CHAPTER 5
JOINT PROBABILITY DISTRIBUTION

L1 - Discrete variables

Learning Objectives:

At the end of the lecture, you will be able to :

- define the joint probability distributions for discrete variables.

- calculate the marginal probability distributions.

- find the mean and the variance of X and Y of the joint probability distributions.

Definition of JPD for discrete variables

If X and Y are discrete random variables, then $p(x,y)$ is called the joint probability distribution (JPD) or joint probability mass function if it satisfies the following properties:

1. $p(x, y) \geq 0$

2. $\sum_{x} \sum_{y} p(x, y) = 1$

3. $p(x, y) = p(X = x, Y = y)$

Marginal Probability Distributions (MPD)

If X and Y are discrete random variables with joint probability mass function, $p(x,y)$, then the marginal probability mass functions for X is:

$$p(x) = p(X = x) = \sum_{y} p(x, y)$$

the marginal probability mass functions for Y is:

$$p(y) = p(Y = y) = \sum_{x} p(x, y)$$

Mean for *X* and *Y*

If X and Y are discrete random variables with joint probability mass function, *p(x,y)*, marginal probabilities *p(x)* , then the mean or expected values for X is:

$$E(x) = \mu_x = \sum_x x p(x)$$

If X and Y are discrete random variables with joint probability mass function, *p(x,y)*, marginal probabilities *p(y)*, then the mean or expected values for Y is:

$$E(y) = \mu_y = \sum_y y p(Y = y)$$

Variance for *X* and *Y*

If X and Y are discrete random variables with joint probability mass function, $p(x,y)$, marginal probabilities $p(x)$, then the variance for X is:

$$V(x) = \sigma_x^2 = \sum_x (x - \mu_x)^2 p(x) = \sum_x x^2 p(x) - (\sum_x xp(x))^2$$

If X and Y are discrete random variables with joint probability mass function, $p(x,y)$, marginal probabilities $p(y)$, then the variance for Y is:

$$V(y) = \sigma_y^2 = \sum_y (y - \mu_y)^2 p(y) = \sum_y y^2 p(y) - (\sum_y yp(y))^2$$

Example

A bus ferrying employees travels from Tronoh to Ipoh routeand back over the same route each day. There are three bus stops on this route. Let *X* be the number of bus stops that bus must stop on the way from Tronoh to Ipoh. Let *Y* be the number of bus stops the bus must stop on the way back from Ipoh to Tronoh. The JPD of *X* and *Y* is given below:

		X			
		0	1	2	3
Y	0	0.01	0.02	0.07	0.01
	1	0.03	0.06	0.10	0.06
	2	0.05	0.12	0.15	0.08
	3	0.02	0.09	0.08	0.05

a. Find the marginal distribution for *X* and *Y*.

b. What is the expected number of bus stops the bus must stop over the entire road, that is going to Ipoh and back to Tronoh?

c. Find also the variance of *X* and the variance of *Y*.

Solution

The JPD of X and Y is given below:

		X			
		0	1	2	3
Y	0	0.01	0.02	0.07	0.01
	1	0.03	0.06	0.10	0.06
	2	0.05	0.12	0.15	0.08
	3	0.02	0.09	0.08	0.05

a. The marginal distribution for X is:

$$P(x) = \sum_{y} p(x, y), \quad x = 0, 1, 2, 3$$

$$P(0) = \sum_{y=0}^{3} p(0, y) = P(0,0) + p(0,1) + p(0,2) + p(0,3)$$

$$= 0.01 + 0.03 + 0.05 + 0.02 = 0.11$$

$$P(1) = \sum_{y=0}^{3} p(1, y) = P(1,0) + p(1,1) + p(1,2) + p(1,3)$$

$$= 0.02 + 0.06 + 0.12 + 0.09 = 0.29$$

$$P(2) = \sum_{y=0}^{3} p(2, y) = P(2,0) + p(2,1) + p(2,2) + p(2,3)$$

$$= 0.07 + 0.10 + 0.15 + 0.08 = 0.4$$

$$P(3) = \sum_{y=0}^{3} p(3, y) = P(3,0) + p(3,1) + p(3,2) + p(3,3)$$

$$= 0.01 + 0.06 + 0.08 + 0.05 = 0.2$$

The JPD of *X* and *Y* is given below:

	X			
	0	1	2	3
Y 0	0.01	0.02	0.07	0.01
Y 1	0.03	0.06	0.10	0.06
Y 2	0.05	0.12	0.15	0.08
Y 3	0.02	0.09	0.08	0.05

a. The marginal distribution for *X* is:

x	0	1	2	3
P(x)	0.11	0.29	0.40	0.20

The JPD of *X* and *Y* is given below:

