Approximate Membership Queries

Part Two

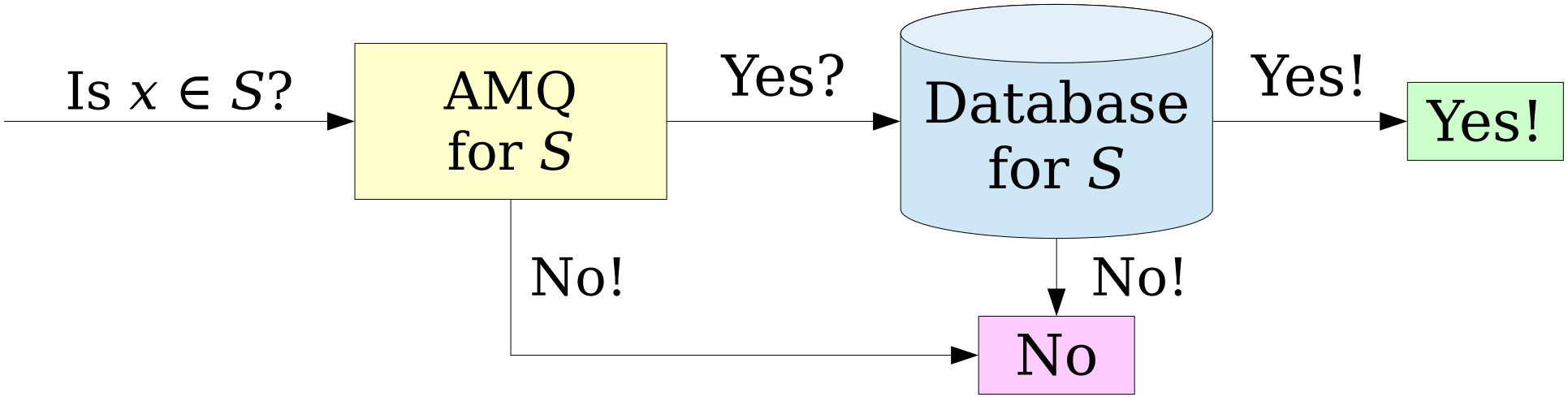
Outline for Today

- ***Recap from Last Time***
  - Where are we, again?
- ***Cuckoo Filters***
  - A practical replacement for Bloom filters.
- ***XOR Filters***
  - A theoretical and practical improvement on the Bloom filter.
- ***Spatial Coupling***
  - Boosting the space usage of XOR filters.

Recap from Last Time

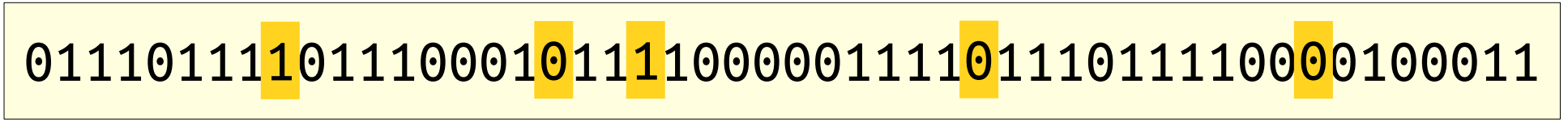
Our Model

- ***Goal:*** Design our data structures to allow for false positives but not false negatives.
- That is:
  - if $x \in S$, we always return true, but
  - if $x \notin S$, we return false with probability $1 - \varepsilon$.
- This is often a good idea in practice, since false positives can be filtered out with a more expensive process.



The Story So Far

- Bloom filters can store an approximation of an n-element set using $1.44 \lg \varepsilon^{-1}$ bits per element.
- Each query computes (up to) $\lg \varepsilon^{-1}$ hashes of an item. A query that answers “no” requires, on expectation, two hashes before a 0 bit is found.
- Bloom filters don’t have very good cache locality (i.e. lookups are scattered through the array).



	Bits Per Element	Hashes Per Query
Bloom Filter (1970)	$1.44 \lg \varepsilon^{-1}$	$\lg \varepsilon^{-1}$

The Story So Far

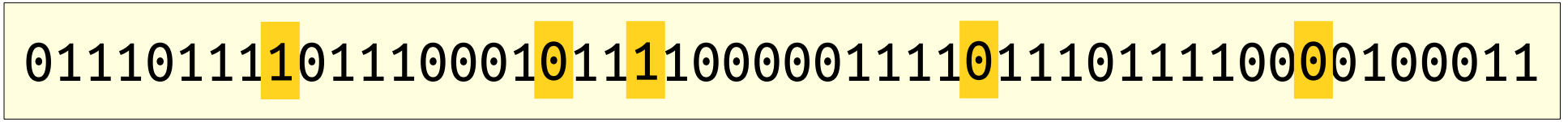
- ***Theorem:*** Assuming $\varepsilon|U| \gg n$, any AMQ structure for storing n items from U with error rate ε requires at least $n \lg \varepsilon^{-1}$ bits.
- The Bloom filter is within a factor of 1.44 of optimal in terms of space.
- If possible, we'd like to knock down the time cost of a query.
- ***Question:*** Can we do better?

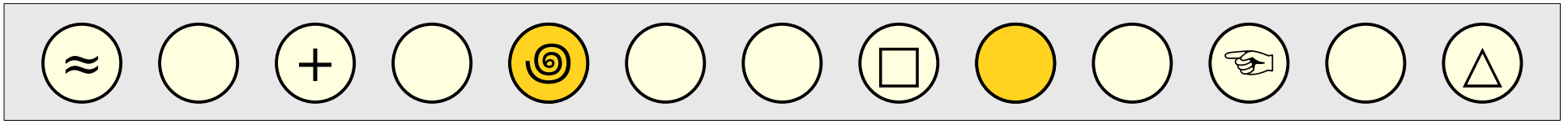
	Bits Per Element	Hashes Per Query
Bloom Filter (1970)	$1.44 \lg \varepsilon^{-1}$	$\lg \varepsilon^{-1}$

Cuckoo Filters

From Bloom to Cuckoo

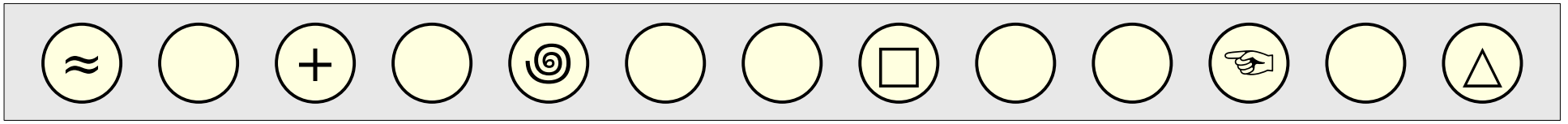
- A Bloom filter works by
  - hashing an item with some fixed number of hash functions,
  - looking at some positions in a table determined by those hash functions, and
  - using those contents to determine whether the item is in the table.
- If you squint at it the right way, this looks a *lot* like what a cuckoo hash table does.
- ***Question:*** Can we adapt cuckoo hashing to work as an approximate membership query?





Cuckoo Space Usage

- ***Recall:*** A vanilla cuckoo hash table with two hash functions holding n items uses $\Theta(n)$ slots.
- ***Question:*** Asymptotically, how many *bits* of memory are needed to store a cuckoo hash table holding n distinct elements?



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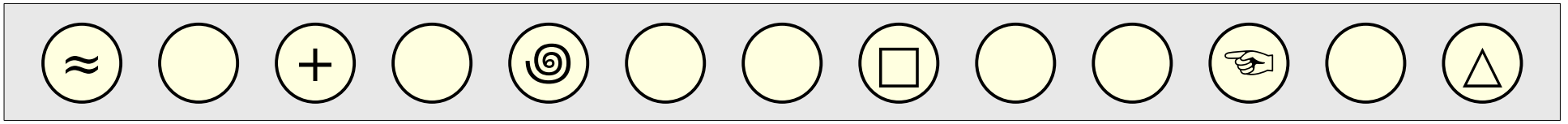

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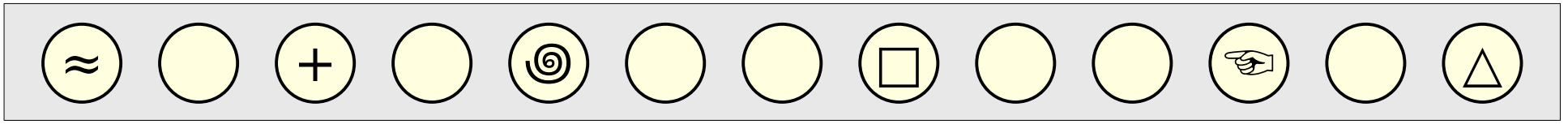
Now, ***private chat me your best guess***. Not sure? Just answer "??."



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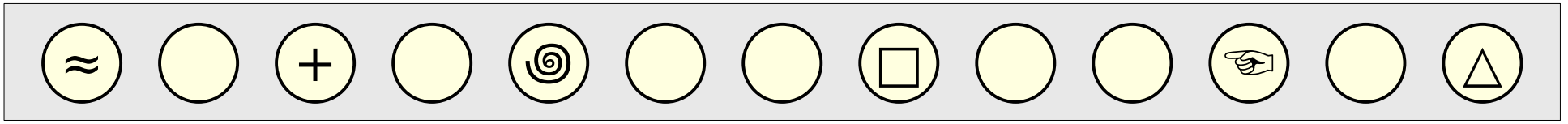
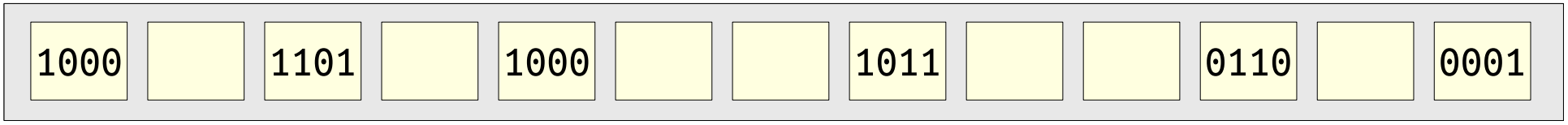
- ***Question:*** Asymptotically, how many *bits* of memory are needed to store a cuckoo hash table holding n distinct elements?
- If we store n distinct elements, then we need $\Omega(\log n)$ bits, on average, for each of those elements.
  - Otherwise, we don't have enough bits to write out n distinct items.
- Total space usage: **$\Omega(n \log n)$** bits.
- ***Goal:*** Reduce this space to $\Theta(n \log \varepsilon^{-1})$ bits, and ideally be as close to the information-theoretic lower bound as possible.



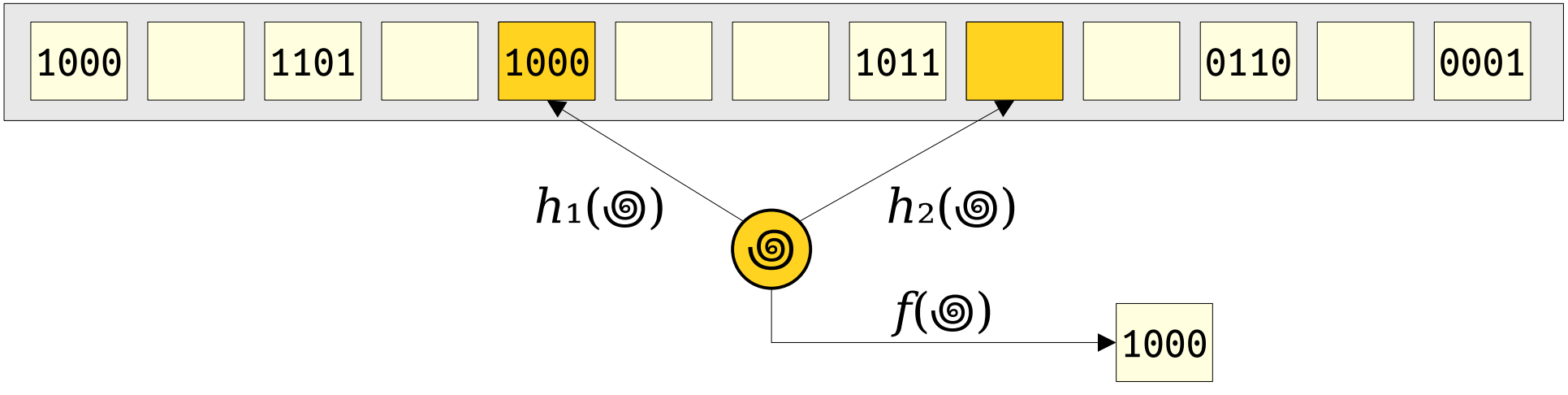
Cuckoo Space Usage

- We need to drop the usage of our cuckoo hash table to $\Theta(n \log \varepsilon^{-1})$.
- ***Intuition:*** That sure looks like what you'd get if you had an array of $\Theta(n)$ slots, each of which was of size $\Theta(\log \varepsilon^{-1})$.
- This isn't enough space to store each of the items in full.
- ***Key Idea:*** Associate each item with a ***fingerprint***, a bit sequence of length L. Then, store fingerprints rather than the items themselves.
  - Ideally, we'll get $L = \Theta(\log \varepsilon^{-1})$.
- We can do this by using a hash function ***f*** that maps from items to L-bit sequences.



Cuckoo Filters

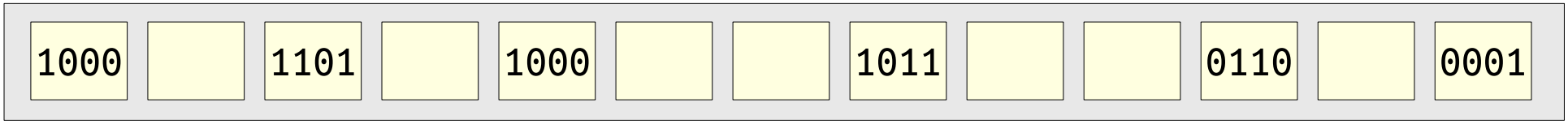
- A ***cuckoo filter*** is a modified cuckoo hash table that stores items' fingerprints, rather than the full items.
- To check whether an item x is present in the cuckoo filter, do the following:
  - Hash the item with two hash functions h_1 and h_2 to get two table indices.
  - Hash the item with f to compute its fingerprint.
  - See whether $f(x)$ is at position $h_1(x)$ or $h_2(x)$ in the table.
- If the item is in the table, this will definitely find it.
- If the item is not in the table, there's a small false positive probability if we coincidentally find a matching fingerprint.



Cuckoo Filters

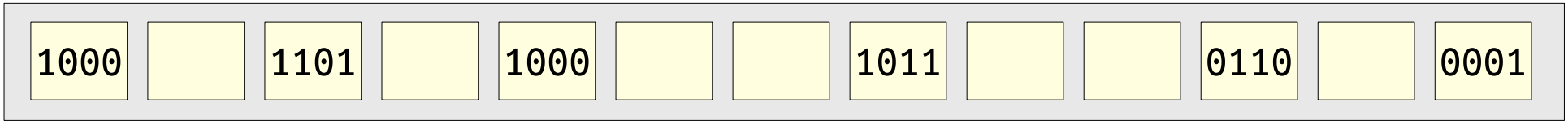
- Our fingerprints have size L. We'd like our false positive rate to be ε. How should we pick L?
- Suppose we query for $x \notin S$. Each lookup tests two positions, each of which has at most a 2^{-L} probability of being $f(x)$.
- We can conservatively bound the false positive rate at 2^{-L+1}.
- Therefore, we should pick

$$\boldsymbol{L = 1 + \lg \varepsilon^{-1}}.$$



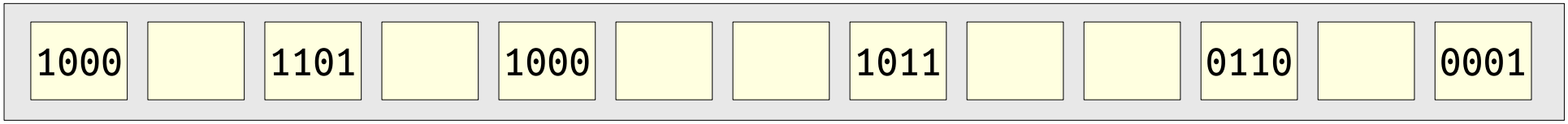
Cuckoo Filters: Two Challenges

- Unfortunately, we're not done just yet. There are two issues we need to address.
  - ***Space utilization:*** Regular cuckoo hash tables max out with a load factor of α = ½.
  - ***Displacement logic:*** There's a subtle issue that pops up when determining how to move items around the table.
- Let's focus on each of these in turn.



Space Utilization

- ***Recall:*** A regular cuckoo hash table (two hash functions, one item per slot) has a maximum load factor of ½.
- If we want to store n elements in a cuckoo filter, we need at least $2n$ slots to keep the load factor below ½.
- Each slot has size $1 + \lg \varepsilon^{-1}$, since it stores an L-bit fingerprint.
- Total space usage: $\mathbf{2n \lg \varepsilon^{-1} + 2n}$. This is worse than a Bloom filter.
- ***Question:*** Can we do better?

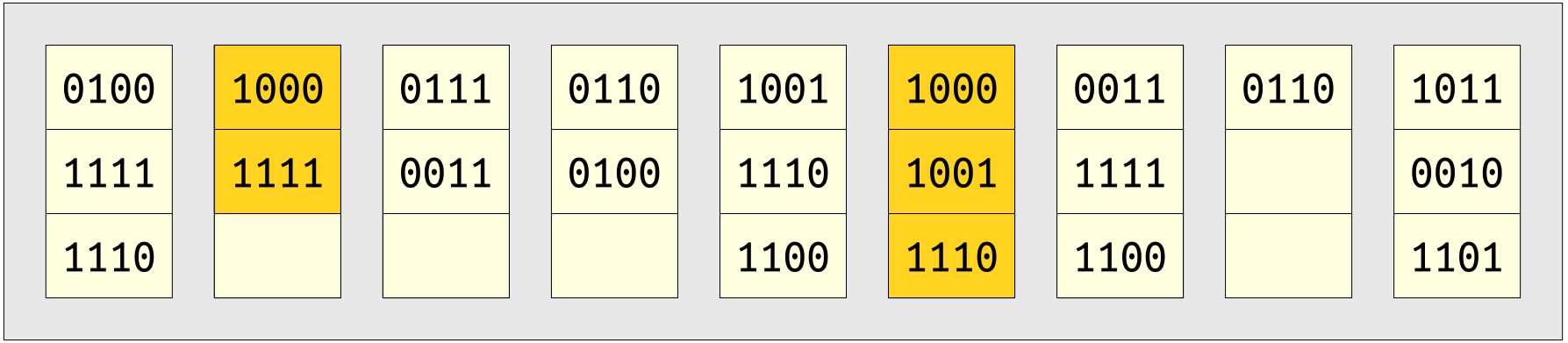


Space Utilization

- In our lecture on cuckoo hashing, we saw two strategies for improving the space utilization of a cuckoo hash table:
  - ***d-Ary Cuckoo Hashing:*** Pick $d \geq 2$ hash functions to determine where to place items.
  - ***Blocked Cuckoo Hashing:*** Each table slot can hold $b \geq 1$ different items.
- In practice, blocked cuckoo hashing is much faster than d-ary cuckoo hashing.
  - We need to evaluate fewer hash functions, and hash functions can be a bottleneck.
  - There's better *locality of reference*, since we only probe two locations.
- ***Idea:*** Improve our space utilization using blocked cuckoo hashing, sticking with two hash functions.

Space Utilization

- Increasing b, the number of items per slot, will
  - increase the probability of false positives, since each query looks at more fingerprints (each query now sees $2b$ fingerprints), but
  - increase the load factor of the table, giving better space utilization.
- ***Question:*** How should we tune the choice of b?



Space Utilization

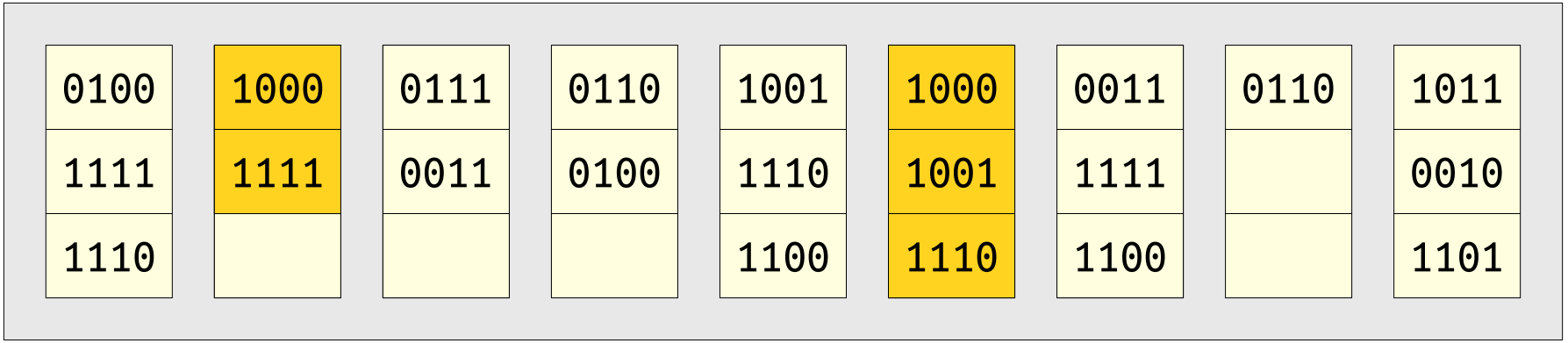
- ***Question:*** With fingerprints of length L and a block size of b, what is the probability of a false positive?
- Suppose we look up a key x that isn't in the set. Each query sees at most $2b$ locations, each of which matches $f(x)$ with probability at most 2^{-L}.
- Probability of a false positive is at most

$$\varepsilon = 2b \cdot 2^{-L},$$

  which solves to

$$\boldsymbol{L = \lg \varepsilon^{-1} + 1 + \lg b}.$$

- Our fingerprint size has to increase as the size of the slot increases, but not by all that much.



Space Utilization

- Here's the empirical maximum α values we saw with regular blocked cuckoo hashing.
- If we have $n = \alpha m$ items in our table, then our table size needs to be $m = \alpha^{-1} n$. Each item requires a $(\lg \varepsilon^{-1} + 1 + \lg b)$-bit fingerprint.
- Here's what that looks like for different values of b:
  - $b = 1$: about $2.21 \lg \varepsilon^{-1} + 2.21$ bits per element.
  - $b = 2$: about $1.15 \lg \varepsilon^{-1} + 2.30$ bits per element.
  - $b = 4$: about $1.05 \lg \varepsilon^{-1} + 3.15$ bits per element.
  - $b = 8$: about $1.02 \lg \varepsilon^{-1} + 4.08$ bits per element.
  - $b = 16$: about $1.01 \lg \varepsilon^{-1} + 5.05$ bits per element.
- For common values of ε, this is minimized when $b = 4$ or $b = 8$.

	$b = 1$	$b = 2$	$b = 4$	$b = 8$	$b = 16$
Empirical max α	**0.453**	**0.870**	**0.966**	**0.992**	**0.997**
α^{-1}	**2.21**	**1.15**	**1.05**	**1.02**	**1.01**

Space Utilization

- Using $b = 4$, we get that the space usage for our cuckoo filter is

$$\mathbf{1.05 \lg \varepsilon^{-1} + 3.15}$$

  bits per element.

- For small ε, this is strictly better than a Bloom filter.

	$b = 1$	$b = 2$	$b = 4$	$b = 8$	$b = 16$
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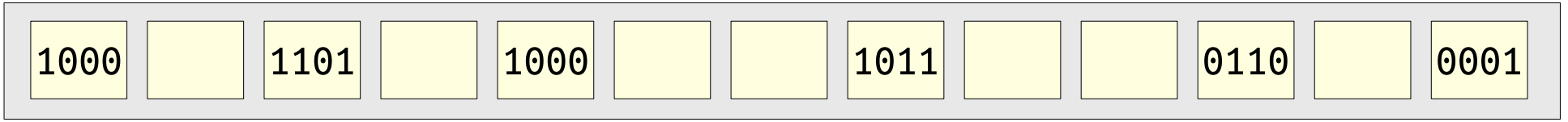
The Catch: Displacements

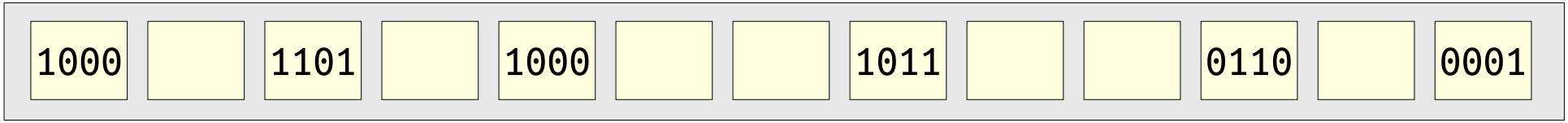
Supporting Displacements

- In a normal cuckoo hash table, insertions can displace items already in the table.
- Specifically, we may need to move an item x from position $h_1(x)$ to $h_2(x)$, or vice-versa.
- ***Problem:*** Our table stores the *fingerprint* of x, rather than x itself. If we need to displace $f(x)$, how do we determine which slot to move it to if we don't know x?



Supporting Displacements

- As you saw on IA4, one way to get hash functions for cuckoo hashing is the following:
  - Pick $h_1 : U \to \{0, 1, 2, \ldots, m - 1\}$, where U is the universe of possible keys, assuming the table has m slots.
  - Pick $h_\Delta : U \to \{1, 2, 3, \ldots, m - 1\}$ as an offset hash function that moves us relative to h_1.
  - Define $h_2(x) = h_1(x) \oplus h_\Delta(x)$.
- To displace an item x at position i:
  - Compute $h_\Delta(x)$ from the value of x in the table.
  - Move the item to index $i \oplus h_\Delta(x)$.
- ***Idea:*** Adapt this approach to work with fingerprints.



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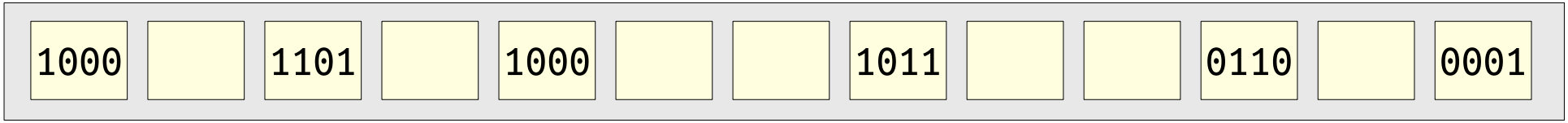
We have $f(x)$, rather than x, stored in the table.

We can't evaluate $h_\Delta(x)$.

However, we can evaluate $h_\Delta(f(x))$.



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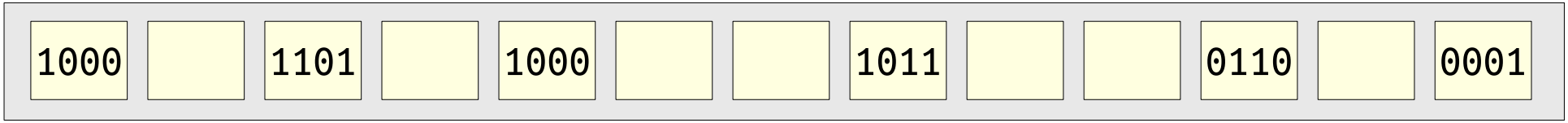
  - Pick $h_1 : U \to \{0, 1, 2, \ldots, m - 1\}$, where U is the universe of possible keys, assuming the table has m slots.
  - Pick $h_\Delta : F \to \{1, 2, 3, \ldots, m - 1\}$, where F is the set of possible fingerprints.
  - Define $h_2(x) = h_1(x) \oplus h_\Delta(f(x))$.
- To displace a fingerprint $f(x)$ at position i:
  - Compute $h_\Delta(f(x))$ from the value of $f(x)$ in the table.
  - Move the item to index $i \oplus h_\Delta(f(x))$.
- This will always move the fingerprint from $h_1(x)$ to $h_2(x)$ or vice-versa, and doesn't require knowing x.



Supporting Displacements

- Look closely at the definition of $h_2(x)$:

$$h_2(x) = h_1(x) \oplus h_\Delta(f(x)).$$

- Do you notice anything potentially problematic about this?
- Take an extreme case: suppose our fingerprints are just a single bit. What can you say about $h_2(x)$?

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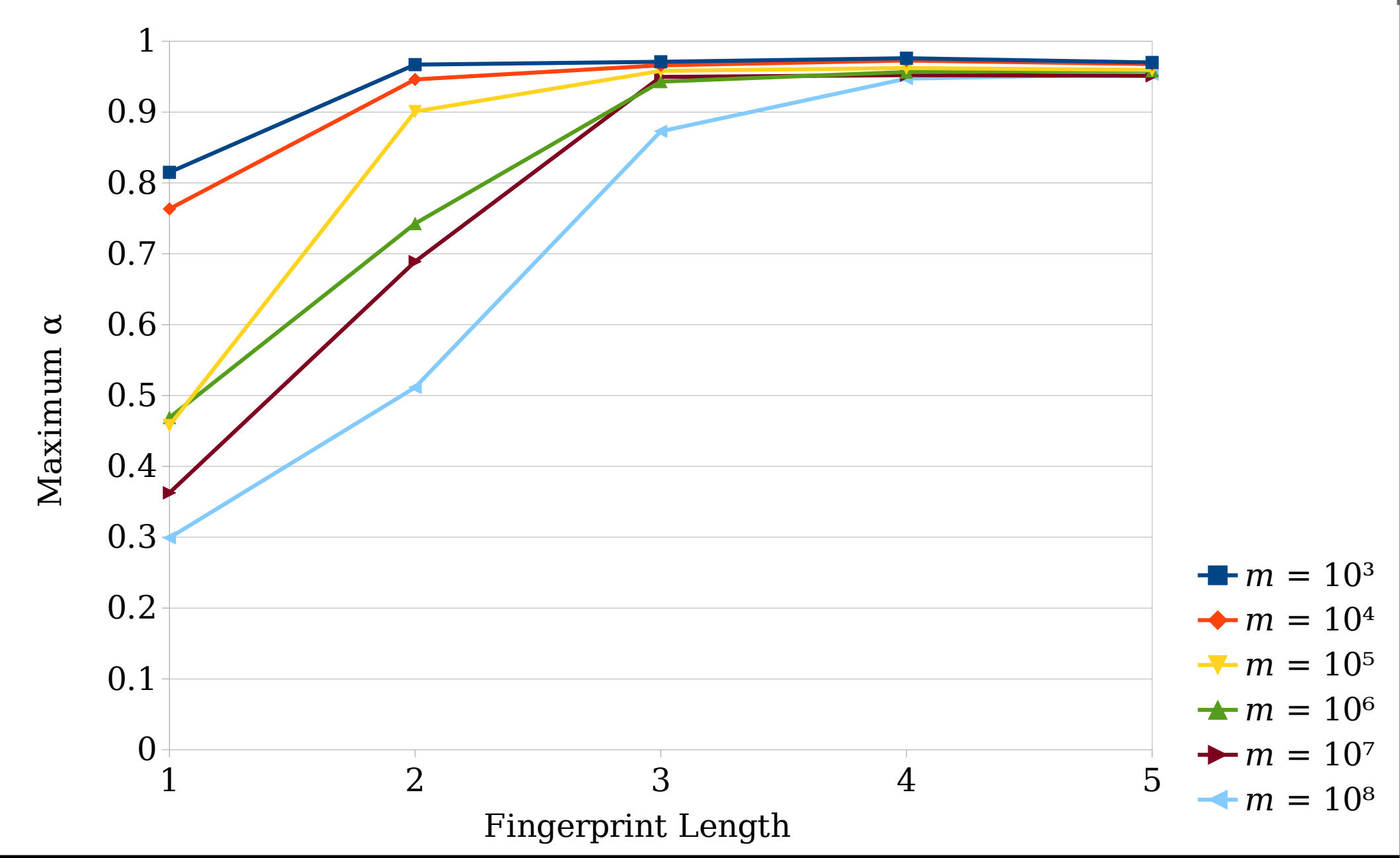
- ***Problem:*** While $h_1(x)$ is uniform over the space of the table, $h_2(x)$ can only take on at most 2^L different values, where 2^L is (probably) much smaller than m, the number of table slots.
- This means that the distribution of possible pairs $(h_1(x), h_2(x))$ is not uniform over the table, invalidating our initial analysis of cuckoo hashing.
- This is not just a theoretical issue.

