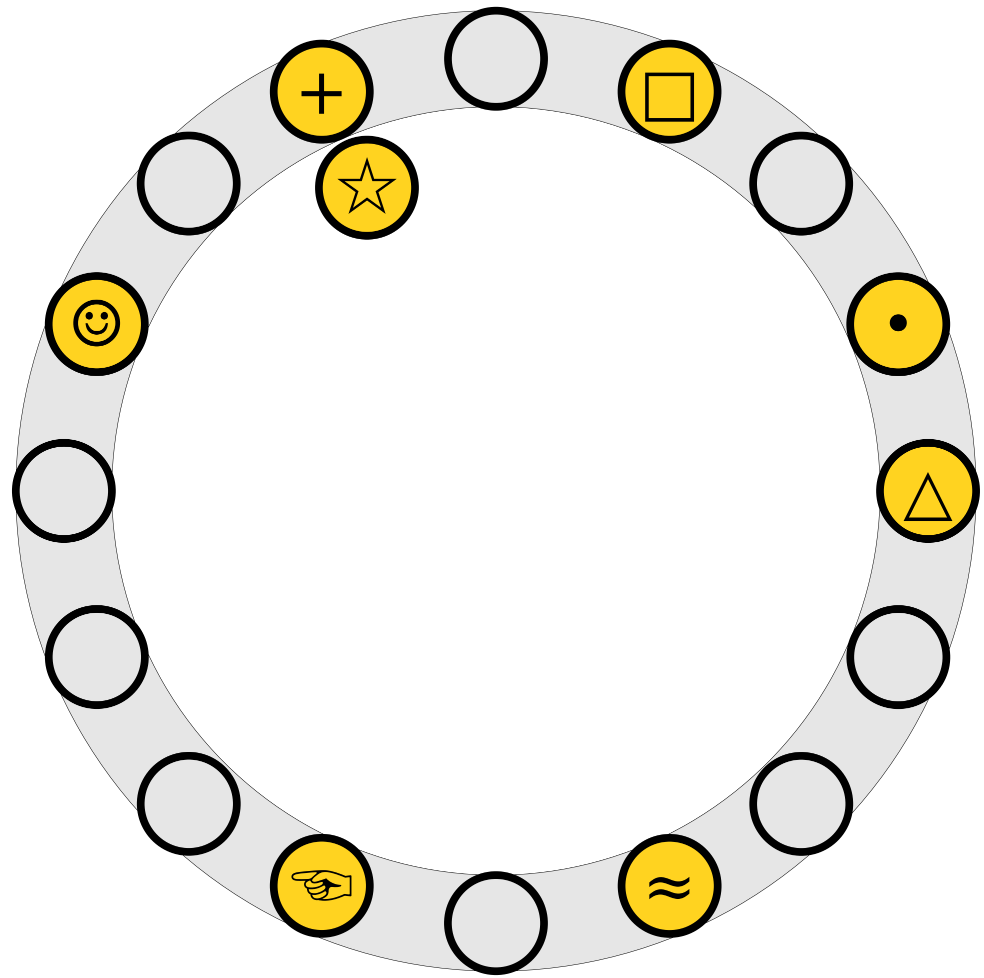

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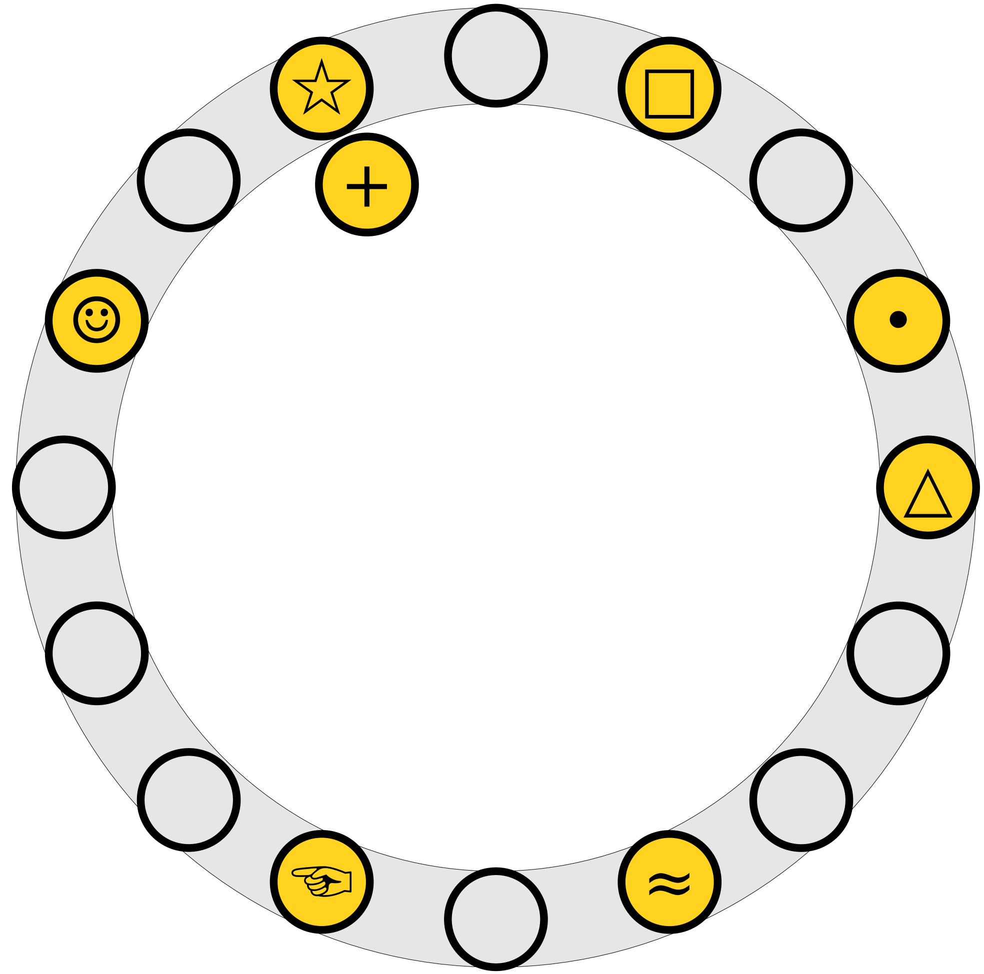

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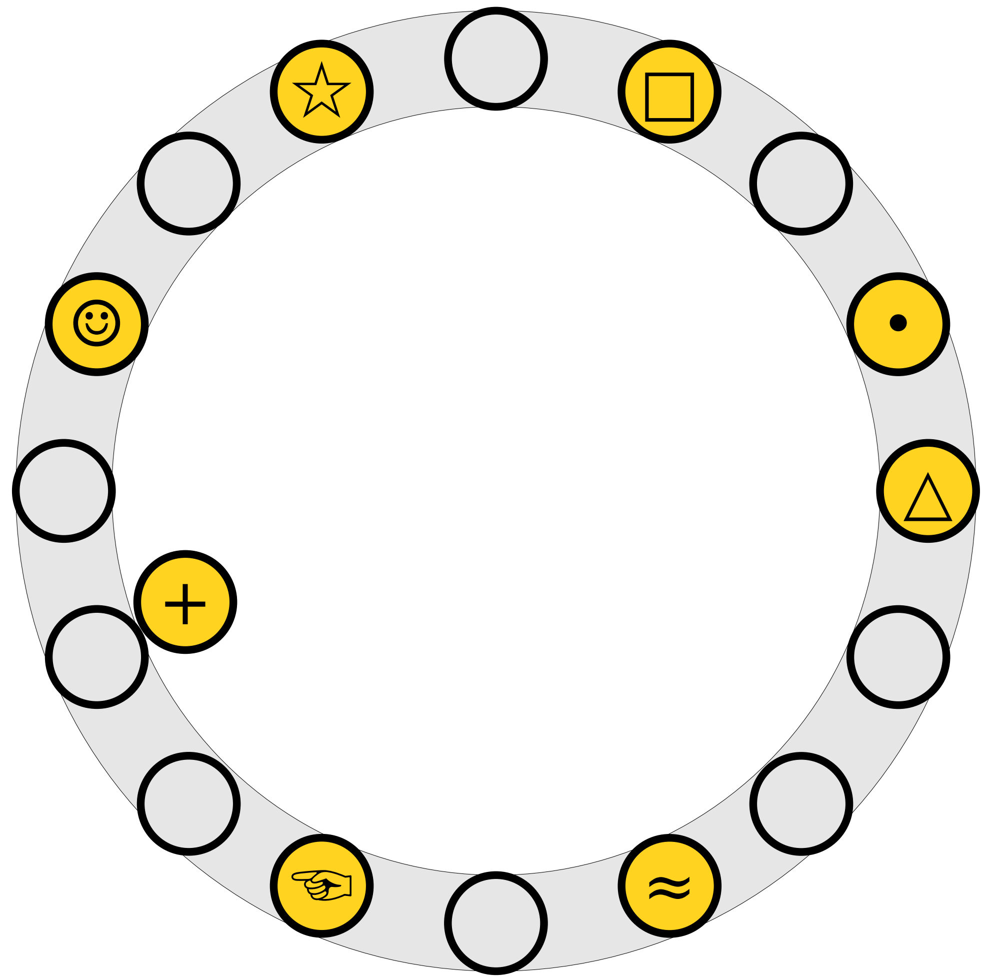

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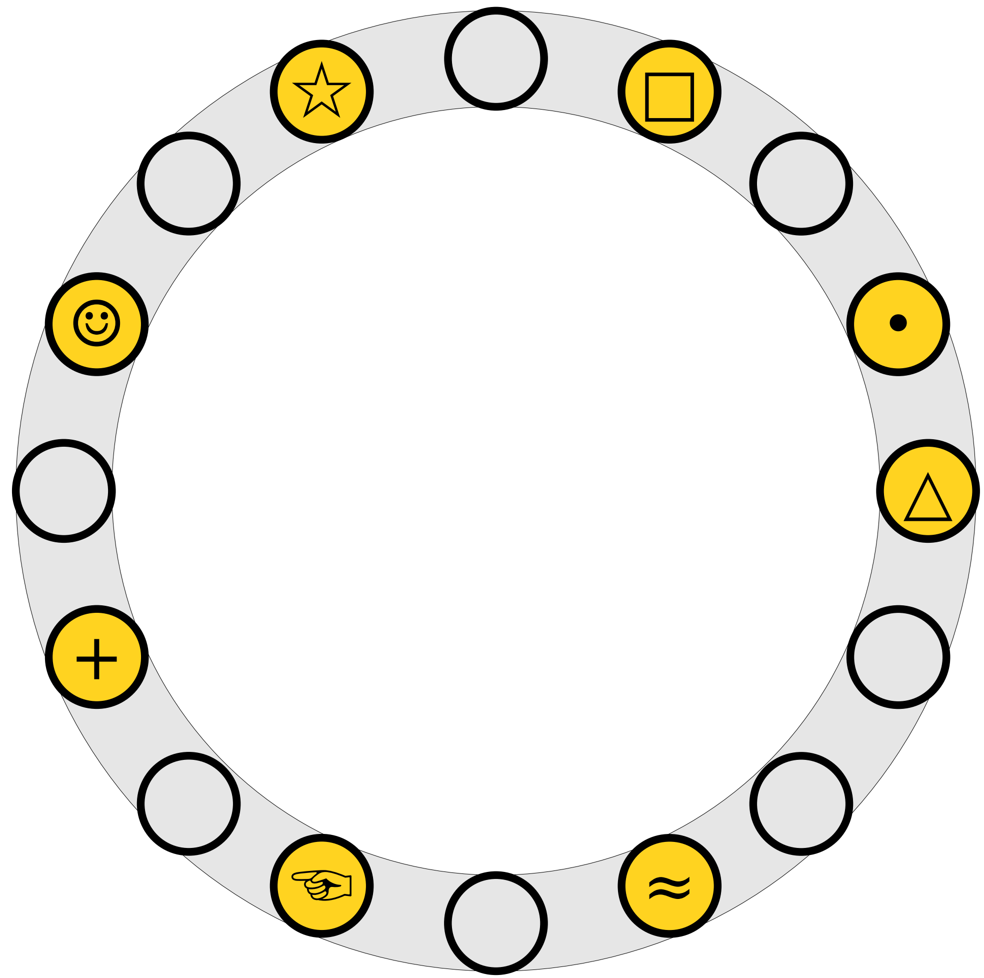

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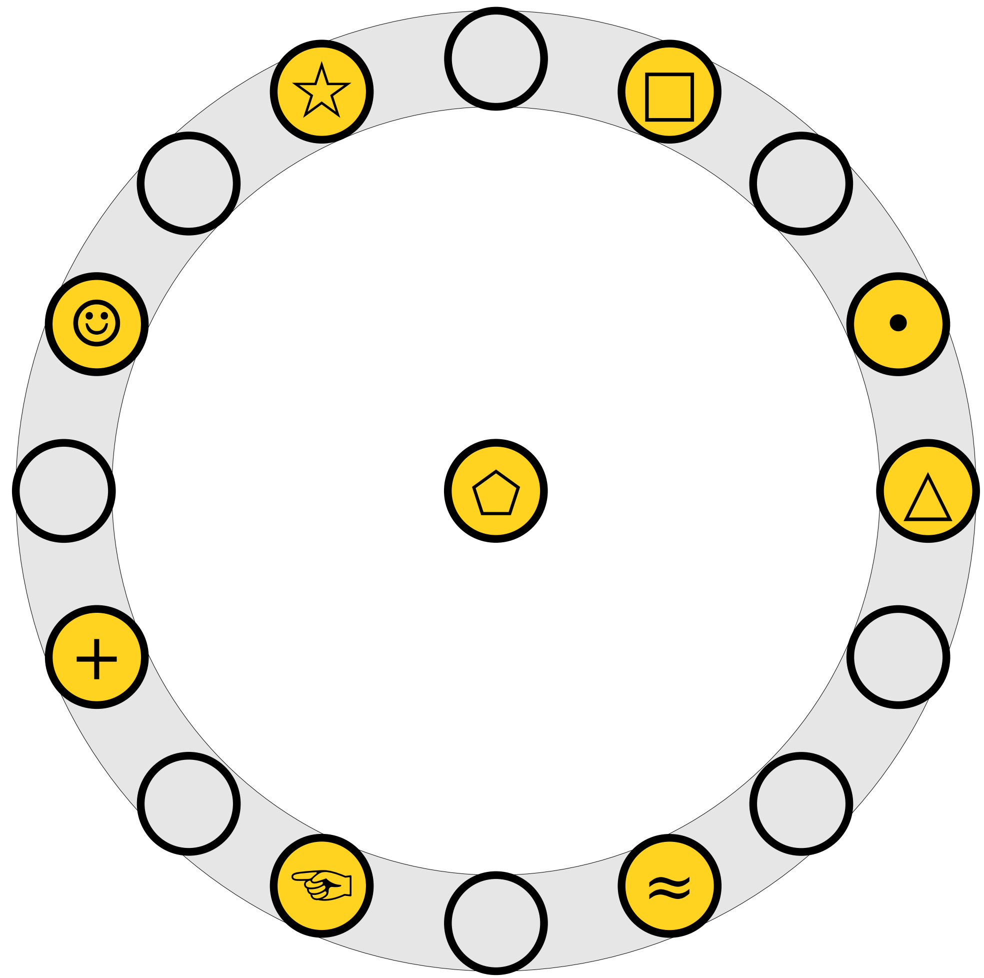

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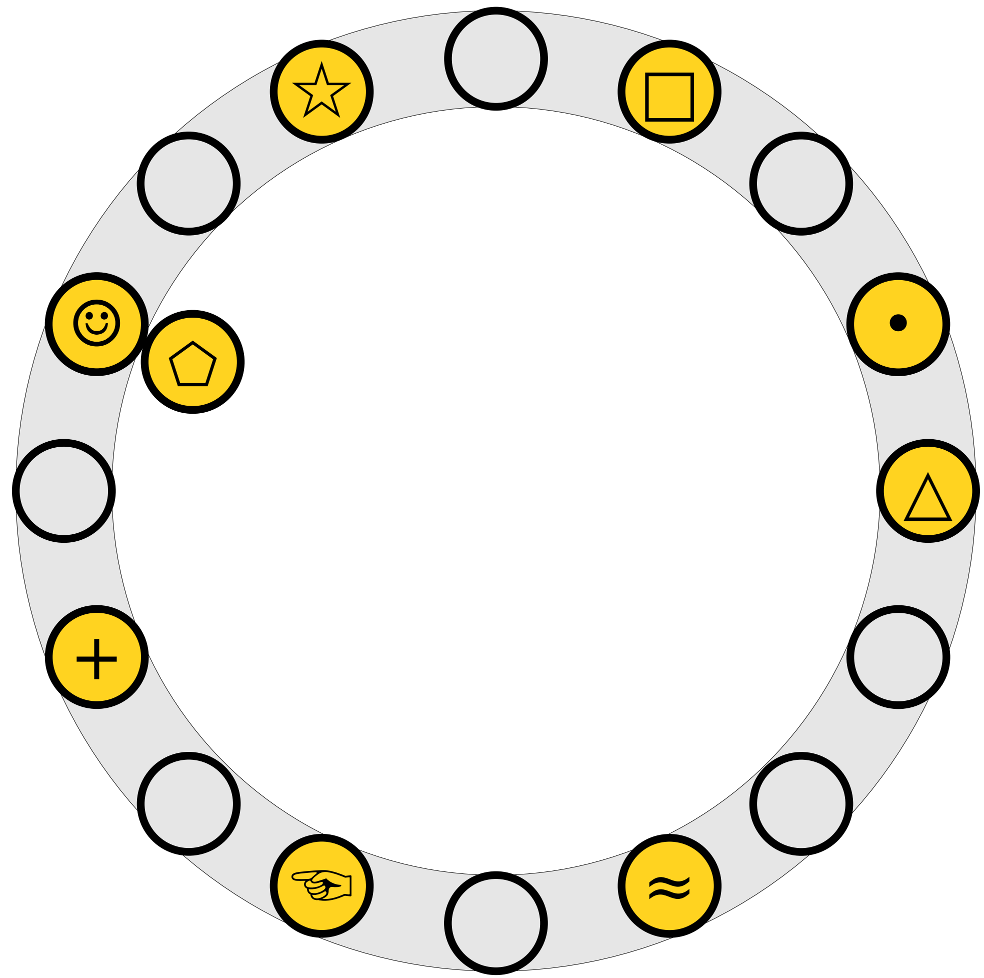

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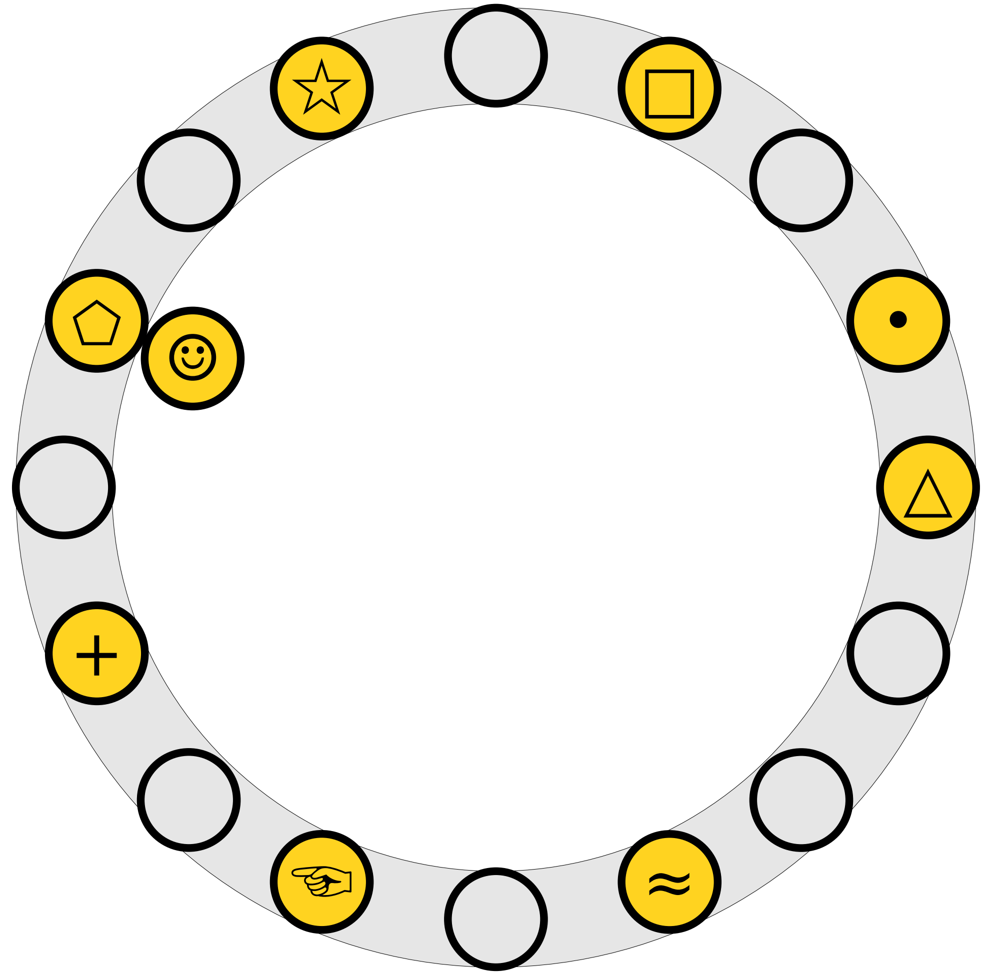

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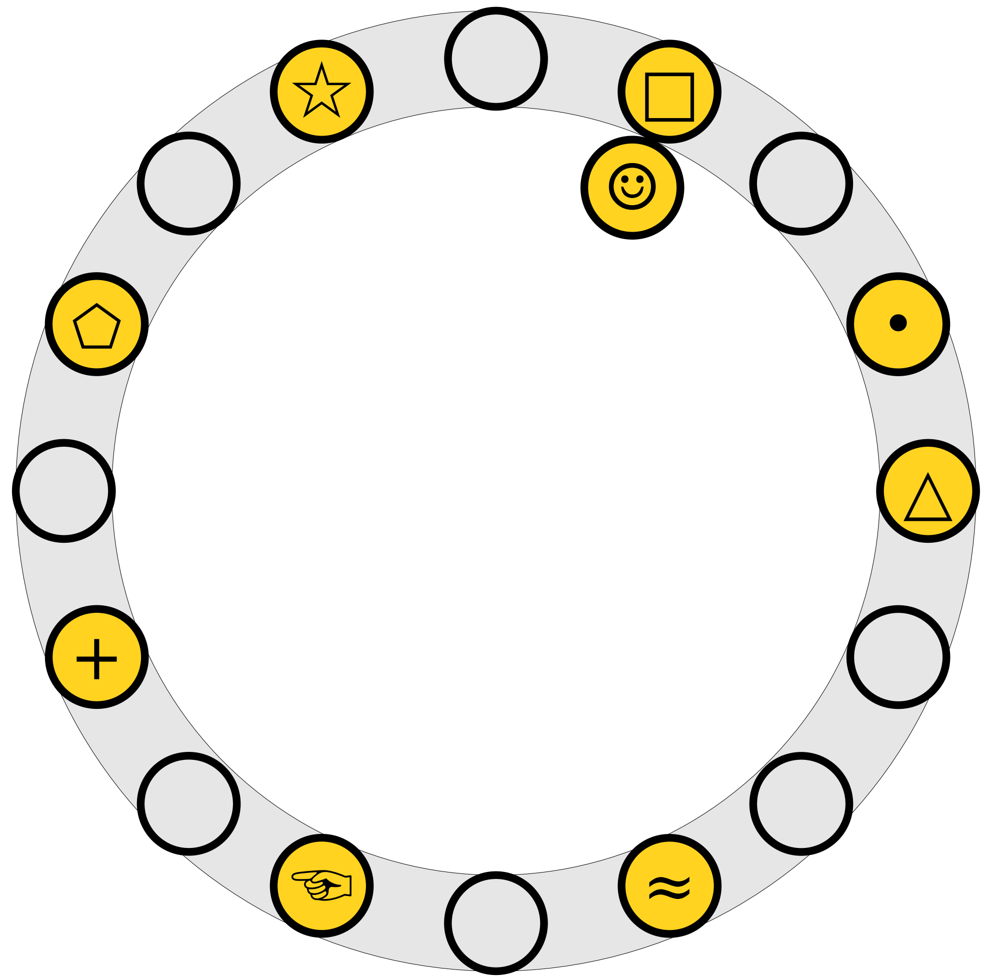

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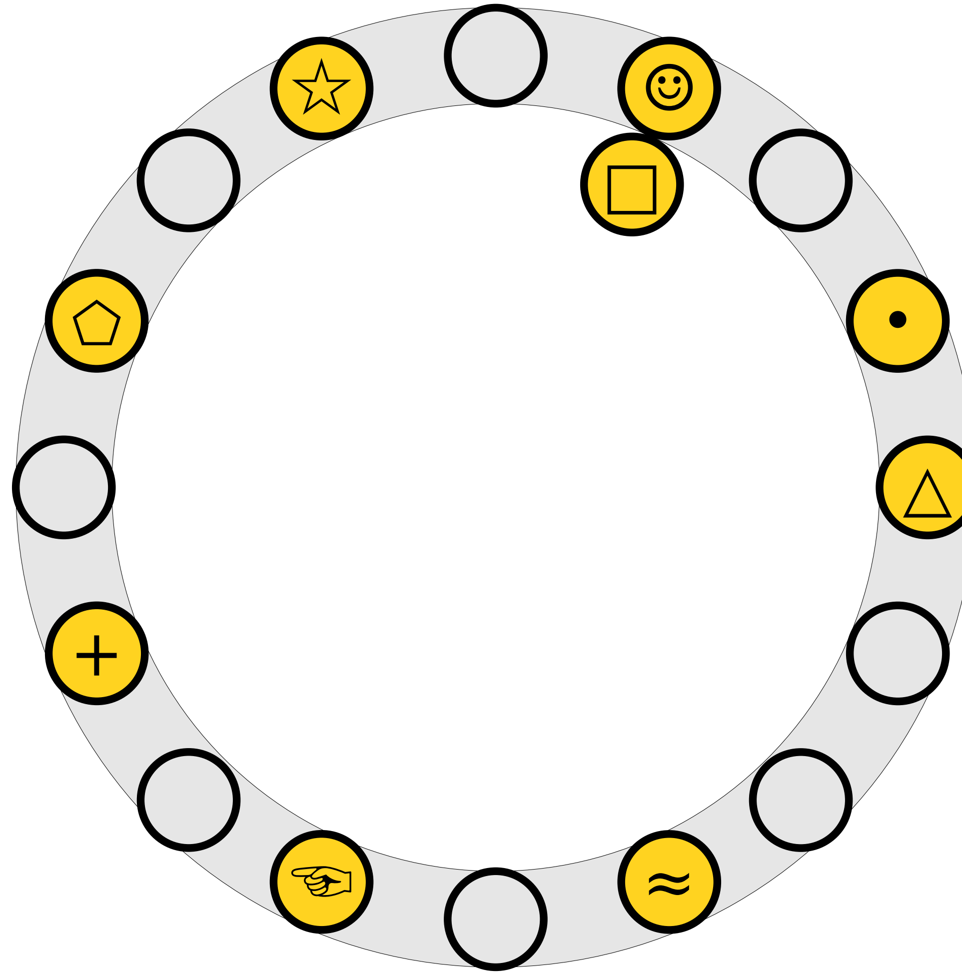

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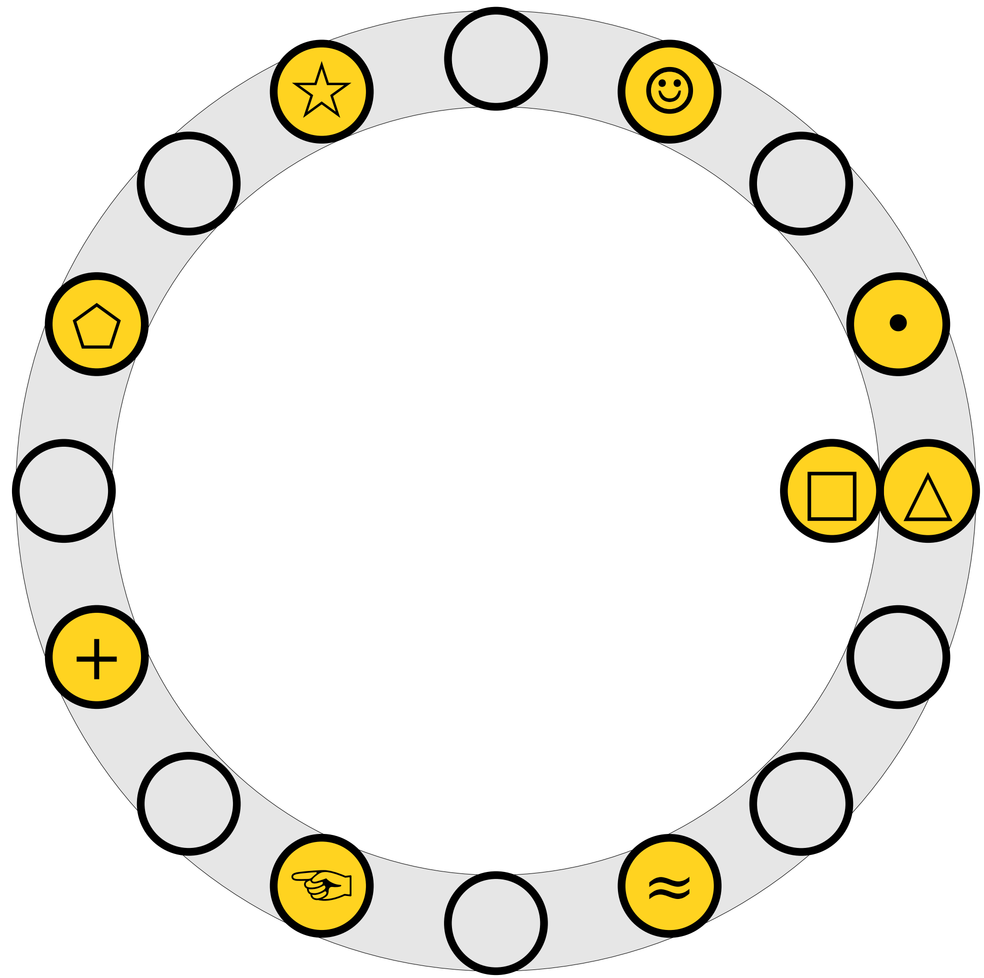

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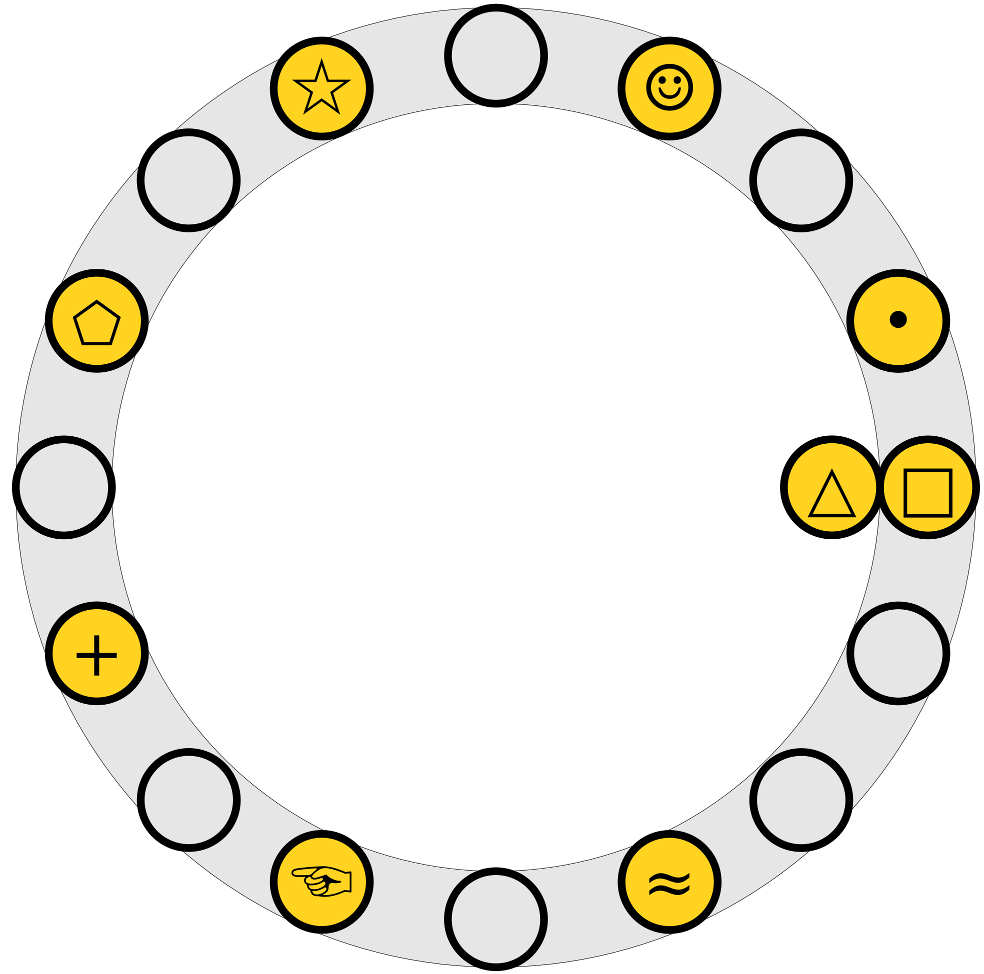

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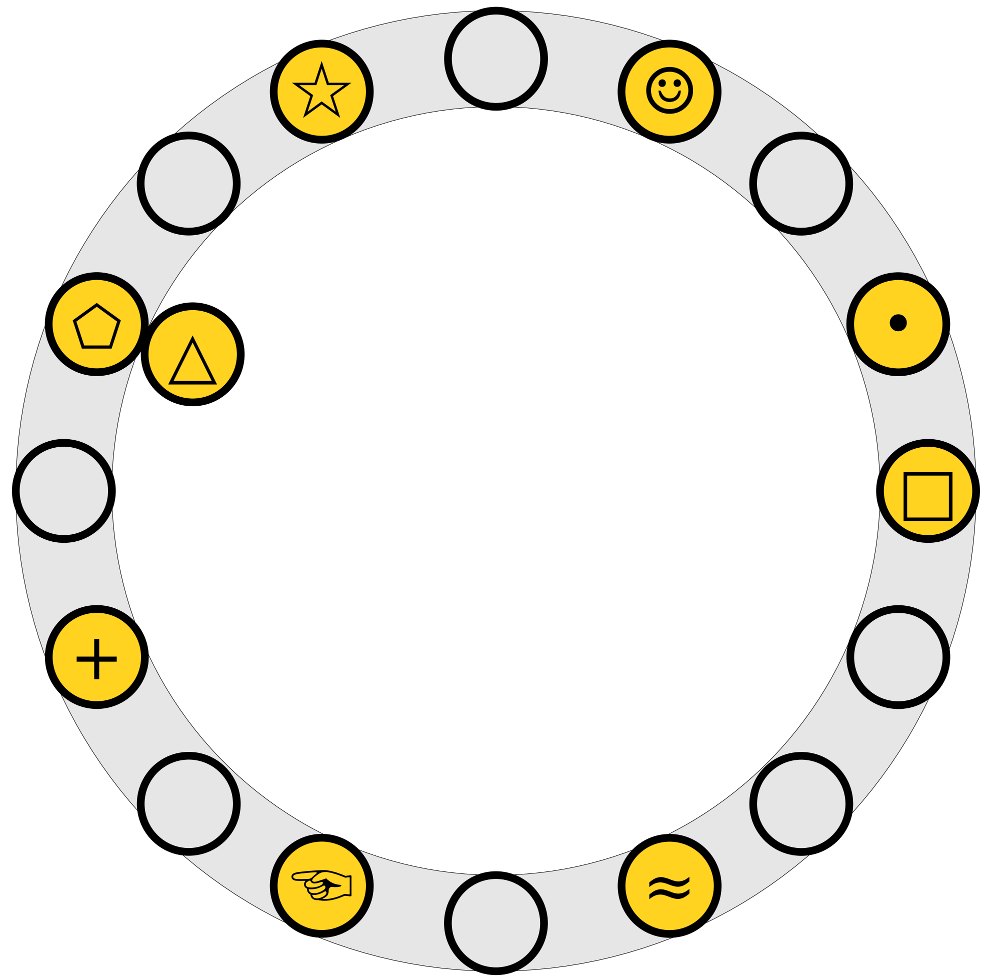

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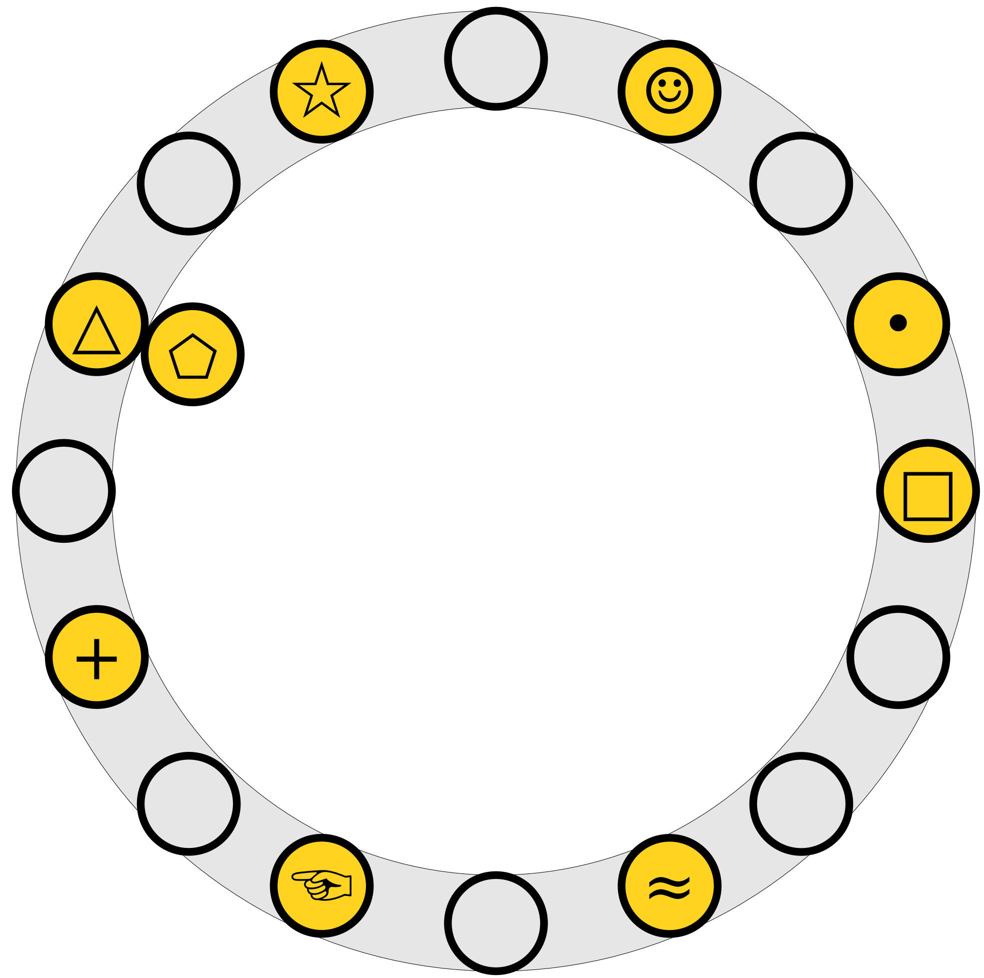

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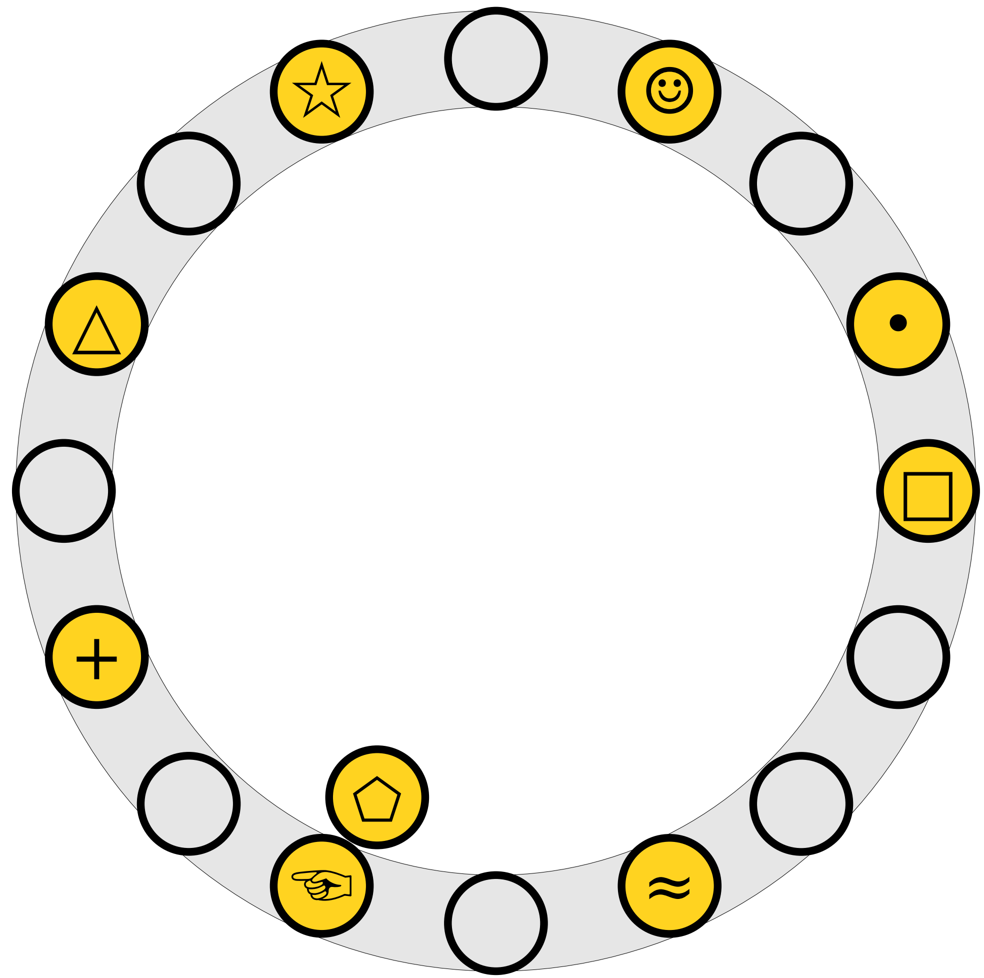

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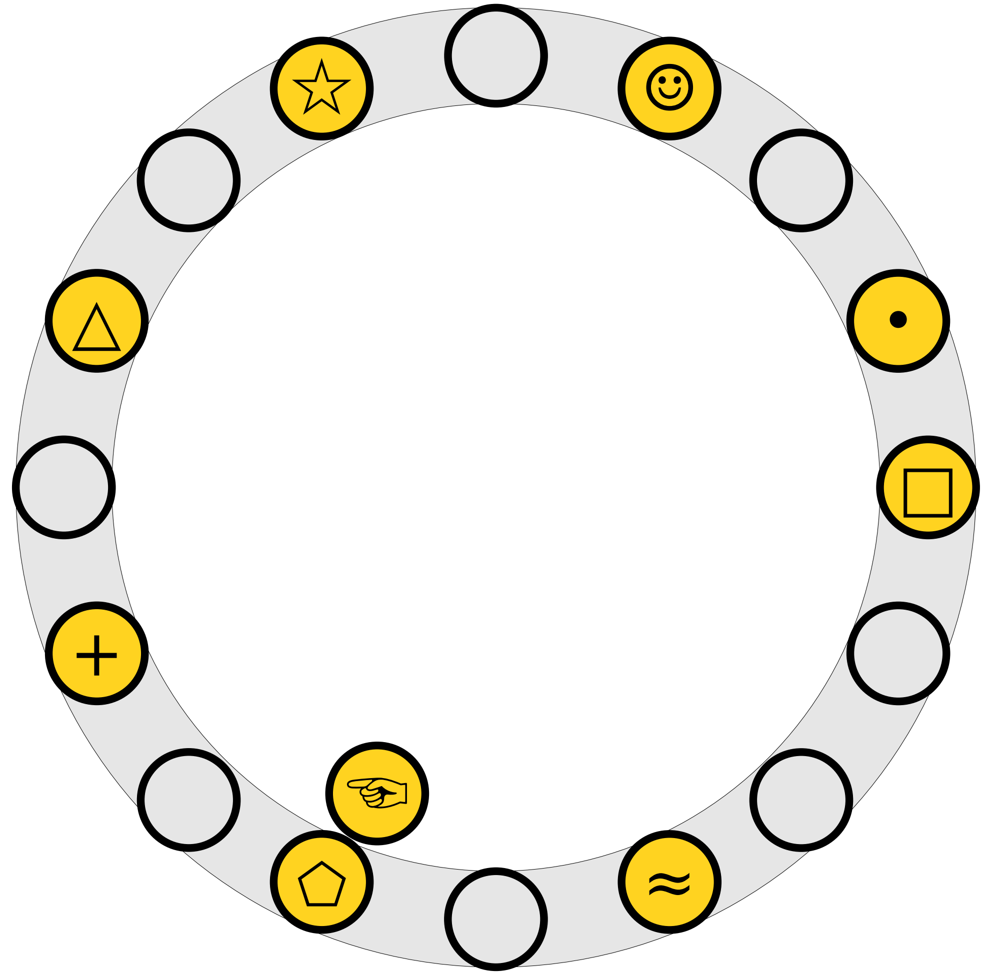

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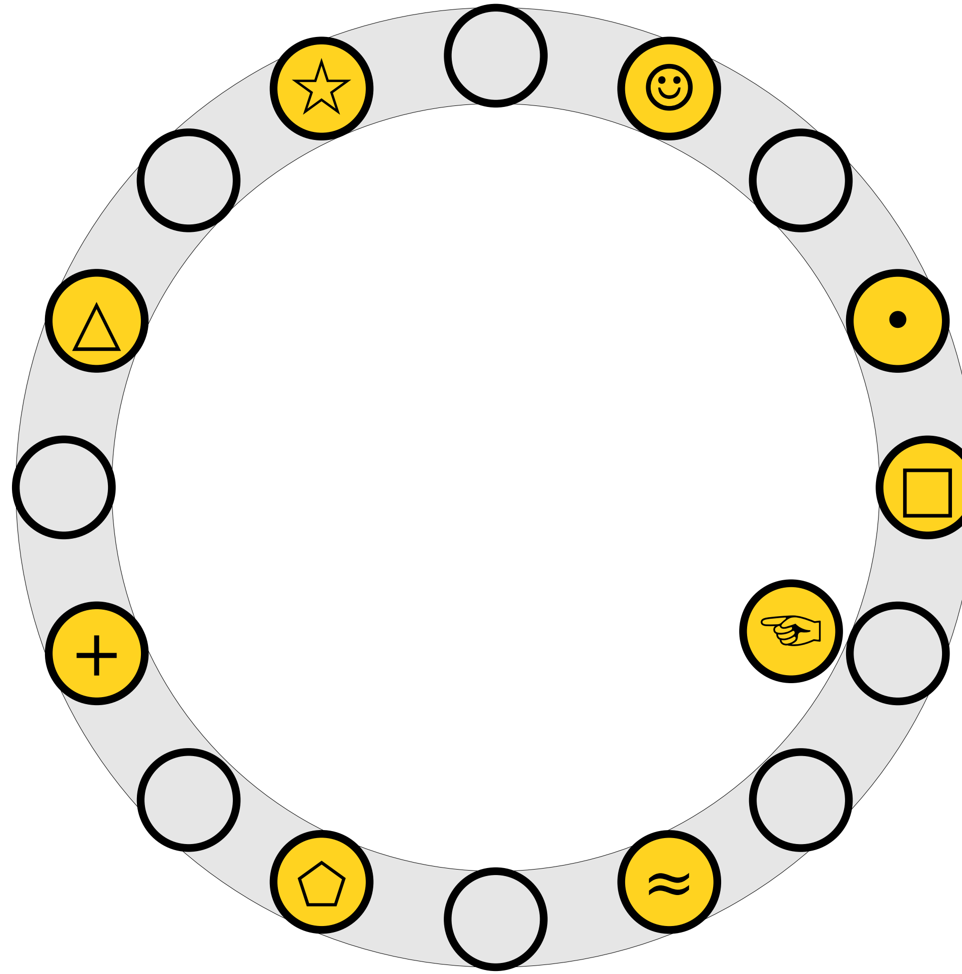

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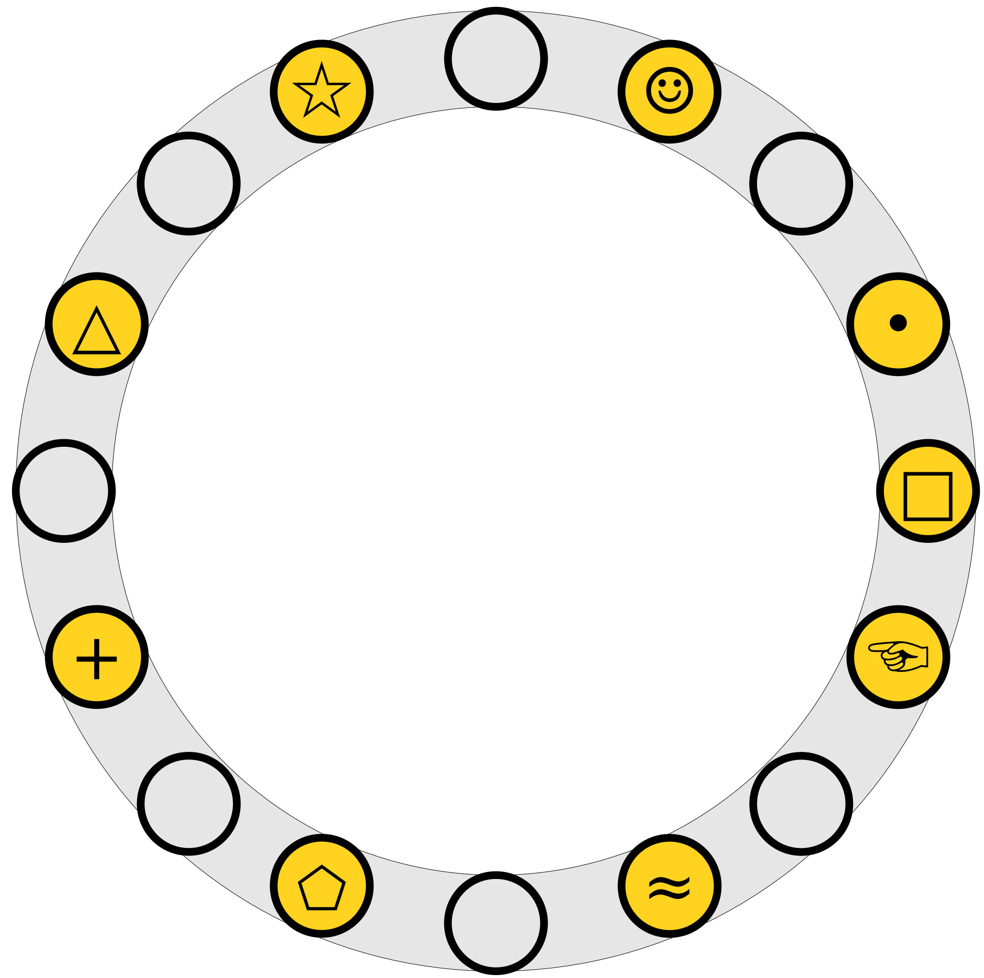

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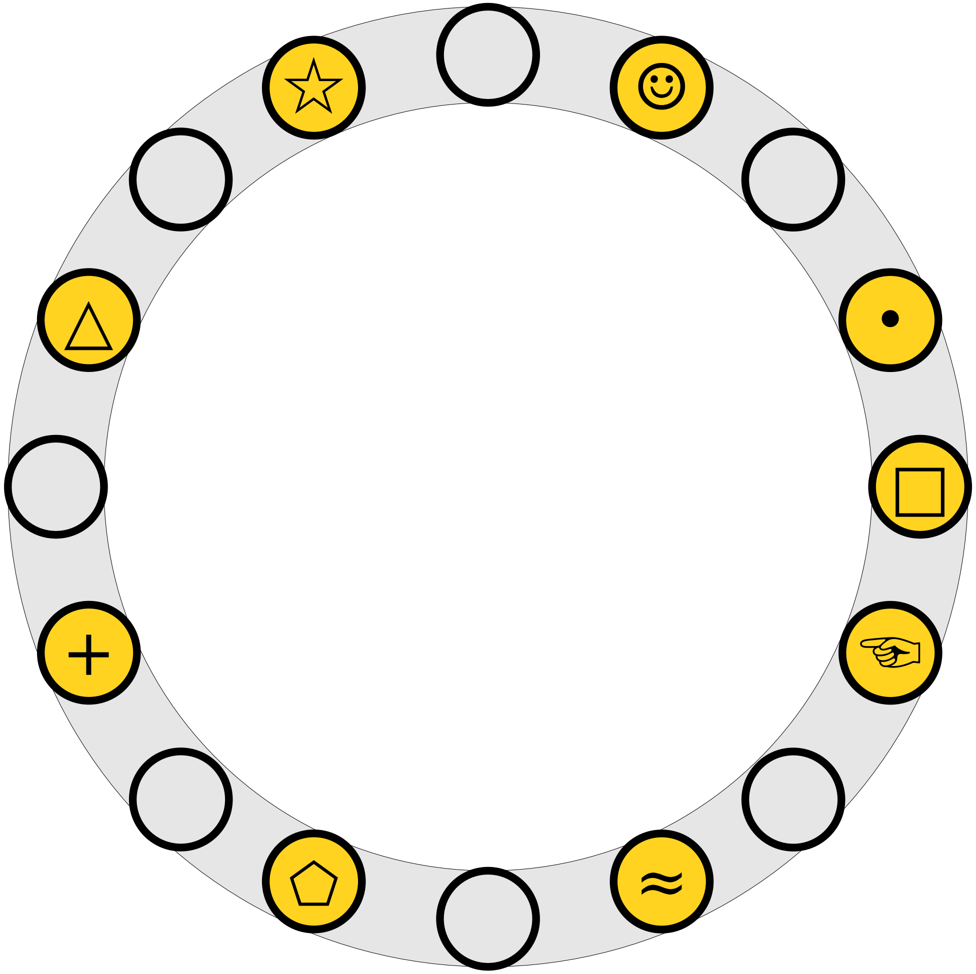

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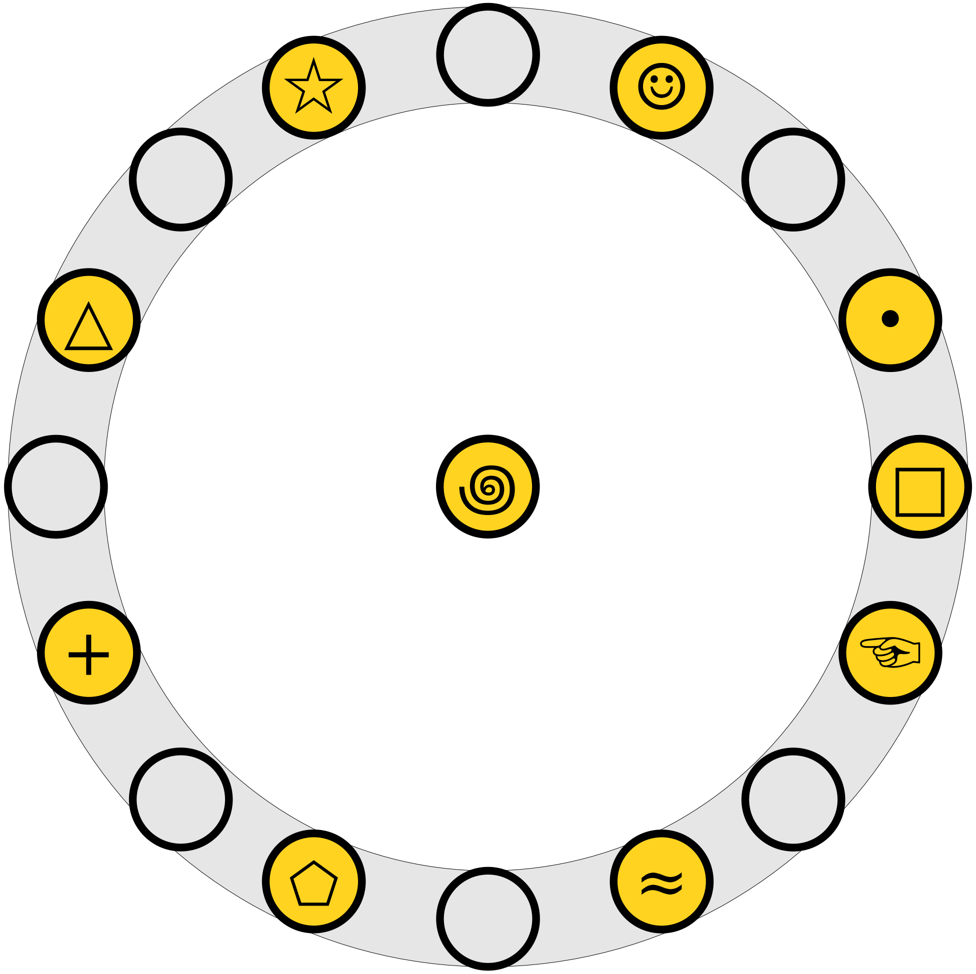

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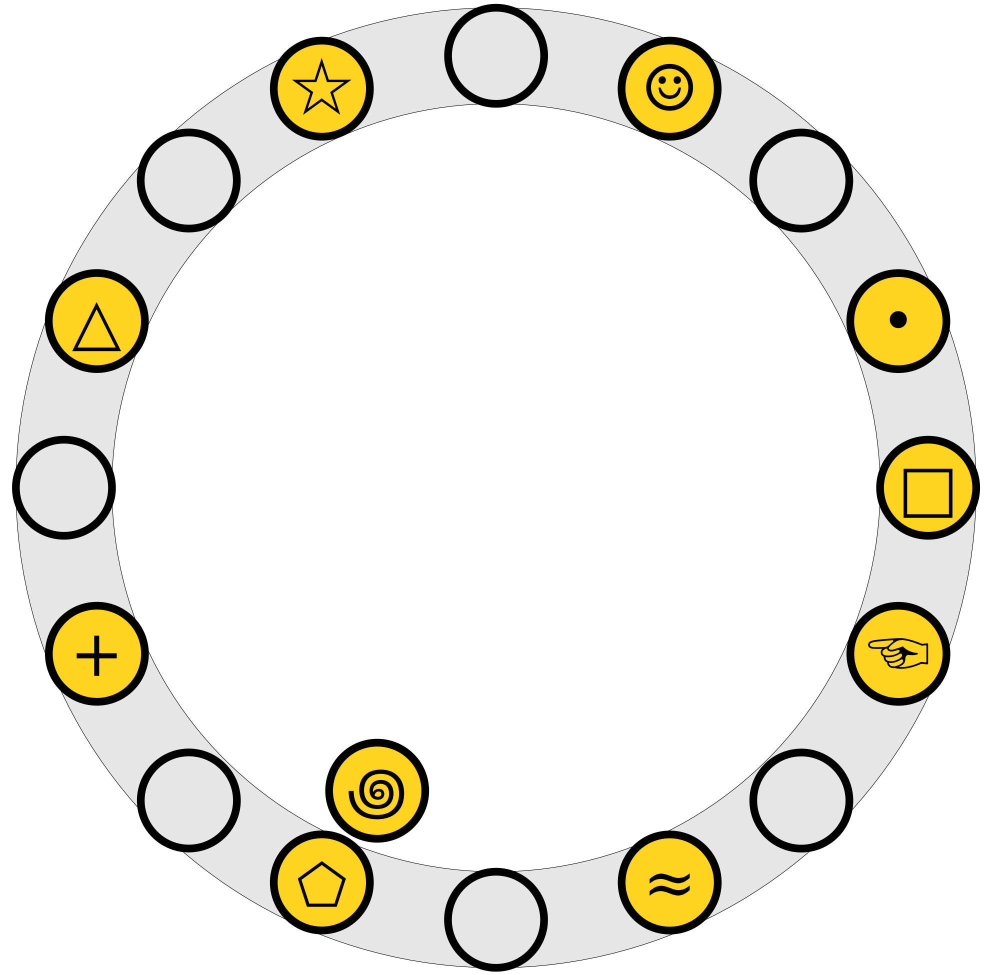

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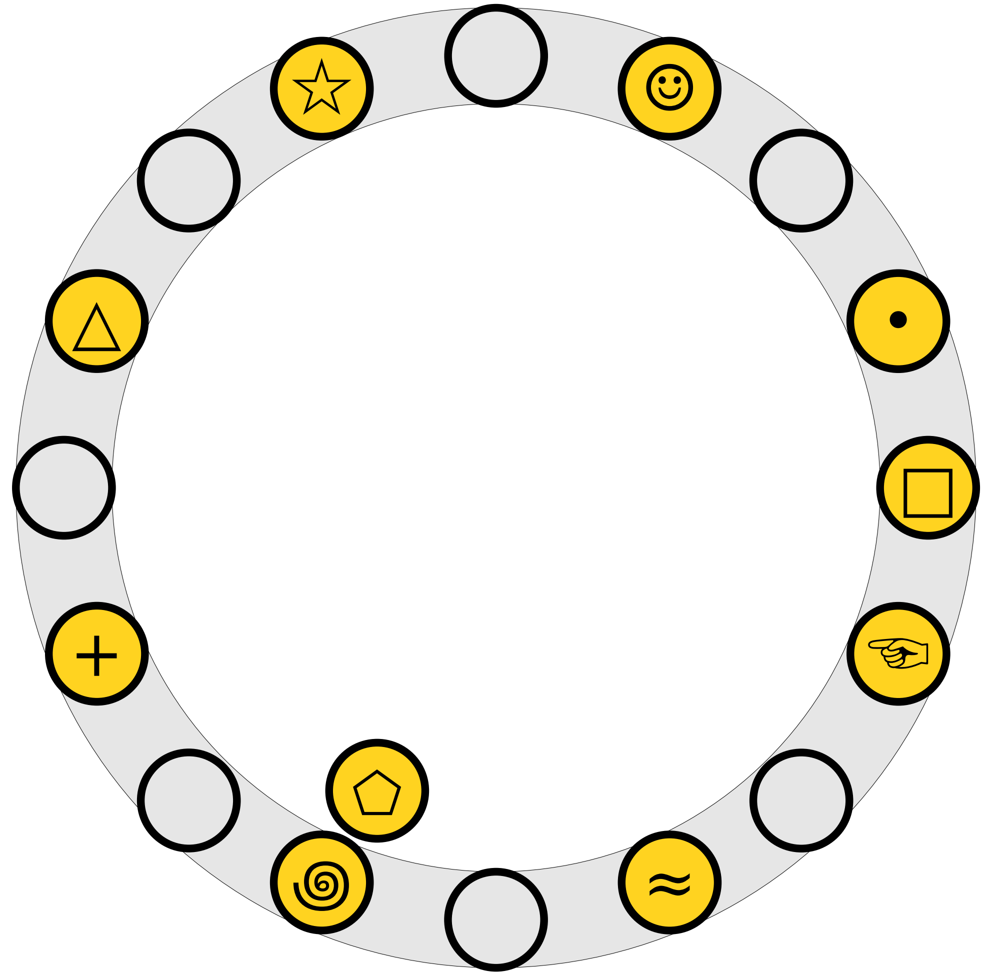

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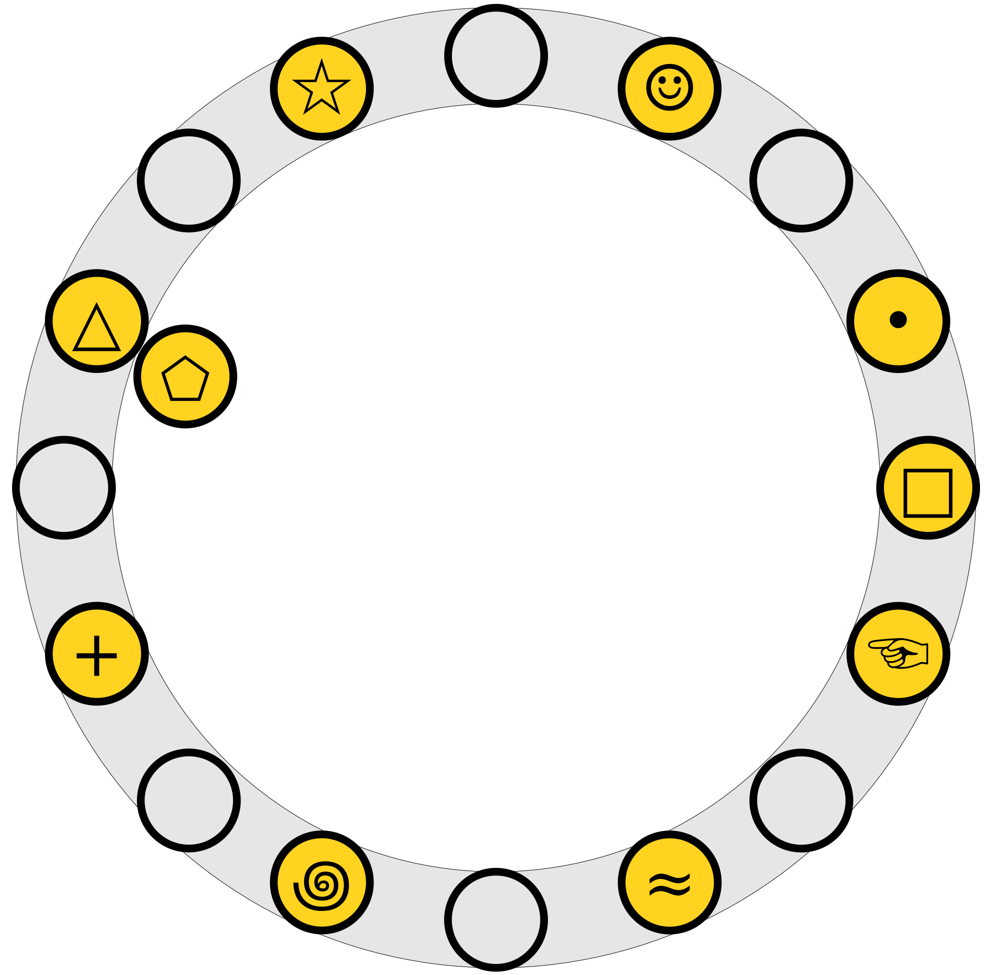

