数学物理方法

Mathematical Methods in Physics

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目录

复变函数部分

第一章　复数

约定：我们认为 $z \in \mathbb{C}$，$x, y \in \mathbb{R}$

1.1　定义以及运算

定义 1.1.1.

$$i = \sqrt{-1}$$

称之为虚数单位。通过虚数单位和『实数单位 (1)』的线性组合，可以得到任意复数的表示方式：

$$z = x + iy,\ z \in \mathbb{C}, x\&y \in \mathbb{R}$$

x, y 分别称为实部和虚部，记为：

$$x = \mathrm{Re}\, z$$

$$y = \mathrm{Im}\, z$$

定义 1.1.2.

$$z^* = x - yi$$

称为 z 的共轭复数。容易得到，

$$z \cdot z^* = x^2 + y^2 = |z|^2 \geq 0$$
$$x = \mathrm{Re}\, z = \frac{z + z^*}{2}$$
$$y = \mathrm{Im}\, z = \frac{z - z^*}{2i}$$

注意到复数的运算与实数的运算存在许许多多的不同之处，例如

例 1.1.3.

$$\lim_{y \to 0} \frac{1}{x + yi} \neq \frac{1}{x}$$

$$\lim_{y\to 0}\frac{1}{x+yi}=\lim_{y\to 0}\frac{x-yi}{x^2+y^2}\to$$

$$\mathrm{Re}\,z=\begin{cases}0,\ x=0\\ \dfrac{1}{x},\ x\neq 0\end{cases}\quad \mathrm{Im}\,z=-i\pi\delta(x)$$

1.2　复数的几何表示

引入复平面可以容易地表示复数的几何形式：即 $z=x+yi$ 在 x 轴 (实轴) 上的投影为 x，在 y 轴 (虚轴) 上的投影为 y。那么，对应向量的 (主) 辐角 θ 以及模 ρ 便定义为：

定义 1.2.1.

$$\theta=\mathrm{Arg}\ z;\quad \rho=\sqrt{x^2+y^2}$$

主辐角记为 $\mathrm{Arg}\ z\in[-\pi,\pi]=\arctan\dfrac{y}{x}$，辐角记为 $\arg z$ 那么得到：

引理 1.2.2.

$$z=\rho(\cos\theta+i\sin\theta)$$

注意到：

$$\frac{1}{z}=\frac{1}{\rho(\cos\theta+i\sin\theta)}=\frac{1}{\rho}(\cos\theta-i\sin\theta)$$

引理 1.2.3. 假设 $1/z=n\in\mathbb{C}$，

$$\rho_n=1/\rho_z;\ \ \arg z=-\arg n$$

同样

引理 1.2.4. 假设

$$z=\prod_{i=1}^{n}z_i\to\rho_z=\prod_{i=1}^{n}\rho_{z_i};\quad \arg z=\sum_{i=1}^{n}\arg z_i$$

$$z_i\in\mathbb{C}$$

定理 1.2.5. *de Moivre' s* 定理：

$$z_1=\rho_1(\cos\theta_1+i\sin\theta_1)\quad z_2=\rho_2(\cos\theta_2+i\sin\theta_2)$$

$$\Rightarrow$$

$$z_1\cdot z_2=\rho_1\rho_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))$$

结合1.2.3和1.2.4，我们可以得到任意个复数的乘法除法公式：

推论 1.2.6.

$$z=\frac{\prod_{i=1}^{n}a_i\in\mathbb{C}}{\prod_{i=1}^{n}b_i\in\mathbb{C}}:\Longrightarrow\rho_z=\frac{\prod_{i=1}^{n}\rho_{a_i}}{\prod_{i=1}^{n}\rho_{b_i}}\quad \arg\ z=\sum_{i=1}^{n}\arg\ a_i-\sum_{i=1}^{n}\arg\ b_i$$

1.3　复数数列

形式如下的序列称为复数数列

$$z_n=x_n+iy_n,\quad n=1,2,3,4,\dots$$

$$z_n\text{ 收敛}\Leftrightarrow x_n,y_n\text{ 收敛}$$

1.4　欧拉公式以及复数的指数函数形式

定理 1.4.1. 欧拉公式：

$$e^{i\theta}=\cos\theta+i\sin\theta$$

证明. 由 Taylor-Sereis

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

得到

$$e^{i\theta}=\sum_{n=0}^{\infty}\frac{i^n\theta^n}{n!}=\left[1-\frac{\theta^2}{2}+\frac{\theta^4}{4!}+\dots\right]+i\left[\theta-\frac{\theta^3}{3!}+\dots\right]$$

考虑到 $\cos\theta$ 和 $\sin\theta$ 的 Taylor-Series，得到：

$$e^{i\theta}=\cos\theta+i\sin\theta$$

□

显然如上的证明并不是一个严格的证明，因为我们没有证明如上的展开适用于复数域，以及在交换次序时没有事先证明它绝对收敛。结合1.2.2得到

$$z=\rho e^{i\theta}$$

称为复数的指数函数形式。

例 1.4.2. 计算无穷级数：$\cos\theta+\cos 2\theta+\cos 3\theta+\dots$

证明. 原式等价于

$$\mathrm{Re}\ \left[e^{i\theta}+e^{i2\theta}+e^{i3\theta}+\dots\right]$$

$$\begin{aligned}&e^{i\theta}+e^{i2\theta}+e^{i3\theta}+\dots\\=&\lim_{n\to\infty}\frac{e^{i\theta}-e^{i(n+1)\theta}}{1-e^{i\theta}}\end{aligned}$$

□

1.5　复数域上的指数函数的反函数

对于 $\forall\ z\in\mathbb{C}$, 如何定义函数 $g=f(z)=e^z$ 的反函数? 即定义一个函数，使得:

$$f^{-1}(g)=z$$

这个函数称为复对数函数，区别于 $\mathbb{R}$ 上的指数函数 $\ln(x)$。

定义 1.5.1. 复对数函数:

$$\mathrm{Ln}\ z=\ln|z|+i\mathrm{arg}\ z+2n\pi i$$

$$s.t.\mathrm{Ln}\ g=\mathrm{Ln}\ |z|e^{i\mathrm{arg}\ z+2ni\pi}$$

其多值性来源于

$$g=e^z=e^{z+2ni\pi}$$

第二章　复变序列

对于某一复数序列 $u_n = x_n + iy_n$，其和前 n 项和 S_n：

$$\sum_{n=0}^{\infty}(x_n + iy_n)$$

$$S_n = X_n + iY_n$$

$$X_n = \sum_{i=0}^{n} x_i, Y_n = \sum_{j=0}^{n} y_j$$

无穷级数收敛的充要条件：$\forall\, \varepsilon > 0,\ \exists\, n > 0;\ n \in \mathbb{Z}\ \ s.t.\ \forall\, p > 0$

$$|u_{n+1} + u_{n+2} + \ldots + u_{n+p}| < \varepsilon$$

级数收敛的必要条件：

$$\text{Preliminary Test:}\ \lim_{n\to\infty} u_n = 0$$

2.1　级数收敛性判别法

Test for alternating series: An alternating series converges if the absolute value of the terms decreases steadily to zero, that is, if $|a_{n+1}| \leq |a_n|$ and $\lim_{n\to\infty} a_n = 0$. **一致递减 至 0**

Comparison Method: If $\exists N \in \mathbb{N}, \forall n > N$, the condition $|u_n| < v_n$ is satisfied. If $\sum_{n=0}^{\infty} v_n$ are convergent, then $\sum_{n=0}^{\infty} |u_n|$ are convergent.

Ratio Method: If there exists a constant ρ (un-correlated with n), and $|u_{n+1}/u_n| < \rho < 1$, then $\sum_{n=0}^{\infty} u_n$ are absolutely convergent.

d'Alembert Method(Criterion): 级数的通项比值 $\left(\frac{u_{n+1}}{u_n}\right)$ 的模的上极限小于 1，则原级数绝对收敛；级数的通项比值的模下极限大于 1，则原级数发散。

Gauss Method: Assume that the ratio between two neighboring terms has the following form: $\dfrac{u_n}{u_{n+1}} = 1 + \dfrac{\mu}{n} + O\left(n^{-\lambda}\right)$ where $\mu = a + ib, \lambda > 1$.

If $a > 1$, $\sum\limits_{n=0}^{\infty} u_n$ absolutely convergent.

If $a \le 1$, $\sum\limits_{n=0}^{\infty} |u_n|$ divergent.