Build a cuckoo filter ($b = 4$) with $m/4$ slots and fingerprints of length L. What is the maximum load factor α where the success probability of inserting $n = \alpha m$ items is at least 99%?



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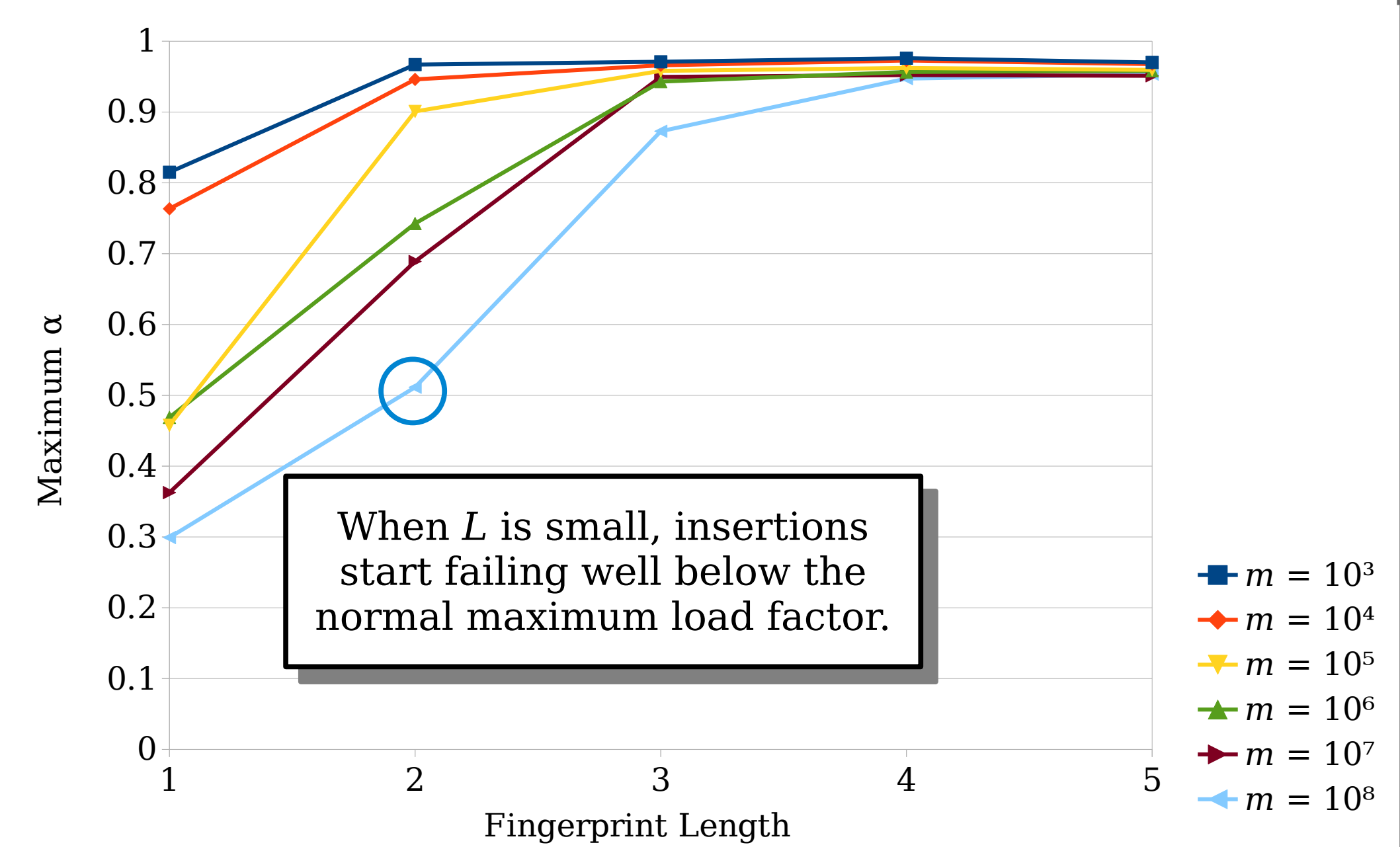

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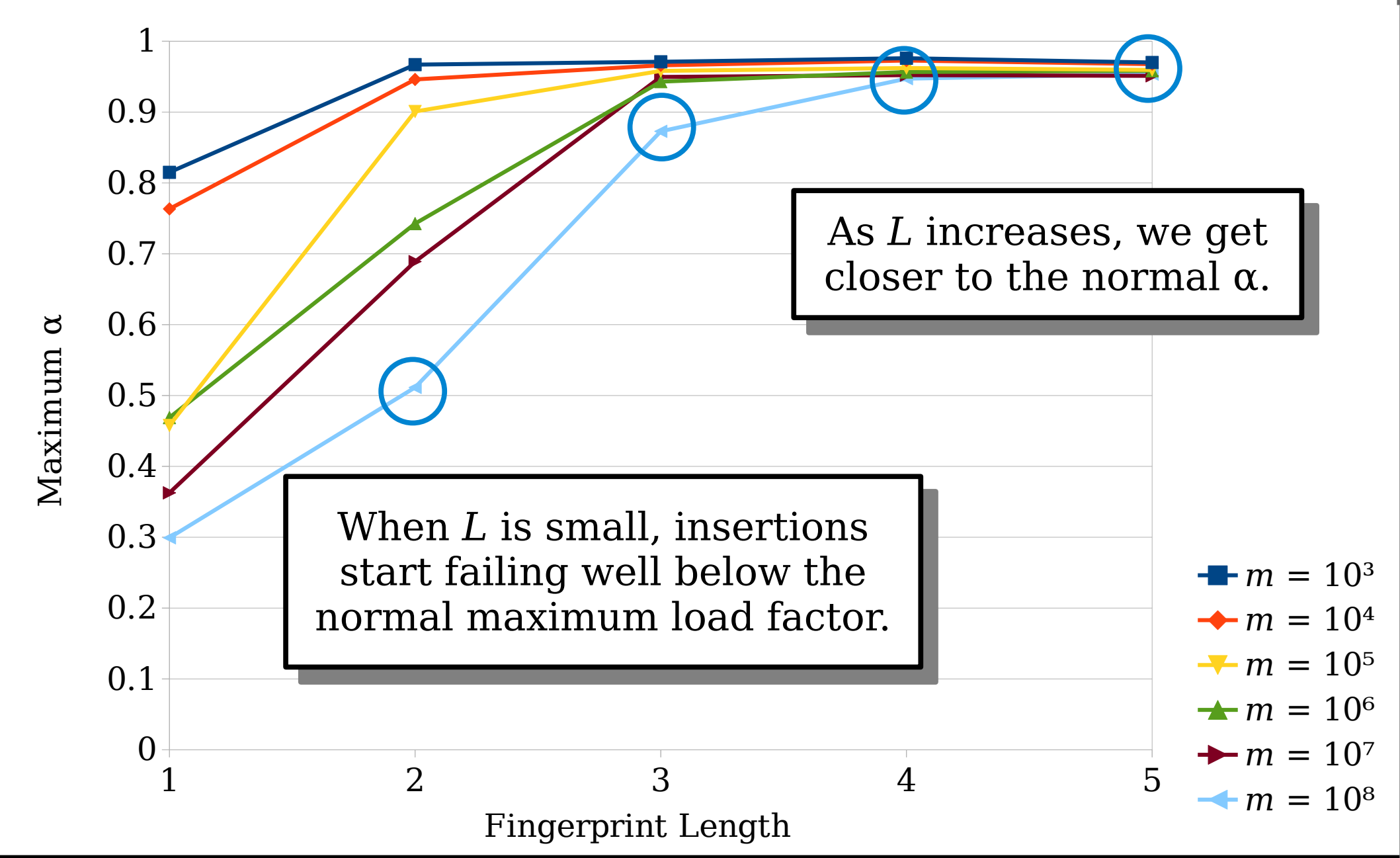

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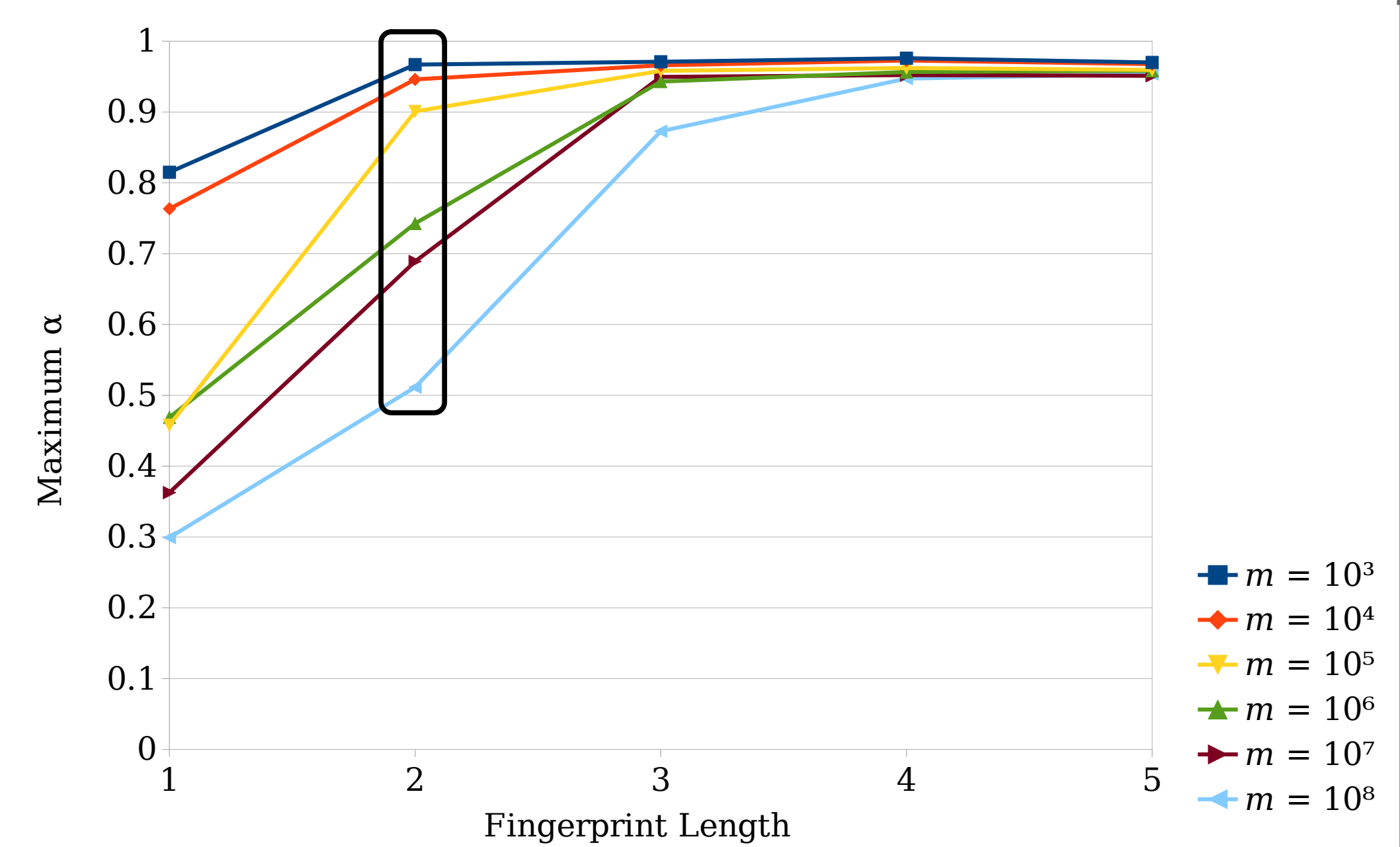

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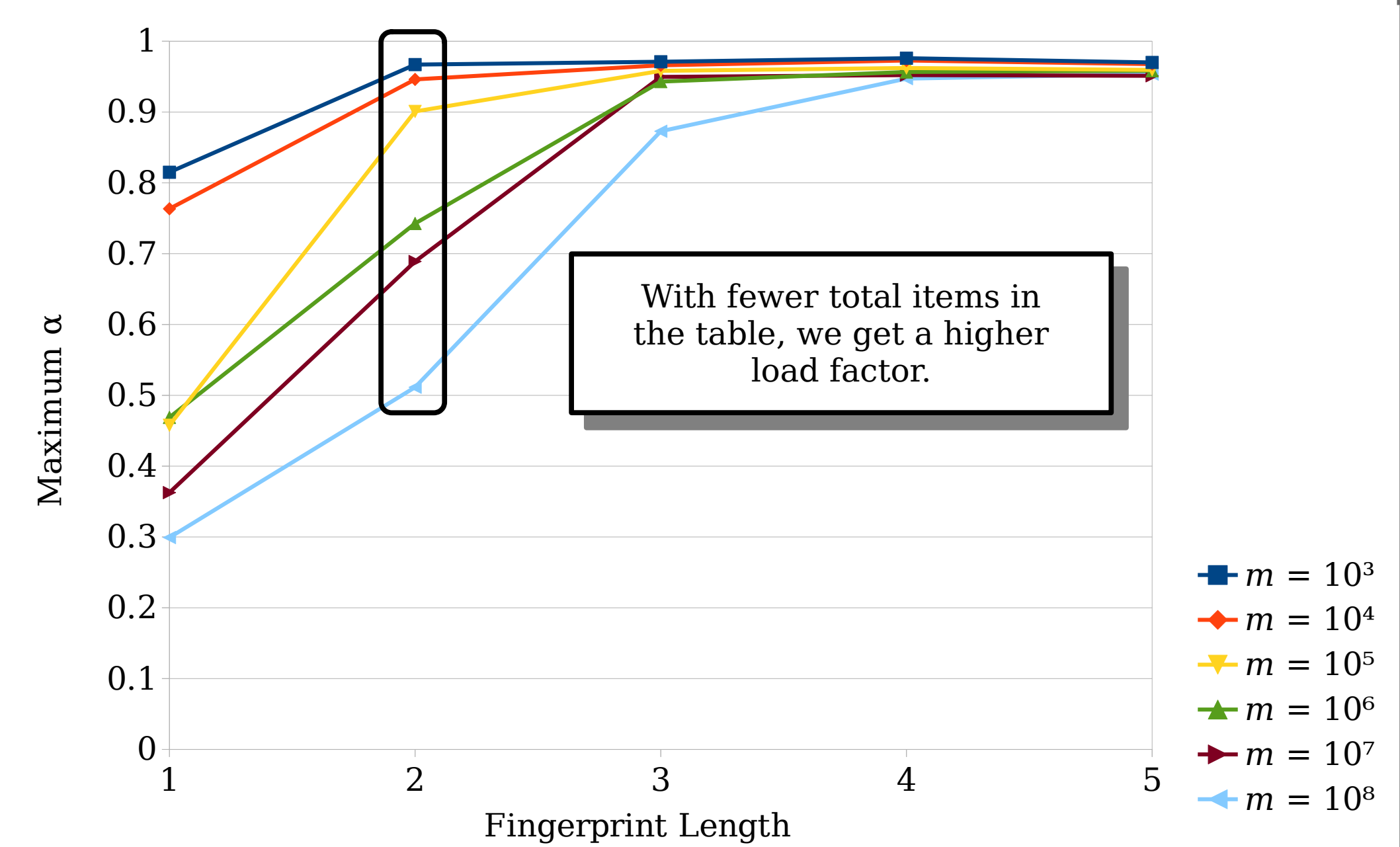

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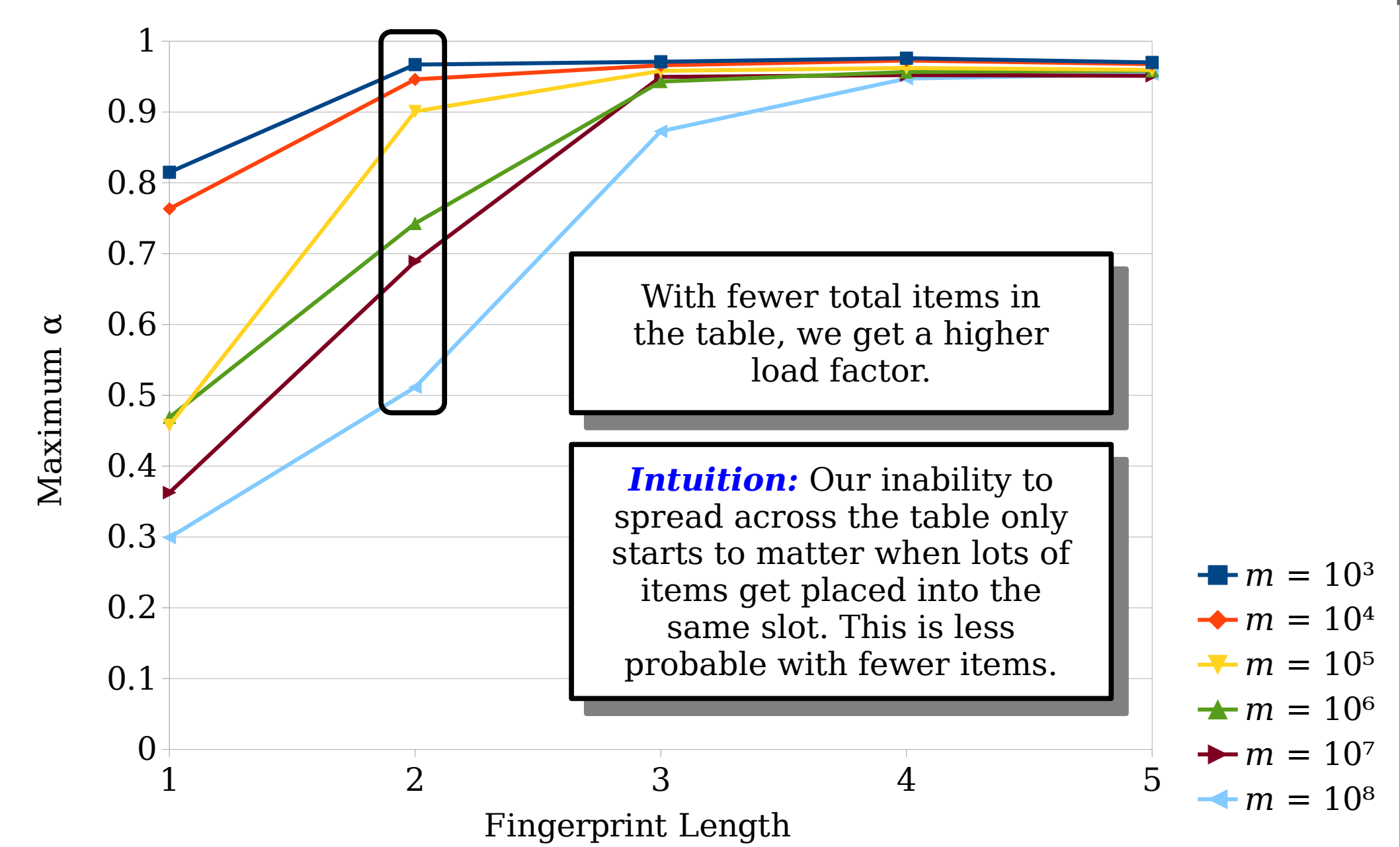

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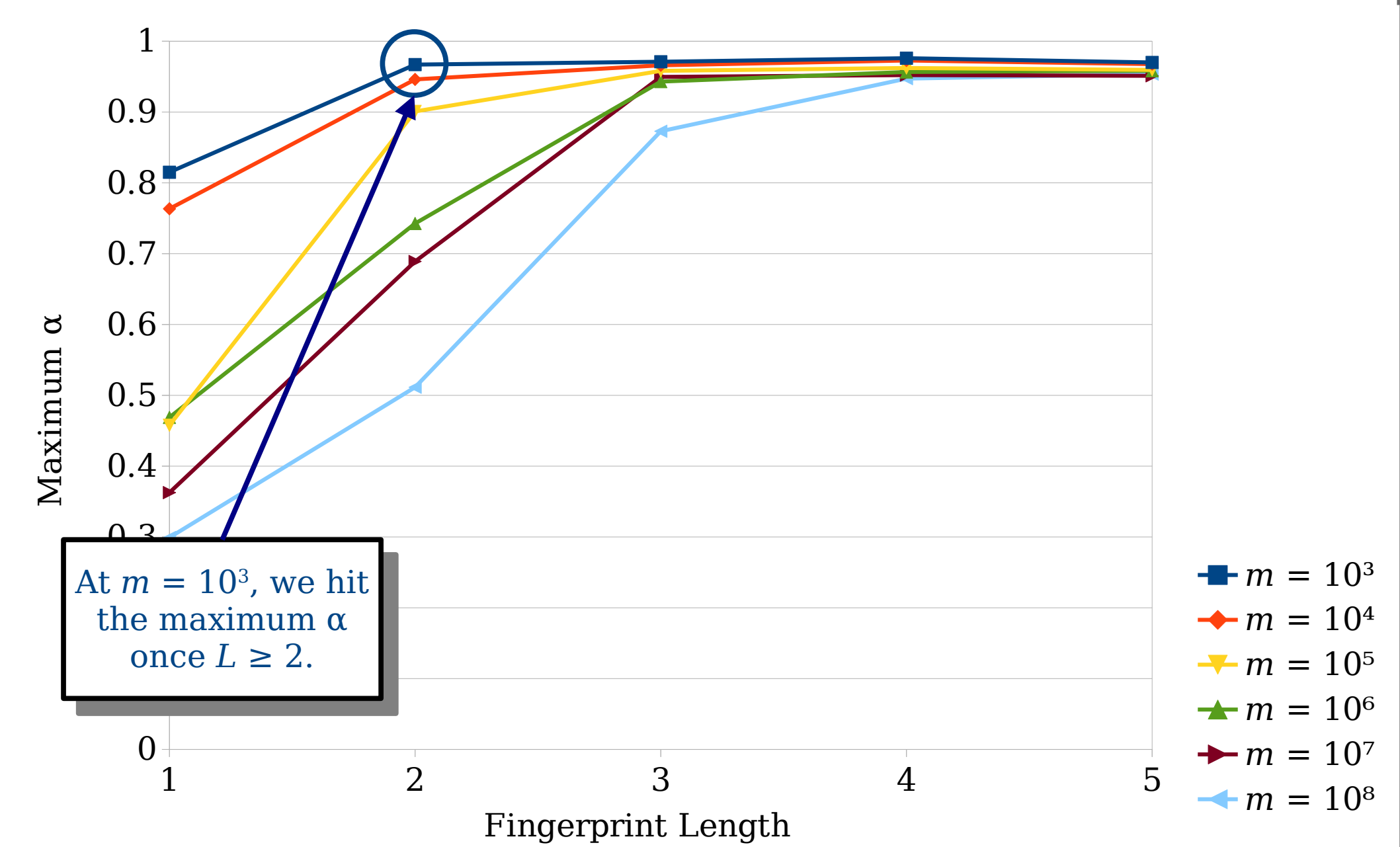

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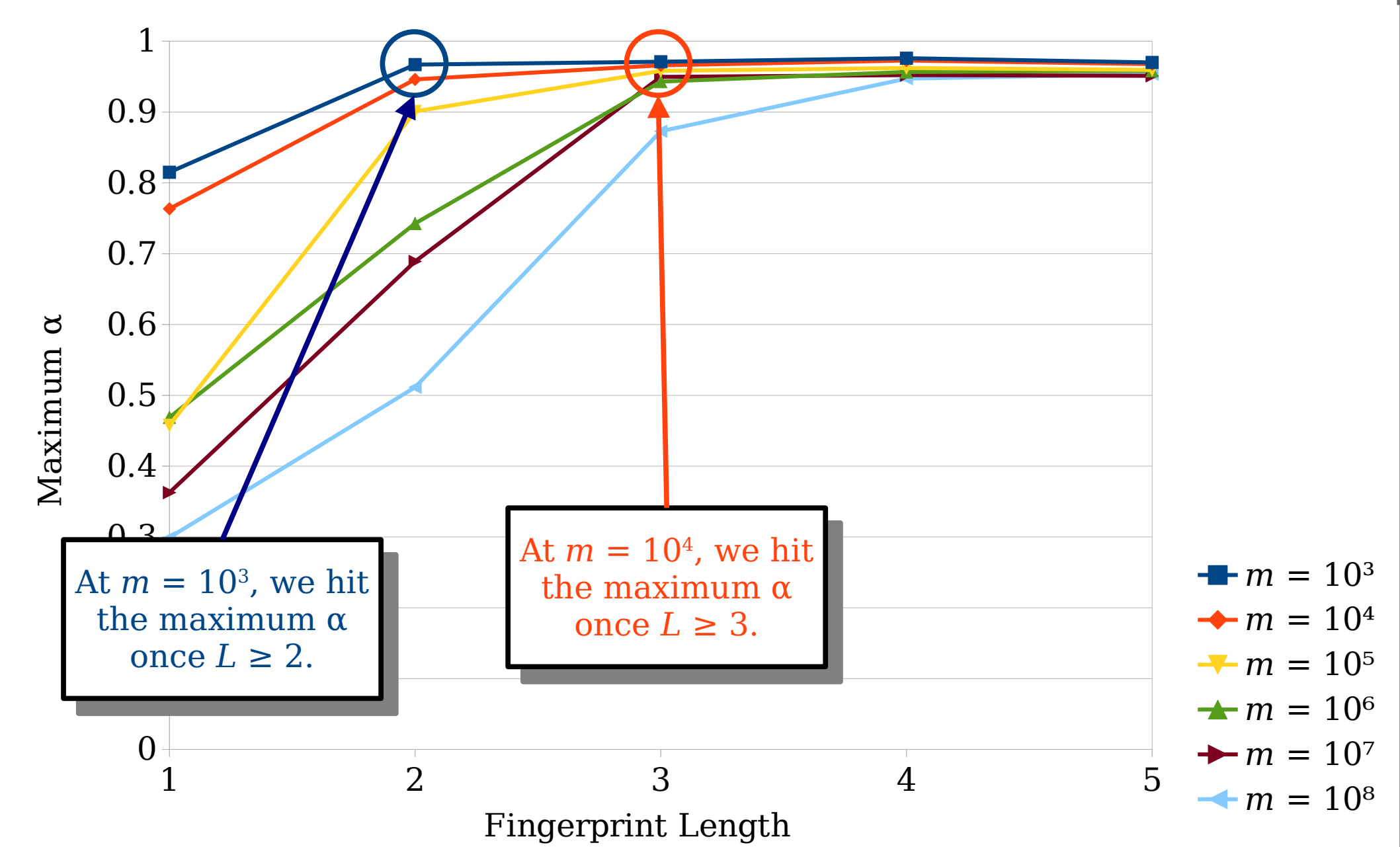

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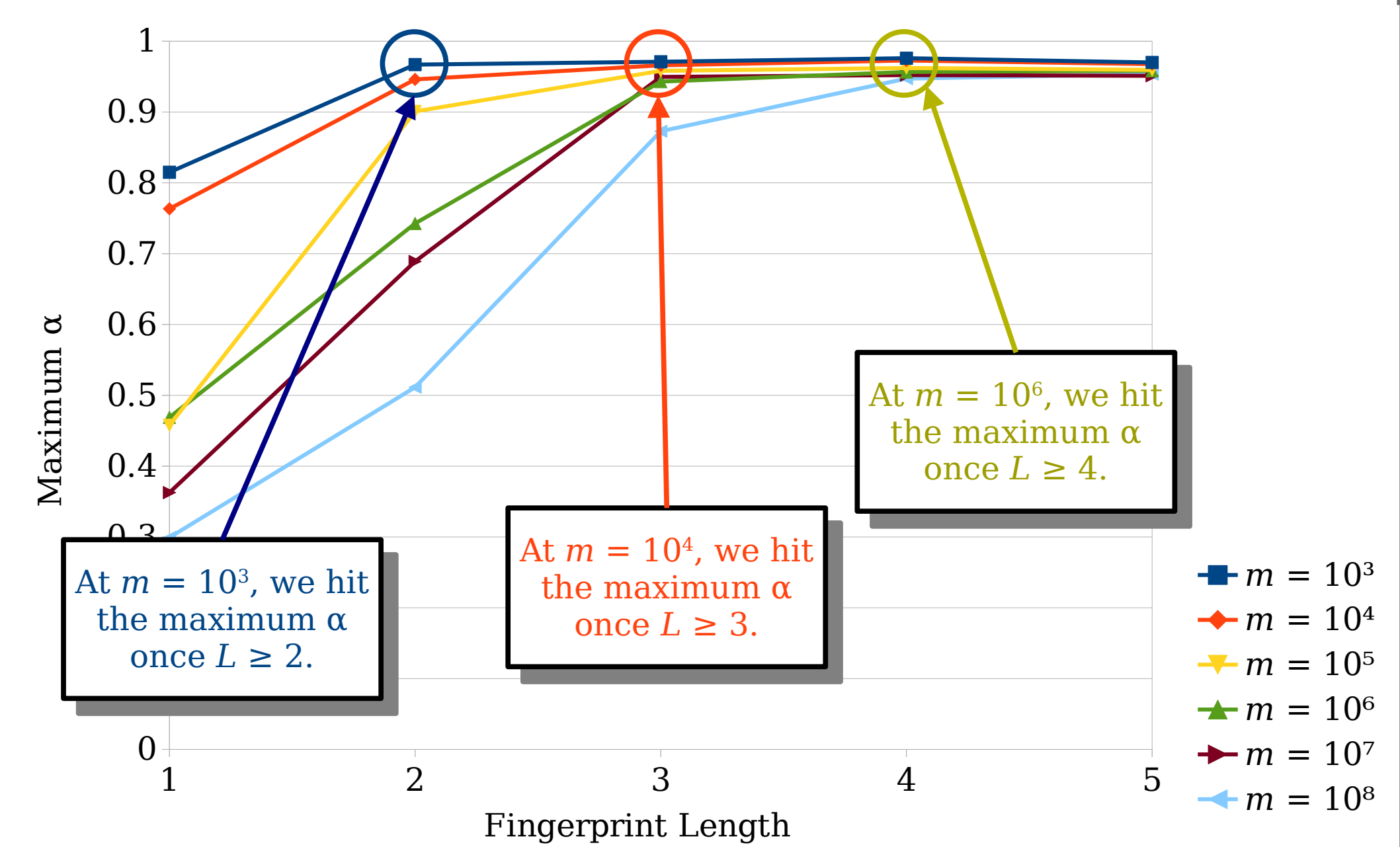

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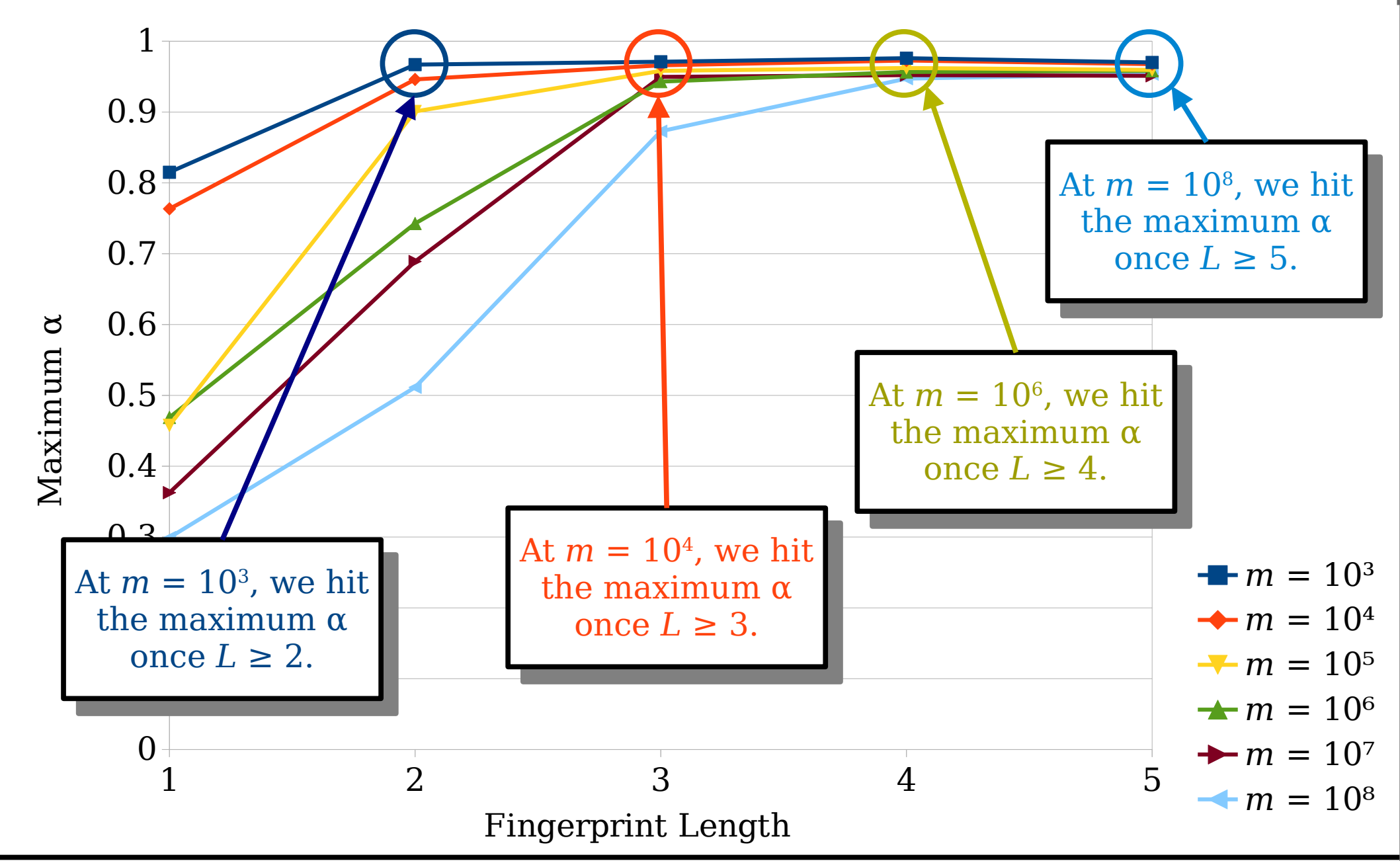

Build a cuckoo filter ($b = 4$) with $m/4$ slots and fingerprints of length L. What is the maximum load factor α where the success probability of inserting $n = \alpha m$ items is at least 99%?

