Chapter 7
Hypothesis Testing
- Two Populations

L 1: Hyp Testing – mean (2 populations, variance known)

L2: Hyp Testing – mean (2 populations, variance unknown but equal)

Learning Objectives

At the end of the lesson student should be able to

- Carry out hypothesis testing for difference in means for two normal populations (variances known)
- Carry out hypothesis testing for difference in means for two normal populations (variances unknown but equal)

Comparing Two Population Means

- Comparison between two different probability distributions.

Purpose of study is to determine if there is a difference between the two probability distributions.

- Experimenter has two sets of data observations.

$x_1, x_2, \ldots., x_n \longrightarrow$ Population A

$y_1, y_2, \ldots., y_n \longrightarrow$ Population B

One aspect of assessment is the comparison between the means μ_A and μ_B

If $\mu_A = \mu_B \longrightarrow$ Population A = Population B (probability distributions are equal)

If $\mu_A \neq \mu_B \longrightarrow$ probability distributions are different

Comparing Two Population Means

Normal distribution,
Known variances,

Normal distribution,
Unknown variances,

Case 1: $\sigma_1^2 = \sigma_2^2$

Case 2: $\sigma_1^2 \neq \sigma_2^2$

TEST ABOUT TWO NORMAL MEANS WHEN TWO VARIANCES ARE KNOWN

Assumptions:

$X_{11}, X_{12}, X_{13}, \ldots, X_{1n_1}$ is a random sample from population 1 with mean μ_1 and variance σ^2_1

2. $X_{21}, X_{22}, X_{23}, \ldots, X_{2n_2}$ is a random sample from population 2 with mean μ_2 and variance σ^2_2

3. The two populations represented by X_1 and X_2 are independent.

4. Both populations are normally distributed.

Point estimator for the difference between two means $\mu_1 - \mu_2 = \bar{x}_1 - \bar{x}_2$

TEST ABOUT TWO NORMAL MEANS WHEN TWO VARIANCES ARE KNOWN

Test about the difference

Null Hypothesis:

$$H_0 : \mu_1 = \mu_2 \quad \textbf{vs} \quad H_1 : \mu_1 \neq \mu_2, \text{or } H_1 : \mu_1 > \mu_2 \text{ or } H_1 : \mu_1 < \mu_2$$

Test statistic:

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 / n_1) + (\sigma_2^2 / n_2)}} \sim N(0,1) \ if \ \mu_1 = \mu_2$$

Alternative Hypothesis	Rejection Criteria (Reject H_0)
$H_1 : \mu_1 \neq \mu_2$	$Z > z_{\alpha/2} \quad or \quad Z < -z_{\alpha/2}$
$H_1 : \mu_1 > \mu_2$	$Z > z_{\alpha}$
$H_1 : \mu_1 < \mu_2$	$Z < -z_{\alpha}$

A $100(1-\alpha)\%$ **confidence interval for** $\mu_1 - \mu_2$ **is**

$$(\bar{x}_1 - \bar{x}_2) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq (\bar{x}_1 - \bar{x}_2) + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

The procedures used in hypothesis testing for 2 populations are the same as in the hypothesis testing for one population.

Conclusion of the test could also be based on the confidence Interval and the p-value.

Example 1:

A random sample of size $n = 25$ taken from a normal population with standard deviation 5.2 has a mean equals 81. A second random sample of size n = 36, taken from a different normal population with standard deviation 3.4, has a mean equals 76. Do the data indicate that the true mean value μ_1 and μ_2 are different? Carry out a test at $\alpha = \underline{0.01}$

Solution

1. From the problem context, identify the parameter of interest

Parameter of interest; the difference between the true average $\mu_1 - \mu_2$

2. State the null hypothesis H_0 and appropriate alternative hypothesis, H_1

$H_0 : \mu_1 = \mu_2$ $\qquad$ $H_1 : \mu_1 \neq \mu_2$

3. Determine the appropriate test statistic

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 / n_1) + (\sigma_2^2 / n_2)}}$$