例 2.1.1. 使用 Gauss Method 判别级数 $S_n = \sum\limits_{n=0}^{\infty} \dfrac{1}{n}$ 的收敛性：

$$\frac{u_n}{u_{n+1}} = \frac{n+1}{n} = 1 + \frac{1}{n}$$

则 $a = 1$，原级数发散。

使用 Gauss Method 判别级数 $S_n = \sum\limits_{n=0}^{\infty} \dfrac{1}{n^2}$ 的收敛性：

$$\frac{u_n}{u_{n+1}} = \left(\frac{n+1}{n}\right)^2 = 1 + \frac{1}{n^2} + \frac{2}{n}$$

则 $a = 2$，原级数绝对收敛。

Cauchy Method: $|u_n|^{1/n}$ 的上极限小于 1，原级数绝对收敛；大于 1，原级数发散。

2.2　复数序列的一致收敛

如果 S_n 一致收敛，则：

- Continuity $u_k(z)$ is continuous in G, and $\sum_{k=1}^{\infty} u_k(z)$ is uniformly convergent, then $S(z) = \sum_{k=1}^{\infty} u_k(z)$ is continuous in G
- $\displaystyle\int_C \sum_{k=1}^{\infty} u_k(z)\mathrm{d}z = \sum_{k=1}^{\infty} \int_C u_k(z)\mathrm{d}z$
- $f(z) = \sum\limits_{k=1}^{\infty} u_k(z)$ is analytic in $G \to f^{(p)}(z) = \sum\limits_{k=1}^{\infty} u_k^{(p)}(z)$

2.3　幂级数与阿贝尔定理

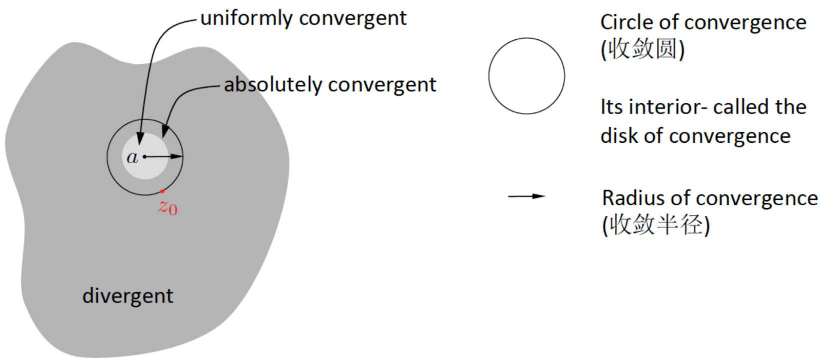
对于幂级数

$$\sum_{n=0}^{\infty} c_n(z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + \dots$$

有

定理 2.3.1. Abel theorem: If the series $\sum_{n=0}^{\infty} c_n(z-a)^n$ are convergent at $z=z_0$, then the series are absolutely convergent in a disk region (with a radius of $|z_0-a|$) surrounding z_0, and are uniformly convergent in the region $|z-a| \leq r\,(r<|z_0-a|)$.

推论 2.3.2. If $\sum_{n=0}^{\infty} c_n(z-a)^n$ are divergent at z_1, then also divergent in $|z-a| > |z_1-a|$.



计算幂级数的收敛半径的方法:

方法 2.3.3. Cauchy-Hadamard Formula:

$$R=\frac{1}{\overline{\lim}_{n\to\infty}|c_n|^{1/n}}=\underline{\lim}_{n\to\infty}\left|\frac{1}{c_n}\right|^{1/n}$$

方法 2.3.4. d'Alembert Critrion:

$$R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

第三章　复变函数

3.1　复变函数的概念

定义 3.1.1. 复变函数是复数区域到复数区域的映射。

$$f : \mathbb{C} \to \mathbb{C}$$

$$f(z) = u(x, y) + iv(x, y) \quad z = x + iy \quad x, y \in \mathbb{R}$$

与实变函数不同，区域与区间是有显著差异的。

定义 3.1.2. 如果复平面上的点集 D 满足以下条件：

1. 开集性：不包含边界。$\forall\, z_0 \in D, \quad \exists\, \epsilon > 0 \quad s.t.\ \{z \mid |z - z_0| < \epsilon\} \subset D$
2. 连通性：任意两点之间可以用区域内的线连通。

那么点集 D 称为（开）区域。闭区域

$$\overline{D} = D + \partial D$$

∂D 是 D 区域的边界。边界具有方向，其正方向定义为使得区域位于运动方向的左手侧的方向。

定义 3.1.3. 双曲函数定义为：

$$\sinh(z) = \frac{e^z - e^{-z}}{2} \quad \cosh(z) = \frac{e^z + e^{-z}}{2} \quad \tanh(z) = \frac{\sinh(z)}{\cosh(z)}$$
$$\coth(z) = \frac{\cosh(z)}{\sinh(z)} \quad \operatorname{sech}(z) = \frac{1}{\cosh(z)} \quad \operatorname{csch}(z) = \frac{1}{\sinh(z)}$$

类比 $\cos z, \sin z$ 可以得到：双曲函数的周期性：

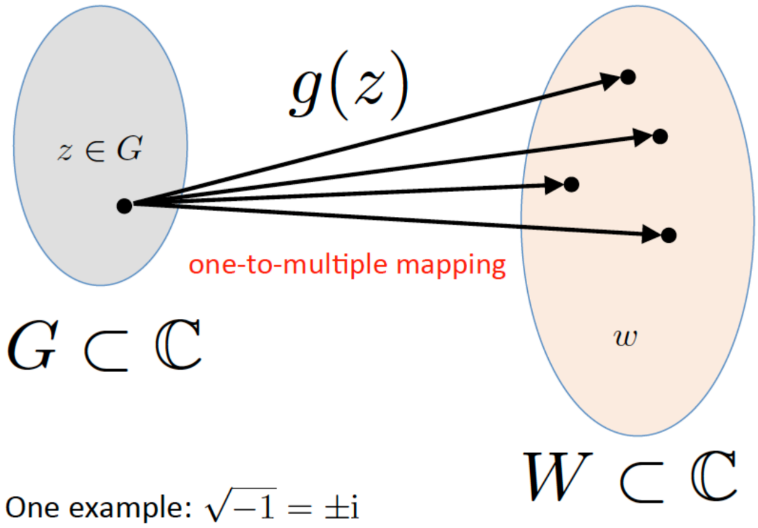
$$\sinh(z) = \sinh(z+i2n\pi) \quad \cosh(z) = \cosh(z+i2n\pi) \quad \tanh(z) = \tanh(z+in\pi) \quad n \in \mathbb{Z}$$

定义 3.1.4. 函数 $f^{-1}(z)$ 称为函数 $f(z)$ 的逆函数，如果：

$$f^{-1}(f(z)) = z$$

3.2 单值性与黎曼面

注意函数的多值性：



$$\text{定义域 } G \quad z = x + iy \quad \xrightarrow{f(z)} \quad \text{值域 } W \quad w = f(z) = u(x,y) + iv(x,y)$$

例 3.2.1. 根式函数：$w = \sqrt[n]{z-a}$。令 $z - a = re^{i\theta}$ 得到 w 有 n 个根：

$$w_1 = \sqrt[n]{r}e^{\theta/n} \quad w_2 = \sqrt[n]{r}e^{\theta/n+2\pi/n} \quad \ldots \quad w_n = \sqrt[n]{r}e^{\theta/n+2(n-1)\pi/n}$$

辐角的多值性

例 3.2.2. 对数函数：

$$w = \ln z = \ln|z| + i(\theta \pm 2n\pi)$$

模的多值性

反三角函数：

$$\begin{aligned}
\arcsin(z) &= \frac{1}{i}\ln\left(iz + \sqrt{1-z^2}\right) \\
\arccos(z) &= \frac{1}{i}\ln\left(z + \sqrt{z^2-1}\right) \\
\arctan(z) &= \frac{1}{2i}\ln\frac{1+iz}{1-iz}
\end{aligned}$$

例 3.2.3. 以 $\arcsin z$ 为例：

$$\sin(w) = \frac{e^{iw} - e^{-iw}}{2i} = z$$

Multiply e^{iw} for both sides, we have

$$\left(e^{iw}\right)^2 - 2iz\left(e^{iw}\right) - 1 = 0$$
$$e^{iw} = \frac{2iz \pm \sqrt{4-4z^2}}{2} = iz + \sqrt{1-z^2}$$
$$\Rightarrow w = \frac{1}{i}\ln\left(iz + \sqrt{1-z^2}\right)$$

复合函数多值性的判断：

例 3.2.4. $\sin\sqrt{z}$ 是多值函数 *(两个值)*，而 $\cos\sqrt{z}$ 是单值函数。

定义 3.2.5. 当自变量 z 围绕某点 z_0 旋转一圈 *(辐角增加 2π)* 之后，若得到的新的函数与原函数不相等，则 z_0 称为一个支点。

例如：

例 3.2.6. $w = \sqrt{z}$：

$$z' = z \cdot e^{2\pi i} \to w' = \sqrt{z} \cdot e^{\pi i} = -\sqrt{z} \neq w$$

所以 $z_1 = 0$，$z_2 = \infty$ 是 w 的两个支点。

方法 3.2.7. 多值函数的单值化*:*

- 限定辐角的范围，例如 $(0, 2\pi]$
- 规定某点 z_0 的值，然后描绘途径该点到目标点 z 的不同路径下的 $f(z)$ 的取值。

3.3　导数及解析函数的定义

定义 3.3.1. $f(z)$ 在 z_0 以及其邻域上有定义，且沿任何路径 $z \to z_0$ 时均有

$$\lim_{z\to z_0} f(z) = f(z_0)$$

则 $f(z)$ 在 z_0 上连续。

定义 3.3.2. 若 $f(z)$ 在其定义域上处处连续，则称其为连续函数。

定义 3.3.3. 若 $f(z)$ 在其 z_0 上连续，且沿任何路径 $\Delta z \to 0$

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

存在且唯一，则称 $f(z)$ 在 z_0 可导。

定义 3.3.4. 若 $f(z)$ 在 z_0 及其邻域各点均可导，则称为在 z_0 解析。

定义 3.3.5. 若 $f(z)$ 在域 D 上处处解析，则称为 D 上的解析函数。

3.4　柯西-黎曼条件

若 $f(z)=u(x,y)+iv(x,y)$ 其中 u,v 均为二元实函数，那么 $f(z)$ 可导的必要条件之一为柯西-黎曼条件：

定义 3.4.1. Cauchy-Riemann Condition*:*

$$\begin{cases} \dfrac{\partial u(x,y)}{\partial x}=\dfrac{\partial v(x,y)}{\partial y} \\ \dfrac{\partial v(x,y)}{\partial x}=-\dfrac{\partial u(x,y)}{\partial y} \end{cases}$$