A Lower Bound

- ***Theorem:*** For cuckoo filters to behave "close enough" to regular cuckoo hashing, and therefore reach the theoretical maximum load factor, we must have

$$L = \Omega((\log n) / b).$$

- ***Recall:*** To achieve false positive rate ε, we need to have

$$L = \lg \varepsilon^{-1} + 1 + \lg b.$$

- In practice, picking a modest choice of ε (say, $\varepsilon = {}^{1}/_{32}$) is sufficient to satisfy the lower bound with $b = 4$ and $n \leq 2^{32}$.
- In theory, cuckoo filters only use $\Theta(\lg \varepsilon^{-1})$ bits per element if either ε drops as n increases, or we increase b and our lookup times are allowed to increase.
- This means that cuckoo filters are "practically" better than a Bloom filter for most choices of ε, but only *theoretically* better than a Bloom filter if $\varepsilon = o(1)$ as a function of n.

To Summarize

The Cuckoo Filter ($b = 4$)

- Maintain an array whose size is a power of two and that has at least $0.27n$ slots, each of which can hold four fingerprints of size $\lg \varepsilon^{-1} + 3$, all initially empty.
- Pick a fingerprinting function f from items to fingerprints.
- Pick a hash function h_1 from items to slots, and a hash function h_Δ from fingerprints to slots (excluding zero). Define $h_2(x) = h_1(x) \oplus h_\Delta(f(x))$.
- For each item x to store, compute its fingerprint $f(x)$ and insert $f(x)$ into the table using the normal cuckoo hashing insertion algorithm.
- To see if x is in the table, compute $f(x)$ and see if it's in the table in either slot $h_1(x)$ or $h_2(x)$.

The Story So Far

- Cuckoo filters require at most three hashes per query, compared with $\lg \varepsilon^{-1}$ for the Bloom filter. They also have markedly better cache locality.
- For $\varepsilon \leq 2^{-7}$, in practice, cuckoo filters use less space than Bloom filters.
- This is a practical, drop-in replacement for a Bloom filter for small choices of ε.

	Bits Per Element	Hashes/Query
Bloom Filter (1970)	$1.44 \lg \varepsilon^{-1}$	$\lg \varepsilon^{-1}$
Cuckoo Filter, $b = 4$ (2014)	$1.05 \lg \varepsilon^{-1} + 3.15$ *(for sufficiently small ε)*	3

The Story So Far

- In Theoryland, cuckoo filters require us to decrease ε or increase b as n increases.
- ***Question:*** Is there a way to improve on Bloom filters, both practically and in Theoryland?

	Bits Per Element	Hashes/Query
Bloom Filter (1970)	$1.44 \lg \varepsilon^{-1}$	$\lg \varepsilon^{-1}$
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XOR Filters

Bloom and Cuckoo Revisited

- From the Bloom filter, we have the following ideas:

Store a large array of tiny items.

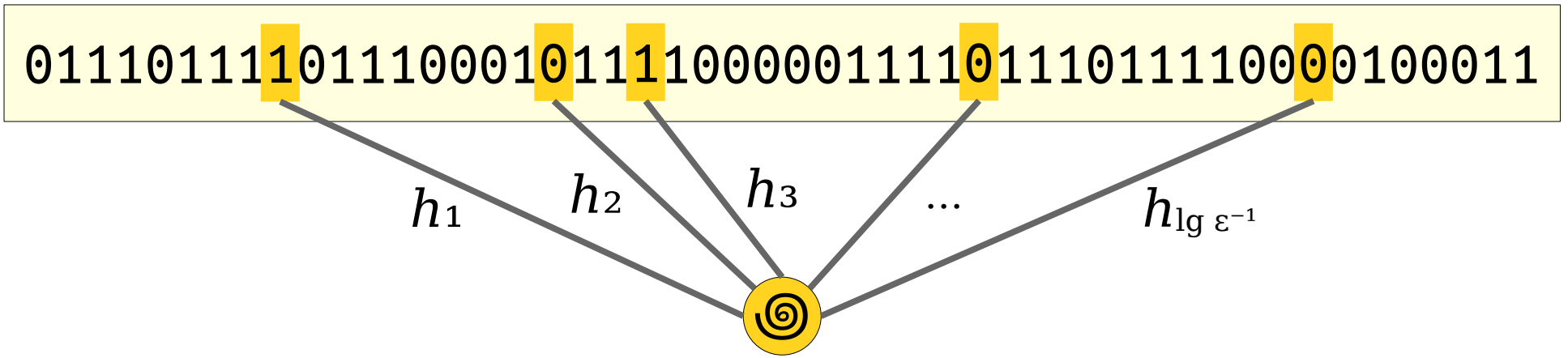
Hash an item to multiple positions in an array, aggregate those array positions together, and see whether you get what you want.

- From the cuckoo filter, we have the following idea:

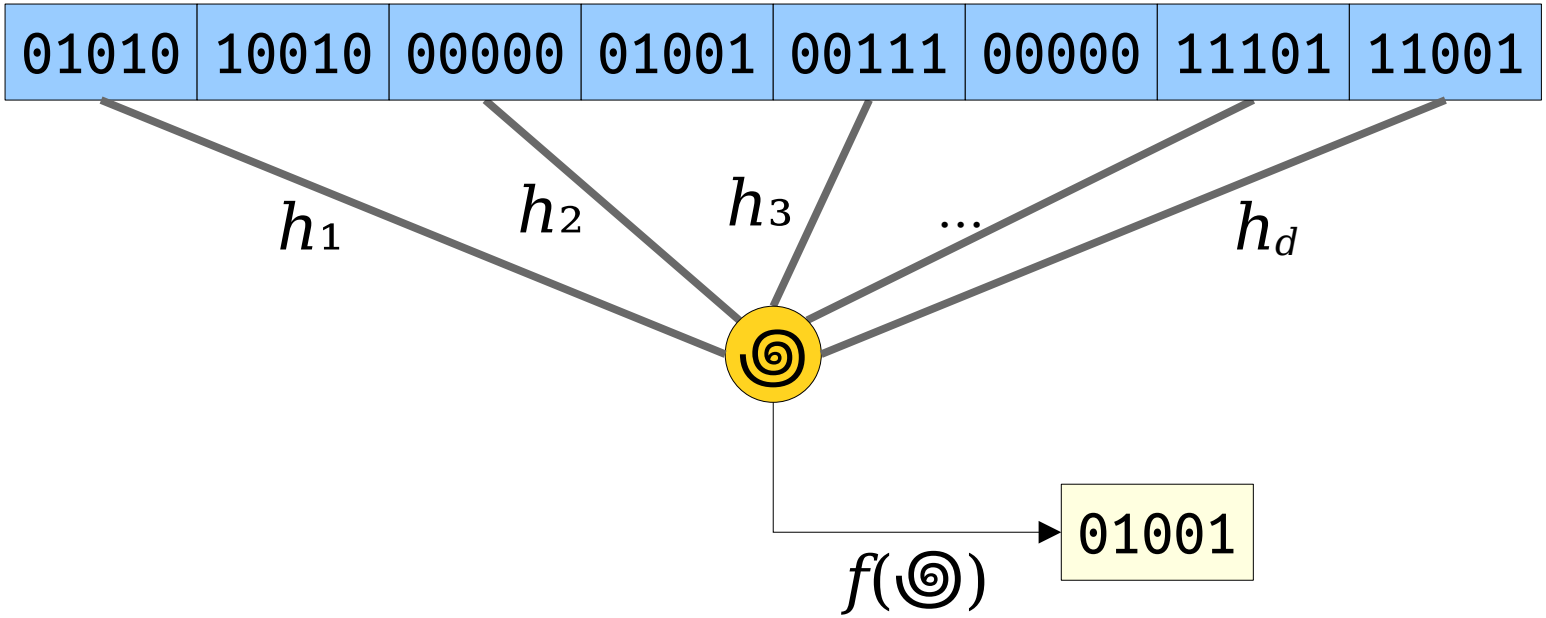
Store a medium-sized array of medium-sized items.

Hash each item to a fingerprint, then store the fingerprints in a space-efficient manner.

- What happens if we combine these ideas together?



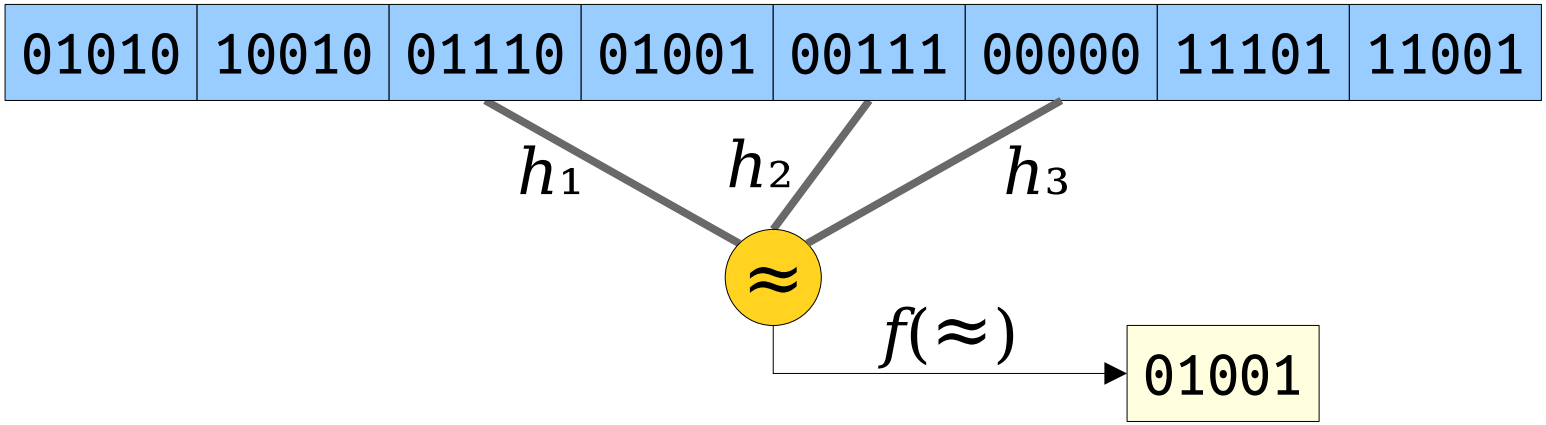
	Bloom Filters	
Store an array of items	Array of bits	
Pick some array slots	Hash item with $\lg \varepsilon^{-1}$ hash functions	
Derive single value from slots	AND	
Compare value against expected	Should be 1	



	Bloom Filters	Something New
Store an array of items	Array of bits	Array of L-bit values
Pick some array slots	Hash item with lg ε^{-1} hash functions	Hash item with d hash functions
Derive single value from slots	AND	XOR
Compare value against expected	Should be 1	Should be item fingerprint

The XOR Filter

- Create an array of $\Theta(n)$ slots, each of which stores a single L-bit number.
- Pick d hash functions h_1, h_2, …, h_d from items to slots.
- Pick a hash function f from items to L-bit fingerprints
- To query whether an item x is in the set:
  - Compute $h_1(x)$, $h_2(x)$, …, $h_d(x)$ and look at those slots.
  - XOR the contents of those slots together.
  - Return whether the XORed value matches $f(x)$.



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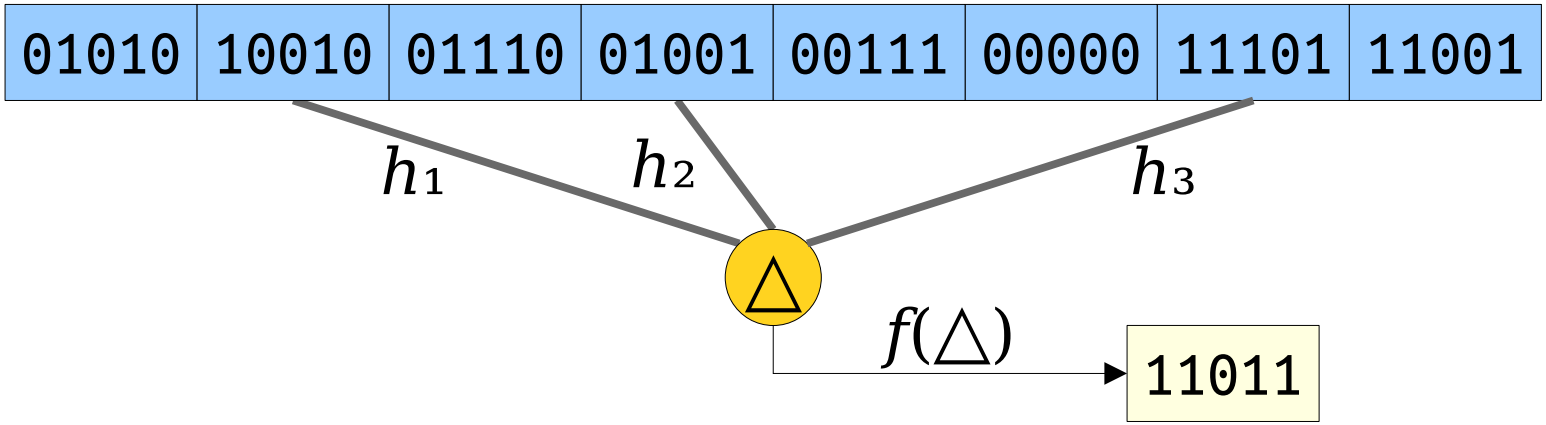
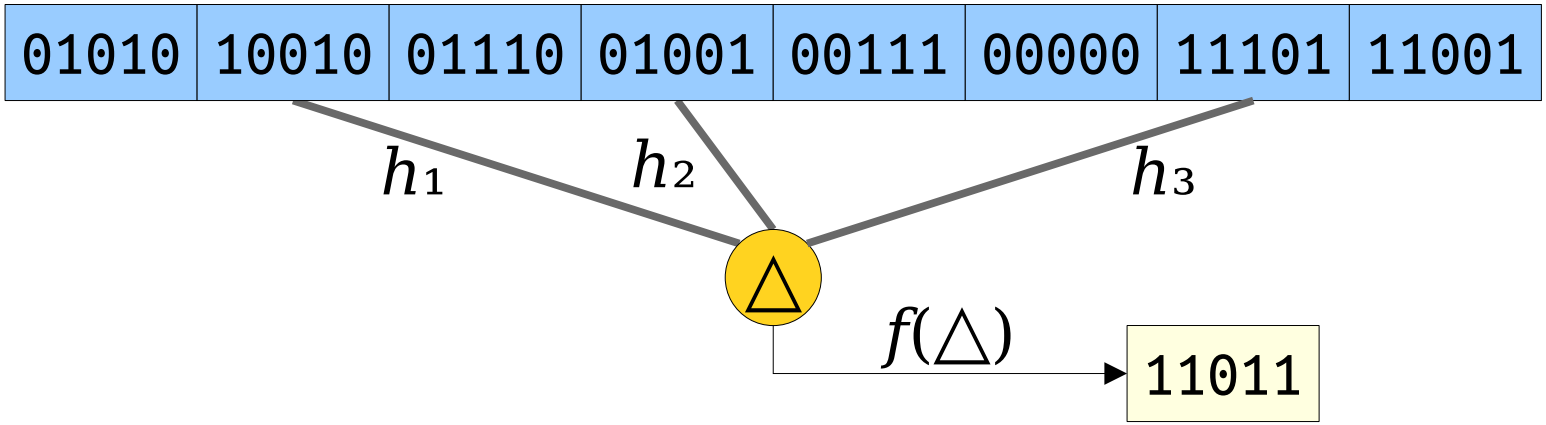


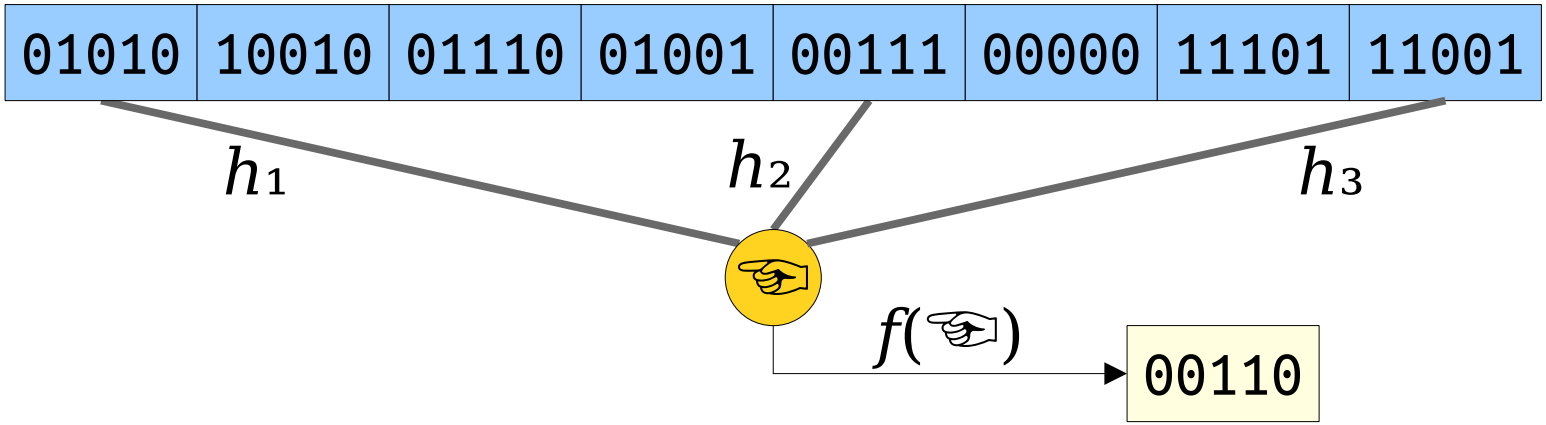
The XOR Filter

- Questions to address:
  - How do we pick L, the fingerprint length?
  - How big should our array be?
  - How do we initialize the array contents?
  - How many hash functions should we use?
- Let's go through each of these in turn.



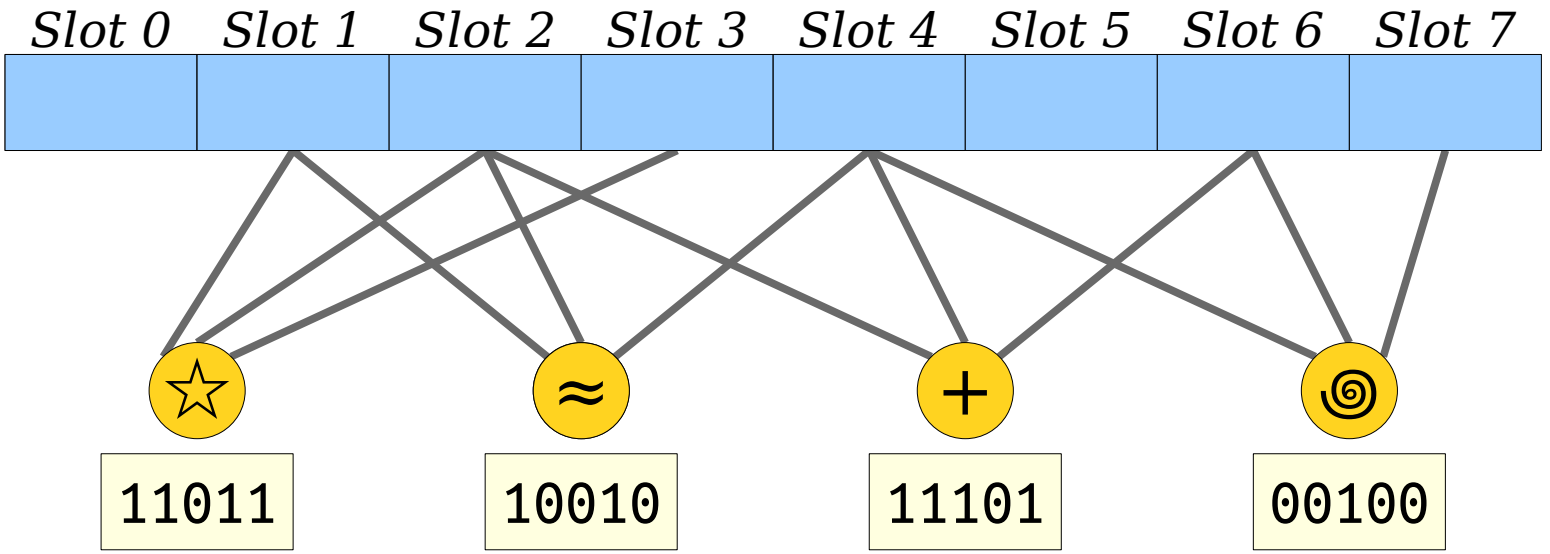
The Fingerprint Size

- Take any item $x \notin S$.
- We hash x to some table locations, then XOR all those locations together to get a value $T[h_1(x)] \oplus \ldots \oplus \ldots T[h_d(x)]$.
- We also compute the fingerprint $f(x)$.
- For simplicity, assume that all hash functions here (h_1, …, h_d and f) are truly random. This makes $f(x)$ independent of the computed value $T[h_1(x)] \oplus \ldots \oplus \ldots T[h_d(x)]$.
- Probability of a false positive is therefore the probability that $f(x) = T[h_1(x)] \oplus \ldots \oplus \ldots T[h_d(x)]$, which is 2^{-L}.
- Setting $L = \mathbf{\lg \varepsilon^{-1}}$ then ensures an appropriate false positive rate.



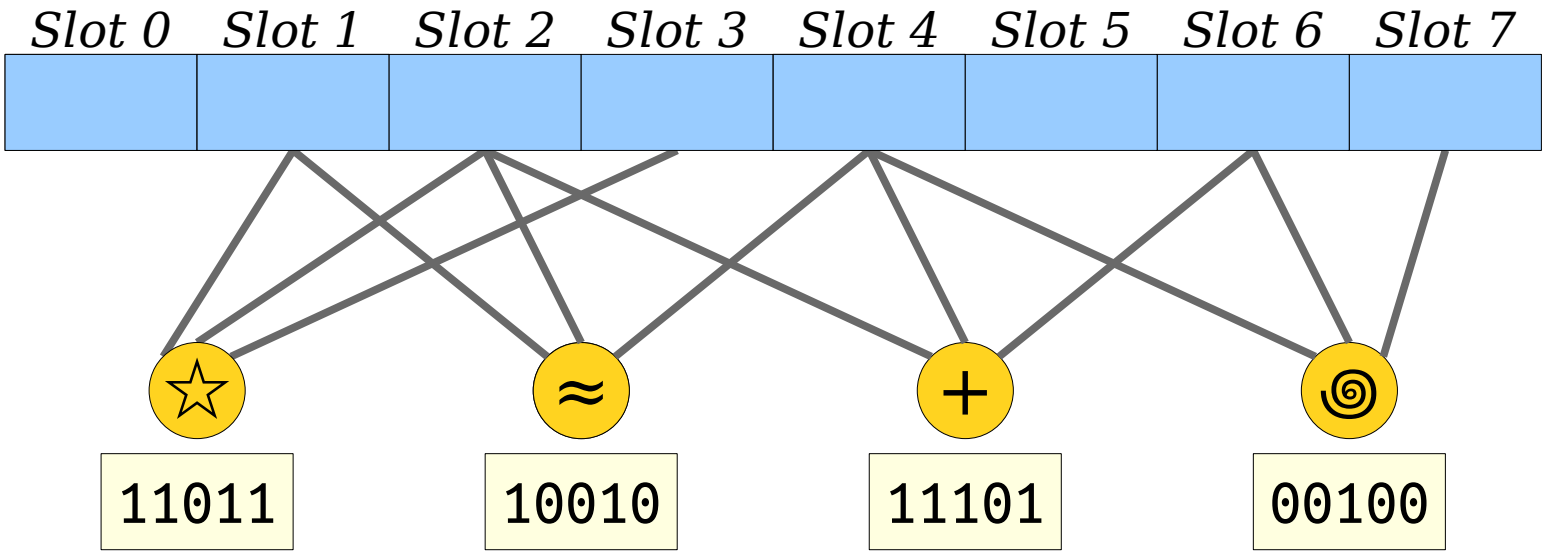
Filling Our Table

- Here's a concrete example of a table to fill in. Lines between items and slots indicate where each item hashes.
- Is it possible to set the bits of this table in a way that makes everything work out?



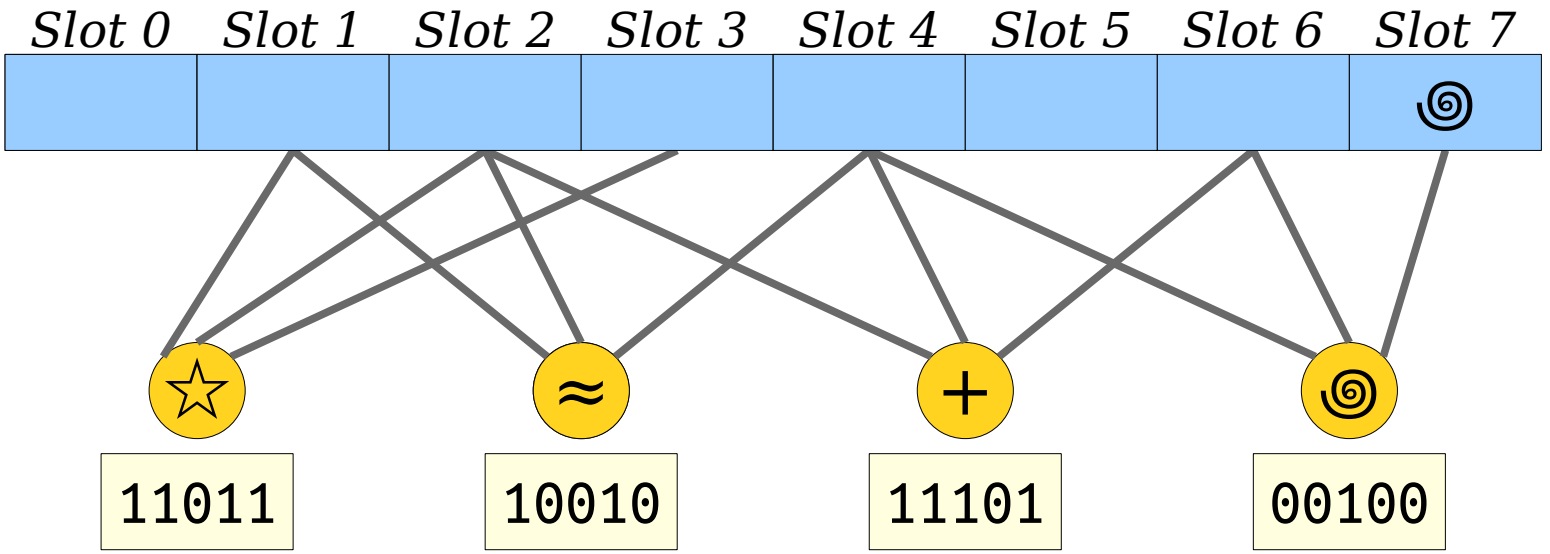
Filling Our Table

- Notice that slot 7 has only one item (namely, 🌀) hashing to it.
- This means that tuning slot 7 can only impact whether 🌀 has the correct XOR of its slots. It has no impact on any other items.
- ***Idea:*** "Assign" 🌀 to slot 7.



Filling Our Table

- Notice that slot 7 has only one item (namely, ◎) hashing to it.
- This means that tuning slot 7 can only impact whether ◎ has the correct XOR of its slots. It has no impact on any other items.
- ***Idea:*** "Assign" ◎ to slot 7.



