Complete Six-Degree-of-Freedom Barrowman Aerodynamic Method

Full Nonlinear Equations, All Stability Derivatives, Higher-Order Terms, and Transfer Functions for Slender Finned Rockets

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Abstract

This document provides a complete, implementation-ready reference for six-degree-of-freedom (6DOF) flight dynamics of slender finned rockets using Barrowman aerodynamic theory. It includes: (i) full nonlinear rigid-body equations of motion in body axes; (ii) complete static aerodynamic coefficients for bodies of revolution and fin sets with explicit derivations from slender-body theory; (iii) all dynamic stability derivatives including damping and cross-coupling terms; (iv) systematic first, second, and selected third-order derivatives in angle of attack, sideslip, Mach number, and angular rates; (v) linearization about trim conditions; (vi) full 12-state linear model with longitudinal, lateral-directional, and coupling dynamics; (vii) transfer functions for all control channels; and (viii) detailed worked examples with numerical values.

Assumptions: small-to-moderate angles of attack and sideslip, attached flow, slender axisymmetric body, thin flat-plate fins, negligible aeroelastic effects. Extended validity considerations and empirical corrections are discussed where appropriate.

Contents

1 Limitations and extensions

Barrowman's method is a *linear, potential-flow* approximation. It is valid for:

- $|\alpha|, |\beta| \lesssim 10^\circ$ (attached flow)
- Slender bodies: length-to-diameter ratio $L/d \gtrsim 5$
- Thin fins: thickness-to-chord ratio $t/c \lesssim 0.05$
- Subsonic to low supersonic Mach numbers (with appropriate corrections)

Outside these regimes, use empirical corrections (e.g., nonlinear $\sin\alpha$ factors, vortex-lift augmentation for bodies at high α) or transition to CFD/wind-tunnel data.

2 Notation, Coordinate Systems, and Conventions

2.1 Body-fixed reference frame

We use a right-handed body-fixed coordinate system with origin at a convenient reference point (typically nose tip or center of gravity):

$$\begin{aligned} &x\text{-axis:} && \text{forward along vehicle symmetry axis (positive toward nose)} \\ &y\text{-axis:} && \text{to the right (starboard) when looking forward} \\ &z\text{-axis:} && \text{downward (completing right-hand system)} \end{aligned} \tag{1}$$

The body angular velocity vector is $\boldsymbol{\omega} = [p,\, q,\, r]^T$ where:

$$p = \text{roll rate (rad/s, about } x), \quad q = \text{pitch rate (rad/s, about } y), \quad r = \text{yaw rate (rad/s, about } z). \tag{2}$$

2.2 Aerodynamic angles and velocity components

Let $\mathbf{V}_{\text{body}} = [u,\, v,\, w]^T$ be the inertial velocity resolved in body axes. The total airspeed, angle of attack, and sideslip angle are

$$V = \sqrt{u^2 + v^2 + w^2}, \qquad \alpha = \arctan\frac{w}{u}, \qquad \beta = \arcsin\frac{v}{V}. \tag{3}$$

For small angles, $\alpha \approx w/u$ and $\beta \approx v/V$.

2.3 Reference quantities and nondimensionalization

Aerodynamic forces and moments are nondimensionalized by:

$$\text{Reference area:} \quad S = \frac{\pi}{4} d^2 \quad \text{(body maximum cross-section)}, \tag{4}$$

$$\text{Reference length:} \quad L_{\text{ref}} = d \quad \text{(body diameter)}, \tag{5}$$

$$\text{Dynamic pressure:} \quad q_\infty = \frac{1}{2}\rho V^2. \tag{6}$$

Force coefficients:

$$C_X = \frac{F_x}{q_\infty S}, \quad C_Y = \frac{F_y}{q_\infty S}, \quad C_Z = \frac{F_z}{q_\infty S}. \tag{7}$$

Moment coefficients (about the center of gravity):

$$C_\ell = \frac{L_x}{q_\infty S d}, \quad C_m = \frac{M_y}{q_\infty S d}, \quad C_n = \frac{N_z}{q_\infty S d}. \tag{8}$$

Here L_x is the rolling moment, M_y the pitching moment, and N_z the yawing moment.

2.4 Nondimensional angular rates

Define nondimensional rate parameters:

$$\hat{p} = \frac{p\,d}{2V}, \qquad \hat{q} = \frac{q\,d}{2V}, \qquad \hat{r} = \frac{r\,d}{2V}. \tag{9}$$

These appear in dynamic derivative definitions.

2.5 Euler angles and transformation

Inertial-to-body transformation uses Euler angles (ϕ, θ, ψ) (roll, pitch, yaw in a 3-2-1 sequence). The kinematic equations relating body rates to Euler rate are:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}. \tag{10}$$

3 Full Nonlinear Six-Degree-of-Freedom Equations of Motion

3.1 Force equations (translational dynamics)

The body-axis force equations for a rigid body are:

$$m(\dot{u} + qw - rv) = F_x + T_x + mg_x, \tag{11}$$
$$m(\dot{v} + ru - pw) = F_y + T_y + mg_y, \tag{12}$$
$$m(\dot{w} + pv - qu) = F_z + T_z + mg_z, \tag{13}$$

where m is mass, F_x, F_y, F_z are aerodynamic forces in body axes, T_x, T_y, T_z are thrust components, and g_x, g_y, g_z are gravitational acceleration components transformed into body axes:

$$\begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{bmatrix} g_0, \tag{14}$$

with $g_0 = 9.80665\,\mathrm{m/s^2}$.

3.2 Moment equations (rotational dynamics)

The body-axis moment equations are:

$$I_x\dot{p} - (I_y - I_z)qr - I_{xz}(\dot{r} + pq) = L_x, \tag{15}$$
$$I_y\dot{q} + (I_x - I_z)pr - I_{xz}(p^2 - r^2) = M_y, \tag{16}$$
$$I_z\dot{r} - (I_x - I_y)pq + I_{xz}(\dot{p} - qr) = N_z, \tag{17}$$

where I_x, I_y, I_z are principal moments of inertia and I_{xz} is the product of inertia (typically $I_{xy} = I_{yz} = 0$ for axisymmetric rockets). For an axisymmetric vehicle with x along the axis, $I_y = I_z$ and the moment equations simplify.

3.3 Kinematic equations

Position in inertial frame $\mathbf{r}_I = [x_I, y_I, z_I]^T$ evolves as:

$$\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{z}_I \end{bmatrix} = \mathbf{R}_{IB} \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \tag{18}$$

where $\mathbf{R}_{IB}$ is the inertial-to-body direction-cosine matrix (DCM) constructed from (ϕ, θ, ψ).

3.4 The 12-state vector

The full nonlinear state is:

$$\mathbf{x} = [u,\, v,\, w,\, p,\, q,\, r,\, \phi,\, \theta,\, \psi,\, x_I,\, y_I,\, z_I]^T. \tag{19}$$

Equations (3.1)–(3.7) define $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ where $\mathbf{u}$ is the control vector (fin deflections, TVC angles, throttle, etc.).