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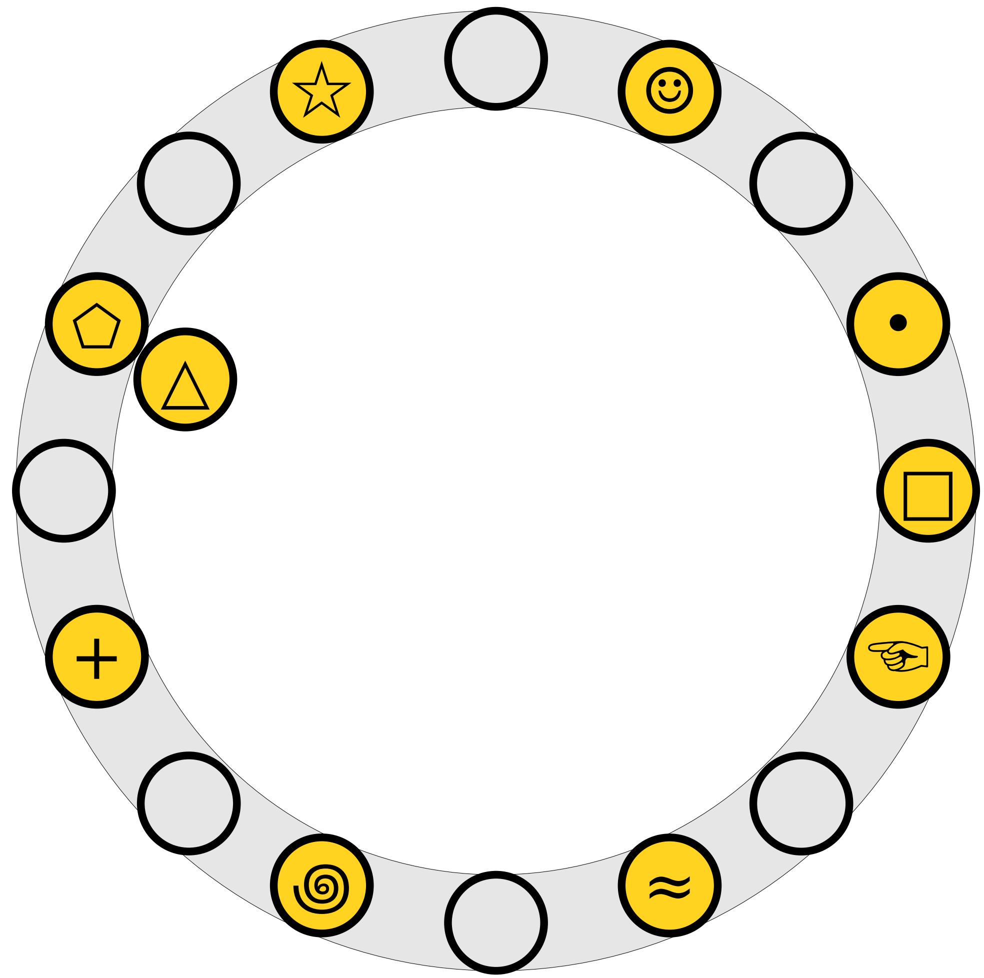

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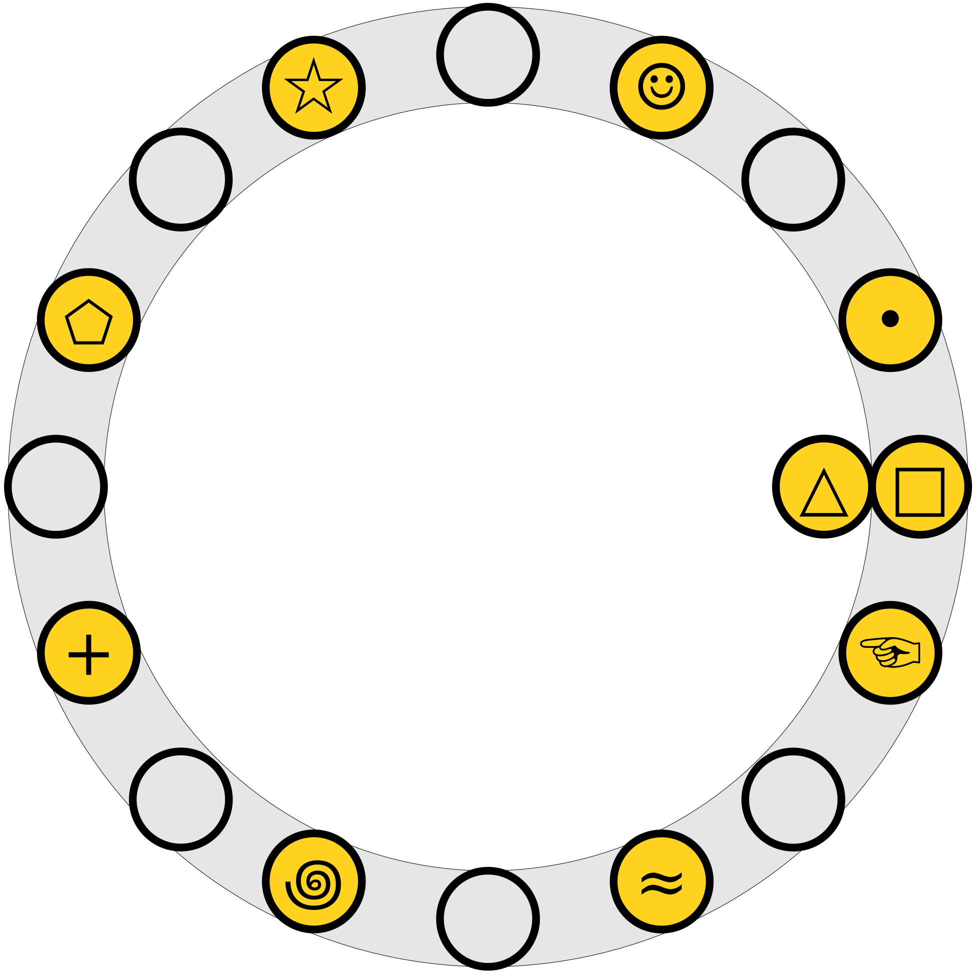

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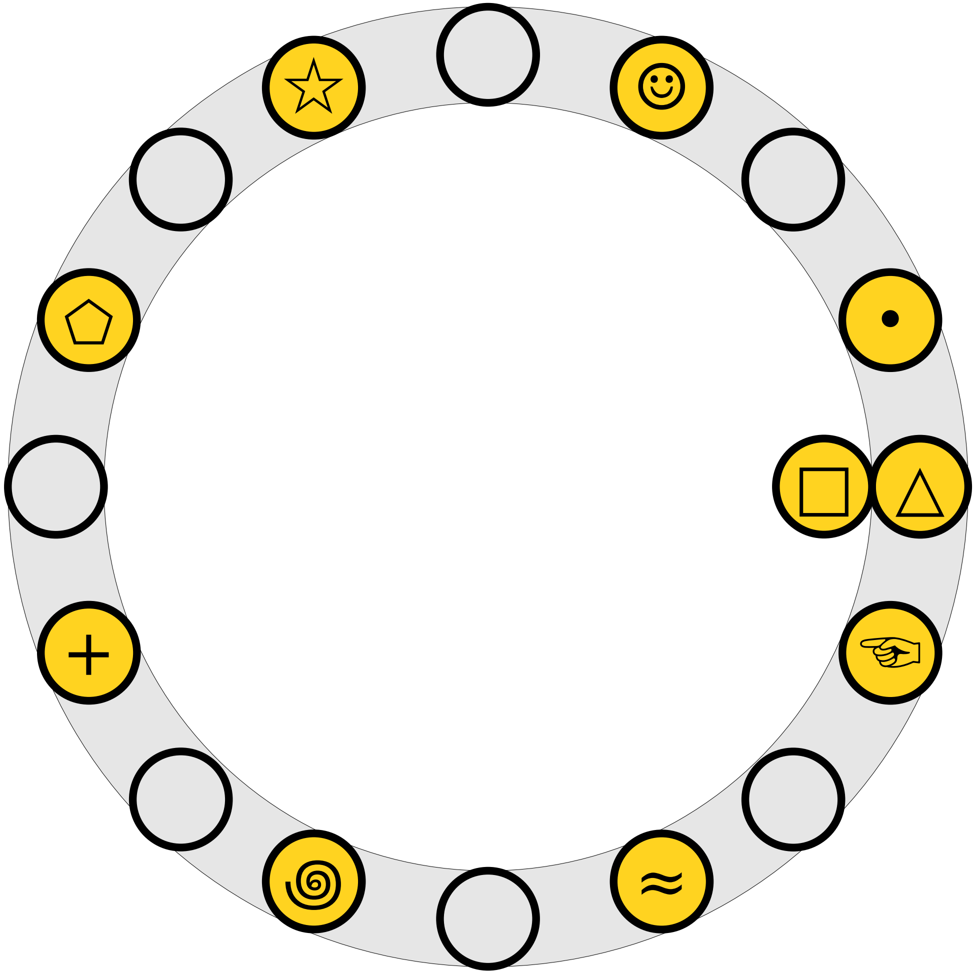

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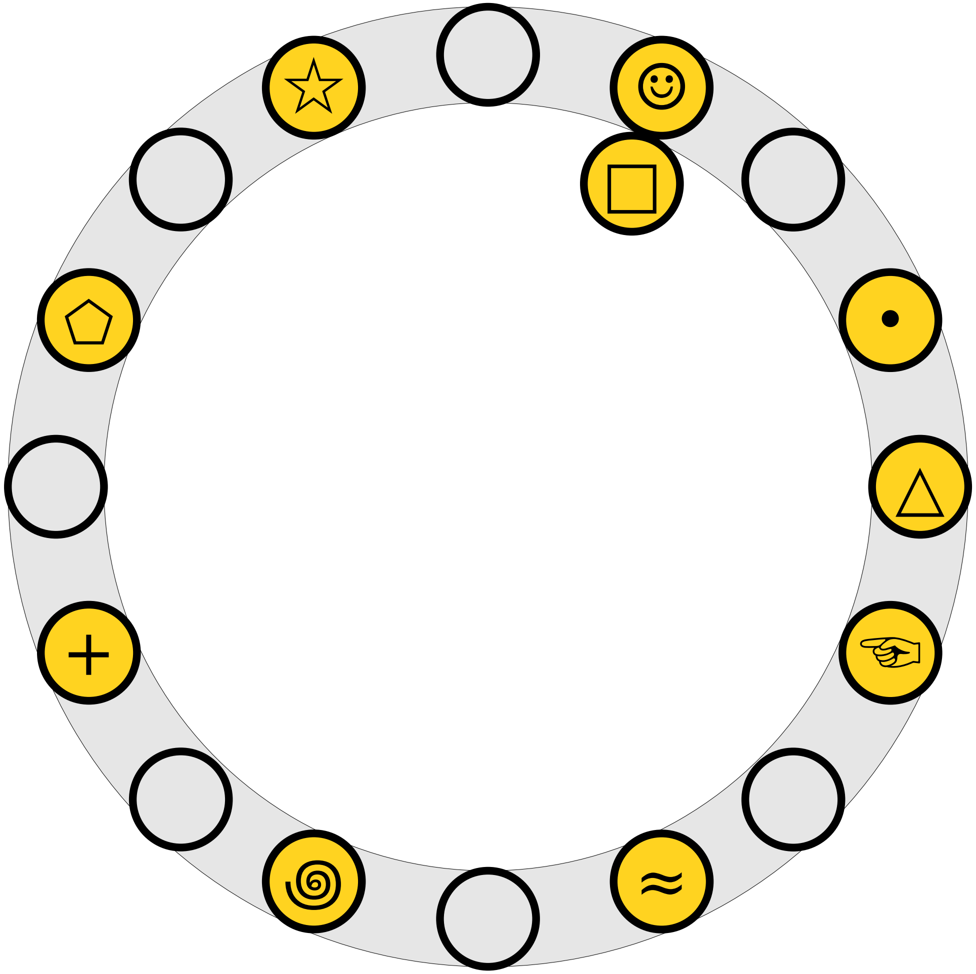

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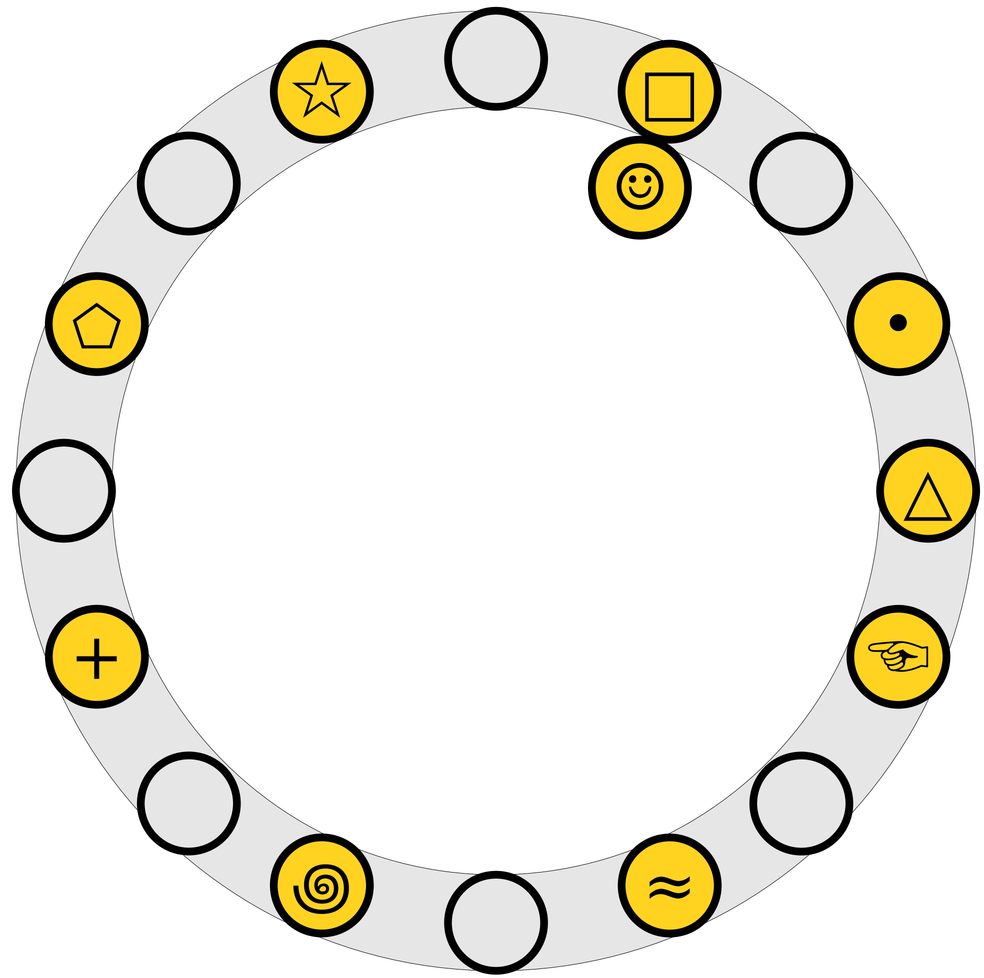

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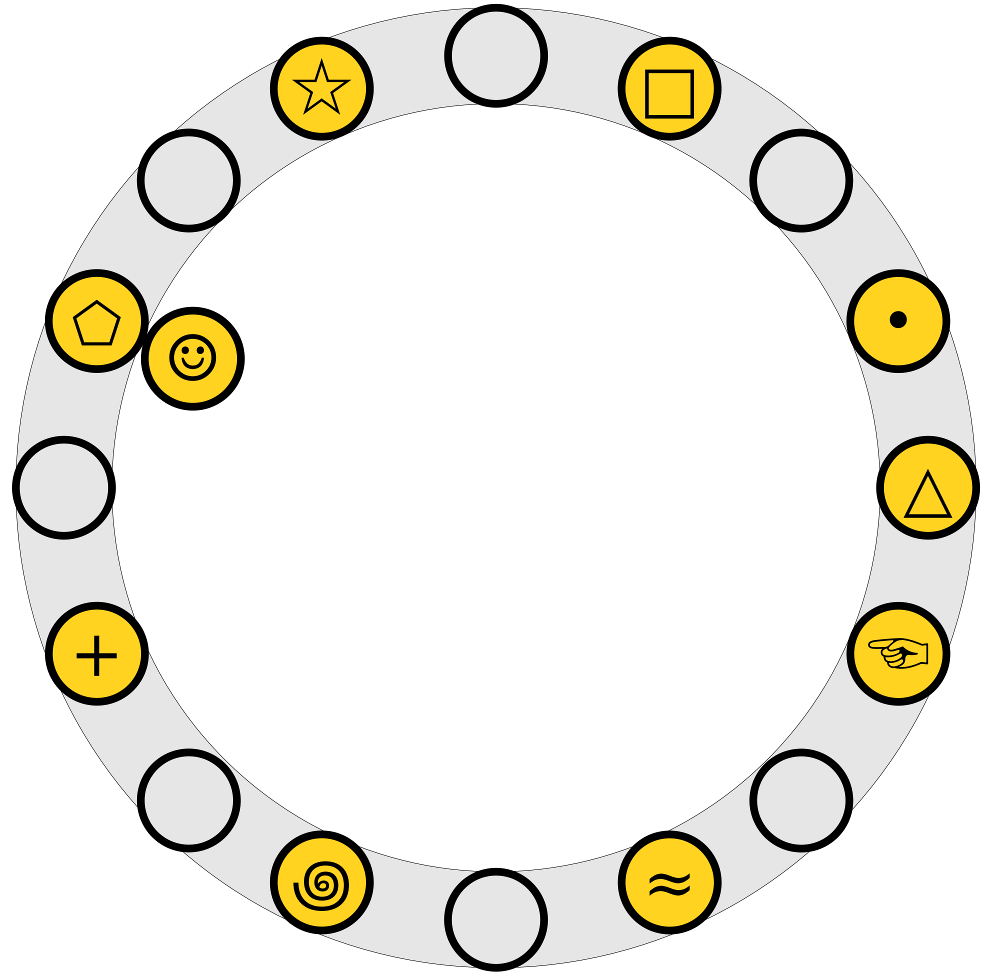

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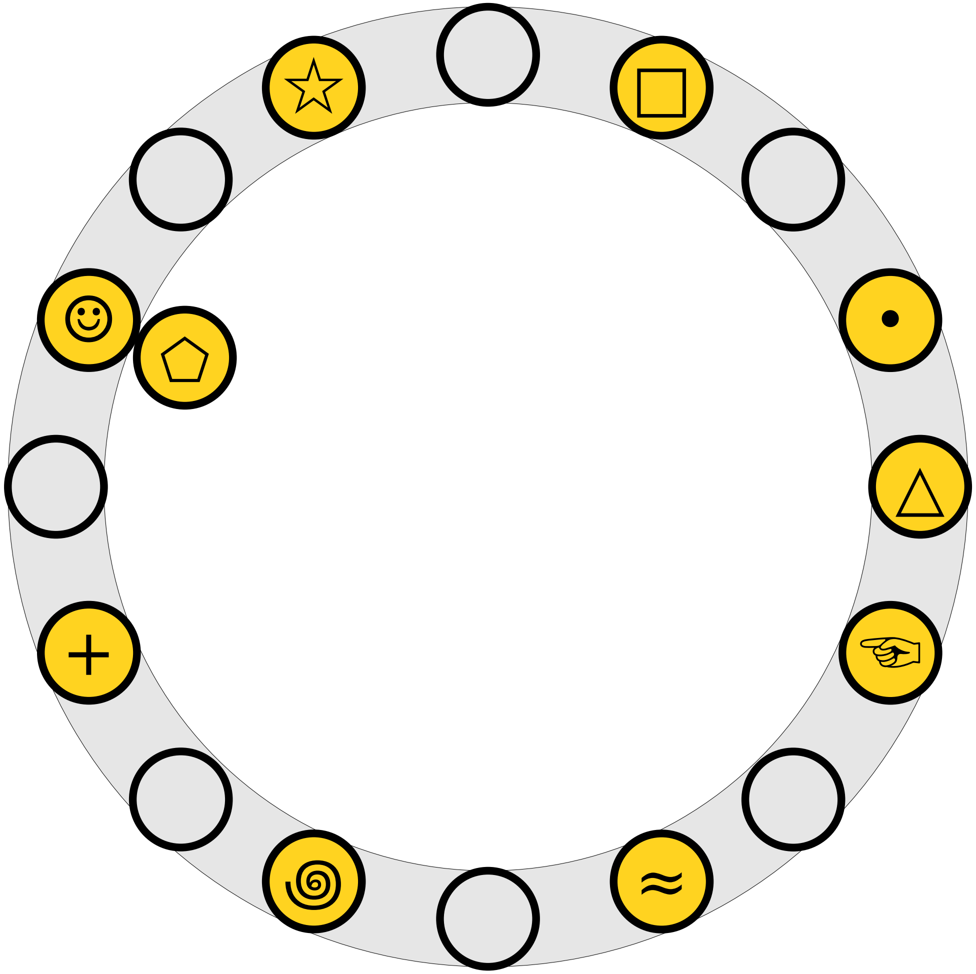

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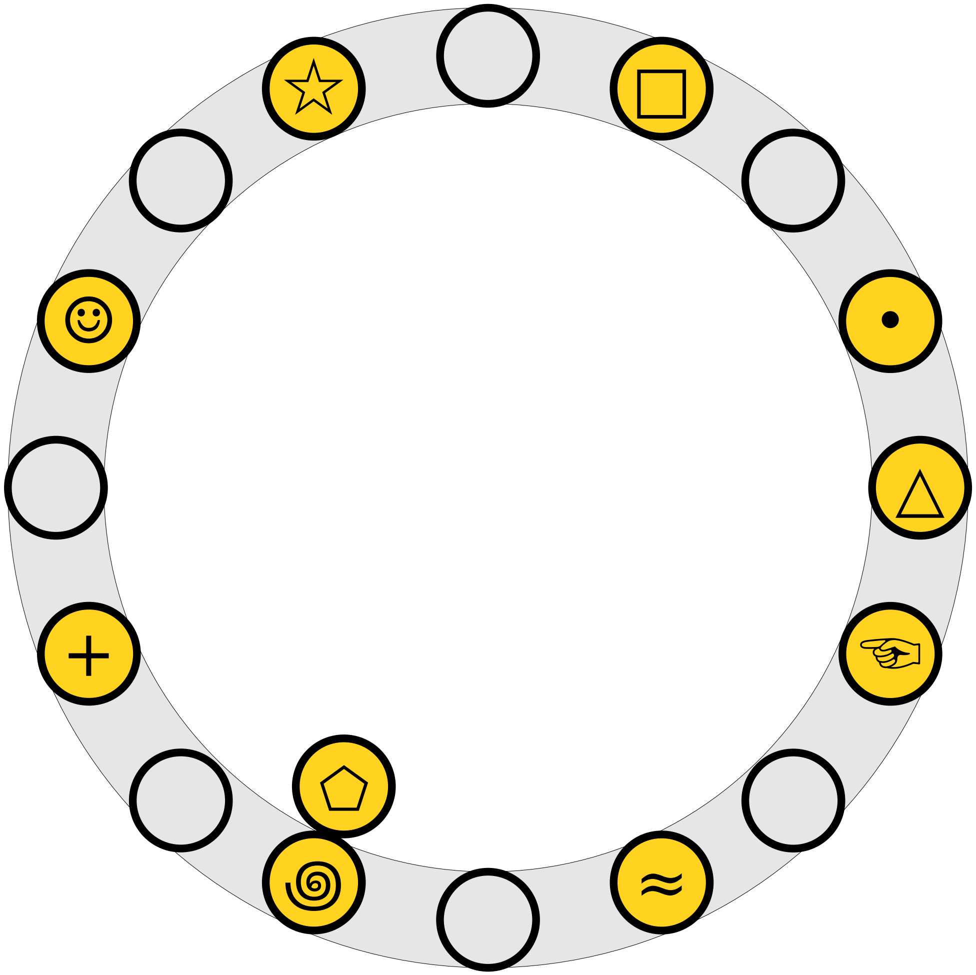

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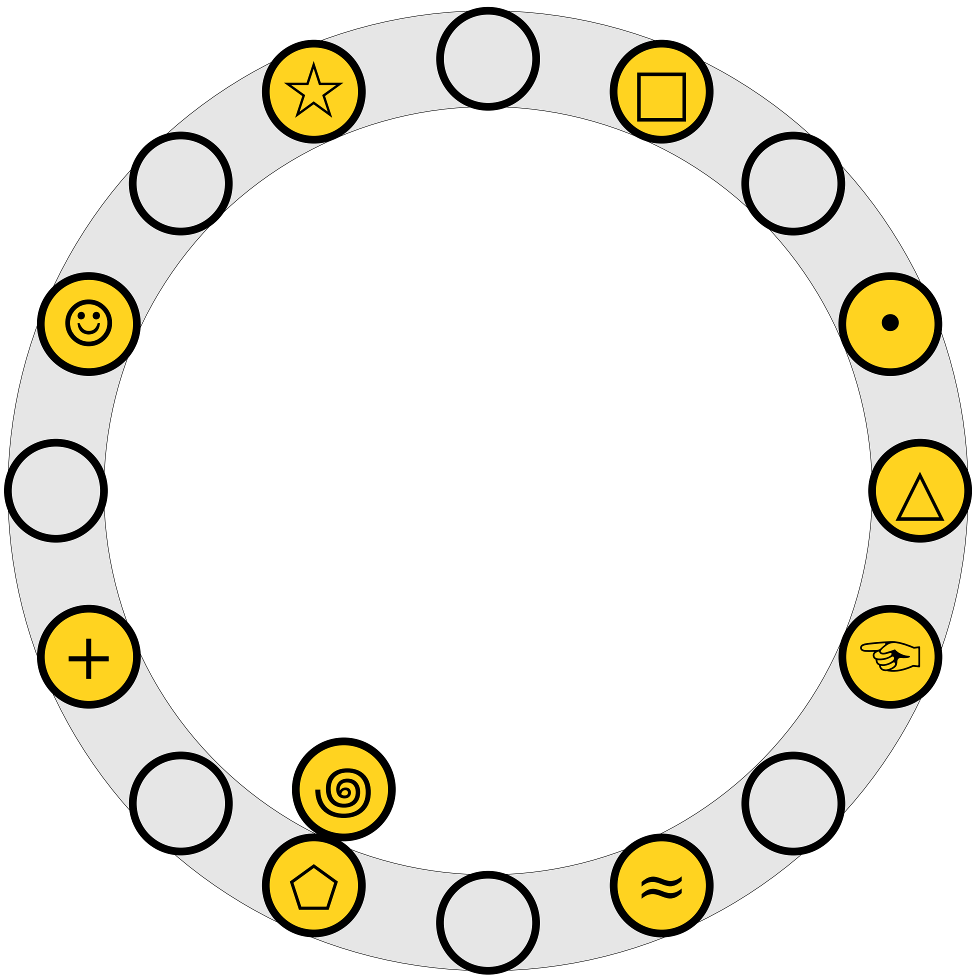

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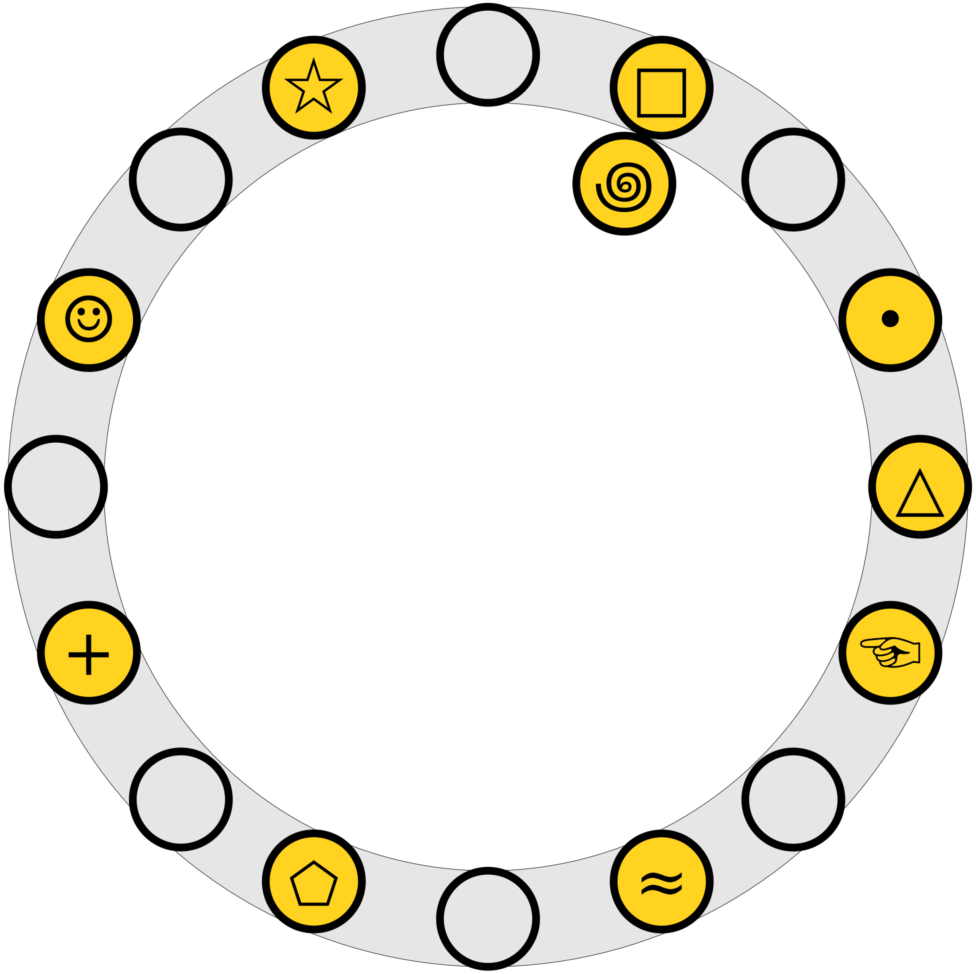

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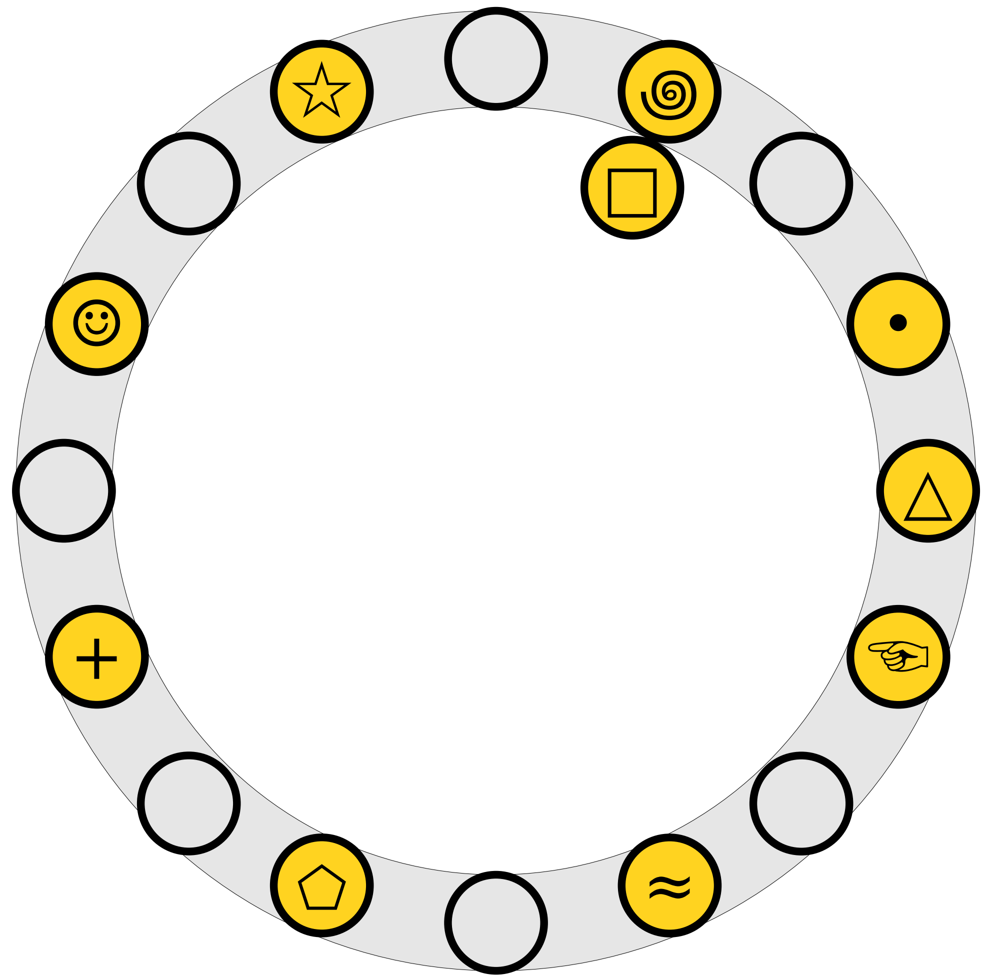

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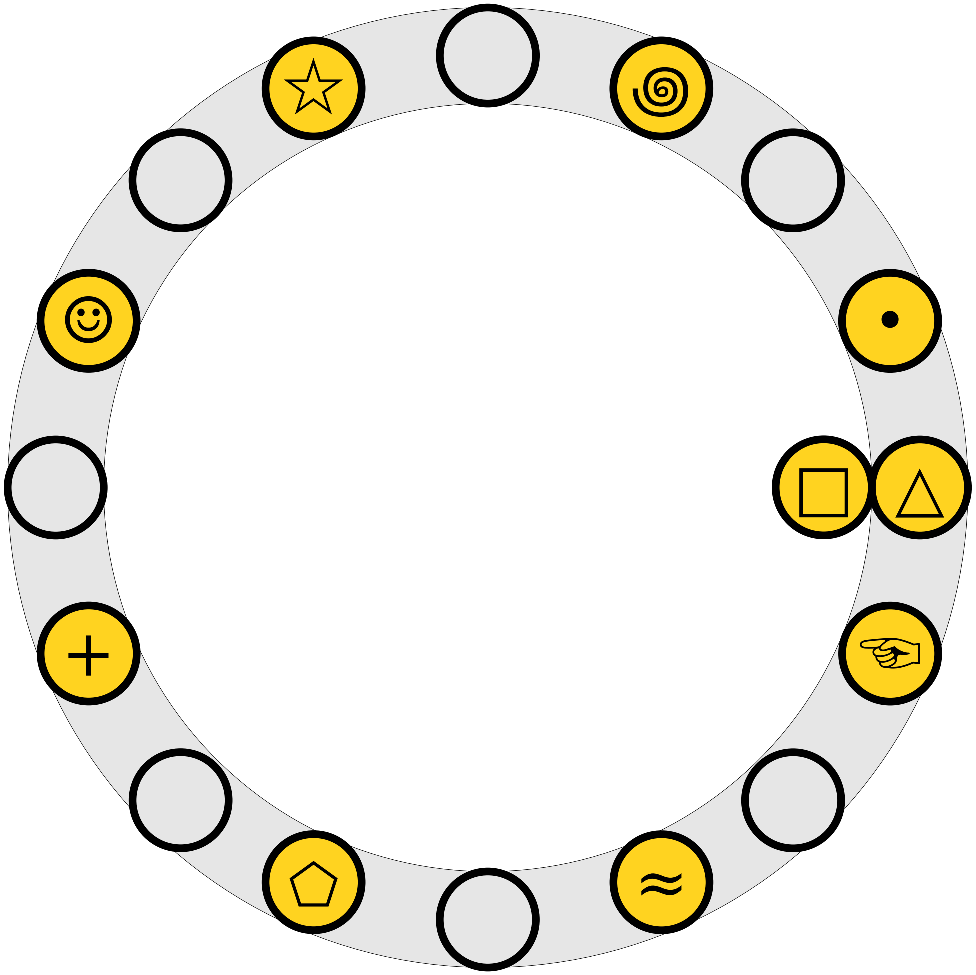

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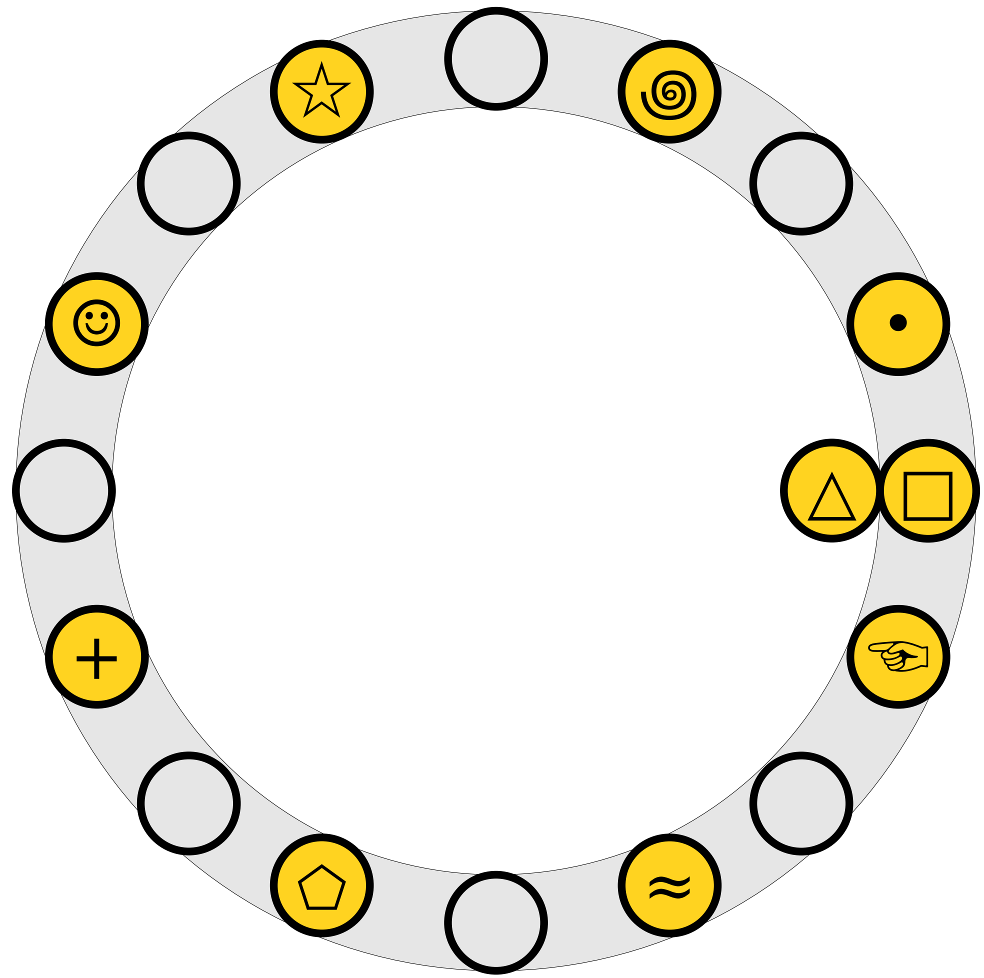

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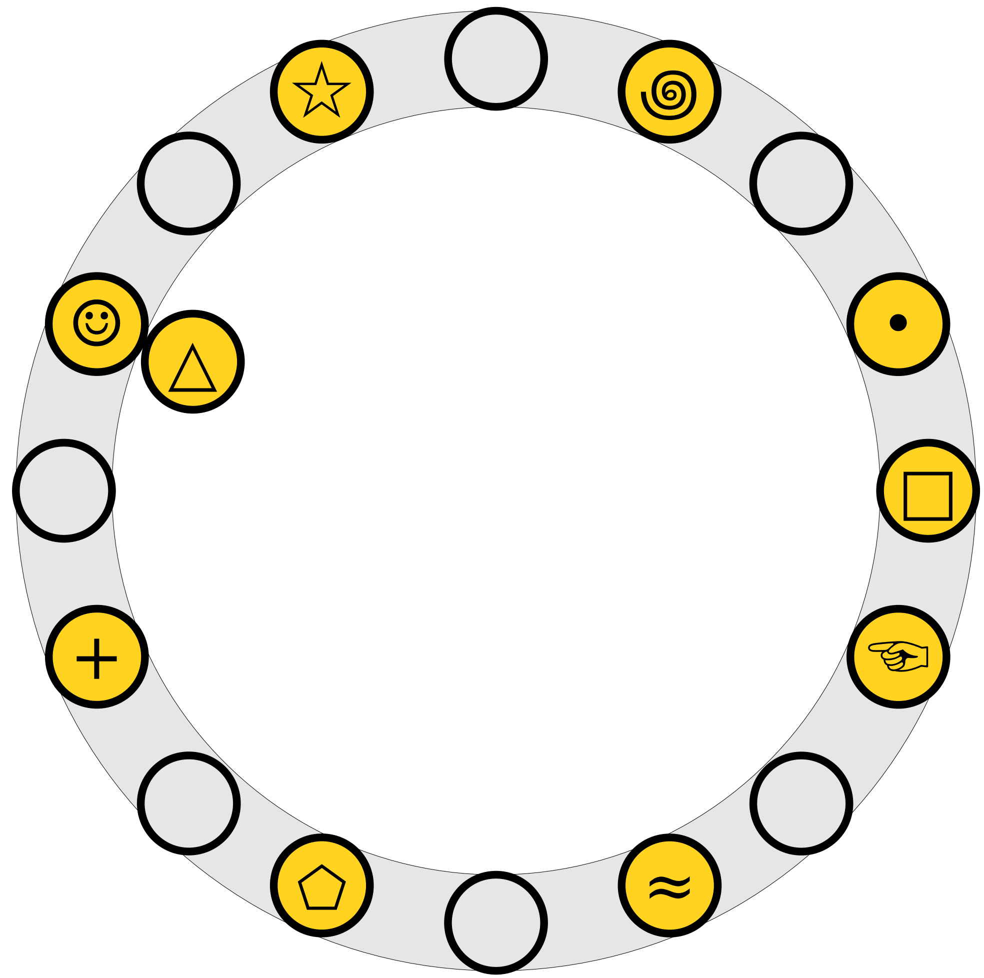

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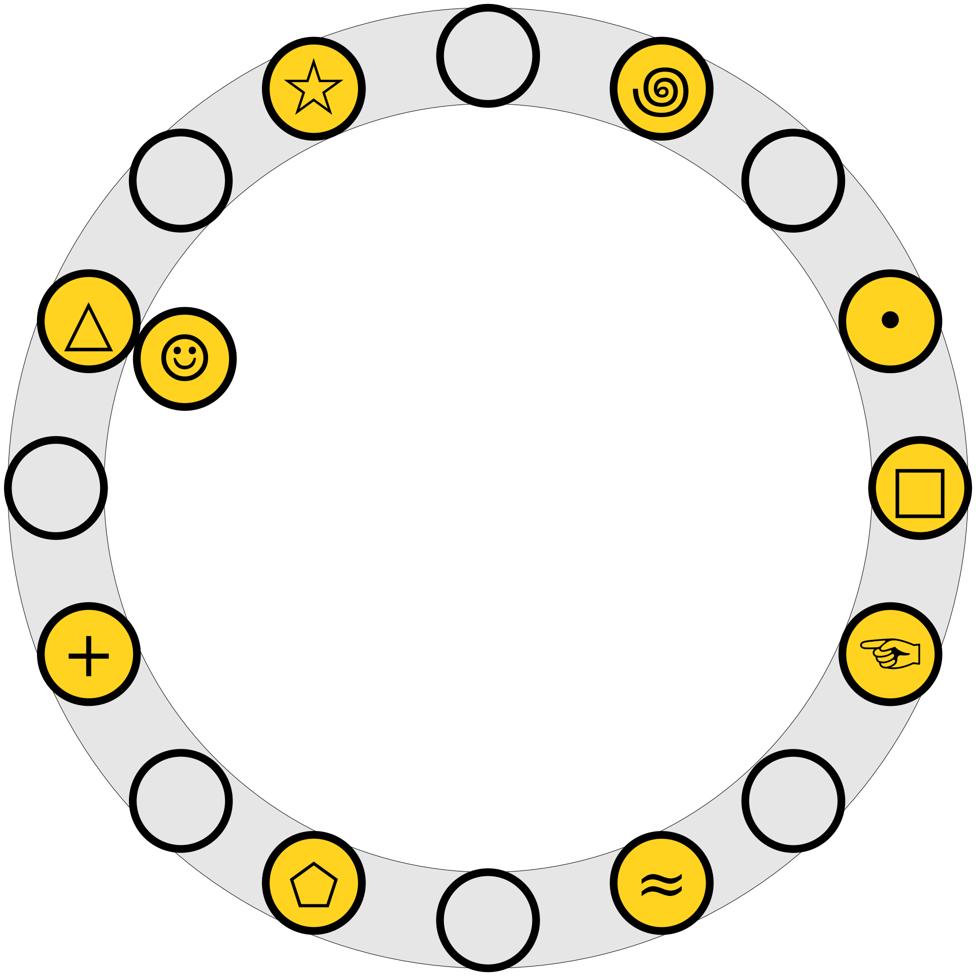

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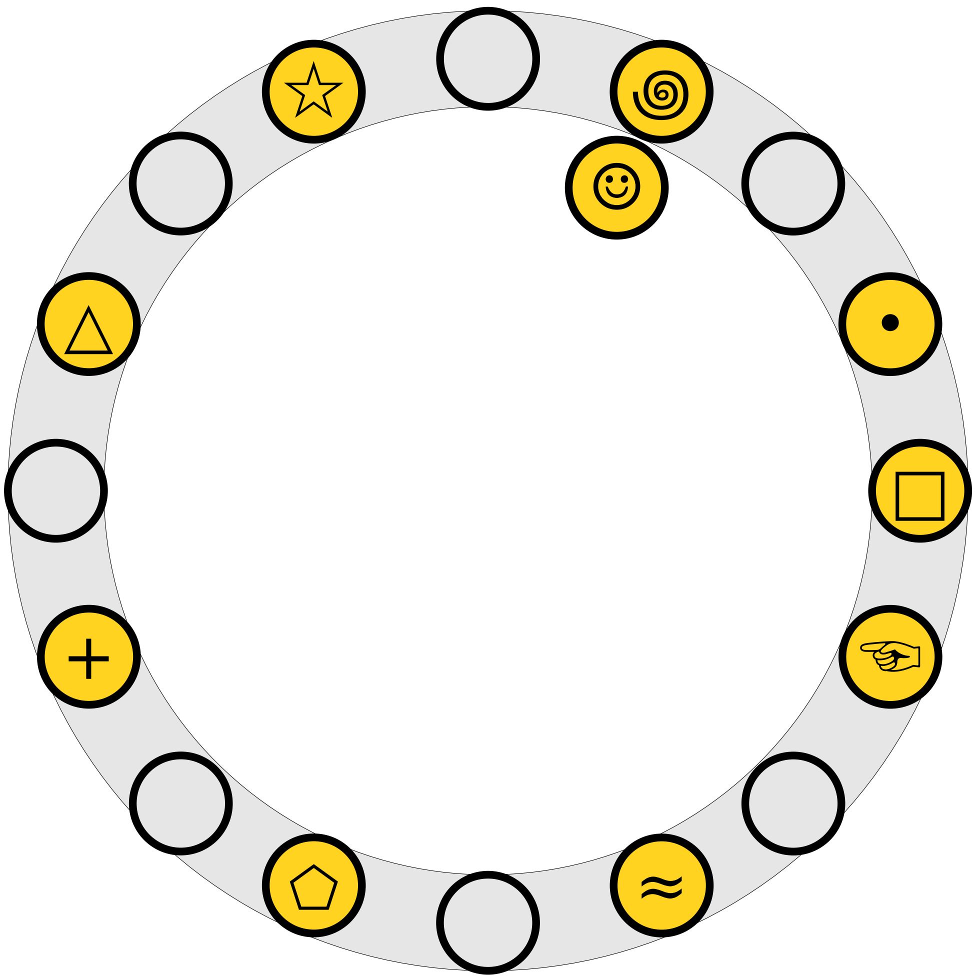

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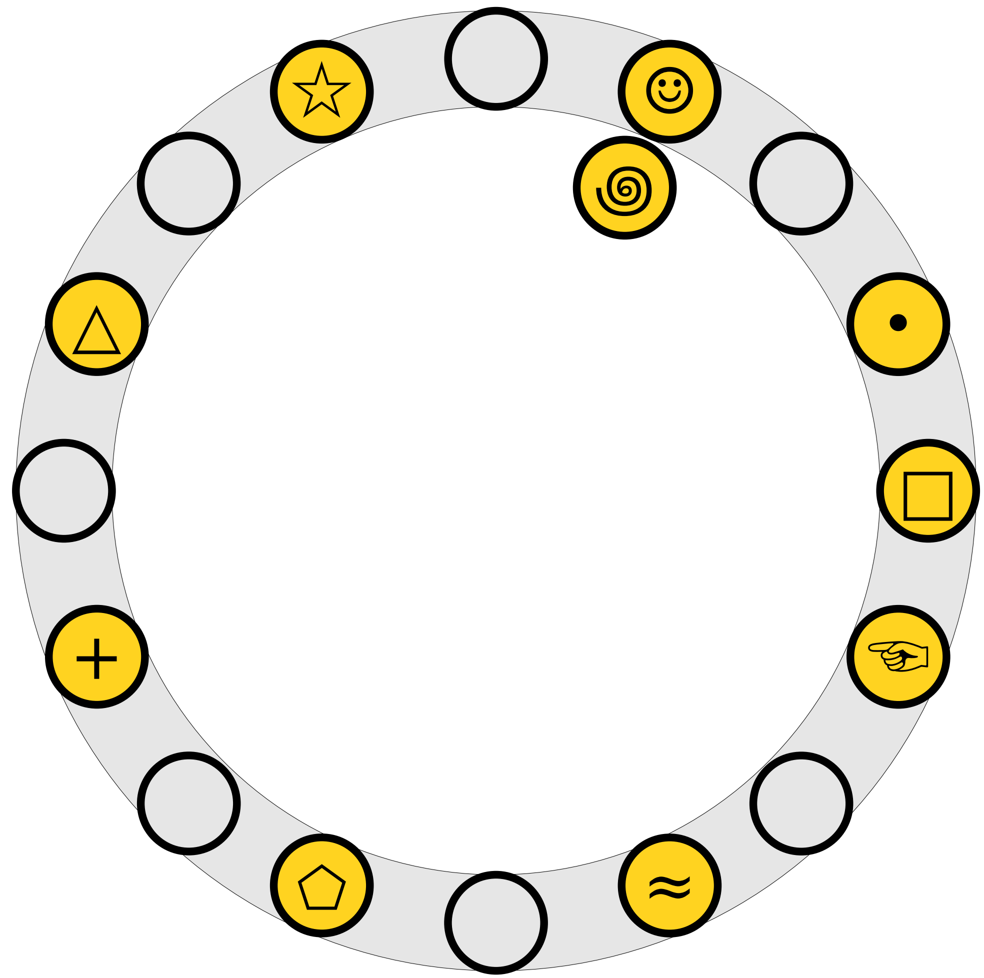

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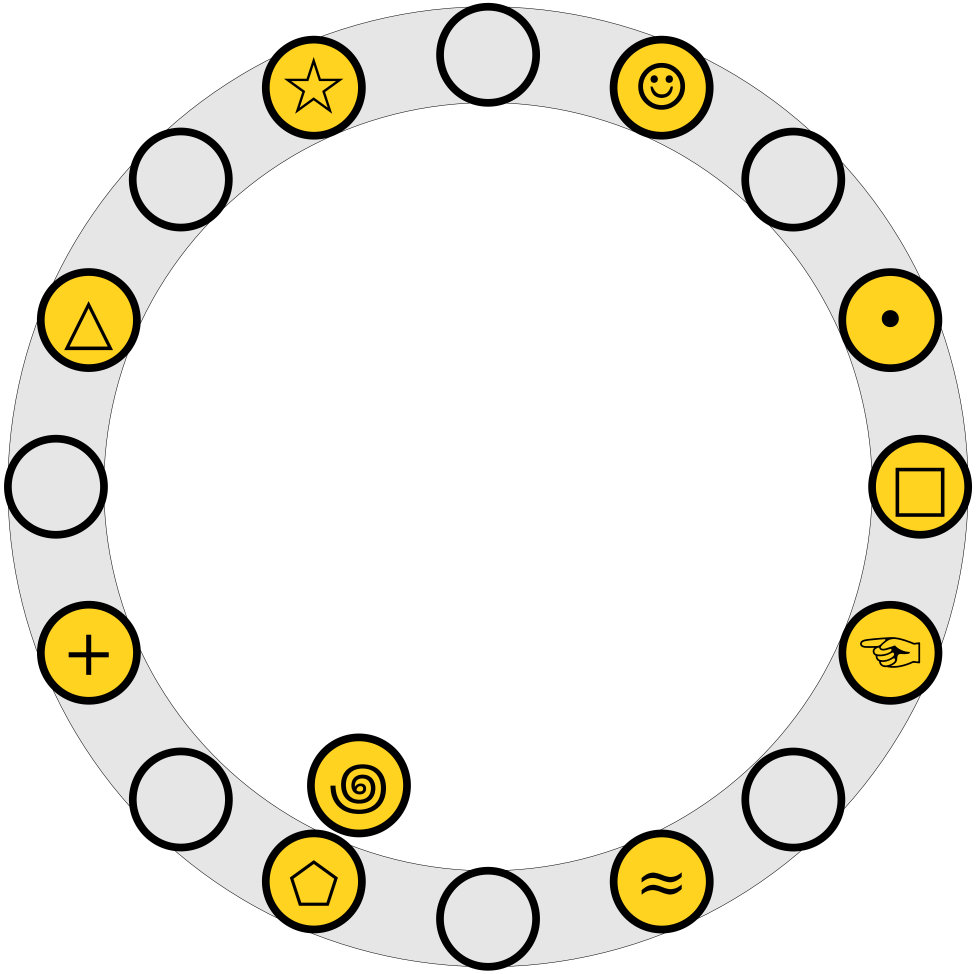

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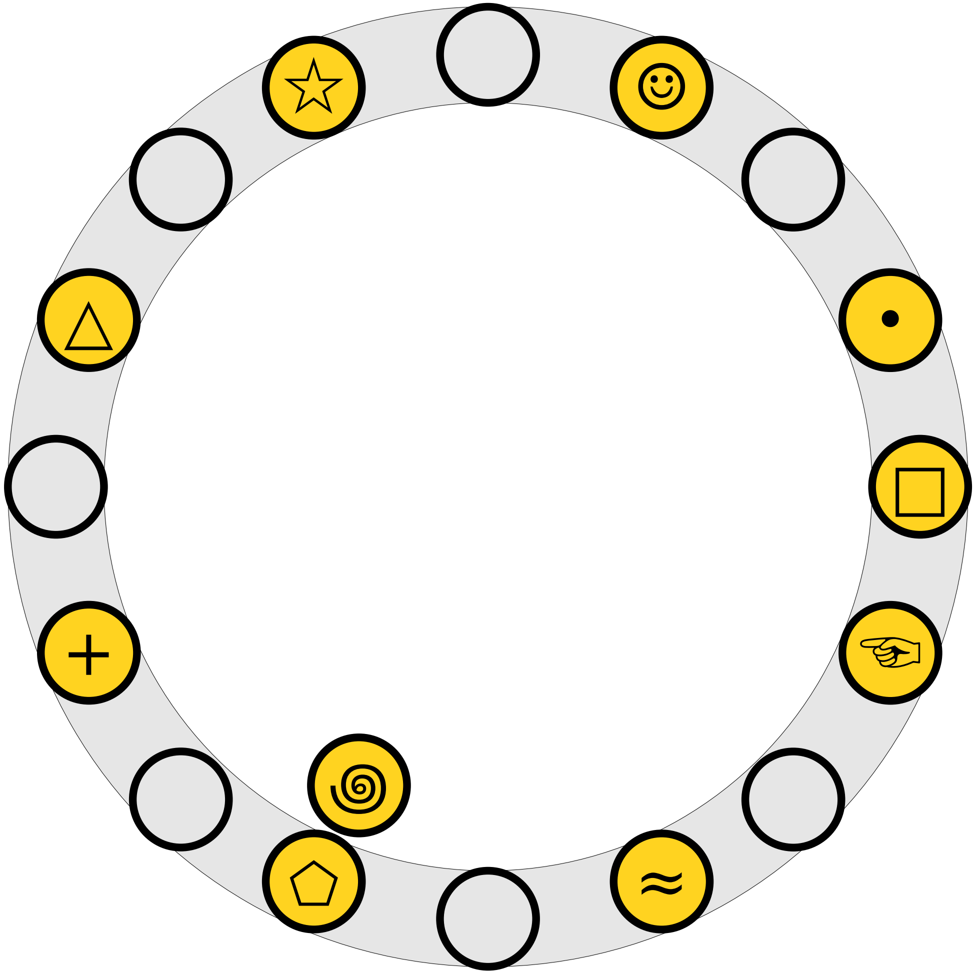

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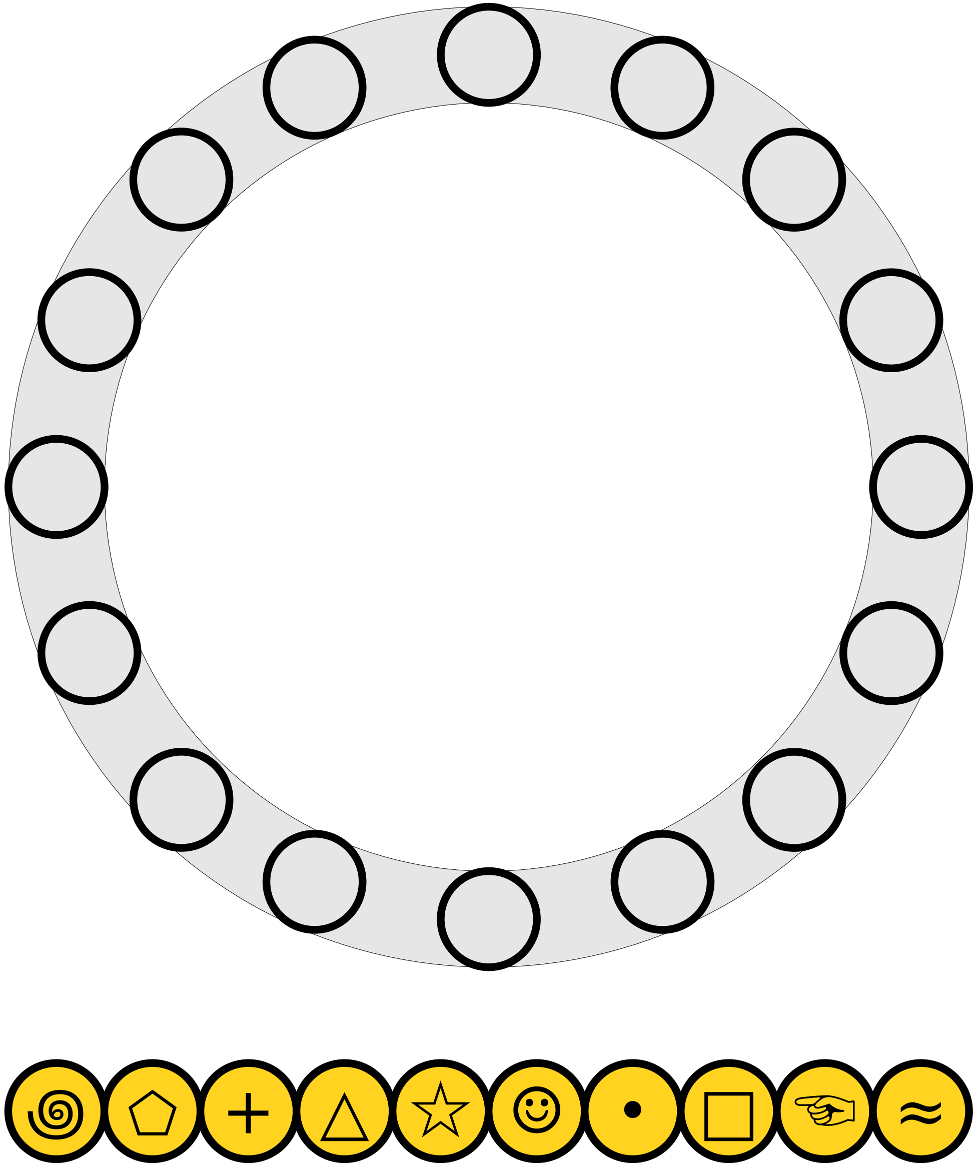

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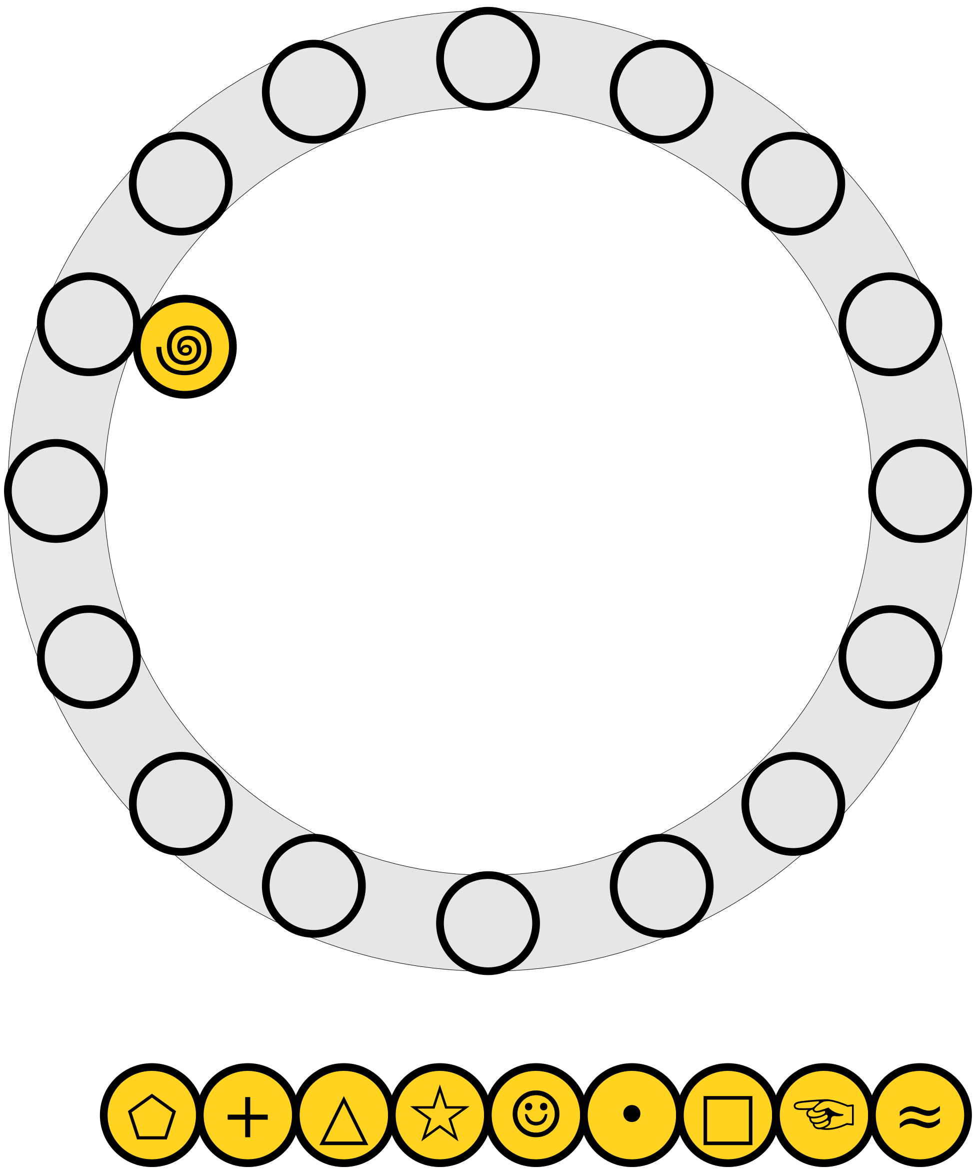

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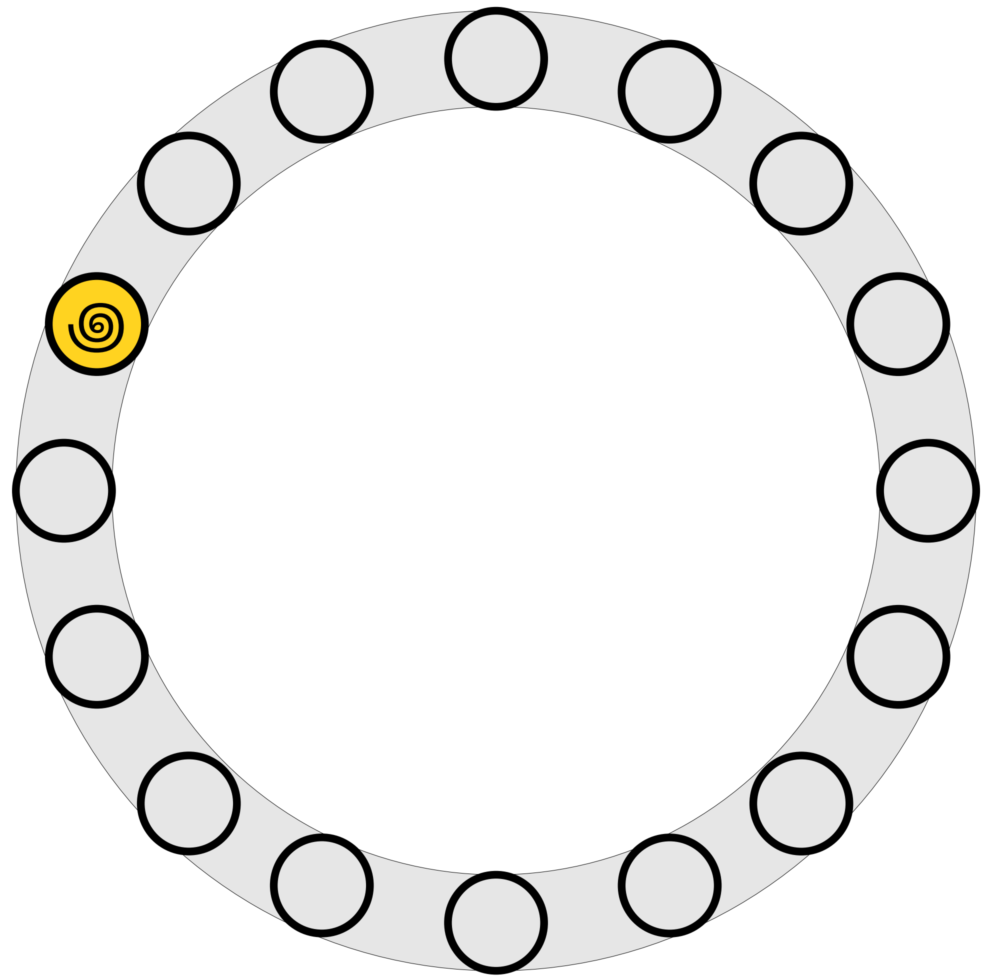