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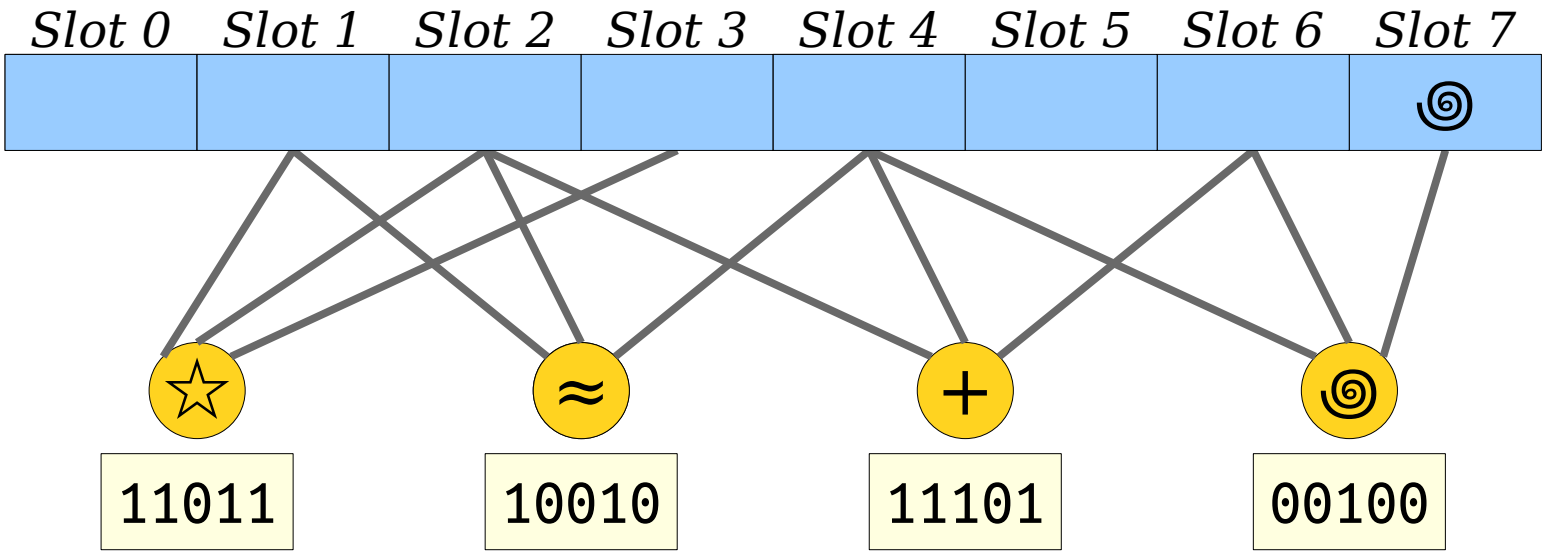
- This means that tuning slot 7 whether ◎ has the correct XO impact on any other items.
- ***Idea:*** "Assign" ◎ to slot 7.

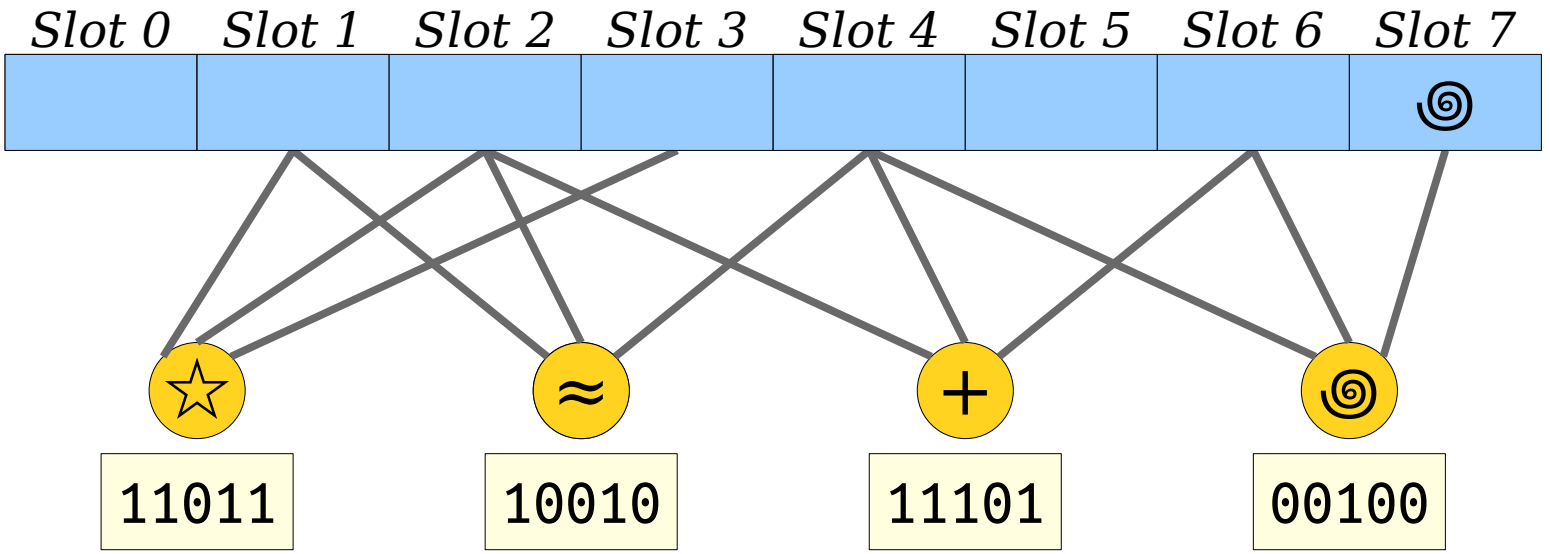
We're not actually storing ◎ here. We're just marking that ◎ owns this slot.



Filling Our Table

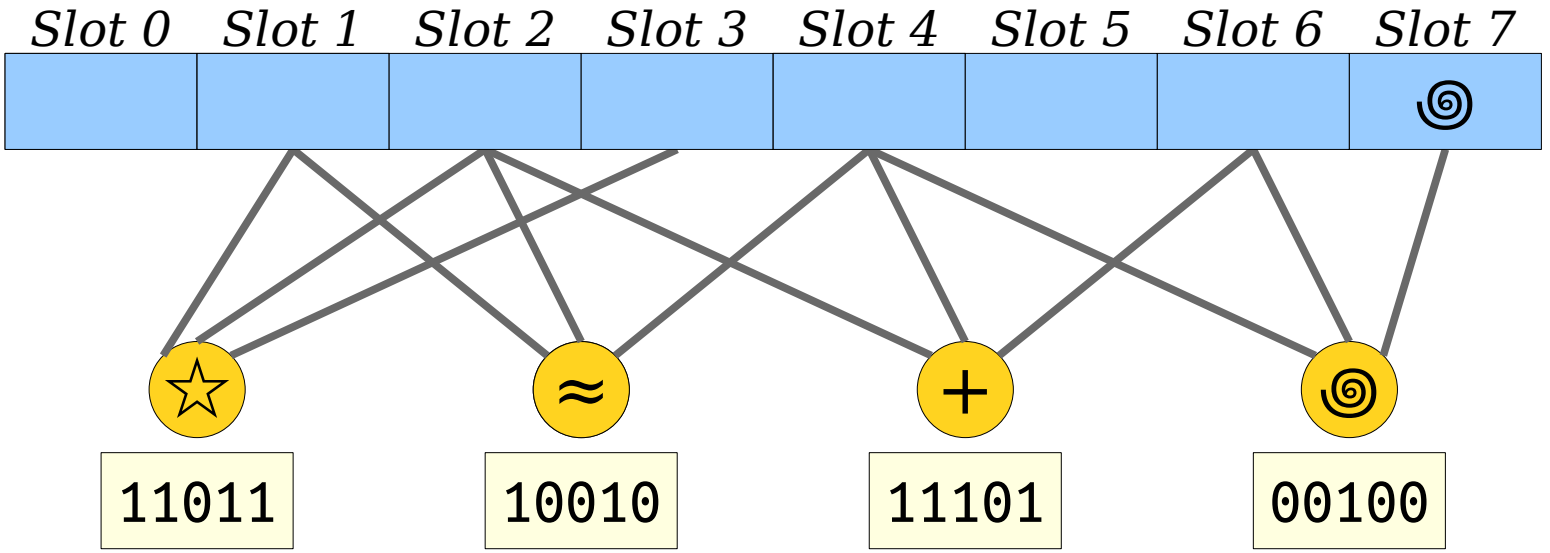
- ***Claim:*** Regardless of the values of the other slots, we can set slot 7's value so that T[4] ⊕ T[6] ⊕ T[7] = `00100`.
- Specifically, set T[7] = T[4] ⊕ T[6] ⊕ `00100`.

$$\begin{aligned} T[4] \oplus T[6] \oplus T[7] &= T[4] \oplus T[6] \oplus (T[4] \oplus T[6] \oplus \texttt{00100}) \\ &= (T[4] \oplus T[4]) \oplus (T[6] \oplus T[6]) \oplus \texttt{00100} \\ &= \texttt{00100}. \end{aligned}$$



Filling Our Table

- Because we've "claimed" slot 7 for 🌀, we can worry about tuning slot 7 for 🌀 after we tune all the other slots.
- ***Idea:*** Remove 🌀 from the set of items to place. Recursively place everything else, then assign slot 7 so 🌀's XORs pan out.



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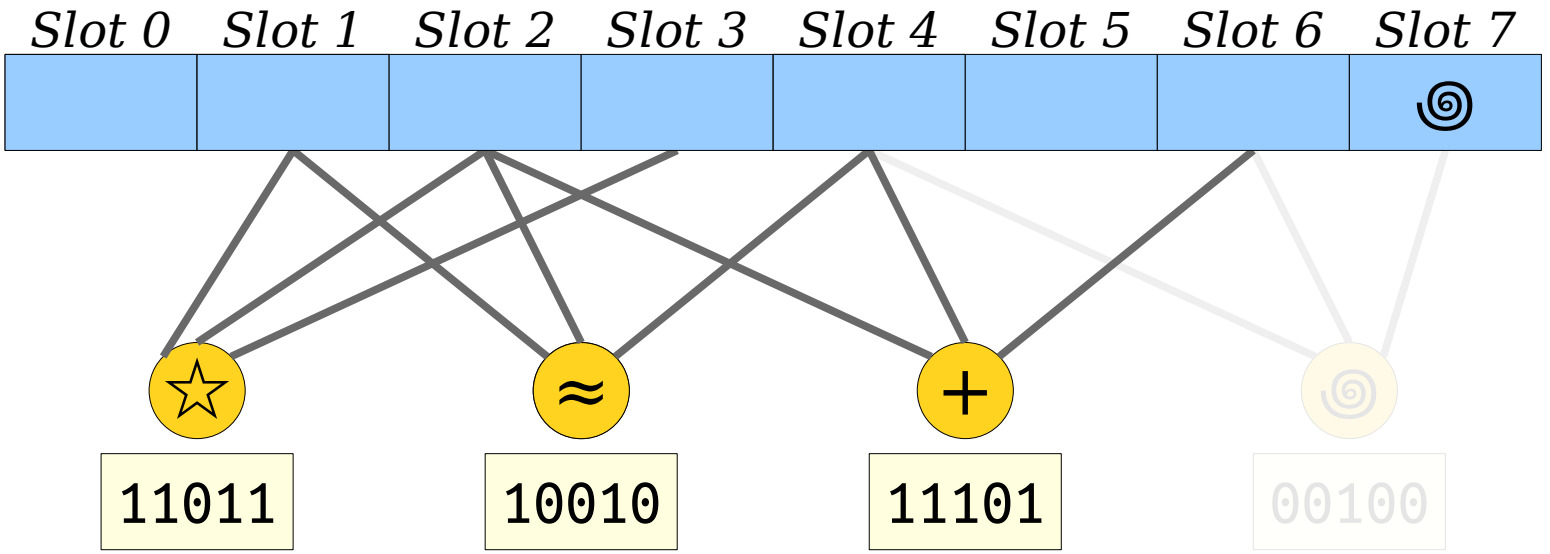
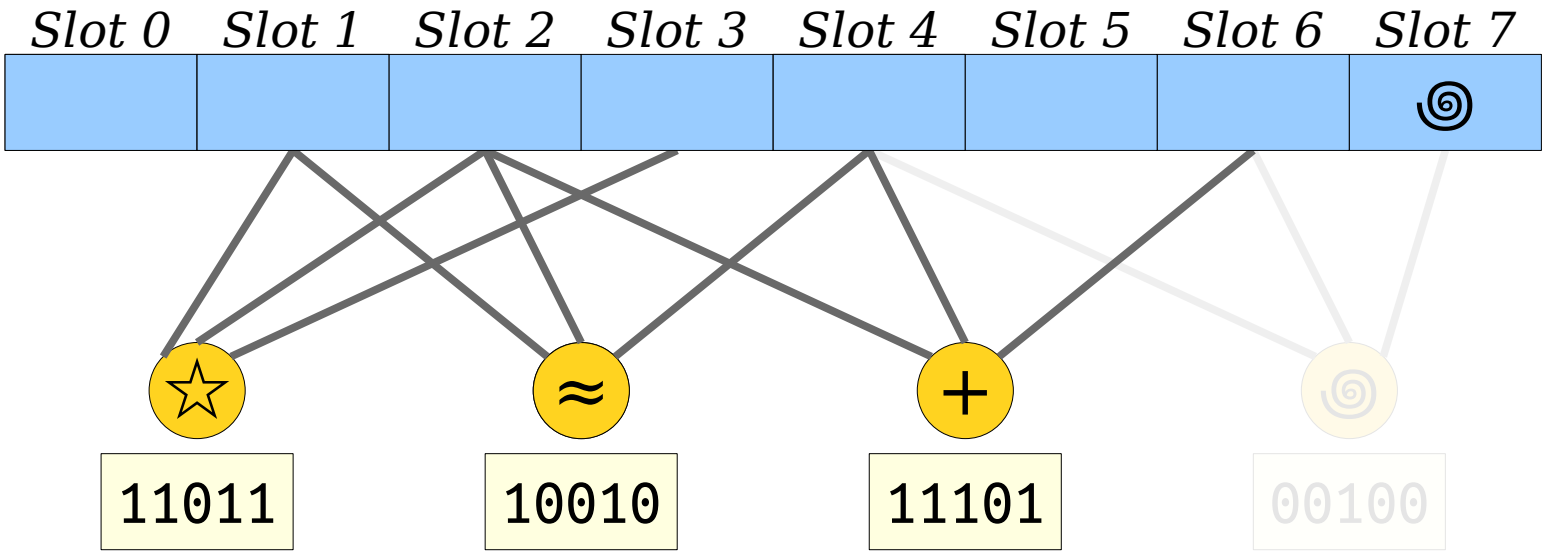


Filling Our Table

- After removing 🌀, notice that + is now the only item that hashes to slot 6.
- As before, we'll "assign" + to that slot. Regardless of what goes in slot 2 or slot 4, we can always tune slot 6 so everything XORs out appropriately.



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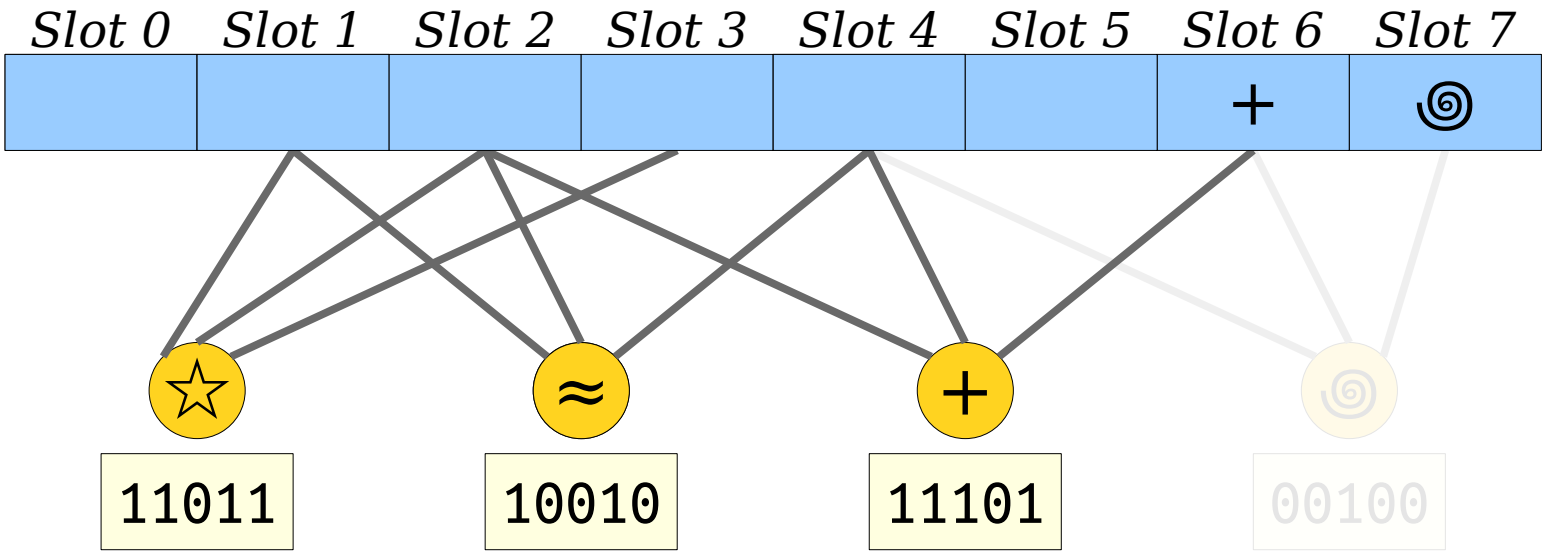

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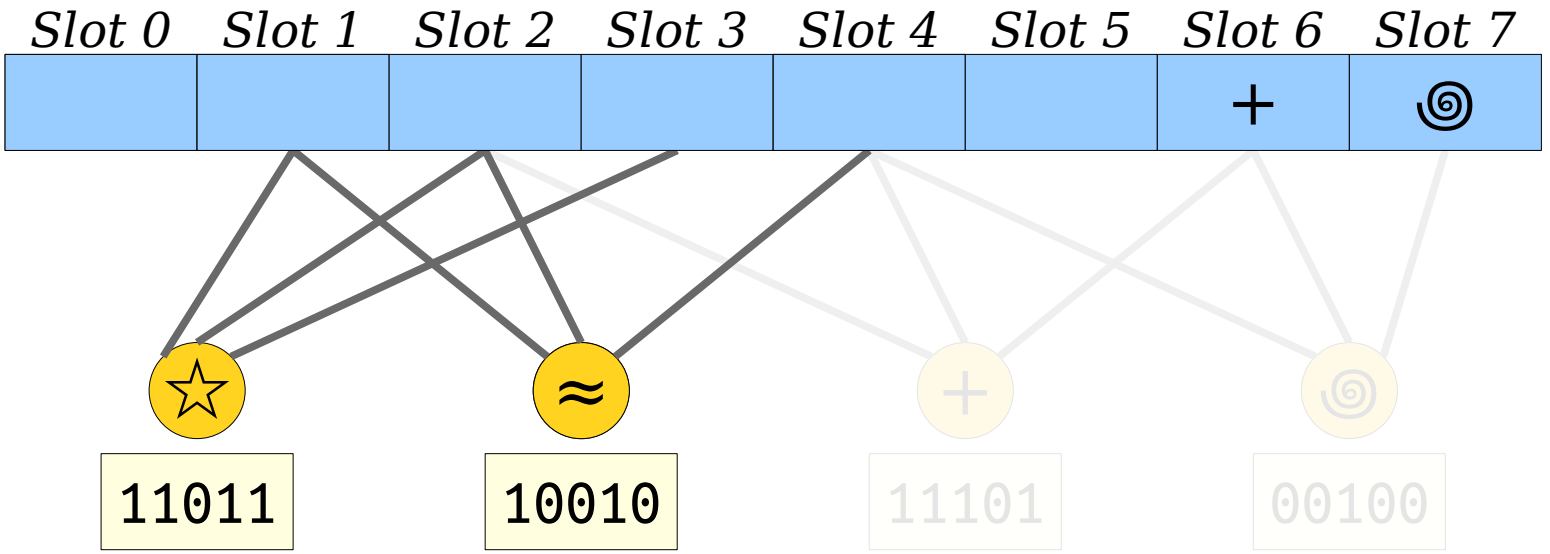

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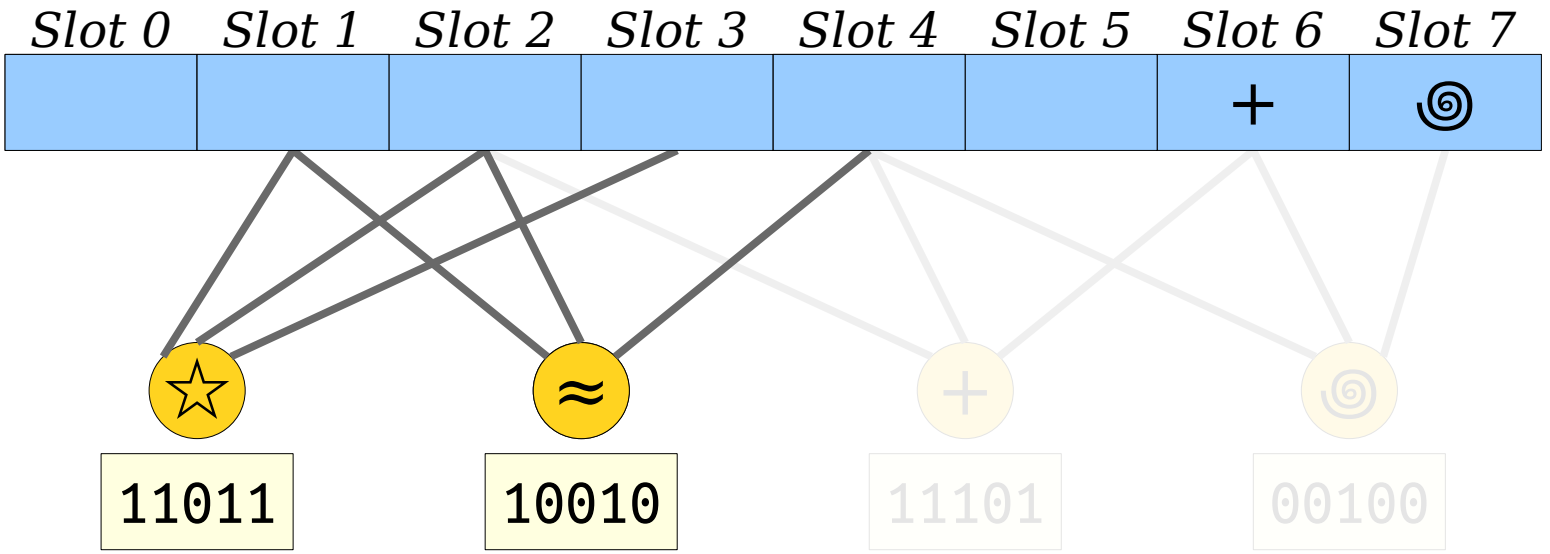
- Now, ☆ has sole ownership of slot 3, so we can assign it there.
- As before, once everything else is placed, we can tune slot 3 to make everything XOR out properly.



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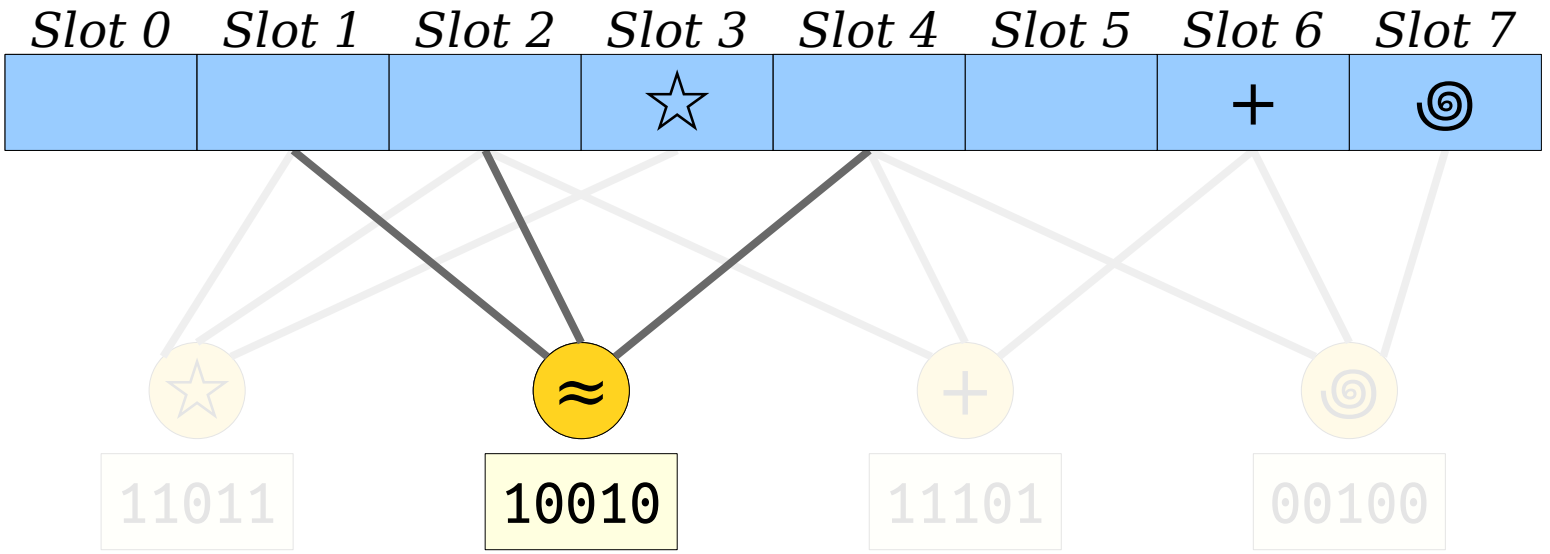
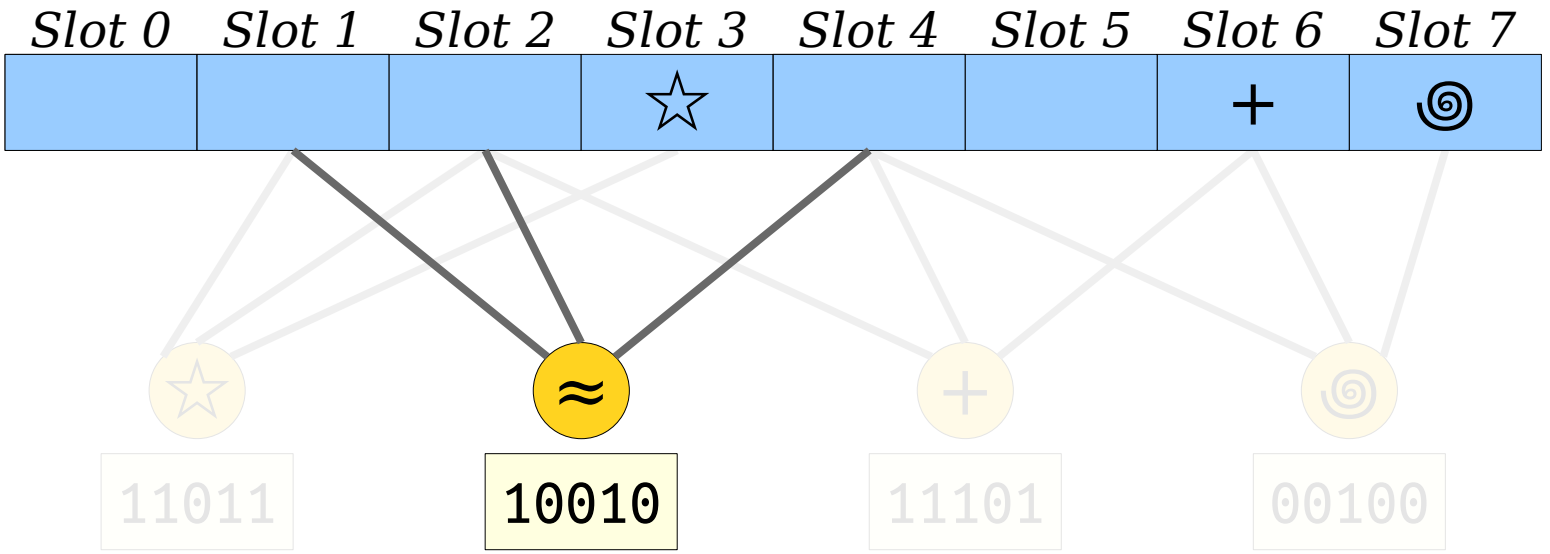


Filling Our Table

- We're left with just ≈ and no other items to place. We can set the values of these slots however we'd like, as long as they XOR to ≈'s fingerprint (`10010`).
- A simple option: Put `10010` in one of them and zeros in the others.



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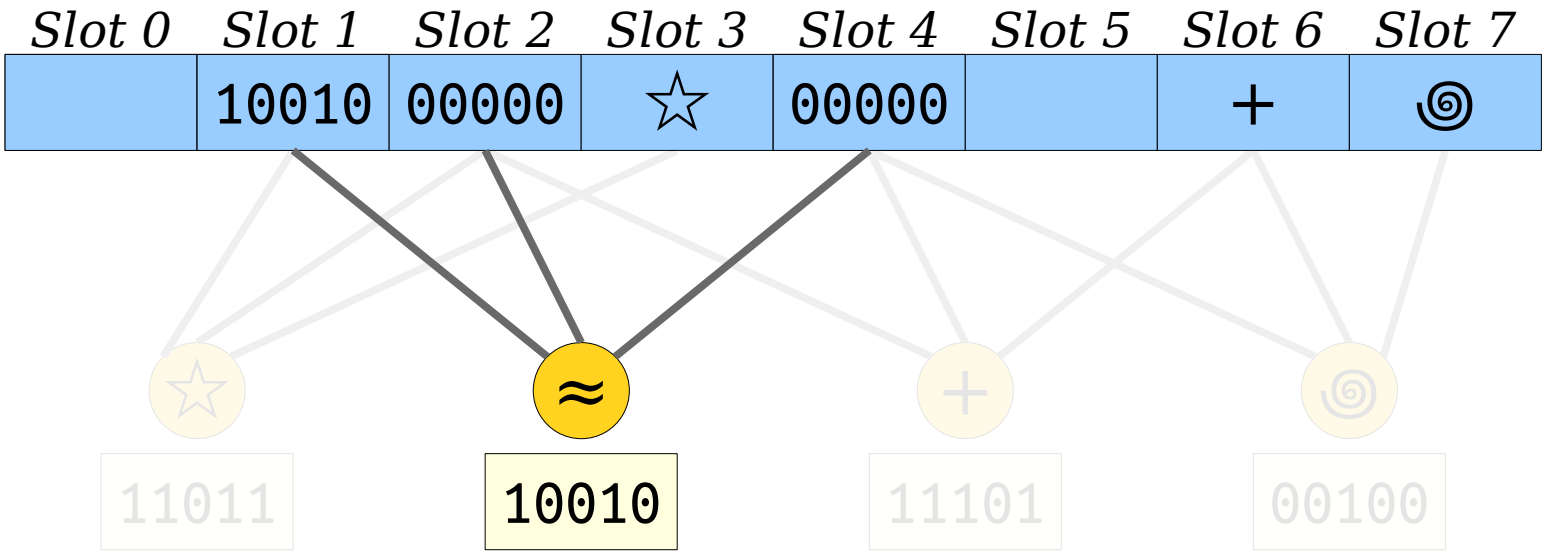
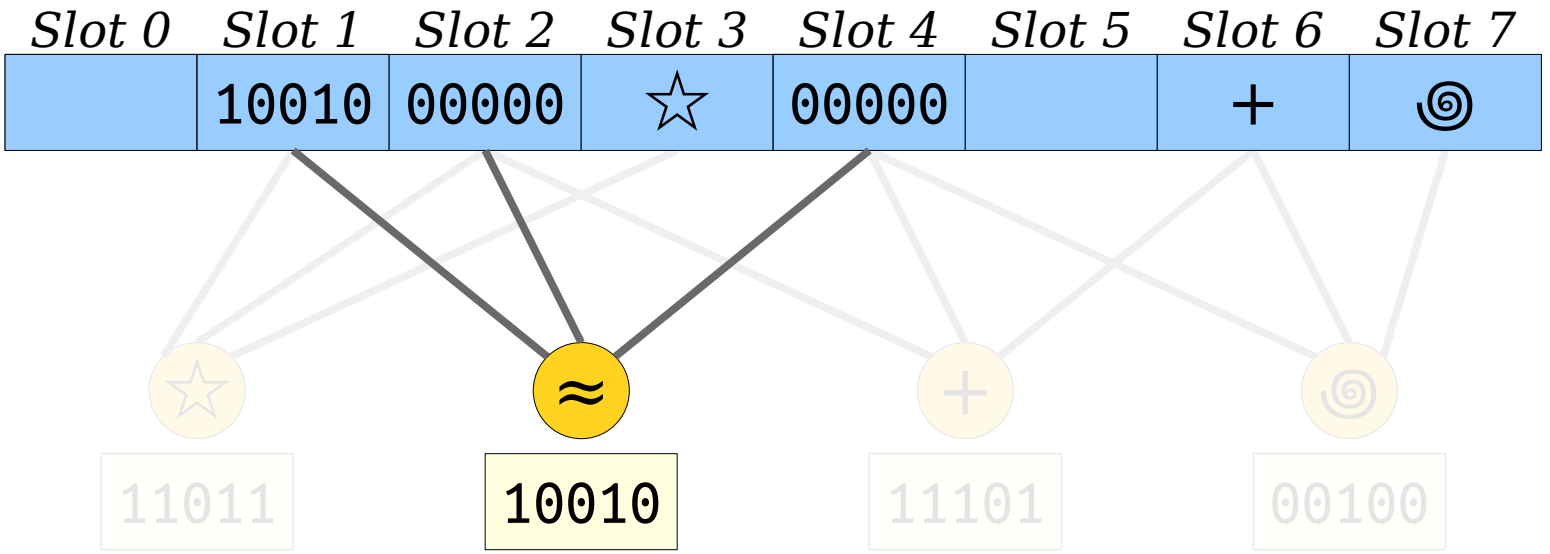


Filling Our Table

- The last item we removed was ☆, so let's add that back in.