4 Body Aerodynamics from Slender-Body Theory

4.1 Axisymmetric components: normal force and pitching moment

For a body of revolution at angle of attack α, slender-body theory gives the local normal-force coefficient per unit length:

$$\frac{dC_N}{dx} = \frac{2}{S} \frac{dA(x)}{dx} \sin\alpha, \tag{20}$$

where $A(x) = \pi r(x)^2$ is the cross-sectional area at station x. Integrating over a component spanning $[0, l]$:

$$C_{N,B}(\alpha) = \frac{2}{S}\big[A(l) - A(0)\big] \sin\alpha \equiv K_B \sin\alpha, \tag{21}$$

with $K_B = 2\Delta A/S$ and $\Delta A = A(l) - A(0)$.

The pitching-moment coefficient about the component's upstream end is the first moment:

$$C_{m,B}(\alpha) = \frac{2}{Sd} \int_0^l x \frac{dA}{dx}\, dx \, \sin\alpha = \frac{2}{Sd}\Big[l\, A(l) - \int_0^l A(x)\, dx\Big] \sin\alpha. \tag{22}$$

Define the body volume $V_B = \int_0^l A(x)\, dx$; then

$$C_{m,B}(\alpha) = K_M \sin\alpha, \qquad K_M = \frac{2}{Sd}\big[lA(l) - V_B\big]. \tag{23}$$

4.2 Body center of pressure

The body-component CP measured from its upstream end is:

$$X_B = \frac{C_{m,B,\alpha}}{C_{N,B,\alpha}}\, d = \frac{l\, A(l) - V_B}{A(l) - A(0)}. \tag{24}$$

4.3 Lateral-directional symmetry

Because the body is axisymmetric, the same expressions apply in the lateral plane:

$$C_{Y,B}(\beta) = K_B \sin\beta, \qquad C_{n,B}(\beta) = K_M \sin\beta, \tag{25}$$

where now β is the sideslip angle. There is no body contribution to rolling moment: $C_{\ell,B} = 0$ for a symmetric body.

4.4 Common body components

4.4.0.1 Conical nose. For a cone of length L_n from apex to base diameter d:

$$A(0) = 0, \quad A(L_n) = \frac{\pi d^2}{4} = S, \quad V_B = \frac{1}{3} S L_n. \tag{26}$$

Then:

$$K_B = 2, \qquad K_M = \frac{2}{3}\frac{L_n}{d}, \qquad X_B = \frac{2L_n}{3}. \tag{27}$$

4.4.0.2 Cylindrical section. $\Delta A = 0 \Rightarrow K_B = 0$, no Barrowman body lift.

4.4.0.3 Boattail. Diameter decreasing from d to d_b over length L_b:

$$\Delta A = \frac{\pi}{4}(d_b^2 - d^2), \qquad K_B = \frac{2\Delta A}{S} = 2\left(\frac{d_b^2 - d^2}{d^2}\right) < 0. \tag{28}$$

The negative contribution shifts the total CP aft.

4.5 Higher-order derivatives in α and β

Retaining $\sin\alpha$:

$$\frac{\partial C_{N,B}}{\partial \alpha} = K_B \cos\alpha, \qquad \frac{\partial^2 C_{N,B}}{\partial \alpha^2} = -K_B \sin\alpha, \qquad \frac{\partial^3 C_{N,B}}{\partial \alpha^3} = -K_B \cos\alpha. \tag{29}$$

Analogous derivatives hold for $C_{Y,B}(\beta)$, $C_{m,B}(\alpha)$, and $C_{n,B}(\beta)$.

5 Fin Aerodynamics: Planform Effects, Compressibility, and Interference

5.1 Fin geometry definitions

Consider a trapezoidal fin with:

- Root chord c_r (at body surface)
- Tip chord c_t
- Exposed semi-span s (from body surface to tip)
- Taper ratio $\lambda = c_t/c_r$

- Leading-edge sweep angle Λ_{LE}
- Mid-chord sweep angle Γ_c

The chord varies linearly with span:

$$c(y) = c_r + m\,y, \qquad m = \frac{c_t - c_r}{s}, \qquad 0 \le y \le s. \tag{30}$$

The planform area of one fin is:

$$A_f = \int_0^s c(y)\,dy = \frac{s}{2}(c_r + c_t). \tag{31}$$

The body radius at the fin trailing-edge station is denoted r_t.

5.2 Subsonic fin normal-force slope (single fin)

For a single thin fin in subsonic inviscid flow, combining Prandtl–Glauert compressibility correction with low-aspect-ratio effects, a widely validated form is:

$$(C_{N\alpha})_1 = \frac{2\pi}{\beta}\,\frac{A_f}{S}\,\frac{\mathcal{R}}{2 + \sqrt{4 + \left(\frac{\mathcal{R}}{\cos\Gamma_c}\right)^2}}, \tag{32}$$

where

$$\beta = \sqrt{1 - M^2}, \qquad \mathcal{R} = \frac{2s^2}{A_f} = \frac{4s}{c_r + c_t} \quad \text{(aspect-ratio parameter)}. \tag{33}$$

Equation (32) interpolates between the 2D thin-airfoil slope $2\pi/\beta$ (high AR) and the slender-body limit (low AR).

5.3 Multiple fin sets: projection factor

For N equally spaced fins (e.g., $N = 3$ or $N = 4$), the effective normal-force slope in pitch (or yaw) sums the projections:

$$(C_{N\alpha})_N = \sum_{k=1}^{N} (C_{N\alpha})_1 \sin^2\phi_k, \tag{34}$$

where ϕ_k is the azimuthal angle of fin k. For $N \ge 3$ equally spaced fins, $\sum_k \sin^2\phi_k = N/2$, so:

$$(C_{N\alpha})_N = \frac{N}{2}\,(C_{N\alpha})_1. \tag{35}$$

Similarly for lateral (sideslip):

$$(C_{Y\beta})_N = \frac{N}{2}\,(C_{Y\beta})_1 = \frac{N}{2}\,(C_{N\alpha})_1. \tag{36}$$

5.4 Fin–body interference factor

The body presence enhances fin effectiveness. A standard analytic interference factor is:

$$K_{T(B)} = 1 + \frac{r_t}{s + r_t}, \tag{37}$$

where r_t is the body radius at the fin station. The final fin-set slopes including interference are:

$$(C_{N\alpha})_{T(B)} = K_{T(B)}\,(C_{N\alpha})_N = K_{T(B)}\,\frac{N}{2}\,(C_{N\alpha})_1, \tag{38}$$

$$(C_{Y\beta})_{T(B)} = K_{T(B)}\,(C_{Y\beta})_N. \tag{39}$$

5.5 Supersonic fin lift (strip theory)

For $M > 1$, linearized supersonic theory gives the pressure coefficient on a thin flat plate at incidence α as:

$$C_p = \pm\frac{2\alpha}{\sqrt{M^2 - 1}}. \tag{40}$$

Integrating over the planform yields the normal-force slope. For swept fins, an effective correction is to replace M with $M_n = M\cos\Lambda$ (component normal to leading edge). A practical supersonic extension of (32) is:

$$(C_{N\alpha})_{1,\mathrm{sup}} = \frac{4}{\sqrt{M^2 - 1}}\,\frac{A_f}{S}\,\eta_{\mathrm{sweep}}, \tag{41}$$

where η_{sweep} accounts for leading-edge sweep (consult empirical correlations or strip integration). For unswept rectangular fins, $\eta_{\mathrm{sweep}} \approx 1$.