极坐标的柯西黎曼条件：

$$z=re^{i\theta}\Rightarrow \Delta z=\frac{\partial z}{\partial r}\Delta r+\frac{\partial z}{\partial \theta}\Delta\theta=e^{i\theta}\Delta r+ire^{i\theta}\Delta\theta$$

(1) Along r direction $(\Delta\theta=0)$

$$\lim_{\Delta r\to 0}\frac{\Delta u+i\Delta v}{\Delta re^{i\theta}}=\frac{1}{e^{i\theta}}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)$$

(2) Along θ direction $(\Delta r=0)$

$$\lim_{\Delta\theta\to 0}\frac{\Delta u+i\Delta v}{r\Delta\theta \mathrm{i}e^{i\theta}}=\frac{1}{rie^{i\theta}}\left(\frac{\partial u}{\partial\theta}+i\frac{\partial v}{\partial\theta}\right)=\frac{1}{e^{i\theta}}\left(\frac{-i}{r}\frac{\partial u}{\partial\theta}+\frac{1}{r}\frac{\partial v}{\partial\theta}\right)$$

$$\Rightarrow\begin{cases} \dfrac{\partial u}{\partial r}=\dfrac{1}{r}\dfrac{\partial v}{\partial\theta} \\ \dfrac{\partial v}{\partial r}=-\dfrac{1}{r}\dfrac{\partial u}{\partial\theta} \end{cases}$$

推论 3.4.2. 在某一个点，$f(z)$ 可导的充分必要条件：

1. 函数的实部和虚部均为二元可微实函数。

2. 满足柯西黎曼条件。

证明. 假设 $f(z)=u(x,y)+iv(x,y)$，由条件 1 得：

$$\Delta u=\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y+\epsilon_1\Delta x+\epsilon_2\Delta y$$
$$\Delta v=\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y+\epsilon_3\Delta x+\epsilon_4\Delta y$$
$$\lim_{\Delta x\to 0,\Delta y\to 0}\epsilon_i=0\quad i=1,2,3,4$$

$$\begin{aligned}\Delta f=\Delta u+i\Delta v=&\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y+\epsilon_1\Delta x+\epsilon_2\Delta y+\\&i\left(\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y+\epsilon_3\Delta x+\epsilon_4\Delta y\right)\\=&\left(i\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\right)\Delta x+\left(i\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\right)\Delta y+(\epsilon_1+i\epsilon_3)\Delta x+(\epsilon_2+i\epsilon_4)\Delta y\\=&\left(i\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\right)\Delta x+i\left(\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}\right)\Delta y+(\epsilon_1+i\epsilon_3)\Delta x+(\epsilon_2+i\epsilon_4)\Delta y\end{aligned}$$

由条件 2 得:

$$\begin{aligned}\Delta f=\left(i\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\right)\Delta z+(\epsilon_1+i\epsilon_3)\Delta x+(\epsilon_2+i\epsilon_4)\Delta y\\z=x+iy\rightarrow\Delta z=\Delta x+i\Delta y\\\lim_{\Delta z\rightarrow 0}\frac{\Delta f}{\Delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\end{aligned}$$

□

推论 3.4.3. *在某一个区域 G 内，$f(z)$ 解析的充分必要条件:*

1. *函数的实部和虚部均为二元可微实函数，且其四个偏导 $\left(\frac{\partial u}{\partial x}\ \frac{\partial v}{\partial x}\ \frac{\partial u}{\partial y}\ \frac{\partial v}{\partial y}\right)$ 连续。*
2. *满足柯西黎曼条件。*

注意：多值函数一定不可导，不解析。

例 3.4.4. *e^z 在 $z\rightarrow\infty$ 时一定不解析，因为其在 $z\rightarrow\infty$ 时是多值的。同理，三角函数和双曲函数在 $z\rightarrow\infty$ 时也是不可导的。*

3.5　解析函数的特性

假设某个复变解析函数：$f(z)=u(x,y)+iv(x,y)\quad u,v\in\mathbb{R}$。由柯西-黎曼条件得到:

引理 3.5.1.

$$\frac{\partial^2u(x,y)}{\partial^2x}+\frac{\partial^2u(x,y)}{\partial^2y}=0\tag{3.1}$$

$$\frac{\partial^2v(x,y)}{\partial^2x}+\frac{\partial^2v(x,y)}{\partial^2y}=0\tag{3.2}$$

3.1和3.2 是拉普拉斯方程。所以解析函数的实部和虚部均为调和函数。即：

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$\Delta v = \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

定理 3.5.2.
$$\frac{\partial f(z)}{\partial z^*} = 0$$
即解析函数与其自变量的共轭无关。

证明.
$$x = \frac{z+z^*}{2}, \quad y = \frac{z-z^*}{2}$$

$$\frac{\partial f(z)}{\partial z^*} = \frac{\partial f(z)}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f(z)}{\partial y}\frac{\partial y}{\partial z^*} =$$
$$\frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + \frac{i}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] = 0$$

□

定理 3.5.3. 解析函数的实部和虚部的等值线的切向量相互垂直。

证明.
$$u(x,y) = C \Rightarrow \mathrm{d}u(x,y) = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y = 0$$
$\mathrm{d}\vec{s}$ 是实部等值线的切向量，则
$$\mathrm{d}\vec{s} = (\mathrm{d}x, \mathrm{d}y) \propto \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right)$$
$$v(x,y) = C' \Rightarrow \mathrm{d}v(x,y) = \frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y = 0$$
$\mathrm{d}\vec{s'}$ 是虚部等值线的切向量，则
$$\mathrm{d}\vec{s'} = (\mathrm{d}x', \mathrm{d}y') \propto \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}\right)$$
$$\mathrm{d}\vec{s} \cdot \mathrm{d}\vec{s'} \propto \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right) \cdot \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}\right) = 0 \Rightarrow \mathrm{d}\vec{s} \perp \mathrm{d}\vec{s'}$$

□

3.6　由部分确定整个解析函数

如果已知某个解析函数的实部 $u(x,y)$ 以及在某点 z_0 的取值，可以确定整个解析函数：

方法 3.6.1. 由于柯西-黎曼条件，

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \to v(x,y) = \int \frac{\partial v}{\partial x}\mathrm{d}x + h(y)$$

$$\frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x} \to v(x,y) = -\int \frac{\partial u}{\partial y}\mathrm{d}x + h(y)$$

$$\Rightarrow$$

$$\frac{\partial v(x,y)}{\partial y} = -\int \frac{\partial^2 u}{\partial y^2}\mathrm{d}x + h'(y) = \frac{\partial u(x,y)}{\partial x}$$

$$\Rightarrow$$

$$h'(y) = \frac{\partial u(x,y)}{\partial x} + \int \frac{\partial^2 u(x,y)}{\partial y^2}\mathrm{d}x \to h(y) = \int h'(y)\mathrm{d}y + C$$

方法 3.6.2. 利用 C-R 条件，先找到解析函数的导数：

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \equiv g(z)$$

$$\Rightarrow$$

$$f(z) = \int g(z)\mathrm{d}z + C$$

方法 3.6.3.

$$f(z) = u(x,y) + iv(x,y), \quad f^*(z) = u(x,y) - iv(x,y)$$

$$\Rightarrow$$

$$u(x,y) = \frac{f(z)+f^*(z)}{2}, \quad v(x,y) = \frac{f(z)-f^*(z)}{2i}$$

通过代数运算，我们可以将 $u(x,y)$ 写成：

$$u(x,y) = u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right) = h(z) + h^*(z) = [h(z)+iC] + [h(z)+iC]^*$$

对比系数可得：

$$f(z) = 2h(z) + 2iC$$

第四章 复变函数的积分

4.1 解析函数的积分特性

定义 4.1.1. 复变函数的积分定义为：

$$\int_L f(z)\mathrm{d}z \equiv \lim_{n\to\infty}\sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1})$$

其中 L 为有向路径。

一些较为常用的性质：

$$\int_L f_1(z) + f_2(z)\mathrm{d}z = \int_L f_1(z)\mathrm{d}z + \int_L f_2(z)\mathrm{d}z$$

$$\int_L f(z)\mathrm{d}z = -\int_{-L} f(z)\mathrm{d}z$$

$$\int_{L_1+L_2} f(z)\mathrm{d}z = \int_{L_1} f(z)\mathrm{d}z + \int_{L_2} f(z)\mathrm{d}z$$

$$\int_C af(z)\mathrm{d}z = a\int_C f(z)\mathrm{d}z \quad \text{where } a \text{ is a constant complex number}$$

$$\left|\int_C f(z)\mathrm{d}z\right| \leq \int_C |f(z)||dz|$$

$$\left|\int_C f(z)\mathrm{d}z\right| \leq Ml \quad \text{where } M \text{ is upper bound of } f(z)$$

定理 4.1.2. 单连通域上解析函数的柯西积分定理：假设 C 是某个单连通域的边界。

$$\oint_C f(z)\mathrm{d}z = 0$$

证明. 假设将复变解析函数 $f(z)$ 沿着某一单连通域做回路积分：

$$\oint_C f(z)\mathrm{d}z = \oint_C [u(x,y) + iv(x,y)]\,(\mathrm{d}x + i\mathrm{d}y) \tag{4.1}$$

其中正方向定义为确保解析区域在左手边的方向。展开4.1得到：

$$\oint_C [u\mathrm{d}x - v\mathrm{d}y] + i\oint_C [u\mathrm{d}y + v\mathrm{d}x]$$