4. Critical value given α = 0.01

$\alpha/2 = 0.01/2 = 0.005,$

So $z_{\alpha/2} = z_{0.005} = 2.58$

5. State the rejection region for the statistic

$$\text{Reject } H_0 \text{ if } \; Z \le -z_{0.005} \quad or \quad Z \ge z_{0.005}$$

$$\Rightarrow \text{Reject } H_0 \text{ if } \; Z \le -2.58 \quad or \quad Z \ge 2.58$$

6. Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute the value

$$\overline{X}_1 = 81,\ \overline{X}_2 = 76,\ \sigma_1 = 5.2,\ \sigma_2 = 3.4,\ n_1 = 25,\ n_2 = 36,$$

Compute the value of the test statistic:

$$Z = \frac{(\overline{x}_1 - \overline{x}_2) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 / n_1) + (\sigma_2^2 / n_2)}} = \frac{81 - 76 - (0)}{\sqrt{(5.2^2 / 25) + (3.4^2 / 36)}} = 4.22$$

7. Make a decision

Decision: since $Z = 4.22 > 2.58$, we reject H_0

the data shows a sufficient evidence that $\mu_1 \neq \mu_2$

Example 1:

Find 99% CI on the difference in mean strength

$$\bar{X}_1 = 81,\ \bar{X}_2 = 76,\ \sigma_1 = 5.2,\ \sigma_2 = 3.4,\ n_1 = 25,\ n_2 = 36,$$

Solution From formula:

$$\boxed{(\bar{x}_1 - \bar{x}_2) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \le (\mu_1 - \mu_2) \le (\bar{x}_1 - \bar{x}_2) + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$\Rightarrow (81-76) - 2.58\sqrt{\frac{(5.2)^2}{25} + \frac{(3.4)^2}{36}} \le (\mu_1 - \mu_2) \le (81-76) + 2.58\sqrt{\frac{(5.2)^2}{25} + \frac{(3.4)^2}{36}}$$

$$\Rightarrow 1.944 \le (\mu_1 - \mu_2) \le 8.055$$

99% Confidence interval is [1.944, 8.055]

Example 2:

The burning rates of two different solid-fuel propellants used in rocket systems are being studied. It is known that both propellants have approximately the same standard deviation of burning rate, that is 3cm/second. Two random samples with the same sample size of 20 specimens are tested and the sample mean burning rates are 18 cm/second and 24 cm/second respectively.

i. Test the hypothesis that both propellants have the same mean burning rate, using the P-value approach.
ii. Construct a 95% two-sided CI on the difference in means, $\mu_1 - \mu_2$. What is the practical meaning of this interval?

Solution

1. From the problem context, identify the parameter of interest

Parameter of interest; the difference between the true average $\mu_1 - \mu_2$

2. State the null hypothesis H_0 and appropriate alternative hypothesis, H_1

$$H_0 : \mu_1 = \mu_2 \qquad H_1 : \mu_1 \neq \mu_2$$

3. Determine the appropriate test statistic

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 / n_1) + (\sigma_2^2 / n_2)}}$$

6. Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute the value

$$\bar{X}_1 = 18,\ \bar{X}_2 = 24,\ \sigma_1^2 = 9,\ \sigma_2^2 = 9,\ n_1 = n_2 = 20,$$

Compute the value of the test statistic:

$$Z = \frac{(18-24)-0}{\sqrt{\frac{9}{20}+\frac{9}{20}}} = -6.32$$

Find the p - value

P-value = $2(1-\Phi(6.32)) = 2(1-1) = 0$

.Since p-value < 0.05, we reject the null hypothesis at α=0.05. Both propellants are not the same mean burning rate.

Find a 95% CI on the difference in means.

From formula:

$$\boxed{(\bar{x}_1 - \bar{x}_2) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \le (\mu_1 - \mu_2) \le (\bar{x}_1 - \bar{x}_2) + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$(18-24) - 1.96\sqrt{\frac{9}{20} + \frac{9}{20}} \le \mu_1 - \mu_2 \le (18-24) + 1.96\sqrt{\frac{9}{20} + \frac{9}{20}}$$

$$\therefore \quad -7.589 \le \mu_1 - \mu_2 \le -4.141$$

Since $\mu_1 - \mu_2 = 0$ is not in the interval then we reject H_0.
Both propellants are not the same mean burning rate.