5.6 Transonic regime

In the range $0.8 \lesssim M \lesssim 1.2$, neither subsonic nor supersonic theory is accurate. Use empirical blending:

$$(C_{N\alpha})_{\mathrm{trans}} = f(M)\,(C_{N\alpha})_{\mathrm{sub}} + \big[1 - f(M)\big]\,(C_{N\alpha})_{\mathrm{sup}}, \tag{42}$$

with a smooth blending function $f(M)$ (e.g., cubic spline from $M = 0.9$ to $M = 1.1$).

5.7 Fin center of pressure (longitudinal)

The fin-set aerodynamic center is near the quarter-chord of the area-weighted mean. For a trapezoid with leading edge at $x_{\mathrm{LE}}(y) = x_{\mathrm{LE},0} + y\tan\Lambda_{\mathrm{LE}}$, the area-weighted quarter-chord location from the root leading edge is:

$$\bar{x}_{\mathrm{qc}} = \frac{\displaystyle\int_0^s \big[x_{\mathrm{LE}}(y) + 0.25\,c(y)\big]\,c(y)\,dy}{\displaystyle\int_0^s c(y)\,dy}. \tag{43}$$

Substituting $c(y) = c_r + my$ and $x_{\mathrm{LE}}(y) = y\tan\Lambda_{\mathrm{LE}}$:

$$\bar{x}_{\mathrm{qc}} = \frac{\tan\Lambda_{\mathrm{LE}}\left(\frac{c_r s^2}{2} + \frac{m s^3}{3}\right) + \frac{1}{4}\left(c_r^2 s + c_r m s^2 + \frac{m^2 s^3}{3}\right)}{\frac{s}{2}(c_r + c_t)}. \tag{44}$$

The fin-set CP measured from the nose tip is:

$$X_T = X_{f,\mathrm{LE}} + \bar{x}_{\mathrm{qc}}, \tag{45}$$

where $X_{f,\mathrm{LE}}$ is the axial station of the fin root leading edge from the nose.

5.8 Lateral center of pressure for roll calculations

The spanwise centroid of the fin normal-force distribution (for roll-moment arm) is:

$$\bar{Y}_T = r_t + \frac{\displaystyle\int_0^s y\, c(y)\, dy}{\displaystyle\int_0^s c(y)\, dy} = r_t + \frac{s}{3}\, \frac{c_r + 2c_t}{c_r + c_t}. \tag{46}$$

5.9 Mach-number derivatives

From (32), $\partial(C_{N\alpha})/\partial M$ arises from $\partial(1/\beta)/\partial M$:

$$\frac{\partial}{\partial M}\left(\frac{1}{\beta}\right) = \frac{M}{\beta^3}, \qquad \frac{\partial^2}{\partial M^2}\left(\frac{1}{\beta}\right) = \frac{1 + 2M^2}{\beta^5}. \tag{47}$$

Propagate these through (32) to obtain:

$$\frac{\partial (C_{N\alpha})_1}{\partial M} = \frac{M}{\beta^3}\, \frac{2\pi A_f}{S}\, \frac{\mathcal{R}}{2 + \sqrt{\cdots}}, \qquad \frac{\partial^2 (C_{N\alpha})_1}{\partial M^2} = \frac{1 + 2M^2}{\beta^5}\, \frac{2\pi A_f}{S}\, \frac{\mathcal{R}}{2 + \sqrt{\cdots}}. \tag{48}$$

5.10 Angle-of-attack higher derivatives for fins

If we retain a $\sin\alpha/\alpha$ factor (modified Barrowman):

$$C_{N,T}(\alpha) = (C_{N\alpha})_{T(B)}\, \alpha\, \frac{\sin\alpha}{\alpha} = (C_{N\alpha})_{T(B)} \sin\alpha, \tag{49}$$

then derivatives are:

$$\frac{\partial C_{N,T}}{\partial \alpha} = (C_{N\alpha})_{T(B)} \cos\alpha, \tag{50}$$

$$\frac{\partial^2 C_{N,T}}{\partial \alpha^2} = -(C_{N\alpha})_{T(B)} \sin\alpha, \tag{51}$$

$$\frac{\partial^3 C_{N,T}}{\partial \alpha^3} = -(C_{N\alpha})_{T(B)} \cos\alpha. \tag{52}$$

For strictly linear models, set $\sin\alpha/\alpha \to 1$ and all higher derivatives vanish.

6 Complete Static Stability Derivatives

6.1 Total normal-force and pitching-moment slopes

Sum body and fin contributions:

$$C_{N\alpha,\mathrm{tot}} = (C_{N\alpha})_B + (C_{N\alpha})_{T(B)}, \tag{53}$$

$$C_{m\alpha,\mathrm{tot}} = C_{m\alpha,B} + C_{m\alpha,T}. \tag{54}$$

The body contribution $(C_{N\alpha})_B$ is the sum over all axisymmetric components (nose, shoulders, boattails). The fin contribution $(C_{N\alpha})_{T(B)}$ includes interference.

6.2 Overall center of pressure and static margin

The total longitudinal CP from the nose tip is the normal-force-weighted average:

$$X_{\mathrm{CP}} = \frac{X_B\,(C_{N\alpha})_B + X_T\,(C_{N\alpha})_{T(B)}}{C_{N\alpha,\mathrm{tot}}}. \tag{55}$$

Given a center-of-gravity location X_{CG} from the nose, the static pitching-moment slope about the CG is:

$$C_{m\alpha} = -\,C_{N\alpha,\mathrm{tot}}\,\frac{X_{\mathrm{CP}} - X_{\mathrm{CG}}}{d}. \tag{56}$$

The **static margin** is:

$$\mathrm{SM} = \frac{X_{\mathrm{CP}} - X_{\mathrm{CG}}}{d} \quad \text{(positive for stability)}. \tag{57}$$

6.3 Lateral-directional static derivatives

By axisymmetry of the body and equal fin spacing:

$$C_{Y\beta} = (C_{Y\beta})_B + (C_{Y\beta})_{T(B)} = K_B \sin\beta + (C_{Y\beta})_{T(B)}, \tag{58}$$

$$C_{n\beta} = -\,C_{Y\beta}\,\frac{Y_{\mathrm{CP}}}{d}, \tag{59}$$

$$C_{\ell\beta} = \text{(body contribution is zero; fin contribution from dihedral/cant)}. \tag{60}$$

For vertical fins (no dihedral), $C_{\ell\beta} \approx 0$ in the linear regime. If fins have dihedral angle Γ_d, a side-force produces a rolling moment:

$$C_{\ell\beta,\mathrm{dihedral}} = -(C_{Y\beta})_{T(B)}\,\sin\Gamma_d\,\frac{\bar{Y}_T}{d}. \tag{61}$$

6.4 Axial-force coefficient C_X

The axial force includes:

- **Zero-lift drag** C_{D0}: skin friction, base drag, wave drag (function of M).
- **Induced drag** from lift: $C_{D,\mathrm{ind}} = \frac{(C_N^2 + C_Y^2)}{\pi e\, AR_{\mathrm{eff}}}$, where e is Oswald efficiency and AR_{eff} an effective aspect ratio.

In body axes with small α, $C_X \approx -C_D - C_N\alpha - C_Y\beta$. Barrowman focuses on C_N, C_m prediction; C_X typically relies on empirical drag polars.