		X			
		0	1	2	3
Y	0	0.01	0.02	0.07	0.01
	1	0.03	0.06	0.10	0.06
	2	0.05	0.12	0.15	0.08
	3	0.02	0.09	0.08	0.05

The marginal distribution for *Y* is:

$$P(y) = \sum_{x} p(x, y), \quad y = 0, 1, 2, 3$$

$$P(0) = \sum_{x=0}^{3} p(x,0) = P(0,0) + p(1,0) + p(2,0) + p(3,0)$$

$$= 0.01 + 0.02 + 0.07 + 0.01 = 0.11$$

$$P(1) = \sum_{x=0}^{3} p(x,1) = P(0,1) + p(1,1) + p(2,1) + p(3,1)$$

$$= 0.03 + 0.06 + 0.10 + 0.06 = 0.25$$

$$P(2) = \sum_{x=0}^{3} p(x,2) = P(0,2) + p(1,2) + p(2,2) + p(3,2)$$

$$= 0.05 + 0.12 + 0.15 + 0.08 = 0.4$$

$$P(3) = \sum_{x=0}^{3} p(x,3) = P(0,3) + p(1,3) + p(2,3) + p(3,3)$$

$$= 0.02 + 0.09 + 0.08 + 0.05 = 0.24$$

The JPD of *X* and *Y* is given below:

	X			
	0	1	2	3
Y 0	0.01	0.02	0.07	0.01
1	0.03	0.06	0.10	0.06
2	0.05	0.12	0.15	0.08
3	0.02	0.09	0.08	0.05

a. The marginal distribution for *Y* is:

y	**0**	**1**	**2**	**3**
P(y)	0.11	0.25	0.40	0.24

b. The expected of X is:

x	0	1	2	3
P(x)	0.11	0.29	0.40	0.20

$$E(x) = \mu(x) = \sum xp(x) = 0(0.11) + 1(0.29) + 2(0.40) + 3(0.20)$$

$$= 0 + 0.29 + 0.80 + 0.60) = 1.69$$

The expected of Y is:

y	0	1	2	3
P(y)	0.11	0.25	0.40	0.24

$$E(y) = \mu(y) = \sum yp(y) = 0(0.11) + 1(0.25) + 2(0.40) + 3(0.24)$$

$$= 0 + 0.25 + 0.80 + 0.72 = 1.77$$

c. The variance of X is:

x	0	1	2	3
$P(x)$	0.11	0.29	0.40	0.20
$xp(x)$	0	0.29	0.80	0.60
$X2p(x)$	0	0.29	1.60	1.80

$$E(x) = \mu(x) = \sum xp(x) = 0(0.11) + 1(0.29) + 2(0.40) + 3(0.20)$$

$$= 0 + 0.29 + 0.80 + 0.60) = 1.69$$

$$E(x^2) = \sum x^2 p(x) = 0^2(0.11) + 1^2(0.29) + 2^2(0.40) + 3^2(0.20)$$

$$= 0 + 0.29 + 1.60 + 1.80 = 3.69$$

$$V(x) = E(x^2) - (E(x))^2 = 3.69 - (1.69)^2 = 3.69 - 2.856$$

$$= 0.834$$

c. The variance of Y is:

y	0	1	2	3
$P(y)$	0.11	0.25	0.40	0.24
$yp(y)$	0	0.25	0.80	0.72
$y2p(y)$	0	0.25	1.60	2.16

$$E(y) = \mu(y) = \sum yp(y) = 0(0.11) + 1(0.25) + 2(0.40) + 3(0.24)$$

$$= 0 + 0.25 + 0.80 + 0.72 = 1.77$$

$$E(y^2) = \sum y^2 p(y) = 0^2(0.11) + 1^2(0.25) + 2^2(0.40) + 3^2(0.24)$$

$$= 0 + 0.25 + 1.60 + 2.16 = 4.01$$

$$V(y) = E(y^2) - (E(y))^2 = 4.01 - (1.77)^2 = 4.01 - 3.133$$

$$= 0.877$$

Exercise

An assignment which consists of two sections was given to students who took the FEM1063 Statistics and Applications course. Let X be the total marks for section 1 and Y be the total marks for section 2. Assume that the JPD for X and Y is given below:

		Y		
		0	5	10
X	0	0.02	0.06	0.12
	5	0.04	0.15	0.30
	10	0.01	0.15	0.15

a. Find the marginal distribution for X and Y.

b. What is the expected number of X and the expected number of Y ?

c. Find also the variance of X and the variance of Y.