TEST ABOUT TWO MEANS WHEN TWO VARIANCES ARE UNKNOWN

❖ **Test about the difference** $\mu_1 - \mu_2 = 0$

Case 1: $\sigma_1^2 = \sigma_2^2$

Null Hypothesis:

$$H_0 : \mu_1 = \mu_2 \quad \textbf{vs} \quad H_1 : \mu_1 \neq \mu_2, \text{or } H_1 : \mu_1 > \mu_2 \text{ or } H_1 : \mu_1 < \mu_2$$

Test statistic:

$$\boxed{T = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}}$$

pooled estimator of σ^2

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

TEST ABOUT TWO NORMAL MEANS WHEN TWO VARIANCES ARE UNKNOWN

- **Test about the difference** $\mu_1 - \mu_2 = 0$

Case 1: $\sigma_1^2 = \sigma_2^2$

Critical region:

Alternative Hypothesis	Rejection Criteria (Reject H_0)
$H_1 : \mu_1 \neq \mu_2$	$T > t_{\alpha/2,\, n_1+n_2-2}$ *or* $T < -t_{\alpha/2,\, n_1+n_2-2}$
$H_1 : \mu_1 > \mu_2$	$T > t_{\alpha,\, n_1+n_2-2}$
$H_1 : \mu_1 < \mu_2$	$T < -t_{\alpha,\, n_1+n_2-2}$

Confidence interval:

A $100(1-\alpha)\%$ **two- sided confidence interval for** $\mu_1-\mu_2$ **when variance is unknown :**

$$(\bar{x}_1-\bar{x}_2)-t_{\alpha/2,n_1+n_2-2}S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\le\mu_1-\mu_2$$

$$\le(\bar{x}_1-\bar{x}_2)+t_{\alpha/2,n_1+n_2-2}S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$$

Example 3:

Professor Adam taught the same large lecture course for two terms. Except for negligible differences the courses were the same. However, one met at 8am and the other met at 11am. The two courses were given final exams of the same degree of difficulty and covering the same material. Both exams were worth 100 points. A random sample of 16 students from the 8am class had an average score 73.2 with standard deviation 8.1. A random sample of 16 students from the 11am class had an average score of 78.1 with standard deviation 10. Assume that the populations variance are the same and the data are drawn from a normal distribution.

i. Do these data indicate that the mean score for the 11am class is higher than mean score for the 8am class?. Use a=0.05.

ii. What is the *P-value* for this test? What is your conclusion?

iii. Construct a 95% two-sided CI for the difference in means scores. Interpret this interval.

Solution

1. From the problem context, identify the parameter of interest

Parameter of interest; the difference between the effect of the score $\mu_1 - \mu_2$

2. State the null hypothesis $\mathbf{H_0}$ and appropriate alternative hypothesis, $\mathbf{H_1}$

$$H_0 : \mu_1 = \mu_2 \qquad\qquad H_1 : \mu_1 < \mu_2$$

3. Determine the appropriate test statistic

$$T = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

4. Critical value given α = 0.05

$$t_{\alpha,\, n_1+n_2-2} = t_{0.05,30} = 1.697$$

5. State the rejection region for the statistic

$$\Rightarrow \text{Reject } H_0 \text{ if } \; T < -1.697$$

6. Compute the value of the test statistic

$$\bar{X}_1 = 73.2,\ \ \bar{X}_2 = 78.1,\ \ s_1 = 8.1,\ \ s_2 = 10.0,\ \ n_1 = 16,\ \ \ n_2 = 16$$

$$S_p = \sqrt{\frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}} = \sqrt{\frac{(15)(8.1)^2 + (15)(10)^2}{16+16-2}} = 9.1$$

$$T = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$\bar{X}_1 = 73.2,\ \bar{X}_2 = 78.1,\ s_1 = 8.1,\ s_2 = 10.0,\ n_1 = 16,\ n_2 = 16$$

Compute the value of the test statistic:

$$T = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{(73.2 - 78.1) - (0)}{9.1\,(\sqrt{\frac{1}{16} + \frac{1}{16}})} = -1.52$$

7. Make a decision

Decision: since $T = -1.52 > -1.697$, we fail to reject H_0. Not enough evidence to say that the mean score at 11am is better than at 8am.