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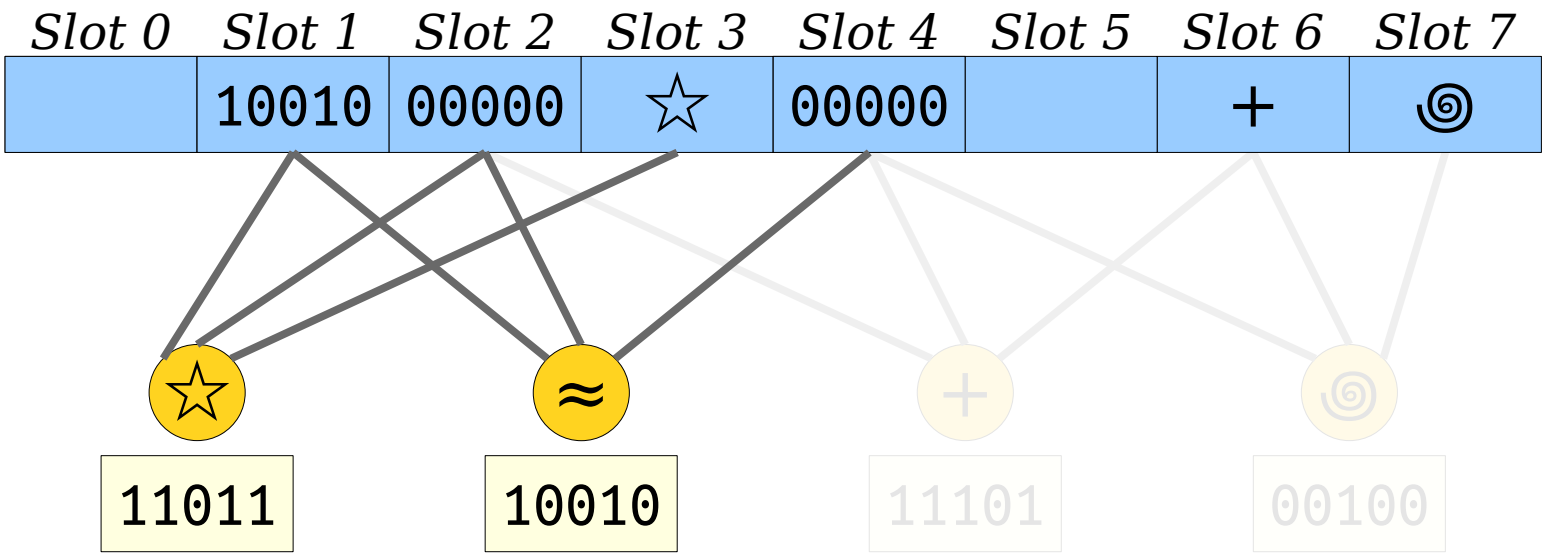

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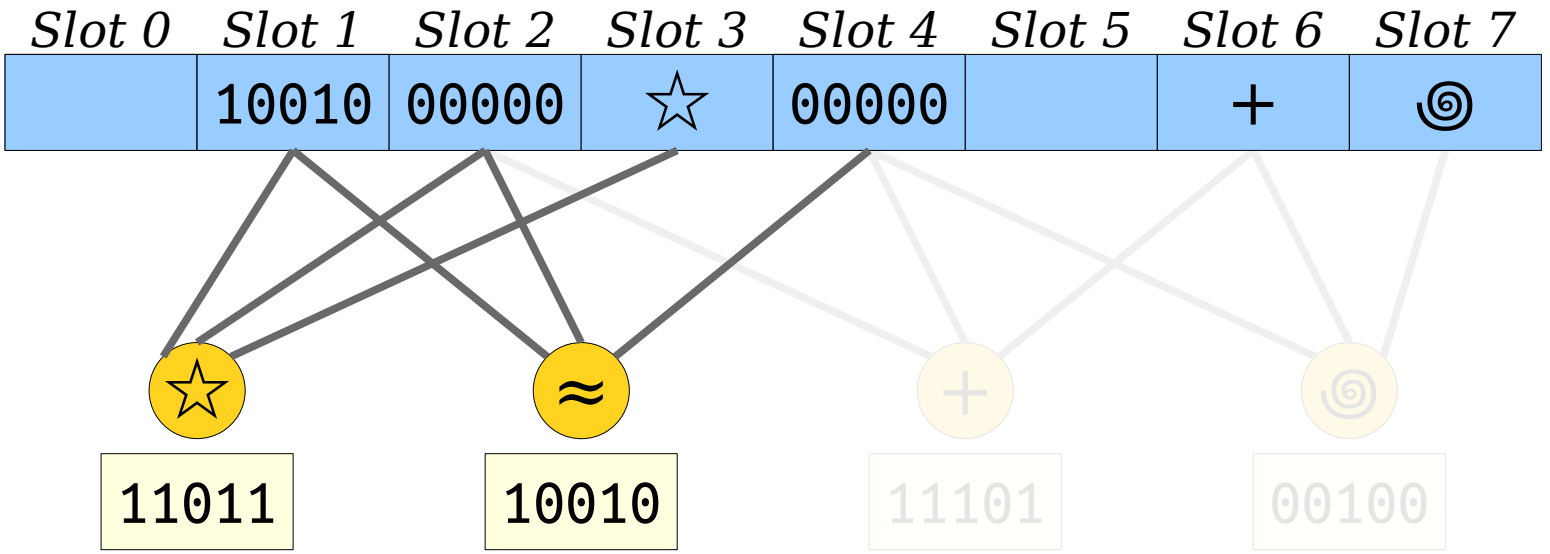
- We now set slot 3 to the XOR of ☆'s fingerprint (11011) and the values in its other two slots.



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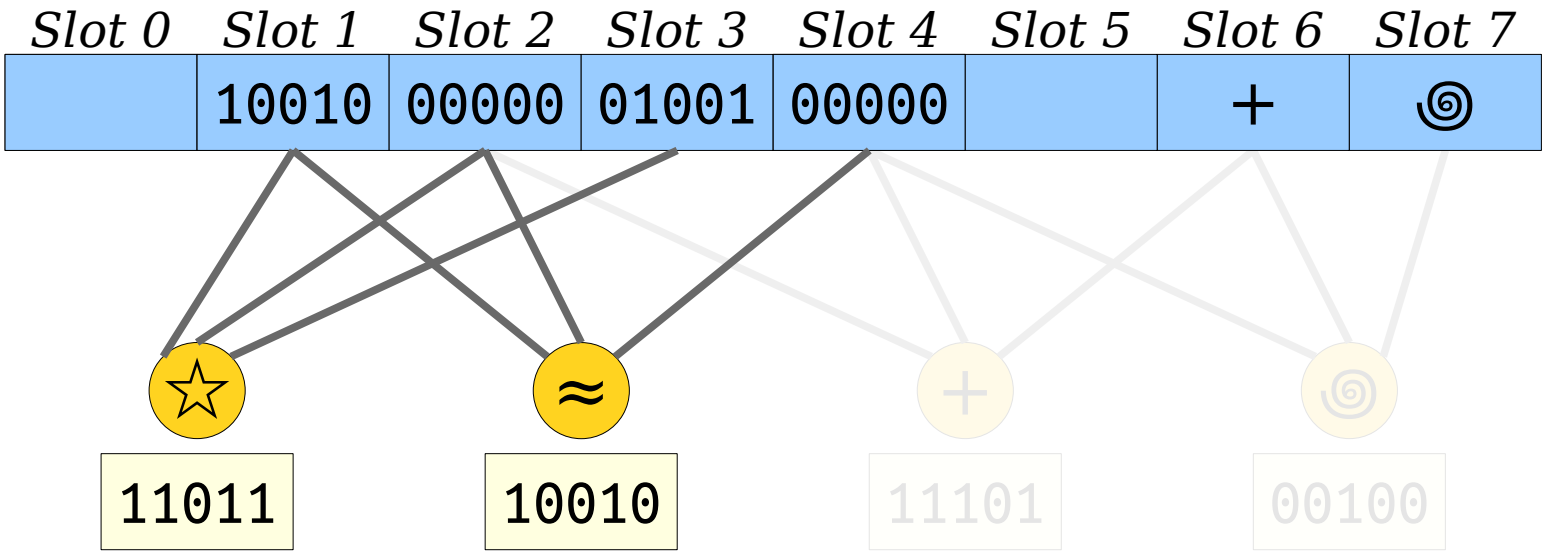
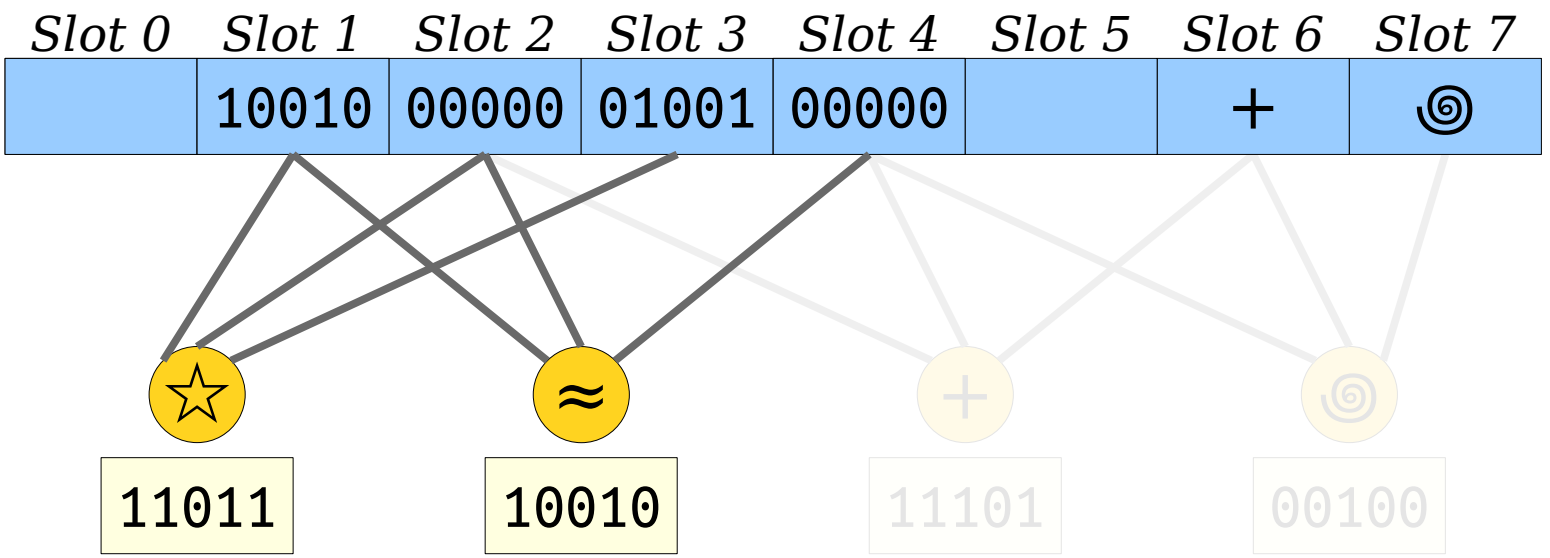


Filling Our Table

- Let’s add the + back in now.



Filling Our Table

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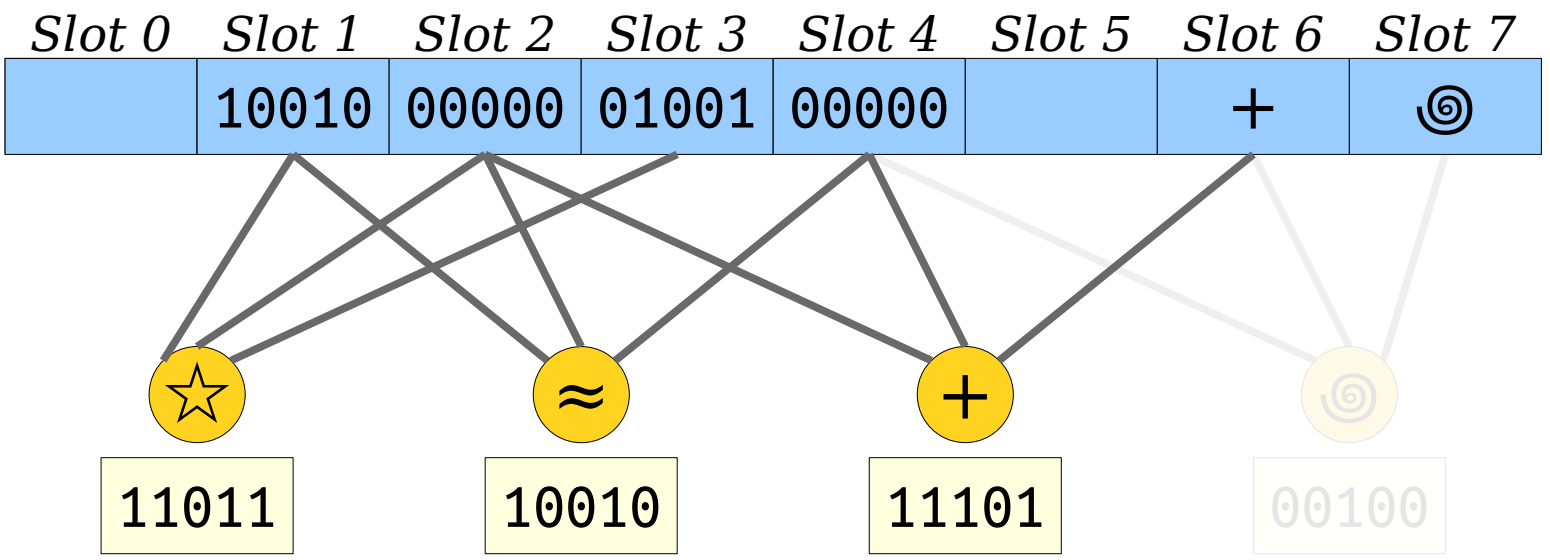

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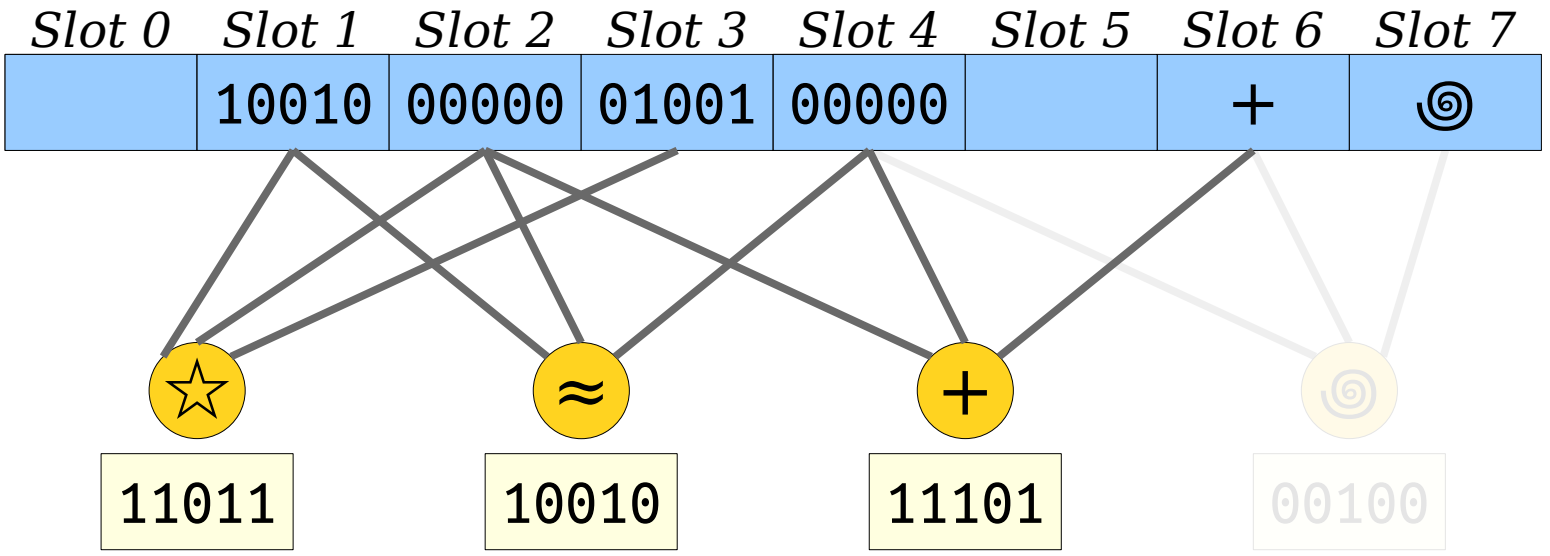

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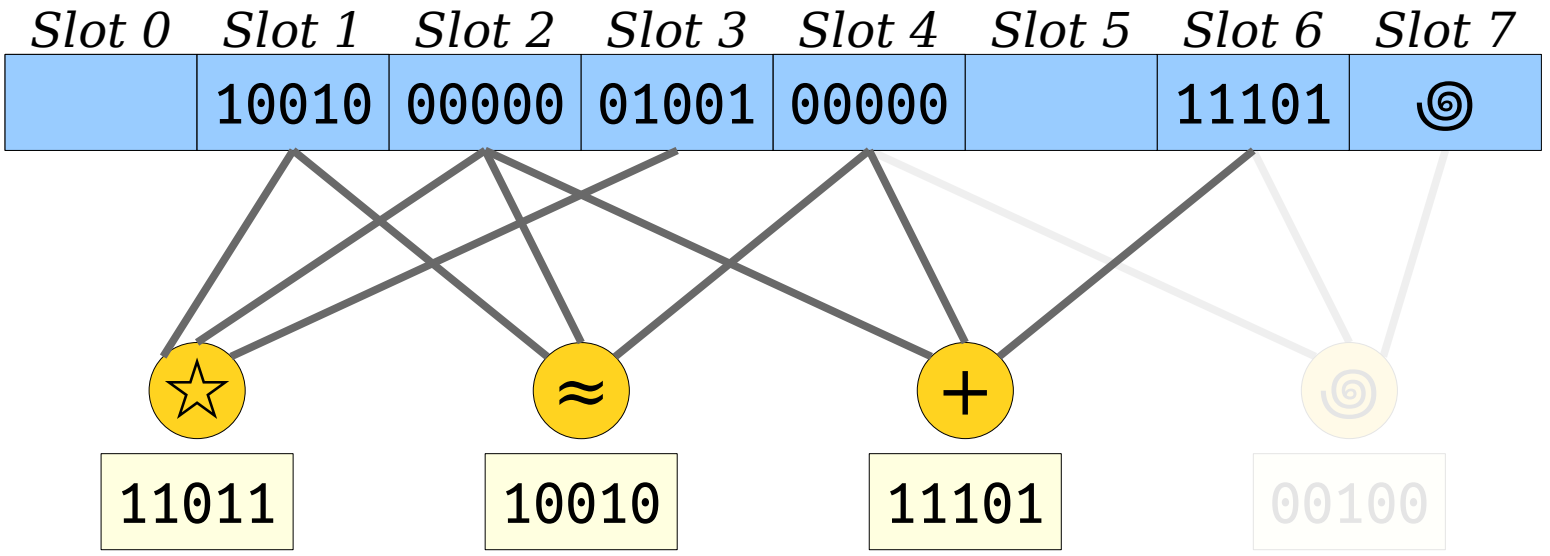

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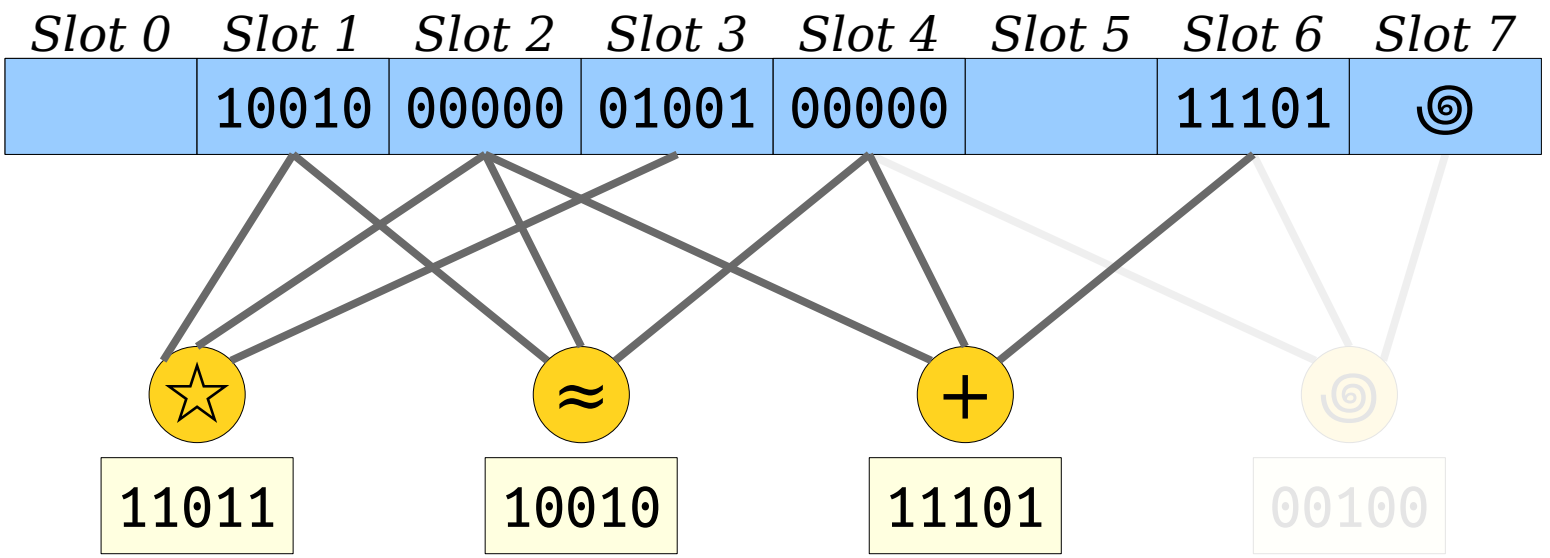

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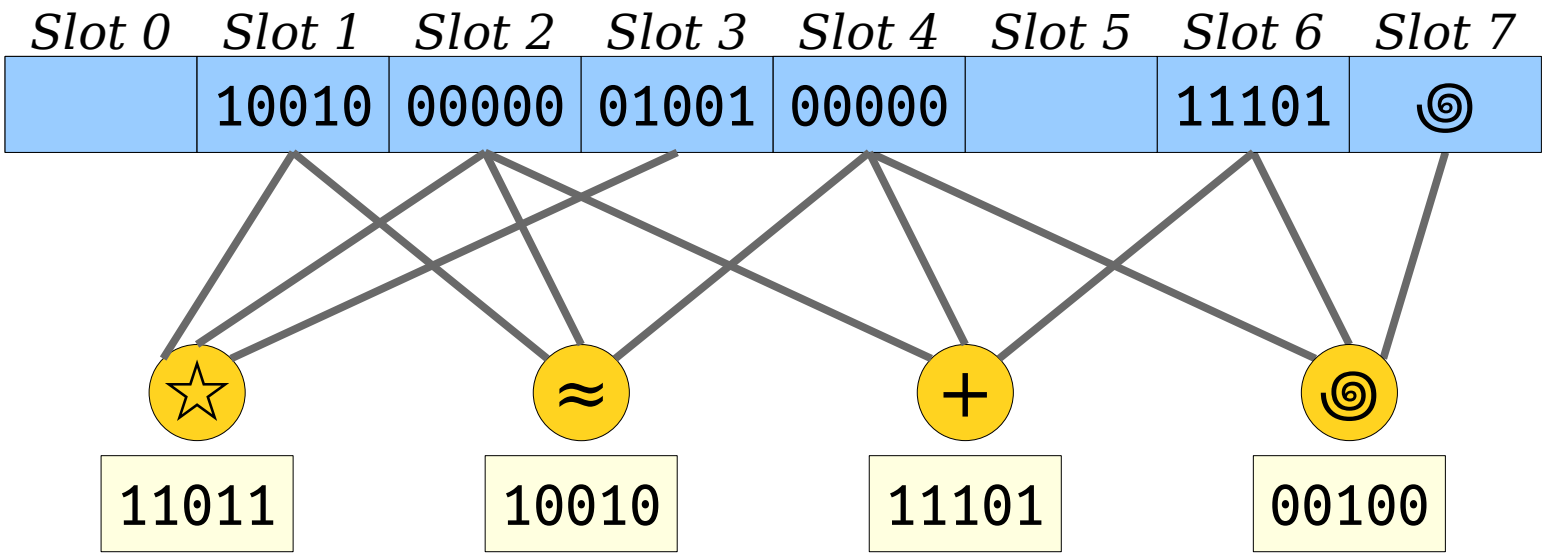

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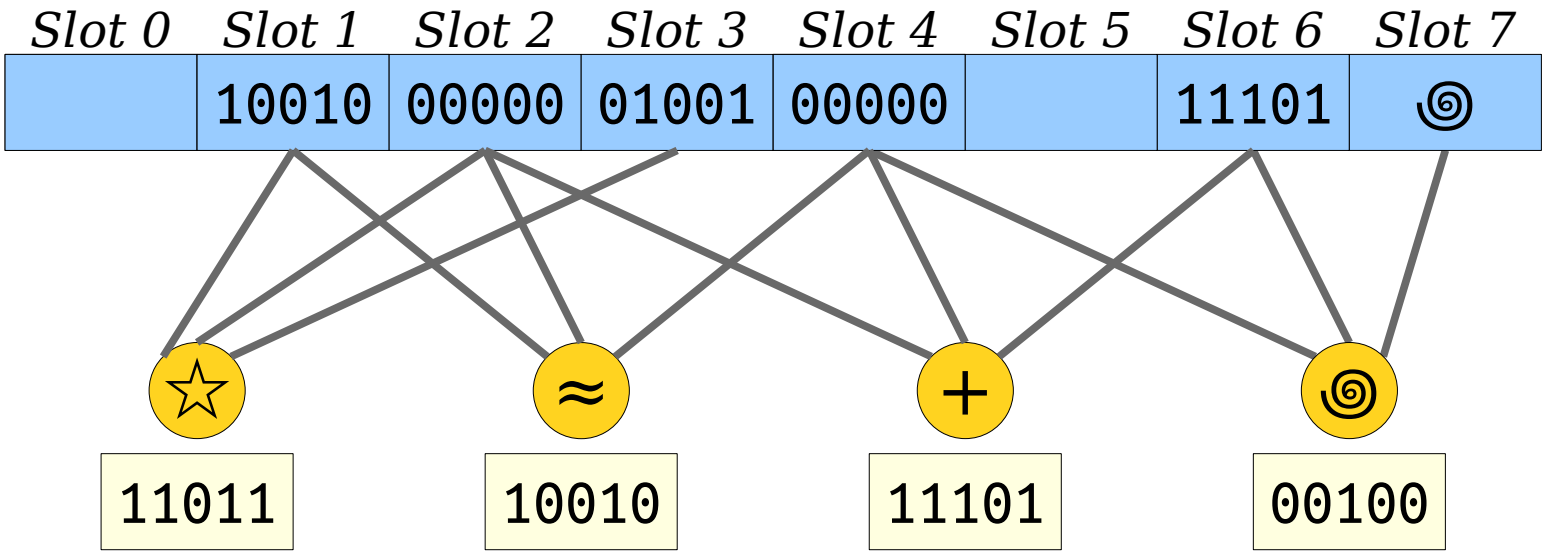

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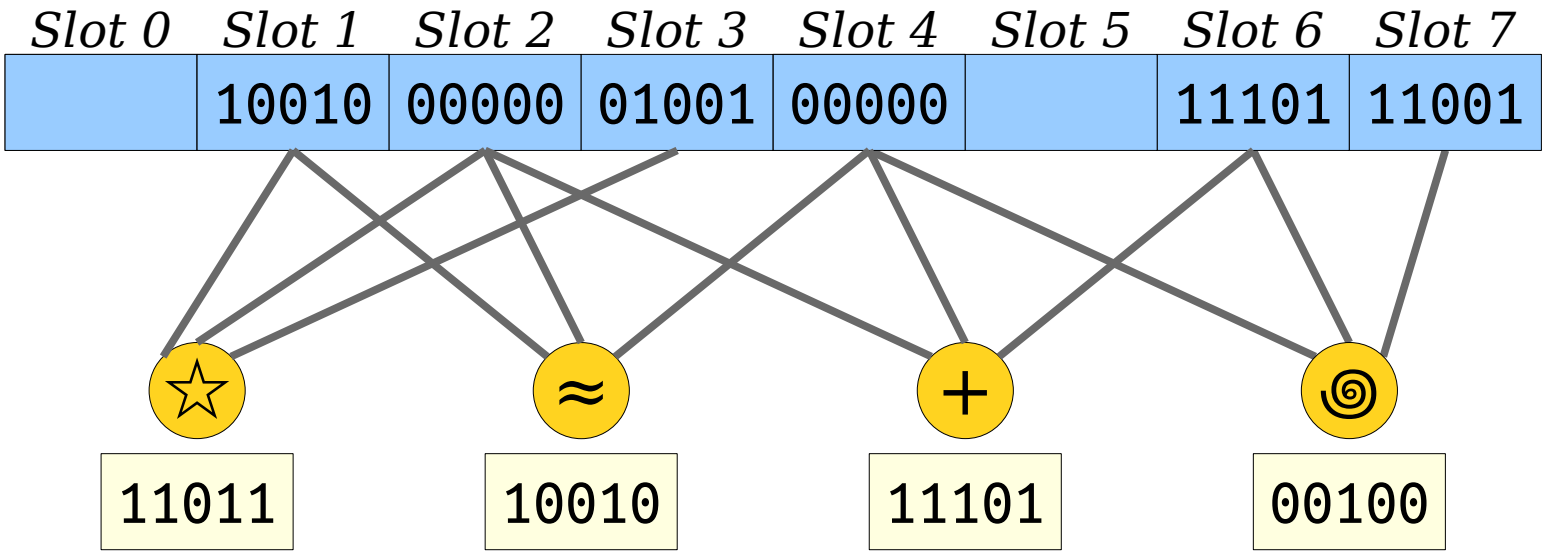