6.5 Summary: static derivative matrix (linear regime)

In small-angle linearization:

$$\begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix} \approx \begin{bmatrix} C_{X0} & 0 & C_{X\alpha} \\ 0 & C_{Y\beta} & 0 \\ C_{Z0} & 0 & C_{Z\alpha} \end{bmatrix} \begin{bmatrix} 1 \\ \beta \\ \alpha \end{bmatrix}, \qquad \begin{bmatrix} C_\ell \\ C_m \\ C_n \end{bmatrix} \approx \begin{bmatrix} 0 & C_{\ell\beta} & 0 \\ 0 & 0 & C_{m\alpha} \\ 0 & C_{n\beta} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \beta \\ \alpha \end{bmatrix}, \tag{62}$$

where $C_{Z\alpha} = -C_{N\alpha,\mathrm{tot}}$ and $C_{m\alpha}$ is given by (56).

7 Dynamic Stability Derivatives: Damping and Rate Effects

7.1 Pitch-rate damping C_{mq} and C_{Zq}

A pitch rate q (rad/s) induces a local effective angle of attack $\Delta\alpha(x) = q(x - X_{\text{CG}})/V$ at station x. The fin set at X_T sees:

$$\Delta\alpha_T = \frac{q(X_T - X_{\text{CG}})}{V} = \frac{qd}{2V}\,\frac{2(X_T - X_{\text{CG}})}{d} = \hat{q}\,\frac{2\Delta x_T}{d}, \tag{63}$$

where $\Delta x_T = X_T - X_{\text{CG}}$ and $\hat{q} = qd/(2V)$. The incremental normal force is:

$$\Delta C_N = (C_{N\alpha})_{T(B)}\,\Delta\alpha_T = (C_{N\alpha})_{T(B)}\,\hat{q}\,\frac{2\Delta x_T}{d}. \tag{64}$$

This contributes to the Z-force derivative:

$$C_{Zq} = \left.\frac{\partial C_Z}{\partial \hat{q}}\right|_{\hat{q}=0} = -\,(C_{N\alpha})_{T(B)}\,\frac{2\Delta x_T}{d}. \tag{65}$$

The incremental pitching moment about the CG is $\Delta C_m = \Delta C_N \cdot (\Delta x_T/d)$, giving:

$$C_{mq} = \left.\frac{\partial C_m}{\partial \hat{q}}\right|_{\hat{q}=0} = (C_{N\alpha})_{T(B)}\left(\frac{2\Delta x_T}{d}\right)\left(\frac{\Delta x_T}{d}\right) = \boxed{\frac{2\Delta x_T^2}{d^2}\,(C_{N\alpha})_{T(B)}}. \tag{66}$$

7.1.0.1 Body contribution. The body itself can contribute pitch damping via unsteady effects (added mass, vortex shedding). A simple empirical augmentation is:

$$C_{mq,\text{body}} \approx -\,K_q\,\frac{V_B}{Sd}, \tag{67}$$

where $K_q \approx 0.5$–1.0 and V_B is the body volume. Include if high fidelity is needed.

7.2 Yaw-rate damping C_{nr} and C_{Yr}

By symmetry, the yaw-rate derivatives mirror the pitch-rate ones:

$$C_{nr} = \left.\frac{\partial C_n}{\partial \hat{r}}\right|_{\hat{r}=0} = \frac{2\Delta x_T^2}{d^2}\,(C_{Y\beta})_{T(B)}, \tag{68}$$

$$C_{Yr} = \left.\frac{\partial C_Y}{\partial \hat{r}}\right|_{\hat{r}=0} = -\,(C_{Y\beta})_{T(B)}\,\frac{2\Delta x_T}{d}. \tag{69}$$

7.3 Roll-rate damping $C_{\ell p}$

A roll rate p produces a spanwise distribution of incidence on the fins: $\Delta\alpha(y) = py/V$. The strip normal force is $dN = q_\infty\,a_f\,c(y)\,(py/V)\,dy$ where a_f is the fin 2D lift-curve slope. The elemental rolling moment is $dL_x = y\,dN$. Summing over all N fins:

$$C_{\ell p} = \left.\frac{\partial C_\ell}{\partial \hat{p}}\right|_{\hat{p}=0} = -\,\frac{2}{Sd}\,N\int_0^s a_f\,c(y)\,y^2\,dy. \tag{70}$$

For a trapezoid $c(y) = c_r + my$, the integral is:

$$\int_0^s c(y)\,y^2\,dy = \frac{c_r s^3}{3} + \frac{ms^4}{4}. \tag{71}$$

Use $a_f = (C_{N\alpha})_1\,(S/A_f)$ (the single-fin slope per unit angle). The negative sign in (70) indicates damping.

7.3.0.1 Explicit form.

$$C_{\ell p} = -\frac{2N}{Sd}\,(C_{N\alpha})_1\,\frac{S}{A_f}\left(\frac{c_r s^3}{3} + \frac{m s^4}{4}\right). \tag{72}$$

Include the interference factor if desired: replace $(C_{N\alpha})_1$ with $(C_{N\alpha})_1 K_{T(B)}$.

7.4 Cross-coupling: C_{Yp}, $C_{\ell r}$, C_{np}

In the linear regime with symmetric fin arrangement, many cross terms vanish:

$$C_{Yp} = 0 \quad \text{(symmetry)}, \tag{73}$$
$$C_{\ell r} \approx 0 \quad \text{(small for slender rockets)}, \tag{74}$$
$$C_{np} \approx 0 \quad \text{(symmetry)}. \tag{75}$$

Nonzero cross-coupling can arise from:

- Asymmetric fin cant or dihedral
- Body asymmetry (e.g., protuberances)
- Large α or β (nonlinear effects)

7.5 Magnus force and moment

For a spinning rocket, Magnus effects can produce side forces and moments. The Magnus force coefficient is:

$$C_{Y,\mathrm{Mag}} = C_{Y,p\alpha}\,\hat{p}\,\alpha, \tag{76}$$

where $C_{Y,p\alpha}$ depends on body shape and is typically determined empirically or via CFD. For preliminary design, Magnus effects are often neglected unless spin rates are high.

7.6 Summary: dynamic derivative matrix (linear)

The rate-dependent contributions are:

$$\begin{bmatrix}\Delta C_X\\ \Delta C_Y\\ \Delta C_Z\end{bmatrix} \approx \begin{bmatrix}0 & 0 & 0\\ 0 & C_{Yp} & C_{Yr}\\ 0 & 0 & C_{Zq}\end{bmatrix}\begin{bmatrix}\hat{p}\\ \hat{q}\\ \hat{r}\end{bmatrix}, \qquad \begin{bmatrix}\Delta C_\ell\\ \Delta C_m\\ \Delta C_n\end{bmatrix} \approx \begin{bmatrix}C_{\ell p} & 0 & C_{\ell r}\\ 0 & C_{mq} & 0\\ C_{np} & 0 & C_{nr}\end{bmatrix}\begin{bmatrix}\hat{p}\\ \hat{q}\\ \hat{r}\end{bmatrix}. \tag{77}$$

For standard rockets, the diagonal terms dominate.