Find the p - value

From t-table with 30 df, $T = 1.52$ is between $t = 1.310$ and $t = 1.697$, which give $0.05 < p < 0.1$.

Since $P > 0.05$, thus we fail to reject H_0 at the 0.05 level of significance and conclude that there is not enough evidence to say that test at 11am is better result from test at 8am.

Find a 95% CI on the difference in means where $t_{0.025,30}$ =2.042 is

$$\bar{x}_1 = 73.2,\ \bar{x}_2 = 78.1,\ s_p = 8.95,\ n_1 = 16,\ n_2 = 16$$

$$(\bar{x}_1 - \bar{x}_2) - t_{\alpha/2, n_1+n_2-2}(s_p)\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \le \mu_1 - \mu_2 \le (\bar{x}_1 - \bar{x}_2) + t_{\alpha/2, n_1+n_2-2}(s_p)\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$(73.2 - 78.1) - (2.042)(9.1)\left(\sqrt{\frac{1}{16} + \frac{1}{16}}\right) \le \mu_1 - \mu_2 \le (73.2 - 78.1) + (2.042)(9.1)\left(\sqrt{\frac{1}{16} + \frac{1}{16}}\right)$$

$$-11.47 \le \mu_1 - \mu_2 \le 1.67$$

Since $\mu_1 - \mu_2 = 0$ is inside the interval then we fail to reject H_0 and conclude that there is not enough evidence to say that test at 11am is better result from test at 8am.

Exercise:

The diameter of steel rods manufactured on two different extrusion machines is being investigated. Two random samples of sizes $n_1 = 15$ and $n_2 = 17$ are selected, and $\bar{x}_1 = 8.73,\ s_1^2 = 0.35$ and $\bar{x}_2 = 8.68,\ s_2^2 = 0.40$ respectively. Assume that data are drawn normal distribution with equal variances. Given that $\alpha = 0.05$

a) Is there evidence to support the claim that the two machines produce rods with different mean diameters ? Use the p – value approach.

b) Construct a 95% two-sided CI on the difference in mean rod diameter.

Solution

1. From the problem context, identify the parameter of interest

Parameter of interest; the difference between the true average $\mu_1 - \mu_2$

2. State the null hypothesis H_0 and appropriate alternative hypothesis, H_1

$$H_0 : \mu_1 = \mu_2 \qquad\qquad H_1 : \mu_1 \neq \mu_2$$

3. Determine the appropriate test statistic

$$\boxed{T = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{S_p \sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}}$$

6. Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute the value

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{14(0.35) + 16(0.40)}{30}} = 0.614$$

$$T = \frac{(8.73 - 8.68) - 0}{0.614\sqrt{\frac{1}{15} + \frac{1}{17}}} = 0.230$$

Find the p - value

$$\text{P-value} = 2P\left(t > 0.23\right) > 2(0.40), \text{ P-value} > 0.80$$

Since p-value > 0.05, we do not reject the null hypothesis and conclude the two machines do not produce rods with significantly different mean diameters

b) 95% confidence interval: $t_{0.025,30} = 2.042$

$$\Delta = t_{\alpha/2, n_1+n_2-2}(s_p)\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} \quad = 2.042(0.614)\sqrt{\frac{1}{15}+\frac{1}{17}}$$

$$(8.73-8.68)-\Delta \leq (\mu_1-\mu_2) \leq (8.73-8.68)+\Delta$$

$$-0.394 \leq (\mu_1-\mu_2) \leq 0.494$$

$\boxed{\text{95\% Confidence interval is [-0.394, 0.494]}}$

Chapter 7
Hypothesis Testing – Two Populations

L3: - Hypothesis Testing – Between two means
(2 populations, variance unknown but not equal)
- Hypothesis Testing – Between two means
(2 populations, variance unknown –large samples)
- Hypothesis Testing- proportion
(2 populations)