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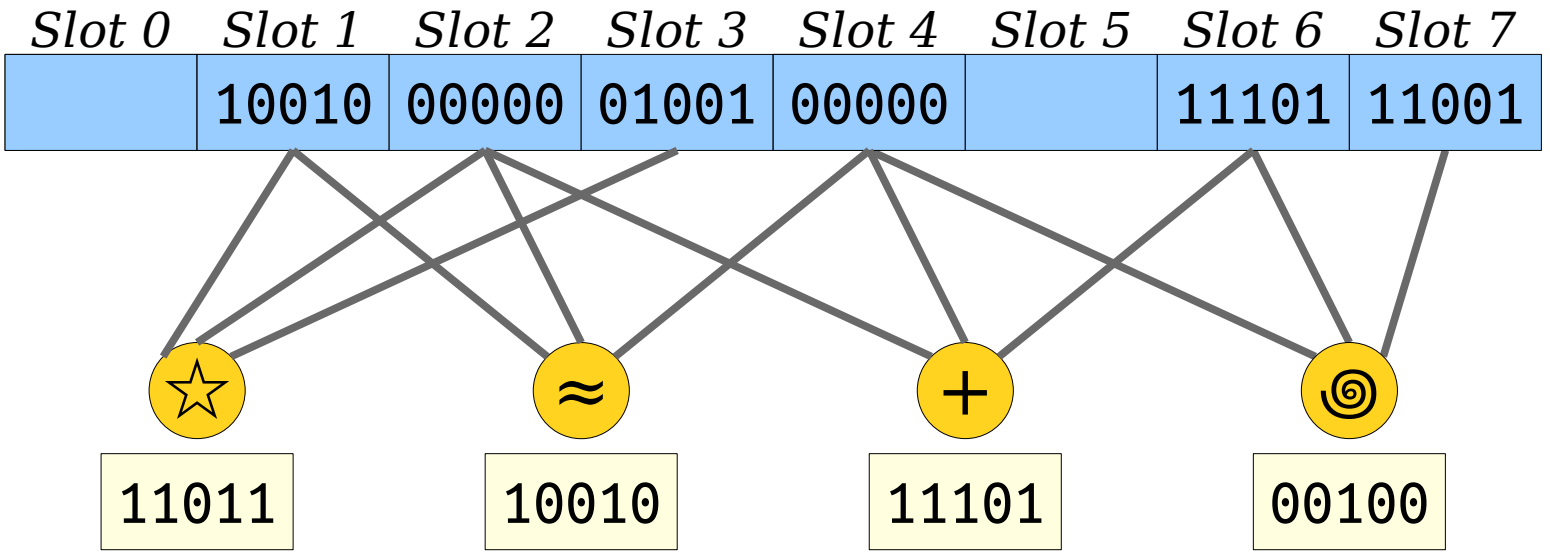

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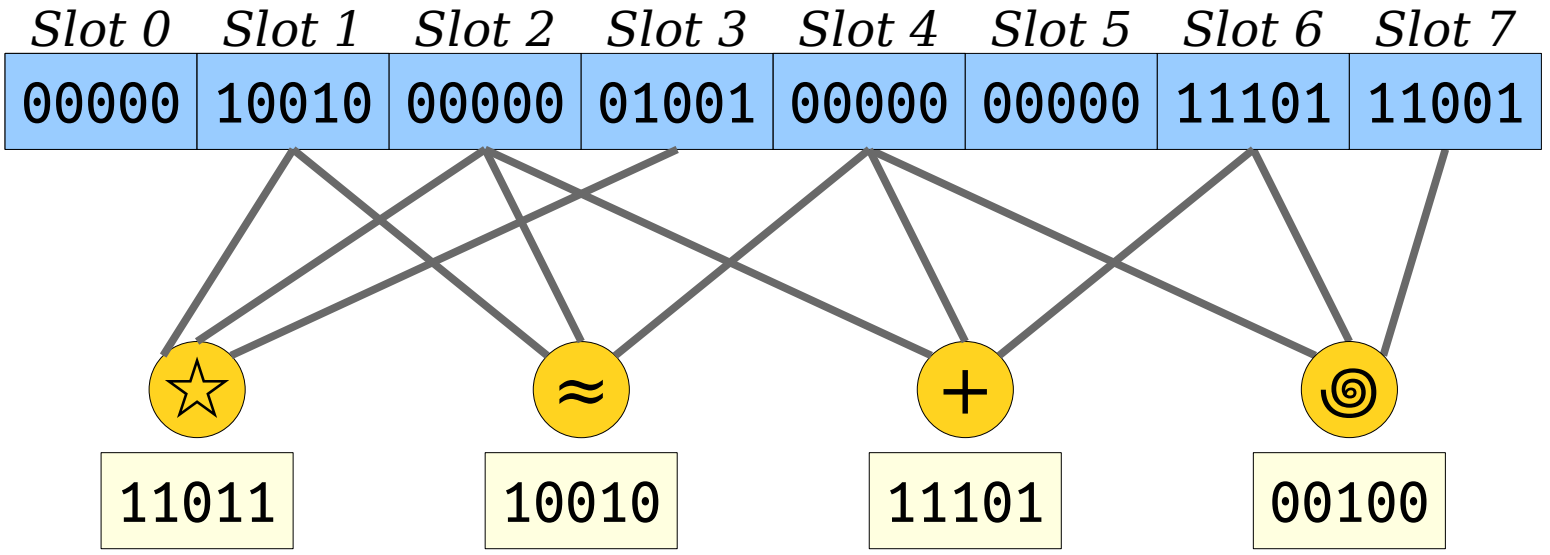
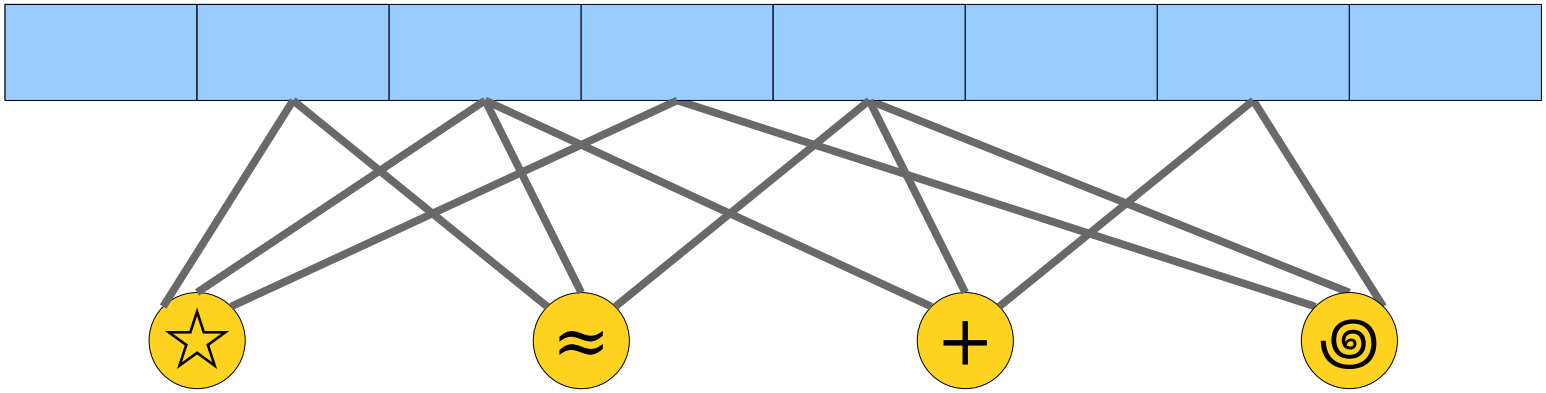


The Peeling Algorithm

- Initialize the table to whatever values you'd like.
- If there are no items to place, there's nothing to do.
- Otherwise:
  - Find a ***peelable item*** x, one that hashes to a slot nothing else hashes to.
  - Recursively fill in the table to place the remaining items.
  - Add x back in, setting the value in the unique slot to the XOR of x's other slots and its fingerprint.
- With the right data structures, this can be implemented in time $O(nd + m)$, where n is the number of items, d the number of hashes, and m the number of slots.

The Peeling Algorithm

- The peeling algorithm can get stuck if there are no peelable items.
- If that happens, the peeling *fails* and we try again using a different set of hash functions.
- ***Question:*** How likely is the peeling algorithm to succeed?



The Peeling Algorithm

- Suppose we are given n items to place in the table. There are two knobs we can turn:
  - We can change m, the number of table slots.
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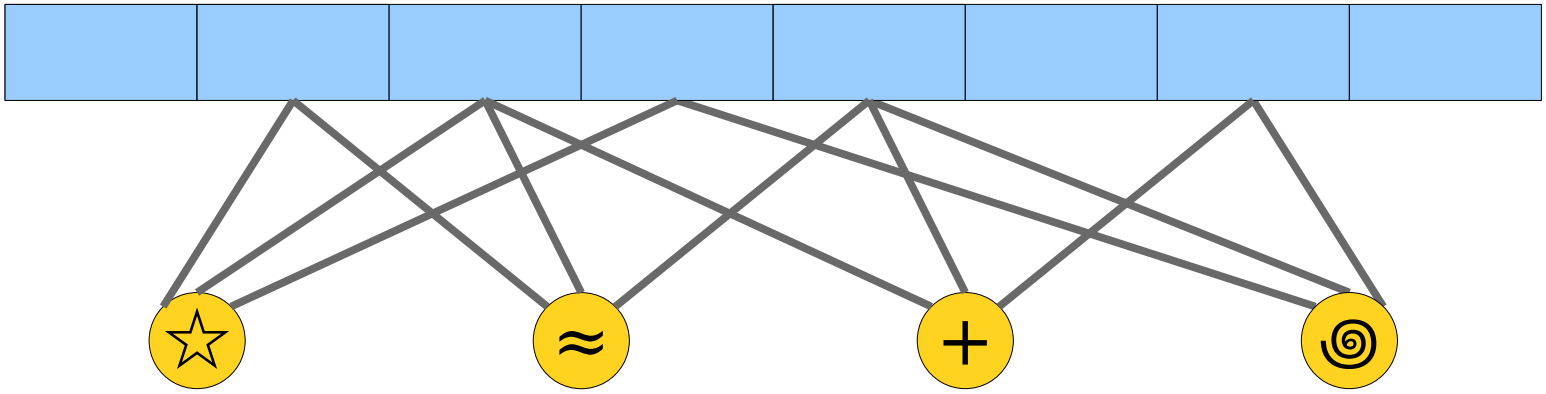
- Intuitively, how does m impact the probability, and how does d impact the probability?



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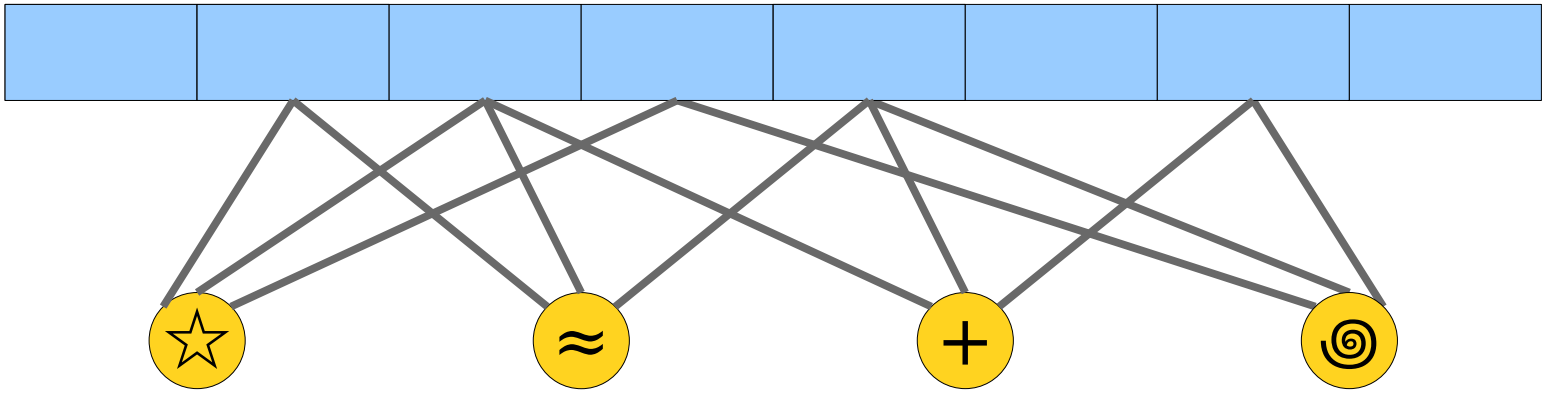
Formulate a hypothesis, but ***don't post anything in chat just yet***.



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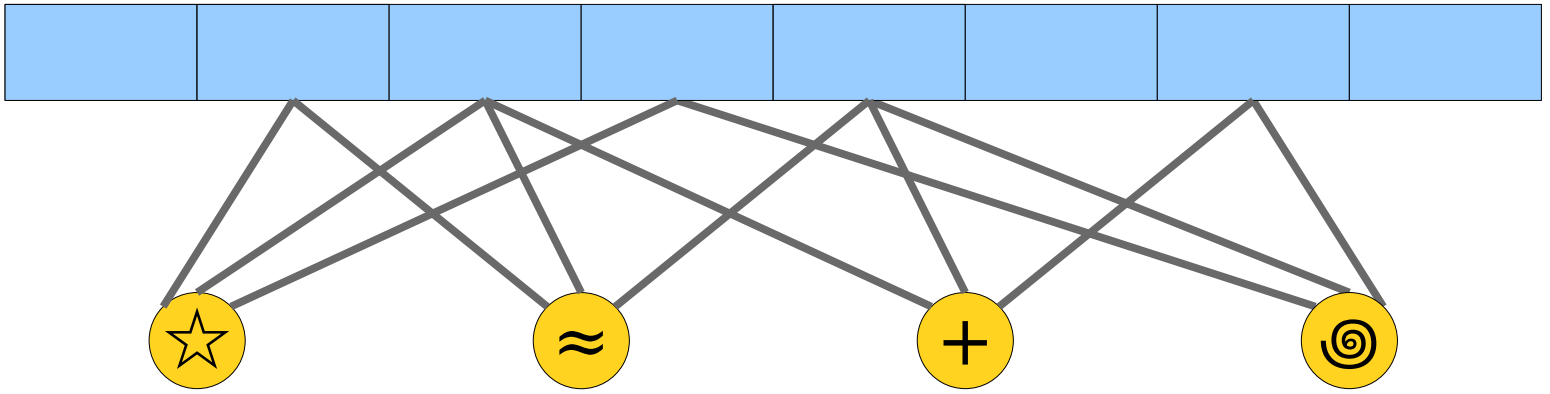
Now, ***private chat me your best guess***. Not sure? Just answer "??."



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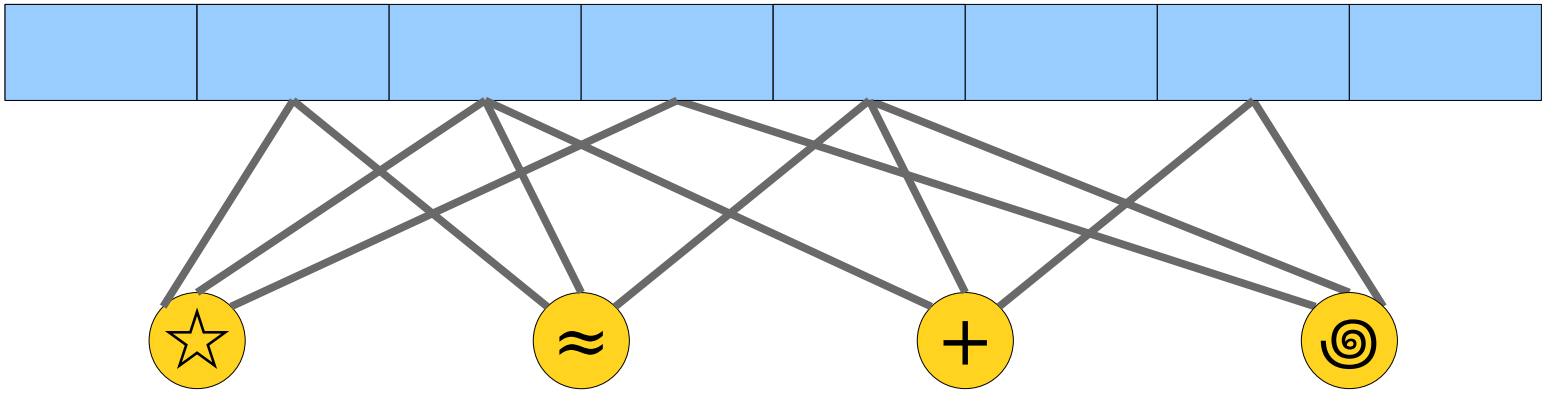
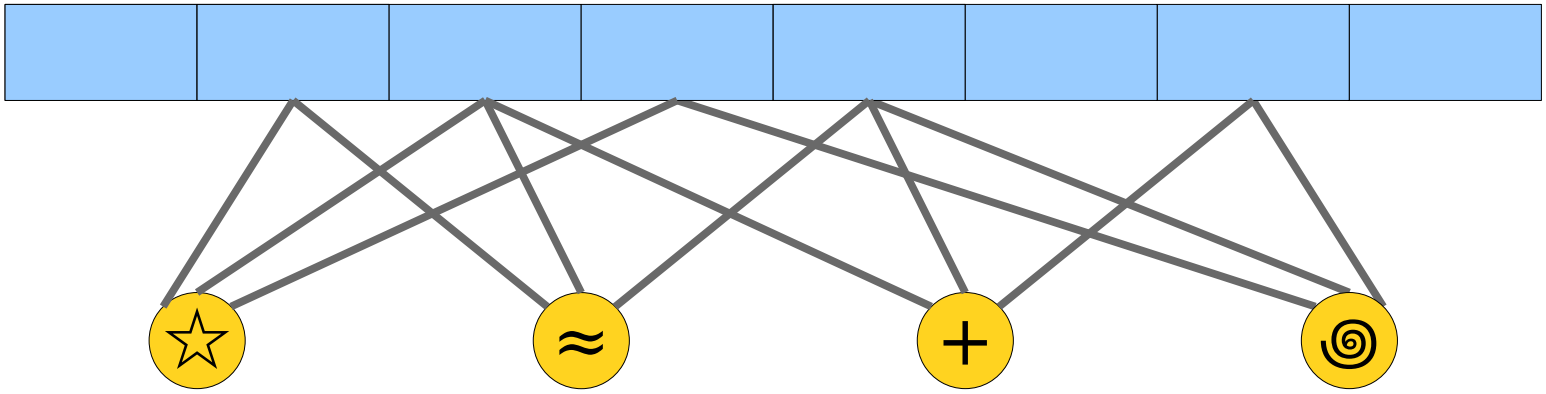


The Peeling Algorithm

- Increasing m (number of table slots) should increase our probability of the peeling algorithm succeding.
  - The more table slots we have, the fewer items we expect to land in each slot.
- Increasing m also increases our memory usage.
  - Ideally, we'd like to keep m as close to n as possible.
- As with cuckoo hashing, let's set $n = \alpha m$, for some $\alpha \leq 1$. We'll need to explore what this choice of α does.



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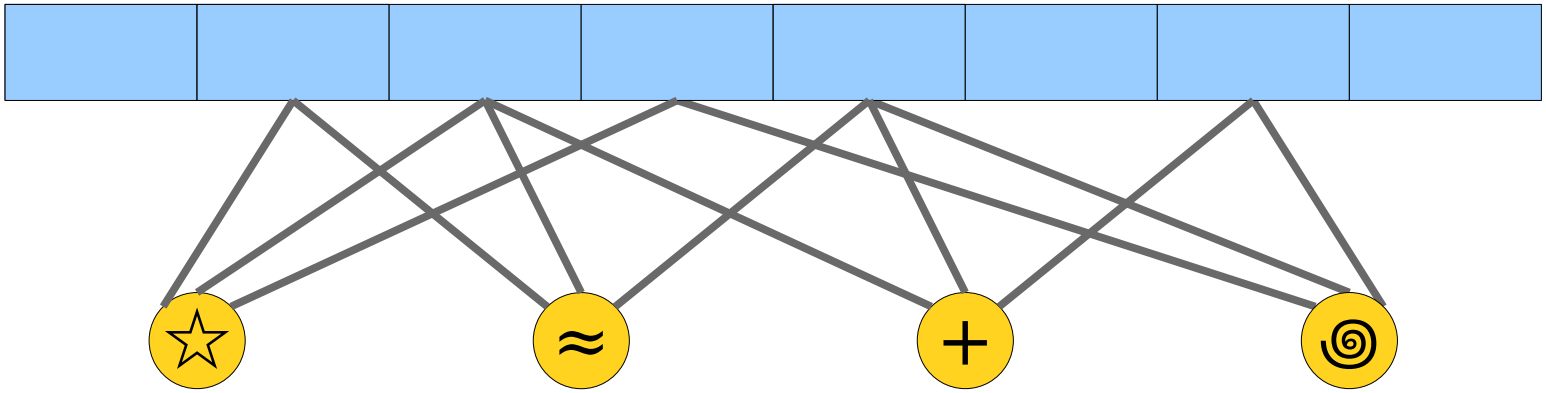

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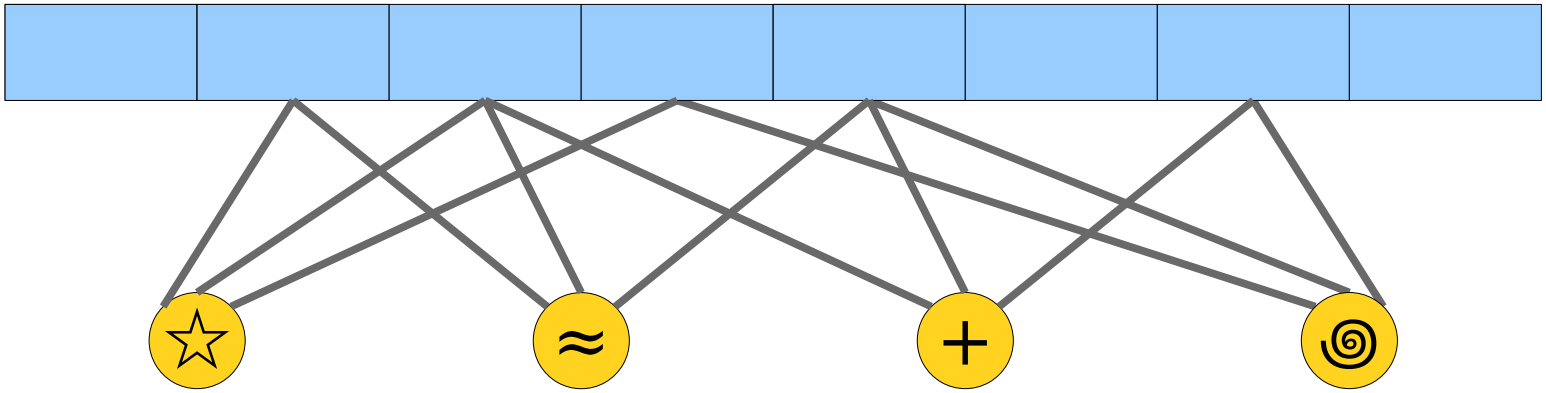

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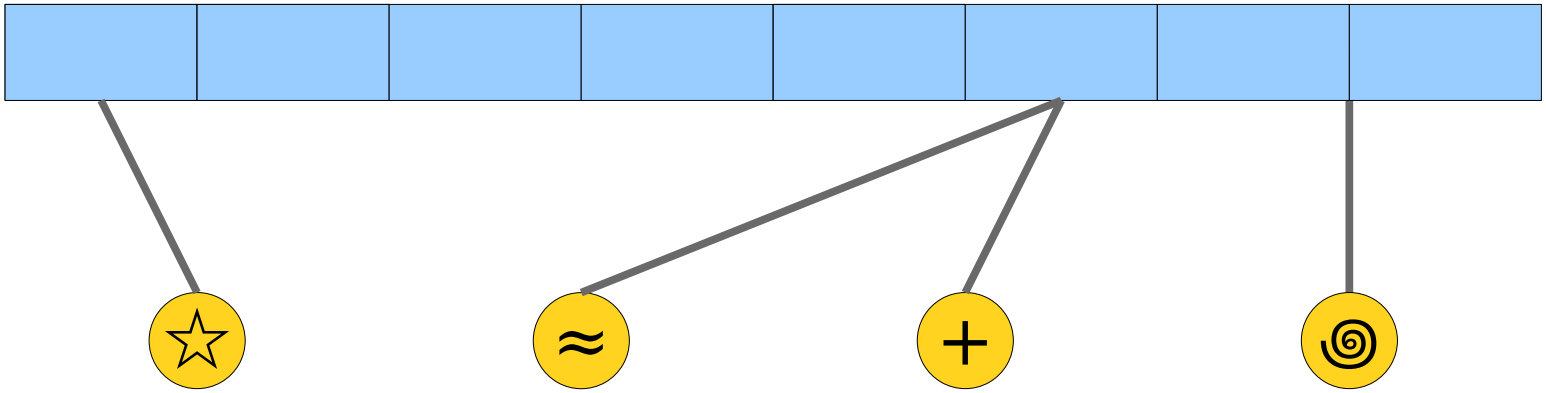

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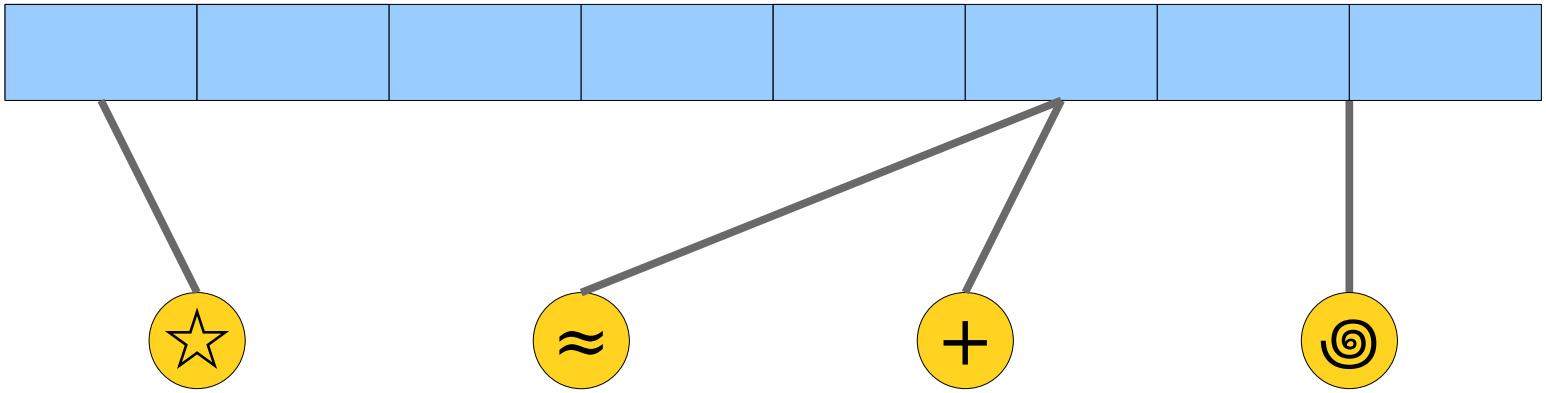

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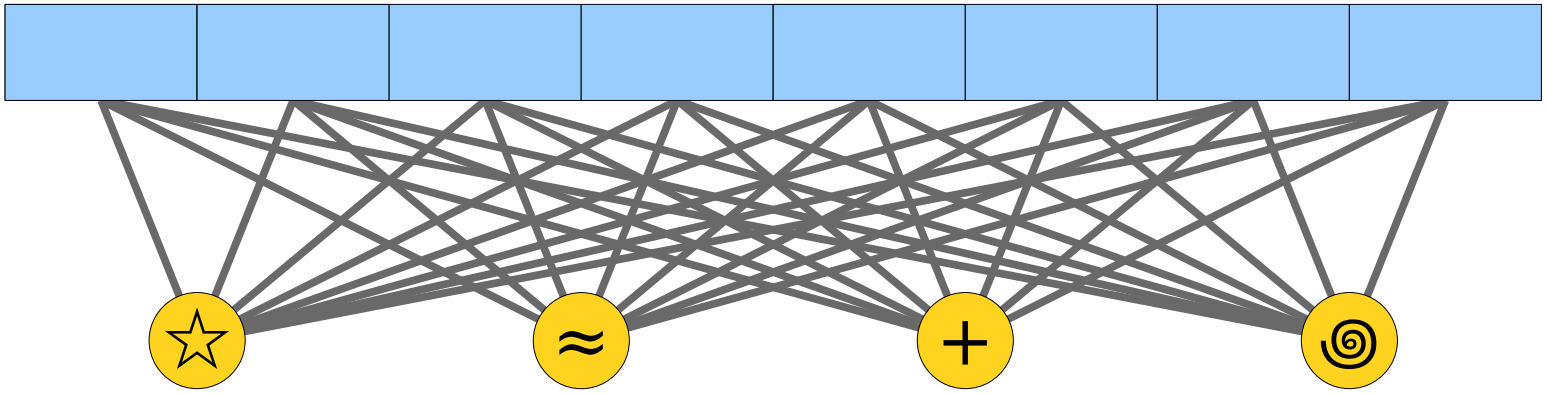

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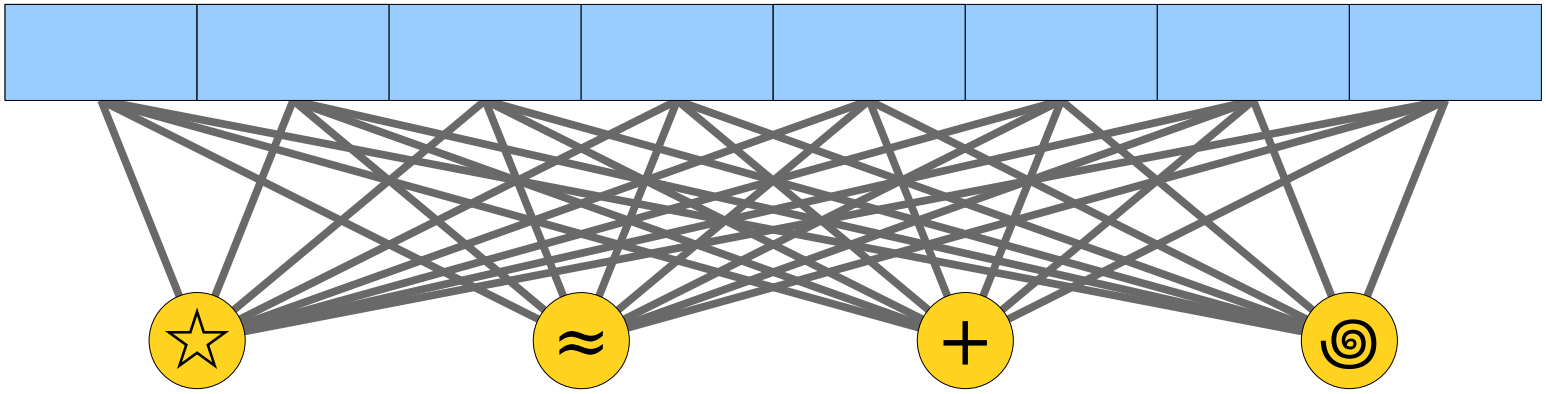
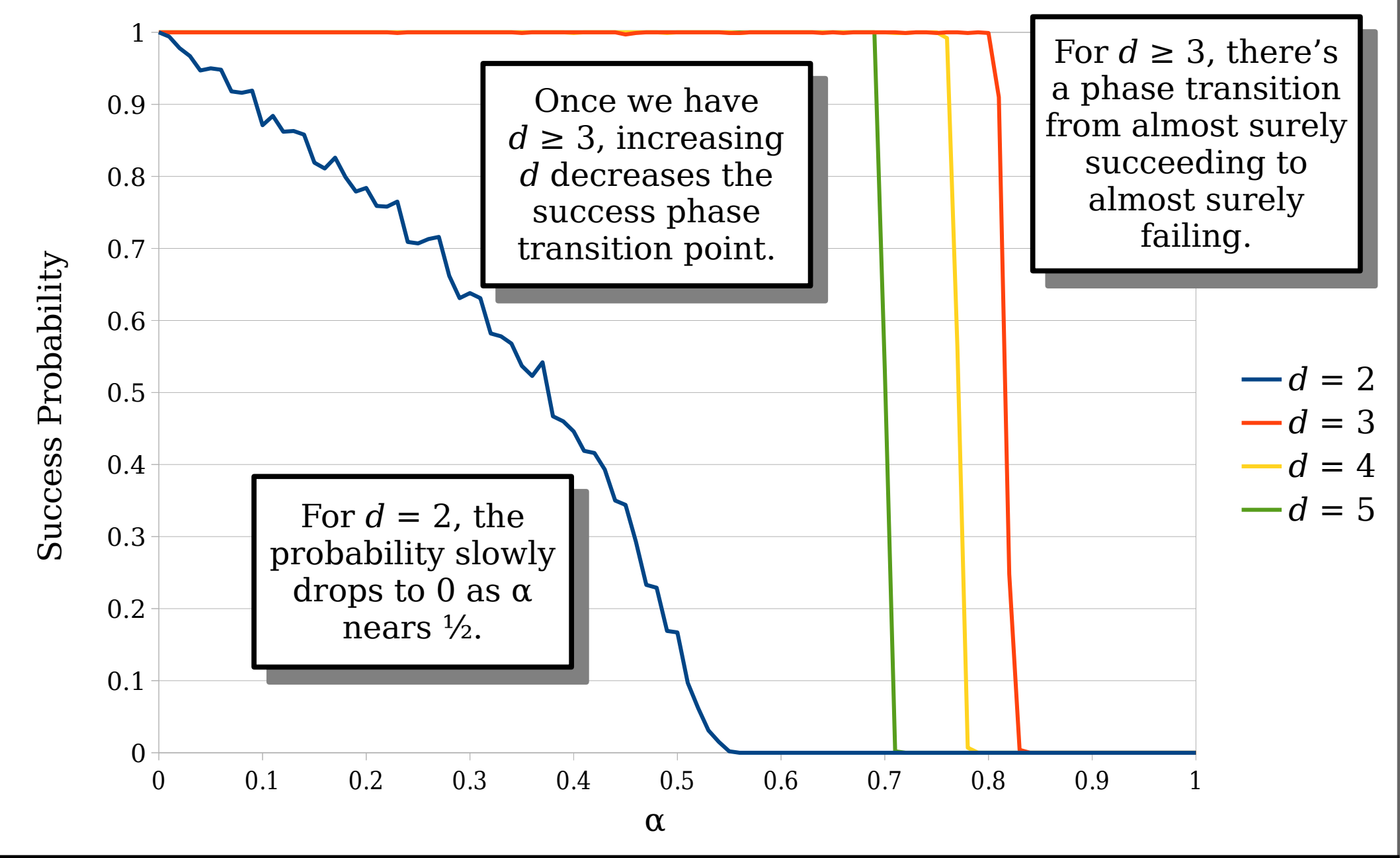
  - If d is too large (say, $d = m$), then there will be too many items hashing to each slot and nothing will be peelable.
- Like the choice of d for Bloom filters, there's probably some optimal choice that's not too big and not too small.