8 Control Derivatives

8.1 Fin deflection (aerodynamic control)

For movable fins or control surfaces with deflection angle δ (positive trailing-edge down), the incremental normal force is approximately:

$$\Delta C_N = \tau\,(C_{N\alpha})_{\mathrm{surf}}\,\delta, \tag{78}$$

where $(C_{N\alpha})_{\text{surf}}$ is the surface's own lift-curve slope and τ is the **control effectiveness factor**:

$$\tau = \frac{\partial C_N/\partial \delta}{\partial C_N/\partial \alpha}. \tag{79}$$

For an all-moving fin, $\tau \approx 1$. For a hinged trailing-edge flap of chord ratio c_f/c, thin-airfoil theory gives:

$$\tau \approx \frac{1 - \sqrt{1 - c_f/c}}{1}. \tag{80}$$

8.1.0.1 Pitch control via symmetric fin deflection. For a set of N fins deflected symmetrically by δ_e (elevator):

$$C_{N\delta_e} = \tau\,(C_{N\alpha})_{T(B)}, \tag{81}$$

$$C_{m\delta_e} = -\,C_{N\delta_e}\,\frac{\Delta x_T}{d}, \tag{82}$$

where $\Delta x_T = X_T - X_{\text{CG}}$.

8.1.0.2 Yaw control via asymmetric fin deflection. For differential deflection (rudder) δ_r:

$$C_{Y\delta_r} = \tau\,(C_{Y\beta})_{T(B)}, \tag{83}$$

$$C_{n\delta_r} = -\,C_{Y\delta_r}\,\frac{\Delta x_T}{d}. \tag{84}$$

8.1.0.3 Roll control via differential aileron deflection. For N fins with alternating deflections $\pm\delta_a$, the rolling-moment derivative is:

$$C_{\ell\delta_a} = N\,\tau\,(C_{N\alpha})_1\,\frac{\bar{Y}_T}{d}, \tag{85}$$

where $\bar{Y}_T$ is the lateral CP of the fins (see Section 5.7).

8.2 Canard surfaces

Canards forward of the CG provide destabilizing normal force but can enhance control authority. Let $(C_{N\alpha})_C$ be the canard normal-force slope (computed as for fins), X_C the canard CP, and $\Delta x_C = X_C - X_{\text{CG}}$ (typically negative). Then:

$$C_{N\alpha,\text{total}} = C_{N\alpha,\text{body+tail}} + (C_{N\alpha})_C, \tag{86}$$

$$C_{m\alpha} = -\left[C_{N\alpha,\text{tail}}\,\frac{\Delta x_T}{d} + (C_{N\alpha})_C\,\frac{\Delta x_C}{d}\right], \tag{87}$$

and for canard deflection δ_c:

$$C_{N\delta_c} = \tau_c\,(C_{N\alpha})_C, \tag{88}$$

$$C_{m\delta_c} = -\,C_{N\delta_c}\,\frac{\Delta x_C}{d}. \tag{89}$$

The canard's negative moment arm (forward of CG) gives $C_{m\delta_c}$ the opposite sign from tail control.

8.3 Thrust vector control (TVC)

If the thrust vector can be gimbaled by angles $\delta_{T,y}$ (pitch plane) and $\delta_{T,z}$ (yaw plane), the force and moment increments are:

$$\Delta F_z = -T\,\delta_{T,y}, \qquad \Delta F_y = T\,\delta_{T,z}, \tag{90}$$

$$\Delta M_y = -T\,\delta_{T,y}\,\ell_T, \qquad \Delta N_z = T\,\delta_{T,z}\,\ell_T, \tag{91}$$

where $\ell_T = X_{\mathrm{CG}} - X_T$ is the (positive) moment arm of the thrust vector about the CG. Nondimensionalizing:

$$C_{Z\delta_{T,y}} = -\frac{T}{q_\infty S}, \qquad C_{m\delta_{T,y}} = -\frac{T\,\ell_T}{q_\infty S d} = C_{Z\delta_{T,y}}\,\frac{\ell_T}{d}, \tag{92}$$

$$C_{Y\delta_{T,z}} = \frac{T}{q_\infty S}, \qquad C_{n\delta_{T,z}} = \frac{T\,\ell_T}{q_\infty S d} = C_{Y\delta_{T,z}}\,\frac{\ell_T}{d}. \tag{93}$$

TVC is especially effective at low dynamic pressures (launch, high altitude).

8.4 Control coupling and limits

- **Fin stall**: For large δ, thin-airfoil theory breaks down. Use nonlinear $C_N(\delta)$ data or a saturation model, e.g., $C_N = C_{N\delta}\sin\delta$.
- **Hinge moments**: For hinged surfaces, the required actuator torque scales with dynamic pressure and surface area. Include a hinge-moment model $C_h = C_{h\alpha}\alpha + C_{h\delta}\delta$ for actuator sizing.
- **Cross-coupling**: Deflecting fins can induce roll (e.g., differential lift on asymmetric deflections). Track all six force/moment components for each control input.

9 Higher-Order Derivatives: Nonlinear Extensions

9.1 Second and third derivatives in α and β

For the body with $\sin\alpha$ retention:

$$\frac{\partial^k C_{N,B}}{\partial\alpha^k} = K_B\,\frac{d^k}{d\alpha^k}\sin\alpha, \qquad k = 1, 2, 3, \ldots \tag{94}$$

Explicitly:

$$\frac{\partial C_{N,B}}{\partial\alpha} = K_B\cos\alpha, \tag{95}$$

$$\frac{\partial^2 C_{N,B}}{\partial\alpha^2} = -K_B\sin\alpha, \tag{96}$$

$$\frac{\partial^3 C_{N,B}}{\partial\alpha^3} = -K_B\cos\alpha, \tag{97}$$

$$\frac{\partial^4 C_{N,B}}{\partial\alpha^4} = K_B\sin\alpha. \tag{98}$$

For fins, the same pattern holds. These derivatives enable Taylor-series expansion:

$$C_N(\alpha) = C_{N0} + C_{N\alpha}\alpha + \frac{1}{2}C_{N\alpha\alpha}\alpha^2 + \frac{1}{6}C_{N\alpha\alpha\alpha}\alpha^3 + \cdots \tag{99}$$

Evaluate at a trim condition α_0 (typically zero) and expand about it.

9.2 Mach-number second derivatives

From $\beta = \sqrt{1 - M^2}$:

$$\frac{d}{dM}\left(\frac{1}{\beta}\right) = \frac{M}{\beta^3}, \tag{100}$$

$$\frac{d^2}{dM^2}\left(\frac{1}{\beta}\right) = \frac{1 + 2M^2}{\beta^5}, \tag{101}$$

$$\frac{d^3}{dM^3}\left(\frac{1}{\beta}\right) = \frac{9M + 6M^3}{\beta^7}. \tag{102}$$

These propagate through any coefficient proportional to $1/\beta$. For example, if $C_{N\alpha} = K/\beta$, then:

$$\frac{\partial C_{N\alpha}}{\partial M} = K\,\frac{M}{\beta^3}, \tag{103}$$

$$\frac{\partial^2 C_{N\alpha}}{\partial M^2} = K\,\frac{1 + 2M^2}{\beta^5}, \tag{104}$$

$$\frac{\partial^3 C_{N\alpha}}{\partial M^3} = K\,\frac{9M + 6M^3}{\beta^7}. \tag{105}$$

9.3 Mixed derivatives: α–M coupling

The mixed derivative $\partial^2 C_N/\partial\alpha\partial M$ describes how the lift-curve slope changes with Mach. For $C_N = K \sin\alpha/\beta$:

$$\frac{\partial^2 C_N}{\partial\alpha\partial M} = K\,\frac{M}{\beta^3}\cos\alpha. \tag{106}$$

This is useful for adaptive control laws that schedule on M and α.