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Learning Objectives

At the end of the lesson student should be able to

- Carry out hypothesis testing for difference in means for two normal populations (variances unknown)
- Carry out hypothesis testing for difference in means for difference in two proportion

TWO VARIANCES ARE UNKNOWN AND UNEQUAL

- **Test about the difference** $\mu_1 - \mu_2 = 0$

Case 2: $\sigma_1^2 \neq \sigma_2^2$

Null Hypothesis is the same as before:

Test statistic:

$$T = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{(S_1^2 / n_1) + (S_2^2 / n_2)}}$$

But the degree of freedom is given by ν

$$\nu = \frac{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}\right)^2}{\dfrac{(s_1^2 / n_1)^2}{n_1 - 1} + \dfrac{(s_2^2 / n_2)^2}{n_2 - 1}}$$

If ν **is not an integer, round down to the nearest integer.**

TEST ABOUT TWO NORMAL MEANS WHEN TWO VARIANCES ARE UNKNOWN

- **Test about the difference** $\mu_1 - \mu_2 = 0$

Case 2: $\sigma_1^2 \neq \sigma_2^2$

Critical region:

Alternative Hypothesis	Rejection Criteria (Reject H_0)
$H_1 : \mu_1 \neq \mu_2$	$T > t_{\alpha/2,v}$ *or* $T < -t_{\alpha/2,v}$
$H_1 : \mu_1 > \mu_2$	$T > t_{\alpha,v}$
$H_1 : \mu_1 < \mu_2$	$T < -t_{\alpha,v}$

Confidence interval:

A $100(1-\alpha)\%$ **two sided confidence interval for** $\mu_1 - \mu_2$ **when variance is unknown :**

Case 2: $\sigma_1^2 \neq \sigma_2^2$

$$(\bar{x}_1 - \bar{x}_2) - t_{\alpha/2,\nu}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \leq \mu_1 - \mu_2$$

$$\leq (\bar{x}_1 - \bar{x}_2) + t_{\alpha/2,\nu}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Value of ν **is from the formula earlier**

Example 1:

Two companies manufacture a rubber material intended for use in an automotive application. 25 samples of material from each company are tested, and the amount of wear after 1000 cycles are observed. For company 1, the sample mean and standard deviation of wear are

$$\bar{x}_1 = 20.12mg/1000\text{cycles and } s_1 = 1.9mg/1000\text{cycles}$$

and for company 2, we obtain

$$\bar{x}_2 = 11.64mg/1000\text{cycles and } s_2 = 7.9mg/1000\text{cycles}$$

Do the sample data support the claim that the two companies produce material with different mean wear. Use 0.02 level of significance. Assume each population is normally distributed but unequal variances?

1. Identify the parameter of interest

Parameter of interest; the difference between the true average $\mu_1 - \mu_2$
Variances unknown and unequal

2. State the null hypothesis H_0 and appropriate alternative hypothesis, H_1

$$H_0 : \mu_1 - \mu_2 = 0 \qquad H_1 : \mu_1 - \mu_2 \neq 0$$

3. Determine the appropriate test statistic

$$T = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{(S_1^2 / n_1) + (S_2^2 / n_2)}}$$

4. Critical value given $\alpha = 0.02$ $t_{\alpha/2,\,v} = t_{0.01,26} = 2.479$

$$v = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2 - 1}} = \frac{\left(\frac{(1.9)^2}{25} + \frac{(7.9)^2}{25}\right)^2}{\frac{\left(\frac{(1.9)^2}{25}\right)^2}{24} + \frac{\left(\frac{(7.9)^2}{25}\right)^2}{24}} = 26.77$$

$$v \cong 26$$

5. State the rejection region for the statistic

$$\Rightarrow \text{Reject } H_0 \text{ if } |T| > 2.479$$

6. Compute the value of the test statistic

$$T = \frac{(20.12 - 11.64) - 0}{\sqrt{\frac{(1.9)^2}{25} + \frac{(7.9)^2}{25}}} = 5.22$$

7. Make a decision

Decision: since $T = 5.22 > 2.479$ we reject H_0

Enough evidence to support that $\mu_1 \neq \mu_2$

Exercise:

The following data represent the running times of films produced by 2 motion-picture companies. Test the hypothesis that the average running time of films produced by company 2 exceeds the average running time of films produced by company 1 by 10 minutes against the one-sided alternative that the difference is less than 10 minutes? Use $\alpha = 0.01$ and assume the distributions of times to be approximately normal with unequal variances.