Create a table of m slots and place $n = \alpha m$ items into it.
Each item has d hashes.
What is the probability that the peeling algorithm succeeds?

Peelable Graphs

- ***Theorem:*** For $d = 2$, the success probability gradually drops to 0 as α approaches ½.
- ***Theorem:*** For $d > 3$, there's an abrupt phase transition at some point. Empirically…
  - … for $d = 3$, we have $\alpha_{max} \approx 0.81$.
  - … for $d = 4$, we have $\alpha_{max} \approx 0.77$.
  - … for $d = 5$, we have $\alpha_{max} \approx 0.70$.
- Therefore, we'll use $\boldsymbol{d = 3}$ hash functions per item stored. With $\alpha \approx 0.81$, a table with n elements needs around $\boldsymbol{1.23n}$ slots.

The XOR Filter

- Putting all the pieces together, here's our construction algorithm:
  - Create an array of **1.23*n*** slots, each of which holds a $(\lg \varepsilon^{-1})$-bit number.
  - Choose ***three*** random hash functions h_1, h_2, and h_3 from items to slots, plus a fingerprinting function f from items to (**$\lg \varepsilon^{-1}$**)-bit hash codes.
  - Initialize the array using the peeling algorithm: find an item that is incident to a slot of degree one, remove it, recursively fill in the rest of the table, then place the item back and put the correct value in the final slot.
- Our query algorithm looks like this:
  - Return whether $T[h_1(x)] \oplus T[h_2(x)] \oplus T[h_3(x)] = f(x)$.

The Story So Far

- Our XOR filter is strictly better than a Bloom filter in terms of space usage, both practically and theoretically.
- It requires fewer hash functions whenever $\varepsilon < 1/16$.
- The query procedure has at most three cache misses.
- ***Question:*** Can we do better?

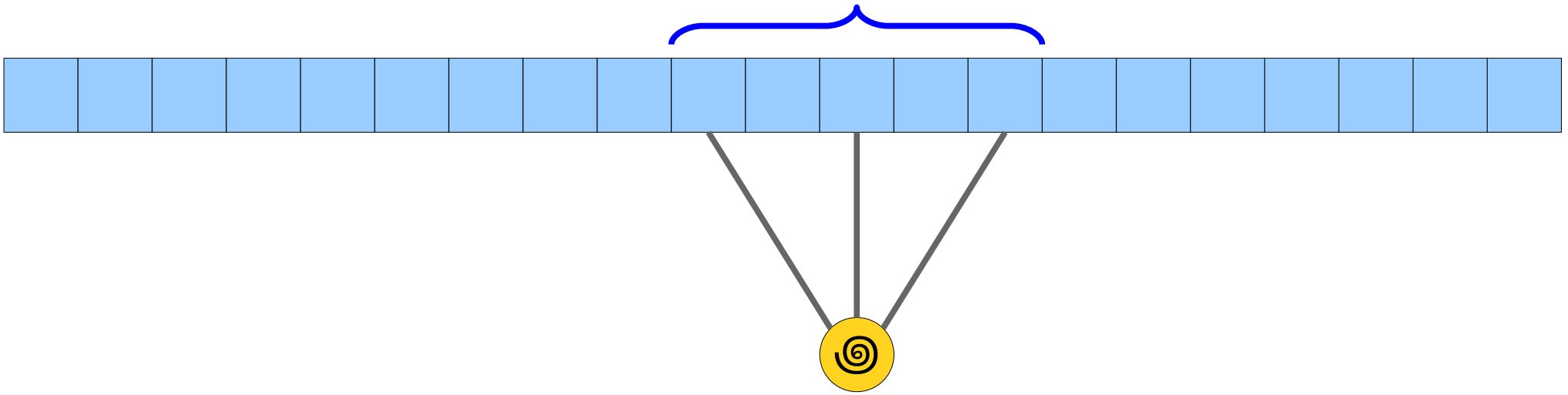
	Bits Per Element	Hashes/Query
Bloom Filter (1970)	$1.44 \lg \varepsilon^{-1}$	$\lg \varepsilon^{-1}$
Cuckoo Filter, $b = 4$ (2014)	$1.05 \lg \varepsilon^{-1} + 3.15$ *(for sufficiently small ε)*	3
XOR Filter (2020)	$1.23 \lg \varepsilon^{-1}$	4

Reducing Space

- The XOR filter uses $1.23n$ table slots to hold n items.
- ***Recall:*** This is because the peeling algorithm stops working for smaller tables.
- This is an inherent property of random graphs made from slots and items, not something particular about our implementation.
- ***Question:*** Can we reduce the space usage further than this?

Spatial Coupling

- ***Idea:*** Choose your hashes in the following way:
  - Pick a window size w.
  - Have an initial hash function h_w that picks a location for that window uniformly at random from the array of slots.
  - Have d follow-up hash functions h_1, h_2, …, h_d that pick locations within that window.



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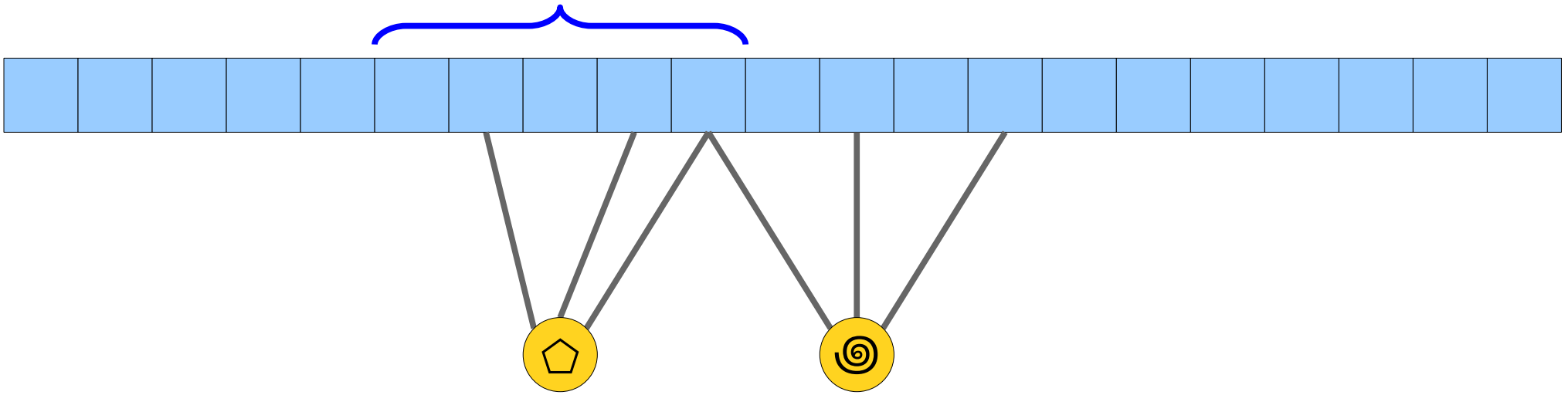

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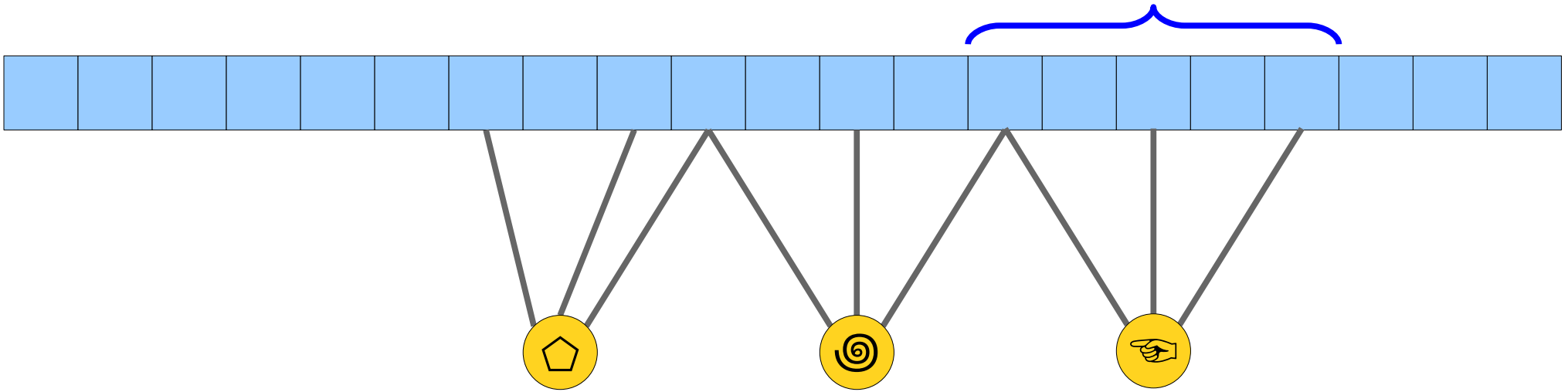

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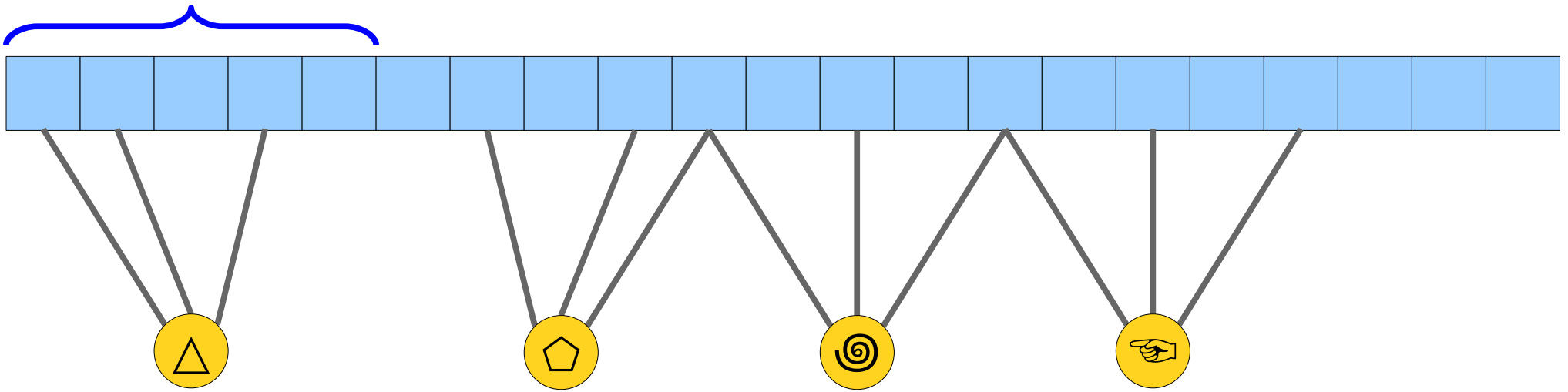

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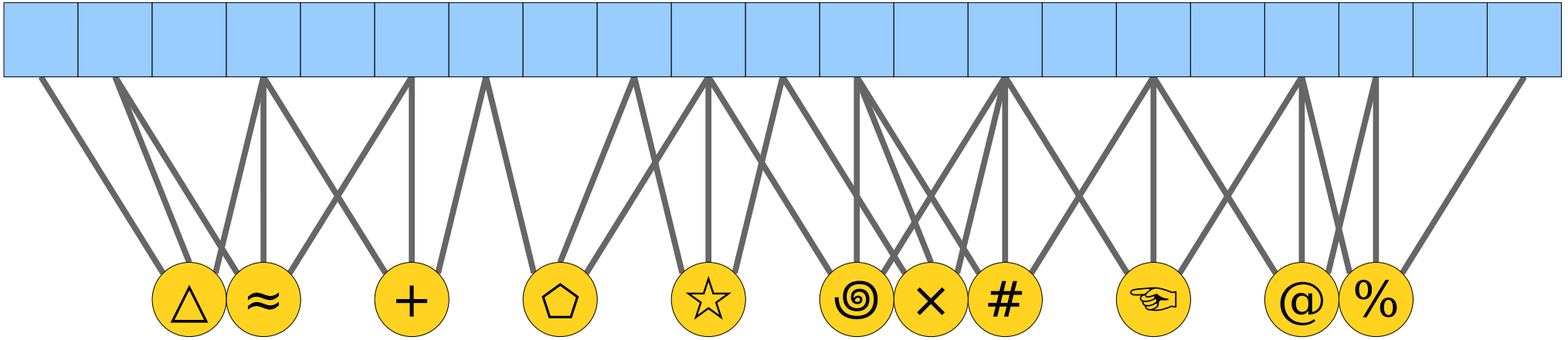

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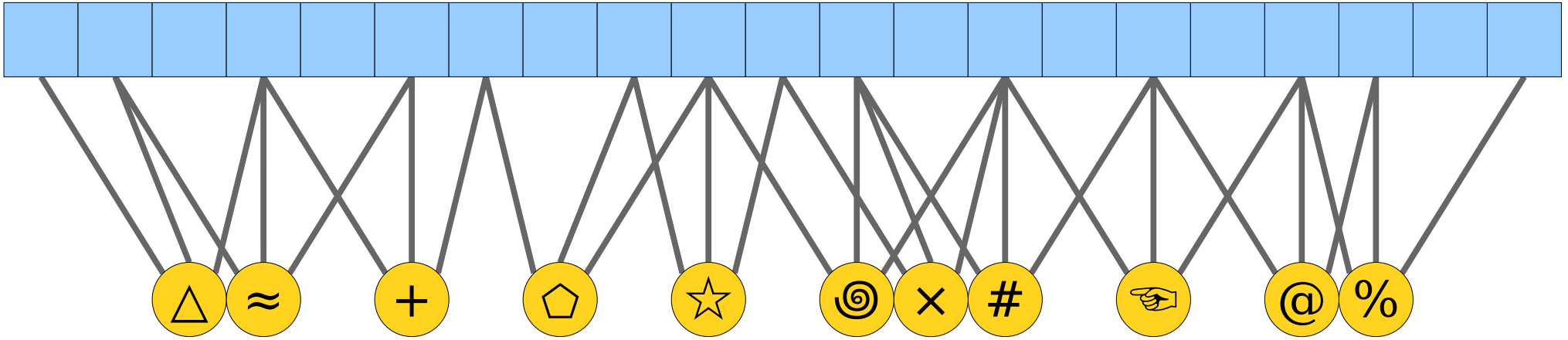
- It’s very unlikely for the slots on the far sides of the table to have elements hashing to them.
- Any items that do hash there will likely be peeled first.



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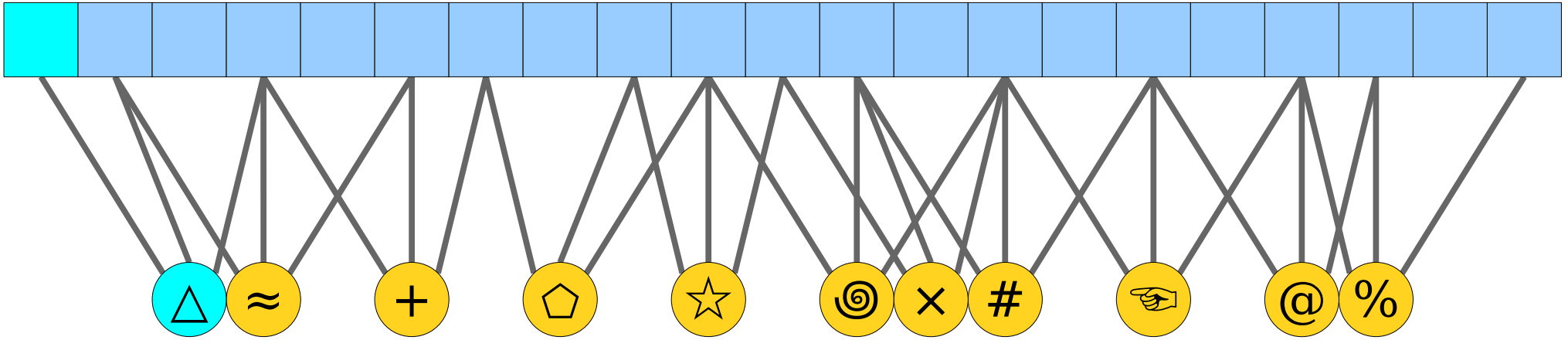

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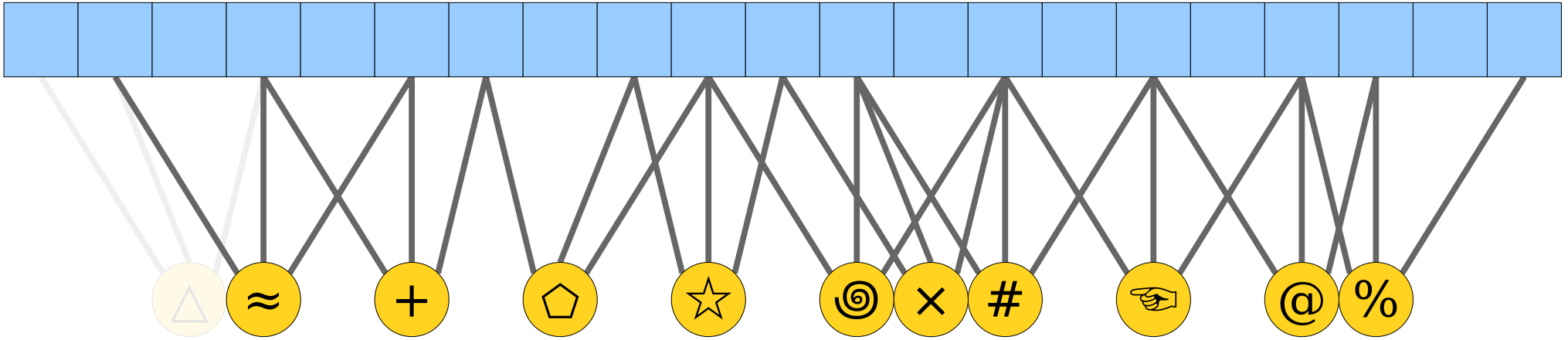

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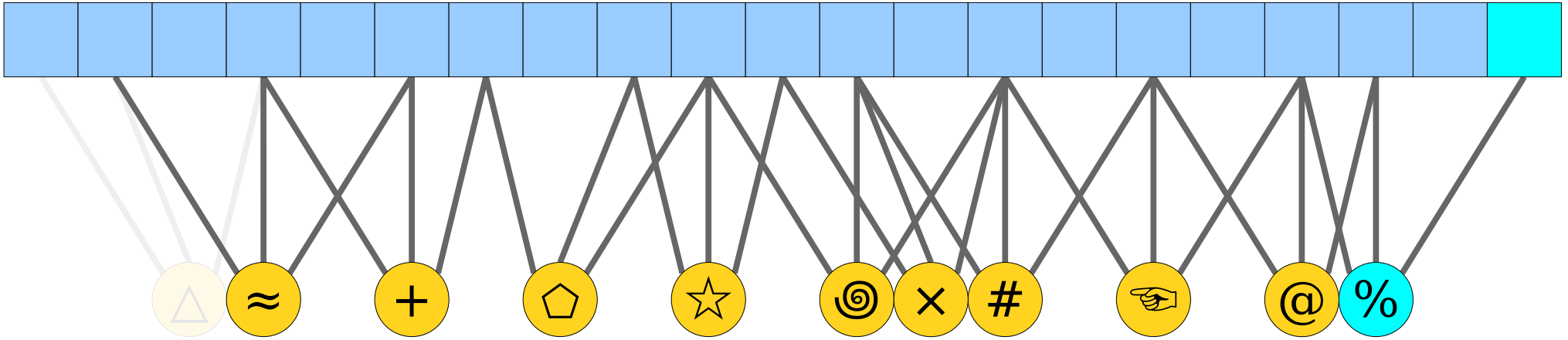

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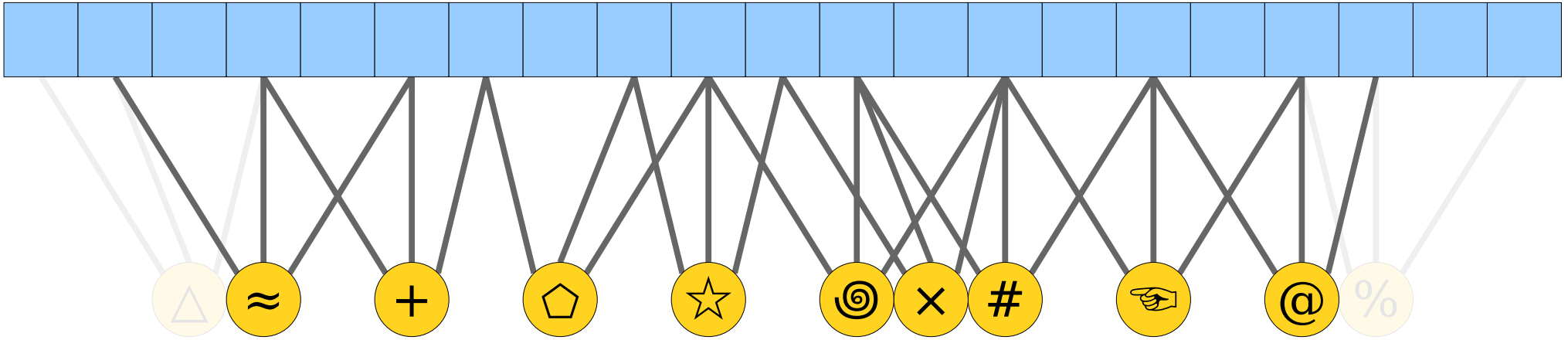

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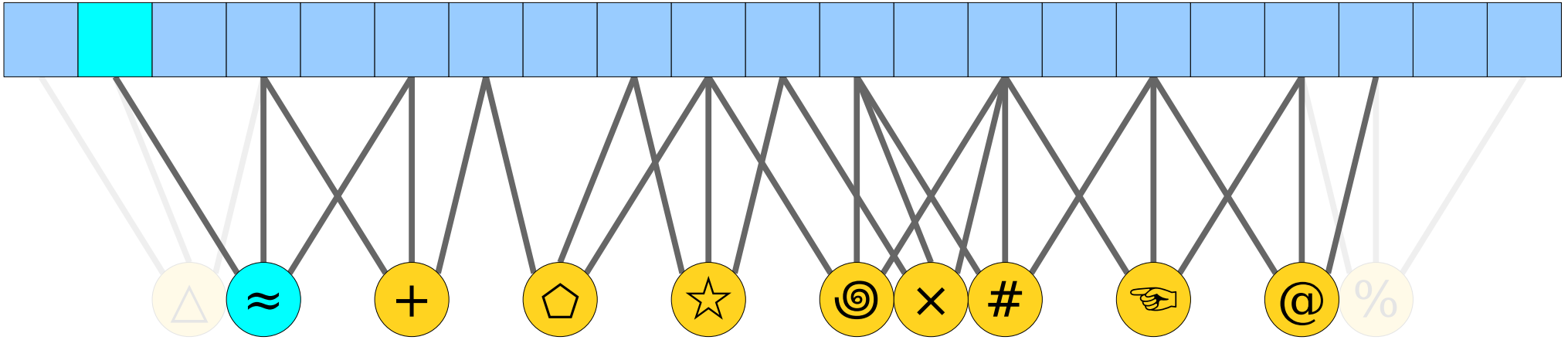

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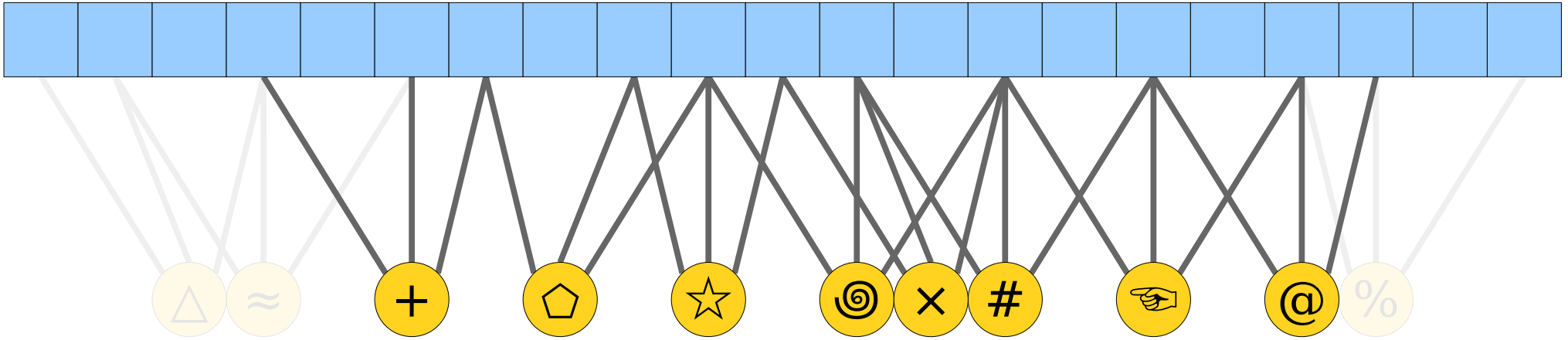

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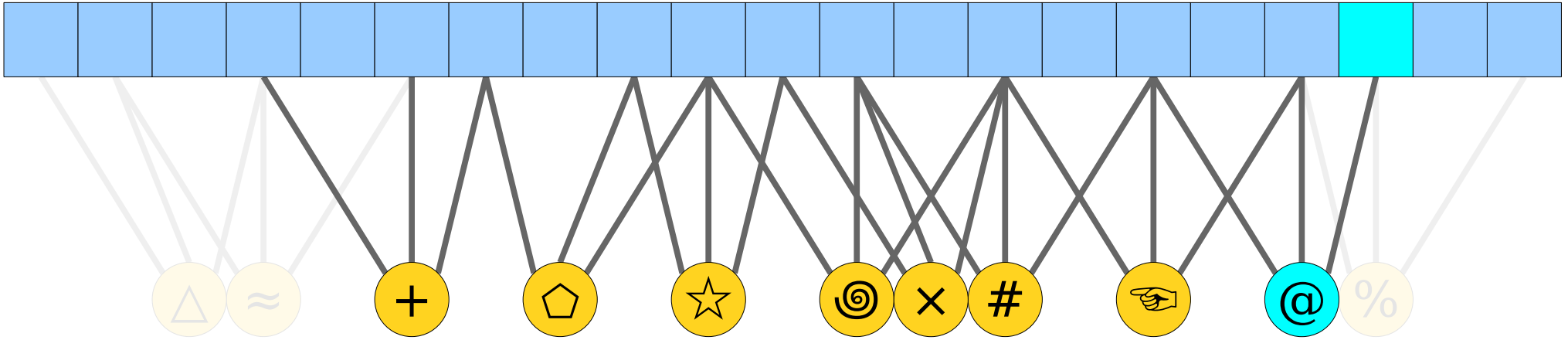