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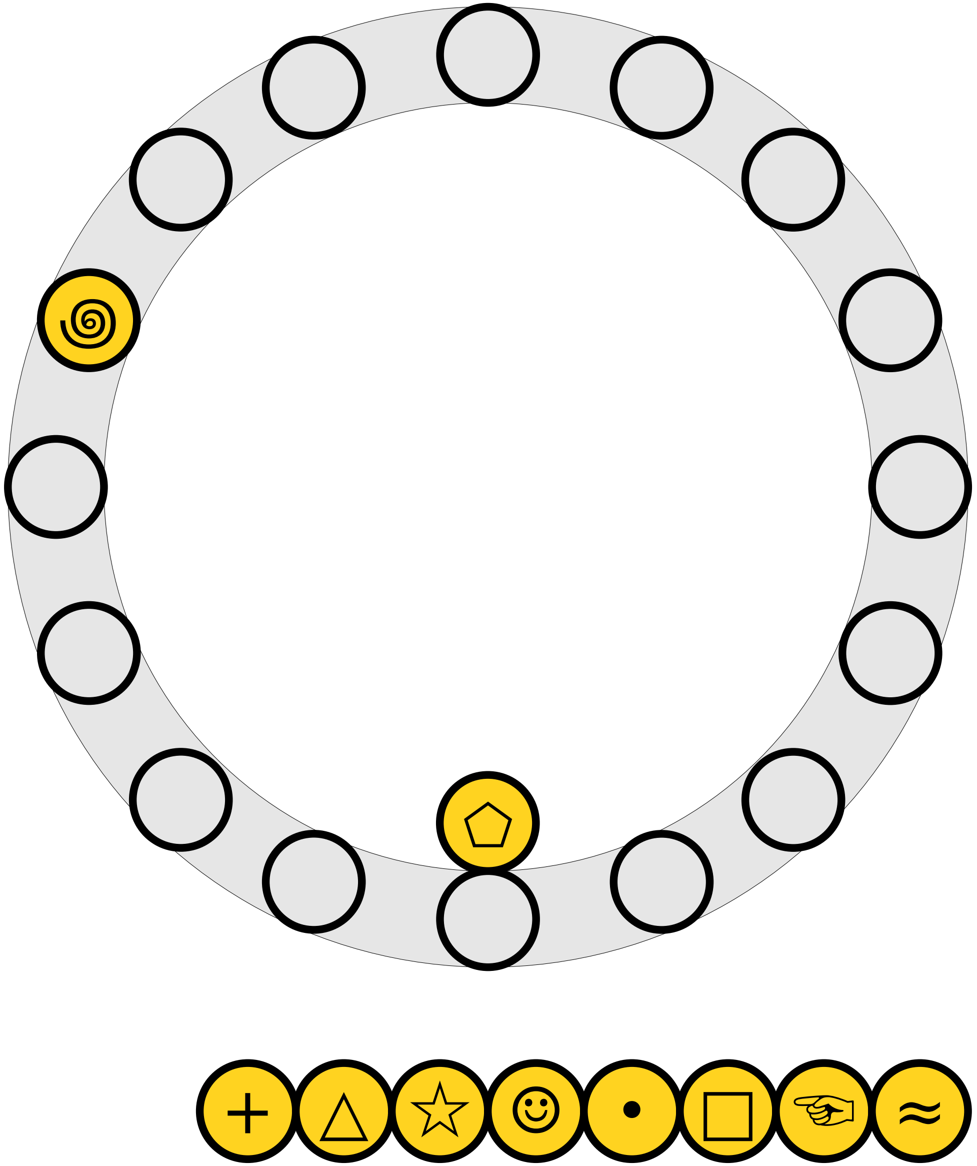

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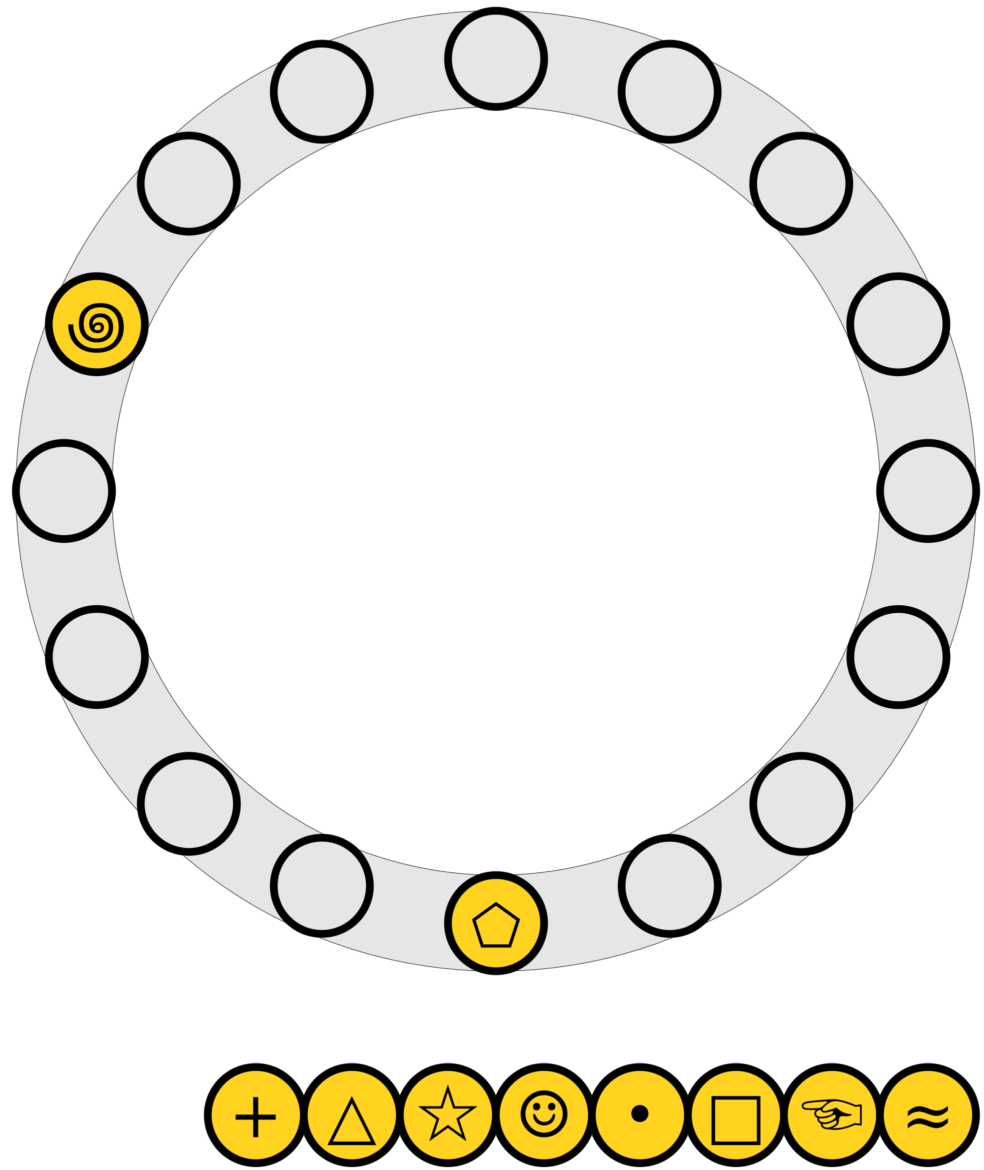

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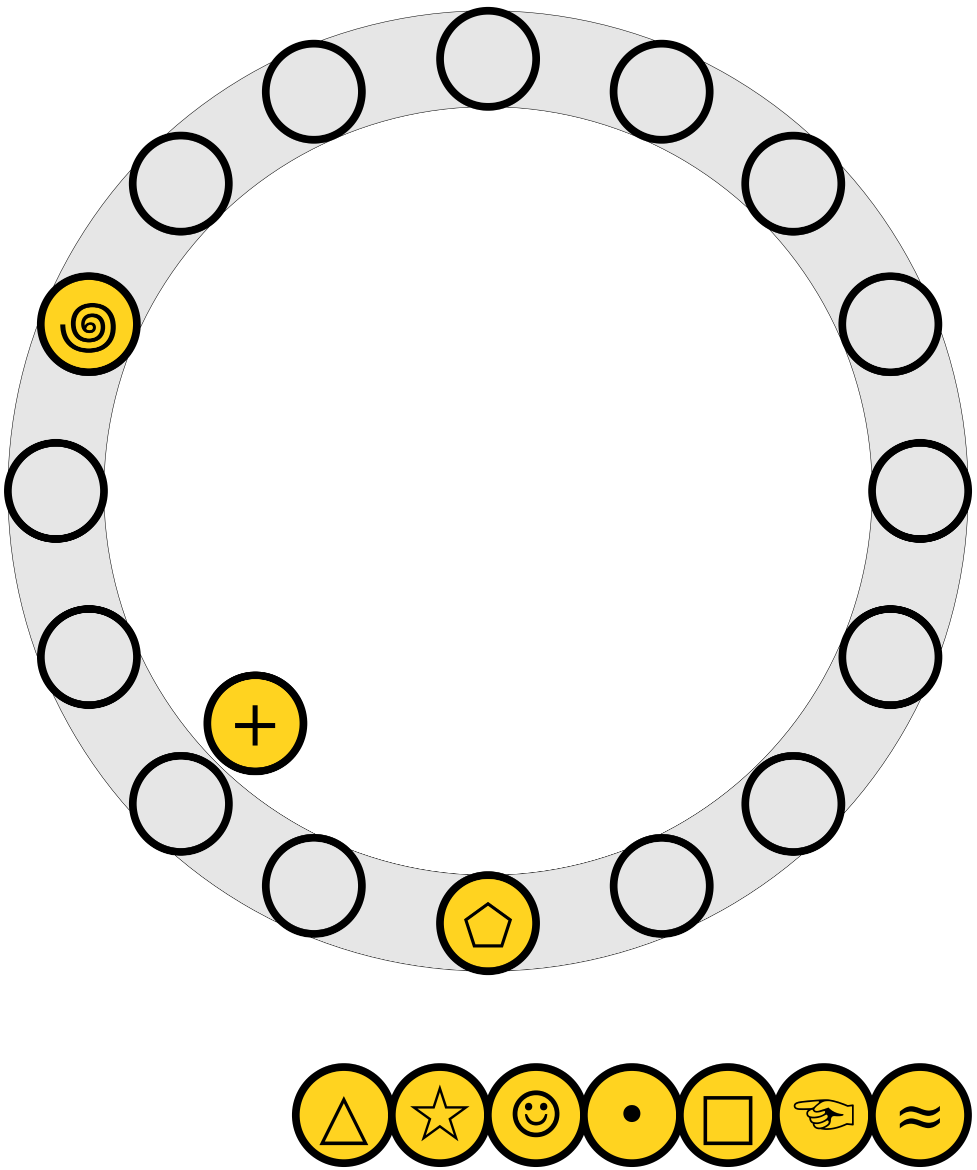

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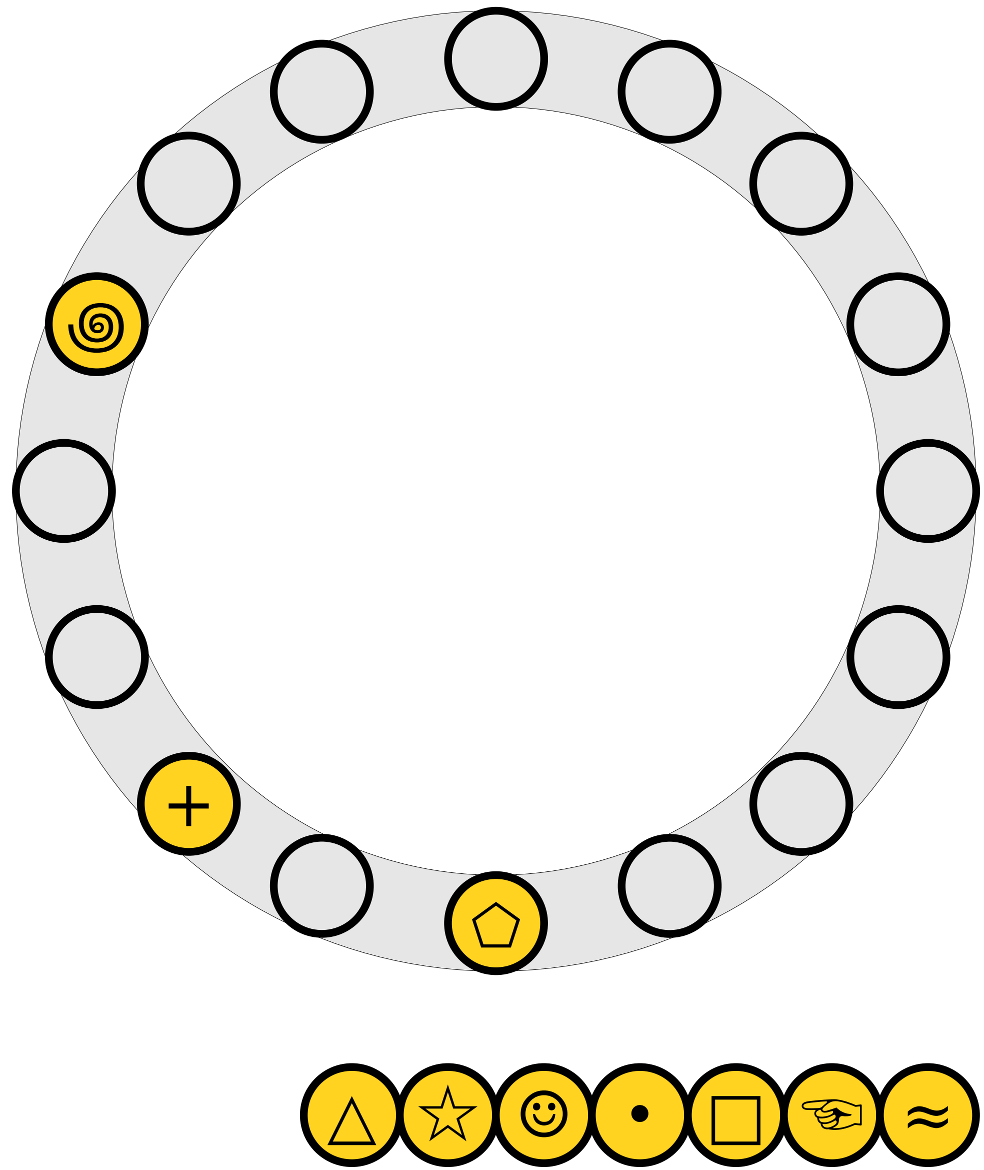

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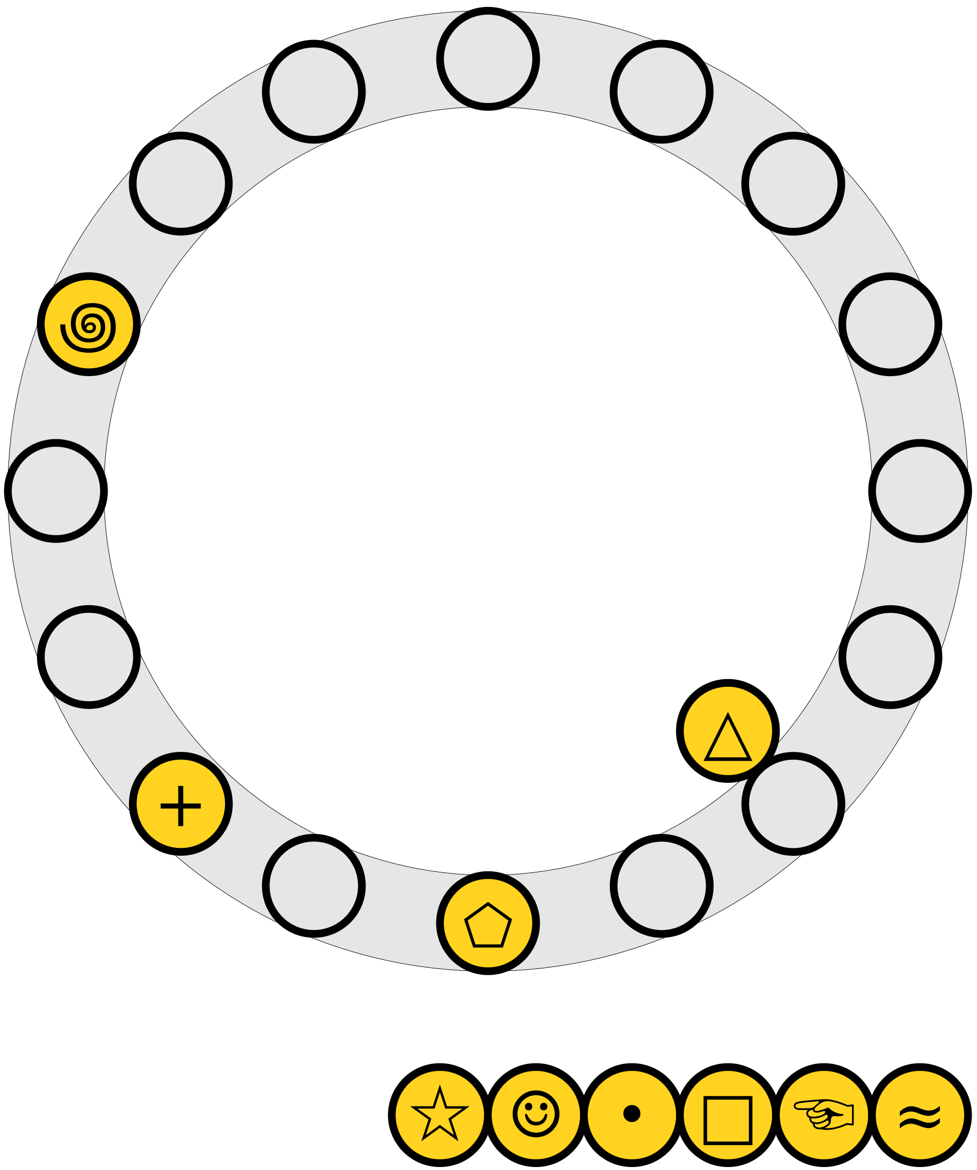

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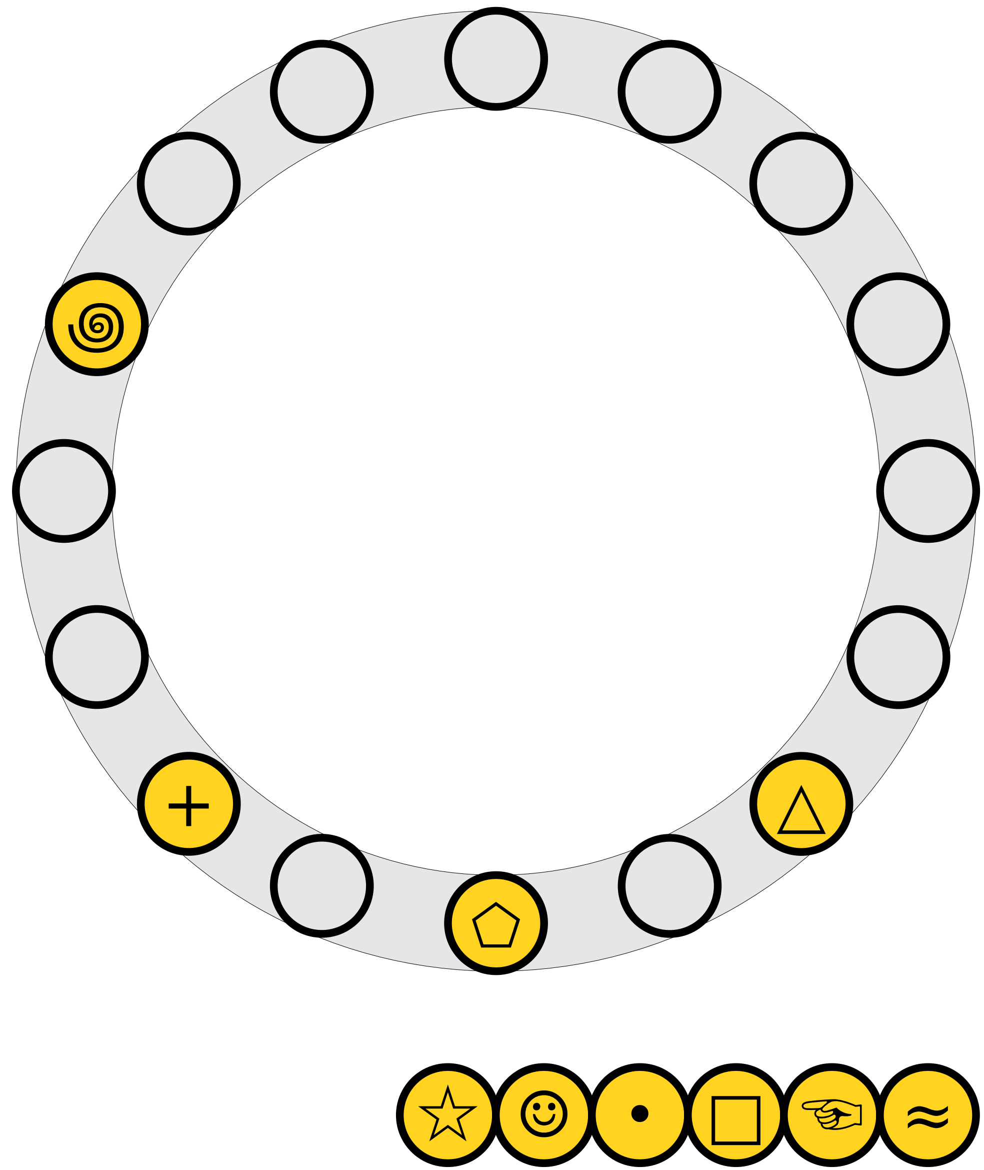

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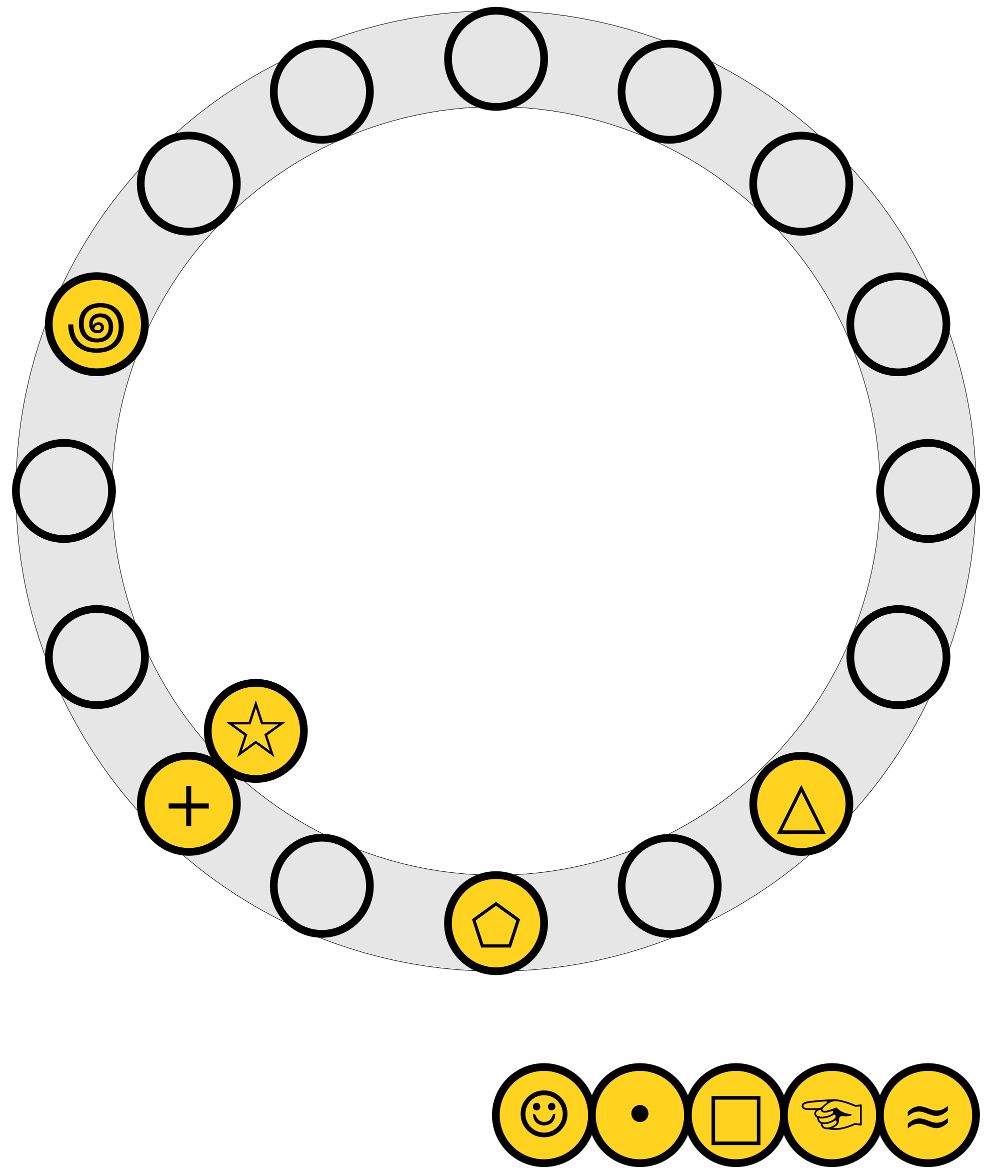

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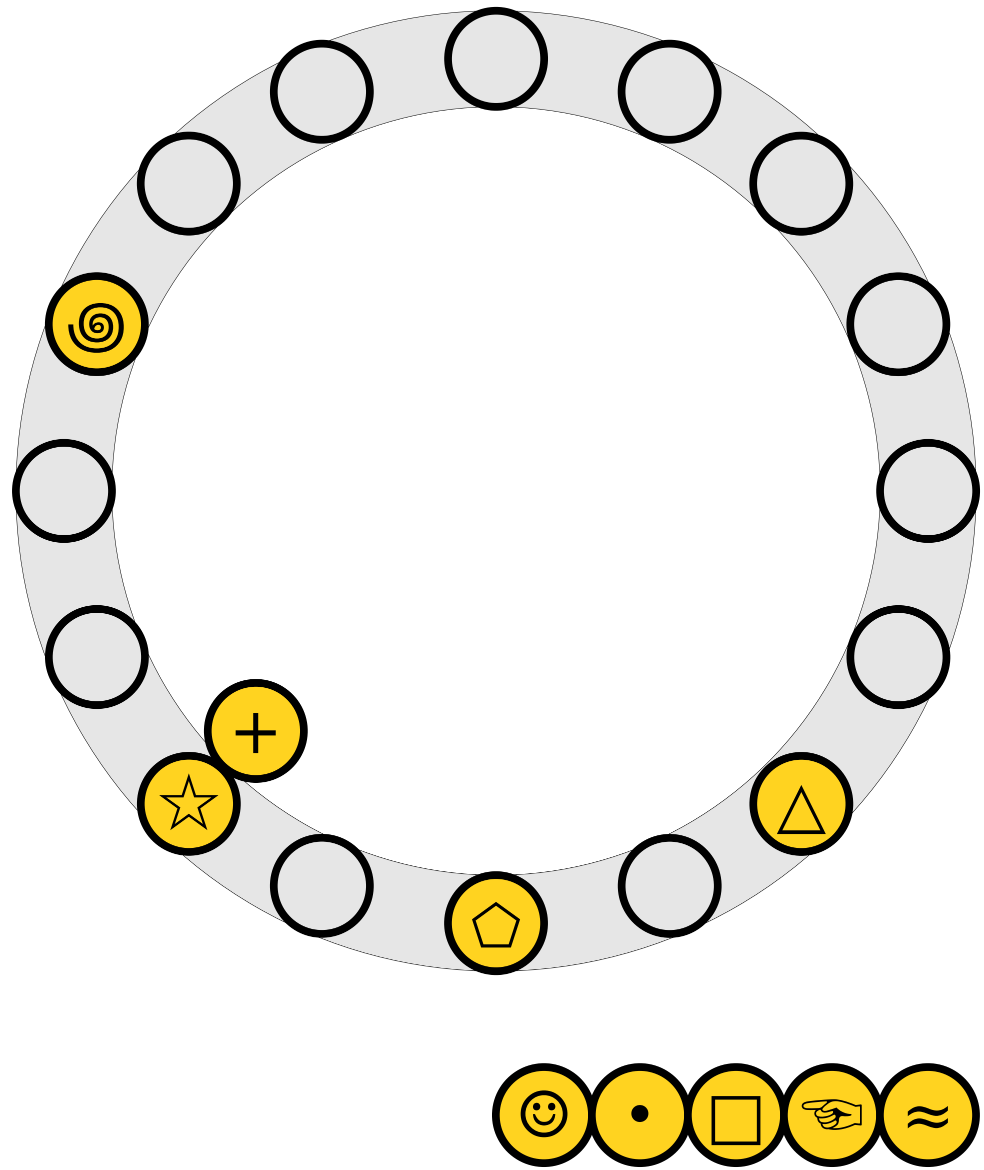

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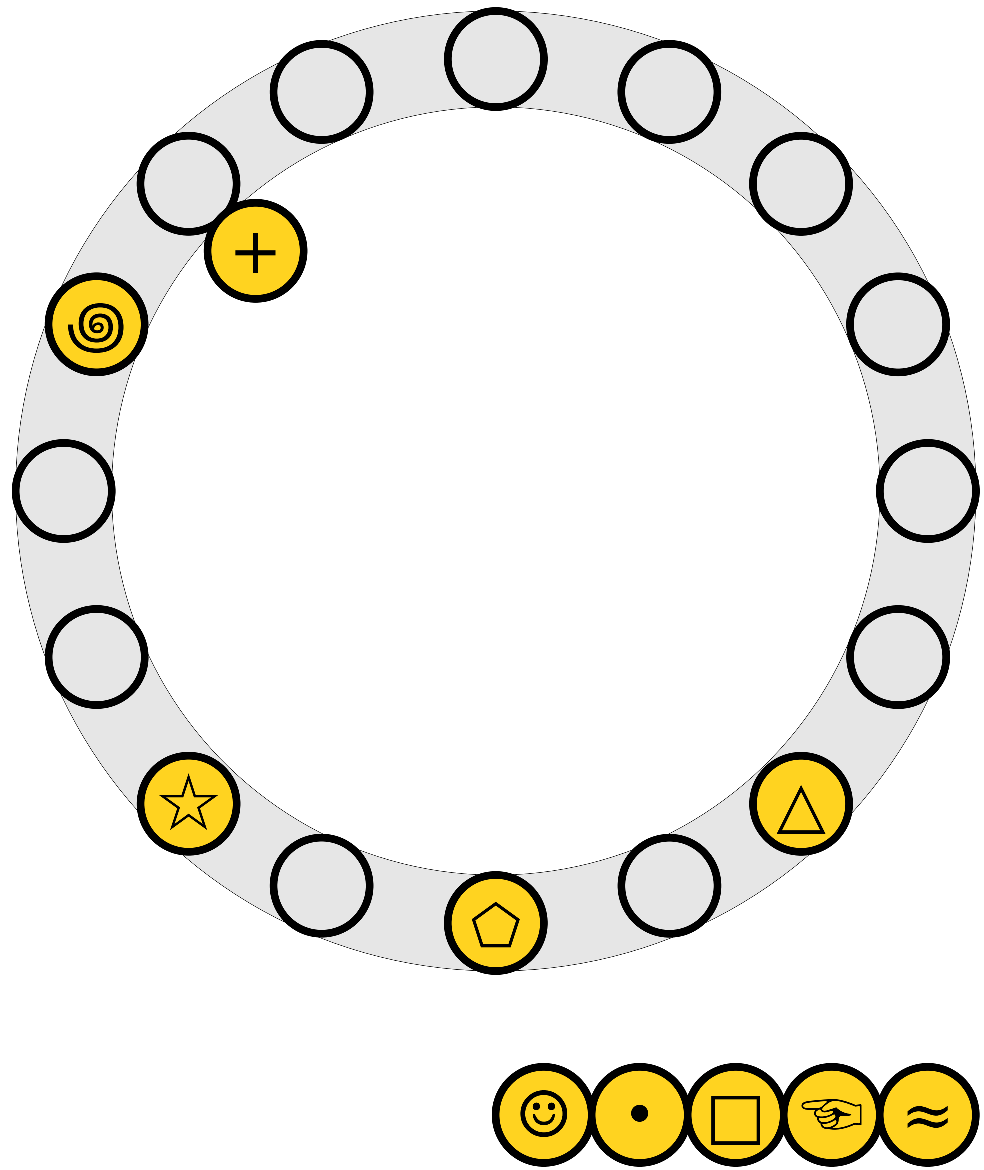

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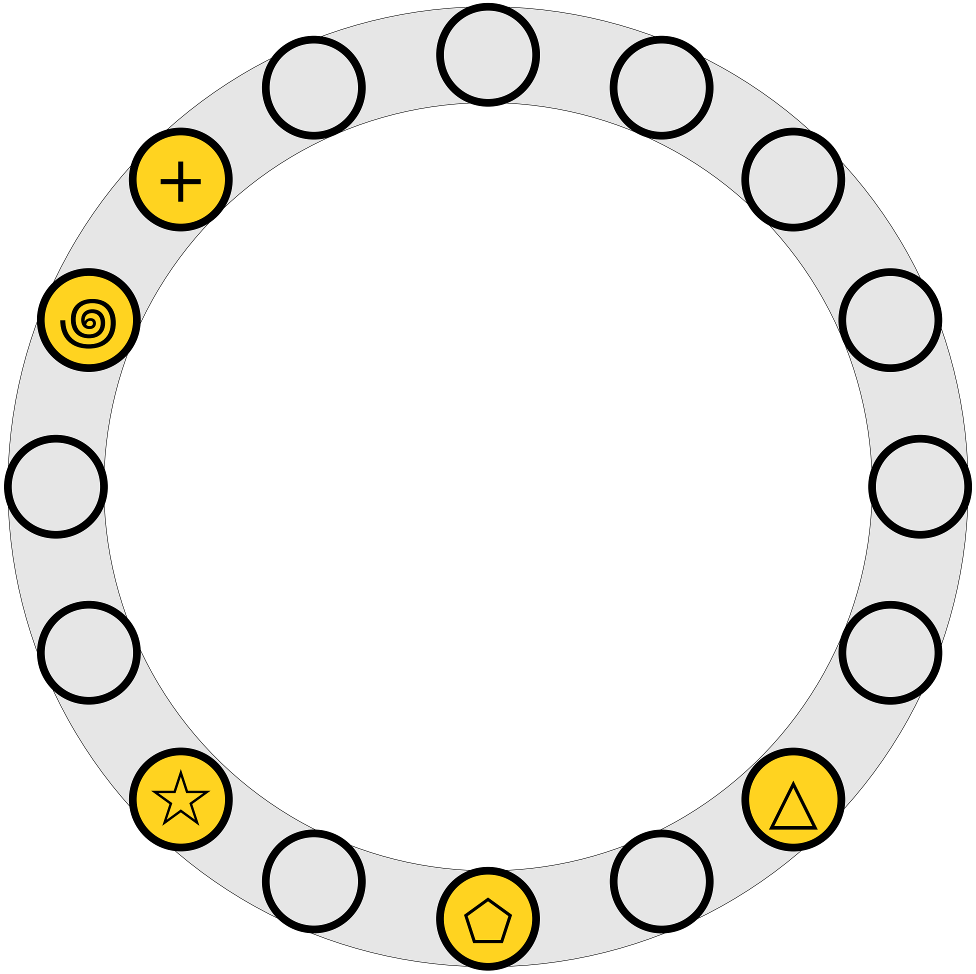

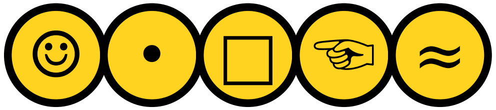

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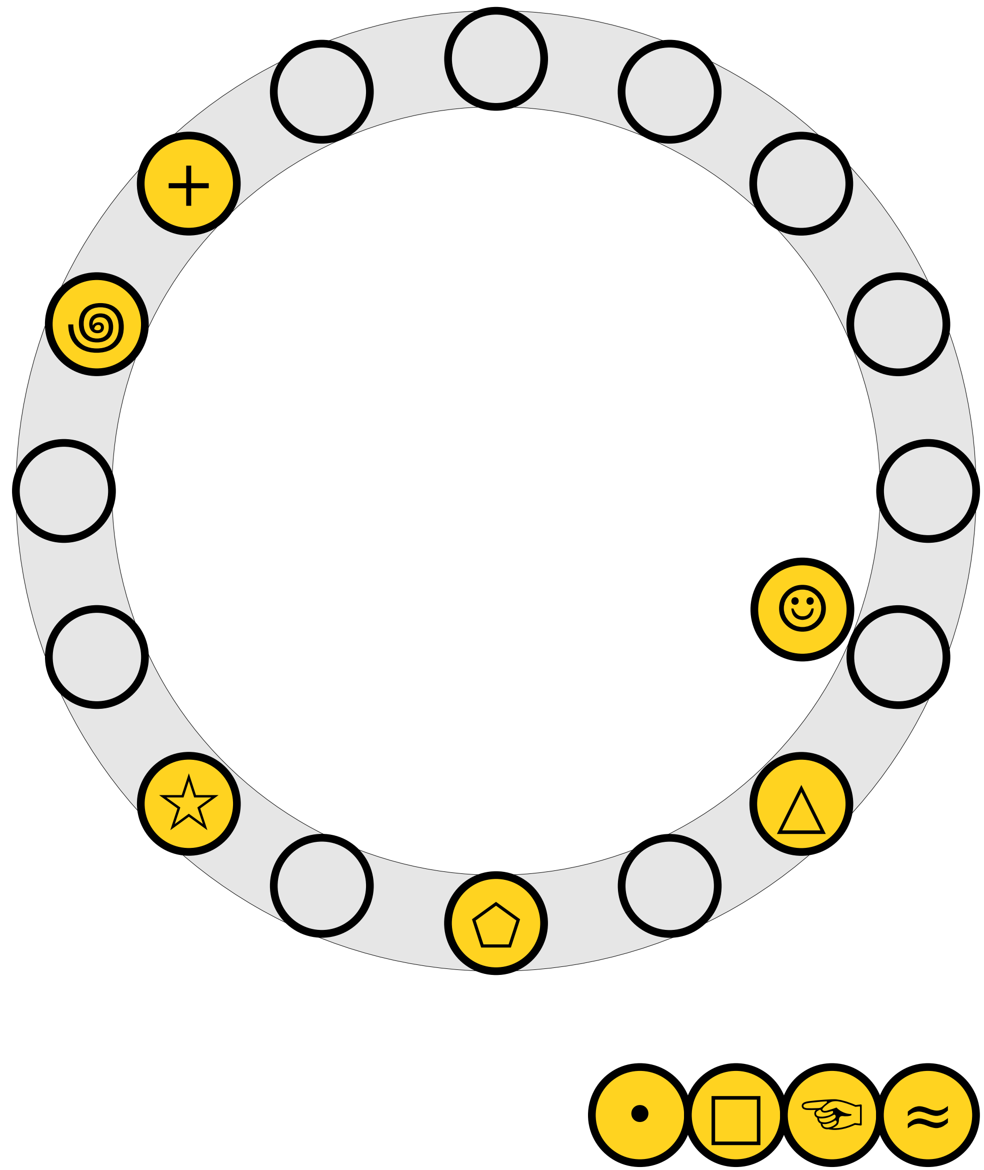

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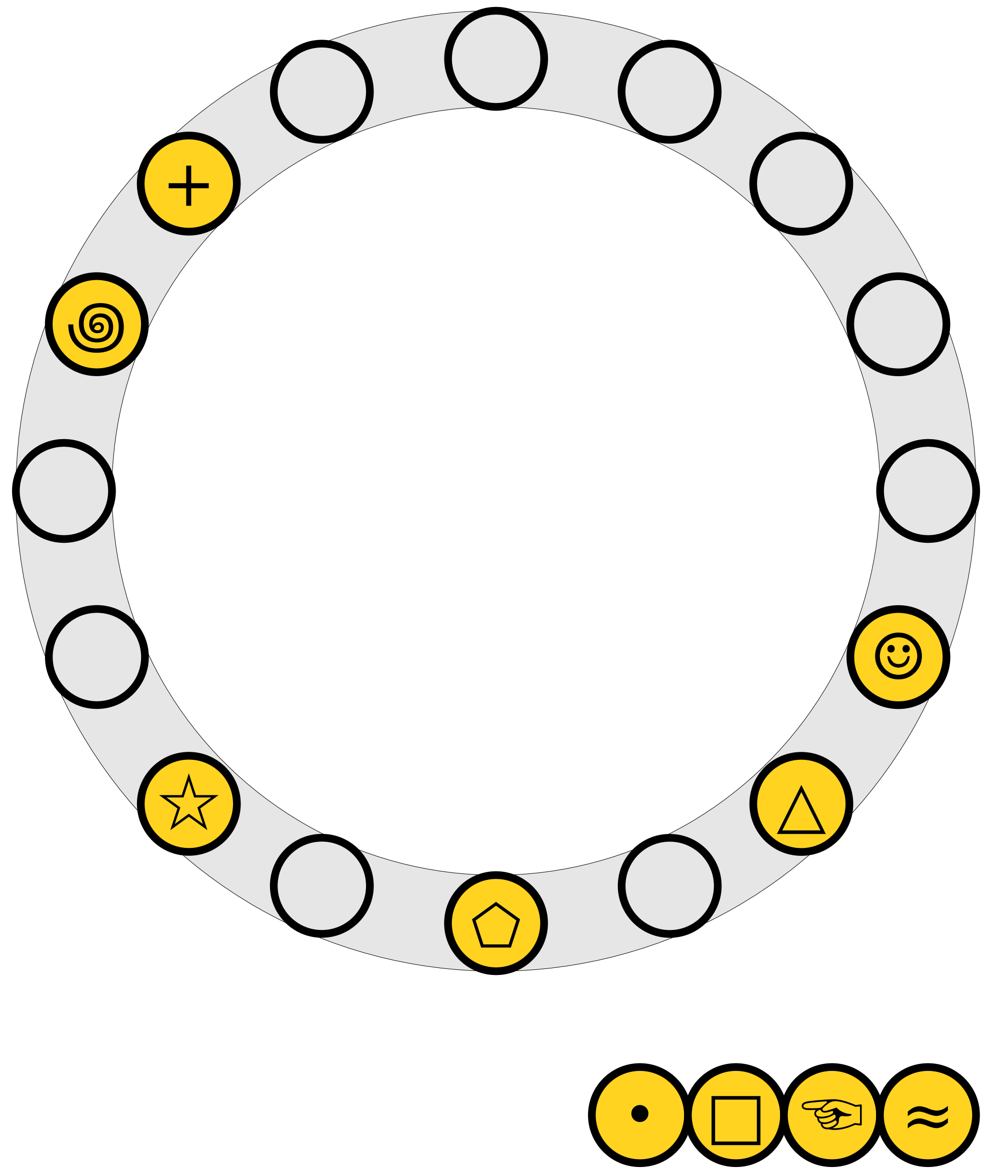

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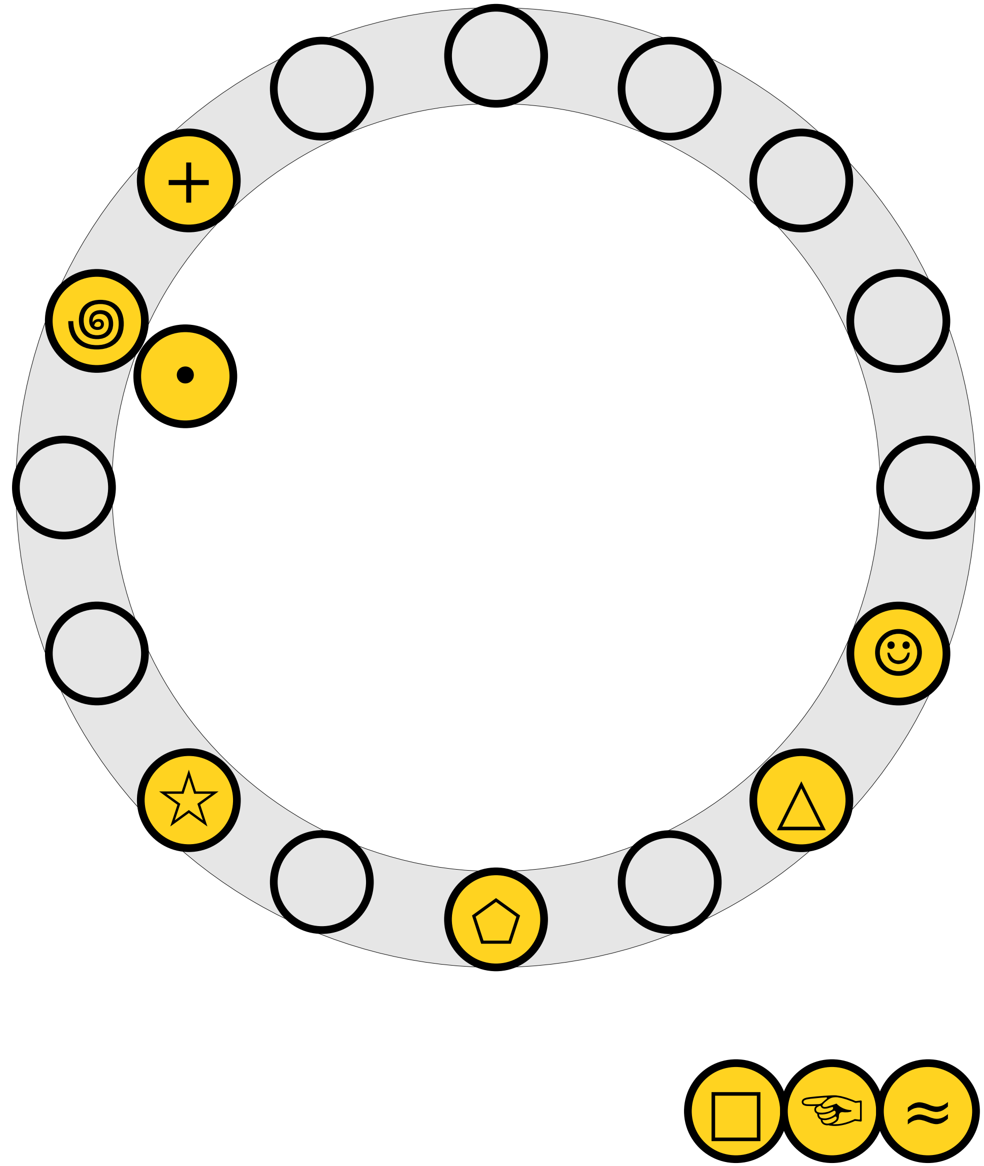

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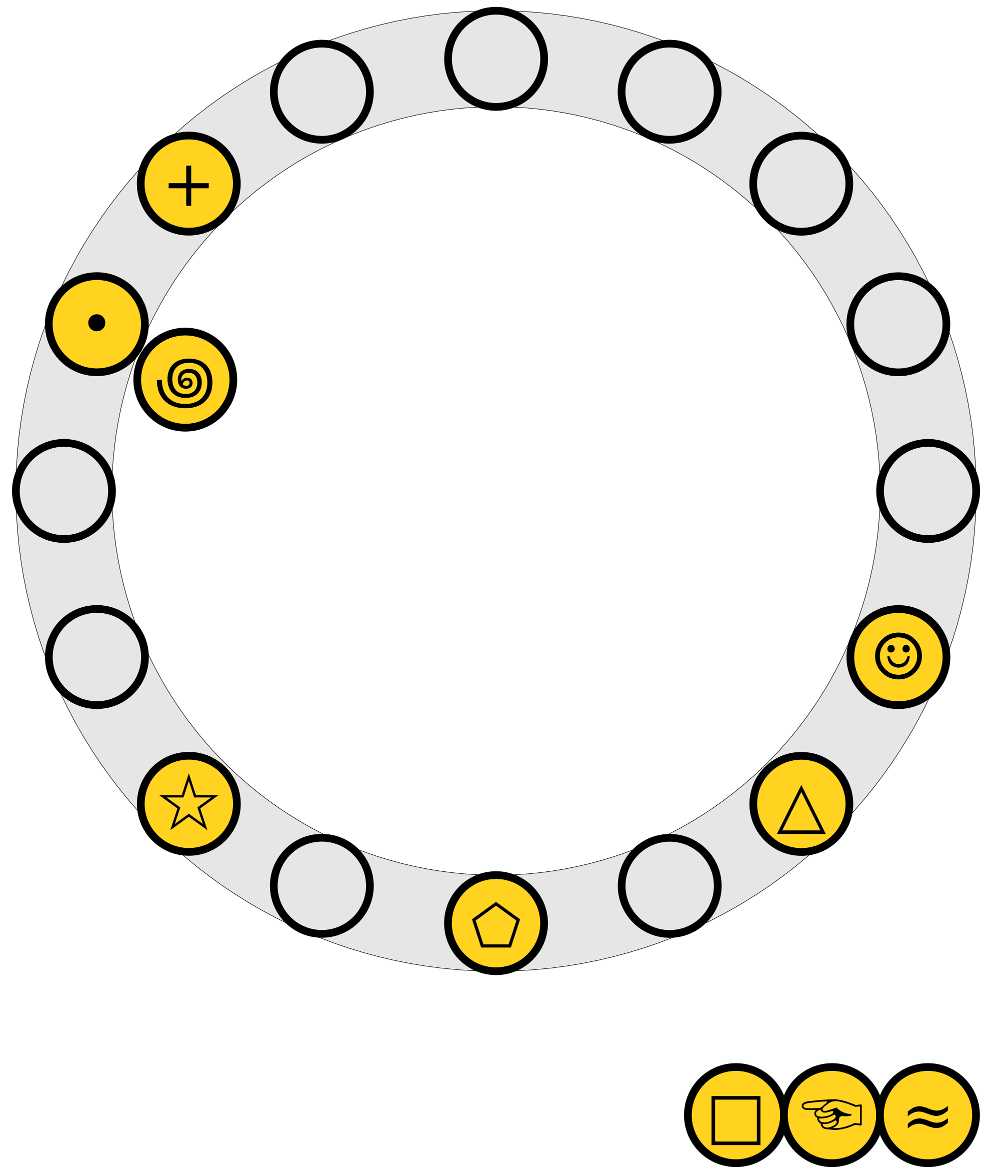

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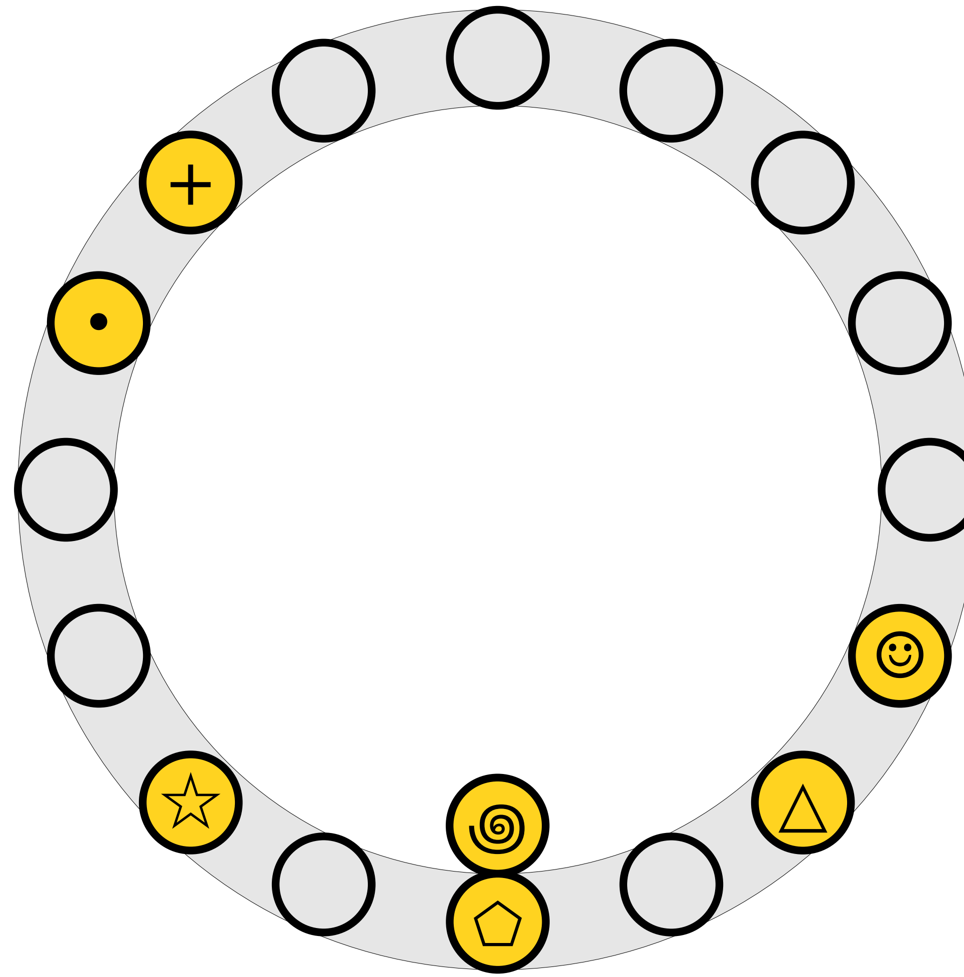

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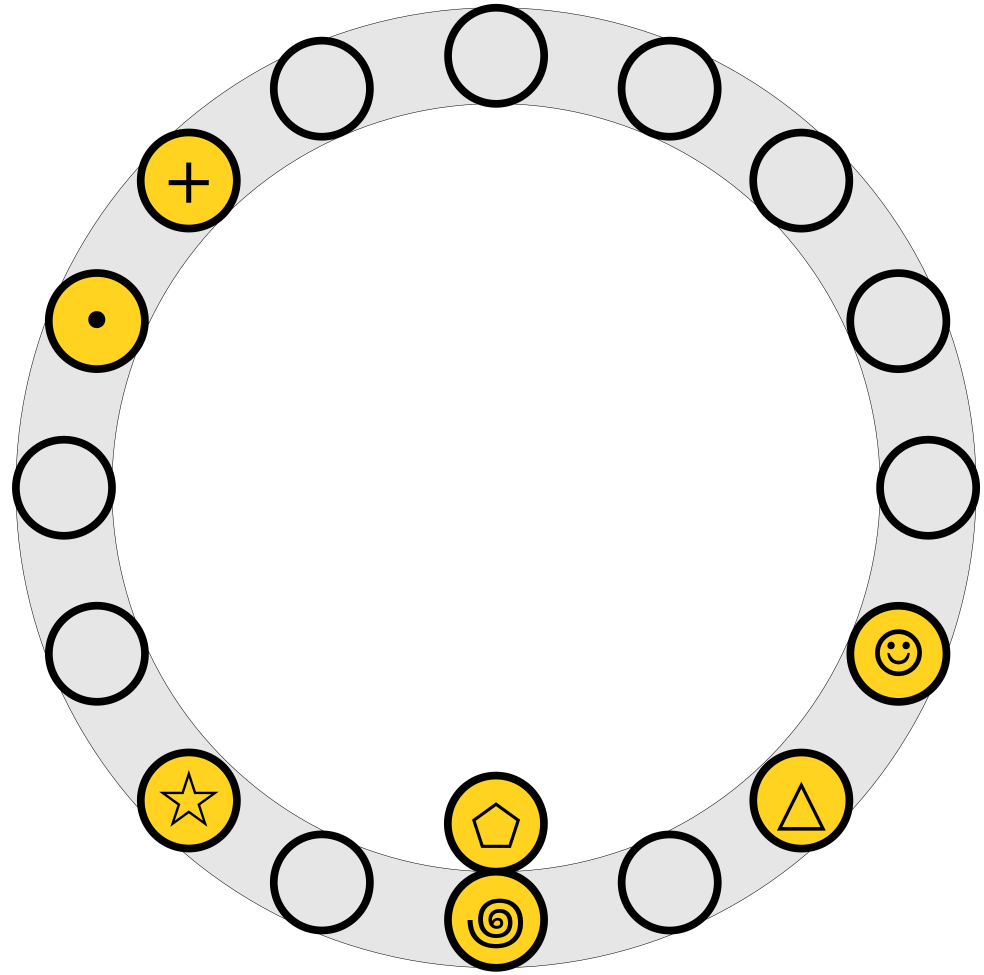

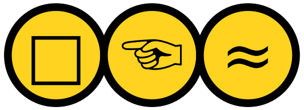

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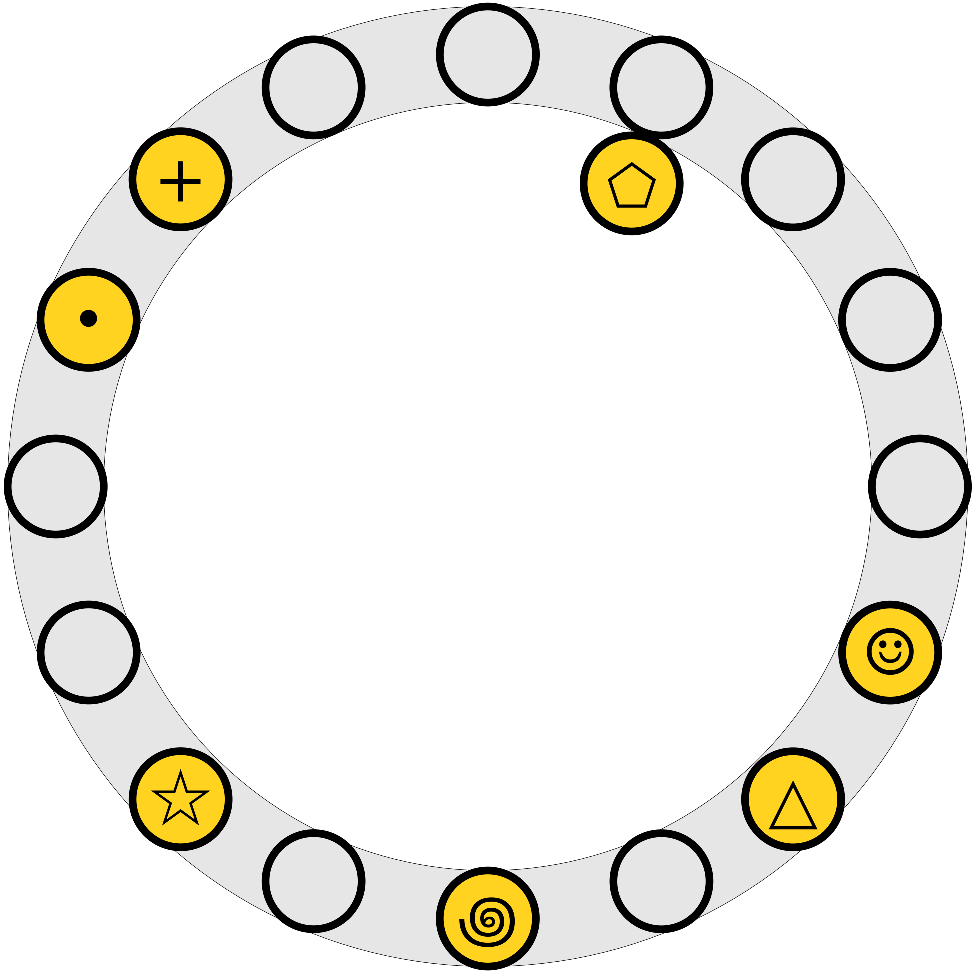

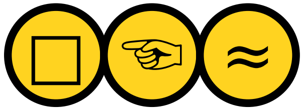

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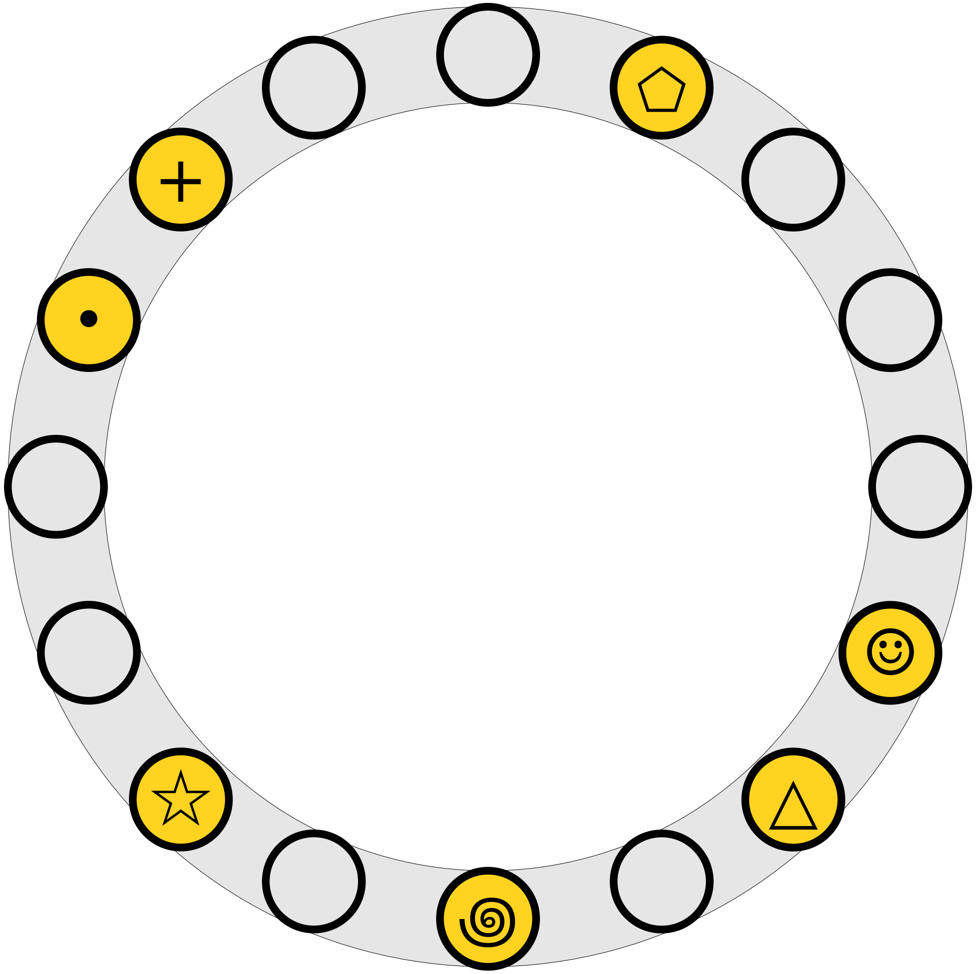

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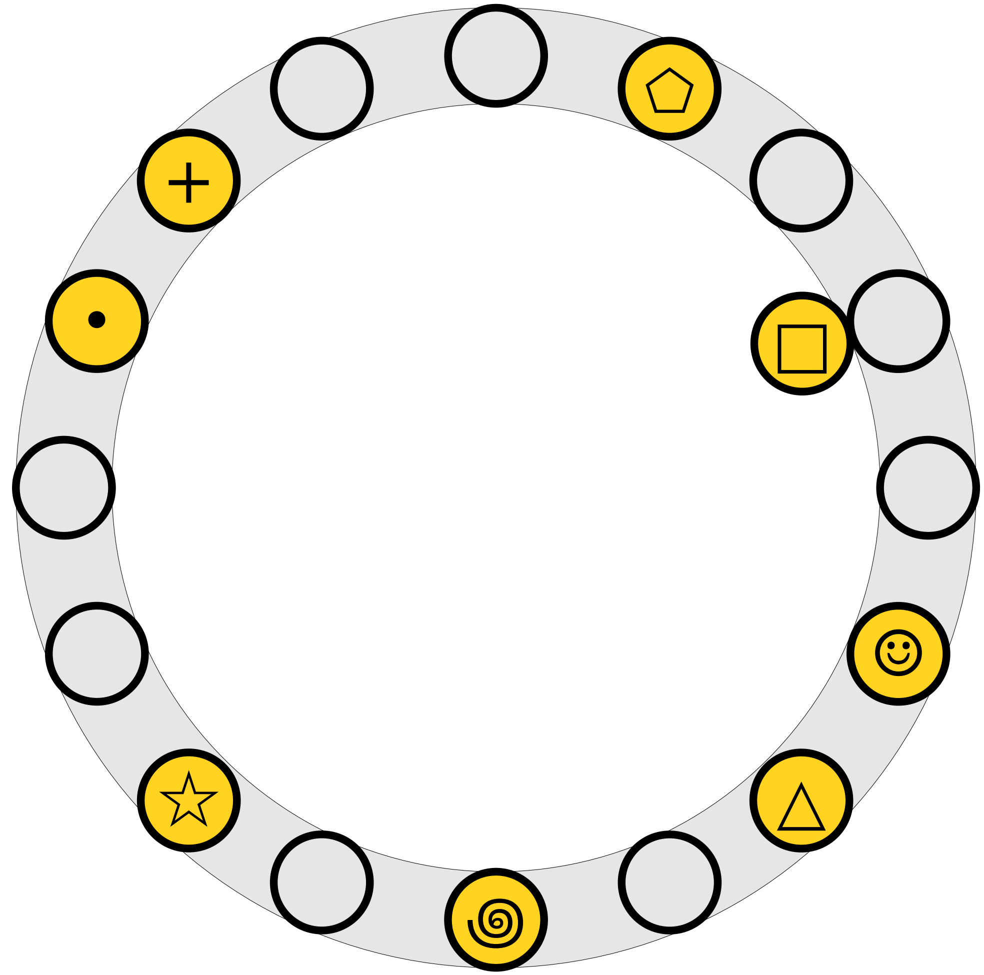