Chapter 5

JOINT PROBABILITY DISTRIBUTIONS

- L2 - Continuous variables

Learning Objectives:

At the end of the lecture, you will be able to :

- define the joint probability distributions for continuous variables.

- calculate the marginal probability distributions.

- find the mean and the variance of X and Y of the joint probability distributions.

Definition of JPD for continuous variables

If X and Y are continuous random variables, then *f(x,y)* is called the joint probability distribution (JPD) or joint probability densenty function if it satisfies the following properties:

1. $f(x, y) \geq 0$ for all x and y

2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1$

3. $P(a < x < b, c < y < d) = \int_{c}^{d}\int_{a}^{b} f(x, y)dxdy$

where a, b, c, d are constants

Marginal Probability Distributions (MPD)

If X and Y are continuous random variables with joint probability density function, $f(x,y)$, then the marginal probability density functions for X is:

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

the marginal probability density functions for Y is:

$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

Mean for *X* and *Y*

If X and Y are continuous random variables with joint probability density function, *f(x,y)*, marginal probabilities *f(x)* , then the mean or expected values for X is:

$$E(x) = \mu_x = \int_{-\infty}^{\infty} xf(x)dx$$

If X and Y are continuous random variables with joint probability density funcetion, *f(x,y)*, marginal probabilities *f(y)*, then the mean or expected values for Y is:

$$E(y) = \mu_y = \int_{-\infty}^{\infty} yf(y)dy$$

Variance for *X* and *Y*

If *X* and *Y* are continuous random variables with joint probability density function, *f(x,y)*, marginal probabilities *f(x)* , then the variance for *X* is:

$$V(x) = \sigma_x^2 = \int_{-\infty}^{\infty} (x-\mu_x)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - (\int_{-\infty}^{\infty} xf(x))^2$$

If *X* and *Y* are continuous random variables with joint probability density function, *f(x,y)*, marginal probabilities *f(y)*, then the variance for *Y* is:

$$V(y) = \sigma_y^2 = \int_{-\infty}^{\infty} (y-\mu_y)^2 f(y)dx = \int_{-\infty}^{\infty} y^2 f(y)dy - (\int_{-\infty}^{\infty} yf(y))^2$$

Example

The JPD for X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{1}{16}xy, & 0 \le x \le 2,\ 0 \le y \le 4 \\ 0 \quad , & \text{otherwise} \end{cases}$$

a. Find the marginal distribution for X and Y.
b. What is the mean of X and the mean of Y?
c. Find also the variance of X and the variance of Y.

Solution

a. The marginal distribution for *X* is:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{0}^{4} \frac{1}{16} xy dy = \left. \frac{xy^2}{32} \right|_0^4 = \frac{x}{2}, \quad 0 \le x \le 2$$

The marginal distribution for *Y* is:

$$f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{0}^{2} \frac{1}{16} xy dx = \left. \frac{x^2 y}{32} \right|_0^2 = \frac{y}{8}, \quad 0 \le y \le 4$$

b. The mean of *X* is:

$$\mu(x) = E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{2} x\frac{x}{2}dx = \left.\frac{x^3}{6}\right|_0^2 = \frac{8}{6} = 1.33$$

The mean of *Y* is:

$$\mu(y) = E(Y) = \int_{0}^{4} yf(y)dy = \int_{0}^{4} y\frac{y}{8}dy = \left.\frac{y^3}{24}\right|_0^4 = \frac{64}{24} = 2.67$$

c. The variance of *X* is:

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^2 x^2 \frac{x}{2} dx = \left. \frac{x^4}{8} \right|_0^2 = \frac{16}{8} = 2$$

$$\therefore V(X) = 2 - (1.33)^2 =$$

The variance of Y is:

$$V(Y) = E(Y^2) - (E(Y))^2$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)dy = \int_0^4 y^2 \frac{y}{8} dy = \left.\frac{y^4}{32}\right|_0^4 = \frac{256}{32} = 8$$

$$\therefore V(Y) = 8 - (2.67)^2 =$$

Exercise

The JPD for X and Y is given by

$$f(x,y)=\begin{cases} kxy, & 0\le x\le 1,\ 0\le y\le 1 \\ 0\ , & \text{otherwise} \end{cases}$$

a. Find the value of the constant k.
b. Find the marginal distribution for X and Y.
c. What is the mean of X and the mean of Y?
d. Find also the variance of X and the variance of Y.