Company \ Time							
X_1	102	86	98	109	92		
X_2	81	165	97	134	92	87	114

1. Identify the parameter of interest

Parameter of interest; the difference between the true average $\mu_2 - \mu_1$
Variances unknown and unequal

2. State the null hypothesis H_0 and appropriate alternative hypothesis, H_1

$$H_0 : \mu_2 - \mu_1 \geq 10 \qquad\qquad H_1 : \mu_2 - \mu_1 < 10$$

3. Determine the appropriate test statistic

$$T = \frac{(\bar{x}_2 - \bar{x}_1) - (\mu_2 - \mu_1)}{\sqrt{(S_1^2 / n_1) + (S_2^2 / n_2)}}$$

4. Critical value given $\alpha = 0.01$

$$t_{\alpha,v} = t_{0.01,7} = 2.998$$

$$v = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}} = \frac{(78.8/5 + 913.333/7)^2}{(78.8/5)^2/4 + (913.333/7)^2/6} = 7.38$$

5. State the rejection region for the statistic

$$\Rightarrow \text{Reject } H_0 \text{ if } T < -2.998$$

6. Compute the value of the test statistic

$$T = \frac{(\bar{x}_2 - \bar{x}_1) - (\mu_2 - \mu_1)}{\sqrt{(S_1^2 / n_1) + (S_2^2 / n_2)}}$$

$$= \frac{(110 - 97.4) - 10}{\sqrt{78.8 / 5 + 913.333 / 7}} = 0.22$$

7. Make a decision

Decision: since $T = 0.22 > -2.998$, we fail to reject H_0

Not enough evidence that $\mu_2 - \mu_1 < 10$

Case 3:

❖ **Test about the difference between two means (Large Samples)**

Null Hypothesis:

$$H_0 : \mu_1 = \mu_2 \quad \textbf{vs} \quad H_1 : \mu_1 \neq \mu_2, \text{or } H_1 : \mu_1 > \mu_2 \text{ or } H_1 : \mu_1 < \mu_2$$

Test statistic:

$$\boxed{Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \sim N(0,1) \text{ under } \mathrm{H}_0}$$

Alternative Hypothesis	Rejection Criteria (Reject H_0)
$H_1 : \mu_1 \neq \mu_2$	$Z > z_{\alpha/2}$ or $Z < -z_{\alpha/2}$
$H_1 : \mu_1 > \mu_2$	$Z > z_{\alpha}$
$H_1 : \mu_1 < \mu_2$	$Z < -z_{\alpha}$

A $100(1-\alpha)\%$ **confidence interval for** $\mu_1 - \mu_2$ **is**

$$(\bar{x}_1 - \bar{x}_2) - z_{\alpha/2}\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} \le \mu_1 - \mu_2 \le (\bar{x}_1 - \bar{x}_2) + z_{\alpha/2}\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$

The procedures used in hypothesis testing for 2 populations are the same as in the hypothesis testing for one population.

Conclusion of the test could also be based on the confidence Interval and the p-value.

Example 3:

A study wants to investigate if the mean price of a new Proton car has changed in 2012 compared to 2011. A sample of cars were taken during these two years and summary information as given below:

Year	2011	2012
Mean car price	RM 45,000	RM 48,000
Standard deviation of car prices	RM 1,500	RM 2,000
Number of cars	150	200

i. Perform the test for the above investigation. What is your conclusion?
ii. Find the two sided 95% CI for the difference between the two means in 2011 and 2012. What is your conclusion comparing with part (i).