9.4 Nonlinear augmentation: vortex lift at high α

For long bodies at $\alpha \gtrsim 10^\circ$, asymmetric vortex shedding can produce additional normal force. A common empirical model is:

$$C_{N,\text{vortex}} = K_v\,A_{\text{plan}}/S\,\sin^2\alpha\cos\alpha, \tag{107}$$

where $K_v \approx 1.2$ and A_{plan} is the body planform area. Add this to the linear Barrowman prediction. Derivatives:

$$\frac{\partial C_{N,\text{vortex}}}{\partial\alpha} = K_v\,A_{\text{plan}}/S\,(2\sin\alpha\cos^2\alpha - \sin^3\alpha). \tag{108}$$

9.5 Summary: polynomial approximations

For implementation in flight code, approximate coefficients as polynomials:

$$C_N(\alpha, M) = a_0(M) + a_1(M)\alpha + a_2(M)\alpha^2 + a_3(M)\alpha^3, \tag{109}$$

$$a_k(M) = b_{k0} + b_{k1}M + b_{k2}M^2 + \cdots \tag{110}$$

Fit $\{a_k, b_{kj}\}$ from the Barrowman formulas or wind-tunnel data.

10 Linearization and the Full 12-State Linear Model

10.1 Trim condition and perturbation variables

Consider a steady-state trim condition (subscript $_0$):

$$\mathbf{x}_0 = [u_0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 0,\, \theta_0,\, 0,\, \cdots]^T, \tag{111}$$

corresponding to straight and level flight at constant velocity $V_0 = u_0$ and pitch angle θ_0 (often zero for rockets).

Define perturbation variables:

$$\Delta u = u - u_0, \quad \Delta v = v, \quad \Delta w = w, \quad \Delta p = p, \quad \Delta q = q, \quad \Delta r = r, \quad \Delta\phi = \phi, \quad \Delta\theta = \theta - \theta_0, \quad \Delta\psi = \psi. \tag{112}$$

For small perturbations, $\alpha \approx w/u_0$, $\beta \approx v/V_0 \approx v/u_0$.

10.2 Linearized force equations

Linearize the body-axis force equations (Section 3.1) about trim:

$$m\,\Delta\dot{u} = \Delta F_x - mg_0 \cos\theta_0\,\Delta\theta, \tag{113}$$

$$m\,\Delta\dot{v} = \Delta F_y + m(u_0\,\Delta r - w_0\,\Delta p) + mg_0 \sin\phi_0 \cos\theta_0\,\Delta\phi + mg_0 \cos\phi_0 \sin\theta_0\,\Delta\theta, \tag{114}$$

$$m\,\Delta\dot{w} = \Delta F_z + m(-u_0\,\Delta q + v_0\,\Delta p) + mg_0 \cos\phi_0 \cos\theta_0\,\Delta\theta. \tag{115}$$

For rockets in near-vertical flight, $\theta_0 \approx 90^\circ$ (nose up), simplifying gravity coupling.

Express aerodynamic forces in terms of derivatives:

$$\Delta F_x = q_\infty S \left(C_{X\alpha}\alpha + C_{Xu}\frac{\Delta u}{u_0} + C_{X\delta}\delta + \cdots \right), \tag{116}$$

$$\Delta F_y = q_\infty S \left(C_{Y\beta}\beta + C_{Yr}\hat{r} + C_{Y\delta}\delta + \cdots \right), \tag{117}$$

$$\Delta F_z = q_\infty S \left(C_{Z\alpha}\alpha + C_{Zq}\hat{q} + C_{Z\delta}\delta + \cdots \right). \tag{118}$$

10.3 Linearized moment equations

For an axisymmetric rocket ($I_y = I_z$, $I_{xz} = 0$), the linearized moment equations are:

$$I_x\,\Delta\dot{p} = \Delta L_x = q_\infty S d \left(C_{\ell\beta}\beta + C_{\ell p}\hat{p} + C_{\ell\delta}\delta + \cdots \right), \tag{119}$$

$$I_y\,\Delta\dot{q} = \Delta M_y = q_\infty S d \left(C_{m\alpha}\alpha + C_{mq}\hat{q} + C_{m\delta}\delta + \cdots \right), \tag{120}$$

$$I_z\,\Delta\dot{r} = \Delta N_z = q_\infty S d \left(C_{n\beta}\beta + C_{nr}\hat{r} + C_{n\delta}\delta + \cdots \right). \tag{121}$$

10.4 Kinematic equations (linearized)

For small Euler angles:

$$\Delta\dot{\phi} = \Delta p + \tan\theta_0\,(\Delta q \sin\phi_0 + \Delta r \cos\phi_0) \approx \Delta p, \tag{122}$$

$$\Delta\dot{\theta} = \Delta q \cos\phi_0 - \Delta r \sin\phi_0 \approx \Delta q, \tag{123}$$

$$\Delta\dot{\psi} = (\Delta q \sin\phi_0 + \Delta r \cos\phi_0)\sec\theta_0 \approx \Delta r / \cos\theta_0. \tag{124}$$

For rockets in vertical flight ($\theta_0 \approx 90^\circ$), special care is needed (Euler singularity); use quaternions or a modified frame.

10.5 State-space form: $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$

Define the 12-state vector (perturbations):

$$\mathbf{x} = [\Delta u,\, \Delta v,\, \Delta w,\, \Delta p,\, \Delta q,\, \Delta r,\, \Delta\phi,\, \Delta\theta,\, \Delta\psi,\, \Delta x_I,\, \Delta y_I,\, \Delta z_I]^T. \tag{125}$$

The control vector (example):

$$\mathbf{u} = [\delta_e,\, \delta_a,\, \delta_r,\, \delta_T]^T, \tag{126}$$

where δ_e (elevator), δ_a (aileron), δ_r (rudder), δ_T (throttle or TVC).

The system matrices A (12×12) and B (12×4) are populated from the dimensional derivatives. For example, the $(3,3)$ element of A (effect of Δw on $\Delta\dot{w}$):

$$A_{33} = \frac{q_\infty S}{m\, u_0}\, C_{Z\alpha}. \tag{127}$$

The $(5,3)$ element (effect of Δw on $\Delta\dot{q}$):

$$A_{53} = \frac{q_\infty S d}{I_y\, u_0}\, C_{m\alpha}. \tag{128}$$

10.5.0.1 Block-diagonal approximation. For slender rockets with $I_x \ll I_y \approx I_z$ and symmetric configuration, the system often decouples into:

- **Longitudinal subsystem**: $[\Delta u,\, \Delta w,\, \Delta q,\, \Delta\theta]^T$ (4 states)
- **Lateral-directional subsystem**: $[\Delta v,\, \Delta p,\, \Delta r,\, \Delta\phi,\, \Delta\psi]^T$ (5 states)
- **Position states**: $[\Delta x_I,\, \Delta y_I,\, \Delta z_I]^T$ (3 states, integrated from velocities)

This simplifies analysis and control design.