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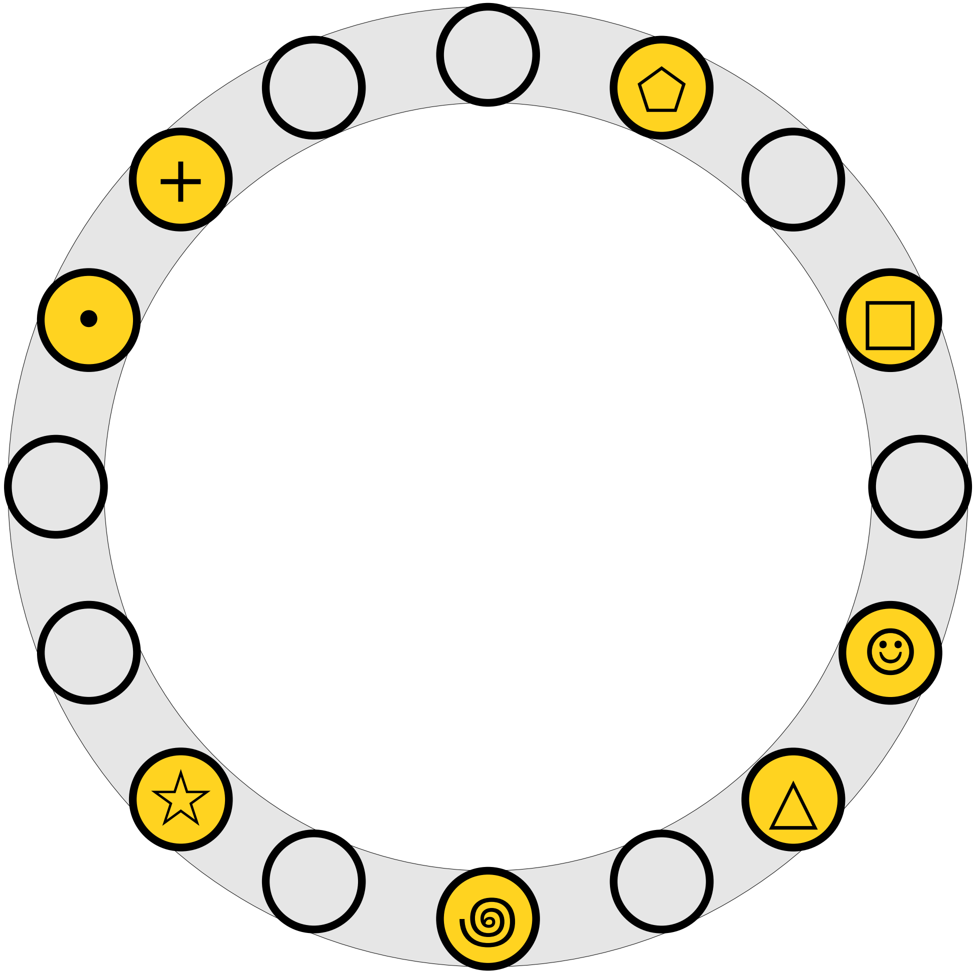

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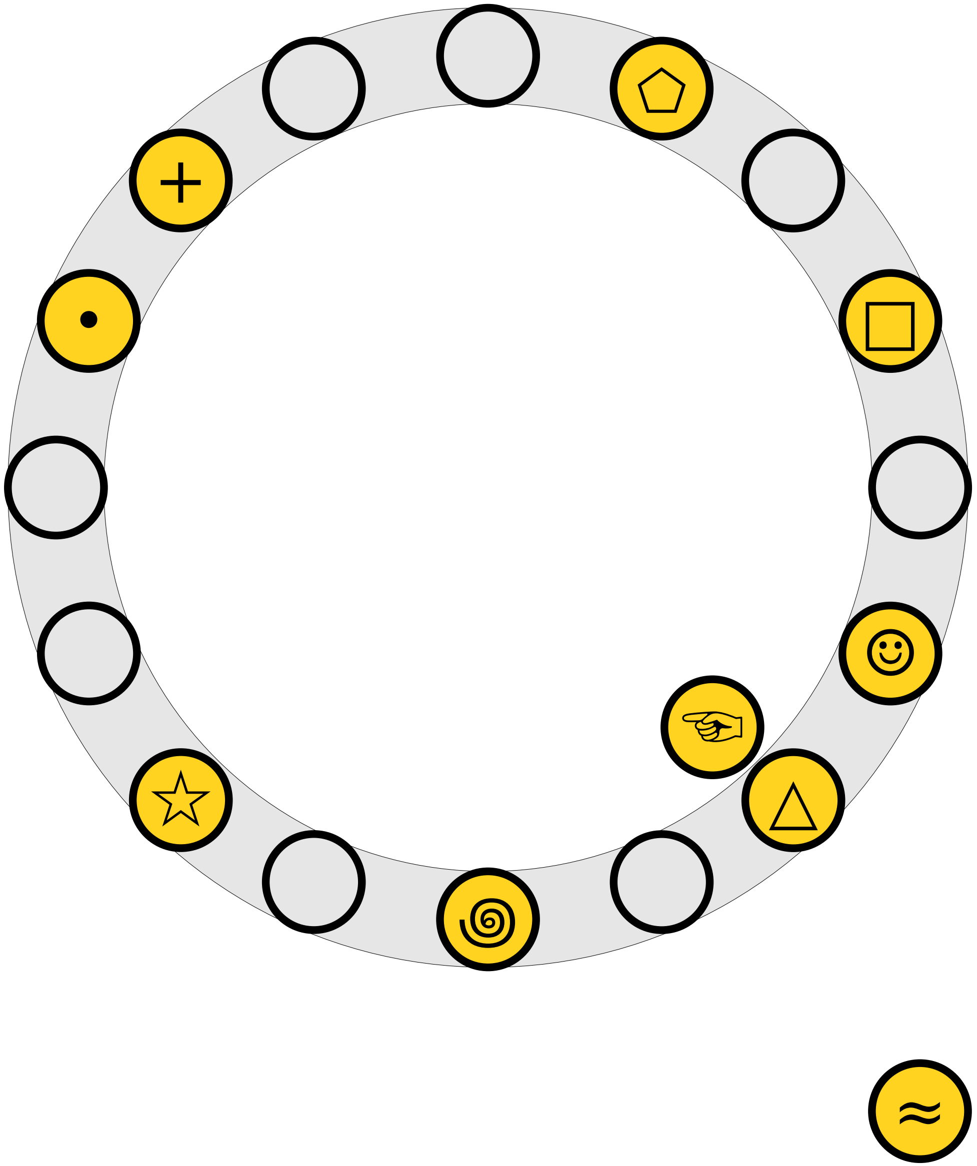

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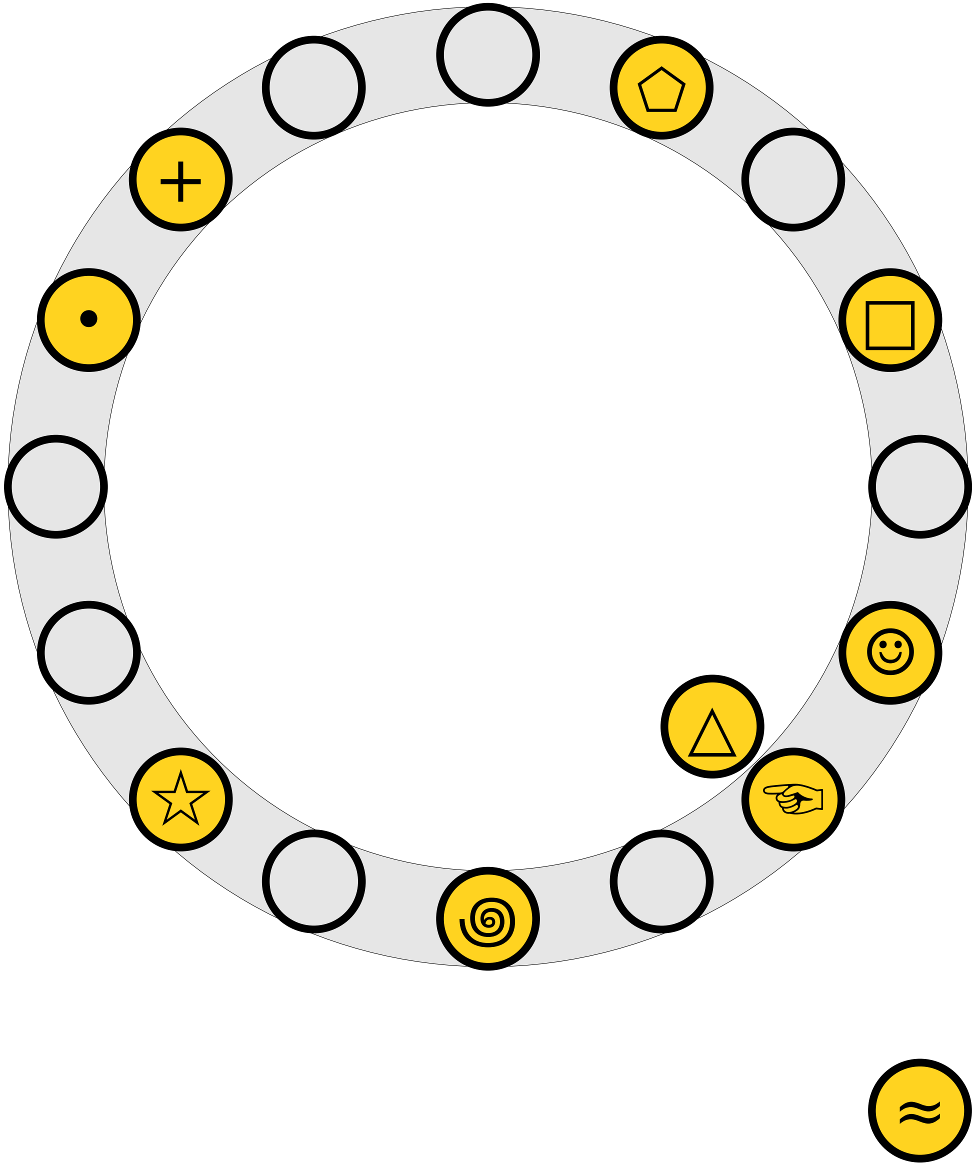

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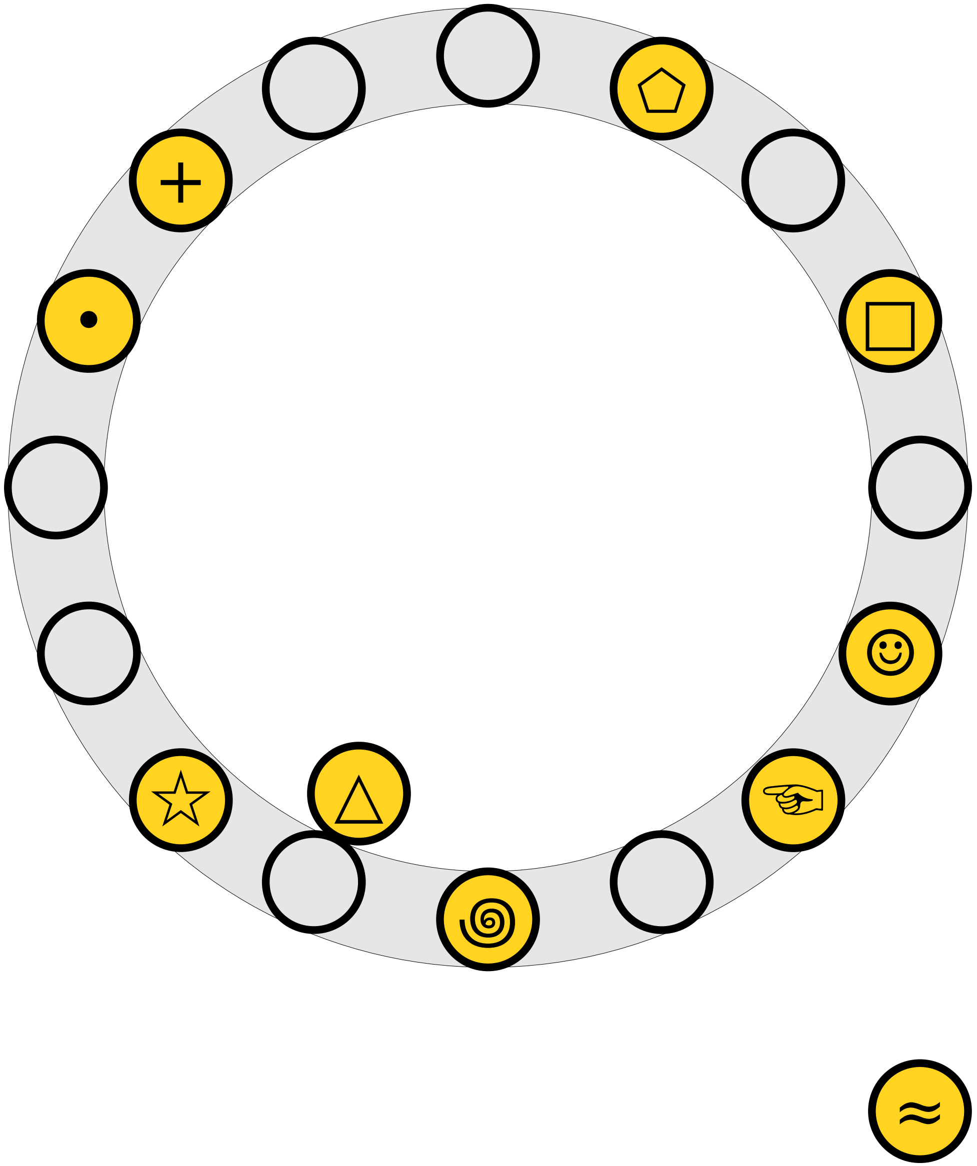

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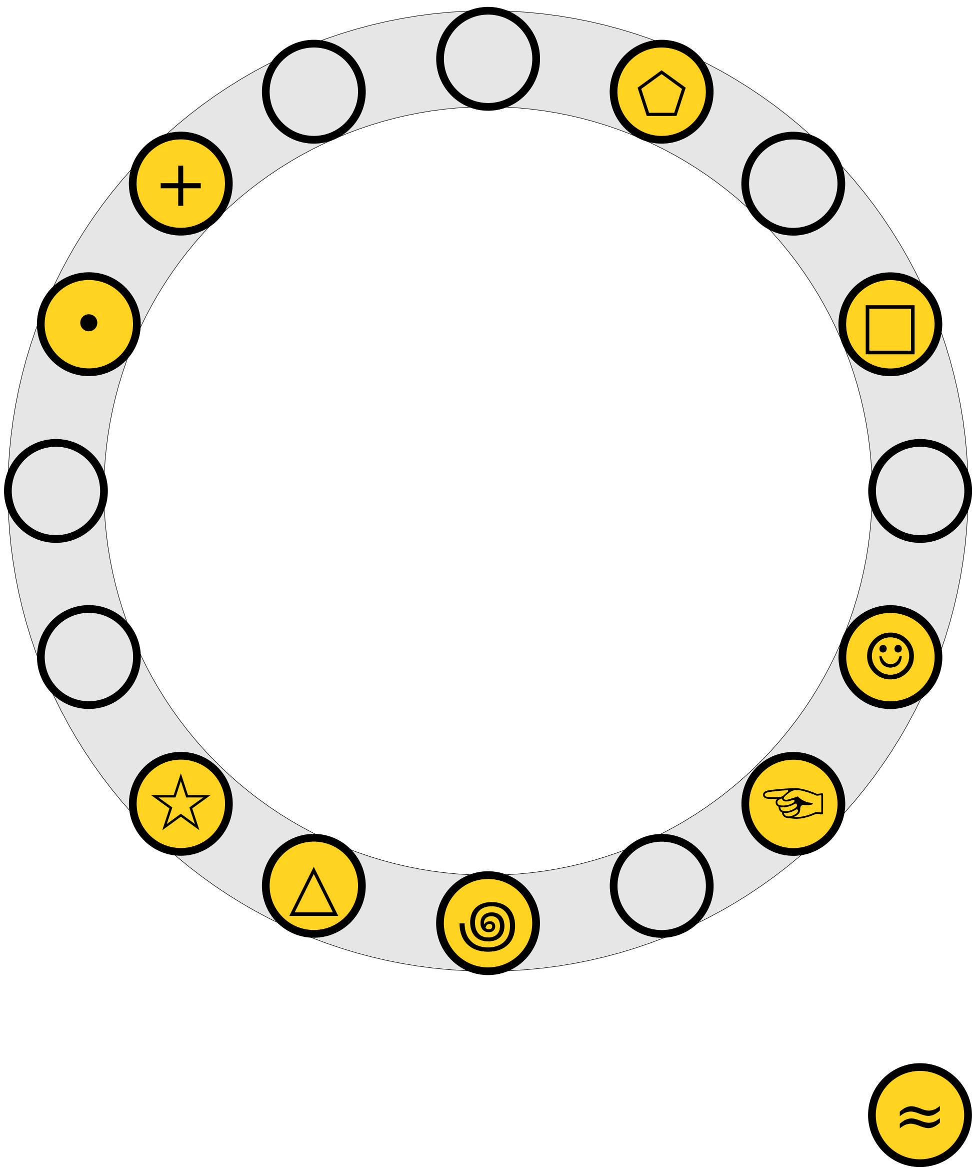

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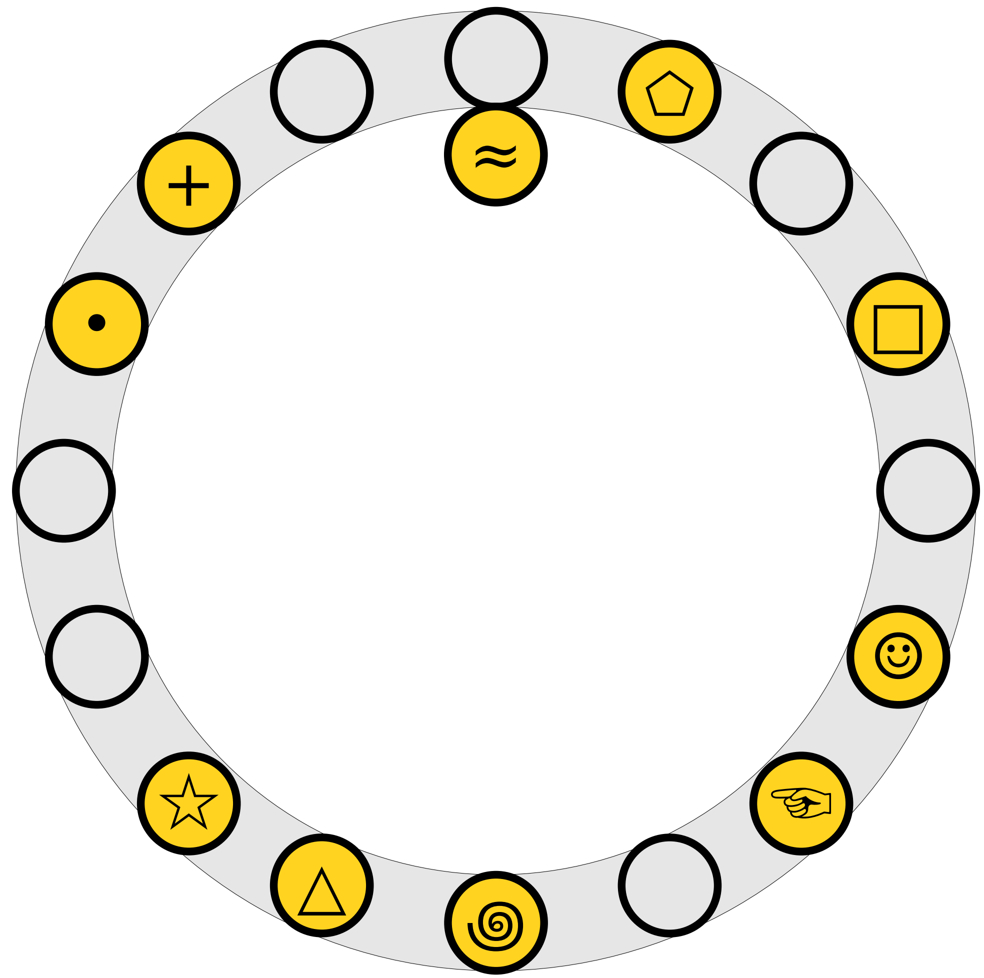

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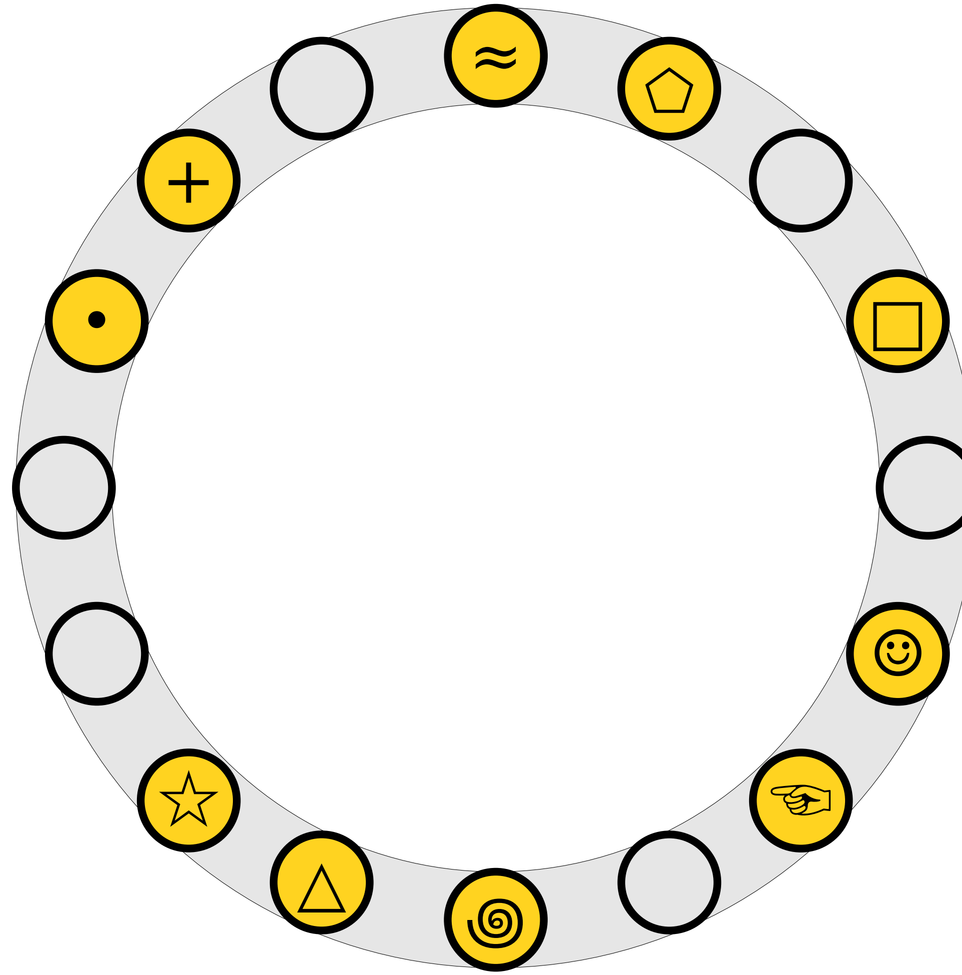
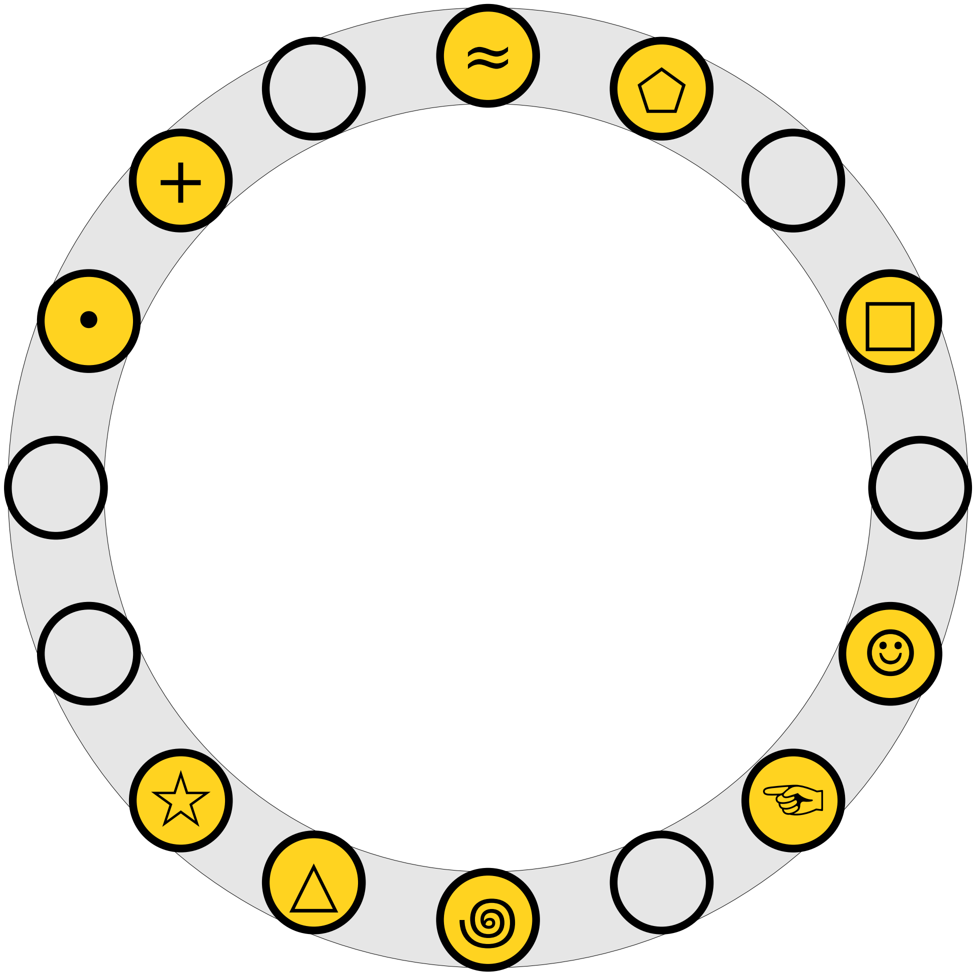


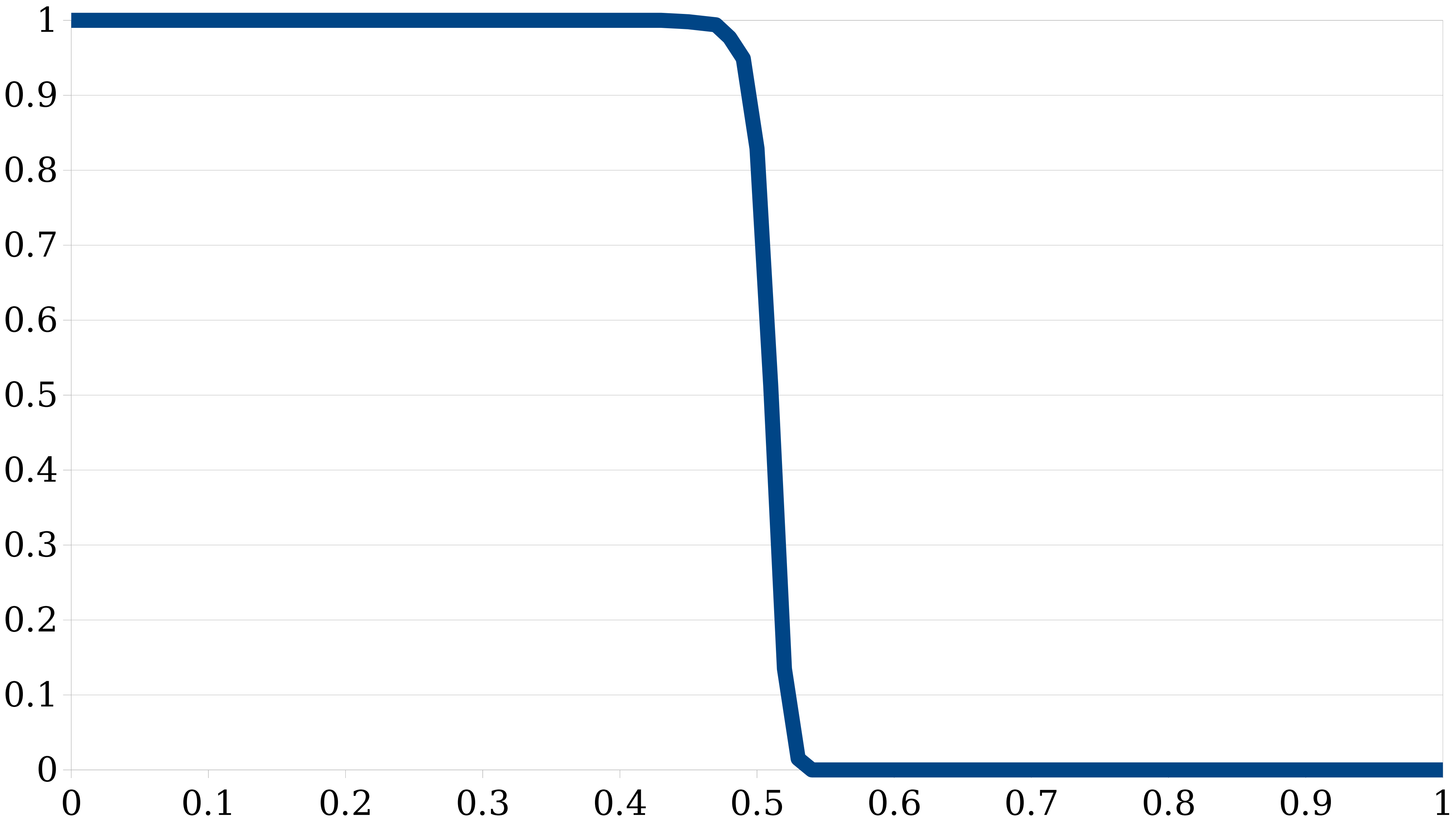
Cuckoo Hashing

- An insertion ***fails*** if the displacements form an infinite cycle.
- If that happens, perform a ***rehash*** by choosing a new h_1 and h_2 and inserting all elements back into the table.
- Multiple rehashes might be necessary before this succeeds – do you see why?

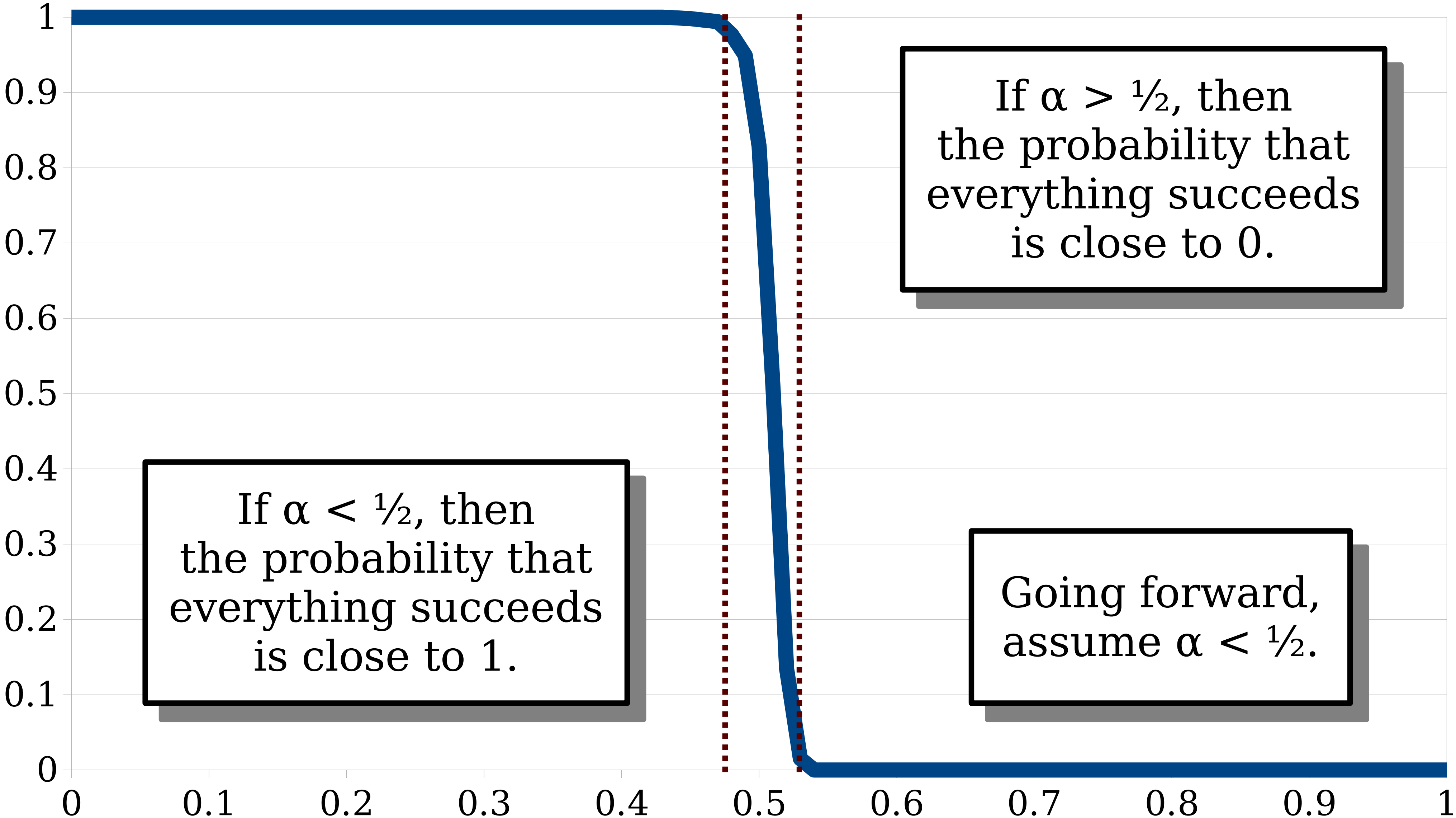


How efficient is cuckoo hashing?

Pro tip: When analyzing a data structure, it never hurts to get some empirical performance data first.



Suppose we have m slots and store n total elements.
What is the probability that all the insertions succeed,
as a function of the ***load factor*** $\alpha = {}^{n}/_{m}$?



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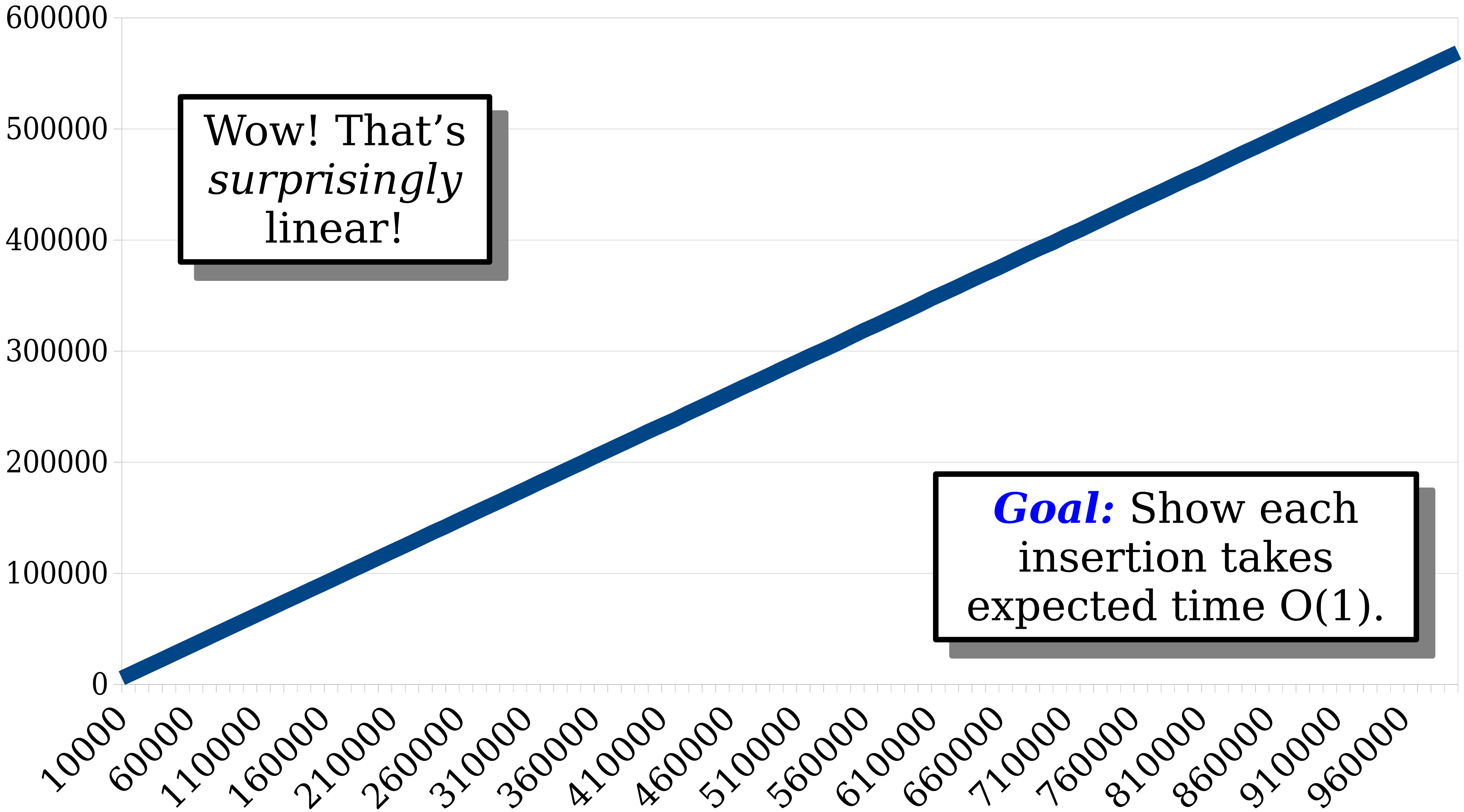


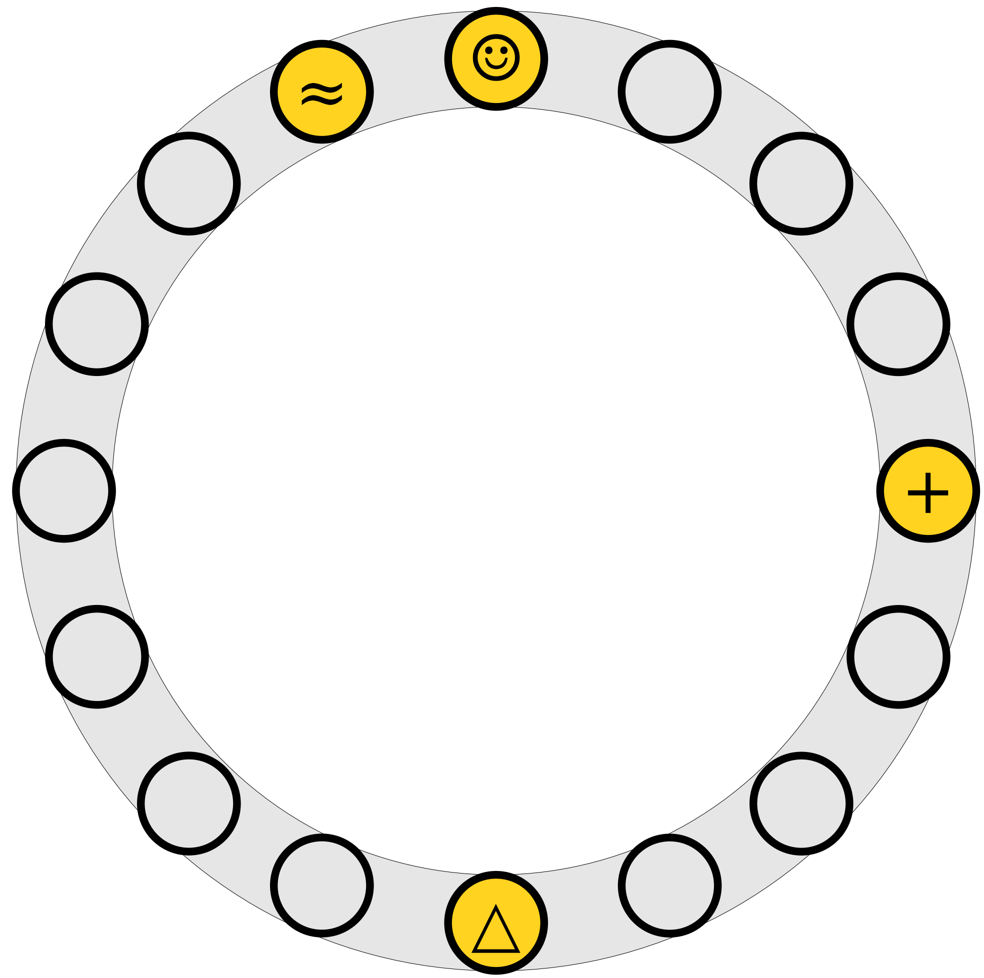
Suppose we store n total elements in a table with m slots, where $n < \frac{1}{2}m$.

How many total displacements occur across all insertions?

Goal: Show that insertions take expected time O(1), under the assumption that $n = \alpha m$ for some $\alpha < \frac{1}{2}$.

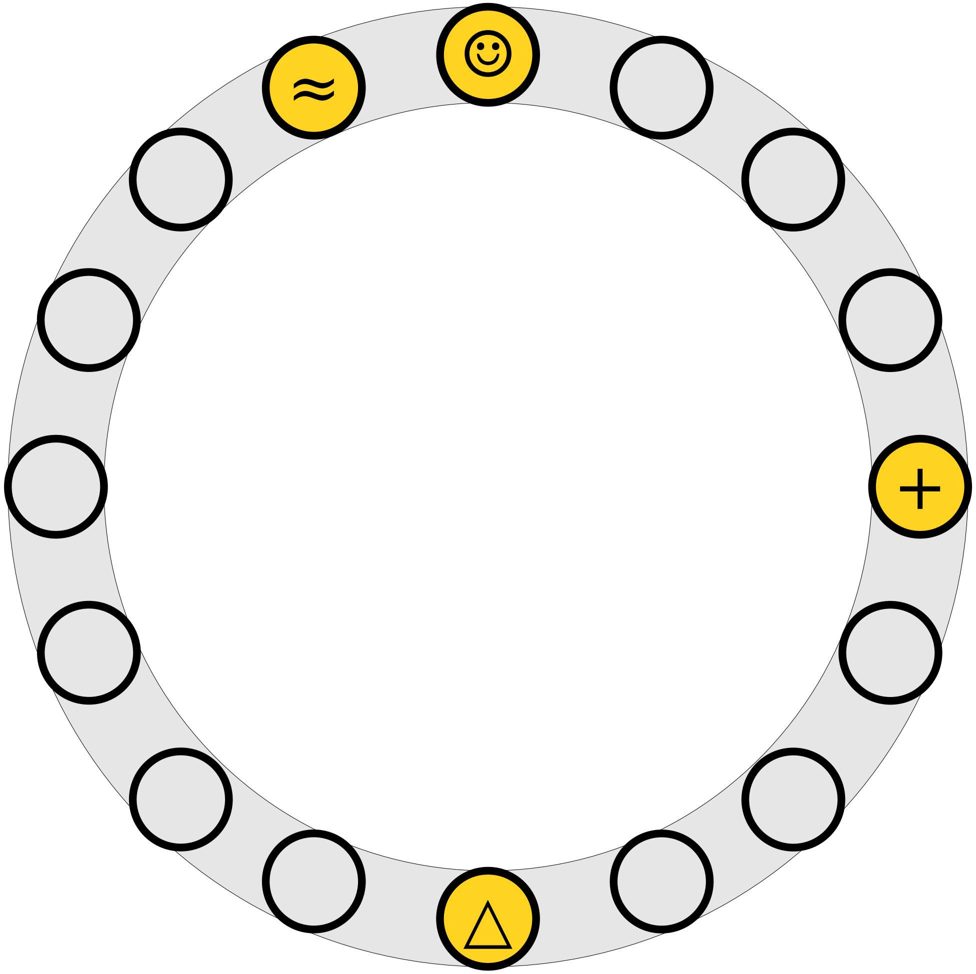
Analyzing Cuckoo Hashing

- The analysis of cuckoo hashing is more difficult than it might at first seem.
- ***Challenge 1:*** We may have to consider hash collisions across multiple hash functions.
- ***Challenge 2:*** We need to reason about chains of displacement, not just how many elements land somewhere.
- To resolve these challenges, we'll need to bring in some new techniques.



The Cuckoo Graph

- The ***cuckoo graph*** is a (multi)graph derived from a cuckoo hash table.



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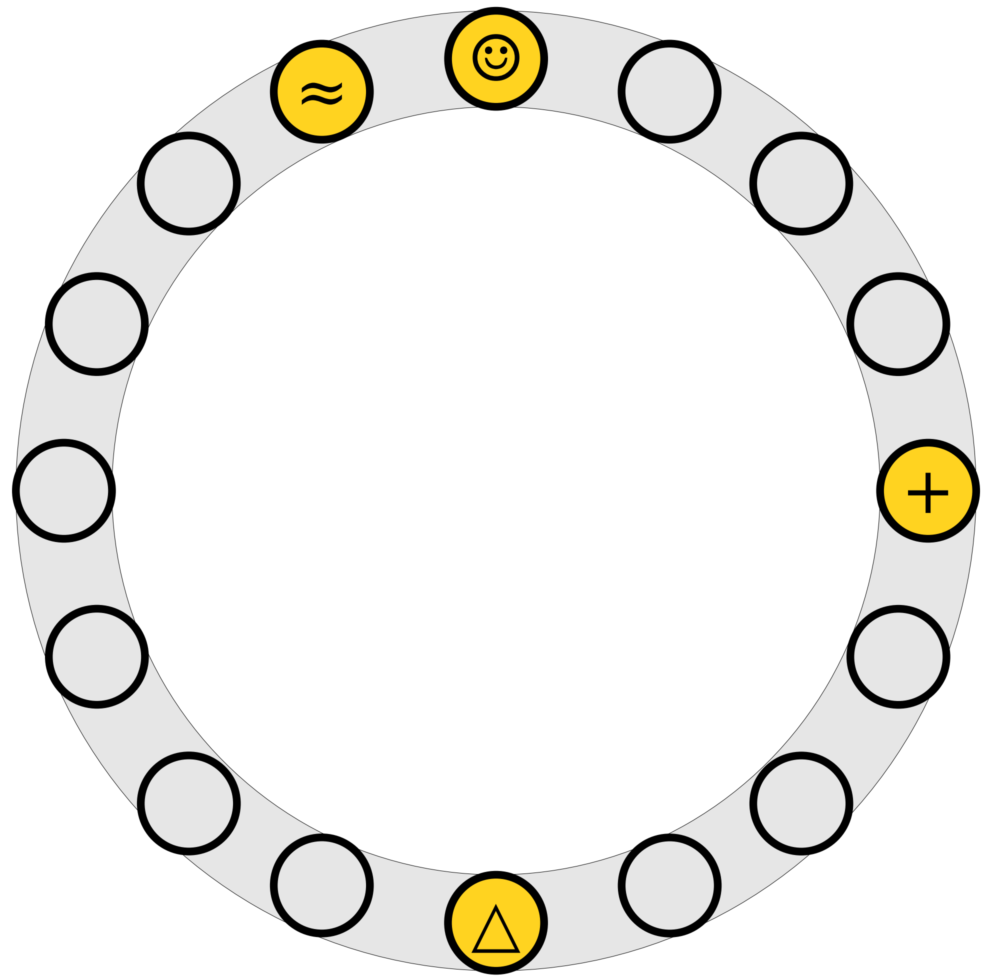

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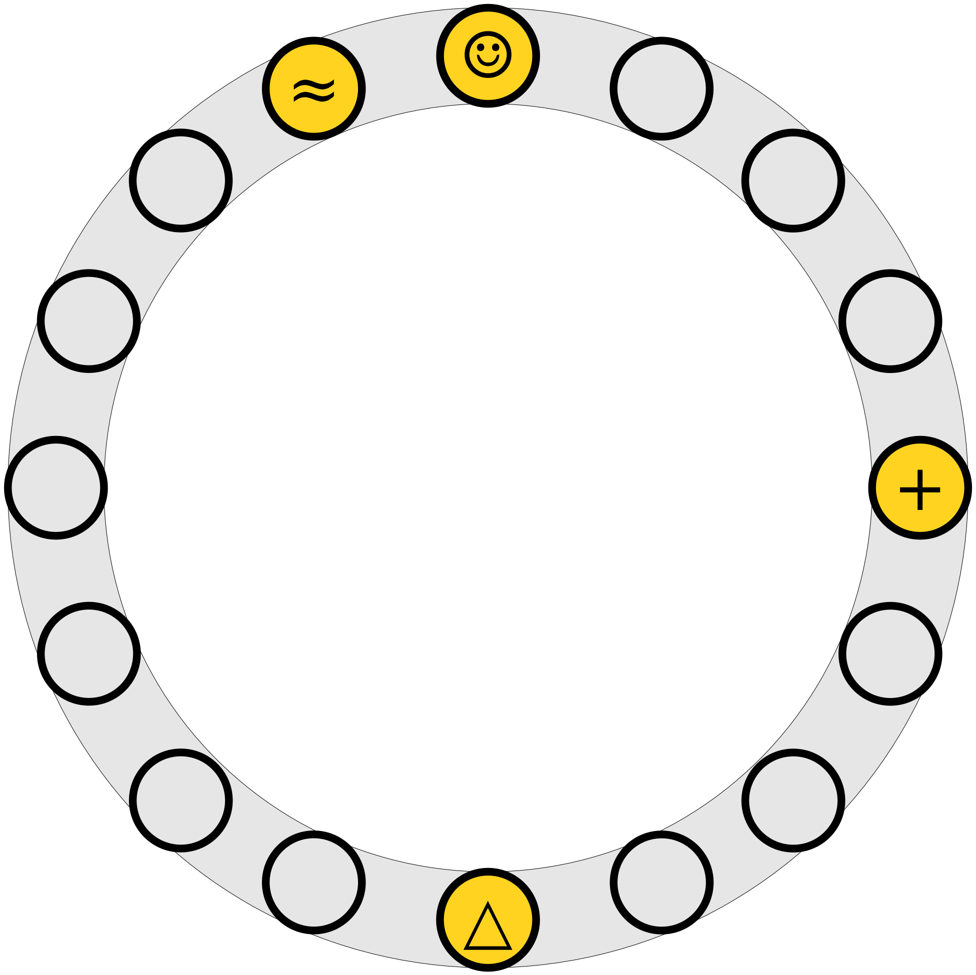
- Each element is an edge linking the slots where it can be placed.



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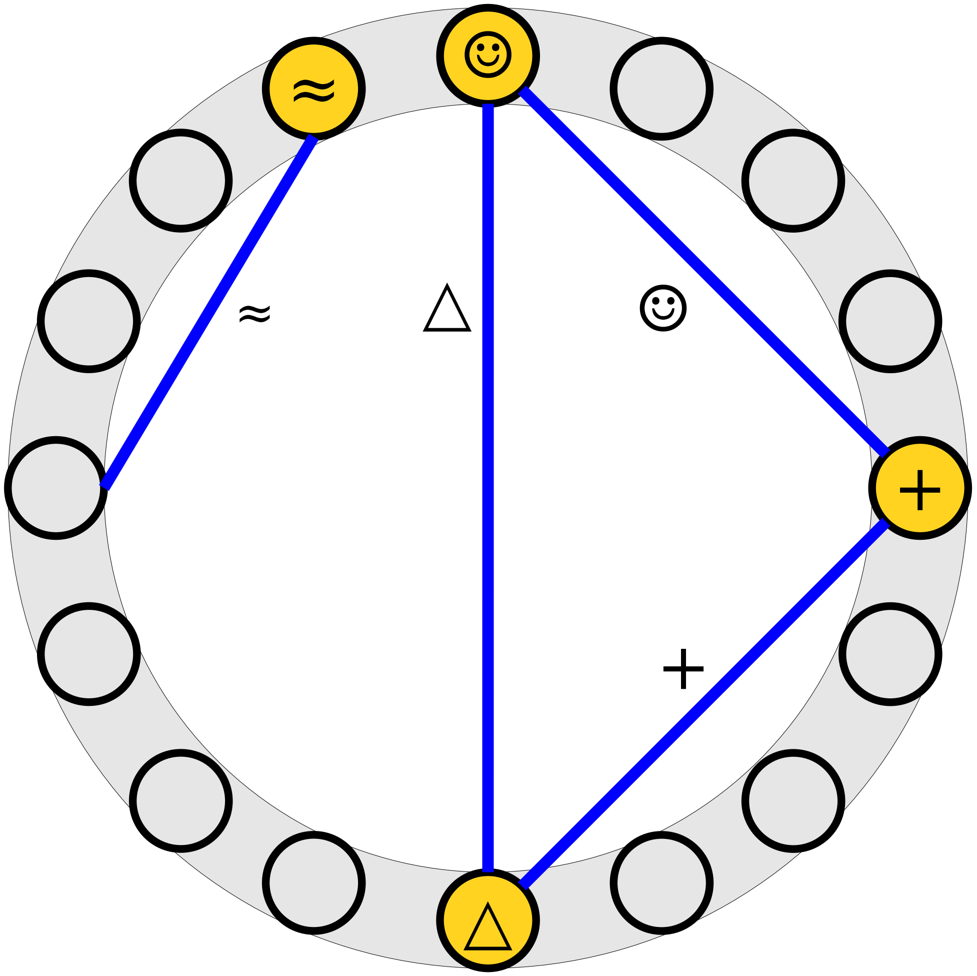

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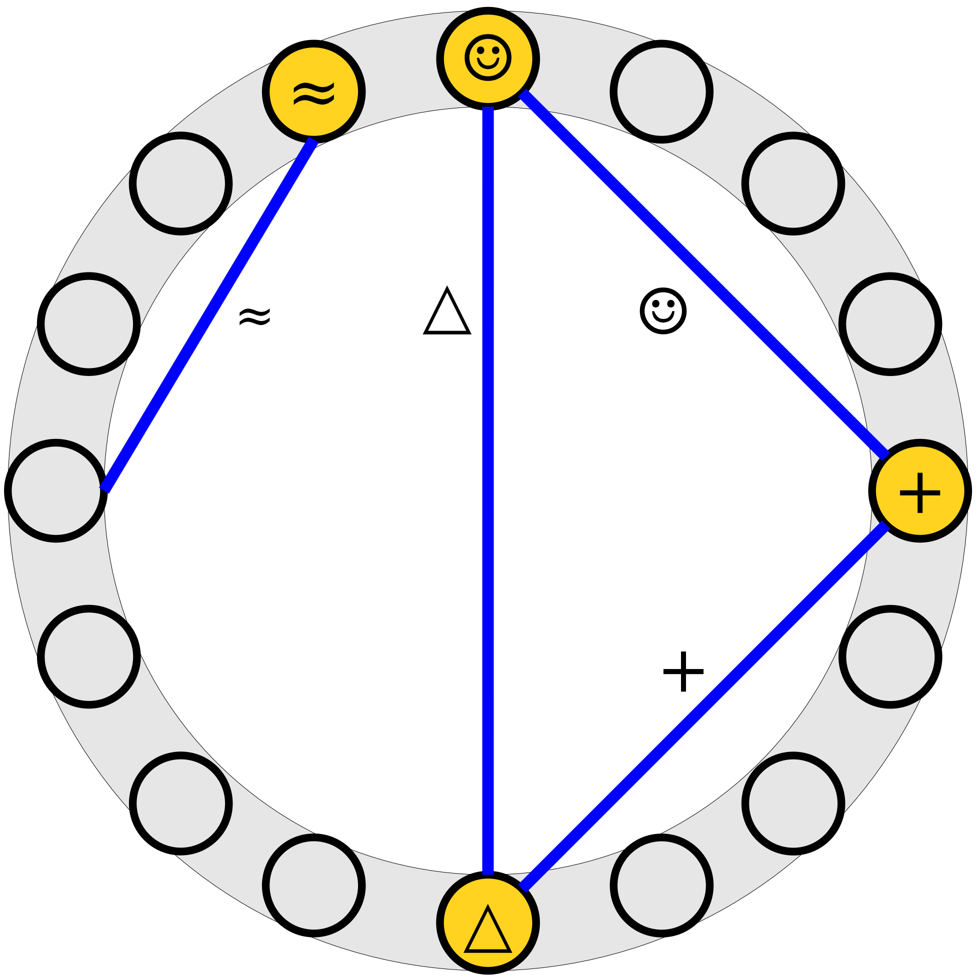

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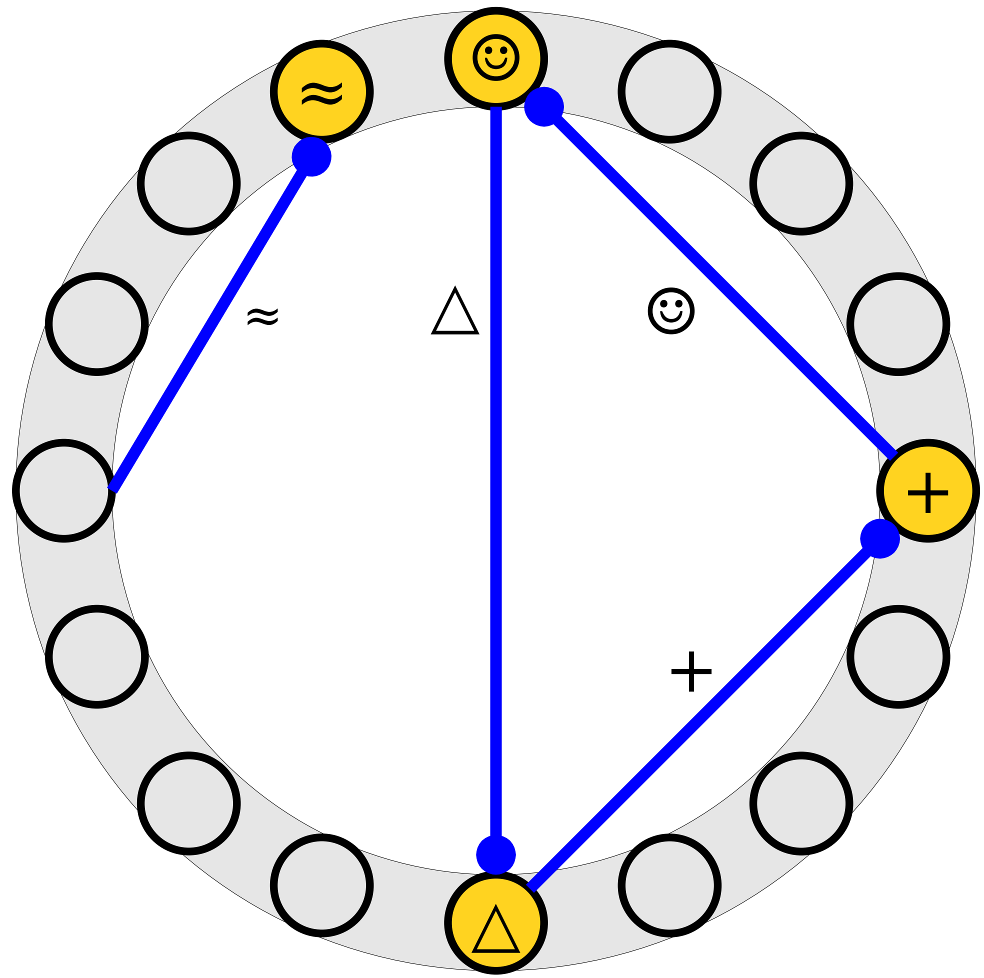

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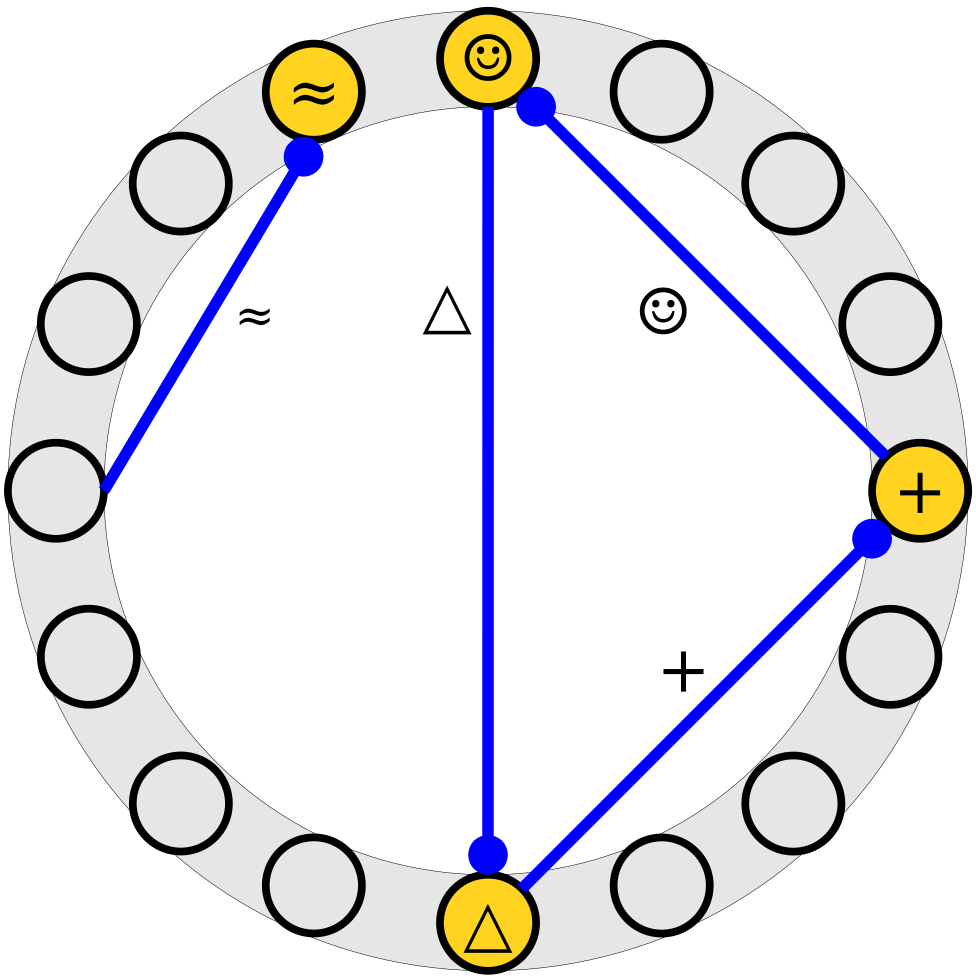
- Each node has at most one dot touching it.



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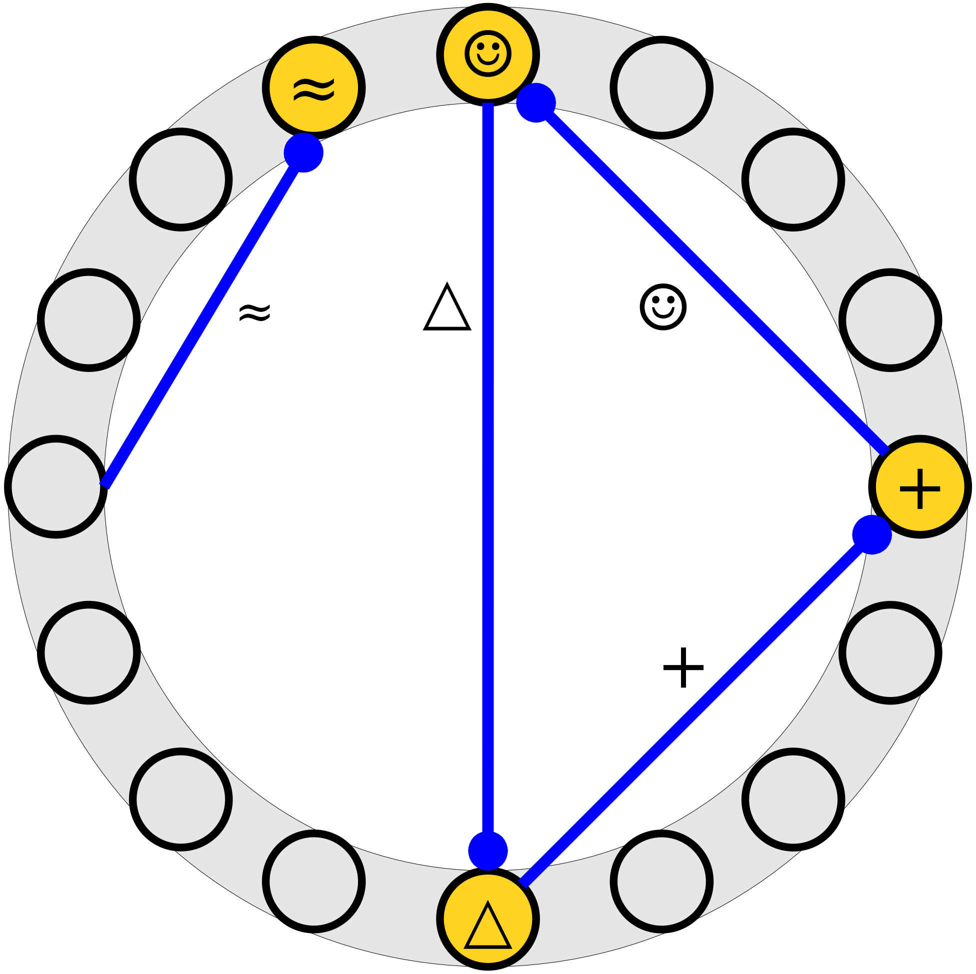
- Inserting an element into a cuckoo hash table adds a new edge to the graph linking two nodes (slots).