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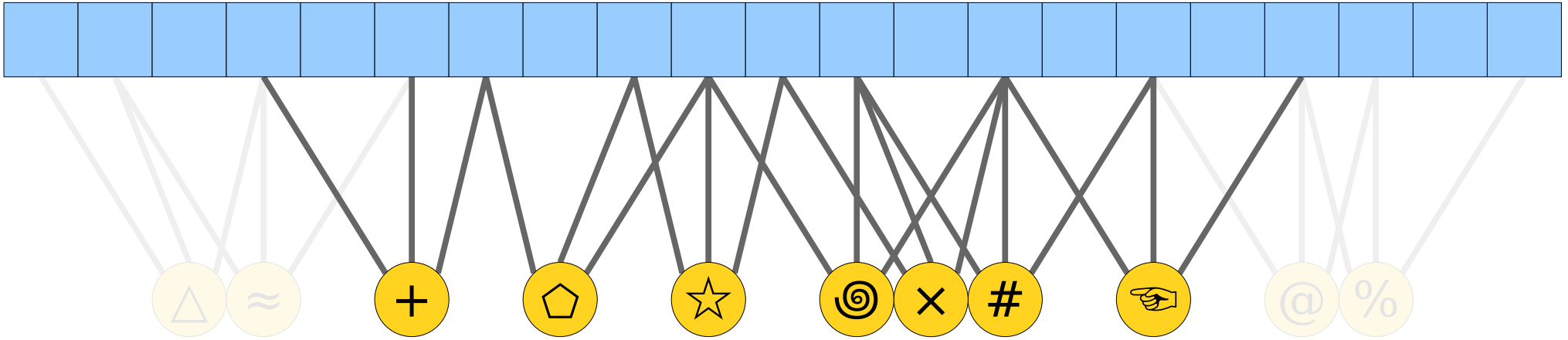

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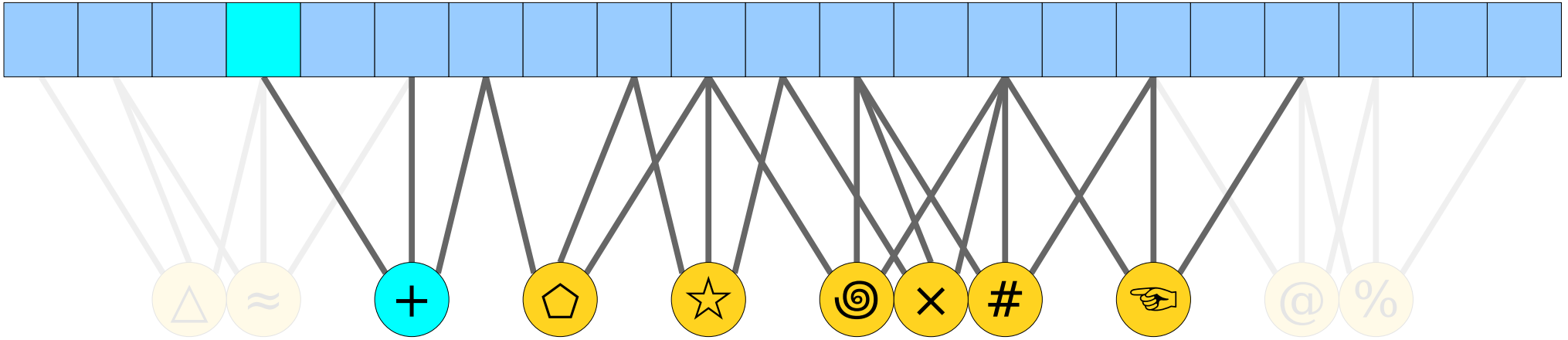

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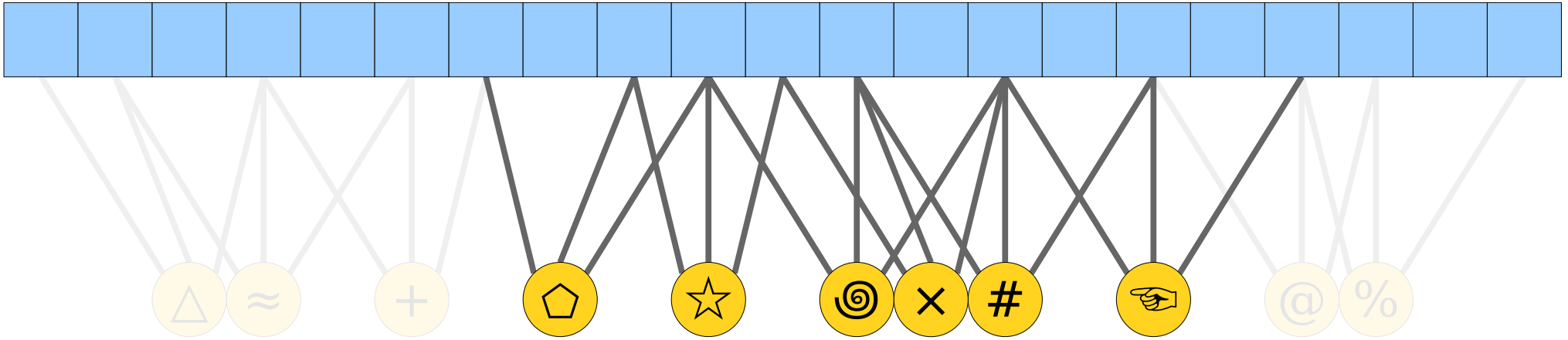

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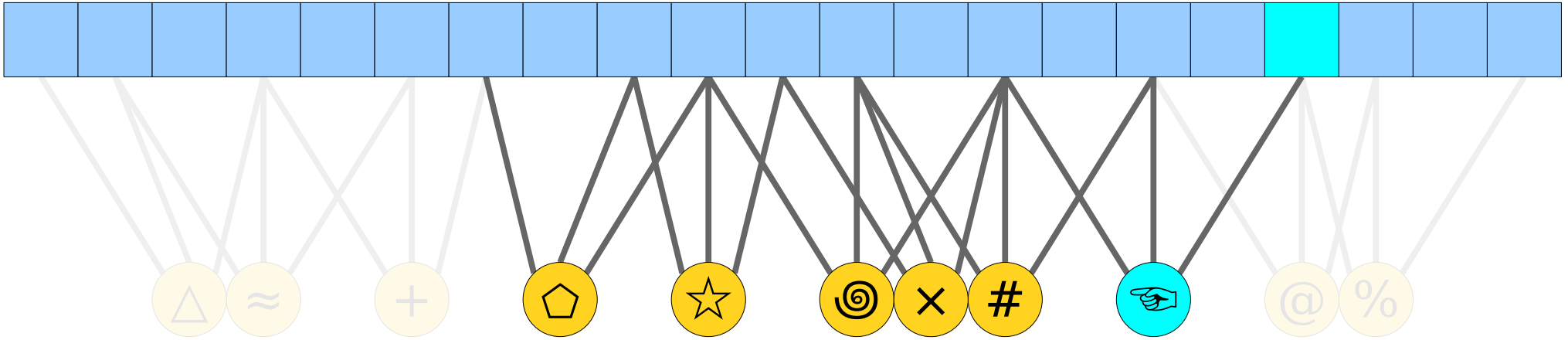

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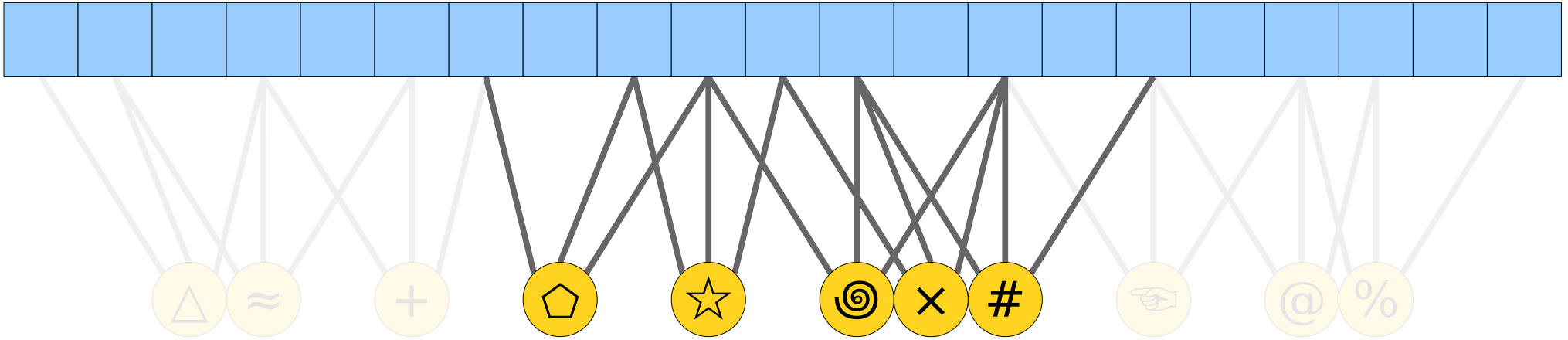

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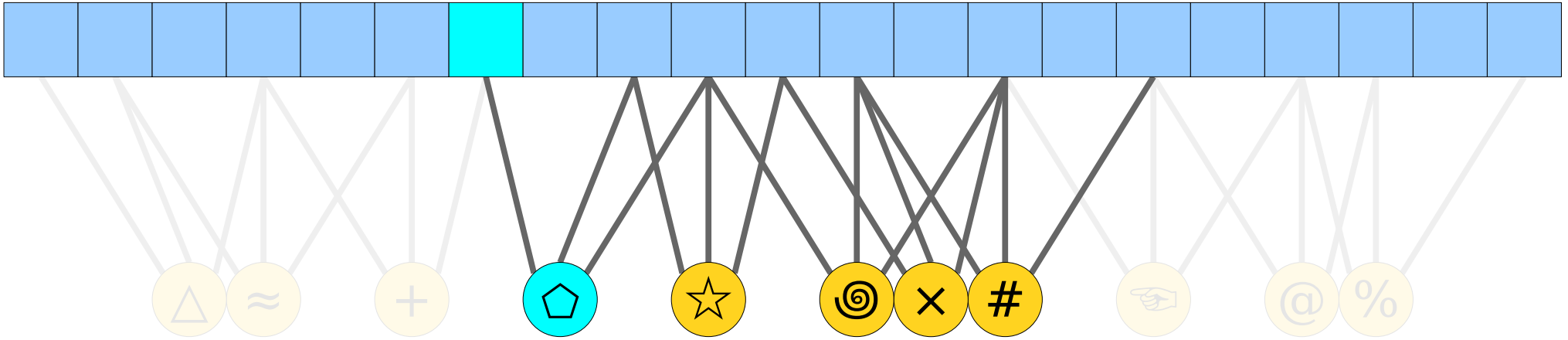

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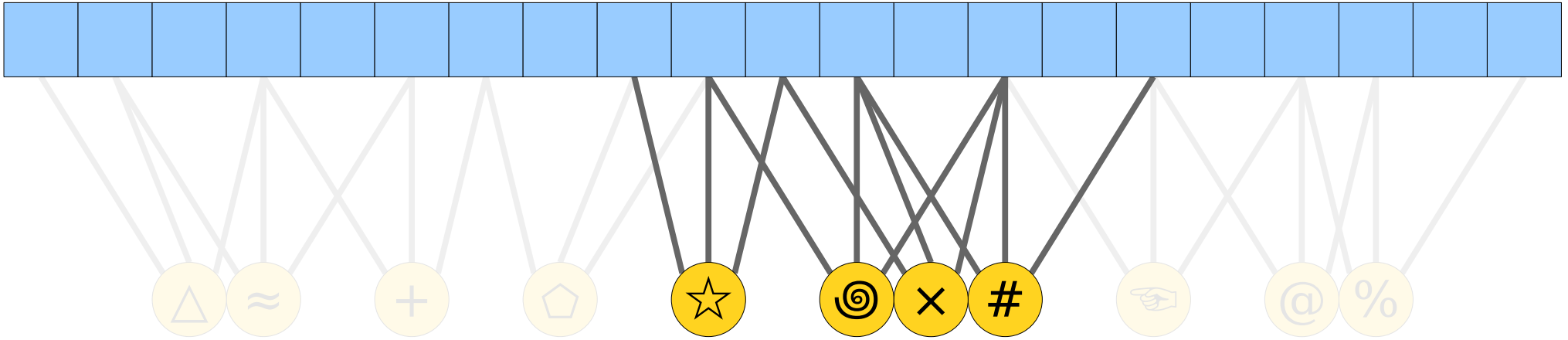

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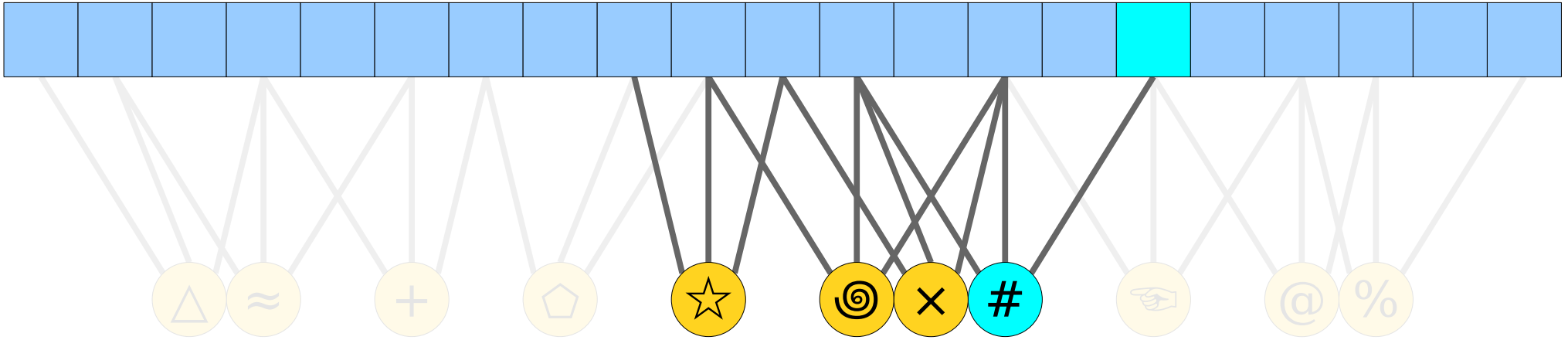

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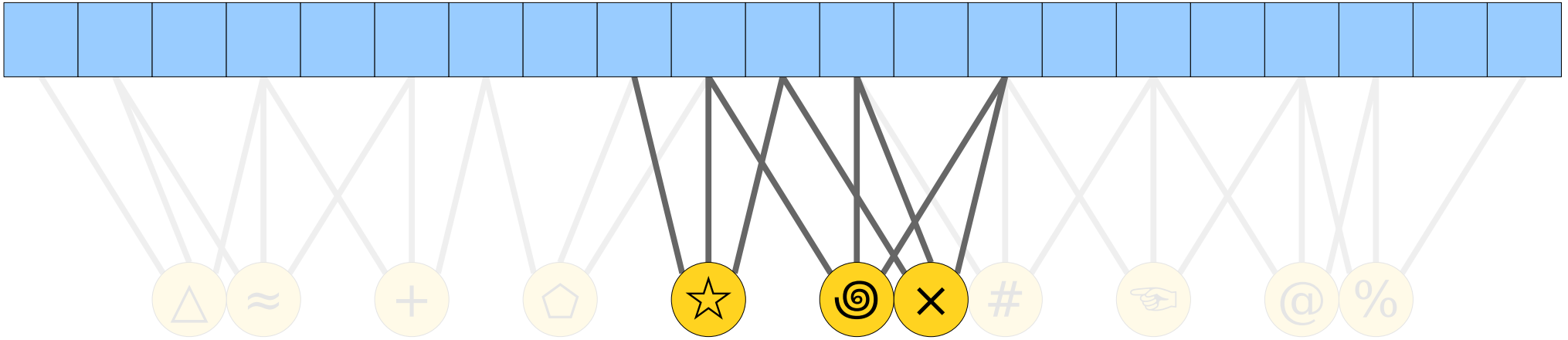

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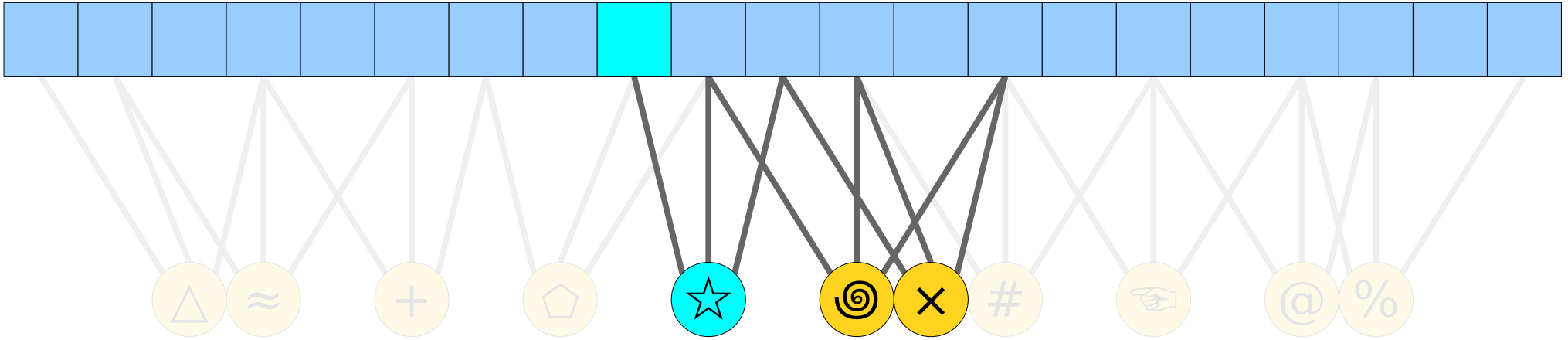

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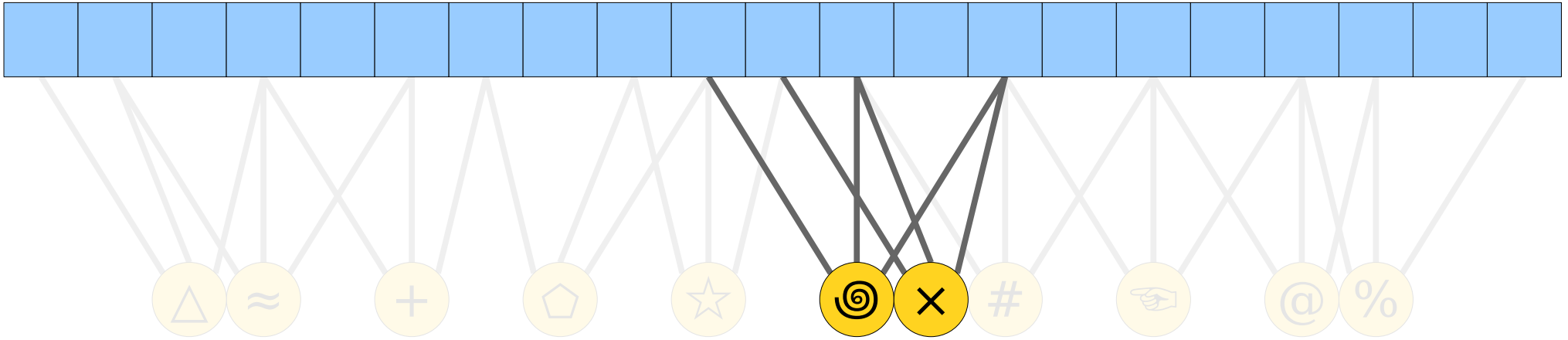

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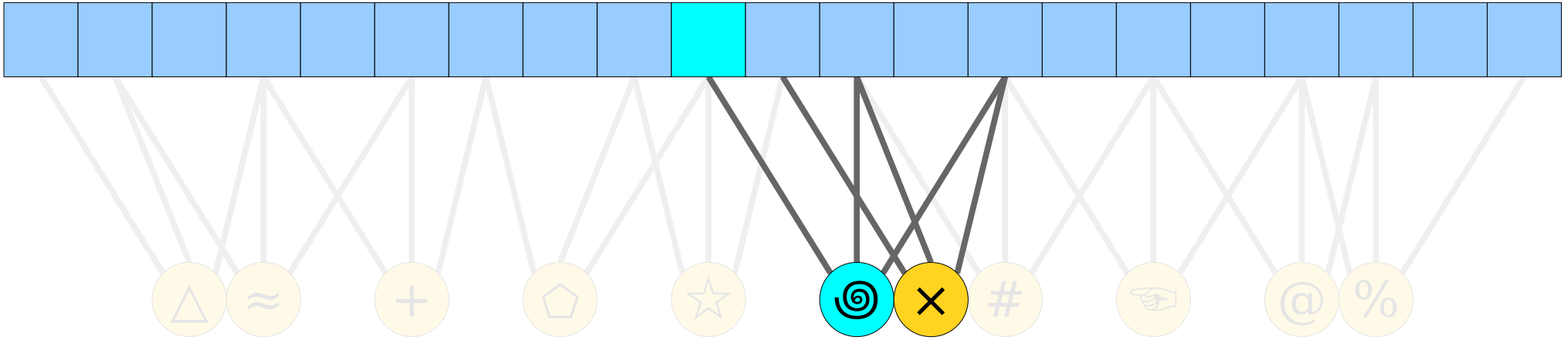

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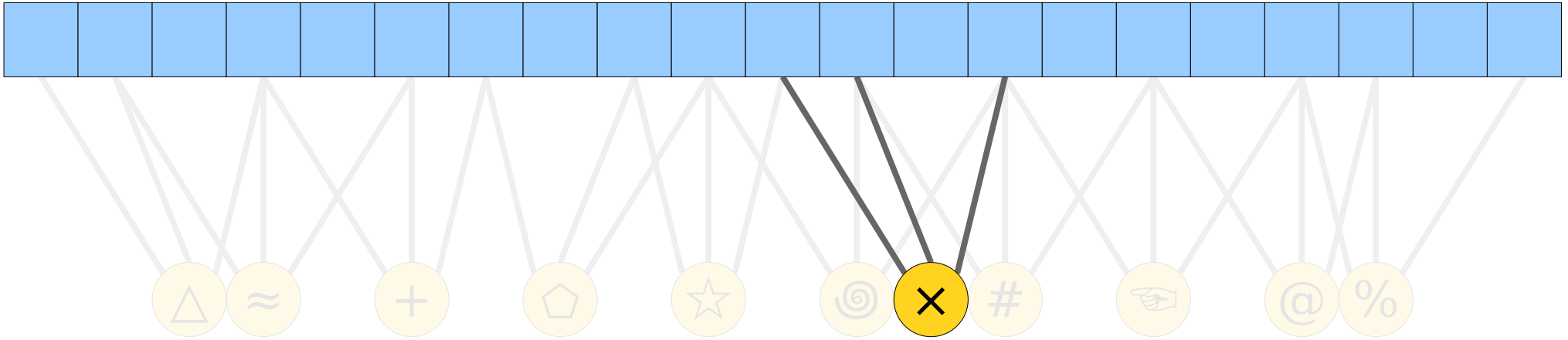

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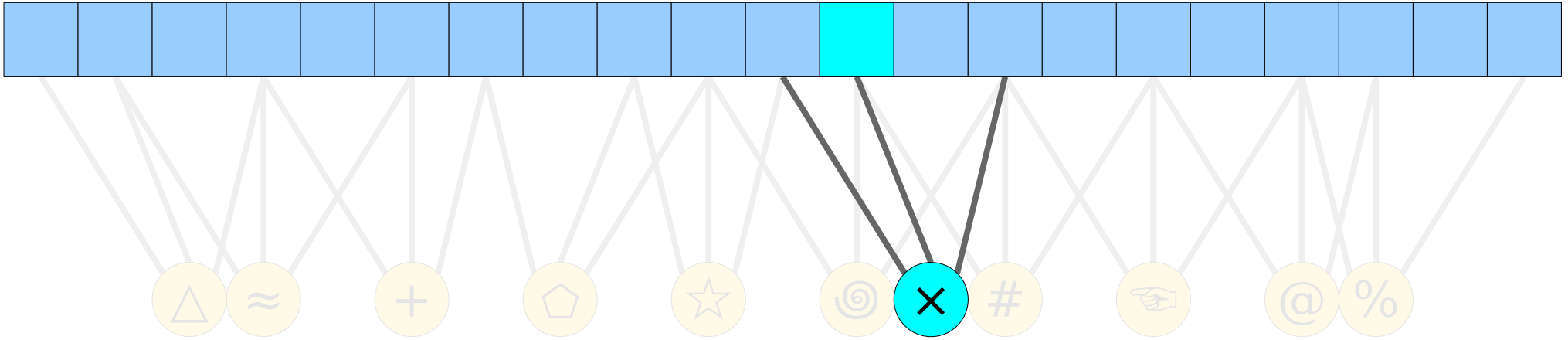

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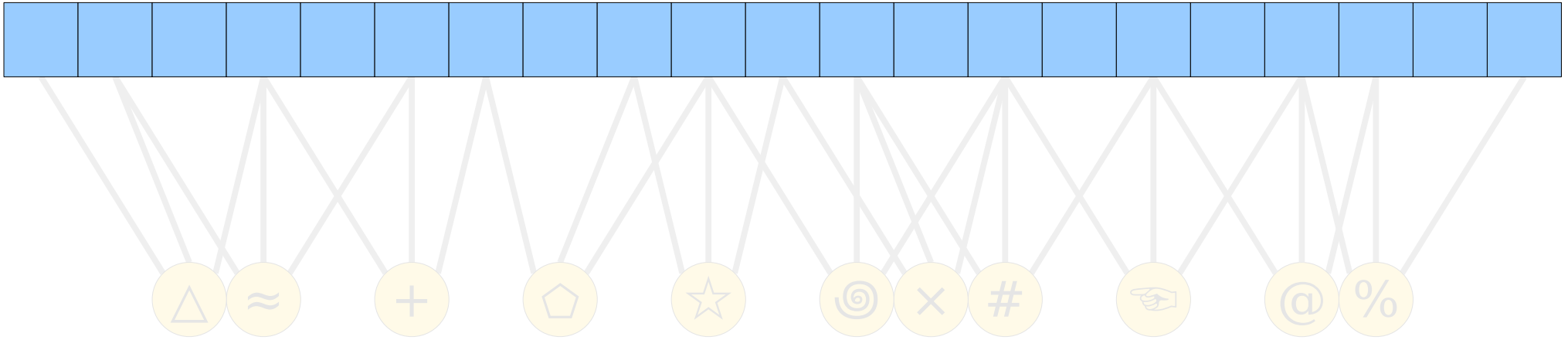
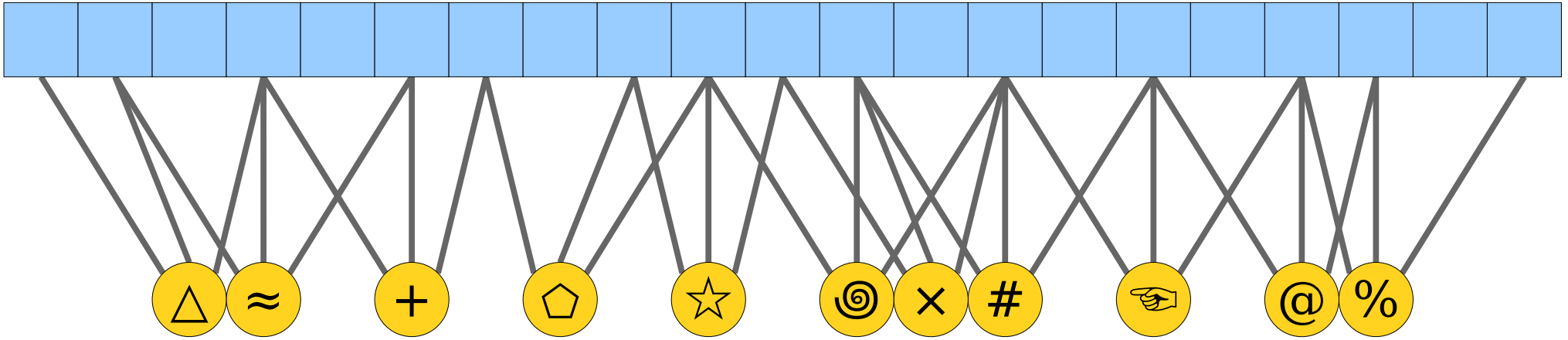


Spatial Coupling

- This strategy of assigning items to slots is called ***spatial coupling*** and is based on ideas first developed in coding theory.
- It allows for significantly more items to be placed into a table while still being peelable.



Spatial Coupling

- The numbers here are the theoretical limiting α values for different hash numbers, as n tends toward infinity.
- ***Caveat 1:*** These are limits as n gets large. Smaller values of n have lower max load factors.
- ***Caveat 2:*** Tuning the window size is currently more of an art than a science. It's conjectured that window sizes grow as $\Theta(n^{1/3})$, but this isn't known for sure.

	XOR Filter	Coupled XOR Filter
3 hashes	0.82	-
4 hashes	0.77	0.91
5 hashes	0.70	0.97
6 hashes	0.64	0.99
7 hashes	0.58	0.99

The Final Scorecard

	Bits Per Element	Hashes/Query
Bloom Filter (1970)	$1.44 \lg \varepsilon^{-1}$	$\lg \varepsilon^{-1}$
Cuckoo Filter, $b = 4$ (2014)	$1.05 \lg \varepsilon^{-1} + 3.15$ *(for sufficiently small ε)*	3
XOR Filter (2020)	$1.23 \lg \varepsilon^{-1}$	4
Coupled XOR Filter, $d = 3$ (2021)	$1.08 \lg \varepsilon^{-1}$ *(for sufficiently large n)*	5
Coupled XOR Filter, $d = 4$ (2021)	$1.03 \lg \varepsilon^{-1}$ *(for sufficiently large n)*	6

More to Explore

Related Data Structures

- There are several theoretically ***optimal AMQ structures*** developed in the past twenty years. They use $(1 + o(1)) \lg \varepsilon^{-1}$ bits per element. The first (I believe?) is due to Pagh, Pagh, and Rao.
- The same techniques we're using to build AMQs can be used to construct ***perfect hash functions***. Instead of storing fingerprints, store an index between 0 and $n - 1$ for the items in question.
- The ***Bloomier filter*** generalizes XOR filters as ways of storing approximate maps rather than approximate sets. They were famously used in a paper about compressing machine learning models (***Weightless***).
- The ***invertible Bloom lookup table*** generalizes XOR filters in a different way to store approximate maps. They use a peeling-type approach to support efficient listing of all items in the table.

Succinct Data Structures

- The idea of aiming for a data structure that uses as few bits as possible gives rise to ***succinct data structures***, which have been developed for several other classes of data structures beyond just AMQ.
- Check out ***wavelet trees*** as a starting point, but that's just the tip of the iceberg.

Random Hypergraph Theory

- We've modeled table slots and items as bipartite graphs. They're more commonly modeled as ***hypergraphs***, and much has been written about them.
- The analysis of *d*-ary cuckoo hashing explores the ***orientability threshold*** for random hypergraphs: at what point does it stop being possible to placing items into slots (taking edges and picking a preferred endpoint such that each node is preferred by at most one edge?)
- The analysis of XOR filters explores the ***peelability threshold*** for random hypergraphs: at what point does it stop being possible to find a node incident to exactly one edge that can be removed? This in turn is related to the idea of ***2-cores*** of hypergraphs.
- The spatial coupling idea combines these together by finding a family of hypergraphs whose peelability thresholds match the (higher) orientability thresholds for random hypergraphs.

Random Matrix Theory

- It's also possible to model the XOR filter and related structures through ***random matrix theory***.
- Imagine the adjacency matrix for the graph of nodes and items as a matrix A, the slots as a vector x, and the fingerprints as a vector b. The goal is to find an x where $Ax = b$, using XOR rather than regular addition.
- Porat's ***matrix filters*** use this idea to solve AMQ using $(1 + o(1)) \lg \varepsilon^{-1}$ bits per item by bringing in some Four Russians speedups and some other clever observations.
- Walzer's ***ribbon filters*** use fast Gaussian elimination solvers and assumptions about the matrix shape to solve AMQ in very low space overhead faster than other techniques.

Next Time

- ***Better than Balanced BSTs***
  - Revisiting some assumptions about efficiency.
- ***The Entropy Property***
  - Beating O(log n) for skewed distributions.
- ***The Dynamic Finger Property***
  - Beating O(log n) for spatial correlations.
- ***The Working Set Property***
  - Beating O(log n) for temporal correlations.