Solution

1. From the problem context, identify the parameter of interest

To test the difference between the true average $\mu_1 - \mu_2$ large sample

2. State the null hypothesis $\mathbf{H_0}$ and appropriate alternative hypothesis, $\mathbf{H_1}$

$$H_0 : \mu_1 = \mu_2 \qquad\qquad H_1 : \mu_1 \neq \mu_2$$

3. Determine the appropriate test statistic

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{(S_1^2 / n_1) + (S_2^2 / n_2)}}$$

6. Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute the value

$$\bar{X}_1 = 45000,\ \bar{X}_2 = 48{,}000,\ S_1 = 1500,\ S_2 = 2000,\ n_1 = 150,\ n_2 = 200,$$

Compute the value of the test statistic:

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{(S_1^2 / n_1) + (S_2^2 / n_2)}} = \frac{45{,}000 - 48{,}000}{\sqrt{(1500^2 / 150) + (2000^2 / 200)}}$$

$$= -16.04$$

Find the p - value

P-value = $$2(1 - \Phi(16.04)) = 2(1 - 1) = 0$$

Since p-value < 0.05, we reject the null hypothesis at $\alpha = 0.05$. We conclude that the mean price of car in 2012 has change significantly compared to 2011.

Find a 95% CI on the difference in means.

From formula:

$$\boxed{(\bar{x}_1 - \bar{x}_2) - z_{\alpha/2}\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} \le (\mu_1 - \mu_2) \le (\bar{x}_1 - \bar{x}_2) + z_{\alpha/2}\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

$$(45000 - 48000) - 1.96\sqrt{\frac{1500^2}{150} + \frac{2000^2}{200}} \le \mu_1 - \mu_2 \le (45000 - 48000) + 1.96\sqrt{\frac{1500^2}{150} + \frac{2000^2}{200}}$$

$$\therefore \quad -3366.67 \le \mu_1 - \mu_2 \le -2633.32$$

Since $\mu_1 - \mu_2 = 0$ is not in the interval then we reject H_0. We conclude that the mean price of car in 2012 has change significantly compared to 2011.

Exercise:

In a test to compare the effectiveness of two drugs designed to lower cholesterol levels, 75 randomly selected patients were given drug A and 100 randomly selected patients were given drug B. Those given drug A reduced their cholesterol levels by an average of 40mg with standard deviation 12mg, and those given drug B reduced their cholesterol levels by an average 42mg with standard deviation of 15mg. Use $\alpha = 0.05$

i. Perform the test for the above investigation. Can you conclude that the mean reduction using drug B is greater than that of drug A?

ii. Find the two sided 95% CI for the difference between the two means of reductions using drug A and drug B. What is your conclusion comparing with part (i).

TEST FOR TWO PROPORTIONS

❖ **Test Problems about two proportions:**

Null Hypothesis: $H_0 : p_1 = p_2 \quad (\text{or } H_0 : p_1 \le p_2,\ H_0 : p_1 \ge p_2\ )$

Test statistic:

$$\mathbf{Z} = \frac{\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2}{\sqrt{\hat{\mathbf{p}}(1-\hat{\mathbf{p}})(\frac{1}{\mathbf{n}_1} + \frac{1}{\mathbf{n}_2})}}\ ;\ \hat{\mathbf{p}}_1 = \frac{\mathbf{X}_1}{\mathbf{n}_1}\ ,\ \hat{\mathbf{p}}_2 = \frac{\mathbf{X}_2}{\mathbf{n}_2}\ ,\ \hat{\mathbf{p}} = \frac{\mathbf{X}_1 + \mathbf{X}_2}{\mathbf{n}_1 + \mathbf{n}_2}$$

$$\mathbf{Z} \approx \mathbf{N}(0,1) \ \ \textbf{when } \mathbf{p}_1 = \mathbf{p}_2$$

Alternative Hypothesis	Rejection Criteria (Reject H_0)
$H_1 : p_1 \ne p_2$	$Z > z_{\alpha/2}$ or $Z < -z_{\alpha/2}$
$H_1 : p_1 > p_2$	$Z > z_{\alpha}$
$H_1 : p_1 < p_2$	$Z < -z_{\alpha}$

Confidence interval:

A $100(1-\alpha)\%$ **two sided confidence interval for** $p_1 - p_2$ **is:**

$$LL = (\hat{p}_1 - \hat{p}_2) - z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$$UL = (\hat{p}_1 - \hat{p}_2) + z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$$LL \le p_1 - p_2 \le UL$$

Example 4:

In a study on the effects of sodium restricted diets on hypertension, 24 out of 55 hypertensive patients were on sodium restricted diets, and 36 out of 149 non-hypertensive patients were on sodium restricted diets.

i. Test the hypothesis that the proportion of patients on sodium restricted diets is higher for hypertensive patients at a=0.05.
ii. What is the P-value for this test?
iii. Construct a 95% two sided CI and comment.