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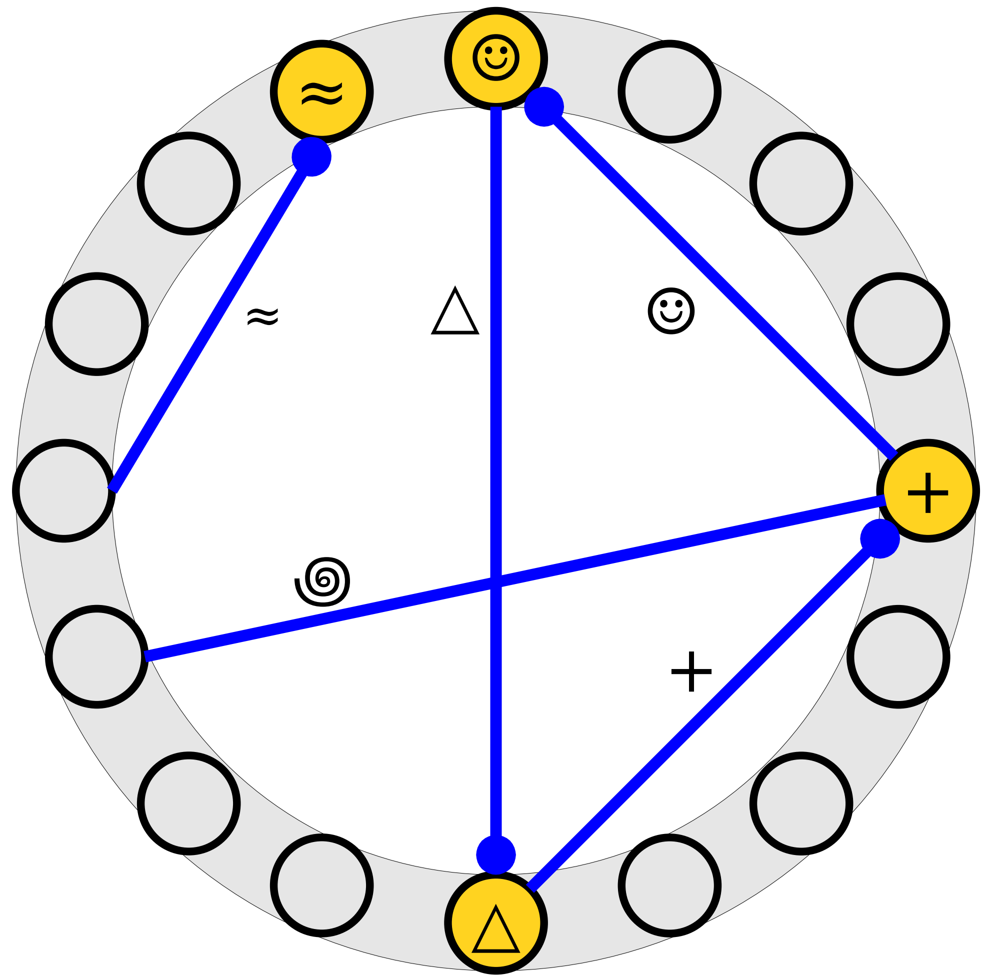

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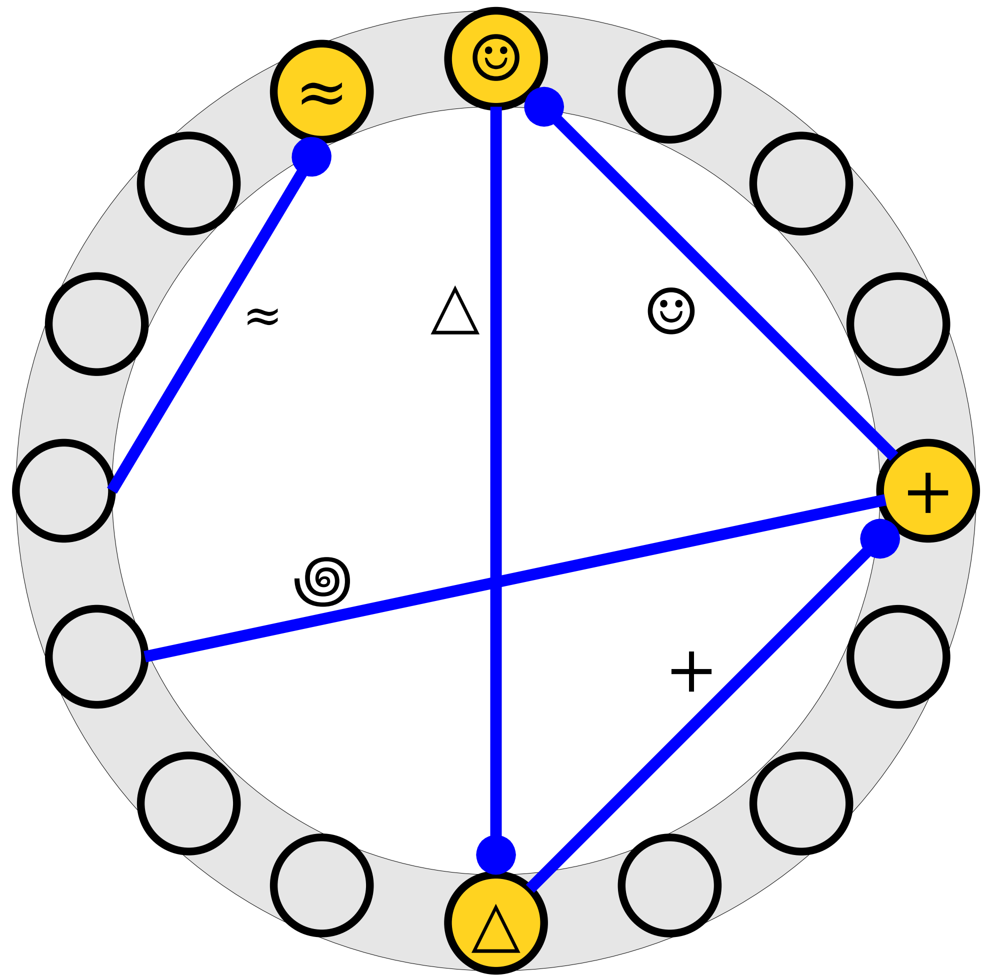
- Inserting an element into a cuckoo hash table adds a new edge to the graph linking two nodes (slots).
- The chain of displacements corresponds to flipping edges.



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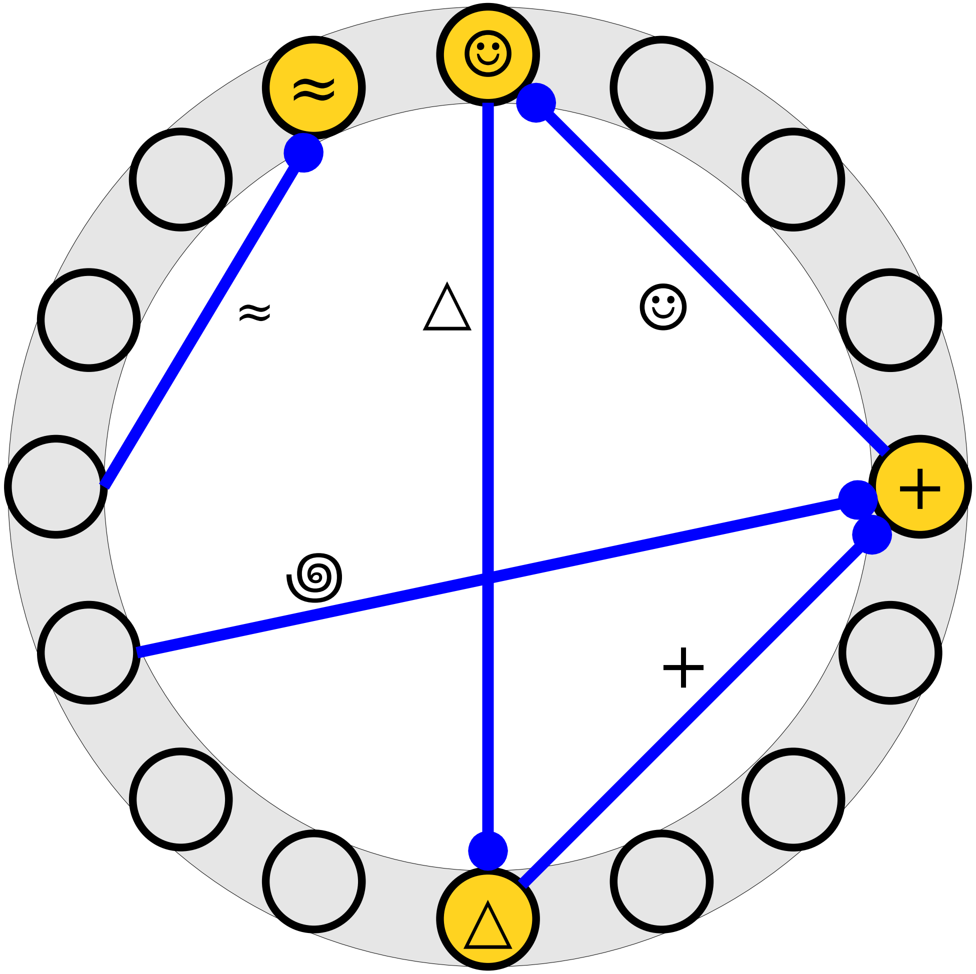

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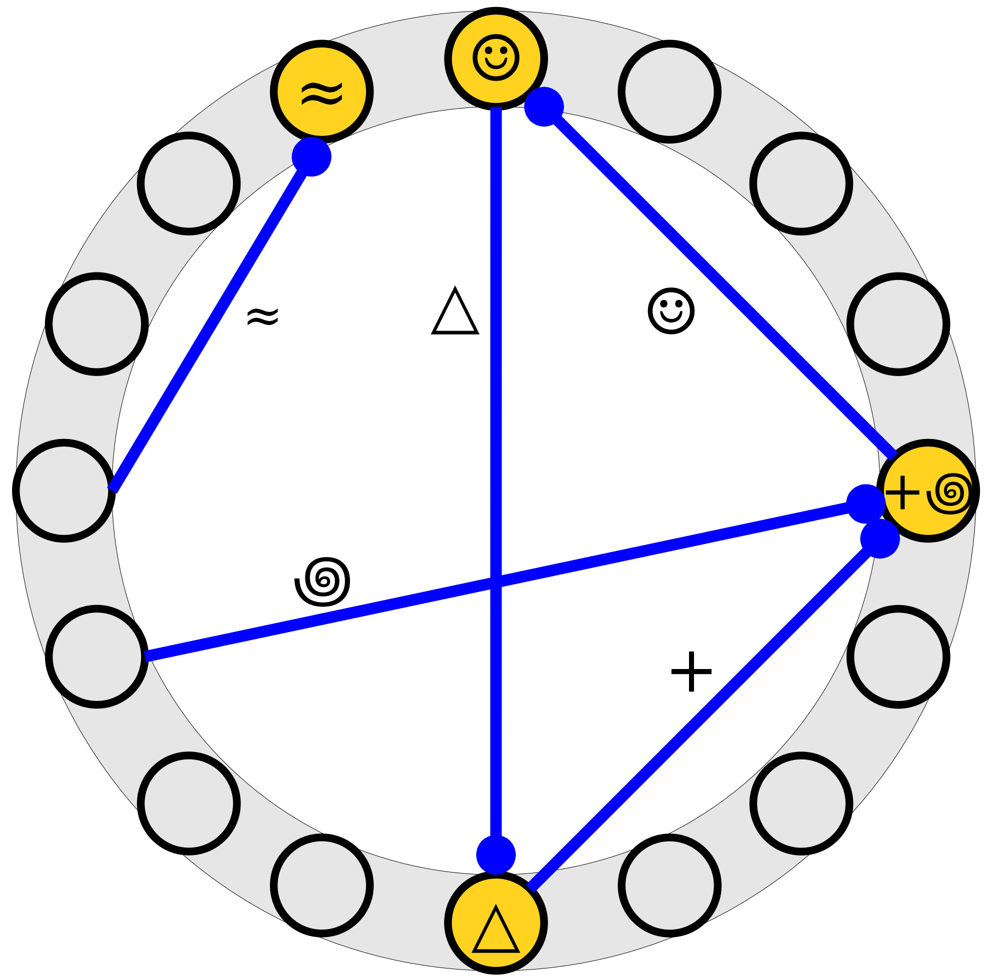

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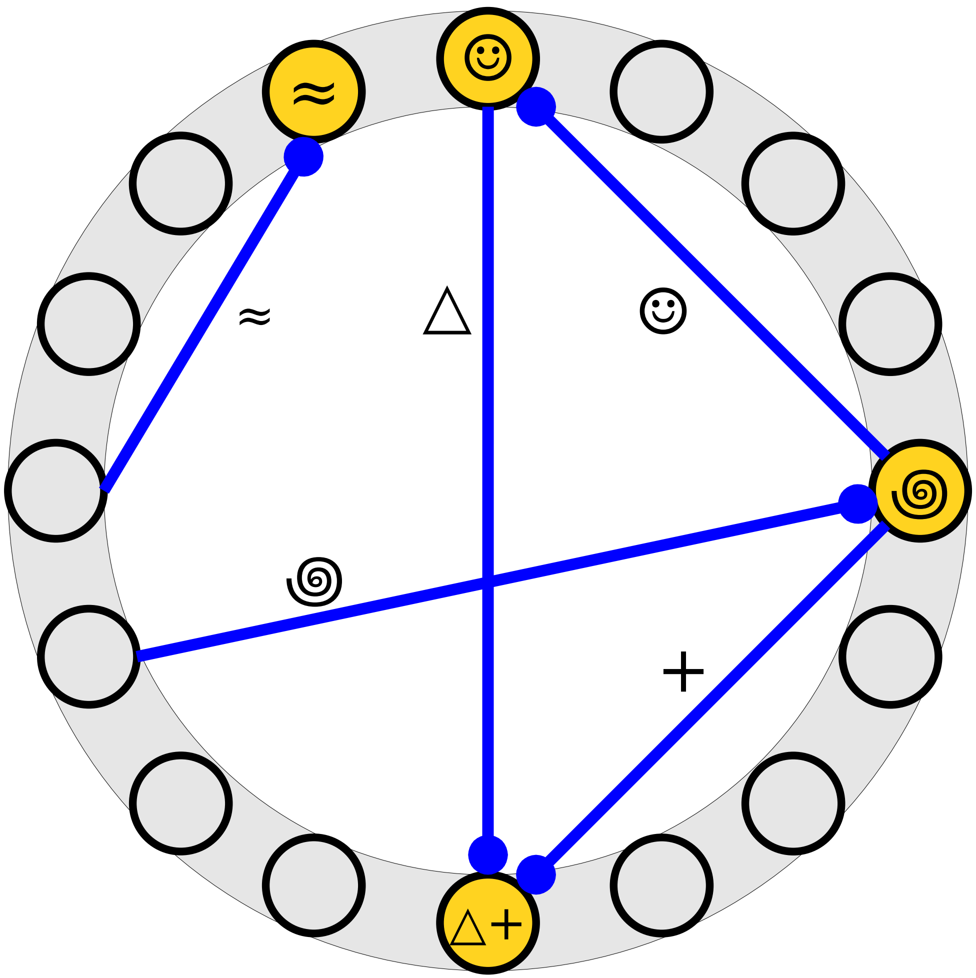

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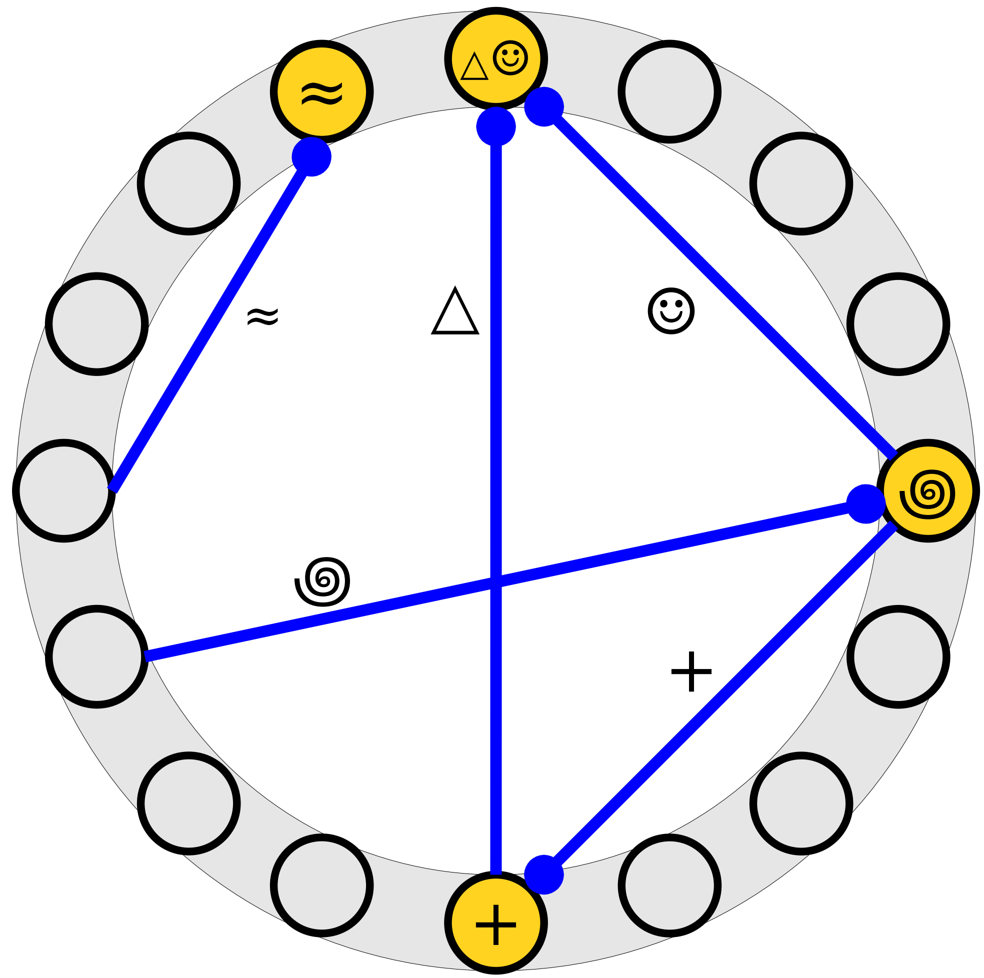

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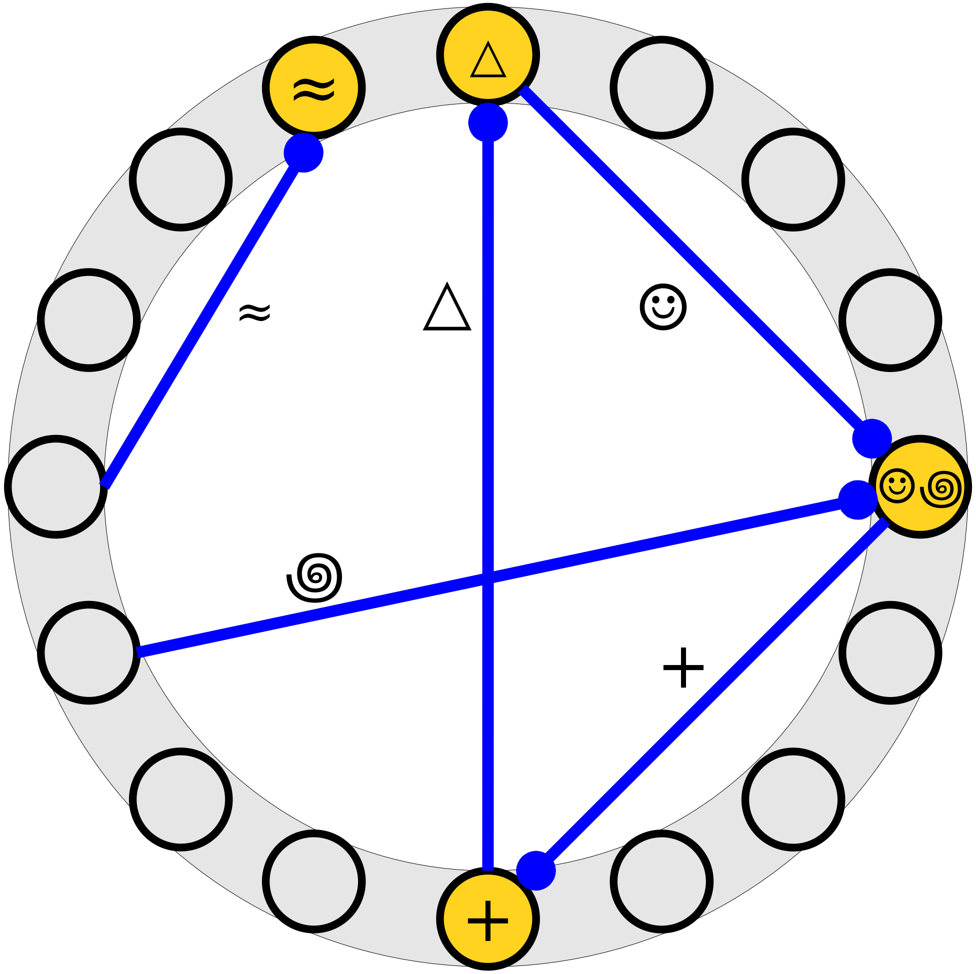

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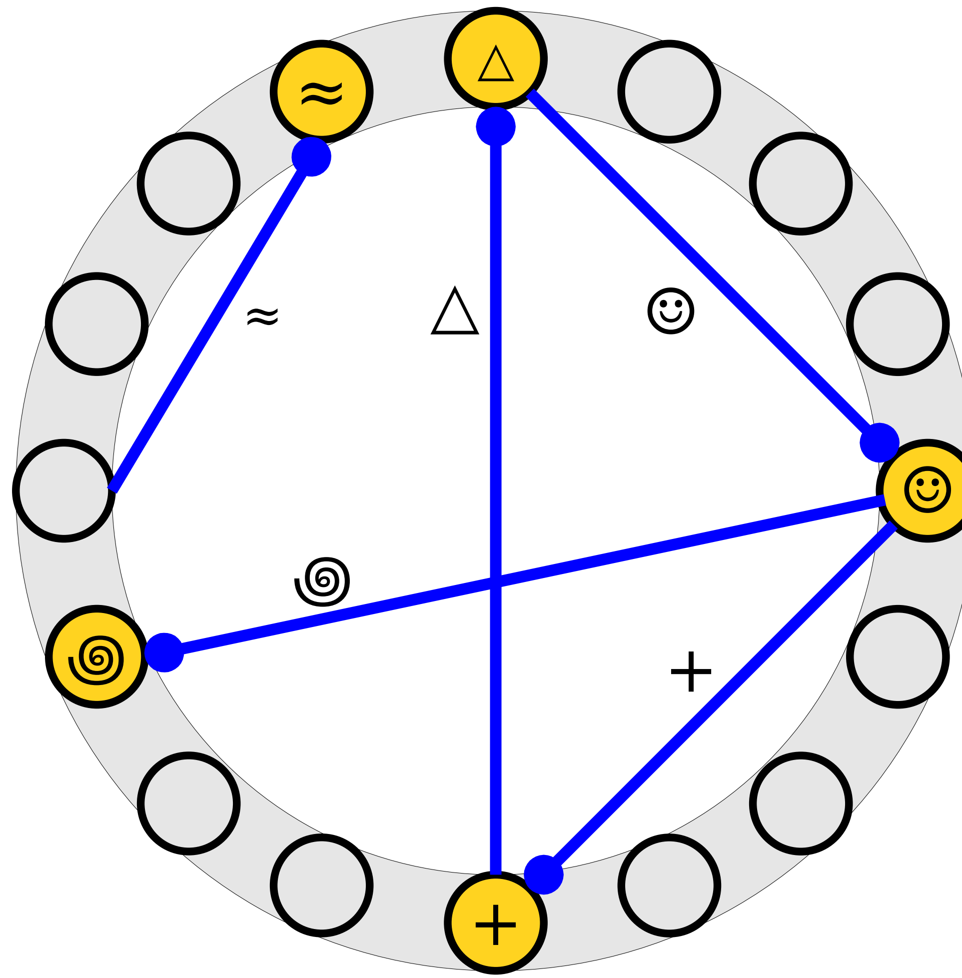

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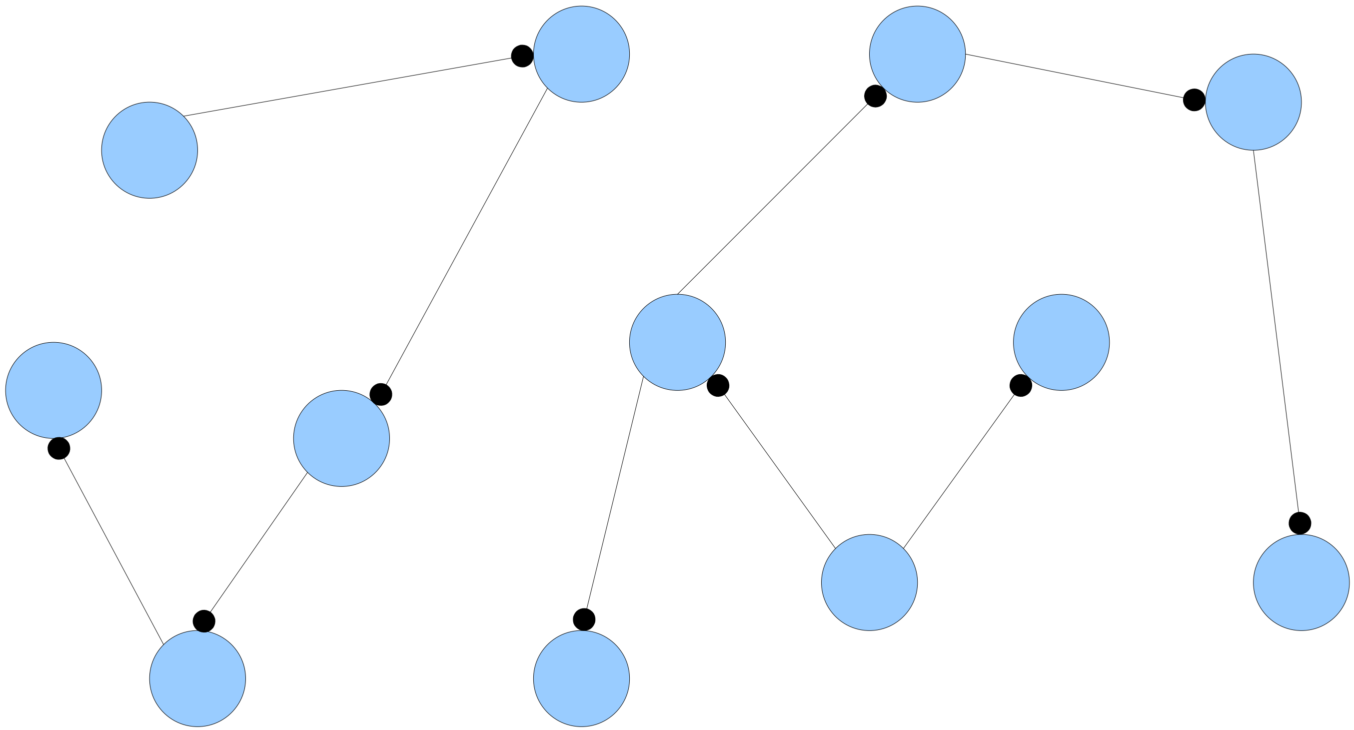

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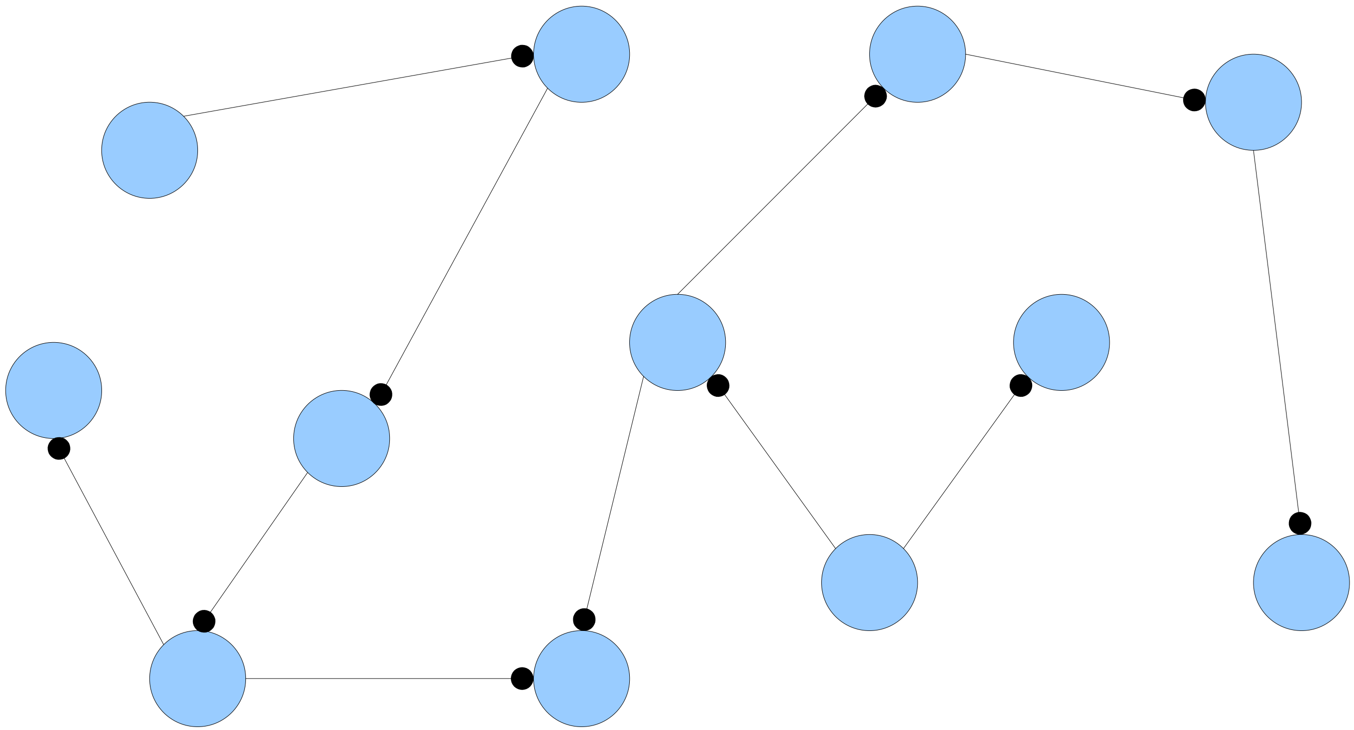

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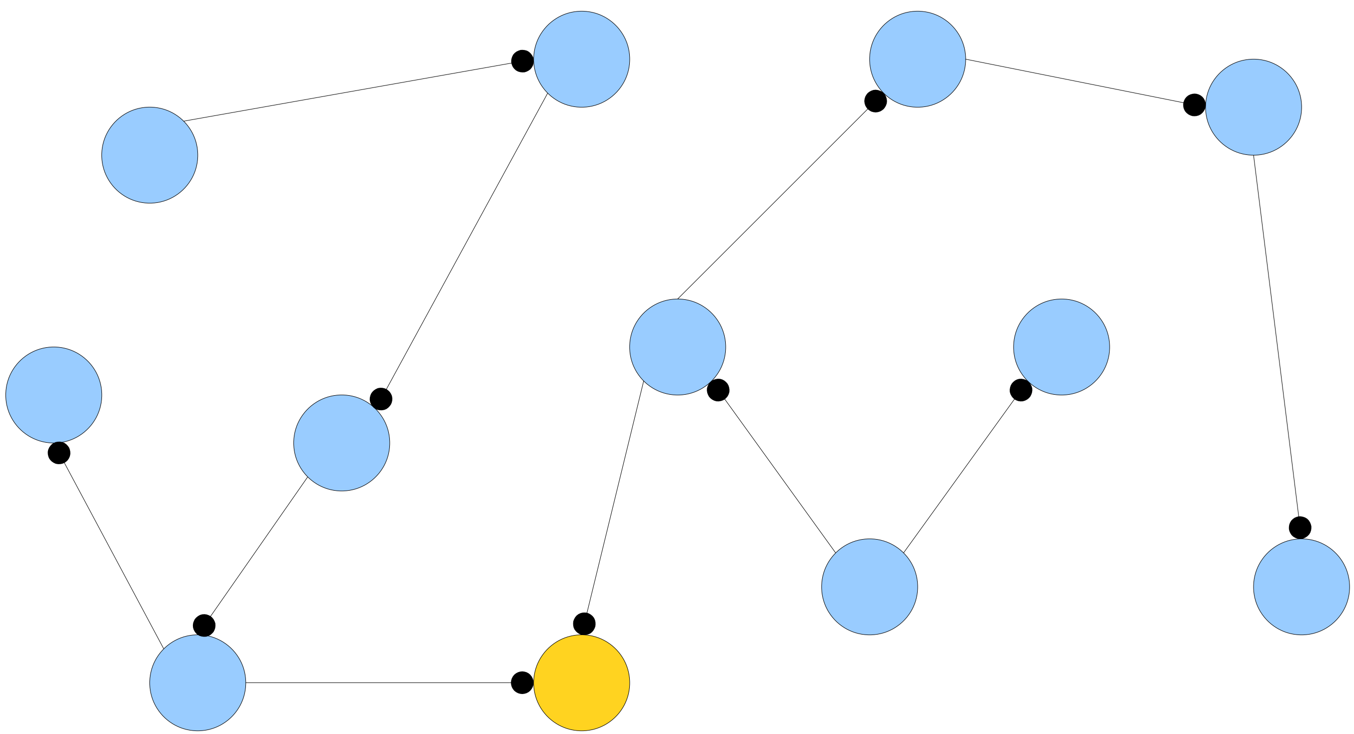

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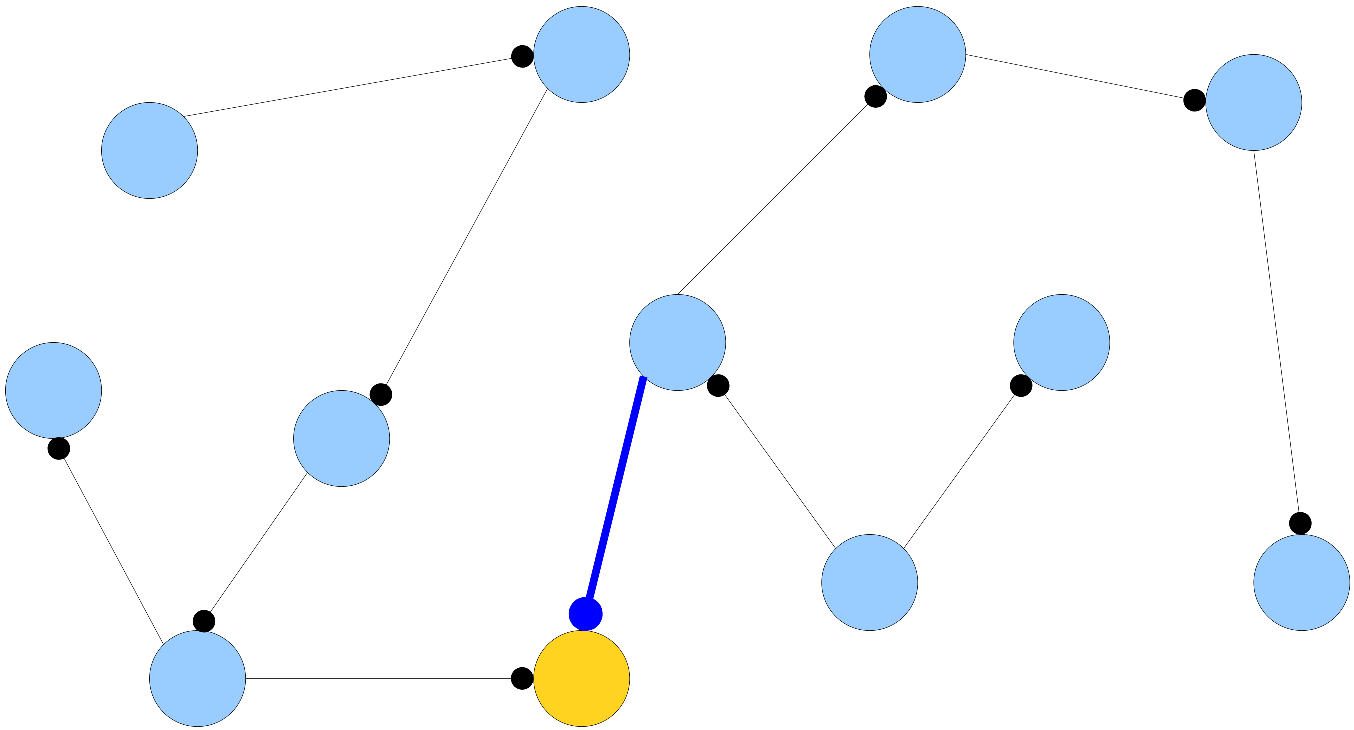

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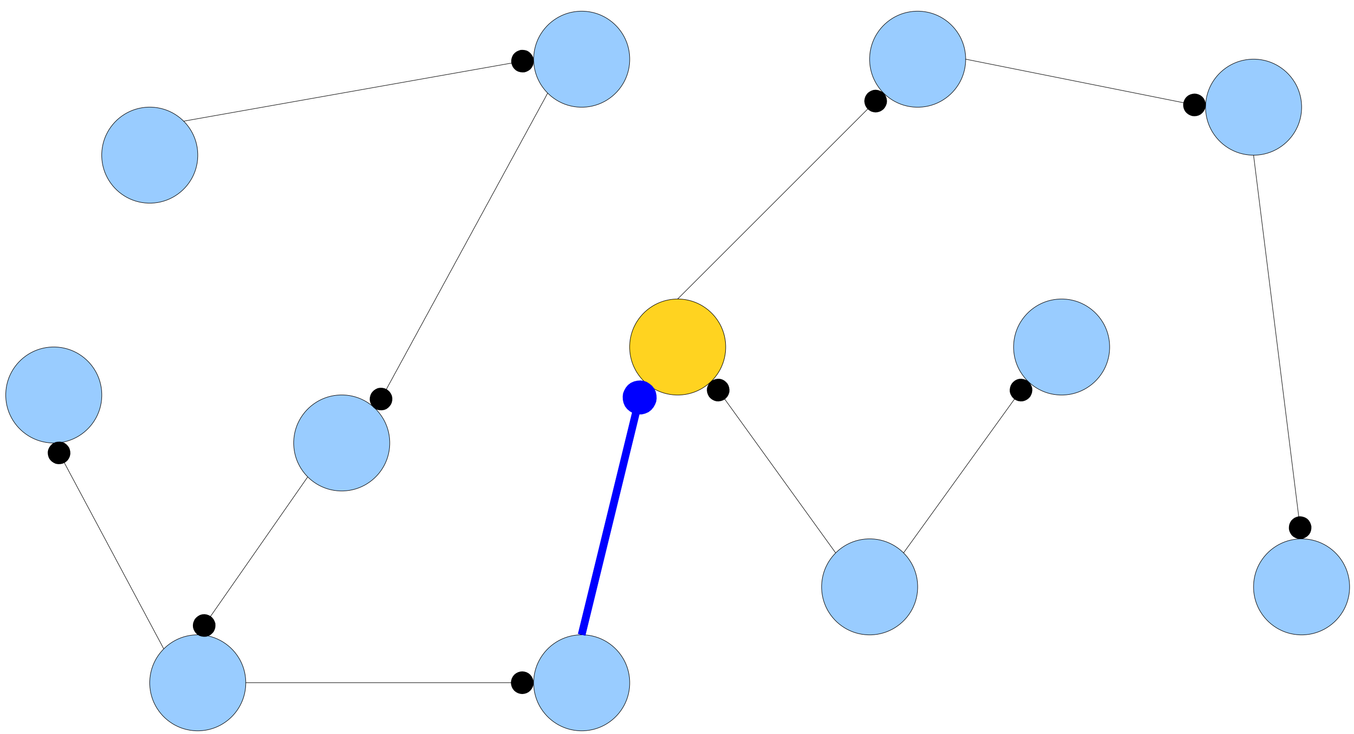

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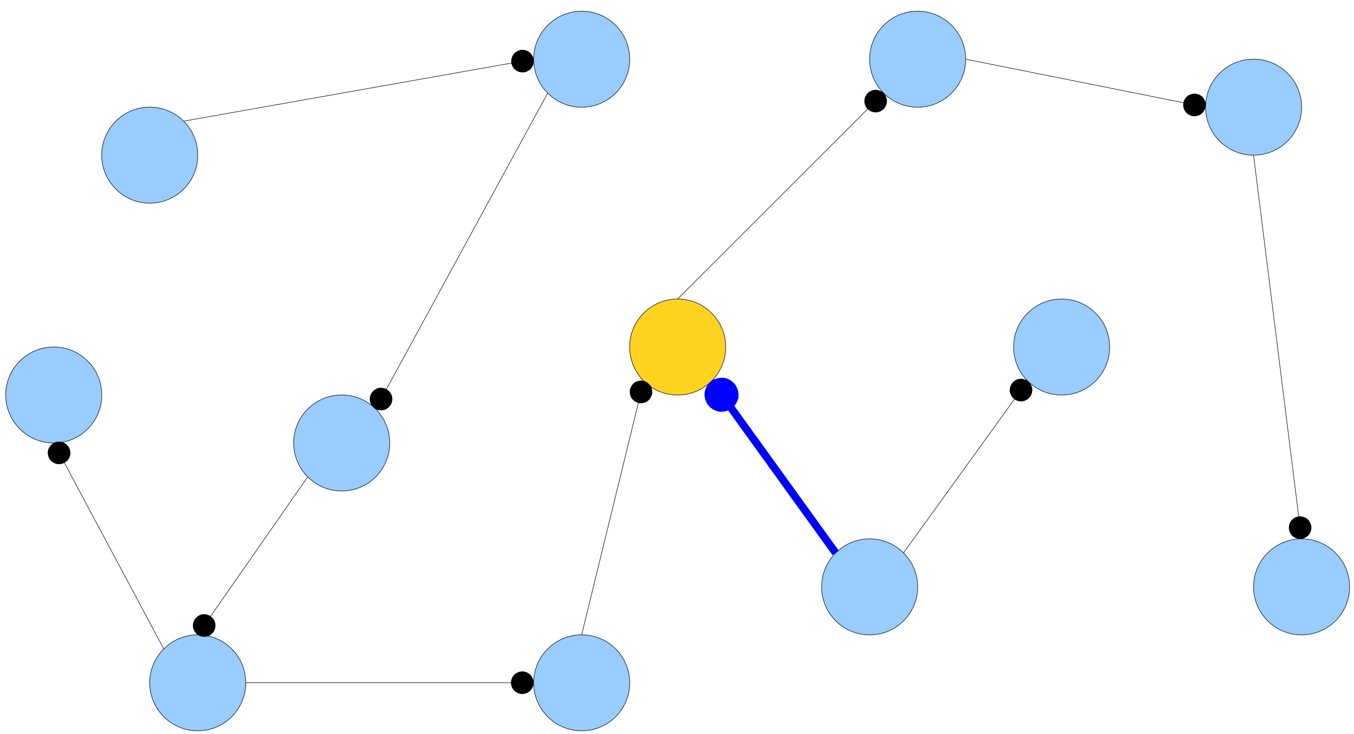

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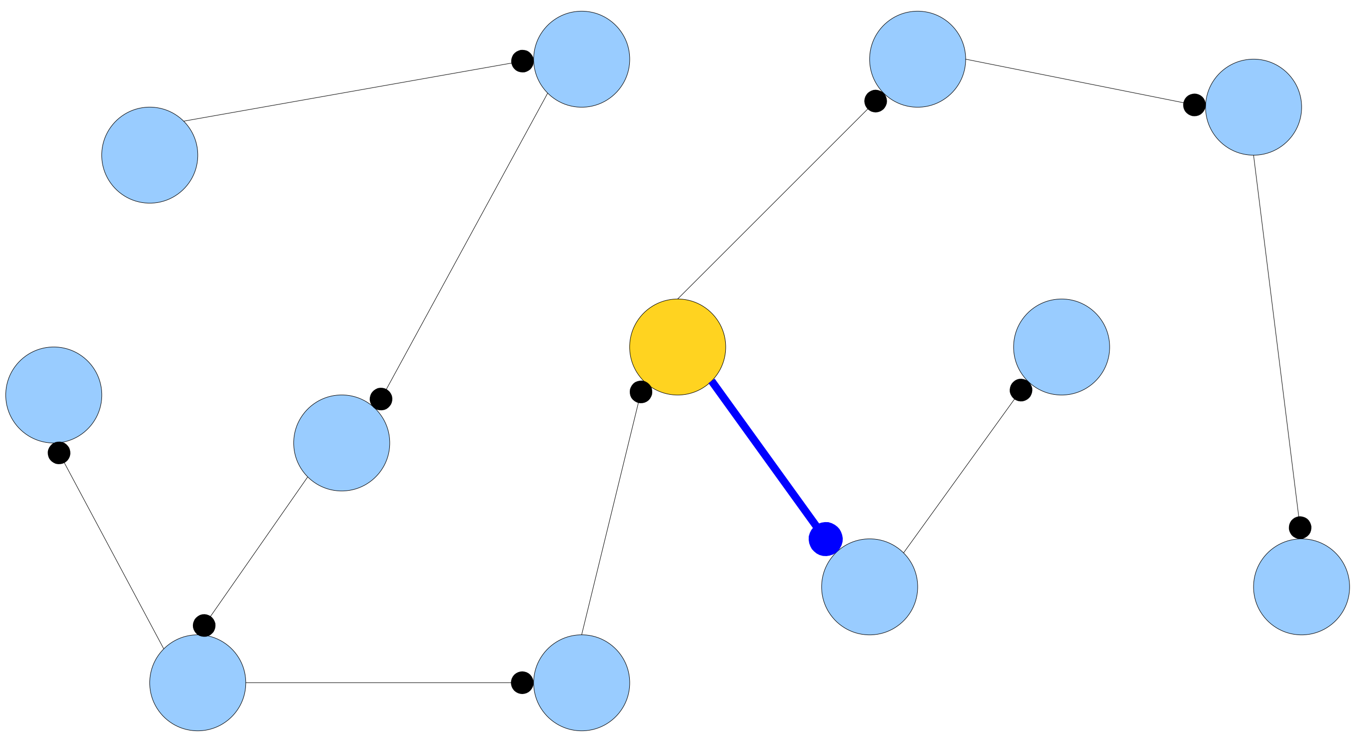

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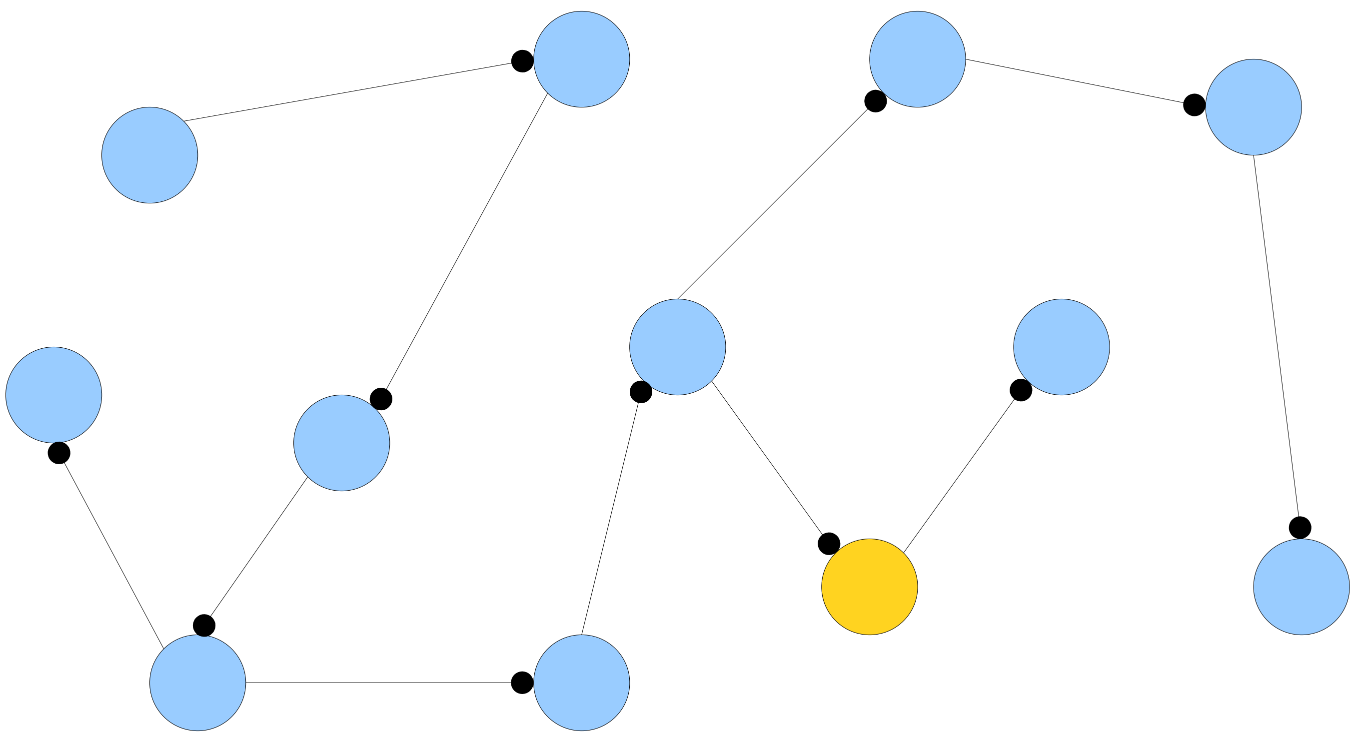

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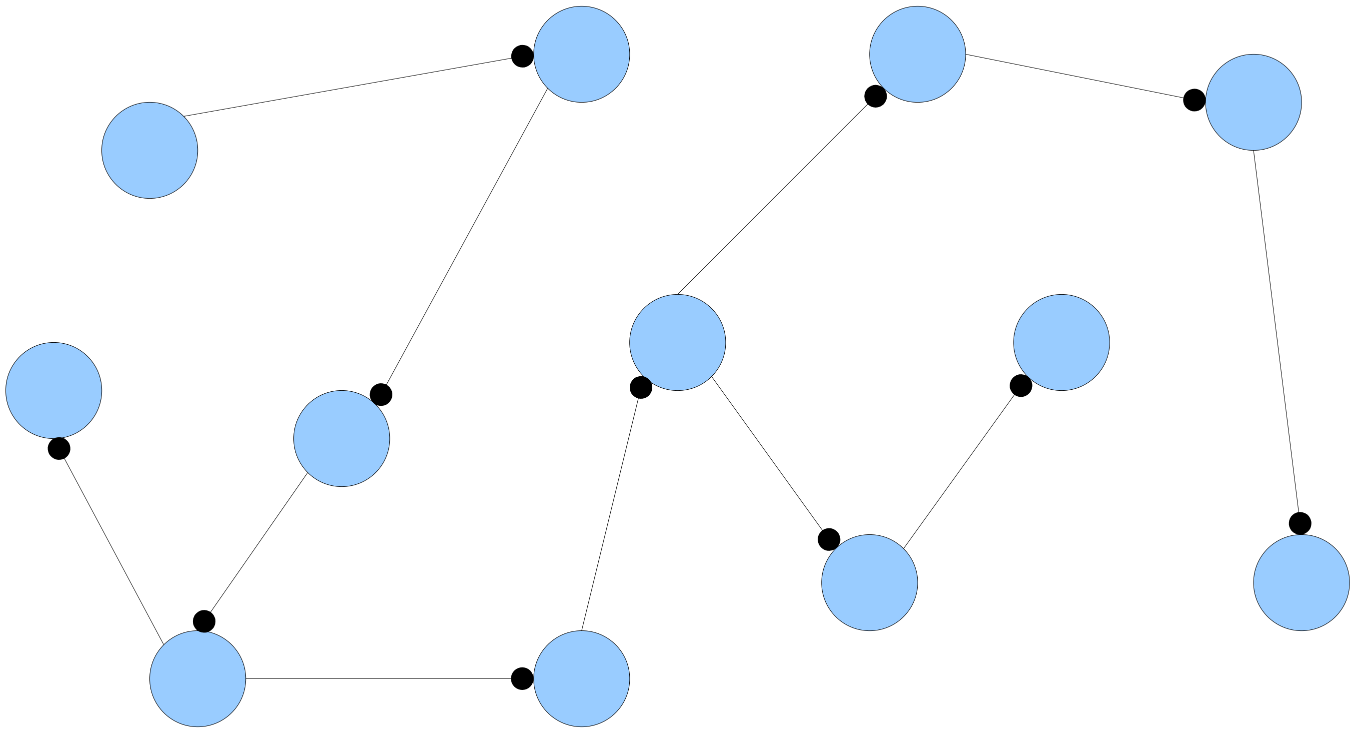

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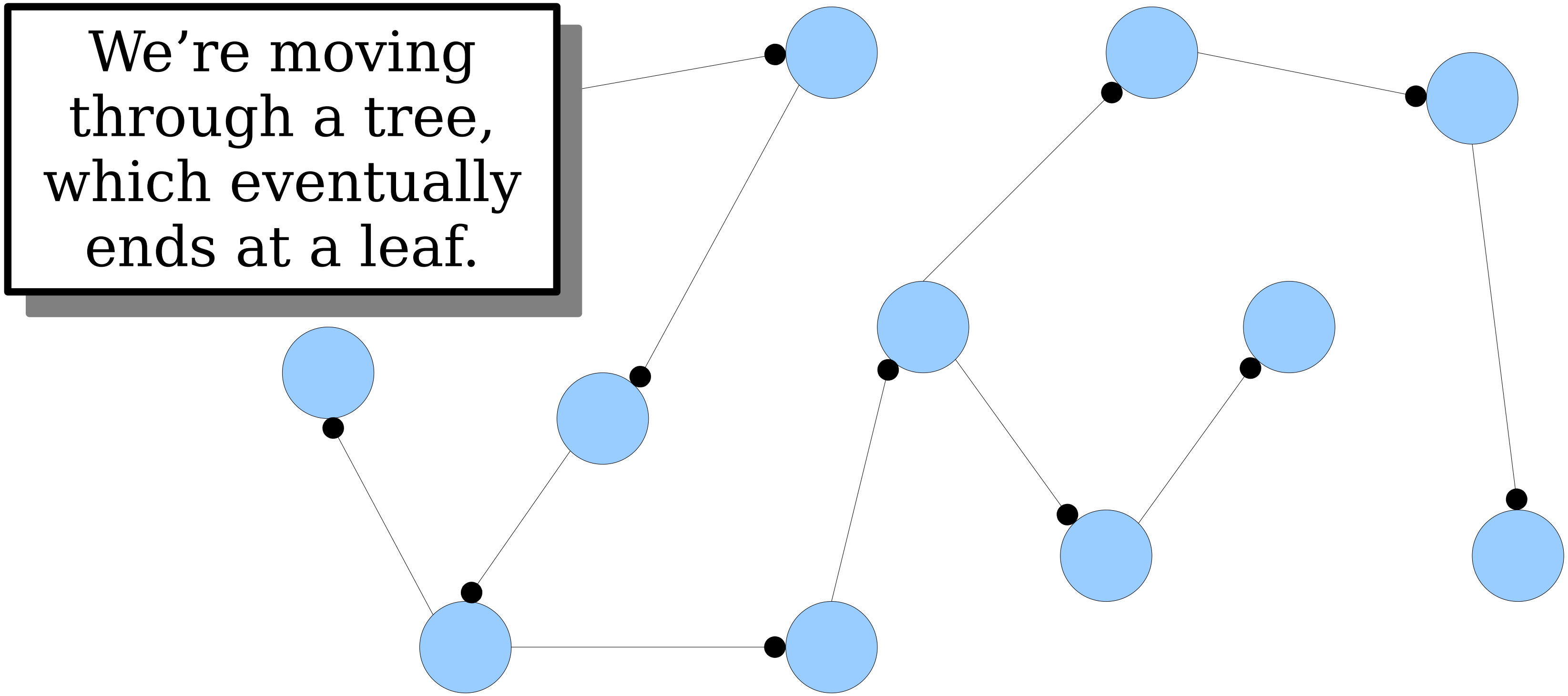

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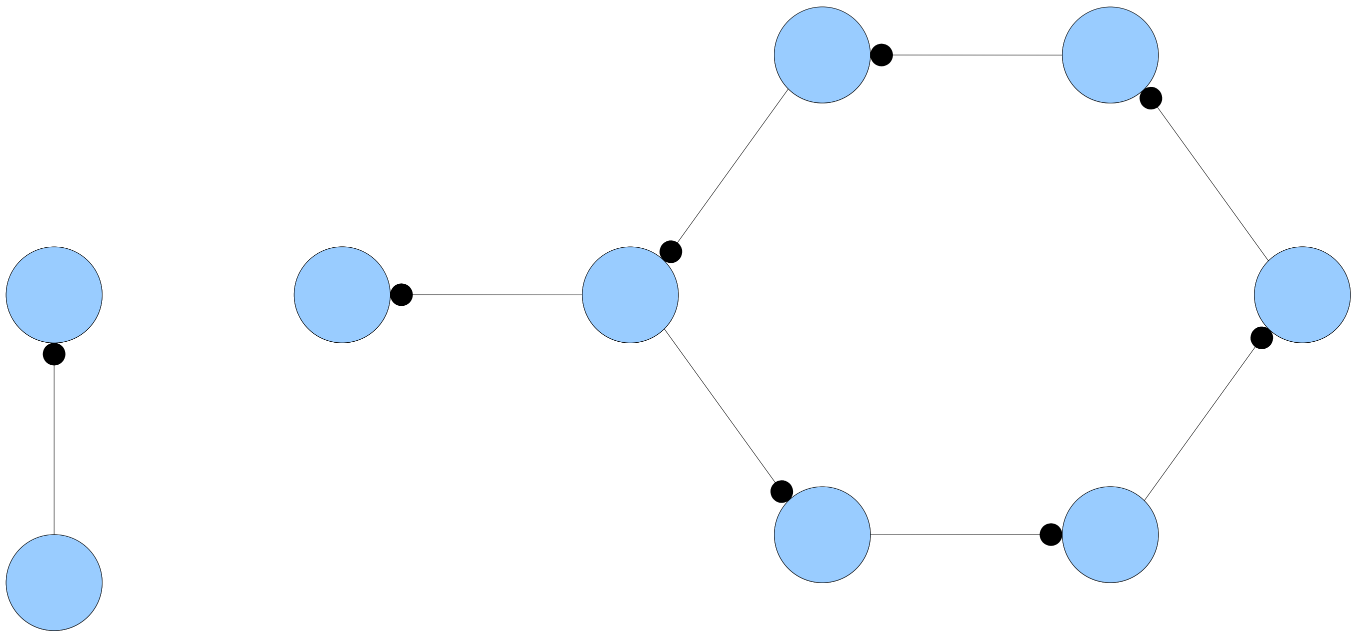

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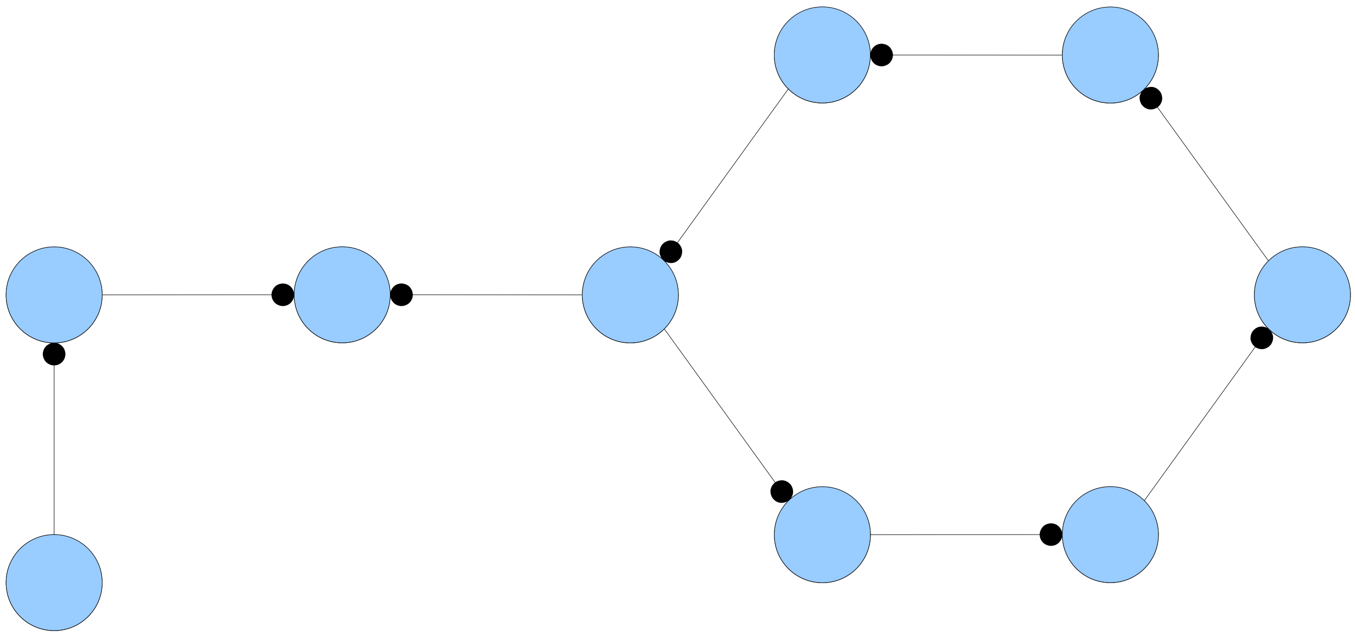

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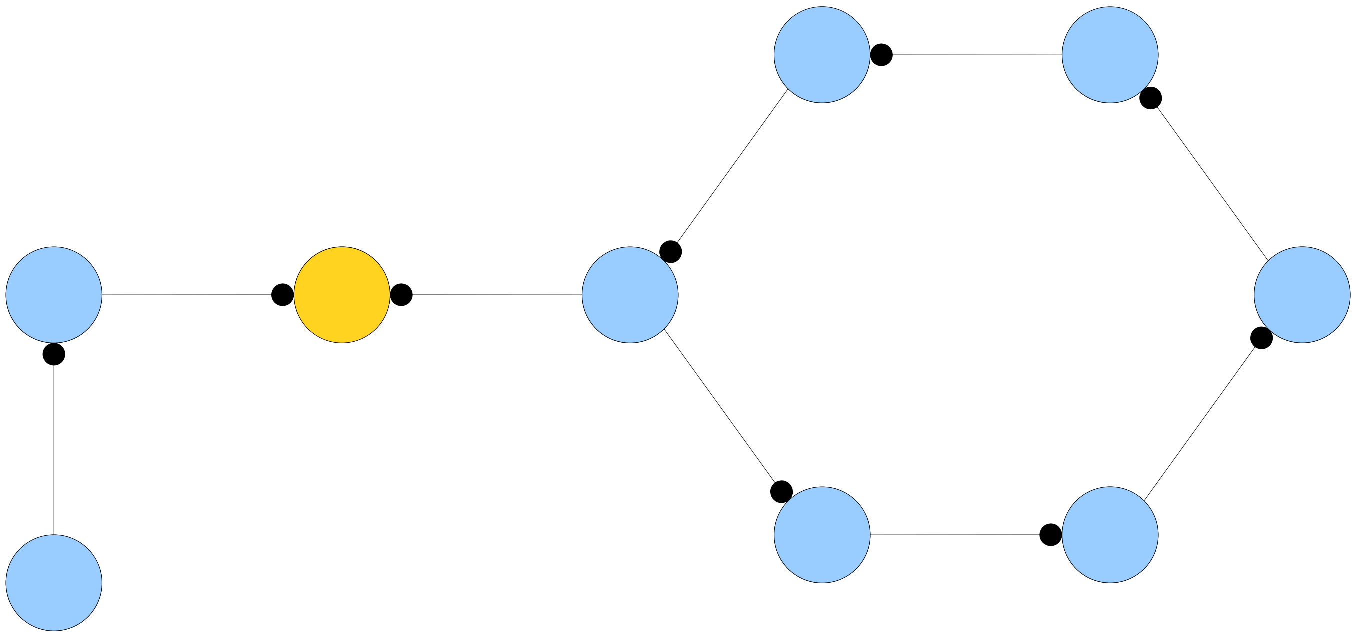

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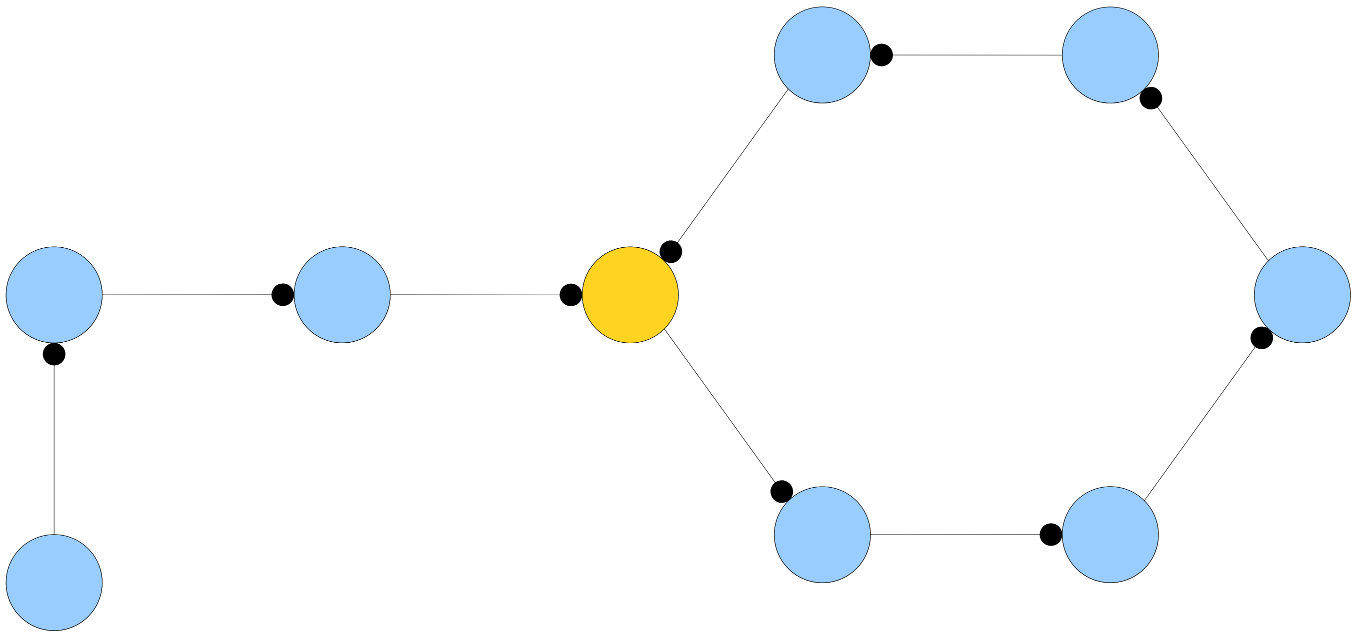