由格林公式

$$\oint_C [P\mathrm{d}x + Q\mathrm{d}y] = \iint_\Sigma \left[-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right] \mathrm{d}x\mathrm{d}y$$

得到：

$$\oint_C f(z)\mathrm{d}z = \iint_\Sigma \left[-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right] \mathrm{d}x\mathrm{d}y + i\iint_\Sigma \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] \mathrm{d}x\mathrm{d}y \tag{4.2}$$

考虑 C-R 条件，

$$4.2 \equiv 0$$

□

定理 4.1.3. *Morera's theorem (莫列拉定理): If $f(z)$ is continuous in $\bar{G}$, if for any closed curve (contour) in $\bar{G}$, $\oint_C f(z)\mathrm{d}z = 0$, then $f(z)$ is analytic in G.*

定理 4.1.4. 复连通域上解析函数的柯西积分定理：假设 C 是一个复连通域的边界，而填上这个复连通域中的 $C_1, C_2, \ldots, C_N$ 所围成的区域可以将该域变为单连通域。那么：

$$\oint_C f(z)\mathrm{d}z = \sum_{n=1}^{N} \oint_{C_n} f(z)\mathrm{d}z$$

例 4.1.5. *Find the value of $\oint_C z^n\mathrm{d}z$, where n is an integer, C is a simply closed curve in $\mathbb{C}$.*

- If n is non-negative, z^n is analytic, then $\oint_C z^n\mathrm{d}z = 0$.
- If n is negative, and if the contour does not enclose $z = 0$, then z^n is analytic inside the region bounded by C, and again we have $\oint_C z^n\mathrm{d}z = 0$.
- If n is negative, and if the contour encloses $z = 0$. We can draw a simple circle around $z = 0$, and apply the Cauchy theorem for a multi-connected

$$\oint_C z^n\mathrm{d}z = \oint_{|z|=\varepsilon} z^n\mathrm{d}z = \int_0^{2\pi} \varepsilon^{n+1}e^{i(n+1)\theta}i\mathrm{d}\theta = \begin{cases} 2\pi i, n = -1; \\ 0, n = -2, -3, -4, \ldots \end{cases}$$

推论 4.1.6. 函数 $f(z)$ 在 Σ_G 内解析，如果 $\Sigma_C \subset \Sigma_G$，其线积分 $\int_C f(z)\mathrm{d}z$ 与路径无关。

引理 4.1.7. 小圆弧引理 *Small Arc Lemma*：

If $f(z)$ is continuous in a small region around $z = a$, and satisfies the following relation: when $\theta_1 \leq \arg(z-a) \leq \theta_2, |z-a| \to 0, (z-a)f(z)$ uniformly approaches k. Then we have

$$\lim_{\delta\to 0}\int_{C_\delta} f(z)\mathrm{d}z = ik\left(\theta_2-\theta_1\right)$$

引理 4.1.8. 大圆弧引理 *Great Arc Lemma*：

If $f(z)$ is continuous in a region around $z = \infty$, and satisfies the following relation: when $\theta_1 \leq \arg(z) \leq \theta_2, z \to \infty, zf(z)$ uniformly approaches K. Then we have

$$\lim_{R\to\infty}\int_{C_R} f(z)\mathrm{d}z = iK\left(\theta_2-\theta_1\right)$$

4.2　柯西积分公式

定理 4.2.1. 柯西积分公式：假设 C 包围了 $f(z)$ 的单连通解析区域，z_0 为区域内一点，则

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\mathrm{d}z$$

证明. 不妨用一个小圆将 z_0 包围，其边界设为 C_r： $\forall\, z \in \Sigma_{C_r} \quad z = z_0 + re^{i\theta}$，则由4.1.4，

$$\oint_C \frac{f(z)}{z-z_0}\mathrm{d}z = \oint_{C_r} \frac{f(z)}{z-z_0}\mathrm{d}z = \\ \int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{re^{i\theta}}ire^{i\theta}\mathrm{d}\theta = i\int_0^{2\pi} f(z_0+re^{i\theta})\mathrm{d}\theta \tag{4.3}$$

再不妨令 $r \to 0$，那么4.3化为

$$i\int_0^{2\pi} f(z_0)\mathrm{d}\theta = 2\pi i f(z_0)$$

即：

$$\oint_C \frac{f(z)}{z-z_0}\mathrm{d}z = 2\pi i f(z_0) \tag{4.4}$$

4.4 同样可以使用小圆弧定理导出：

As $z \to z_0$, $(z-z_0)\cdot f(z)/(z-z_0) \to f(z_0)$. And when $r \to 0$,

$$\oint_{C_r} \frac{f(z)}{z-z_0}\mathrm{d}z = 2i\pi^{1} f(z_0)$$

□

[1]为什么不是 $2ni\pi$? 因为该函数被单值化了

定理 4.2.2. Cauchy Integration Equation for Unbounded Region:

If $f(z)$ is a single-valued analytic function defined along and beyond a simply closed curve C (including the infinity), then we have the following relation

$$\frac{1}{2\pi i}\left[\oint_{C_R}\frac{f(z)}{z-a}\mathrm{d}z+\oint_{C}\frac{f(z)}{z-a}\mathrm{d}z\right]=f(a)$$

where the integration is done along the positive direction of C (which is clockwise) and the positive direction of C_R (anti-clockwise).

由4.2.2，使用大圆弧引理：

$$\text{当 } K=0\text{，即 }\lim_{z\to\infty}f(z)=0: f(a)=\frac{1}{2\pi i}\oint_C\frac{f(z)}{z-a}\mathrm{d}z$$

退化到了4.2.1

引理 4.2.3. 令*4.4*中的 $f(z)=1$，推出公式：

$$\frac{1}{2\pi i}\oint_C\frac{1}{z-z_0}\mathrm{d}z=\begin{cases}1 & z_0\in\Sigma_C\\ 0 & z_0\notin\Sigma_C\end{cases}$$

可以利用4.4计算解析函数的导数：

引理 4.2.4.

$$f(z)=\frac{1}{2\pi i}\oint_C\frac{f(\xi)}{\xi-z}\mathrm{d}\xi\to$$
$$f'(z)=\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}z}\oint_C\frac{f(\xi)}{\xi-z}\mathrm{d}\xi=\frac{1}{2\pi i}\oint_C\frac{f(\xi)}{(\xi-z)^2}\mathrm{d}\xi\to$$
$$f''(z)=\frac{2!}{2\pi i}\oint_C\frac{f(\xi)}{(\xi-z)^3}\mathrm{d}\xi$$
$$\cdots$$
$$f^{(n)}(z)=\frac{n!}{2\pi i}\oint_C\frac{f(\xi)}{(\xi-z)^{n+1}}\mathrm{d}\xi$$

这说明解析函数是任意阶可导的。

定义 4.2.5. 柯西型积分 *Cauchy-type Integral:*

If $\phi(\zeta)$ is a continuous function defined along curve C (piece-wisely smooth) , the

$$f(z)=\frac{1}{2\pi \mathrm{i}}\int_C\frac{\phi(\zeta)}{\zeta-z}\mathrm{d}\zeta,\quad z\notin C$$

is an analytic function defined outside C. And we have

$$f^{(p)}(z)=\frac{p!}{2\pi \mathrm{i}}\int_C\frac{\phi(\zeta)}{(\zeta-z)^{p+1}}\mathrm{d}\zeta,\quad z\notin C$$

方法 4.2.6. *Integral that Contains a Parameter*

1. $f(t,z)$ is a continuous function of t and $z, t \in [a,b], z \in \bar{G}, \bar{G}$ is bounded

2. For any value $t \in [a,b], f(t,z)$ is a single-valued analytic function defined in $\bar{G}$. Then $F(z) = \int_a^b f(t,z)\mathrm{d}t$ is analytic in G, and

$$F'(z) = \int_a^b \frac{\partial f(t,z)}{\partial z}\mathrm{d}t, z \in G$$

4.3 最大模定理

定理 4.3.1. 最大模定理：设 $f(z)$ 在闭区域上解析，则其模 $|f(z)|$ 的最大值只能出现在该区域的边界上，除非 $f(z)$ 是一个常函数。

证明.

$$\begin{aligned} f^n(z) &= \frac{1}{2\pi i}\oint \frac{f^n(\xi)}{\xi - z}\mathrm{d}\xi \\ |f(z)|^n = |\,[f(z)]^n\,| &= \left|\frac{1}{2\pi i}\oint \frac{f^n(\xi)}{\xi - z}\mathrm{d}\xi\right| \\ \leq \frac{1}{2\pi}\oint \frac{|f(\xi)^n|}{|\xi - z|}|\mathrm{d}\xi| \leq \frac{M^n}{2\pi d}\oint_C |\mathrm{d}\xi| &= \frac{M^n}{2\pi d} l \end{aligned}$$

d 为 z 至边界的最短距离，$\forall\, z$： $|z-\xi| \geq d$

M 为 $|f(\xi)|$ 的最大值，$\forall\, z : |f(\xi)| \leq M,\ \xi \in C$

即

$$|f(z)| \leq M\left[\frac{l}{2\pi d}\right]^{1/n} \to |f(z)| \leq \lim_{n\to\infty} M\left[\frac{l}{2\pi d}\right]^{1/n} = M$$

即 $f(z),\ z \in \overline{\Sigma_C}$ 的最大值便是 $f(\xi),\ \xi \in C$ 的最大值。 □

第五章　复变函数的级数

5.1　复变函数在其解析圆域上的泰勒级数展开

$f(z)$ 在 z_0 为圆心的圆域内解析，则对于任意一圆域内点 z，有

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint \frac{f(\xi)}{(\xi - z_0)^{n+1}} \mathrm{d}\xi = \frac{f^{(n)}(z_0)}{n!}$$

证明.