Solution

Step 1 **Problem: Test about two proportions large samples**

Step 2 $H_0 : p_A = p_B \quad H_1 : p_A > p_B$

Step 3 **Determine the appropriate test statistic**

$$Z = \frac{\hat{p}_A - \hat{p}_B}{\sqrt{\hat{p}(1 - \hat{p})(\frac{1}{n_A} + \frac{1}{n_B})}} \approx N(0,1) \quad \text{if} \quad p_A = p_B$$

$$\text{where} \quad \hat{p}_A = \frac{X_A}{n_A}, \quad \hat{p}_B = \frac{X_B}{n_B}, \quad \hat{p} = \frac{X_A + X_B}{n_A + n_B}$$

Step 4 **Critical value given $\alpha = 0.05$,** $\mathbf{z_{?} = z_{0.05} = 1.645}$

Step 5 **Rejection region for the statistic**

$$\Rightarrow \text{Reject } H_0 \text{ if } Z > 1.645$$

Step 6 **Compute the value of the test statistic:**

$$n_A = 55, n_B = 149, x_A = 24, x_B = 36, \hat{p}_A = 0.44, \hat{p}_B = 0.24$$

$$\Rightarrow z = \frac{0.44 - 0.24}{\sqrt{0.30(1-0.30)\left(\frac{1}{55} + \frac{1}{149}\right)}} = 4.9455, \text{ where } \hat{p} = \frac{24+36}{55+149} = 0.30$$

Step 7. Make a decision

Since $z_0 > 1.65$, then we reject H_0. It means that enough evidence to claim that the proportion of patients on hypertension is higher than non hypertension patients

Example 5:

A vote is to be taken among residents of a town and the surrounding county to determine whether a proposed chemical plant should be constructed. If 120 of 200 town voters favour the proposal and 240 of 500 county residents favour it, would you agree that the proportion of town voters favouring the proposal is higher than the proportion of county voters? Use $\alpha = 0.05$

Solution

Step 1 **Problem: Test about two proportions large samples**

Step 2 $H_0 : p_1 = p_2 \quad H_1 : p_1 > p_2$

Step 3 **Determine the appropriate test statistic**

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \approx N(0,1) \quad \text{if} \quad p_1 = p_2$$

$$\text{where} \quad \hat{p}_1 = \frac{X_1}{n_1}, \quad \hat{p}_2 = \frac{X_2}{n_2}, \quad \hat{p} = \frac{X_1 + X_2}{n_1 + n_2}$$

Step 4 **Critical value given $\alpha = 0.05$** $z_{\alpha} = z_{0.05} = 1.645$

Step 5 **Rejection region for the statistic**

$$\Rightarrow \text{Reject } H_0 \text{ if } Z > 1.645$$

Step 6 **Compute the value of the test statistic:**

$$X_1 = 120,\ X_2 = 240,\ n_1 = 200,\ n_2 = 500$$

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \approx N(0,1) \quad \text{if } p_1 = p_2$$

$$\text{where} \quad \hat{p}_1 = \frac{X_1}{n_1},\ \hat{p}_2 = \frac{X_2}{n_2},\ \hat{p} = \frac{X_1 + X_2}{n_1 + n_2}$$

$$\hat{p}_1 = \frac{120}{200},\ \hat{p}_2 = \frac{240}{500},\ \hat{p} = \frac{120 + 240}{200 + 500}$$

$$\Rightarrow Z = 2.9$$

Step 7. **Make a decision**

Decision: since $Z = 2.9 > 1.645$ **, we reject H_0 and agree that the proportion of town voters favouring the proposal is higher than the proportion of county voters.**