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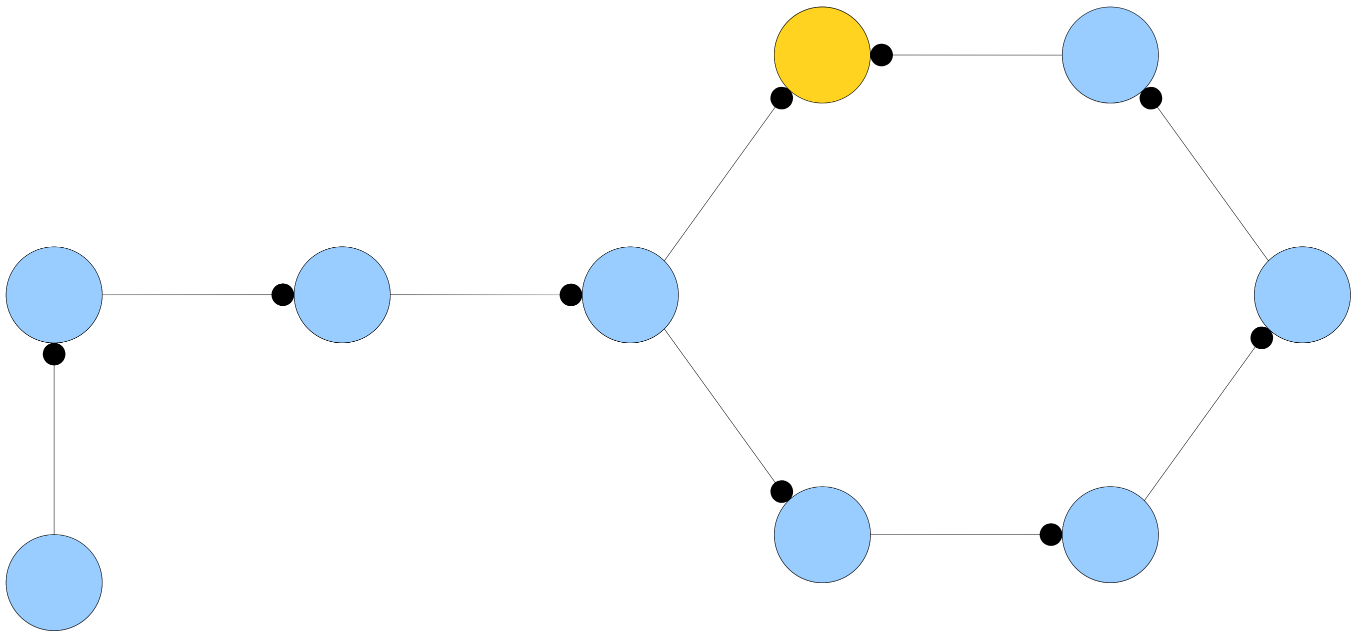

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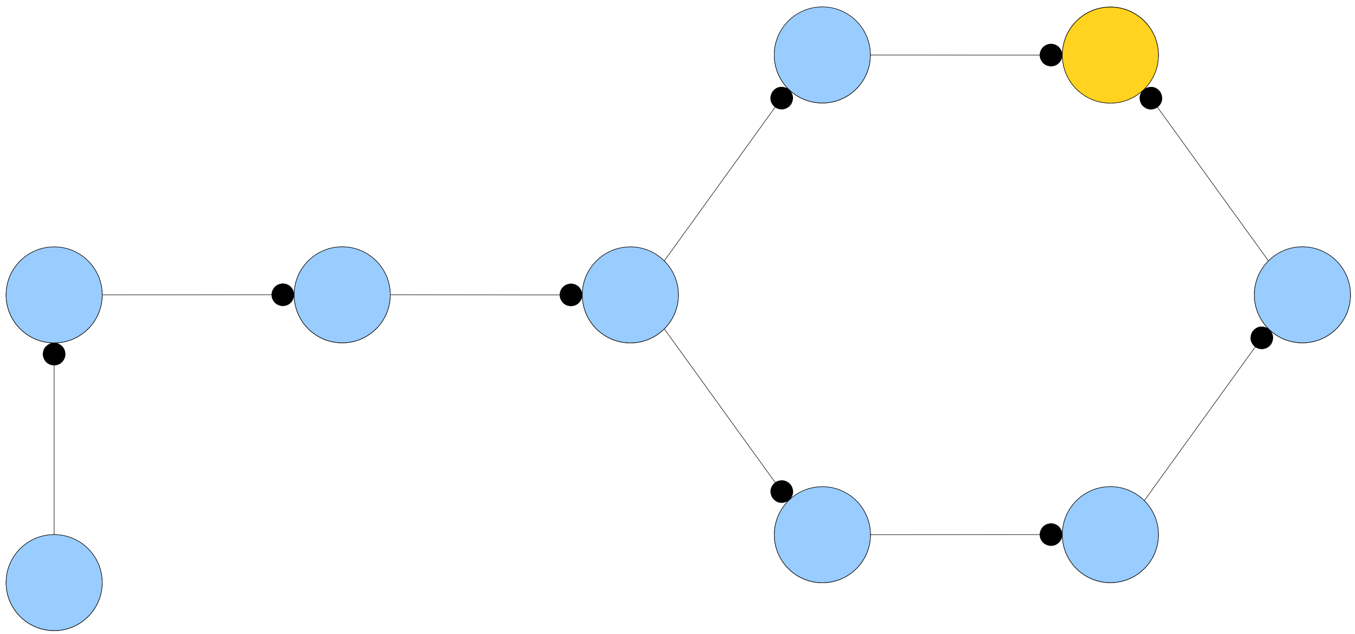

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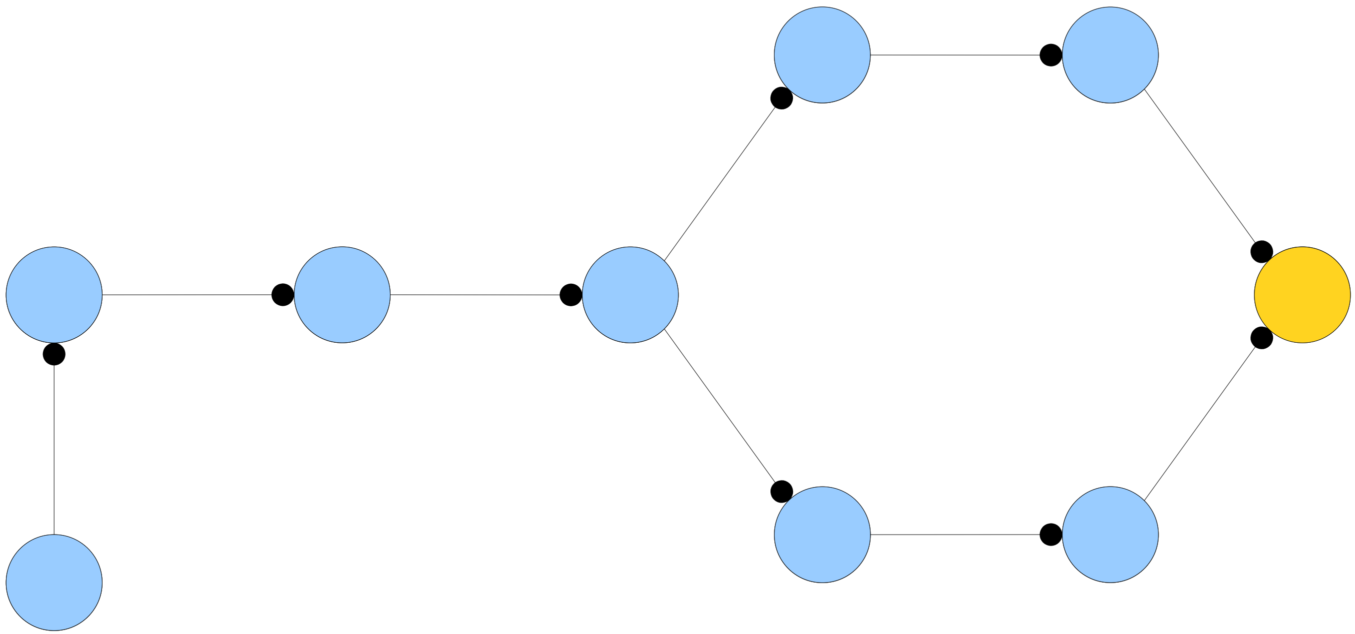

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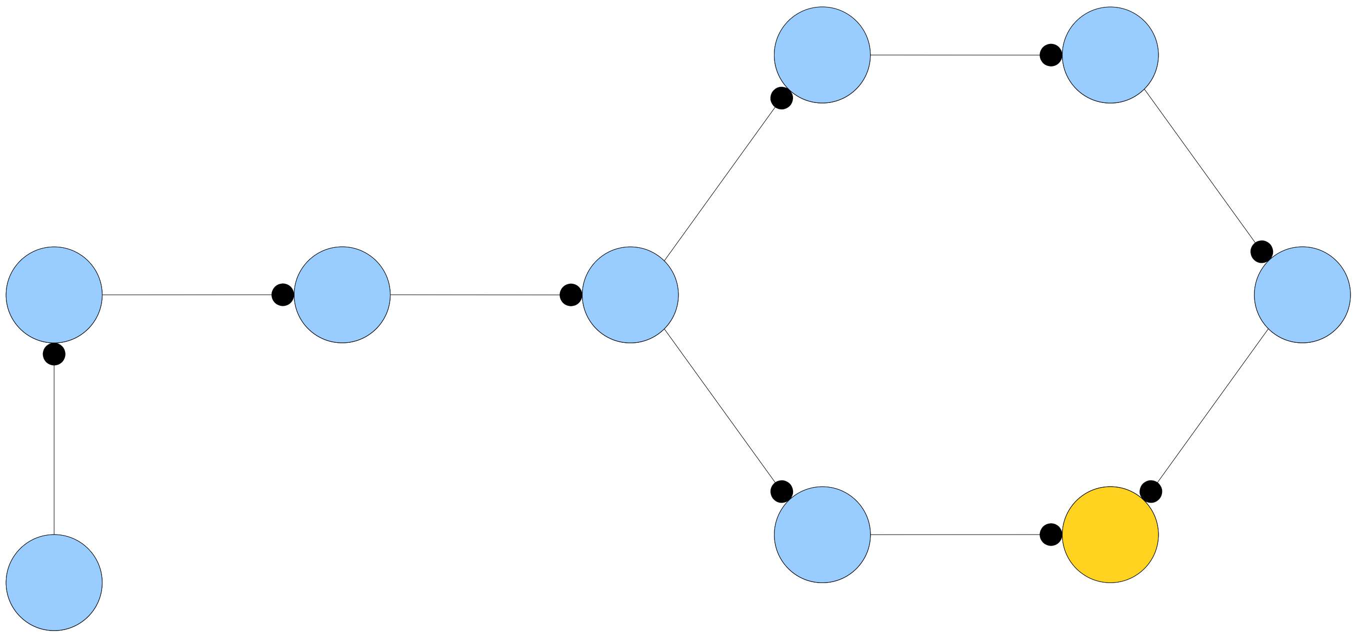

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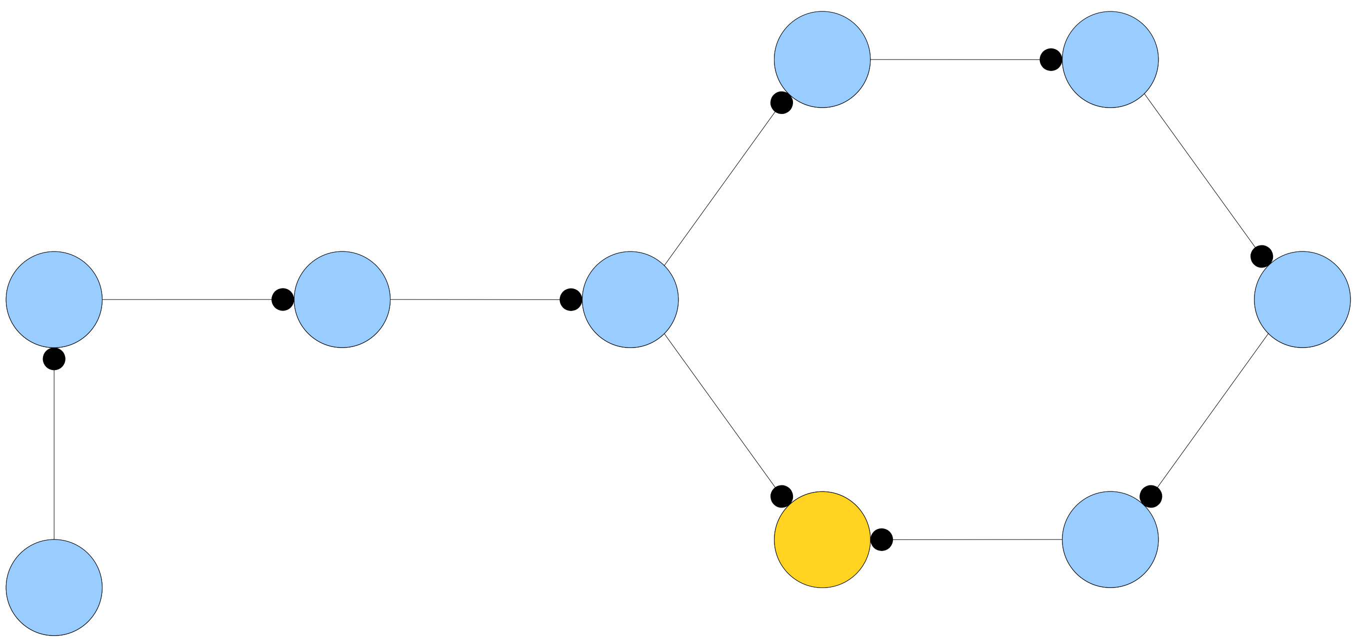

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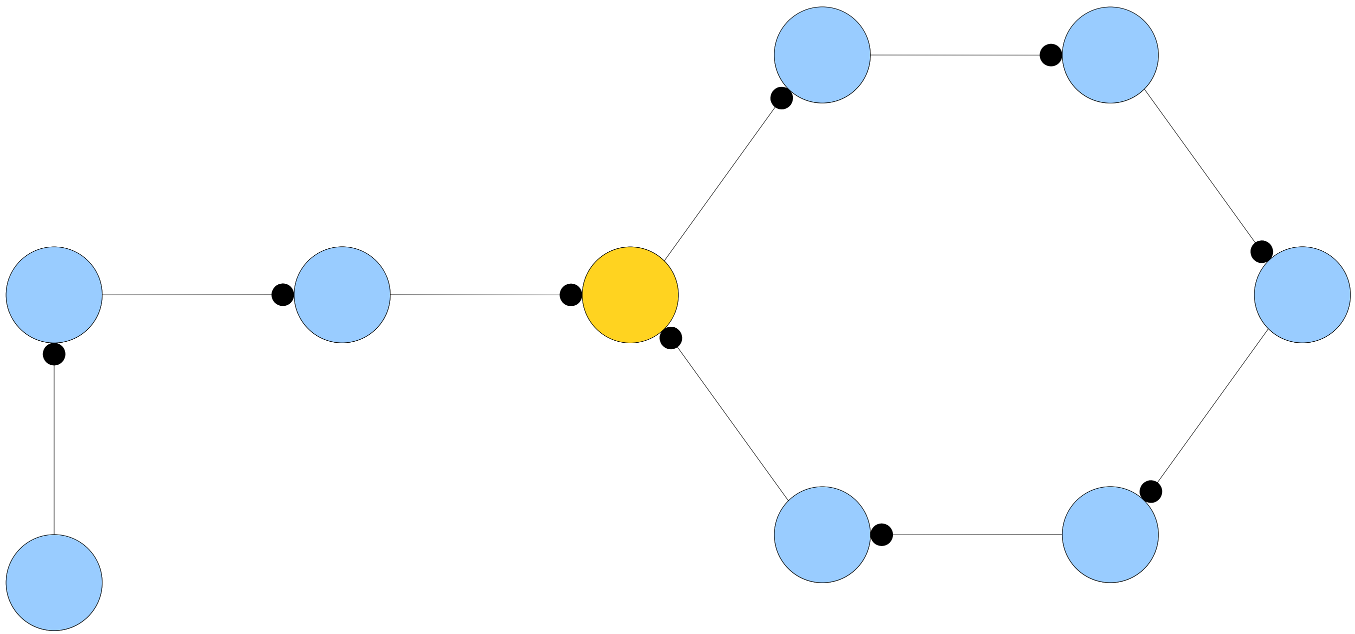

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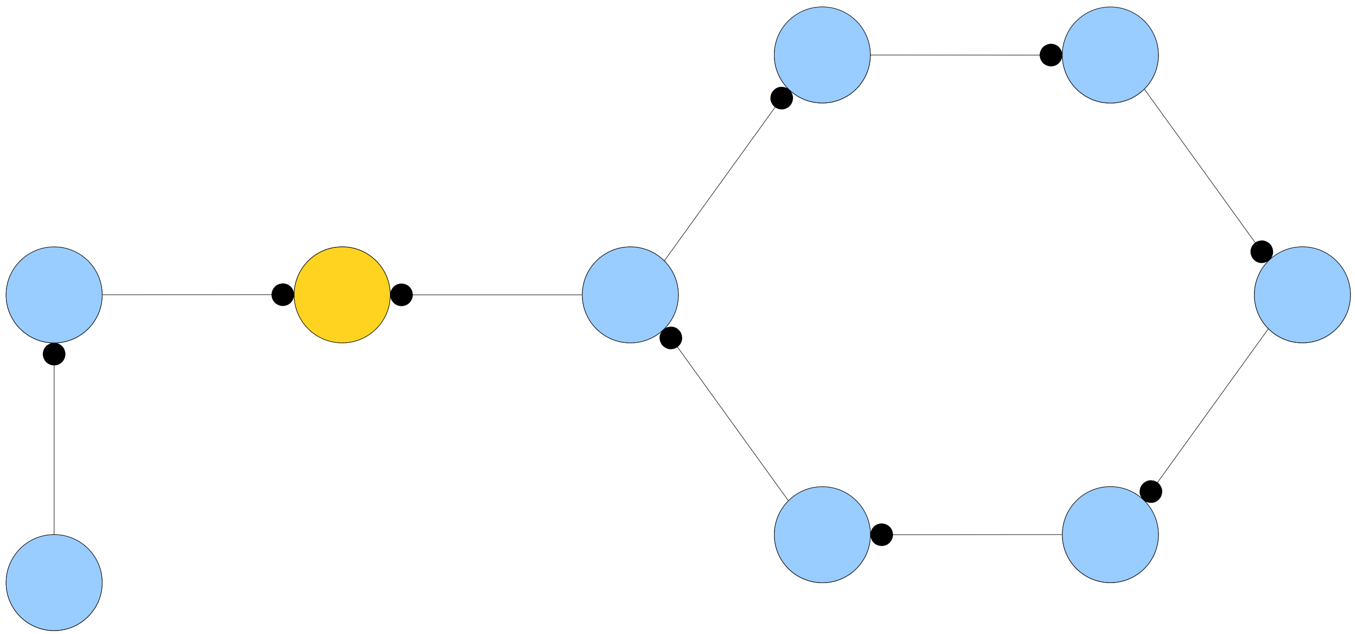

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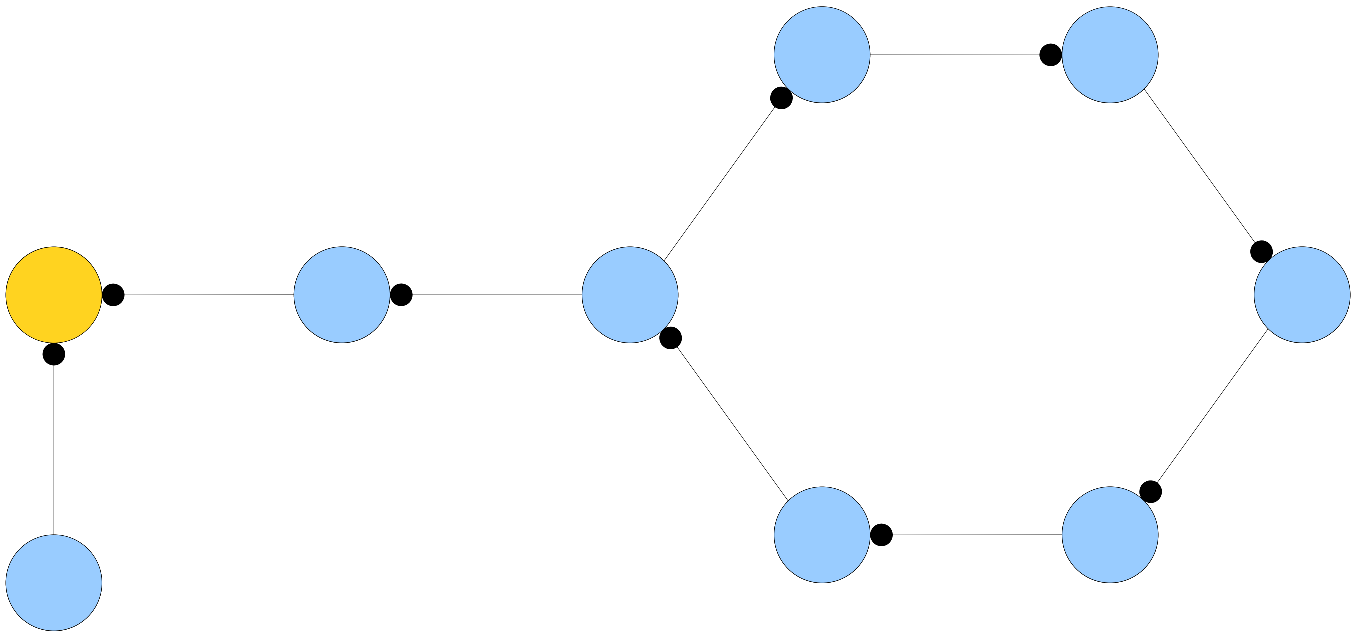

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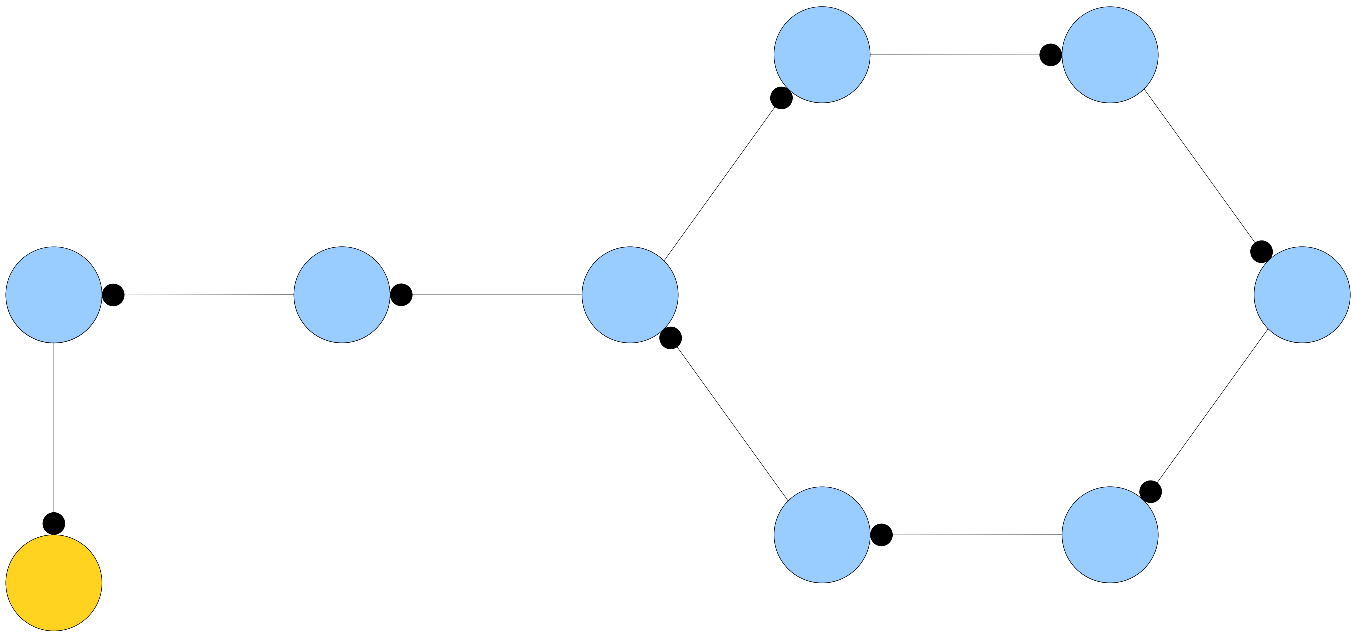

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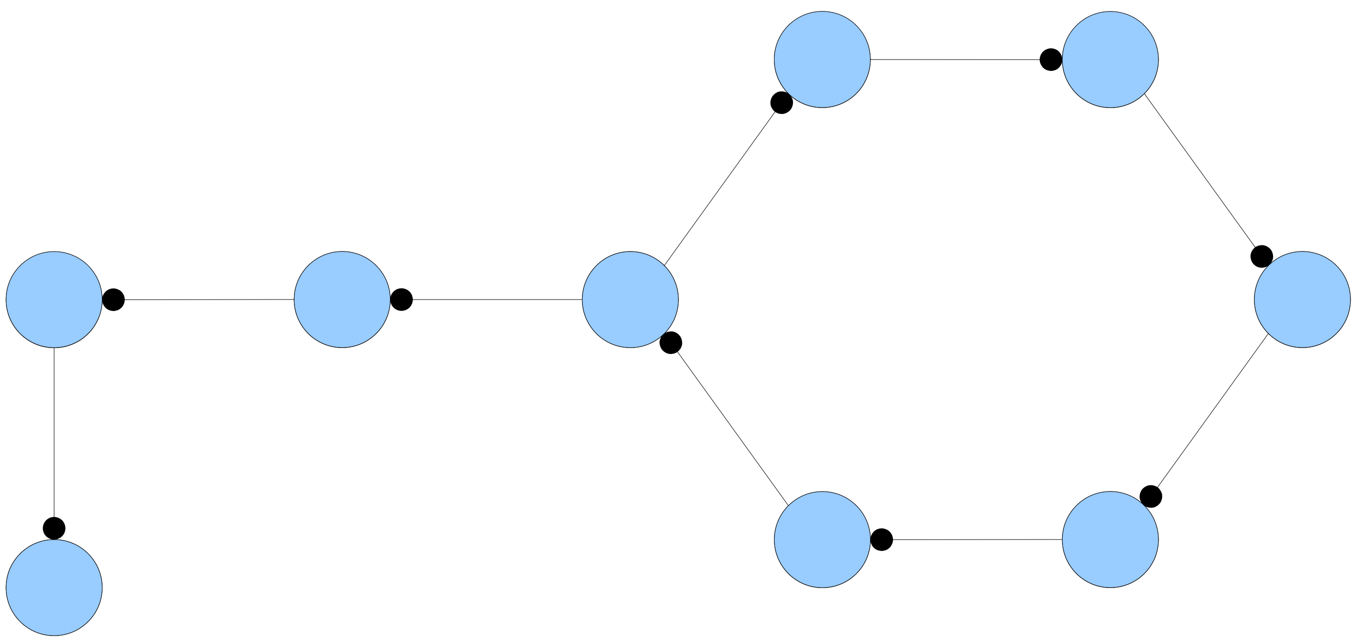

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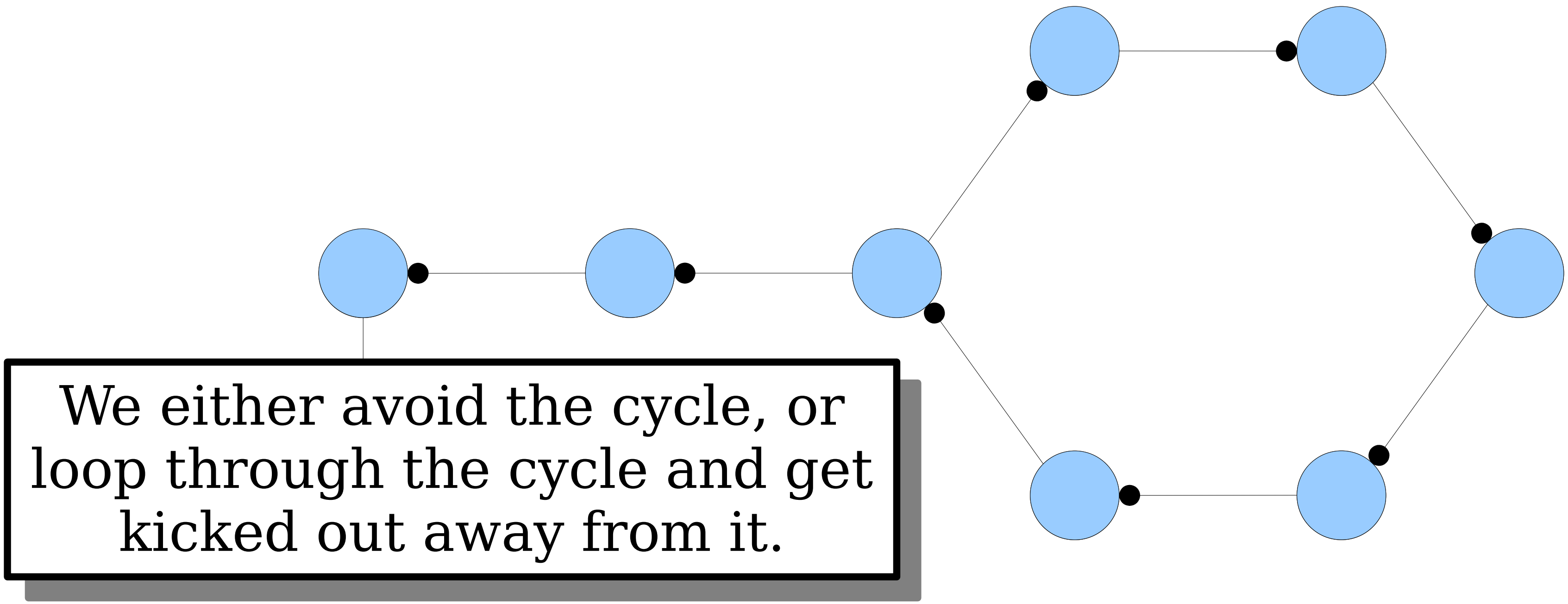


The Cuckoo Graph

- ***Claim 2:*** If x is inserted into a cuckoo hash table, the insertion fails if the connected component containing x contains more than one cycle.

Why?

Formulate a hypothesis, but ***don't post anything in chat just yet***.

The Cuckoo Graph

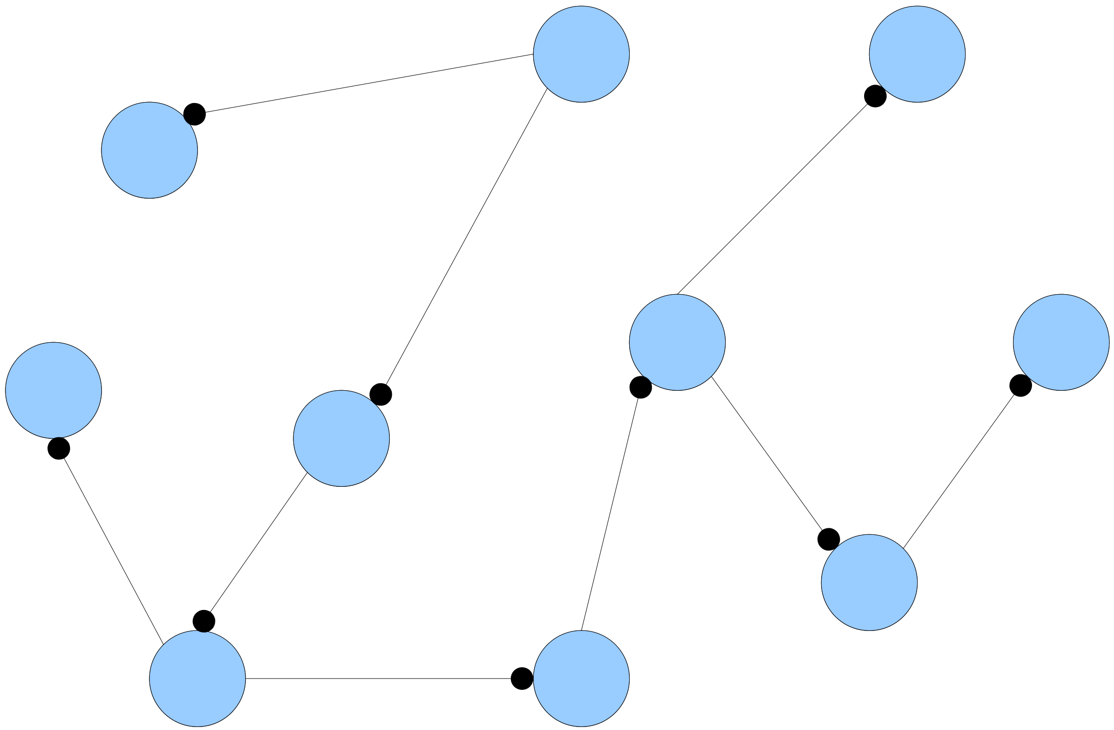
- ***Claim 2:*** If x is inserted into a cuckoo hash table, the insertion fails if the connected component containing x contains more than one cycle.

Why?

Now, ***private chat me your best guess***. Not sure? Just answer “??”.

The Cuckoo Graph

- ***Claim 2:*** If x is inserted into a cuckoo hash table, the insertion fails if the connected component containing x contains more than one cycle.



No cycles: The graph is a directed tree. A tree with k nodes has k – 1 edges.

The Cuckoo Graph

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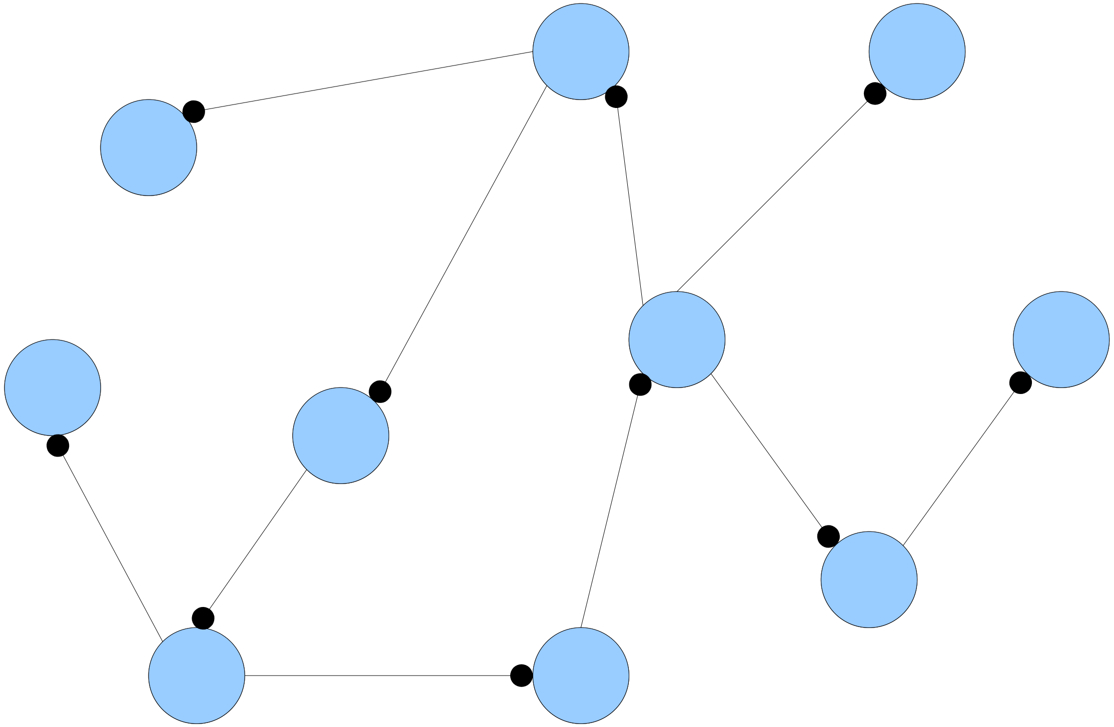

One cycle: We've added an edge, giving k nodes and k edges.

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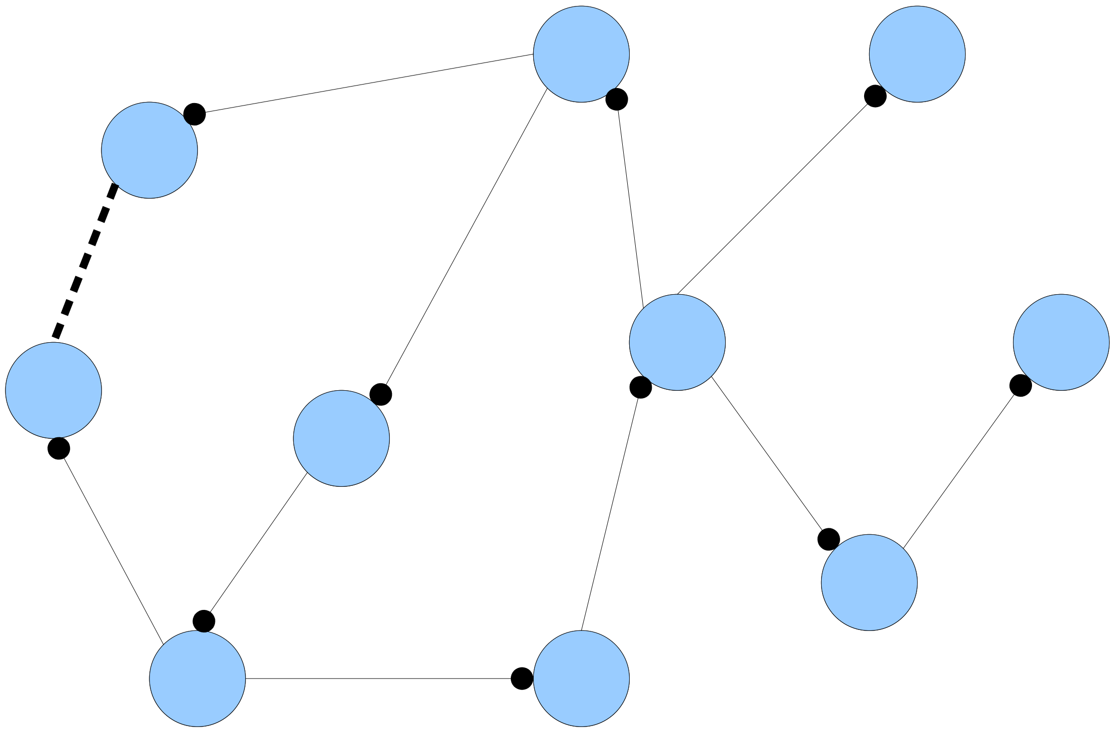
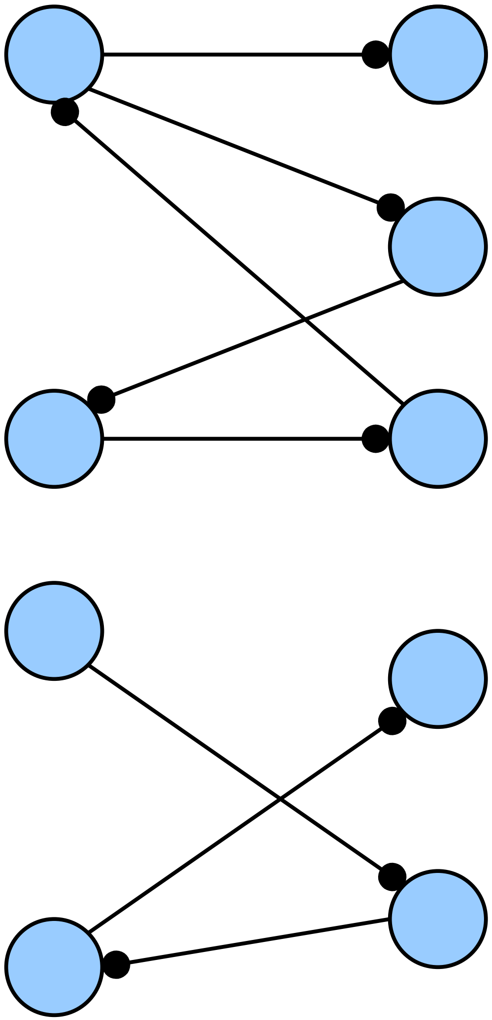


Two cycles: There are k nodes and $k+1$ edges. There are too many edges to place at most one item per node.

The Cuckoo Graph

- A connected component of a graph is called ***complex*** if it contains two or more cycles.
- ***Theorem:*** Insertion into a cuckoo hash table succeeds if and only if the resulting cuckoo graph has no complex connected components.



What is the probability that a connected component in the cuckoo graph is complex?

How big are the connected components in the cuckoo graph?

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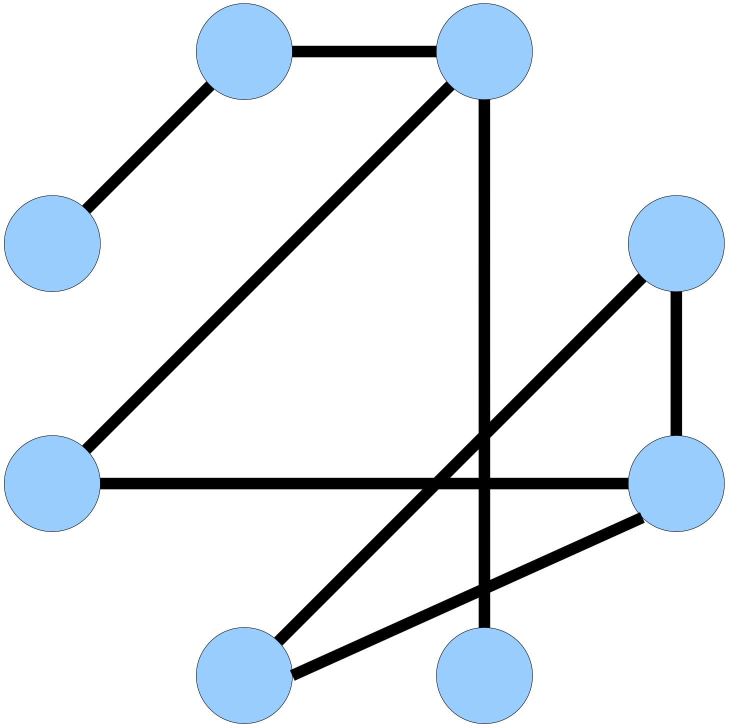
How big are the connected components in the cuckoo graph?

(This tells us how much work we do on a successful insertion.)

The Erdős–Rényi model

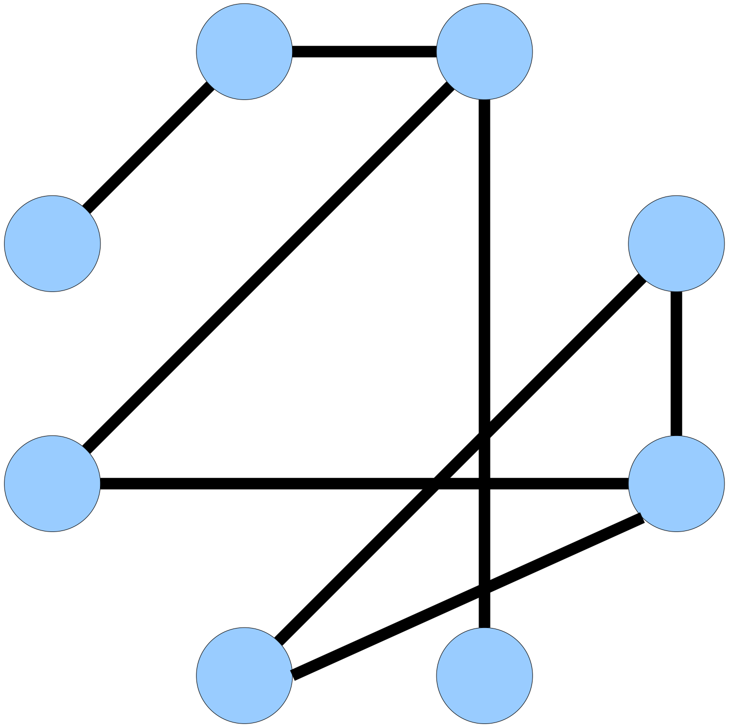
Random Graph Evolution

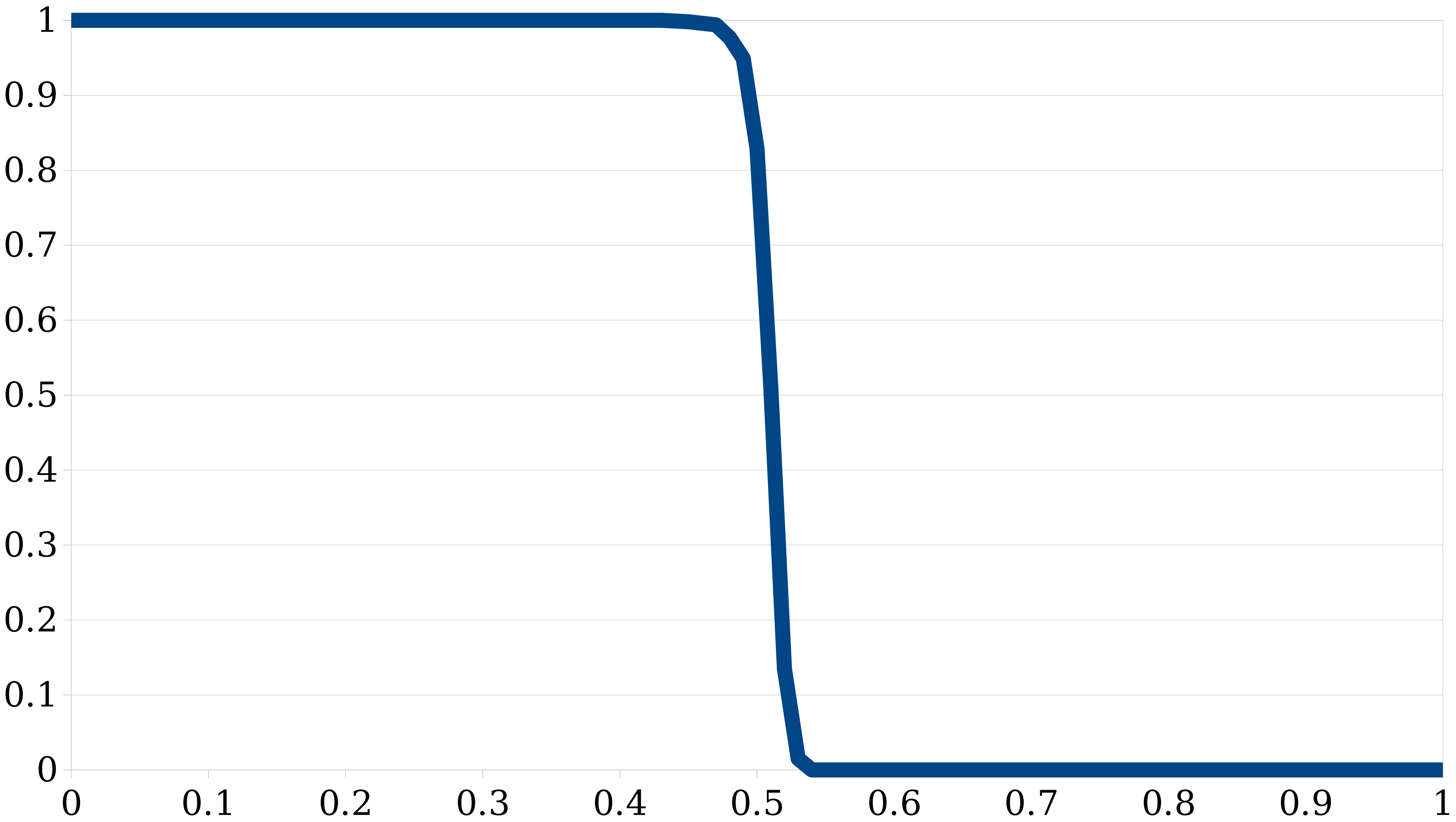
- Consider a graph with V nodes and no edges.
- Incrementally add E edges to the graph, each chosen uniformly at random, possibly with repetition.
- ***Question:*** What properties will this graph (probably) have?



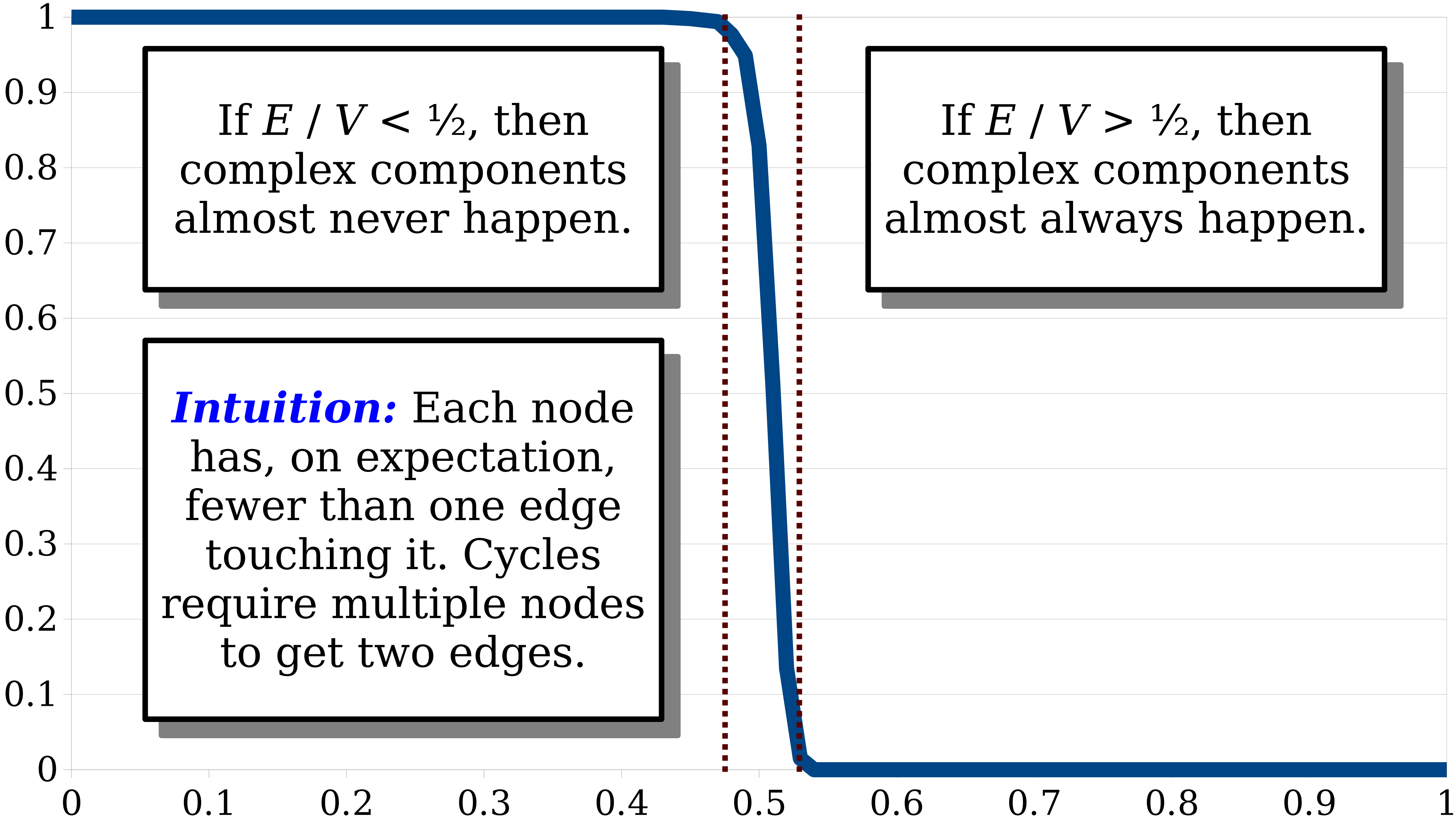
Random Graph Evolution

- ***Claim:*** The phenomena we're observing with cuckoo hashing are, in large part, due to properties of random graphs.
- ***Good News:*** This is a well-studied field! All the results we need were first proved by Erdős and Rényi in 1960.
- This model of incrementally constructing a graph is therefore called the ***Erdős–Rényi*** model.





Consider a random (multi)graph G with V nodes and E edges. What is the the probability that every connected component in G is simple, as a function of the ratio E / V?



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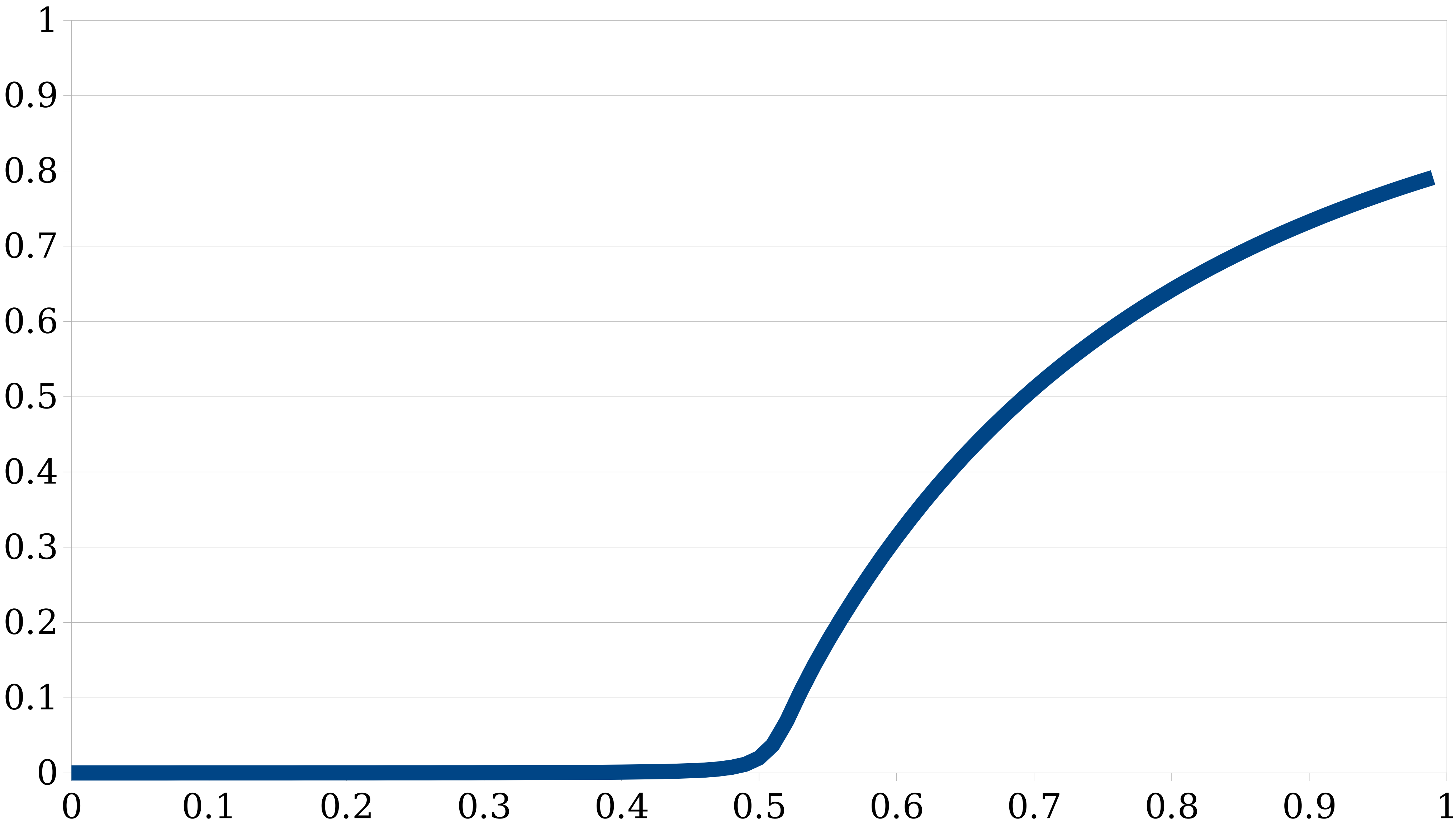

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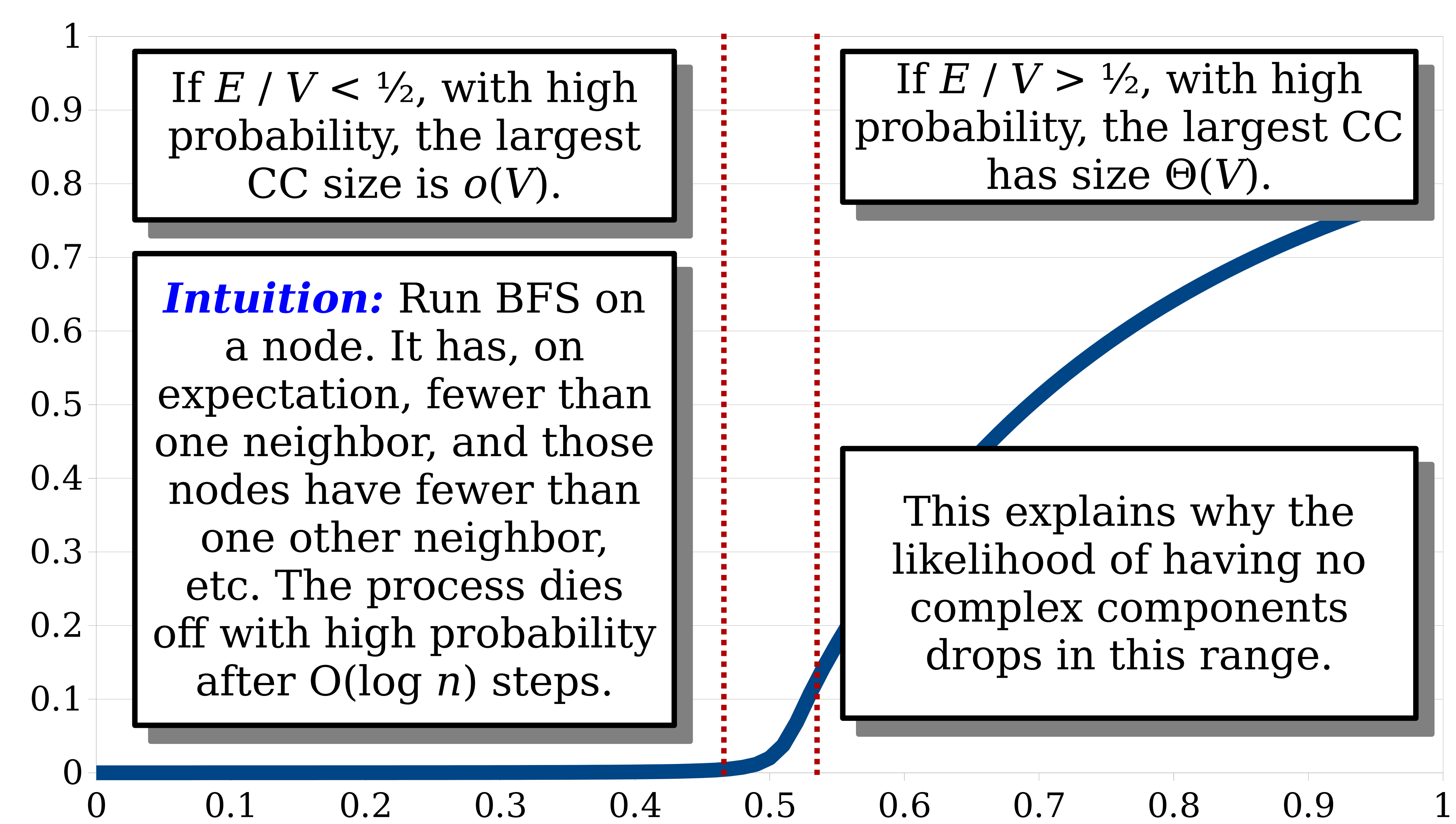
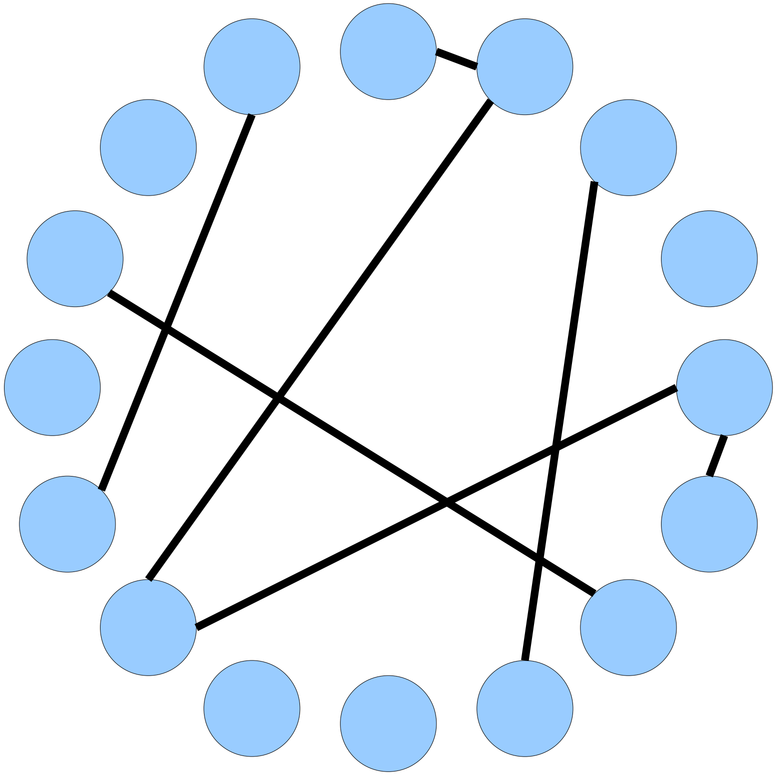


Consider a random (multi)graph G with V nodes and E edges. What fraction of the nodes are in the largest connected component of G, as a function of E / V?

Random Graph Theory

- ***Theorem:*** Let $n = \alpha m$ with $\alpha < ½$. Then the expected size of a connected component in a randomly-chosen graph with m nodes and n edges is O(1), and with high probability the largest connected component has size O(log n).
- ***Corollary:*** Using cuckoo hashing with m slots and $n = \alpha m$ items, for $\alpha < ½$, assuming all inserts succeed, the expected cost of each insert is O(1), and the worst-case cost of each insert is O(log n) with high probability.



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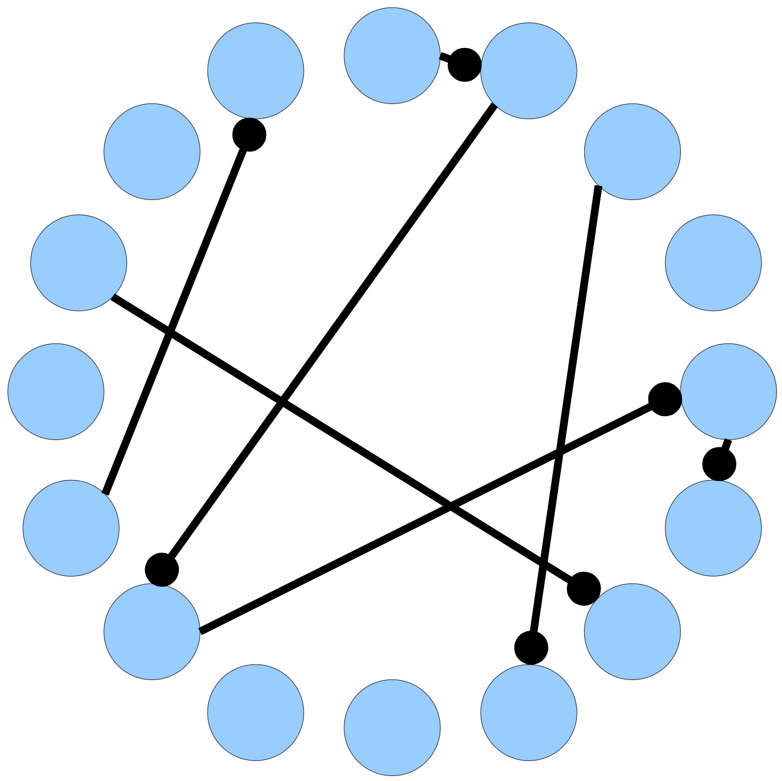
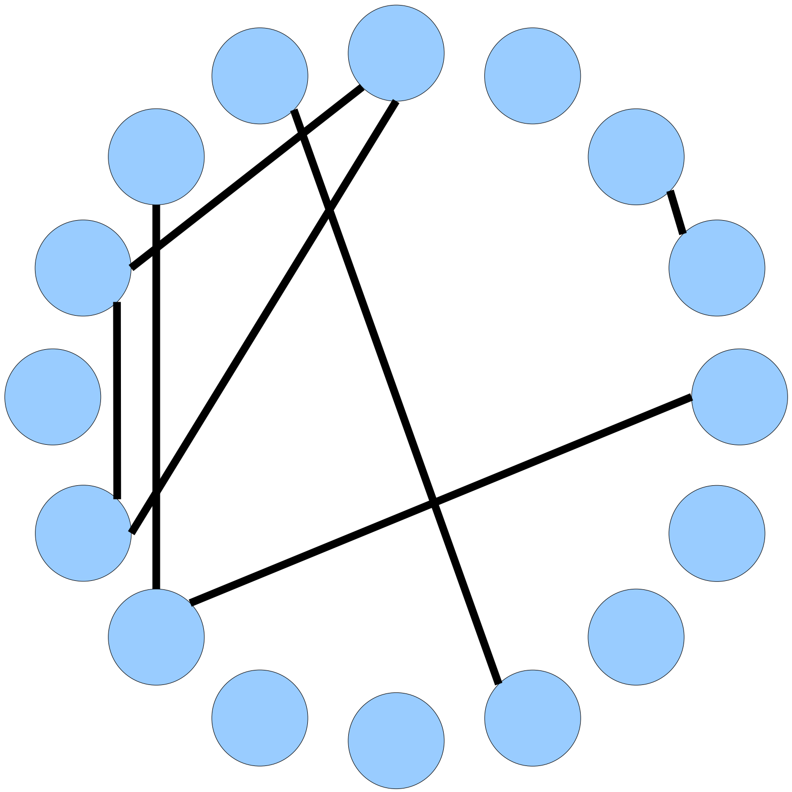


Random Graph Theory

- ***Theorem:*** Let $n = \alpha m$ for some $\alpha < \frac{1}{2}$. Then the probability that any connected component is complex is $O(1/n)$.
- ***Corollary:*** Using cuckoo hashing with m slots and $n = \alpha m$ items, the probability that a series of n insertions fails is $O(1/n)$, and the expected number of times a rehash is required before it succeeds is $O(1)$.