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} \mathrm{d}\xi = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0) - (z - z_0)} \mathrm{d}\xi$$
$$= \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z_0} \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} \mathrm{d}\xi$$

由于

$$\frac{z - z_0}{\xi - z_0} \le 1, \quad \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n, \quad |t| < 1 \tag{5.1}$$

$$\frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z_0} \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} \mathrm{d}\xi = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z_0} \sum_{n=0}^{\infty} \left[\frac{z - z_0}{\xi - z_0}\right]^n \mathrm{d}\xi =$$
$$\sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} \mathrm{d}\xi\right] (z - z_0)^n = f(z)$$

□

复变函数在其解析圆域上的泰勒级数展开的收敛半径为:

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \quad or \quad R = |z_0 - z_1| \quad z_1 \text{ 是离 } z_0 \text{ 最近的奇点}$$

例 5.1.1. 计算 $f(z)=\frac{1}{1-z^2}$ 的泰勒展开，求出收敛半径。

$$\text{By equation } 5.1,$$
$$\frac{1}{1-z^2}=\sum_{n=0}^{\infty}t^n\bigg|_{t=z^2}=\sum_{n=0}^{\infty}z^{2n}$$

引理 5.1.2. 对于给定的 $f(z)$、z_0，其泰勒展开 *(系数)* 唯一。

5.2　利用泰勒级数讨论最大模定理

定义 5.2.1. *Kronecker-δ* 符号：

$$\delta_{mn}\equiv\frac{1}{2\pi}\int_0^{2\pi}e^i(n-m)\theta\mathrm{d}\theta=\begin{cases}0, & m\neq n\\ 1, & m=n\end{cases}$$

假设最大模定理不成立，即：

$$\exists\ z_0\in\Sigma,\ z_0\notin\partial\Sigma\ \ s.t.\ |f(z_0)|=\max|f(z)|$$

那么以 z_0 为中心做泰勒展开：

$$f(z)=\sum_{n=1}^{\infty}a_n(z-z_0)^n\to a_0=f(z_0)$$

由于

$$z-z_0=re^{i\theta}$$

$$\begin{aligned}|a_0|^2&=\frac{1}{2\pi}\int_0^{2\pi}|a_0|^2\mathrm{d}\theta=\frac{1}{2\pi}\int_0^{2\pi}|f(z_0)|^2\mathrm{d}\theta\geq\frac{1}{2\pi}\int_0^{2\pi}f^*(z)\cdot f(z)\mathrm{d}\theta\\&=\frac{1}{2\pi}\int_0^{2\pi}\sum_{m=1}^{\infty}a_m^*[(z-z_0)^*]^m\cdot\sum_{n=1}^{\infty}a_n(z-z_0)^n\mathrm{d}\theta\\&=\sum_{m,n=0}^{\infty}a_m^*a_nr^{m+n}\frac{1}{2\pi}\int_0^{2\pi}e^{i(n-m)\theta}\mathrm{d}\theta\\&=\sum_{m,n=0}^{\infty}a_m^*a_nr^{m+n}\delta_{mn}=\sum_{n=0}^{\infty}a_n^*a_nr^{2n}=\sum_{n=0}^{\infty}|a_n|^2r^{2n}\\&=|a_0|^2+\sum_{n=1}^{\infty}|a_n|^2r^{2n}\end{aligned}\tag{5.2}$$

考虑到

$$\sum_{n=1}^{\infty}|a_n|^2r^{2n}\geq0\Rightarrow|a_0|^2+\sum_{n=1}^{\infty}|a_n|^2r^{2n}\geq|a_0|^2$$

若想要5.2成立，那么

$$\sum_{n=1}^{\infty}|a_n|^2r^{2n}=0\to a_n=0\to f(z)=constant.$$

定理 5.2.2. 刘维尔定理：在全复平面内解析且有界的复变函数必为常函数。

证明. 以 $z_0=0$ 为中心做泰勒展开：

$$f(z)=\sum_{n=0}^{\infty}a_nz^n,\quad a_n=\frac{1}{2\pi i}\oint_C\frac{f(\xi)}{\xi^{n+1}}\mathrm{d}\xi$$

由于

$$\xi\in C\to\xi=re^{i\theta}\to\mathrm{d}\xi=ire^{i\theta}\mathrm{d}\theta$$

则 $|a_n|$ 可以化为

$$|a_n|\le\frac{1}{2\pi}\oint_C\frac{|f(\xi)|}{|\xi^{n+1}|}|\mathrm{d}\xi|\le\frac{1}{2\pi}\int_0^{2\pi}\frac{M}{r^n}\mathrm{d}\theta=\frac{M}{r^n}$$

由于 $f(z)$ 在整个复平面上解析，即其泰勒展开的收敛半径 $R=\infty$，那么

$$|a_n|\le\lim_{r\to\infty}\frac{M}{r^n}=0\to\forall n\neq0:\ a_n=0$$

□

5.3 解析函数的零点及其孤立性

定义 5.3.1. $f(z)$ 在 z_0 点有 $f(z_0)=0$，且在以 z_0 为圆心的圆域内的泰勒级数展开式最低幂次 (最小的使得 $a_n\neq0$ 的 n) 为 k 次，则称 z_0 为 $f(z)$ 的 $k-$ 阶零点。

由定义得到，若 z_0 是 $f(z)$ 的 k-阶零点，则 $\forall\,k>n>0:\ f^{(n)}(z_0)=0$

定理 5.3.2. 零点的孤立性：假设 z_0 为 $f(z)$ 的一个零点，则

$$\exists\,r>0\ s.t.\ \forall z\in\{z\mid|z-z_0|<r\},\ f(z)\neq0$$

即零点不能构成区域。

证明. 假设 z_0 为 $f(z)$ 的一个 k-阶零点：

$$f(z)=\sum_{n=k}^{\infty}a_n(z-z_0)^n=(z-z_0)^k\sum_{m=0}^{\infty}a_{m+k}(z-z_0)^m=(z-z_0)^k\varphi(z)$$

$$\varphi(z_0) \equiv a_k \neq 0$$

由于函数解析，函数必定连续，则

$$\forall\, \epsilon > 0 :\ \exists\, z \neq z_0\ s.t.\ |\varphi(z_0) - \varphi(z)| < \epsilon$$

令 $\epsilon = |\varphi(z_0)|/2$：

$$\begin{aligned}
&|\varphi(z_0)| - |\varphi(z)| < |\varphi(z_0) - \varphi(z)| < |\varphi(z_0)|/2 \\
&|\varphi(z)| > |\varphi(z_0)|/2 > 0 \\
&f(z) = (z - z_0)^k \varphi(z) \neq 0
\end{aligned}$$

即总可以在 z_0 为中心找到一圆域使得在该圆域内除圆心 z_0 外的所有点 z 满足 $f(z) \neq 0$ □

5.4　解析环域上的洛朗级数展开

$f(z)$ 在以 z_0 为圆心的环域内解析，则对于该环域内任何一点 z，有

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad a_n = \frac{1}{2\pi i} \oint f(\xi)(\xi - z_0)^{-n-1} \mathrm{d}\xi$$

证明. 将环域的外环和内环建立一微小链接，使得 $L = C_1 + C_2 + \partial L - \partial L = C_1 + C_2$ 为一单连通区域的边界，

$$\begin{aligned}
f(z) = \frac{1}{2\pi i} \oint_L \frac{f(\xi)}{\xi - z} \mathrm{d}\xi &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi + \frac{1}{2\pi i} \oint_{C_2} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi \\
&= \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi - \frac{1}{2\pi i} \oint_{C_2} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi
\end{aligned}$$

回想起证明泰勒级数时的过程，不妨将 $\xi - z_0$ 设为 r，$z - z_0$ 设为 R，不难发现：对于 C_1，$r > R$，对于 C_2，$r < R$。

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{r - R} \mathrm{d}\xi \\
&= \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{1 - R/r} \mathrm{d}\xi \\
&= \frac{1}{2\pi i} \oint_{C_1} \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^n \frac{f(\xi)}{\xi - z_0} \mathrm{d}\xi \\
&= \frac{1}{2\pi i} \oint_{C_1} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^n \frac{f(\xi)}{\xi - z_0} \mathrm{d}\xi
\end{aligned}$$

同理可得

$$
\begin{aligned}
-\frac{1}{2\pi i}\oint_{C_2}\frac{f(\xi)}{\xi-z}\mathrm{d}\xi &= -\frac{1}{2\pi i}\oint_{C_2}\frac{f(\xi)}{r-R}\mathrm{d}\xi \\
&= \frac{1}{2\pi i}\oint_{C_2}\frac{f(\xi)}{1-r/R}\mathrm{d}\xi \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\oint_{C_2}\left(\frac{r}{R}\right)^n\frac{f(\xi)}{z-z_0}\mathrm{d}\xi \\
&= \frac{1}{2\pi i}\oint_{C_2}\sum_{n=0}^{\infty}\left(\frac{\xi-z_0}{z-z_0}\right)^n\frac{f(\xi)}{z-z_0}\mathrm{d}\xi
\end{aligned}
$$