10.6 Longitudinal subsystem (detailed)

State vector $\mathbf{x}_{\text{long}} = [\Delta u,\, \Delta w,\, \Delta q,\, \Delta\theta]^T$, control $u_{\text{long}} = \delta_e$. The equations are:

$$\Delta\dot{u} = X_u\, \Delta u + X_w\, \Delta w - g_0 \cos\theta_0\, \Delta\theta + X_{\delta_e}\, \delta_e, \tag{129}$$

$$\Delta\dot{w} = Z_u\, \Delta u + Z_w\, \Delta w + (Z_q + u_0)\, \Delta q - g_0 \sin\theta_0\, \Delta\theta + Z_{\delta_e}\, \delta_e, \tag{130}$$

$$\Delta\dot{q} = M_u\, \Delta u + M_w\, \Delta w + M_q\, \Delta q + M_{\delta_e}\, \delta_e, \tag{131}$$

$$\Delta\dot{\theta} = \Delta q, \tag{132}$$

where the dimensional derivatives are:

$$X_u = \frac{q_\infty S}{m u_0} C_{Xu}, \quad X_w = \frac{q_\infty S}{m u_0} C_{X\alpha}, \quad Z_w = \frac{q_\infty S}{m} C_{Z\alpha}, \tag{133}$$

$$Z_q = \frac{q_\infty S d}{2 m V_0} C_{Zq}, \quad M_w = \frac{q_\infty S d}{I_y u_0} C_{m\alpha}, \quad M_q = \frac{q_\infty S d^2}{2 I_y V_0} C_{mq}, \tag{134}$$

$$X_{\delta_e} = \frac{q_\infty S}{m} C_{X\delta_e}, \quad Z_{\delta_e} = \frac{q_\infty S}{m} C_{Z\delta_e}, \quad M_{\delta_e} = \frac{q_\infty S d}{I_y} C_{m\delta_e}. \tag{135}$$

The state matrix:

$$A_{\text{long}} = \begin{bmatrix} X_u & X_w & 0 & -g_0\cos\theta_0 \\ Z_u & Z_w & Z_q + u_0 & -g_0\sin\theta_0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B_{\text{long}} = \begin{bmatrix} X_{\delta_e} \\ Z_{\delta_e} \\ M_{\delta_e} \\ 0 \end{bmatrix}. \tag{136}$$

10.7 Lateral-directional subsystem (detailed)

State vector $\mathbf{x}_{\text{lat}} = [\Delta v,\, \Delta p,\, \Delta r,\, \Delta\phi,\, \Delta\psi]^T$, control $\mathbf{u}_{\text{lat}} = [\delta_a,\, \delta_r]^T$. The equations:

$$\Delta\dot{v} = Y_v\,\Delta v + Y_p\,\Delta p + (Y_r - u_0)\,\Delta r + g_0\cos\theta_0\,\Delta\phi + Y_{\delta_a}\,\delta_a + Y_{\delta_r}\,\delta_r, \tag{137}$$

$$\Delta\dot{p} = L_v\,\Delta v + L_p\,\Delta p + L_r\,\Delta r + L_{\delta_a}\,\delta_a + L_{\delta_r}\,\delta_r, \tag{138}$$

$$\Delta\dot{r} = N_v\,\Delta v + N_p\,\Delta p + N_r\,\Delta r + N_{\delta_a}\,\delta_a + N_{\delta_r}\,\delta_r, \tag{139}$$

$$\Delta\dot{\phi} = \Delta p, \tag{140}$$

$$\Delta\dot{\psi} = \Delta r/\cos\theta_0, \tag{141}$$

where:

$$Y_v = \frac{q_\infty S}{mV_0}C_{Y\beta}, \quad Y_p = \frac{q_\infty Sd}{2mV_0}C_{Yp}, \quad Y_r = \frac{q_\infty Sd}{2mV_0}C_{Yr}, \tag{142}$$

$$L_v = \frac{q_\infty Sd}{I_xV_0}C_{\ell\beta}, \quad L_p = \frac{q_\infty Sd^2}{2I_xV_0}C_{\ell p}, \quad L_r = \frac{q_\infty Sd^2}{2I_xV_0}C_{\ell r}, \tag{143}$$

$$N_v = \frac{q_\infty Sd}{I_zV_0}C_{n\beta}, \quad N_p = \frac{q_\infty Sd^2}{2I_zV_0}C_{np}, \quad N_r = \frac{q_\infty Sd^2}{2I_zV_0}C_{nr}. \tag{144}$$

The state and input matrices:

$$A_{\text{lat}} = \begin{bmatrix} Y_v & Y_p & Y_r - u_0 & g_0\cos\theta_0 & 0 \\ L_v & L_p & L_r & 0 & 0 \\ N_v & N_p & N_r & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/\cos\theta_0 & 0 & 0 \end{bmatrix}, \quad B_{\text{lat}} = \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{145}$$

10.8 Eigenvalue analysis and natural modes

The eigenvalues of A_{long} and A_{lat} reveal the vehicle's natural modes:

10.8.0.1 Longitudinal modes (typical for a stable rocket):

- **Short-period mode**: High-frequency, well-damped oscillation in α and q. $\omega_{n,\text{SP}} \approx \sqrt{-M_w}$, $\zeta_{\text{SP}} \approx -M_q/(2\omega_{n,\text{SP}})$.
- **Phugoid mode**: Low-frequency, lightly damped exchange of kinetic and potential energy (altitude/velocity oscillation). Often negligible for powered rockets.

10.8.0.2 Lateral-directional modes:

- **Roll mode**: Fast, heavily damped pure-roll response. Time constant $\tau_{\text{roll}} \approx -1/L_p$.
- **Dutch-roll mode**: Oscillatory yaw-sideslip coupling. $\omega_{n,\text{DR}} \approx \sqrt{-N_v}$, $\zeta_{\text{DR}} \approx -N_r/(2\omega_{n,\text{DR}})$.
- **Spiral mode**: Very slow, often unstable divergence in heading. Eigenvalue $\lambda_{\text{spiral}} \approx \frac{L_vN_r - N_vL_r}{L_v}$ (can be positive, indicating instability).

For rockets, the roll mode is typically very fast, Dutch-roll moderate, and spiral mode is often unstable but slow enough to be controlled.

11 Transfer Functions for Control Design

11.1 General form

From the state-space model $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$, $\mathbf{y} = C\mathbf{x} + D\mathbf{u}$, the transfer function matrix is:

$$G(s) = C(sI - A)^{-1}B + D. \tag{146}$$

For SISO channels (single input, single output), extract scalar transfer functions.

11.2 Longitudinal transfer functions

11.2.0.1 Pitch attitude to elevator: $\Theta(s)/\Delta_e(s)$. With output matrix $C_\theta = [0\ \ 0\ \ 0\ \ 1]$ (selecting $\Delta\theta$ from $\mathbf{x}_{\text{long}}$):

$$\frac{\Theta(s)}{\Delta_e(s)} = C_\theta(sI - A_{\text{long}})^{-1}B_{\text{long}}. \tag{147}$$

For the short-period approximation (neglecting speed dynamics), the 2-state model $[\Delta w,\ \Delta q]^T$ gives:

$$\frac{\Theta(s)}{\Delta_e(s)} \approx \frac{K_\theta(s + z_\theta)}{s(s^2 + 2\zeta_{\text{SP}}\omega_{n,\text{SP}}s + \omega_{n,\text{SP}}^2)}, \tag{148}$$

where the numerator zero z_θ depends on control location and the $1/s$ pole arises from integrating q to θ.