The Overall Analysis

- Cuckoo hashing gives worst-case lookups and deletions.
- Insertions are expected, amortized O(1).
  - The amortization kicks in because we need to periodically double the sizes of the table as the number of elements increases.
- The hidden constants are small, and this is a practical technique for building hash tables.

Cuckoo Hashing:

- ***lookup***: O(1)
- ***insert***: O(1)*
- ***delete***: O(1)

* *expected, amortized*

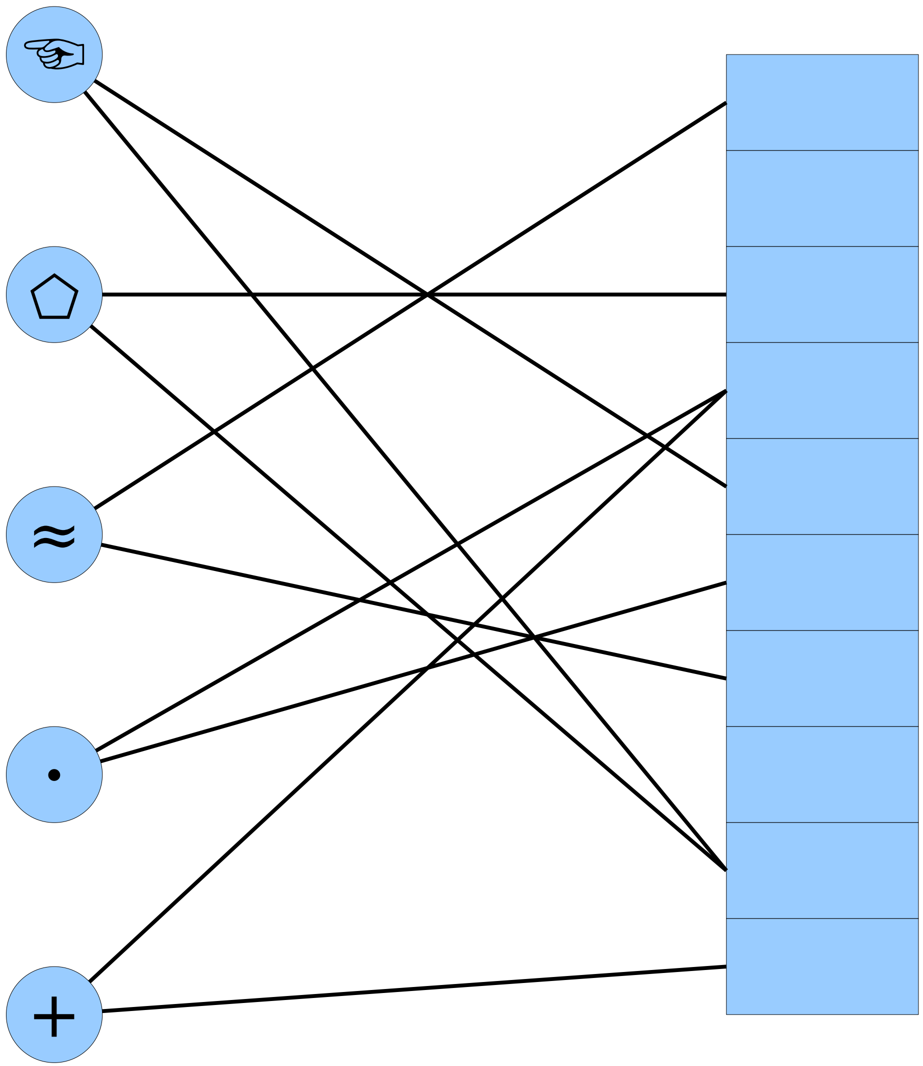
Improving Our Space Usage

Improving Space Usage

- A cuckoo hash table with n elements requires a table of size n / α, with $\alpha < ½$.
- This means at least 50% of the table slots will be empty.
- The root cause is a fundamental property of random graphs; exceeding this threshold makes failure almost certain.
- ***Question:*** How can we push past this to improve cuckoo hashing space usage?

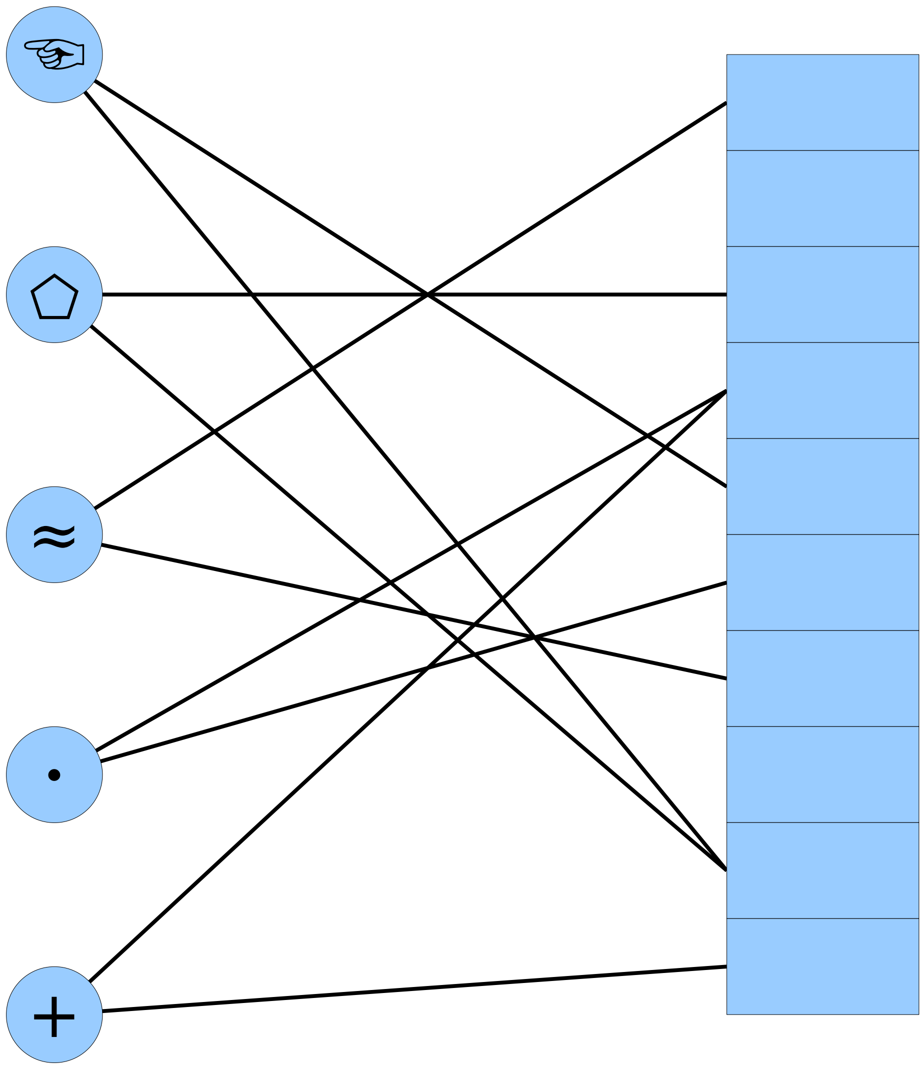
Rethinking Cuckoo Hashing

- Earlier, we used the ***cuckoo graph*** to model cuckoo hashing.
- There's another graph we can use to model cuckoo hashing.
  - Create two groups of nodes: one for items, and one for table slots.
  - Add edges from each item to the table slots that can store it.
- ***Key idea:*** When hashing items to multiple positions, draw this bipartite graph and figure out which of its properties you want to study.



Rethinking Cuckoo Hashing

- Each item needs to go into a table slot.
- ***Goal:*** Choose a group of edges where
  - each node on the left is adjacent to *exactly one* edge, and
  - each node on the right is adjacent to *at most one* chosen edge.
- This models our assignment of items to slots.



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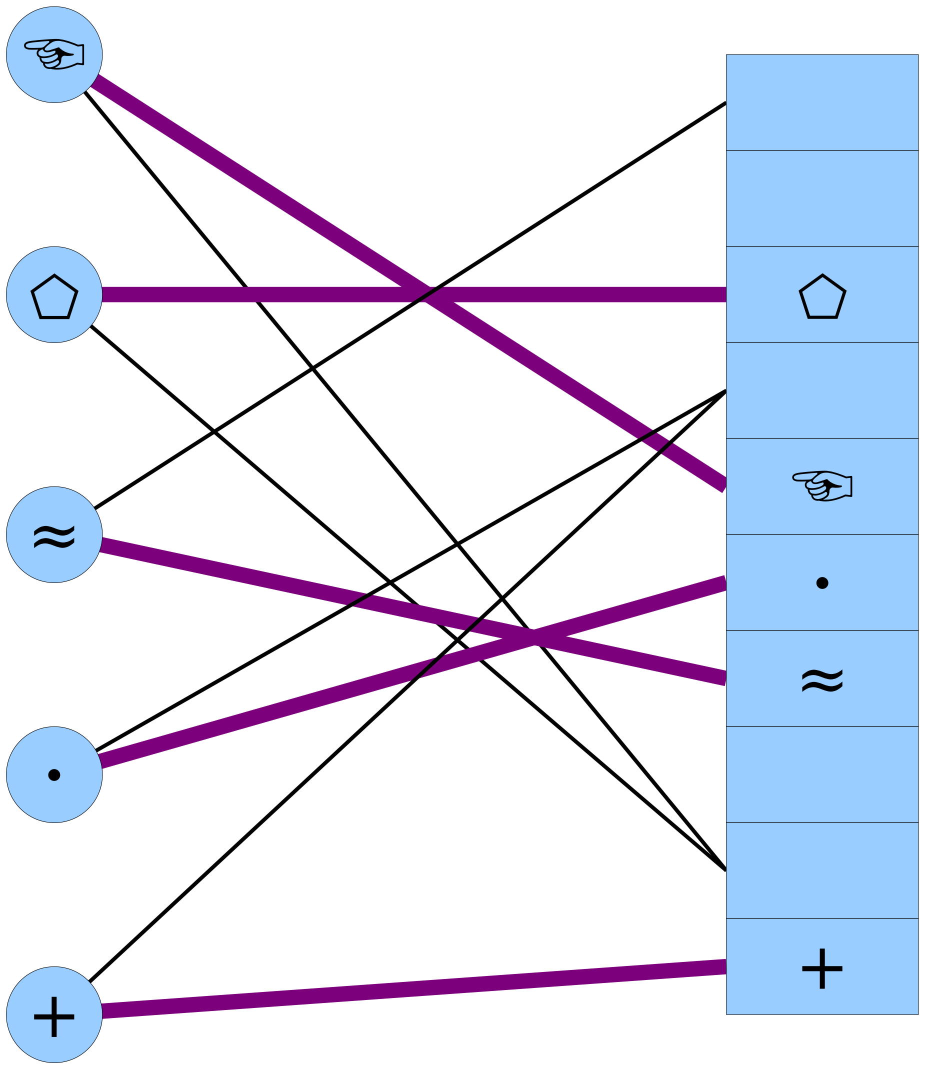

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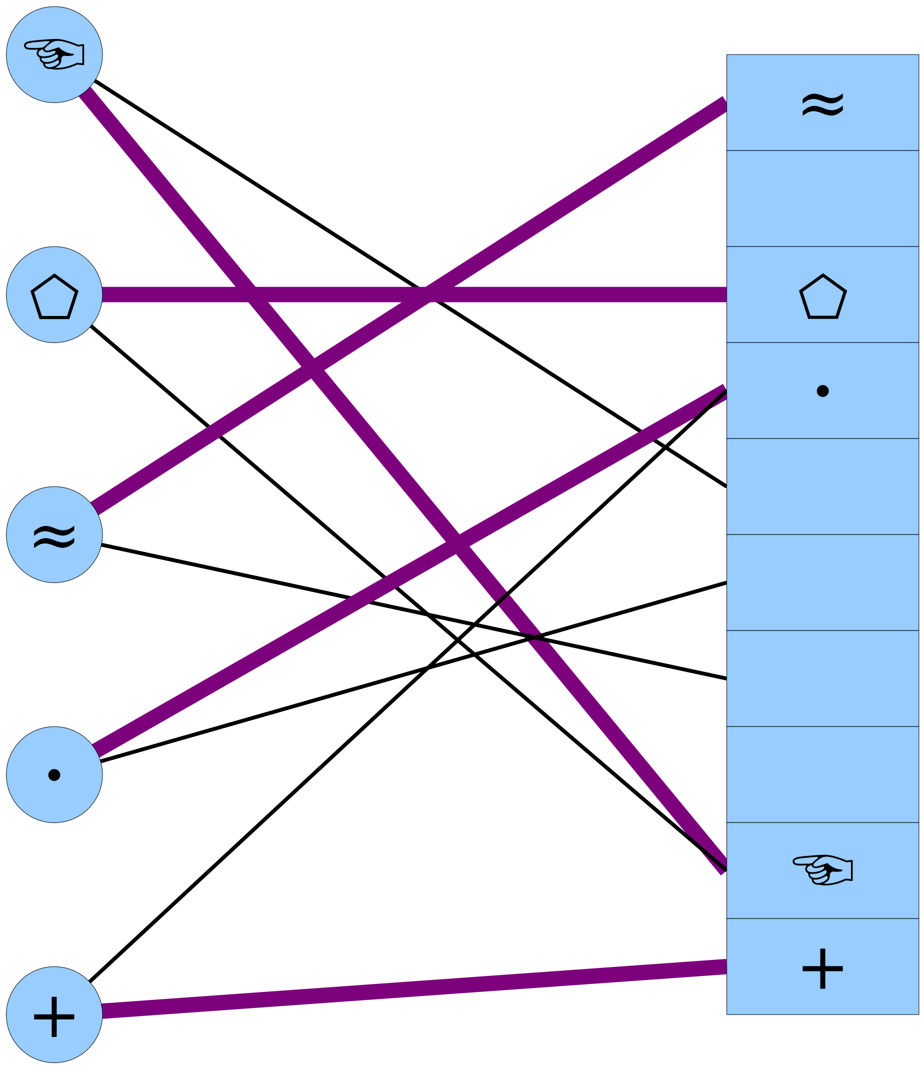

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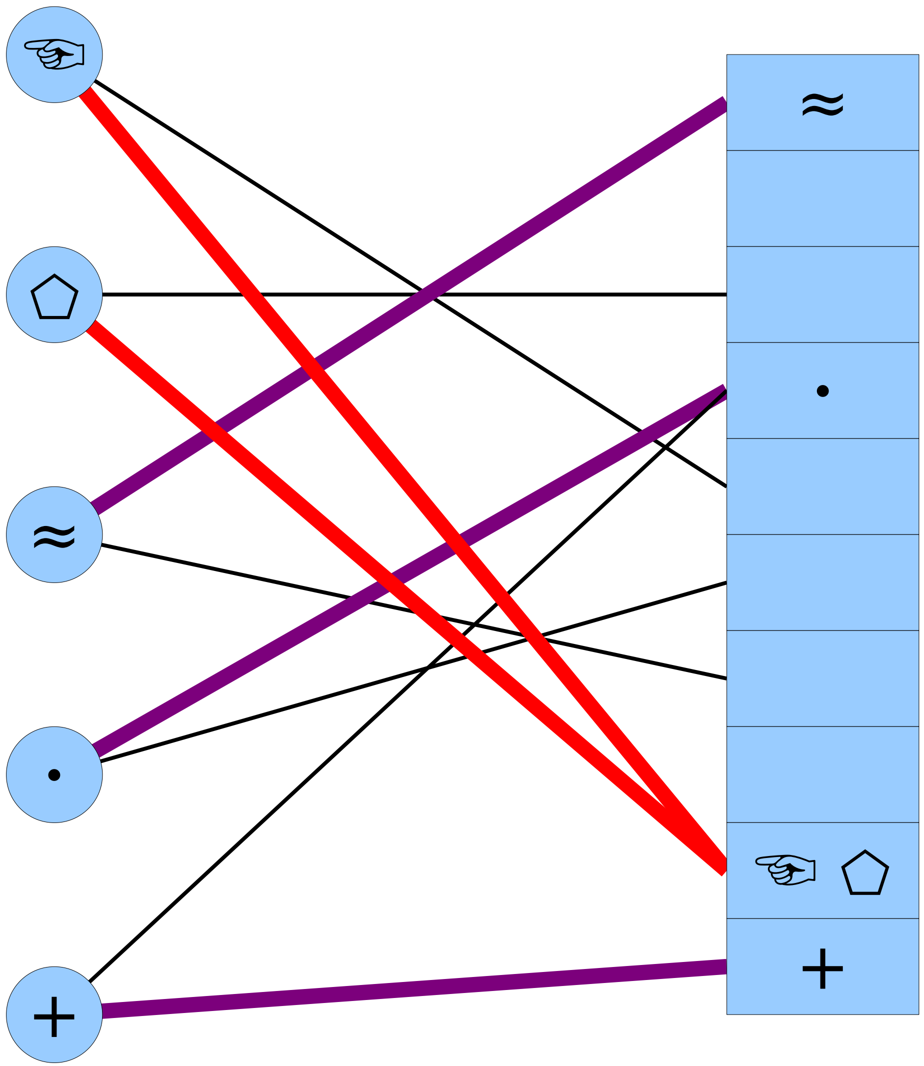

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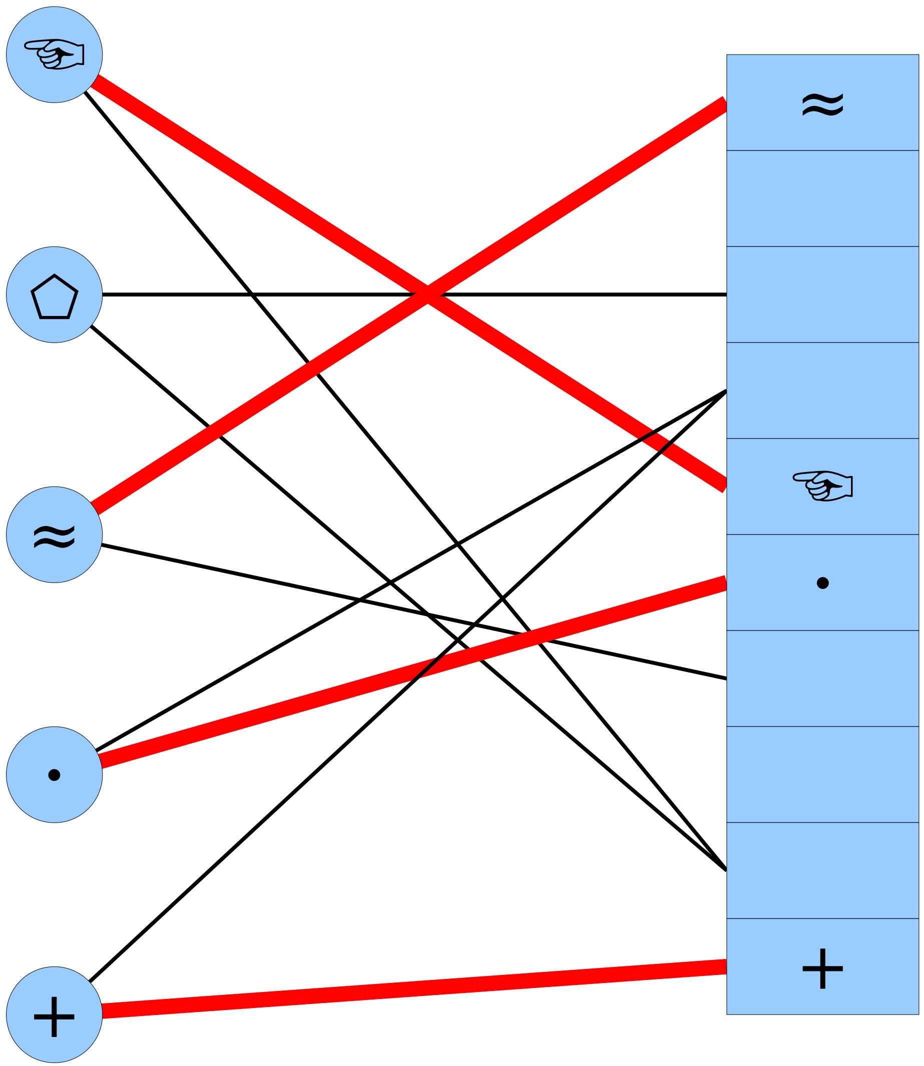
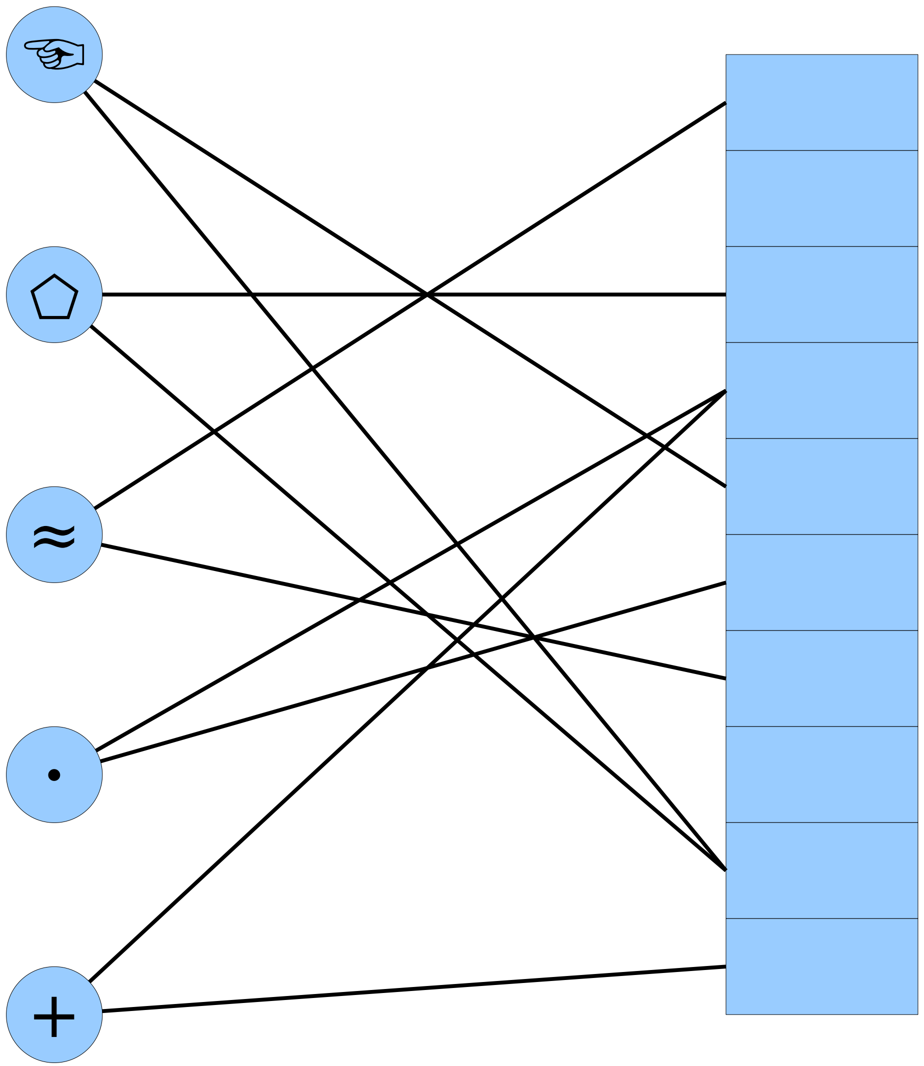


Rethinking Cuckoo Hashing

- We're now left with these constraints:
  - Each item has two outgoing edges.
  - Each item must have one of its outgoing edges selected.
  - Each slot must have at most one of its incoming edges selected.
- We have to relax at least one of these constraints if we want to push past the 50% space utilization limit.



Which of these constraints could we relax, and what might it look like if we did?

Formulate a hypothesis, but ***don't post anything in chat just yet***.

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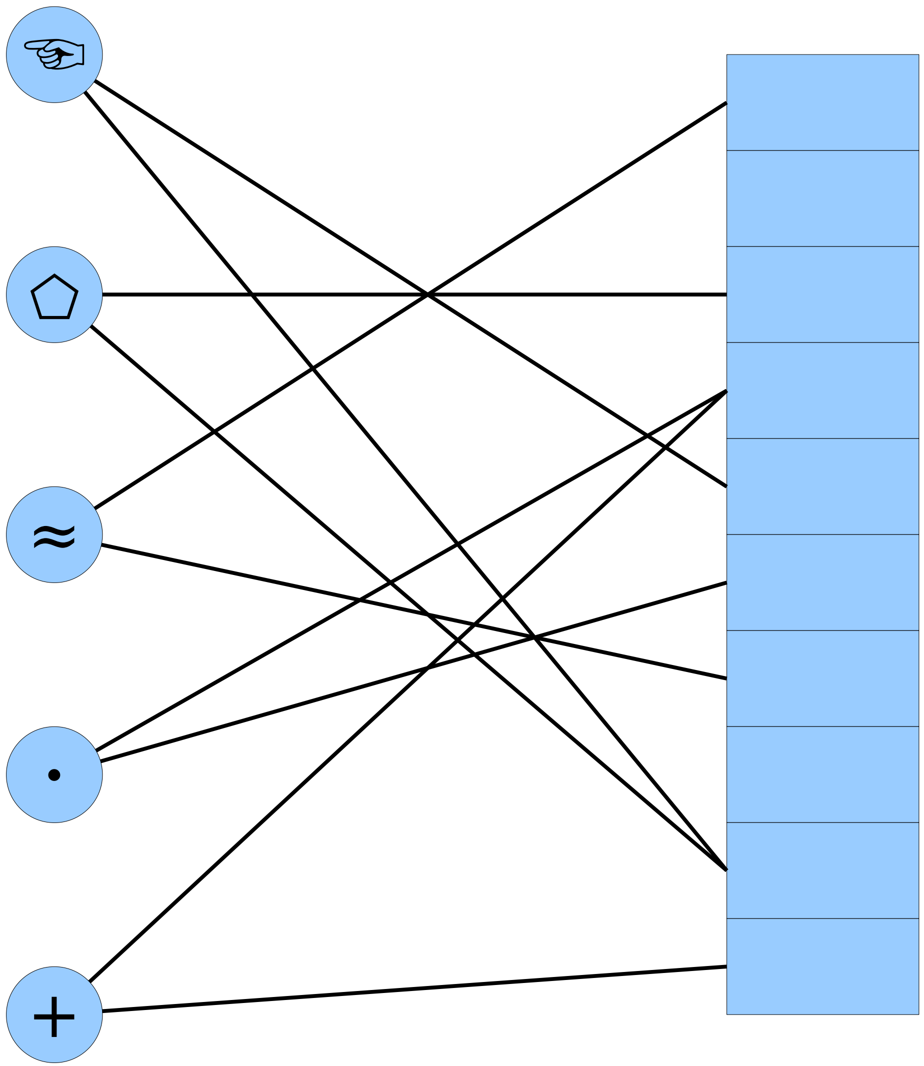

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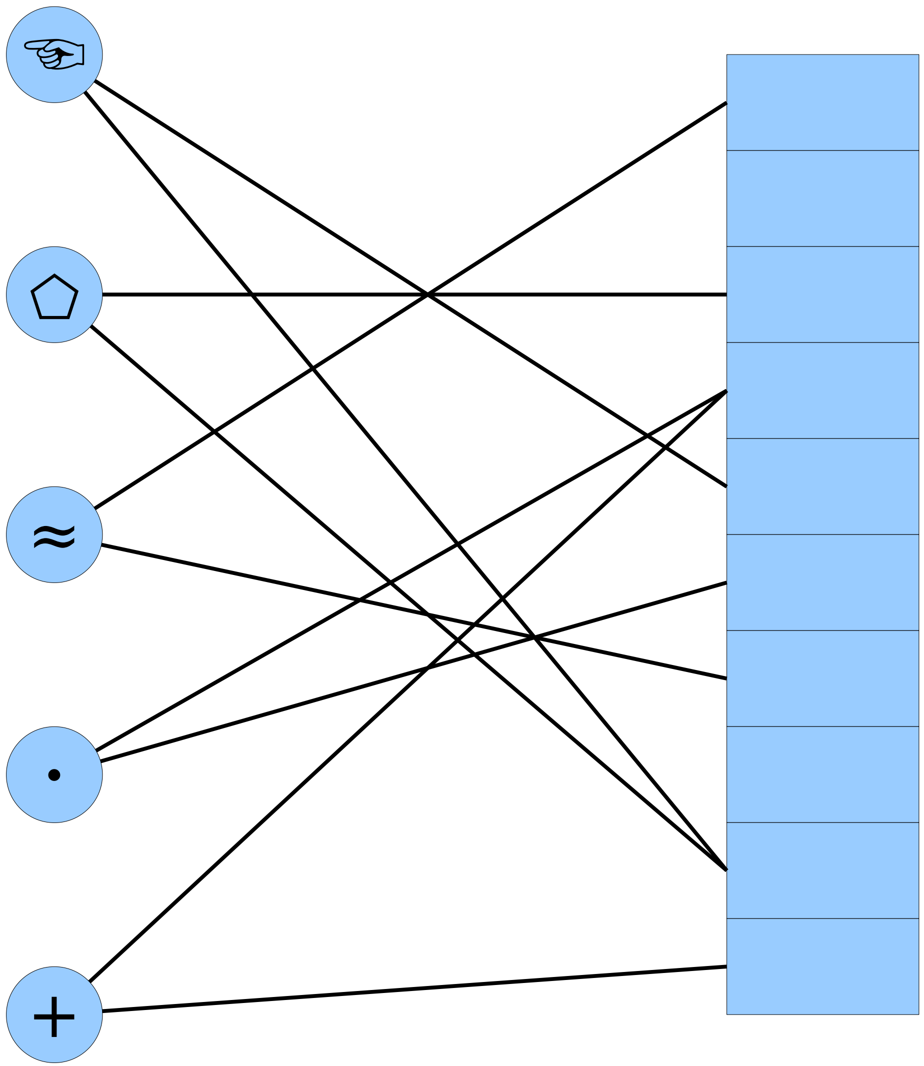

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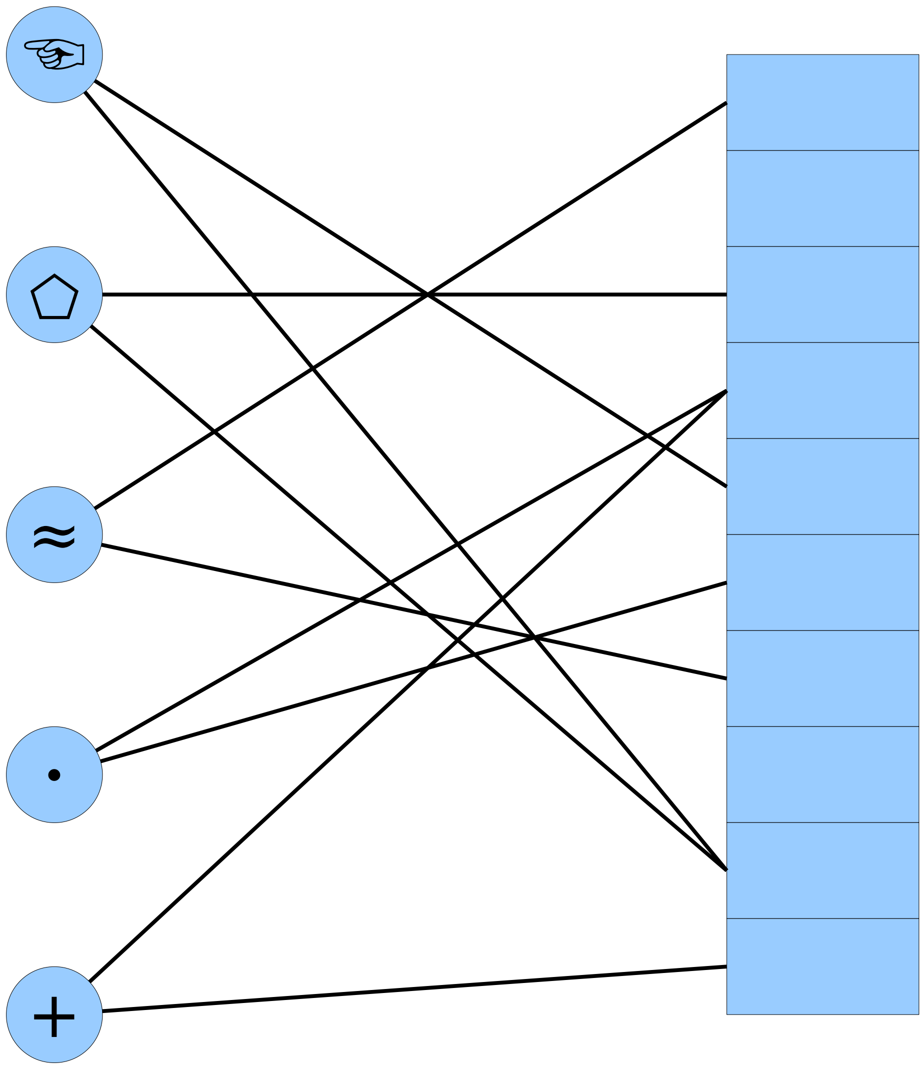

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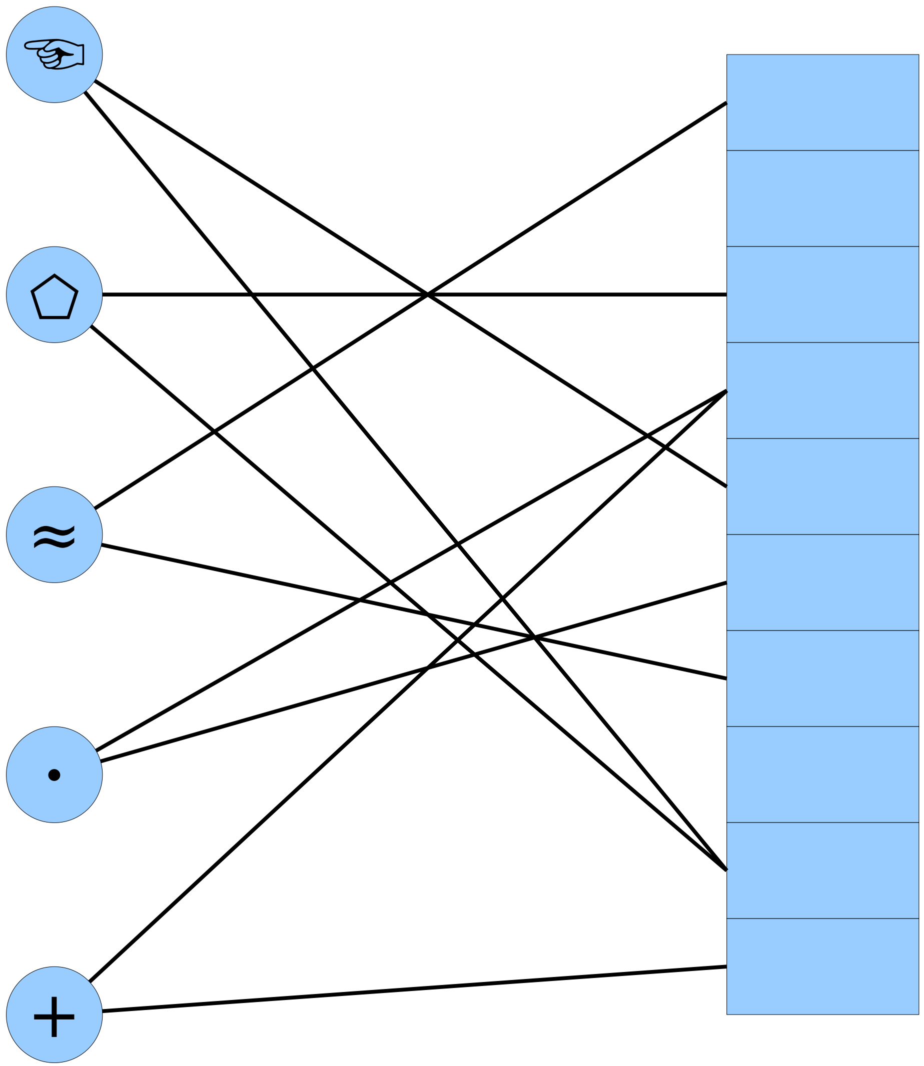

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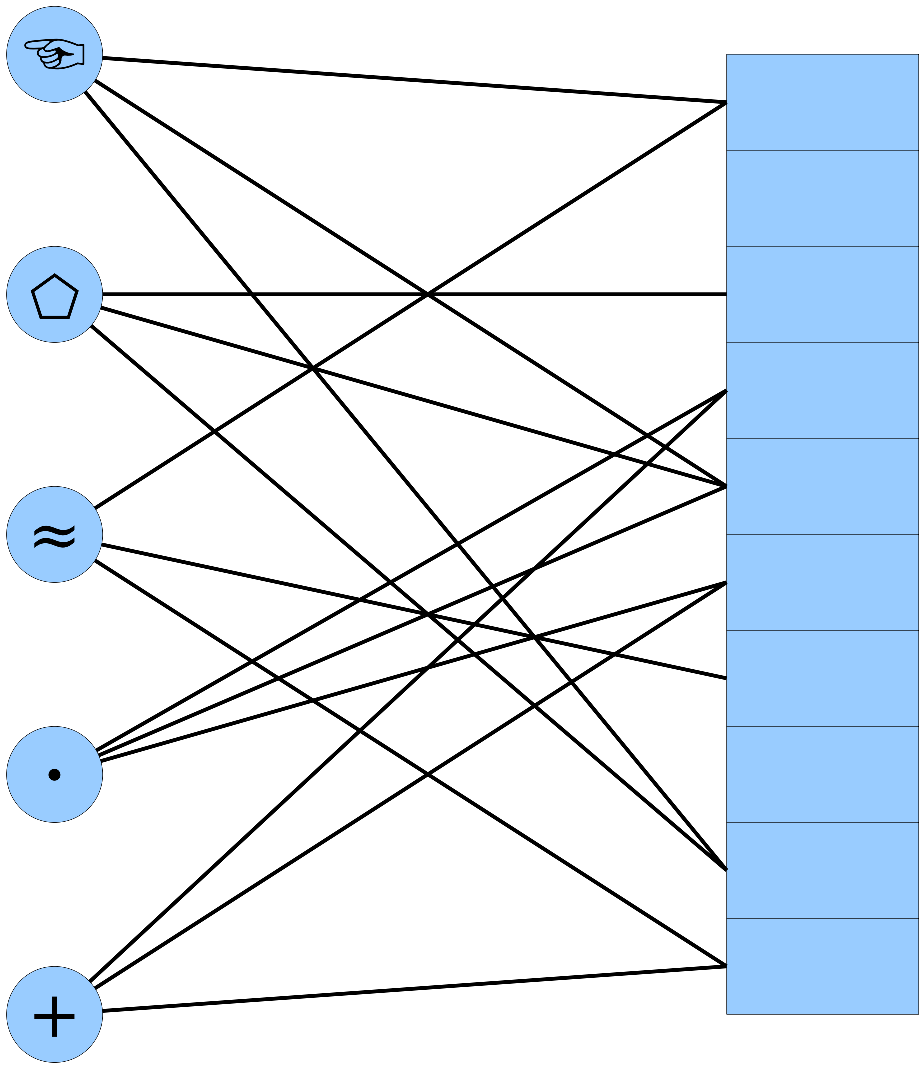

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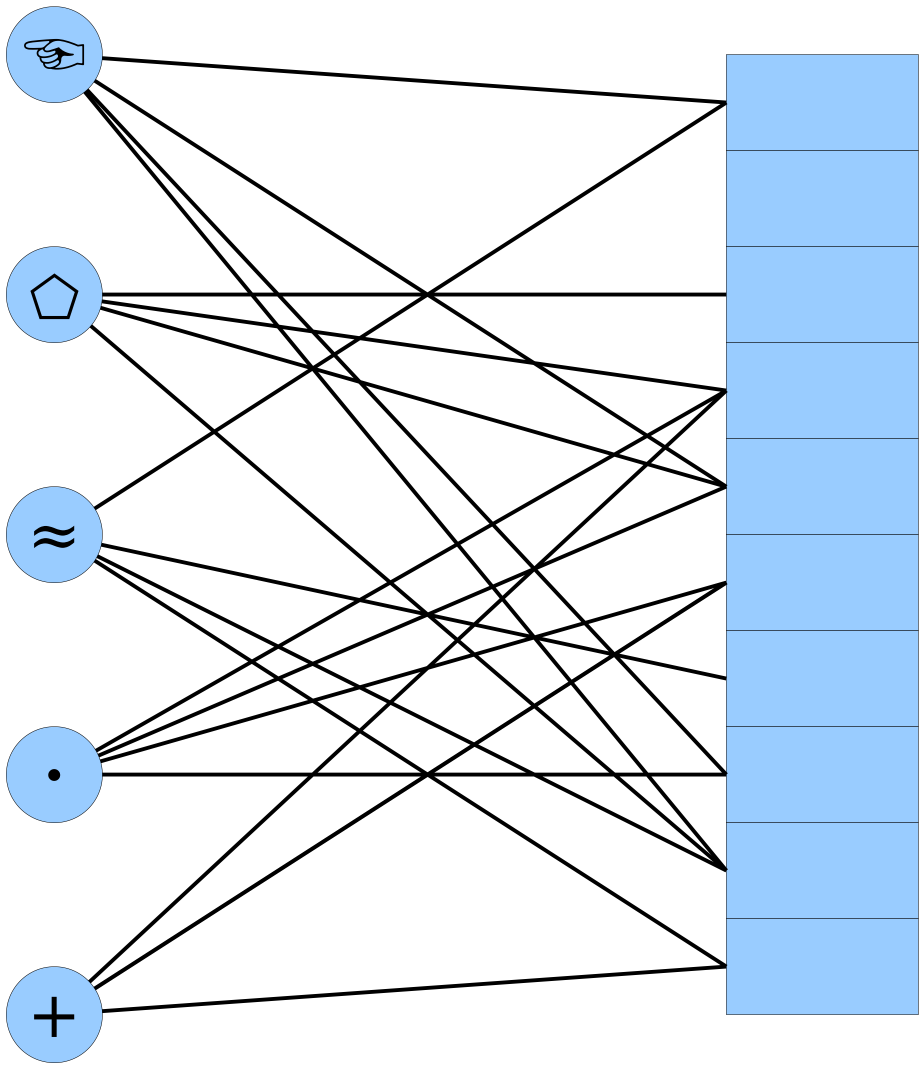

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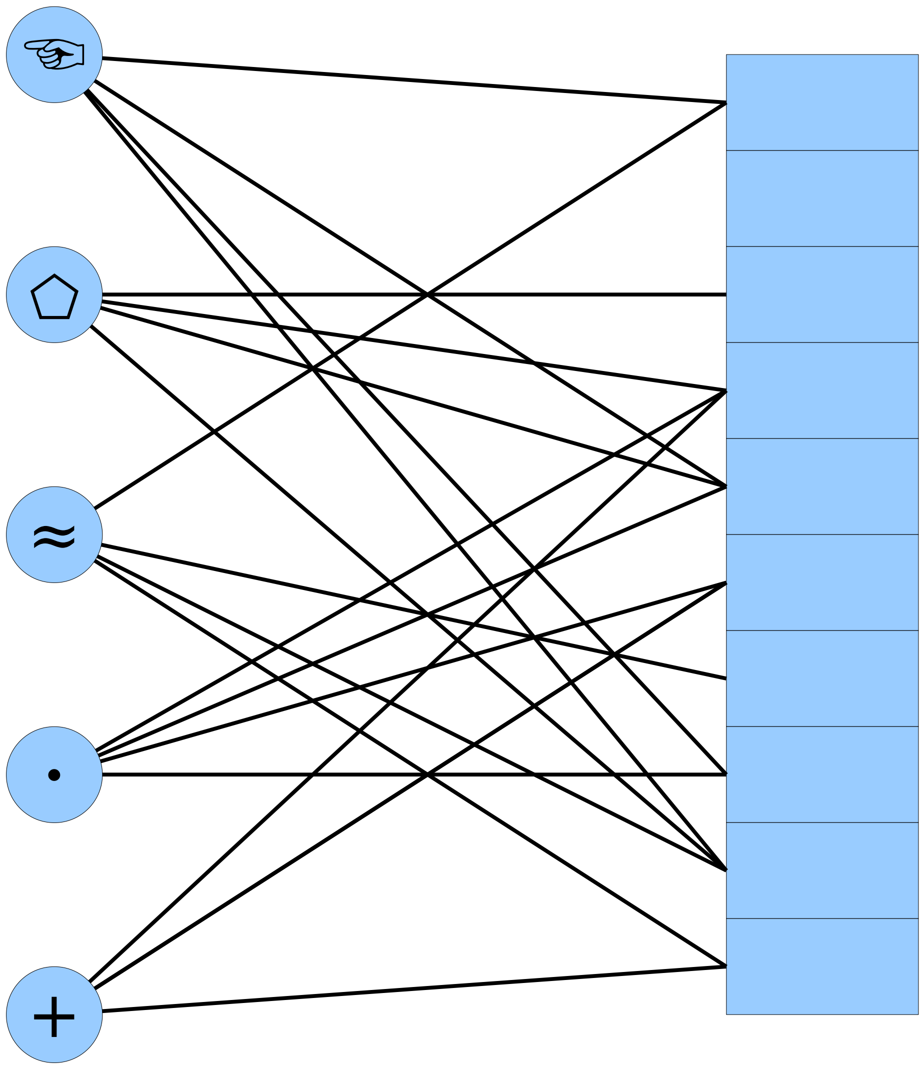
Idea 1: Use more hash functions to give each item more places to land.



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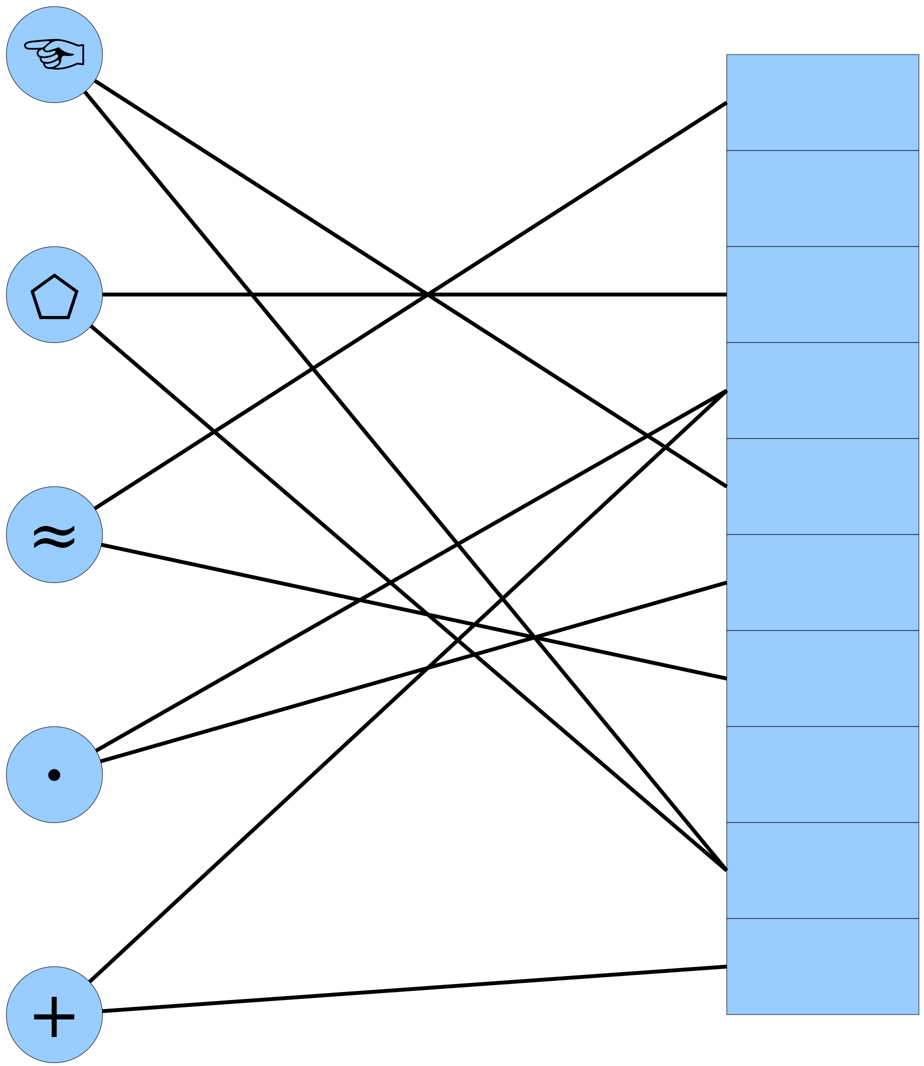

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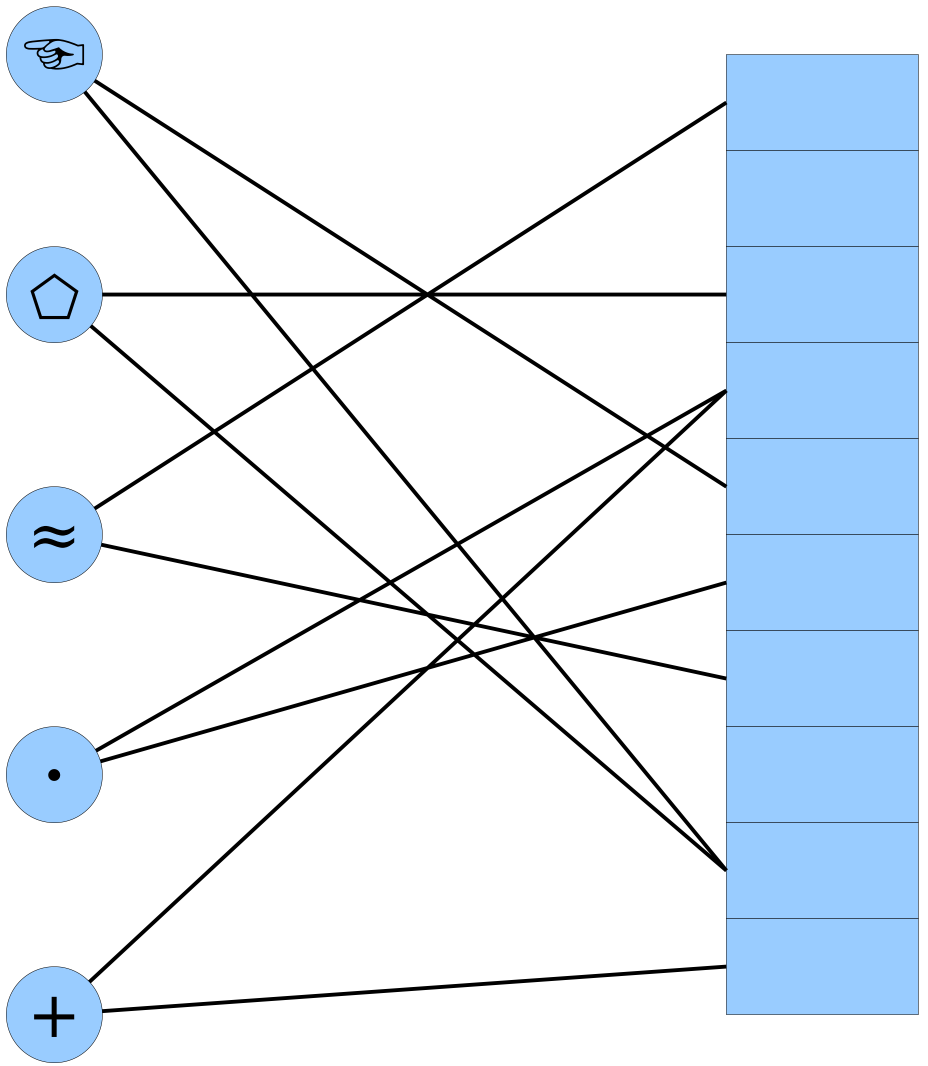

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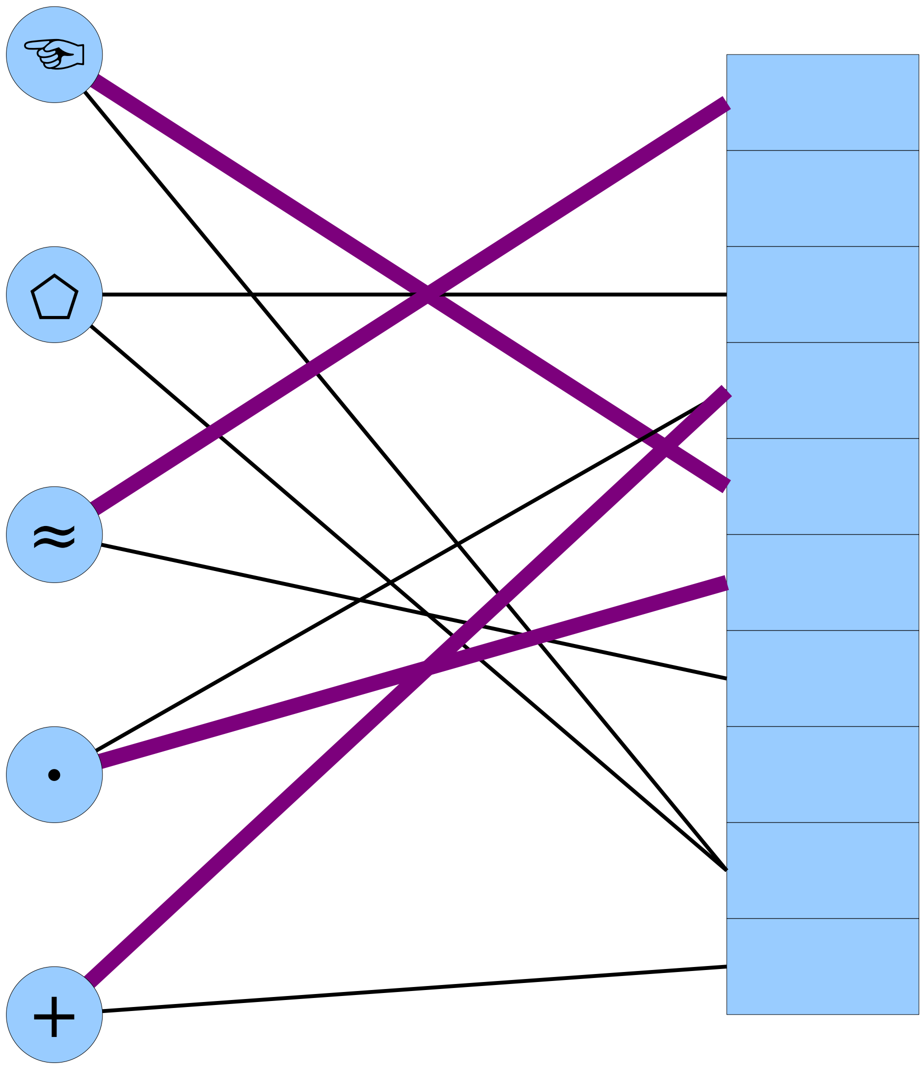

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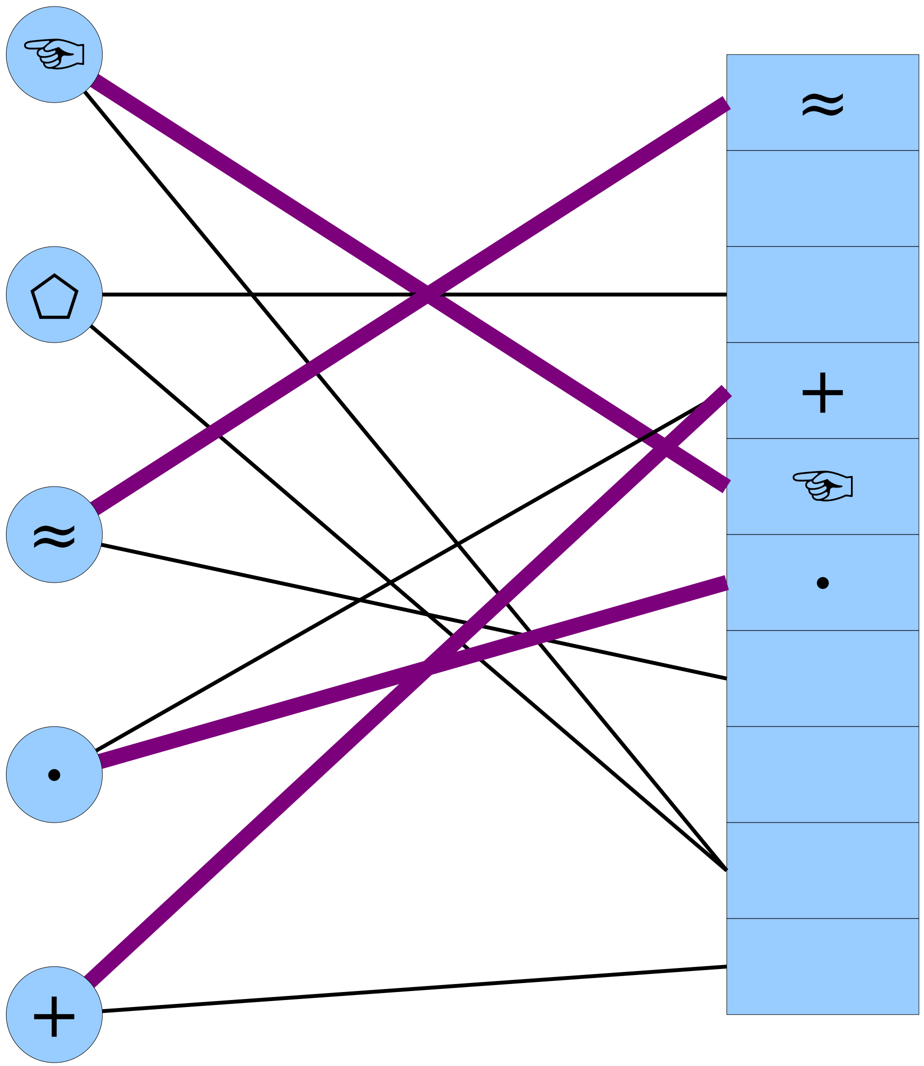

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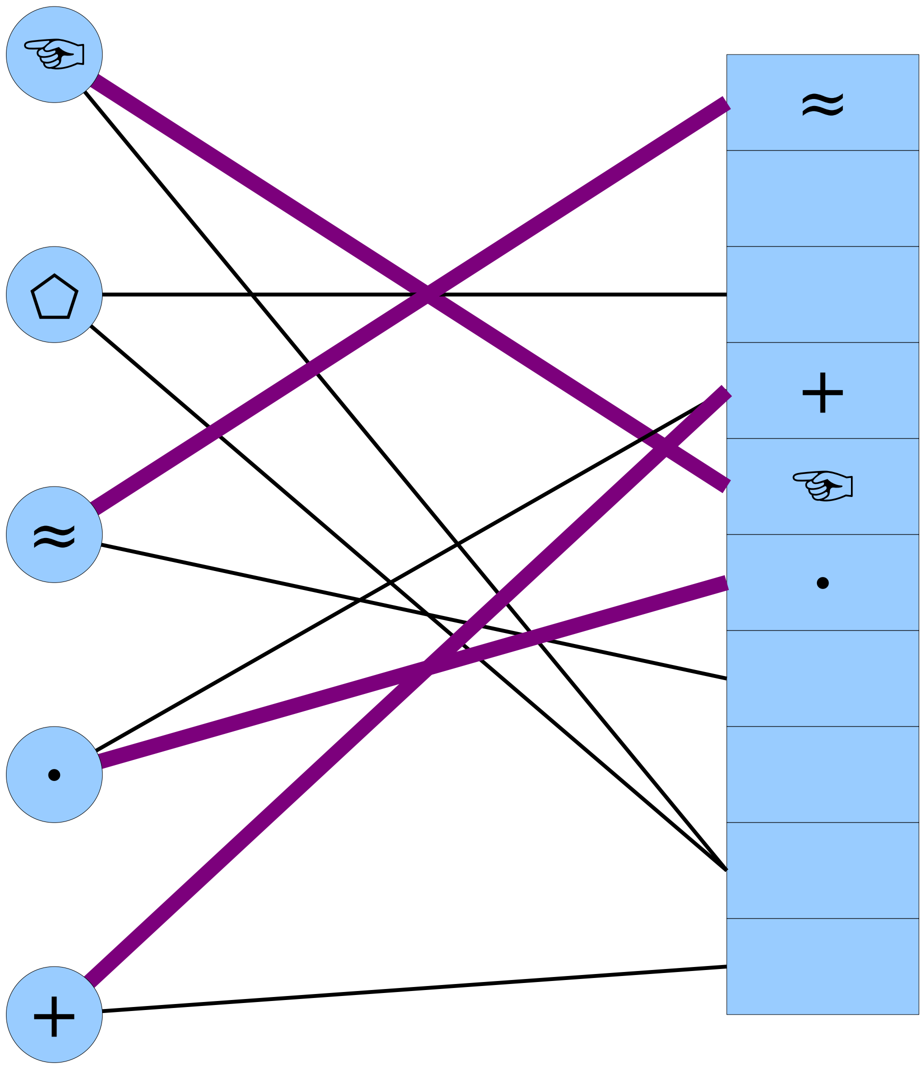

Idea 2: Don't put all items in the table. Let some elements live in another data structure.