代入 $f(z)$ 中得到

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{C_1}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^n\frac{f(\xi)}{\xi-z_0}\mathrm{d}\xi + \frac{1}{2\pi i}\oint_{C_2}\sum_{n=0}^{\infty}\left(\frac{\xi-z_0}{z-z_0}\right)^n\frac{f(\xi)}{z-z_0}\mathrm{d}\xi \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left[\oint\frac{f(\xi)}{(\xi-z_0)^{n+1}}\mathrm{d}\xi\right](z-z_0)^n \\
&+ \frac{1}{2\pi i}\sum_{n=-\infty}^{-1}\left[\oint\frac{f(\xi)}{(\xi-z_0)^{n+1}}\mathrm{d}\xi\right](z-z_0)^n \\
&= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\left[\oint\frac{f(\xi)}{(\xi-z_0)^{n+1}}\mathrm{d}\xi\right](z-z_0)^n
\end{aligned}
$$

□

洛朗级数的收敛半径:

$$
R_1 < |z-z_0| < R_2, \quad R_1 = \lim_{n\to-\infty}\left|\frac{a_{n-1}}{a_n}\right| \quad R_2 = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|
$$

或者可以认为

$$R_1 := \text{以 } z_0 \text{ 为圆心的包含考察点 } z \text{ 的最大解析环域的内径}$$

$$R_2 := \text{以 } z_0 \text{ 为圆心的包含考察点 } z \text{ 的最大解析环域的外径}$$

例 5.4.1. *Find the Laurent expansion of* $\frac{1}{z(z-1)}$ *when (i)* $0<|z|<1$ *and (ii)* $|z|>1$ *You should refer to the previous exercises. convergent in* $0<|z|<1$

1.

$$
0<|z|<1: \quad \frac{1}{z(z-1)} = -\frac{1}{z}\frac{1}{1-z} = -\frac{1}{z}\sum_{n=0}^{\infty}z^n = -\sum_{n=-1}^{\infty}z^n
$$

And we can confirm that $\frac{1}{z(z-1)}$ *has singular point at* $z=0$.

2.

$$|z|>1: \quad \frac{1}{z(z-1)}=\frac{1}{z^2}\frac{1}{1-\frac{1}{z}}=\frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n=\sum_{n=-2}^{\infty}z^n$$

And we can confirm that $\frac{1}{z(z-1)}$ *has singular point along* $|z|=1$.

第六章　留数定理

6.1　留数定理

定义 6.1.1. 孤立奇点：若 $f(z)$ 在 z_0 处不解析，但在其邻域内全都可导 *($\exists\, r > 0$ s.t. $\forall\, z:\ 0 < |z - z_0| < r,\ f'(z_0)$exists.)*，那么 z_0 是 $f(z)$ 的一个孤立奇点。若 z_0 是一个孤立奇点，那么我们可以在其邻环域内对 $f(z)$ 做洛朗展开，若：

1. 若其洛朗级数展开不包含负数项，z_0 称为一个可去奇点 *(removable singular point)*
2. 若其洛朗级数展开包含有限个负数项，z_0 称为一个极点 *(pole)*
3. 若
$$\begin{aligned} f(z) &= a_{-m}(z-b)^{-m} + a_{-m+1}(z-b)^{-m+1} + \ldots + a_0 + a_1(z-b) + \ldots \\ &= (z-b)^{-m}\left[a_{-m} + a_{-m+1}(z-b) + \ldots\right] \\ &= (z-b)^{-m}\phi(z), \quad 0 < |z-b| < R \end{aligned}$$
$\phi(z)$ is analytic for a region around $z = b$. If $\phi(b) = a_{-m} \neq 0$, then b is said to be the m-th order pole of $f(z)$.
$$\frac{1}{f(z)} = (z-b)^m \frac{1}{\phi(z)}$$
4. 若其洛朗级数展开包含无限个负数项，z_0 称为一个本性奇点 *(essential singular point)*

如果 z_0 是 f 的一个本质奇点，那么 $\lim_{z\to z_0} f(z)$ 不存在。

例 6.1.2. 判断 $z_0 = 0$ 是 $f(z) = e^{1/z}$ 的何种奇点。

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{-\infty} \frac{1}{(-n)!} z^n$$

则，$z_0 = 0$ 是其本质奇点，且不难验证，其趋于本质奇点的极限不存在。

定理 6.1.3. *Assume that G is a bounded region, and its boundary C is a smooth, simply closed curve. If except for a finite number of isolated singular points $b_k, k = 1, 2, 3 \ldots, n, f(z)$ is single-valued and analytic in G, and is continuous in $\bar{G}$ (including along C). Then we have the following relation:*

$$\oint_C f(z)\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{n}\operatorname{res} f\left(b_k\right)$$

$\operatorname{res} f\left(b_k\right)$ *called the residue (留数) of $f(z)$ at b_k*

$$f(z) = \sum_{l=-\infty}^{\infty} a_l^{(k)}\left(z-b_k\right)^l, 0<\left|z-b_k\right|<r, \quad \operatorname{res} f(b_k) := a_{-1}^{(k)}$$

证明.

$$\oint_C f(z)\mathrm{d}z = \sum_{k=1}^{n}\oint_{\gamma_k} f(z)\mathrm{d}z$$

Recalling that resault of

$$\oint_C z^n \mathrm{d}z = \oint_{|z|=\varepsilon} z^n \mathrm{d}z = \int_0^{2\pi} \varepsilon^{n+1} e^{i(n+1)\theta} \mathrm{i}\mathrm{d}\theta = \begin{cases} 2\pi i, n=-1; \\ 0, n=-2,-3,-4,\ldots \end{cases}$$

we concluded before, and we can say that:

$$O.E. = 2\pi\mathrm{i}\sum_{k=1}^{n} a_{-1}^{(k)} = 2\pi\mathrm{i}\sum_{k=1}^{n}\operatorname{res} f\left(b_k\right)$$

□

方法 6.1.4. *Computation Method of Residue:*

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-b)^m f(z)\bigg|_{z=b}$$

$$a_{-1} = \frac{1}{(m-1)!}\lim_{z\to a}[(z-a)^m f(z)]^{m-1}$$

The general form of $f(z)$ is $\frac{P(z)}{Q(z)}$. If $P(z)$ and $Q(z)$ are analytic around b, and $P(b) \neq 0, z=b$ is the first-order zero point of $Q(z)$.

$$a_{-1} = \lim_{z\to b}(z-b)f(z) = \lim_{z\to b}(z-b)\frac{P(z)}{Q(z)} = \frac{P(b)}{Q'(b)}$$

$$f(z) = \frac{1}{(z-1)^2(z-2)(z-3)} = \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-2} + \frac{D}{z-3}$$

You can perform decomposition of partial fraction to figure out values for $A-D$. Let's try a different approach.

$$(z-1)f(z) = \frac{1}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + B + \frac{C(z-1)}{z-2} + \frac{D(z-1)}{z-3}$$

Then

$$
\begin{aligned}
A &= \operatorname{res}(z-1)f(z)|_{z=1} = \frac{1}{2}\\
B &= \operatorname{res} f(z)|_{z=1} = \frac{3}{4}\\
C &= \operatorname{res} f(z)|_{z=2} = -1\\
D &= \operatorname{res} f(z)|_{z=3} = \frac{1}{4}
\end{aligned}
$$

例 6.1.5. 如果 ∞ 不是一个 $f(z)$ 的非孤立奇点，那么我们可以定义 f 在 ∞ 处的留数：

$$\operatorname{res} f(\infty) = \frac{1}{2\pi\mathrm{i}}\oint Cf(z)\mathrm{d}z \quad \text{顺时针积分可以包含 } \infty$$

$$
\begin{aligned}
\operatorname{res} f(\infty) &= \frac{1}{2\pi\mathrm{i}}\oint Cf(z)\mathrm{d}z = -\frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(1/t)}{t^2}\mathrm{d}t\\
&= -\frac{f(1/t)}{t^2} \quad \textit{coefficient of term } t^{-1} \textit{ around 0}\\
&= -f(1/t) \quad \textit{coefficient of term } t^{1} \textit{ around 0}\\
&= -f(z) \quad \textit{coefficient of term } z^{-1} \textit{ around } \infty \textit{ Note } z^{-1} \textit{ is analytic at } \infty
\end{aligned}
$$

也就是说，f 有可能在 ∞ 处不解析的同时，在 ∞ 处有非零留数。

同样，∞ 可能是 f 的一个孤立奇点，但是留数为 *0.*

6.2　留数定理的应用: 无穷积分

方法 6.2.1. 使用留数定理计算积分

$$\int_0^{2\pi} R(\sin\theta,\cos\theta)\mathrm{d}\theta \Leftrightarrow I = \oint_{|z|=1} R\left(\frac{z^2-1}{2\mathrm{i}z}, \frac{z^2+1}{2z}\right)\frac{\mathrm{d}z}{\mathrm{i}z}$$

例 6.2.2. *Compute*

$$I = \int_0^{\pi} \frac{1}{1+\varepsilon\cos\theta}\mathrm{d}\theta, |\varepsilon| < 1$$

$$
\begin{aligned}
I &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{1}{1+\varepsilon\cos\theta}\mathrm{d}\theta = \frac{1}{2}\oint_{|z|=1} \frac{1}{1+\varepsilon\frac{z^2+1}{2z}}\frac{\mathrm{d}z}{\mathrm{i}z}\\
&= \frac{1}{2}\oint_{|z|=1} \frac{2}{\varepsilon z^2+2z+\varepsilon}\frac{\mathrm{d}z}{\mathrm{i}} = \pi\sum_{|z|<1}\operatorname{res}\frac{2}{\varepsilon z^2+2z+\varepsilon}\\
&= \pi\frac{2}{2\varepsilon z+2}\bigg|_{z=\left(-1+\sqrt{1-\varepsilon^2}\right)/\varepsilon} = \frac{\pi}{\sqrt{1-\varepsilon^2}}
\end{aligned}
$$