11.2.0.2 Angle of attack to elevator: $(s)/\Delta_e(s)$. Since $\alpha \approx \Delta w/u_0$, output $C_\alpha = [0\ 1/u_0\ 0\ 0]$:

$$\frac{(s)}{\Delta_e(s)} = C_\alpha(sI - A_{\text{long}})^{-1}B_{\text{long}} \approx \frac{K_\alpha(s + z_\alpha)}{s^2 + 2\zeta_{\text{SP}}\omega_{n,\text{SP}}s + \omega_{n,\text{SP}}^2}. \tag{149}$$

This is a second-order response with no integrator (stable steady-state α for step input).

11.2.0.3 Pitch rate to elevator: $Q(s)/\Delta_e(s)$. Output $C_q = [0\ 0\ 1\ 0]$:

$$\frac{Q(s)}{\Delta_e(s)} = C_q(sI - A_{\text{long}})^{-1}B_{\text{long}} \approx \frac{K_q s}{s^2 + 2\zeta_{\text{SP}}\omega_{n,\text{SP}}s + \omega_{n,\text{SP}}^2}. \tag{150}$$

The numerator s indicates derivative action (rate response leads angle response).

11.2.0.4 Altitude control (with outer loop). For altitude hold, cascade: altitude error $\rightarrow$ pitch command $\rightarrow$ elevator. The altitude rate is $\dot{h} = -w\cos\theta + u\sin\theta \approx u_0\theta$ (small angles), so:

$$\frac{H(s)}{\Delta_e(s)} = \frac{u_0}{s}\,\frac{\Theta(s)}{\Delta_e(s)}. \tag{151}$$

11.3 Lateral-directional transfer functions

11.3.0.1 Roll angle to aileron: $\Phi(s)/\Delta_a(s)$. Output $C_\phi = [0\ 0\ 0\ 1\ 0]$ from $\mathbf{x}_{\text{lat}}$:

$$\frac{\Phi(s)}{\Delta_a(s)} = C_\phi (sI - A_{\text{lat}})^{-1} B_{\text{lat},a}, \tag{152}$$

where $B_{\text{lat},a}$ is the first column of B_{lat} (aileron). For the roll-dominated approximation:

$$\frac{\Phi(s)}{\Delta_a(s)} \approx \frac{K_\phi}{s(s - L_p)}, \tag{153}$$

where $L_p < 0$ (damping) so the pole is in the left half-plane. The $1/s$ pole integrates roll rate to angle.

11.3.0.2 Roll rate to aileron: $P(s)/\Delta_a(s)$. Output $C_p = [0\ 1\ 0\ 0\ 0]$:

$$\frac{P(s)}{\Delta_a(s)} \approx \frac{L_{\delta_a}}{s - L_p}. \tag{154}$$

This is a first-order response (time constant $\tau_{\text{roll}} = -1/L_p$).

11.3.0.3 Heading angle to rudder: $\Psi(s)/\Delta_r(s)$. Output $C_\psi = [0\ 0\ 0\ 0\ 1]$:

$$\frac{\Psi(s)}{\Delta_r(s)} = C_\psi (sI - A_{\text{lat}})^{-1} B_{\text{lat},r}. \tag{155}$$

The full form includes Dutch-roll and spiral modes:

$$\frac{\Psi(s)}{\Delta_r(s)} = \frac{K_\psi (s + z_1)(s + z_2)}{s(s^2 + 2\zeta_{\text{DR}}\omega_{n,\text{DR}} s + \omega_{n,\text{DR}}^2)(s - \lambda_{\text{spiral}})}. \tag{156}$$

The $1/s$ pole integrates yaw rate; the spiral pole λ_{spiral} is typically small and may be positive (instability).

11.3.0.4 Sideslip to rudder: $(s)/\Delta_r(s)$. Since $\beta \approx \Delta v/V_0$, output $C_\beta = [1/V_0\ 0\ 0\ 0\ 0]$:

$$\frac{(s)}{\Delta_r(s)} = \frac{K_\beta (s + z_\beta)}{(s^2 + 2\zeta_{\text{DR}}\omega_{n,\text{DR}} s + \omega_{n,\text{DR}}^2)(s - \lambda_{\text{spiral}})}. \tag{157}$$

No pure integrator (stable β for step input).

11.4 Cross-coupling transfer functions

For coordinated turns (e.g., ϕ to δ_r or β to δ_a), non-diagonal elements of $G(s)$ matter:

$$\frac{\Phi(s)}{\Delta_r(s)}, \quad \frac{\Psi(s)}{\Delta_a(s)}, \quad \text{etc.} \tag{158}$$

These arise from off-diagonal terms in A_{lat} and B_{lat}. For symmetric rockets with vertical fins, cross-coupling is small.

11.5 Multivariable control considerations

For full 6DOF autopilot design:

- **Inner loops**: Stabilize fast modes (roll rate, pitch rate).
- **Outer loops**: Track commands (attitude, velocity, position).
- **Gain scheduling**: Update A, B as q_∞, M, X_{CG} vary with altitude and fuel burn.
- **Decoupling**: For tight formation flight or precision landing, add cross-feed to cancel coupling terms.

11.6 Bode and Nyquist analysis

Evaluate $G(j\omega)$ for frequency-domain design:

$$|G(j\omega)| = \text{gain margin}, \qquad \angle G(j\omega) = \text{phase margin}. \tag{159}$$

Ensure adequate margins (typically GM$>6\,$dB, PM$>45°$) across the flight envelope.

11.7 Root-locus design

For proportional control $\delta = -k\,e$, the closed-loop poles are roots of:

$$1 + k\,G(s) = 0. \tag{160}$$

Plot the locus of poles as k varies. Barrowman derivatives provide the open-loop poles and zeros for initial design.

A Closed-Form Integrals for Trapezoidal Fins

For a trapezoid with chord $c(y) = c_r + my$, $m = (c_t - c_r)/s$, $0 \le y \le s$:

$$A_f = \int_0^s c(y)\,dy = c_r s + \frac{ms^2}{2} = \frac{s}{2}(c_r + c_t), \tag{161}$$

$$\int_0^s y\,c(y)\,dy = \frac{c_r s^2}{2} + \frac{ms^3}{3}, \tag{162}$$

$$\int_0^s y^2\,c(y)\,dy = \frac{c_r s^3}{3} + \frac{ms^4}{4}, \tag{163}$$

$$\int_0^s c(y)^2\,dy = c_r^2 s + c_r m s^2 + \frac{m^2 s^3}{3}. \tag{164}$$

These are used in CP calculations (Section 5.6) and roll-damping integrals (Section 7.3).

B Direction-Cosine Matrix (DCM) and Euler Angles

The 3-2-1 Euler sequence (yaw ψ, pitch θ, roll ϕ) gives the inertial-to-body DCM:

$$\mathbf{R}_{IB} = \mathbf{R}_z(\psi)\,\mathbf{R}_y(\theta)\,\mathbf{R}_x(\phi) = \begin{bmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ -c\phi s\psi + s\phi s\theta c\psi & c\phi c\psi + s\phi s\theta s\psi & s\phi c\theta \\ s\phi s\psi + c\phi s\theta c\psi & -s\phi c\psi + c\phi s\theta s\psi & c\phi c\theta \end{bmatrix}, \tag{165}$$

where $s = \sin$, $c = \cos$. For small angles, linearize to first order in ϕ, θ, ψ.

References

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