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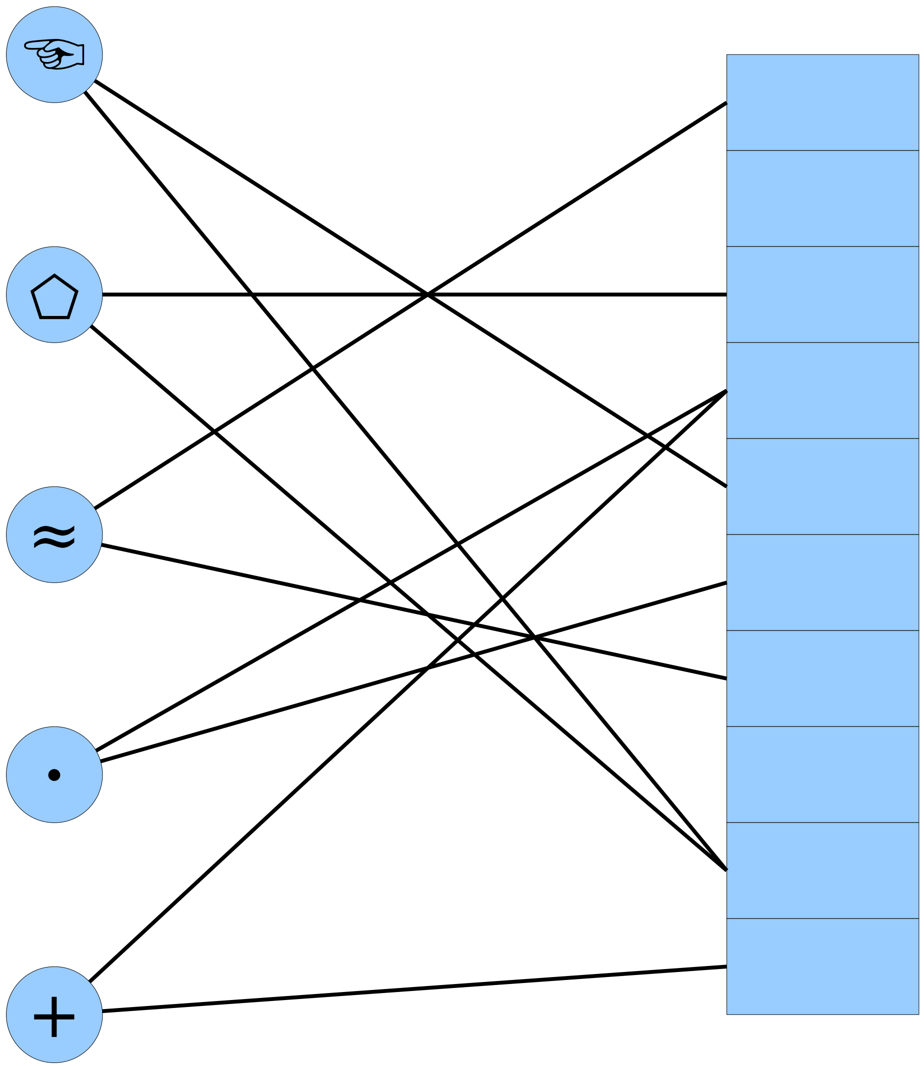

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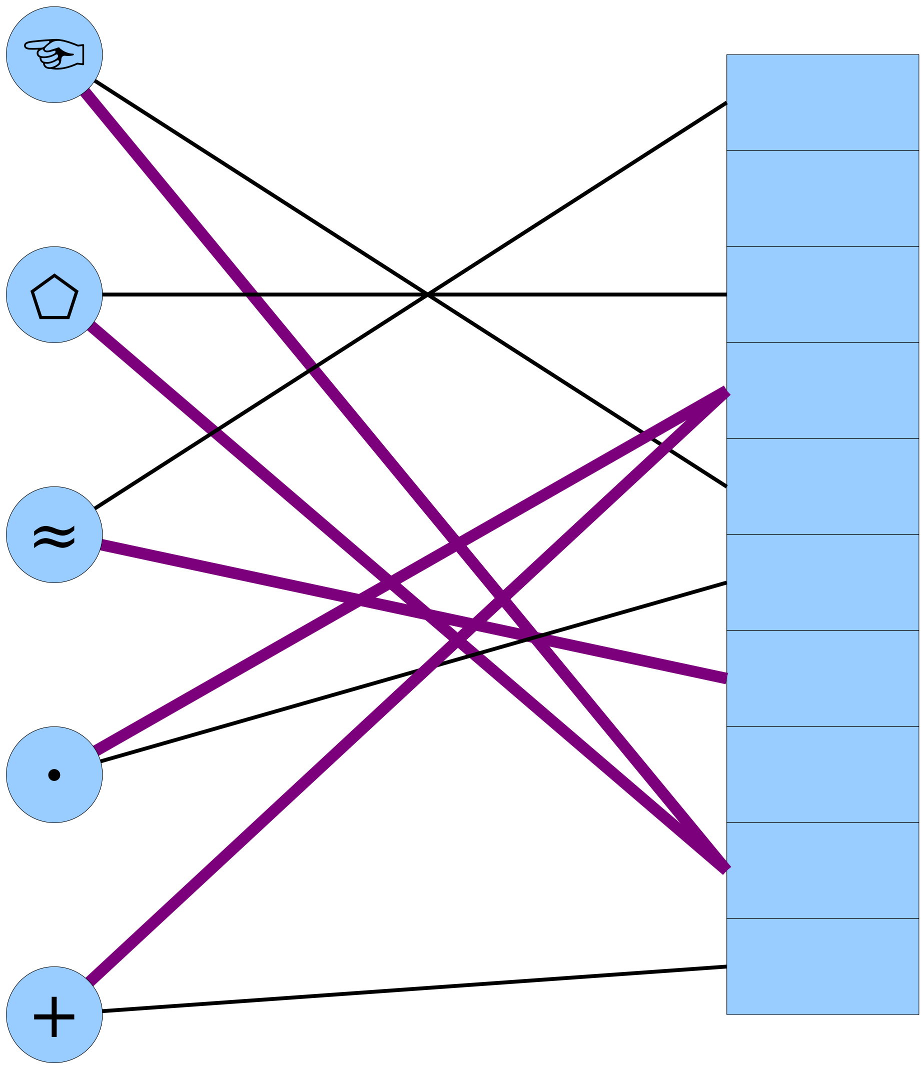

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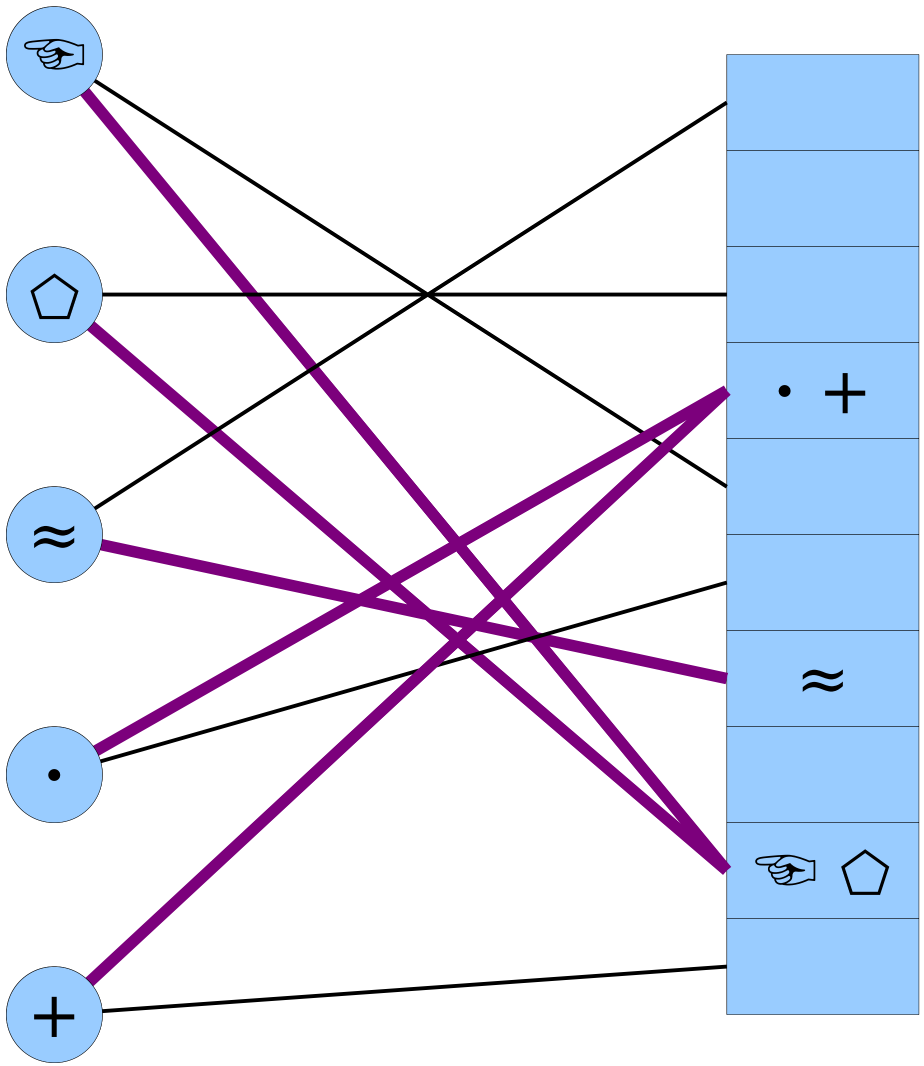

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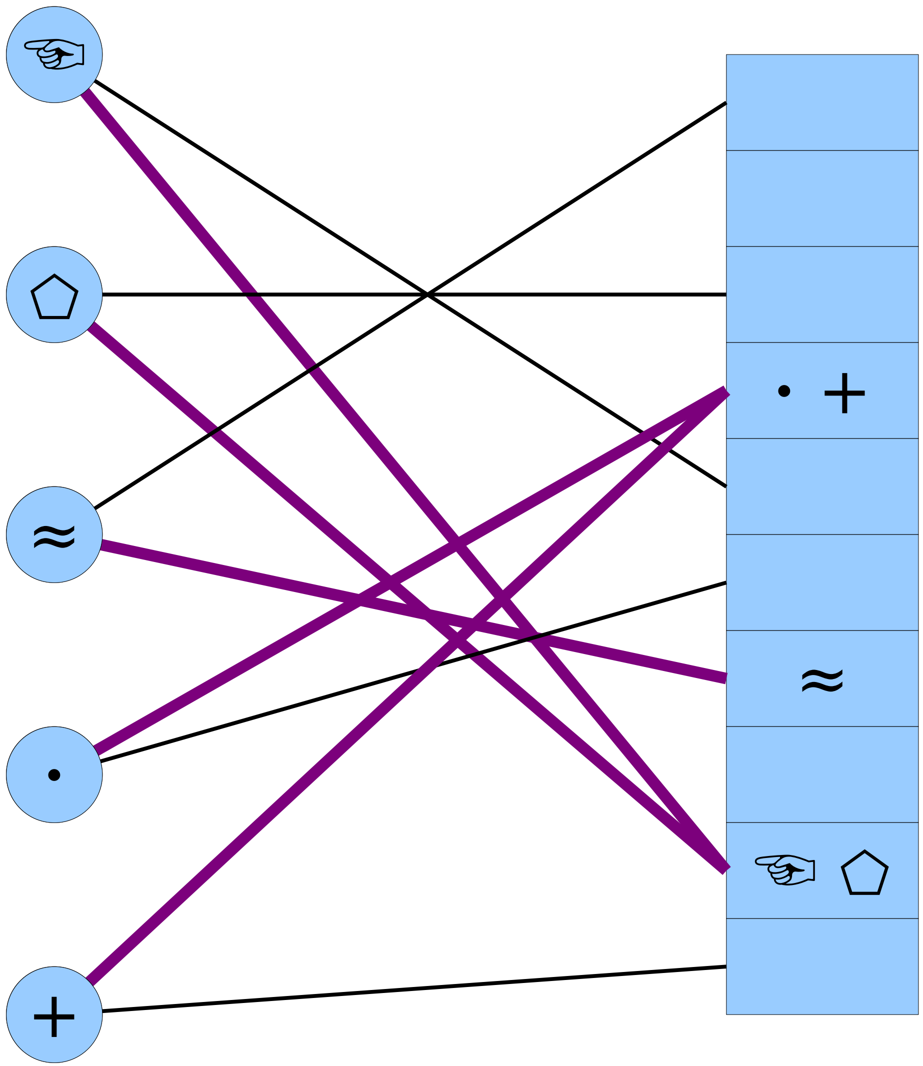
Idea 3: Allow multiple items to be stored in each slot.



Cuckoo Hashing Revisited

- We now have three ideas for improving our basic cuckoo hash table.
  - ***Idea 1:*** Use multiple hash functions to give items more wiggle room about where to go.
  - ***Idea 2:*** Don't place all items in the table; let some of them go somewhere else.
  - ***Idea 3:*** Allow for multiple items to be placed in each slot.
- Each of these ideas has been explored. We'll do a quick survey of what these options look like.

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d-ary Cuckoo Hashing

- In *d*-ary cuckoo hashing, we pick an integer $d \geq 2$ and choose *d* different hash functions.
- Each item can be stored in one up to *d* slots, with choices given by the hash functions.
  - You could do extra work to ensure there are *d* separate locations, or be okay with duplicates if the hashes collide.
- To check if an item is in the table, hash it *d* times and see if it's in any of those slots.

30		51	62		87	73	15		11

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- To check if an item is in the table, hash it d times and see if it's in any of those slots.

46

30		51	62		87	73	15		11

d-ary Cuckoo Hashing

- In d-ary cuckoo hashing, we pick an integer $d \geq 2$ and choose d different hash functions.
- Each item can be stored in one up to d slots, with choices given by the hash functions.
  - You could do extra work to ensure there are d separate locations, or be okay with duplicates if the hashes collide.
- To check if an item is in the table, hash it d times and see if it's in any of those slots.

46

30		51	62		87	73	15		11

d-ary Cuckoo Hashing

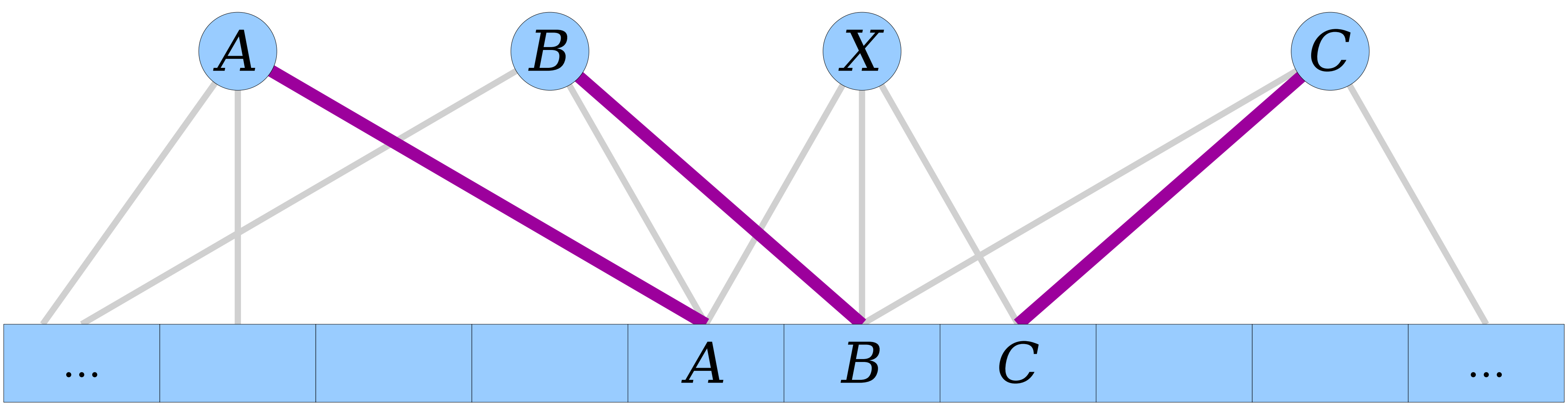
- As with regular cuckoo hashing, when inserting an item, if any of its slots are free, just place the item there.
- If not, we have to displace an existing item.
- ***Question:*** How do you decide which item to pick?

46

30		51	62		87	73	15		11

d-ary Cuckoo Hashing

- Here's our bipartite graph view of our table. Edges in ***purple*** represent where items were placed. Edges in ***gray*** represent unused alternates.
- Let's insert X into this table.



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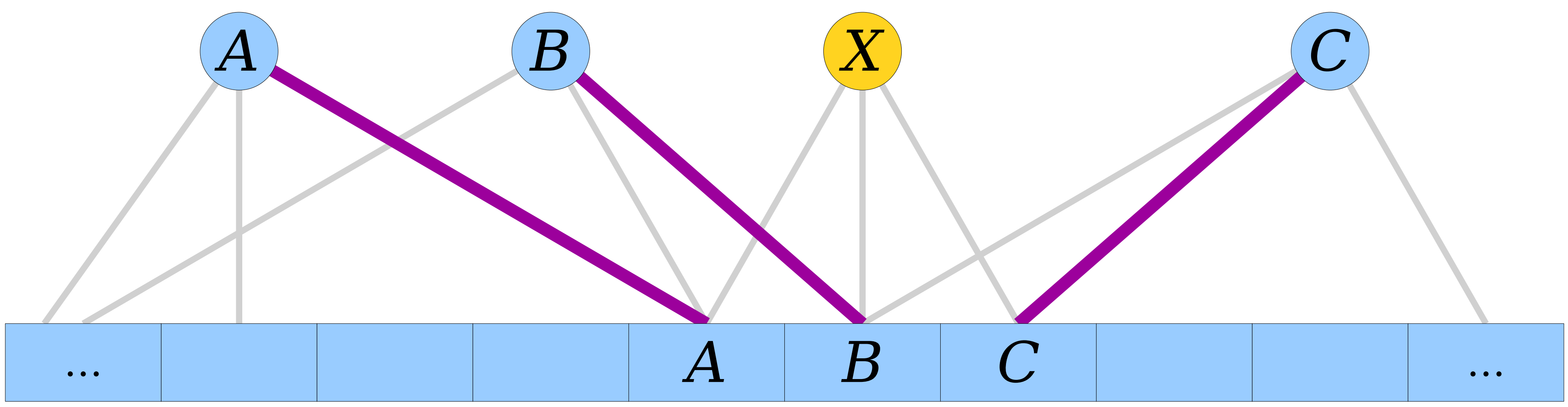

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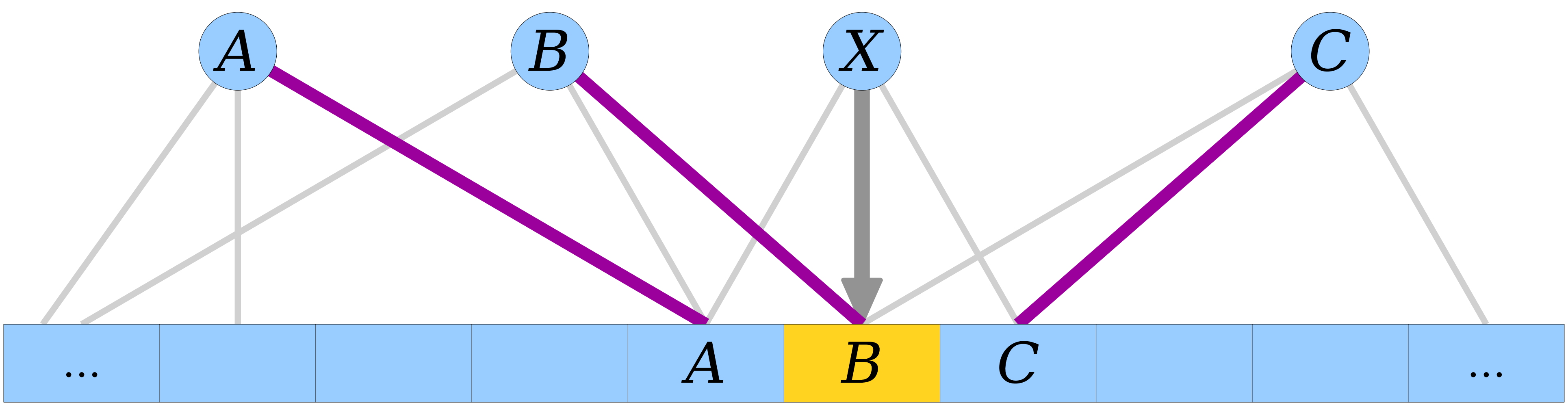

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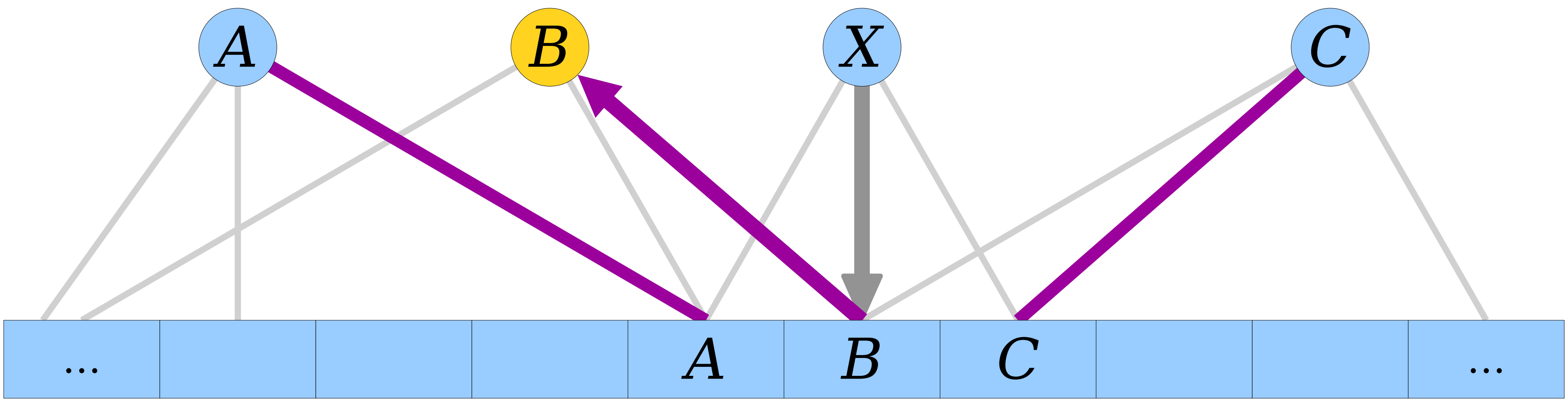

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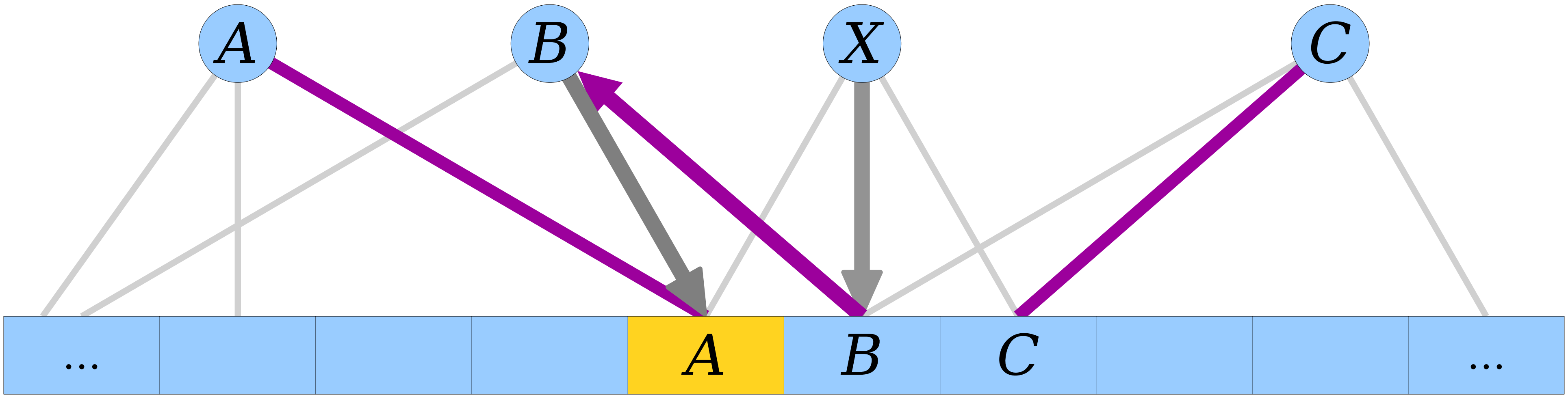

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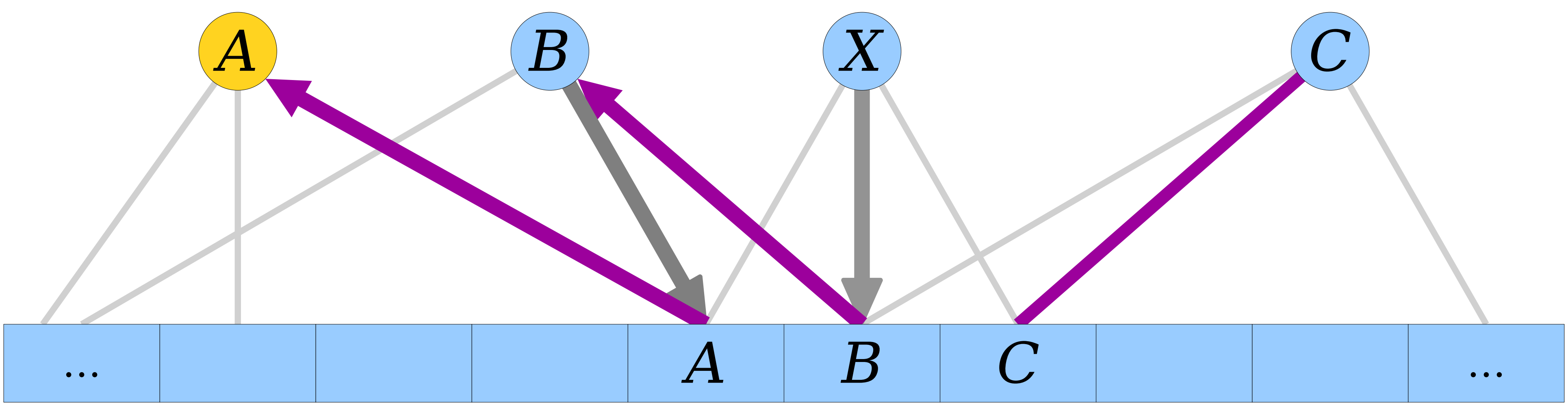

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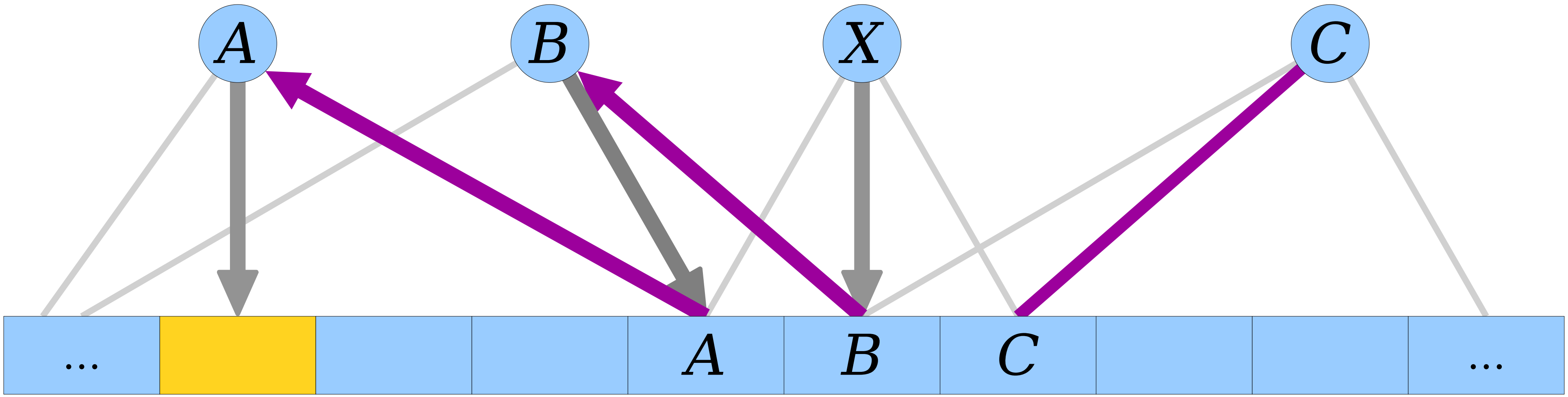

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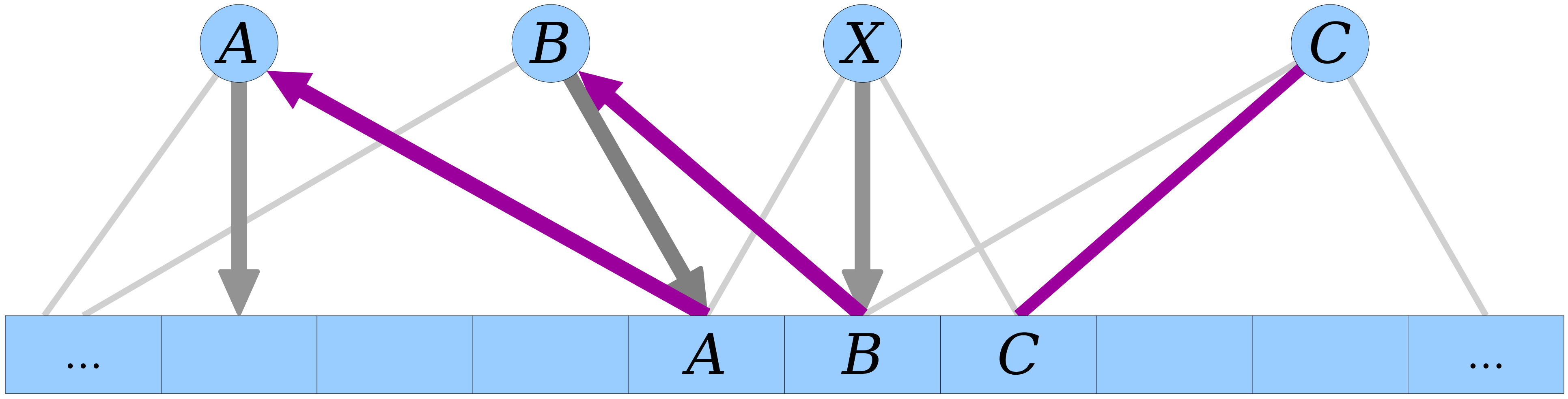

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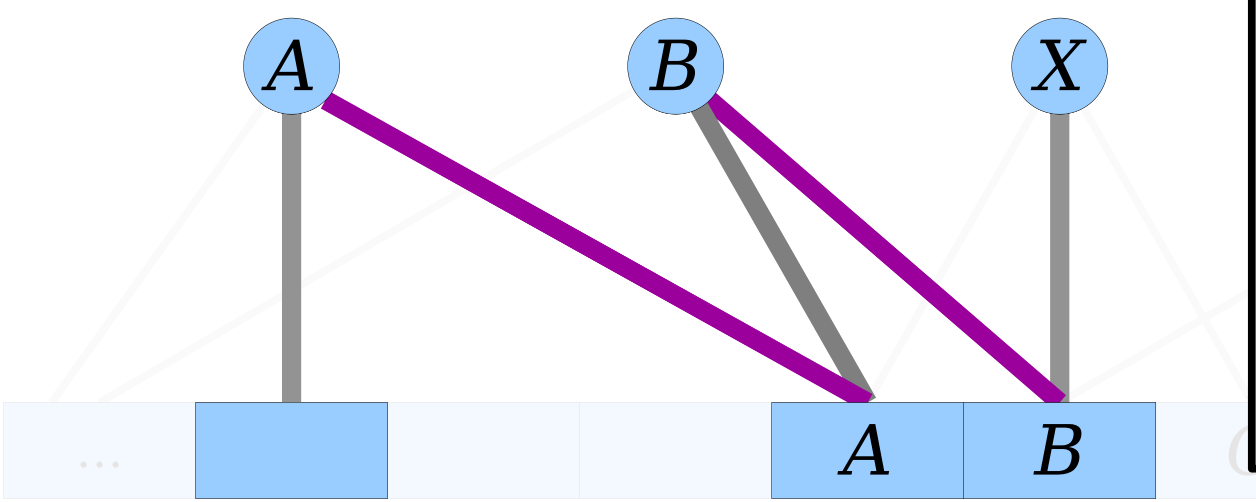

This path starts with an unmatched item, ends with an unmatched slot, and alternates between ***matched*** and ***unmatched*** edges. It's called an ***augmenting path***.

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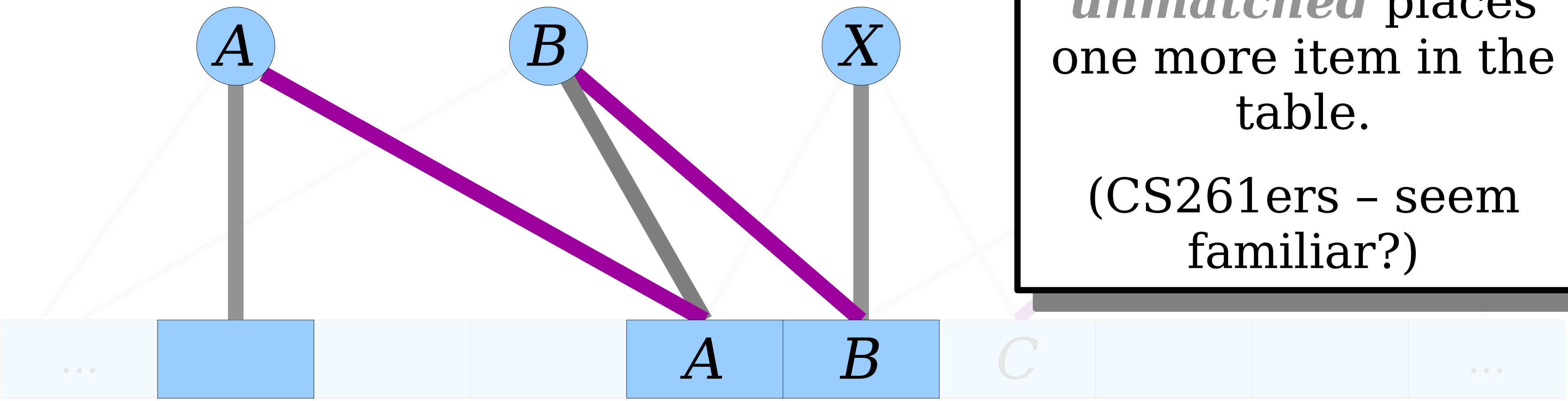

Toggling which edges are ***matched*** and ***unmatched*** places one more item in the table.

(CS261ers – seem familiar?)

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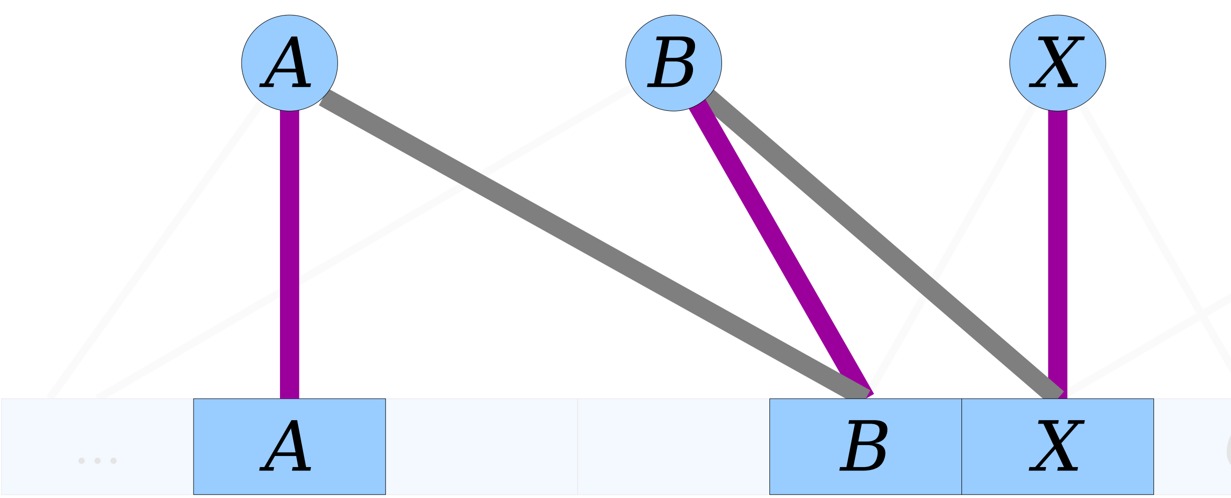

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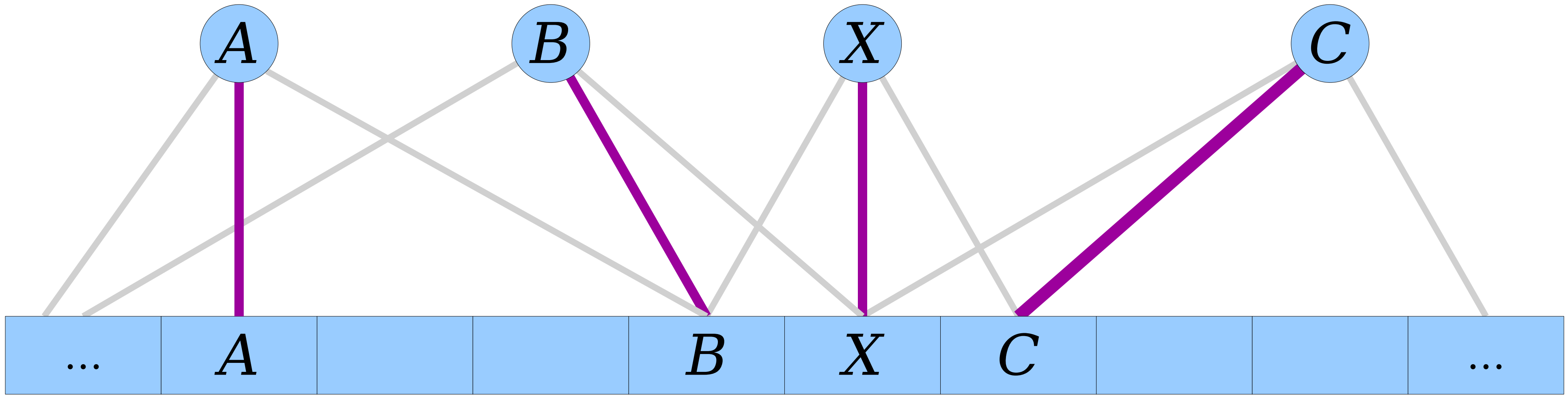
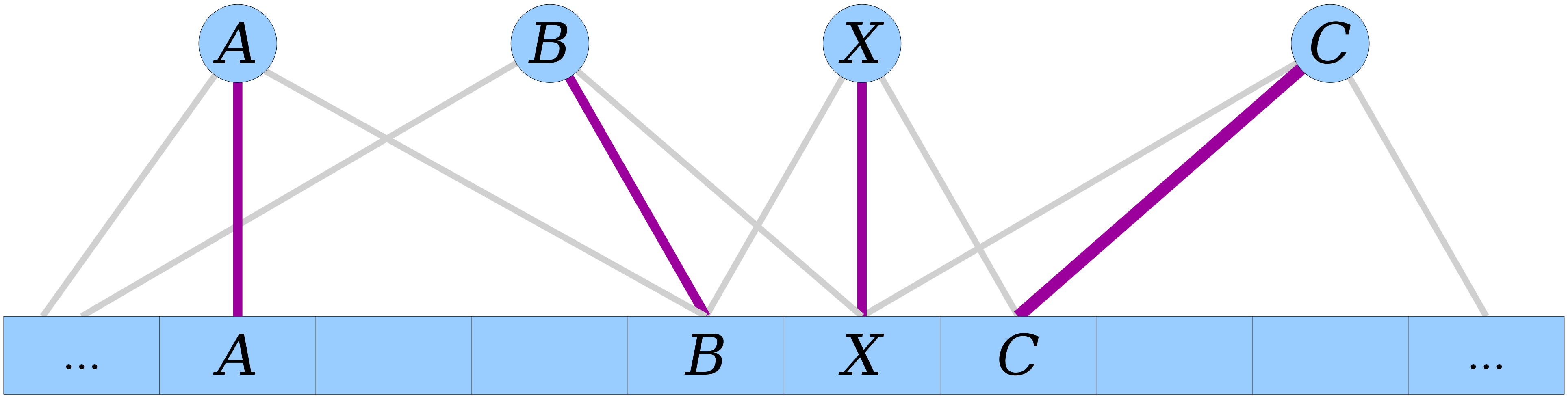


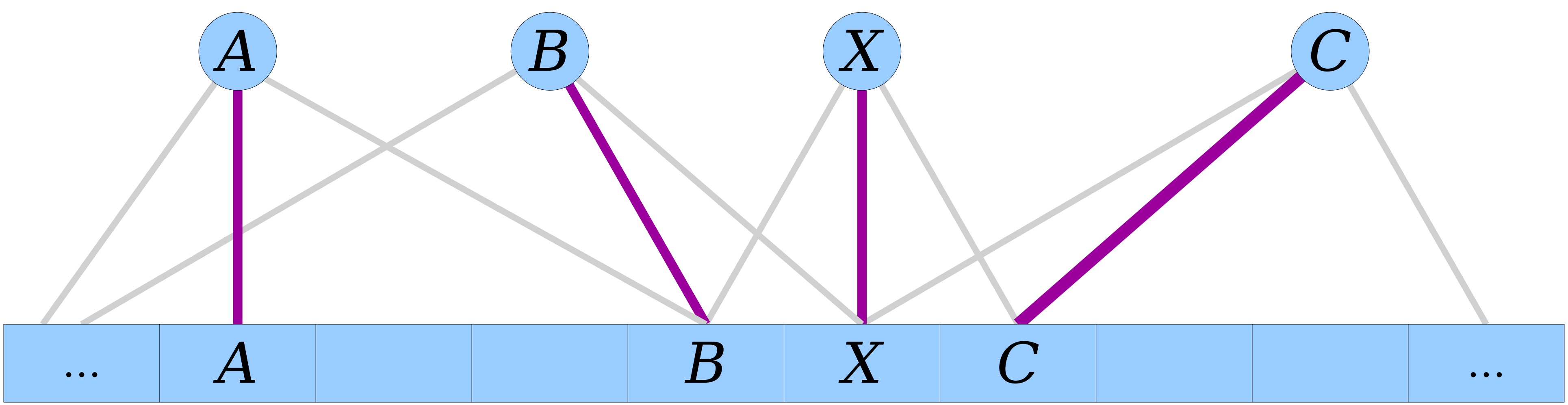
d-ary Cuckoo Hashing

- An ***augmenting path*** is a path starting at an unmatched item, ending at an unmatched slot, alternating between ***matched*** and ***unmatched*** edges.
- Flipping the edges in an alternating path produces a new matching with one more matched node.
- ***Claim:*** A chain of displacements is successful if and only if it finds an augmenting path.

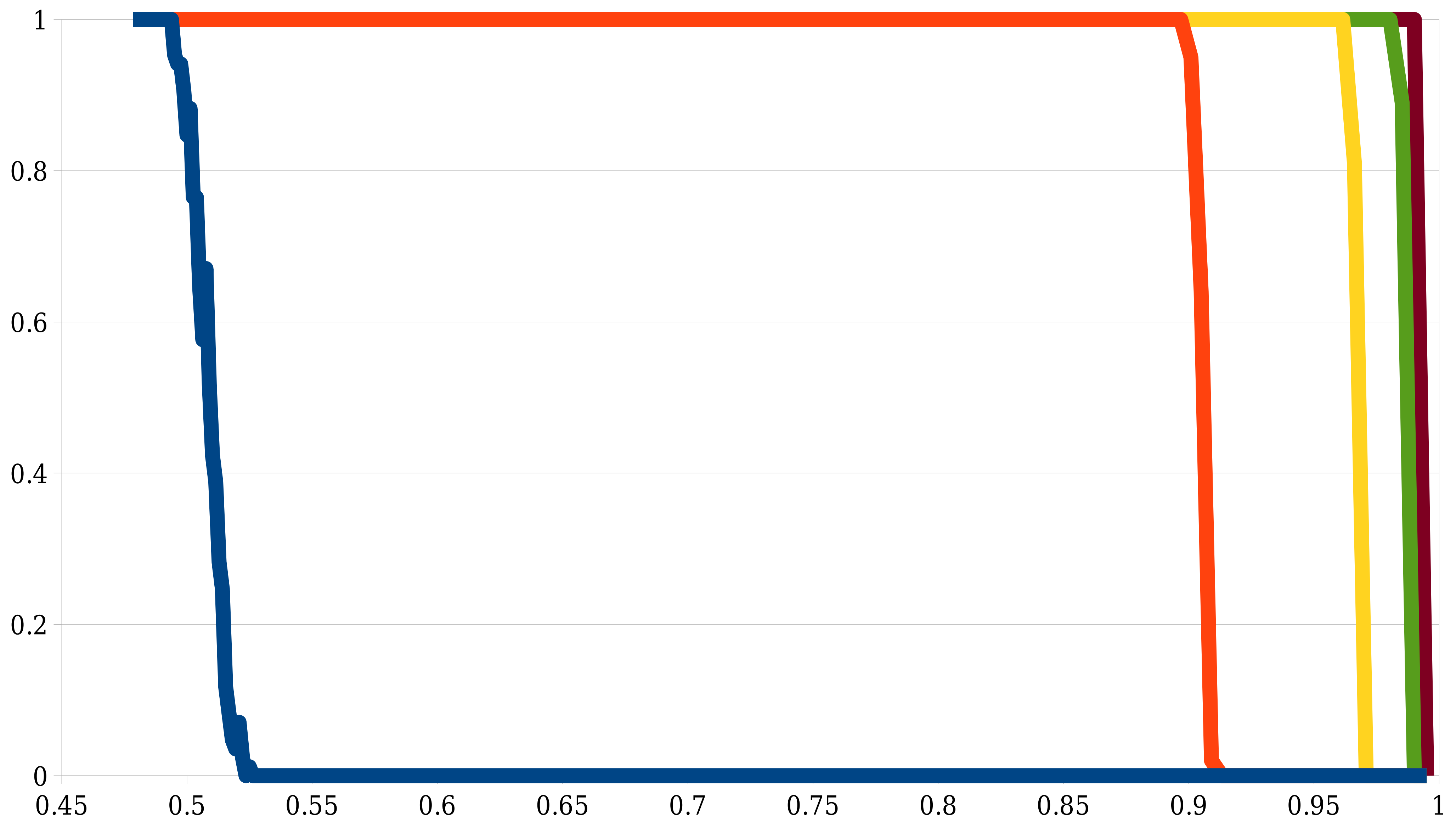


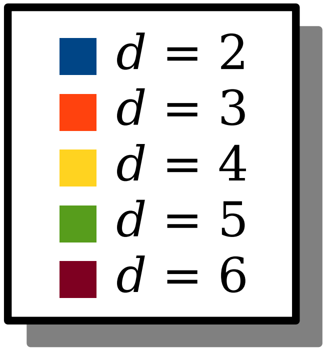
d-ary Cuckoo Hashing

- We can find an augmenting path using a ***random walk***:
  - Kick the first item out by picking any of the d options uniformly at random.
  - From that point forward, kick out a uniformly-random option of any of the d – 1 slots that wasn't the one you came from.
- Alternatively, we can use ***BFS***:
  - Start a BFS at the item to insert. Walk forwards using unmatched edges and backwards using matched edges. Stop when you find an augmenting path.
- Random walks are faster on average, but can be slower in the worst case and have trouble finding cycles. BFS is usually slower in practice but has better worst-case bounds and can find cycles.



How space-efficient is this approach?





Suppose we insert $n = \alpha m$ elements into a hash table with m slots. What is the probability that all insertions succeed?

d-ary Cuckoo Hashing

- ***Theorem:*** For each $d \geq 2$, there is a load factor α_d such that
  - for load factors $\alpha < \alpha_d$, there is a high chance that the table can be built, and
  - for load factors $\alpha > \alpha_d$, the odds that the table can be built are close to zero.
- The proof involves exploring the phase transitions in when matchings in randomly-chosen bipartite graphs start becoming increasingly rare.

d-ary Cuckoo Hashing

- The exact phase transitions for $d \geq 3$ was worked out in the late 2000s, and we have a solution courtesy of Fountoulakis and Panagiotou: the phase transition value α_d is

$$\alpha_d = \frac{x}{d\left(1 - e^{-x}\right)^{d-1}}, \text{ where } x \text{ solves } x = \frac{d\left(e^x - 1 - x\right)}{e^x - 1}$$

- These closely track the numbers I got from my simulations.

	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$
Theoretical α_d	**0.500**	**0.917**	**0.976**	**0.992**	**0.997**
Empirical α_d	**0.500**	**0.901**	**0.966**	**0.985**	**0.990**

Cuckoo Hashing Revisited

- We now have three ideas for improving our basic cuckoo hash table.
  - ***Idea 1:*** Use multiple hash functions to give items more wiggle room about where to go.
  - ***Idea 2:*** Don't place all items in the table; let some of them go somewhere else.
  - ***Idea 3:*** Allow for multiple items to be placed in each slot.
- Each of these ideas has been explored. We'll do a quick survey of what these options look like.

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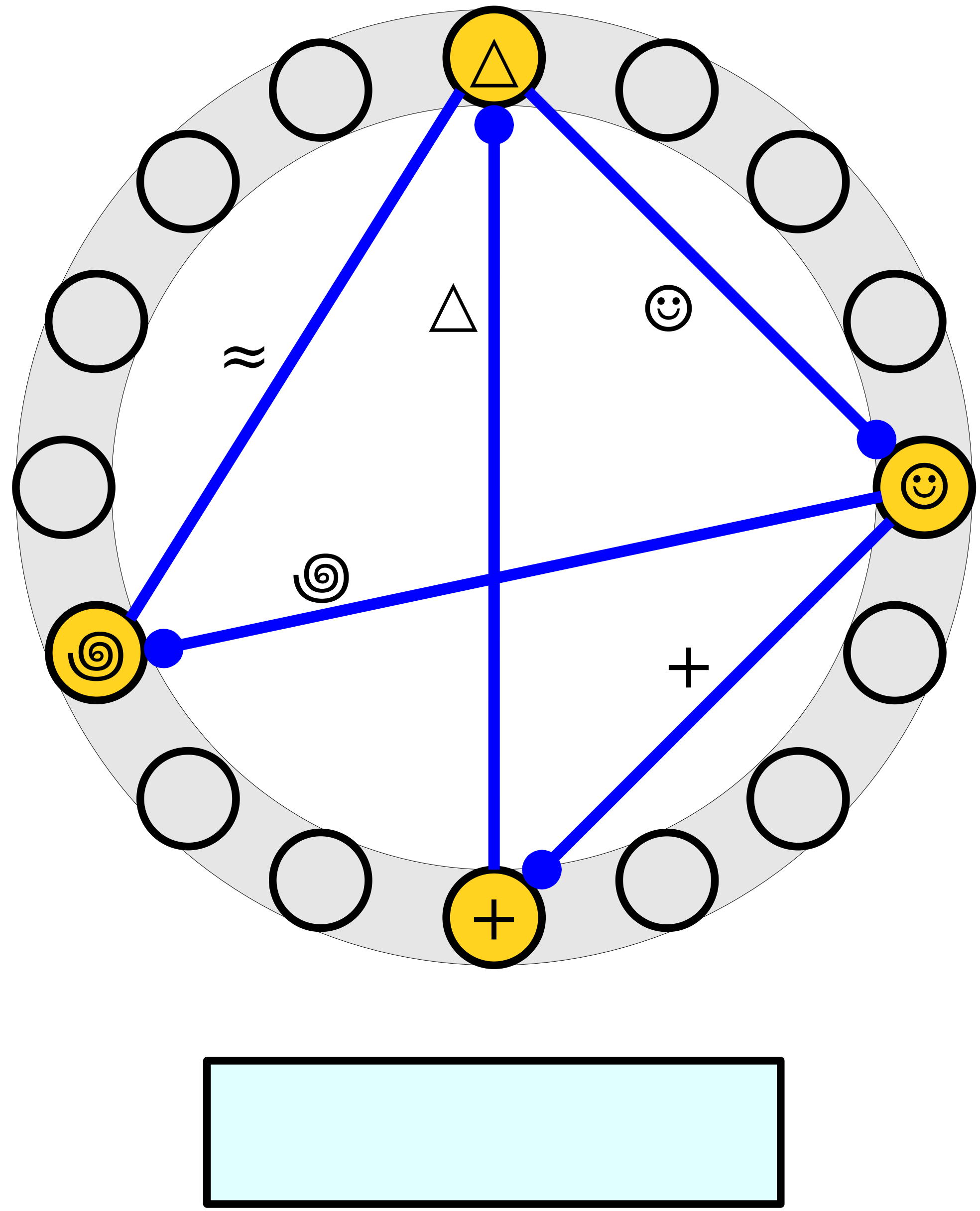
Hashing With Stashing

- Insertions fail when the newly-added node adds a second cycle into a connected component in the cuckoo graph.
- ***Idea:*** When this happens, pick an edge in that connected component that forms a cycle and store it in a secondary hash table called the ***stash***.



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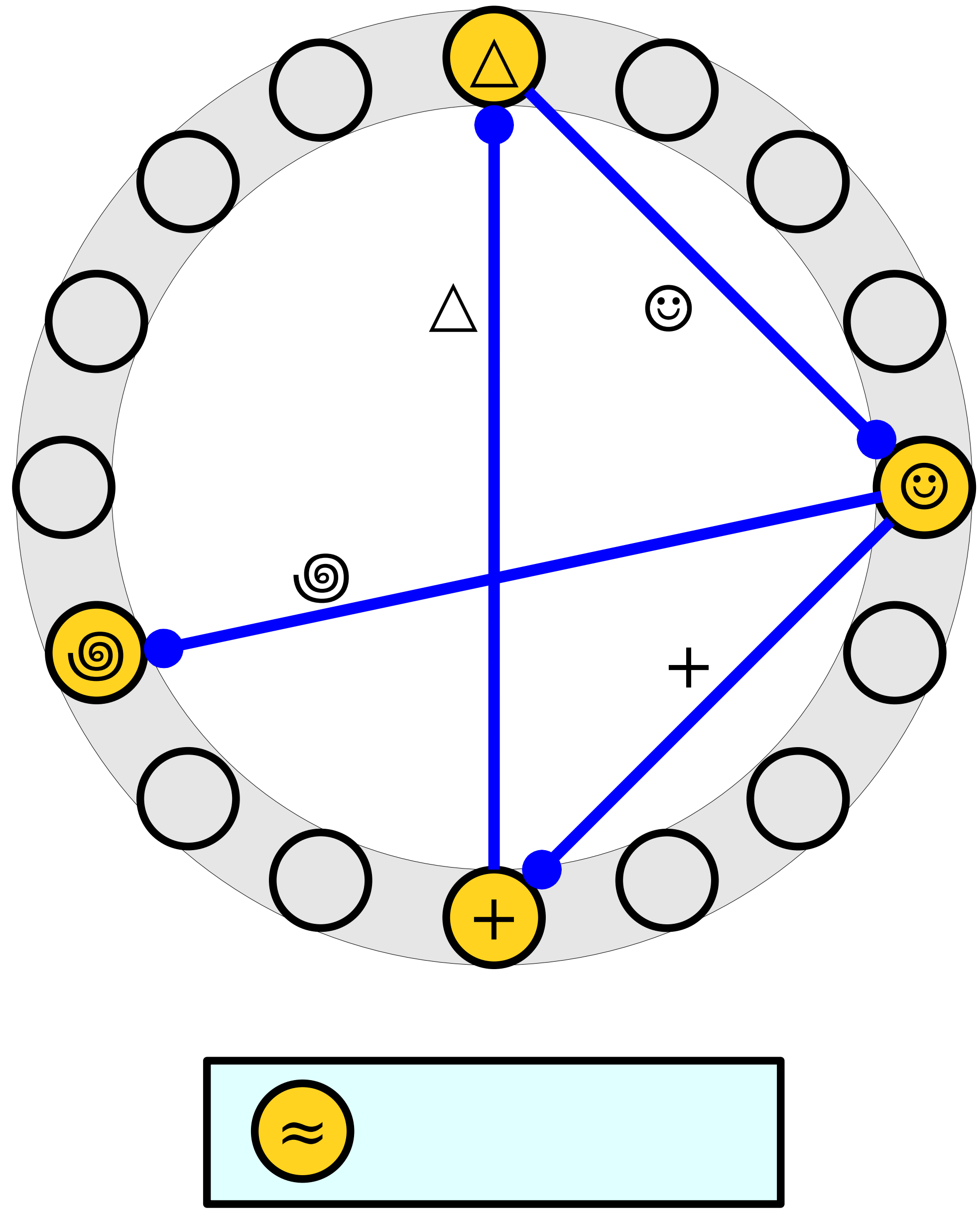
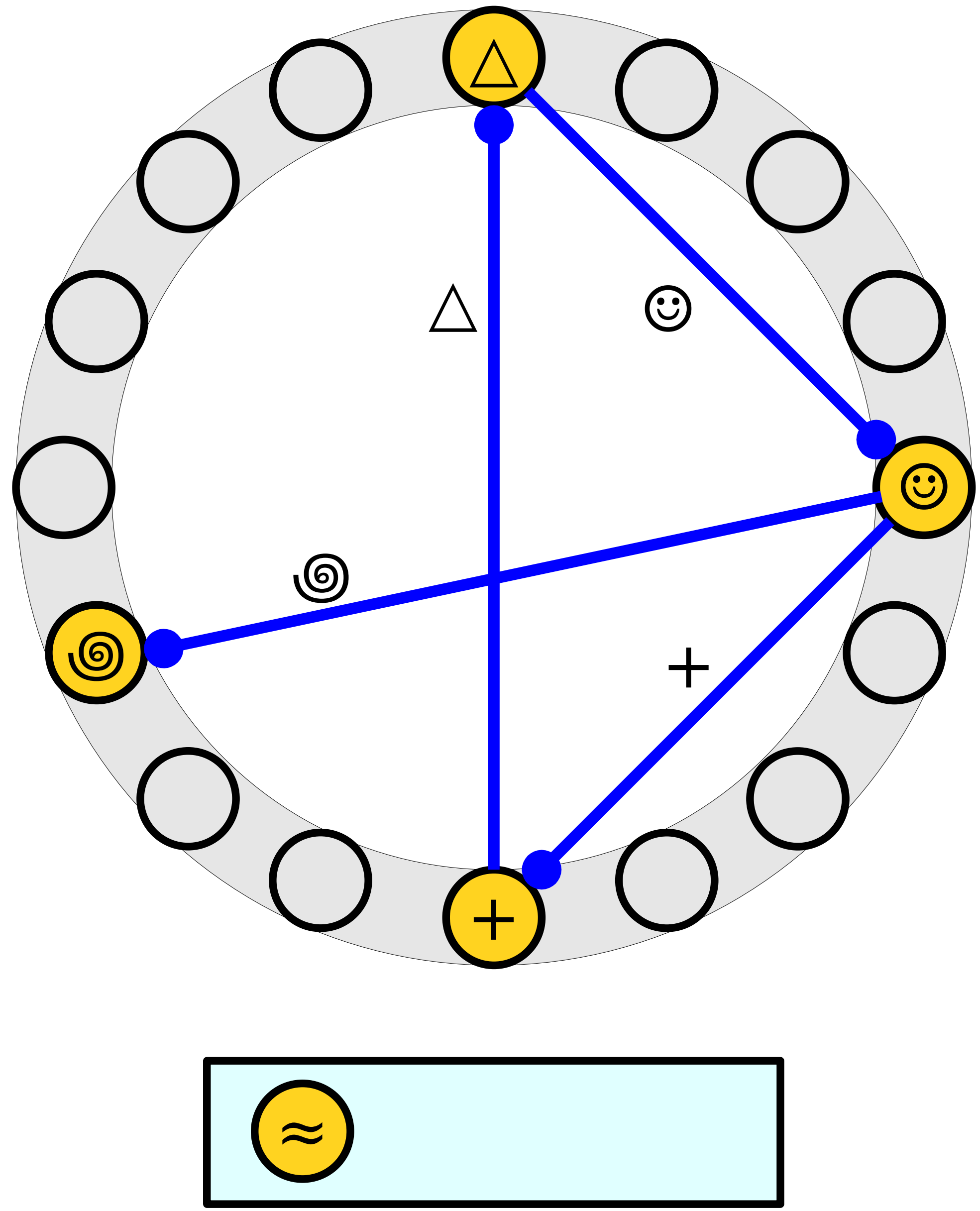


Hashing With Stashing

- Adding a stash doesn't increase the maximum load factor we can use.
- However, it significantly decreases the likelihood that we need to rehash.
- ***Theorem:*** With a stash of size s, the probability of having to rehash after inserting n items is $O(n^{-s})$.



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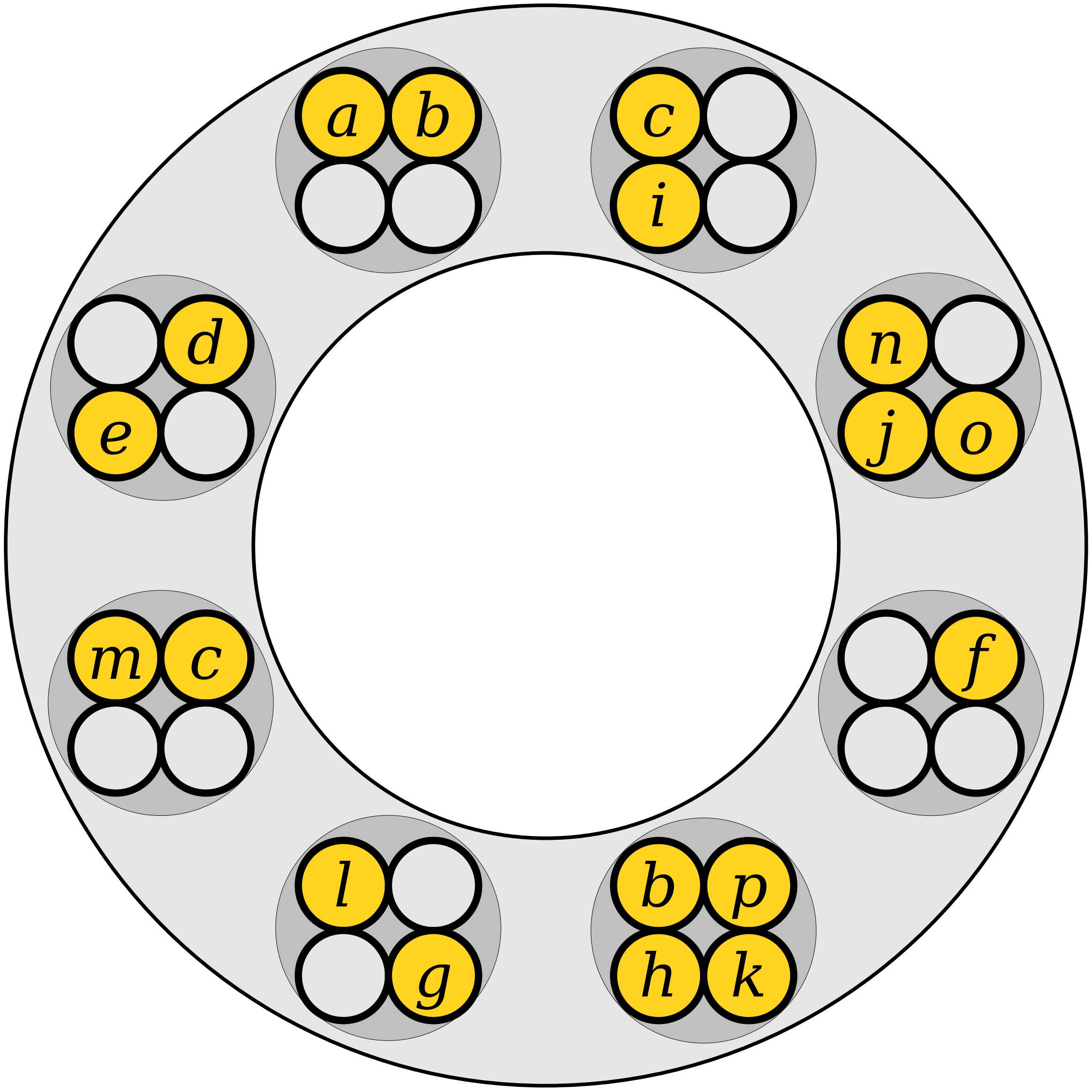
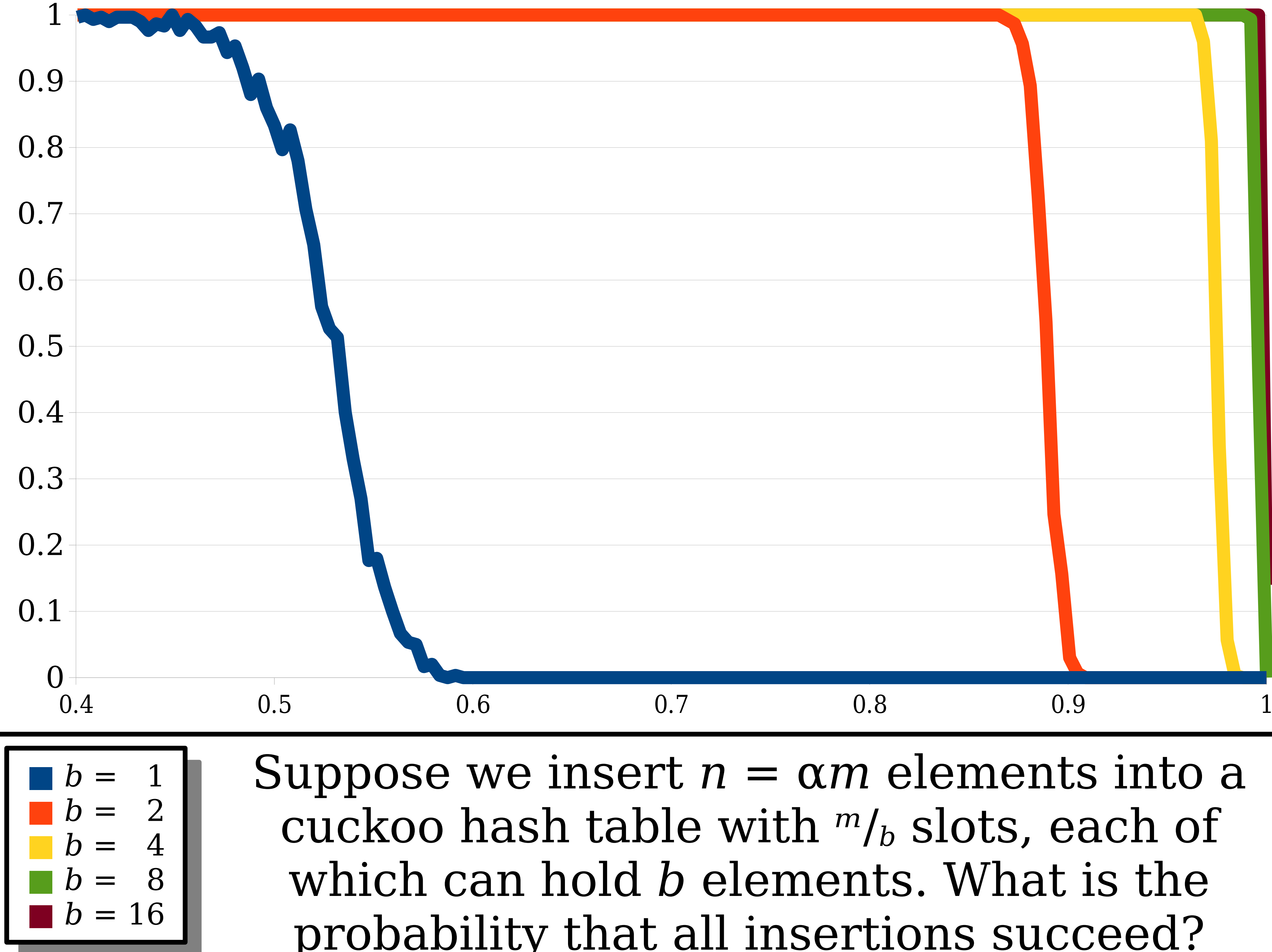
Blocked Cuckoo Hashing

- In ***blocked cuckoo hashing***, each table slot can hold $b \geq 1$ items.
- The parameter b is often chosen so that one block fits perfectly into a cache line, improving locality of reference.
- When inserting an item, place it in one of the two slots it hashes to if there's free space in either.
- If there's no room left, displace an element. Use either a random walk or BFS to select which item to kick.
- Increasing b decreases the likelihood that insertions fail, but increases the cost of lookups and deletions.





Suppose we insert $n = \alpha m$ elements into a cuckoo hash table with $^m/_b$ slots, each of which can hold b elements. What is the probability that all insertions succeed?

Blocked Cuckoo Hashing

- Suppose we have a table with m/b slots, each of which hold b items. Assume $n = m\alpha$ and we use two hash functions.
- ***Theorem:*** Blocked cuckoo hashing succeeds with high probability as long as

$$b \geq 1 + \frac{\ln\left(\frac{\alpha}{\alpha-1}\right)}{1 - \ln 2}.$$

- There is a phase transition result here, but, empirically, the theory hasn't fully captured it.

	$b = 1$	$b = 2$	$b = 4$	$b = 8$	$b = 16$
Predicted max α	**0.500**	**0.576**	**0.715**	**0.895**	**0.990**
Empirical max α	**0.453**	**0.870**	**0.966**	**0.992**	**0.997**

To Summarize

Summary of Cuckoo Hashing

- Cuckoo hashing is a fast and powerful way to build perfect hash tables.
- We can increase the number of hash functions to increase the load factor, though at a cost to lookup and insert times.
- We can use a stash to hold "meddlesome" items, which increases the likelihood that the table can be built.
- We can increase the number of items per slot to increase the load factor, though at a cost to lookup and insert times.

Major Ideas for Today

- Randomized data structures using multiple hash functions can often be analyzed from a graph-theoretic perspective.
- Many properties of random graphs exhibit sharp phase transitions.
- Running experiments is a great way to learn more about randomized data structures.
- Modeling multiple hashes into a table as a bipartite graph is a useful tool.

Next Time

- ***k-Independent Hash Functions***
  - Quantifying how random we need our hash functions to be.
- ***Frequency Estimation***
  - Estimating frequency counts in sublinear space.
- ***Count-Min Sketches***
  - Finding frequent items without actually storing frequencies.