定义 6.2.3. 当我们在计算形如

$$I=\int_{-\infty}^{\infty}f(x)\mathrm{d}x$$

的广义积分时，如果发现最终结果发散，我们常常可以定义一个主值积分：

$$I=\lim_{R_1,R_2\to\infty}\int_{-R_1}^{R_2}f(x)\mathrm{d}x$$

，其主值积分定义为：

$$v.p.\ \ I=\lim_{R\to\infty}\int_{-R}^{R}f(x)\mathrm{d}x$$

例 6.2.4. *Compute*

$$\int_0^{\infty}\frac{\mathrm{d}x}{1+x^4}$$

$$\begin{aligned}\oint_C\frac{1}{1+z^4}\mathrm{d}z&=\int_0^R\frac{1}{1+x^4}\mathrm{d}x+\int_{C_R}\frac{1}{1+z^4}\mathrm{d}z+\int_R^0\frac{\mathrm{i}\mathrm{d}y}{1+(\mathrm{i}y)^4}\\&=(1-\mathrm{i})\int_0^R\frac{dx}{1+x^4}+\int_{c_R}\frac{dz}{1+z^4}\\&=2\pi\,\mathrm{ires}\left.\frac{1}{1+z^4}\right|_{z=e^{\mathrm{i}\pi/4}}=\frac{\pi}{2}\frac{1-\mathrm{i}}{\sqrt{2}}\end{aligned}$$

The red integral is calculated by Large Arc Lemma: Let $R\to\infty$

$$\lim_{R\to\infty}\frac{z}{1+z^4}=0$$

Thus,

$$O.E.=\frac{\sqrt{2}}{4}\pi$$

方法 6.2.5. 计算

$$I=\int_{-\infty}^{\infty}f(x)\cos px\mathrm{d}x,I=\int_{-\infty}^{\infty}f(x)\sin px\mathrm{d}x$$

我们使用：

$$\oint_C f(z)e^{\mathrm{i}pz}\mathrm{d}z=\int_{-R}^{R}f(x)(\cos px+\mathrm{i}\sin px)\mathrm{d}x+\int_{C_R}f(z)e^{\mathrm{i}pz}\mathrm{d}z$$

引理 6.2.6. *Jordan's Lemma:*

Assume that in $0\le\arg z\le\pi$, when $|z|\to\infty,Q(z)$ uniformly converges to 0 . then

$$\lim_{R\to\infty}\int_{C_R}Q(z)e^{\mathrm{i}pz}\mathrm{d}z=0$$

where $p>0,C_R$ is an arc centered at origin, in the upper half space.

证明. Along the arc, we have $z = Re^{\mathrm{i}\theta}$. The condition of uniform convergence implies that

$$\forall \varepsilon > 0, \exists M(\varepsilon) > 0 \text{ (not related to } z\text{)}$$

when $|z| = R > M$ and $0 \le \arg z \le \pi, |Q(z)| < \varepsilon$

$$\begin{aligned}
\left|\int_{C_R} Q(z)e^{\mathrm{i}pz}\mathrm{d}z\right| &= \left|\int_0^{\pi} Q\left(Re^{\mathrm{i}\theta}\right)e^{\mathrm{i}pR(\cos\theta + \mathrm{i}\sin\theta)}Re^{\mathrm{i}\theta}\mathrm{i}\mathrm{d}\theta\right| \\
&\le \int_0^{\pi} \left|Q\left(Re^{\mathrm{i}\theta}\right)\right| e^{-pR\sin\theta}R\mathrm{d}\theta \\
&< \varepsilon R\int_0^{\pi} e^{-pR\sin\theta}\mathrm{d}\theta = 2\varepsilon R\int_0^{\pi/2} e^{-pR\sin\theta}\mathrm{d}\theta \\
&< 2\varepsilon R\int_0^{\pi/2} e^{-2pR\theta/\pi}\mathrm{d}\theta = \frac{\varepsilon\pi}{p}\left(1 - e^{-pR}\right) \to 0
\end{aligned}$$

□

例 6.2.7. *Compute*

$$\int_0^{\infty} \frac{x\sin x}{x^2 + a^2}\mathrm{d}x, a > 0$$

We choose one integral path:

$$(0,0) \to (R,0) \to (0,R) \to (-R,0) \to (0,0)$$

and let $R \to \infty$

$$\begin{aligned}
\oint_C \frac{ze^{\mathrm{i}z}}{z^2 + a^2}\mathrm{d}z &= \int_{-R}^{R} \frac{xe^{\mathrm{i}x}}{x^2 + a^2}\mathrm{d}x + \int_{C_R} \frac{ze^{\mathrm{i}z}}{z^2 + a^2}\mathrm{d}z \\
&= 2\pi\mathrm{i}\,\mathrm{res}\left.\frac{ze^{\mathrm{i}z}}{z^2 + a^2}\right|_{z = a\mathrm{i}} \\
&= \pi\mathrm{i}e^{-a}
\end{aligned}$$

And the red integral is calculated by Jordan lemma:

$$\lim_{R\to\infty} \frac{z}{z^2 + a^2} = 0$$

Thus,

$$O.E. = \frac{\pi}{2}e^{-a}$$

引理 6.2.8. *A Complementary Lemma of Jordan Lemma:*

Assume that $Q(z)$ only has a limited number of singular points. In the lower half space, when $|z| \to \infty, Q(z)$ converges to 0 uniformly. Then

$$\lim_{R\to\infty} \int_{C_R} Q(z)e^{-\mathrm{i}pz}\mathrm{d}z = 2\pi\mathrm{i} \cdot \sum_{\text{whole space}} \mathrm{res}\left\{Q(z)e^{-\mathrm{i}pz}\right\}$$

where $p > 0, C_R$ is a half circle centered at origin, in the upper half space.

6.3　路径上含有奇点的积分

定义 6.3.1. 对于路径上有奇点的积分，我们同样定义主值积分：

$$\int_a^b f(x)dx = \lim_{\delta_1\to 0}\int_a^{c-\delta_1} f(x)dx + \lim_{\delta_2\to 0}\int_{c+\delta_2}^b f(x)dx$$

$$v.p. \int_a^b f(x)dx = \lim_{\delta\to 0}\left[\int_a^{c-\delta} f(x)dx + \int_{c+\delta}^b f(x)dx\right]$$

对于一阶极点，我们可以使用 Large&Small Arc Lemma 轻松搞定，然而，考虑如下积分：

例 6.3.2. *Compute*

$$v.p. \int_{-\infty}^{\infty}\frac{dx}{x\left(1+x+x^2\right)}$$

这个积分，我们尝试如下积分路径：

$$(\delta, 0)\to(R,0)\to(0,R)\to(-R,0)\to(-\delta,0)\to(0,\delta)\to(\delta,0)$$

We let $R\to\infty, \delta\to 0$.

$$\oint_C \frac{dz}{z\left(1+z+z^2\right)} = \int_{-R}^{-\delta}\frac{dx}{x\left(1+x+x^2\right)} + \int_{C_\delta}\frac{dz}{z\left(1+z+z^2\right)} + \int_{\delta}^{R}\frac{dx}{x\left(1+x+x^2\right)} + \int_{C_R}\frac{dz}{z\left(1+z+z^2\right)}$$

Red Integral:

$$R = v.p. \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x(1+x+x^2)}$$

Blue Integral:

$$\int_{C_\delta}\text{clockwise} = -\int_{C_\delta}\text{anti-clockwise}$$

so,

$$\lim_{z\to 0} z\cdot\frac{1}{z(1+z+z^2)} = 1 \Rightarrow B = -i\pi(\text{by small arc lemma})$$

Violet Integral:

$$\lim_{z\to\infty} z\cdot\frac{1}{z(1+z+z^2)} = 0 \Rightarrow V = 0(\text{by large arc lemma})$$

$$L.H.S = 2\pi\mathrm{i}\ \mathrm{res}\ \left.\frac{1}{z\left(1+z+z^2\right)}\right|_{z=e^{\mathrm{i}2\pi/3}} = -\frac{\pi}{\sqrt{3}} - \mathrm{i}\pi$$

Thus, we have:

$$O.E. = -\frac{\pi}{\sqrt{3}}$$

6.4 包含割线的积分

方法 6.4.1. z^{s-1} 是一个多值函数，其割线为 $(0 \to (\infty, 0))$,$Q(z)$ 是一个单值函数。计算：

$$\int_0^\infty x^{s-1} Q(x) dx$$

我们使用如下路径：

$$\delta \to 0, R \to \infty: \quad (\delta, 0^-) \to (0, -\delta) \to (-\delta, 0) \to (0, \delta) \to$$
$$(\delta, 0^+) \to (R, 0^+) \to (0, R) \to (-R, 0) \to (0, -R) \to (R, 0^-) \to (\delta, 0^-)$$

$$\oint_C z^{s-1} Q(z) dz = \int_{C_\delta} z^{s-1} Q(z) dz + \int_\delta^R x^{s-1} Q(x) dx + \int_{C_R} z^{s-1} Q(z) dz + \int_R^\delta \left(x e^{\mathrm{i}2\pi}\right)^{s-1} Q(x) dx$$

例 6.4.2. *Compute*

$$\int_0^\infty \frac{x^{a-1}}{x + e^{\mathrm{i}\phi}} dx, 0 < a < 1, -\pi < \phi < \pi$$

$$\oint_C \frac{z^{a-1}}{z + e^{\mathrm{i}\phi}} dz = \int_\delta^R \frac{x^{a-1}}{x + e^{\mathrm{i}\phi}} dx + \int_{C_R} \frac{z^{a-1}}{z + e^{\mathrm{i}\phi}} dz + \int_R^\delta \frac{\left(x e^{\mathrm{i}2\pi}\right)^{a-1}}{x + e^{\mathrm{i}\phi}} dx + \int_{C_\delta} \frac{z^{a-1}}{z + e^{\mathrm{i}\phi}} dz$$

Red Integral gives:

$$\left(1 - e^{\mathrm{i}2\pi a}\right) \int_\delta^R \frac{x^{a-1}}{x + e^{\mathrm{i}\phi}} dx$$

Cyan Integral gives:

$$\lim_{z \to 0} \frac{z^a}{z + e^{i\phi}} = 0 \Rightarrow Origin.Integral. = 0$$

Blue Integral gives:

$$\lim_{z \to \infty} \frac{z^a}{z + e^{i\phi}} = 0 \Rightarrow Origin.Integral. = 0$$

$$L.H.S = 2\pi \mathrm{i} \sum \operatorname{res} \frac{z^{a-1}}{z + e^{\mathrm{i}\phi}} z = e^{\mathrm{i}(\phi+\pi)} \quad (0 < \phi + \pi < 2\pi)$$

Thus,

$$\int_0^\infty \frac{x^{a-1}}{x + e^{\mathrm{i}\phi}} dx = -\frac{2\pi \mathrm{i}}{1 - e^{\mathrm{i}2\pi a}} e^{\mathrm{i}\pi a} e^{\mathrm{i}\phi(a-1)} = \frac{\pi}{\sin \pi a} e^{\mathrm{i}\phi(a-1)}$$

微分方程部分

第七章　偏微分方程

7.1 Some Basic Assumptions

定义 7.1.1. *Order of differential equation:* Equals to the highest order of the equation.

定义 7.1.2. *Linear:*

$$a_0 y + a_1 y' + a_2 y'' + a_3 y''' + \cdots = b$$

Where a_i and b_i are not function of y.

$$\begin{aligned} y' &= \cot y \qquad (\text{not linear because of the term } \cot y) \\ y' &= 1 \qquad (\text{not linear because of the product } yy') \\ y'^2 &= xy \qquad (\text{not linear because of the term } y'^2). \end{aligned}$$

定义 7.1.3. *A solution of an PDE:* A solution of a differential equation (in the variables x and y) is a relation between x and y which, if substituted into the differential equation, gives an identity.

定义 7.1.4. *General solutions and particular solutions:*

$$\frac{\partial^2 U}{\partial x \partial y} = 2y - x$$

One particular solution is $U(x,y) = xy^2 - \frac{1}{2}x^2 y$

The general solution is $U(x,y) = xy^2 - \frac{1}{2}x^2 y + A(x) + B(y)$

定义 7.1.5.

$$\hat{L}[u] = f \quad \hat{L}\ :\ \textit{linear operator}$$

If $f = 0$, we say that the equation is homogeneous.

If $f \neq 0$, we say that the equation is inhomogeneous.

Suppose $\hat{L}$ is a linear operator, we have:

$$\hat{L}\left[c_1u_1+c_2u_2\right]=c_1\hat{L}\left[u_1\right]+c_2\hat{L}\left[u_2\right],\ (c_1,c_2=constant)$$

If both u_1 and u_2 are solutions to a homogeneous equation $\hat{L}[u]=0$, then we have

$$\hat{L}\left[c_1u_1+c_2u_2\right]=0$$

If both u_1 and u_2 are solutions to an inhomogeneous equation $\hat{L}[u]=f$, then we have

$$\hat{L}\left[u_1-u_2\right]=0$$

Because

$$u_1=(u_1-u_2)+u_2$$

this means the summation of a solution to a homogeneous equation and a solution to an inhomogeneous equation is still a solution to the original inhomogeneous equation.

7.2　偏微分方程的解法——几种情况

对于如下偏微分方程：

$$A_0\frac{\partial^n u}{\partial x^n}+A_1\frac{\partial^n u}{\partial x^{n-1}\partial y}+\ldots+A_n\frac{\partial^n u}{\partial y^n}+B_0\frac{\partial^{n-1}u}{\partial x^{n-1}}+\ldots+M\frac{\partial u}{\partial x}+N\frac{\partial u}{\partial y}+Pu=f(x,y)$$

如果引入记号:

$$\text{Denote } \hat{D}_x\equiv\partial/\partial x,\hat{D}_y\equiv\partial/\partial y$$

那么原式可以表示为：

$$\begin{aligned}\hat{L}(\hat{D}_x,\hat{D}_y)u=(A_0\hat{D}_x^n+A_1\hat{D}_x^{n-1}\hat{D}_y+\ldots+A_n\hat{D}_y^n+\\ B_0\hat{D}_x^{n-1}+\ldots+M\hat{D}_x+N\hat{D}_y+P)u=f(x,y)\end{aligned}$$

7.2.1　Homogenueous Case

齐次情况时，原式写成：

$$\left(A_0\hat{D}_x^n+A_1\hat{D}_x^{n-1}\hat{D}_y+\ldots+A_n\hat{D}_y^n\right)u=0$$

对左侧做多项式分解，可以得到：

$$\hat{L}\left(\hat{D}_x,\hat{D}_y\right)=A_0\left(\hat{D}_x-a_1\hat{D}_y\right)\left(\hat{D}_x-a_2\hat{D}_y\right)\ldots\left(\hat{D}_x-a_n\hat{D}_y\right)$$

如果我们假定原方程有一个 Trial Solution，记为:

$$u = \phi(y + ax)$$

注意到, 由链式法则

$$\hat{D}_x^k = a^k \phi^{(k)}(y + ax), \hat{D}_y^k = \phi^{(k)}(y + ax), \hat{D}_x^r \hat{D}_y^s = a^r \phi^{(r+s)}(y + ax)$$

将上述等式代入原方程，得到

$$\begin{aligned} \left(A_0 a^n + A_1 a^{n-1} + \ldots + A_n\right) \phi^{(n)}(y + ax) = 0 \Rightarrow \\ \left(A_0 a^n + A_1 a^{n-1} + \ldots + A_n\right) = 0 \end{aligned} \tag{7.1}$$

7.1 的解记为

$$a_1, \ a_2, \ a_3, \ , \ldots, \ a_n \ \ \text{(Assume No Multiple Roots)}$$

那么原式的解:

$$u = \phi_1 (y + a_1 x) + \phi_2 (y + a_2 x) + \ldots + \phi_n (y + a_n x)$$

对于重根：Generally, if a is n multiple root, i.e.

$$\hat{L}\left(\hat{D}_x, \hat{D}_y\right) u = \left(\hat{D}_x - a\hat{D}_y\right)^n u = 0$$

then the general solution is

$$u = x^{n-1}\phi_1(y + ax) + x^{n-2}\phi_2(y + ax) + \ldots + x\phi_{n-1}(y + ax) + \phi_n(y + ax)$$

7.2.2 Inhomogeneous Case——1-st Order

原式:

$$(\hat{D}_x - a\hat{D}_y - b)u = 0$$

我们假定原式有如下形式的解:

$$u = f(x)\phi(y + ax)$$

$$\begin{aligned} &\left(\hat{D}_x - a\hat{D}_y - b\right)[f(x)\phi(y + ax)] = \\ &f(x)\underbrace{\left(\hat{D}_x - a\hat{D}_y\right)\phi(y + ax)}_{=0} + \phi(y + ax)\left(\hat{D}_x - b\right) f(x) = 0 \Rightarrow \\ &f' - bf = 0 \Rightarrow f = Ce^{bx} \Rightarrow u = Ce^{bx}\phi(y + ax) = e^{bx}\phi(y + ax) \end{aligned}$$

7.2.3 General Solutions to Linear inhomo-PDEs

inhome-PDE 的解可以写成通解与特解之和的形式。

$f = e^{ax+by}$, $\hat{L}\left(\hat{D}_x, \hat{D}_y\right) = \hat{D}_y + \hat{D}_x$:

If so,then, it is easy to verify:

$$\hat{L}^{-1}\left(\hat{D}_x, \hat{D}_y\right) e^{ax+by} = \frac{1}{L(a,b)} e^{ax+by}, L(a,b) \neq 0$$

where $L(a,b)$ is polynomial of a and b. $L(a,b)$ is calculated by this in this case:

$$\hat{D}_x f = af; \quad \hat{D}_y f = bf; \quad L(a,b) = a+b$$

So, a particular solution of this PDE is:

$$u_0 = \frac{f(x,y)}{\hat{L}\left(\hat{D}_x, \hat{D}_y\right)} = \frac{f(x,y)}{L\left(a,b\right)} = \frac{f(x,y)}{a+b}$$

IF $L(a,b) = 0$: Without losing generality, we can assume:

$$\hat{L}\left(\hat{D}_x, \hat{D}_y\right) = b\hat{D}_x - a\hat{D}_y, \quad [L(a,b) = b \cdot a - a \cdot b = 0]$$

$$\left(b\hat{D}_x - a\hat{D}_y\right) u = e^{ax+by}$$

we assume the special solution as

$$u_0(x,y) = f(x,y)e^{ax+by}$$

$$\left(b\hat{D}_x - a\hat{D}_y\right) f(x,y) = 1$$

Let's further assume

$$f(x,y) = \alpha x + \beta y + \gamma \Rightarrow b\alpha - a\beta = 1$$

- If we take $\beta = \gamma = 0, \alpha = 1/b \quad u_0 = (x/b) \cdot e^{ax+by}$
- If we take $\alpha = \gamma = 0, \beta = -1/a \quad u_0 = (-y/a) \cdot e^{ax+by}$