Mathematical Methods in Physics

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Review

- Taylor series
- Laurent series
- Residue theorem
- Definite integrals using contour integration

Chapter – 03: Partial Differential Equations (PDEs)

- From this week, we will be switching from pure mathematics to problems with physical meanings.

- We will first discuss partial differential equations (偏微分方程). But sometimes we will also discuss ordinary differential equations (常微分方程), to help you understand some basic concepts.

- You should recall the knowledge of derivative, partial derivative, and differential equations learned in calculus.

How does a partial differential equation look like?

- By partial, a function of **multiple variables** must be involved.

- The operation of **partial derivative** is involved.

- There has to be an equation.

Navier-Stokes equation – Fluid dynamics

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \mathbf{u}$$

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right)$$

Gradient **operator (算符):** vector-like

$$\nabla^2 u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Laplace operator: scalar-like

Common PDEs and the corresponding **operators** $\hat{L}$

• Wave equation	$\dfrac{\partial^2 u}{\partial t^2} - a^2\nabla^2 u = f$ u : displacement	$\hat{L} \equiv \dfrac{\partial^2}{\partial t^2} - a^2\nabla^2$ 2nd order in time and space
• Heat equation	$\dfrac{\partial u}{\partial t} - \kappa\nabla^2 u = f$ u : temperature	$\hat{L} \equiv \dfrac{\partial}{\partial t} - \kappa\nabla^2$ 1st order in time and 2nd order in space
• Poisson's equation When f=0, Laplace's equation	$\nabla^2 u = f$ u : electric potential (in the static case)	$\hat{L} \equiv \nabla^2$
• Helmholtz's equation	$\nabla^2 u + k^2 u = f$ u : electric or magnetic field (in the dynamic case)	$\hat{L} \equiv \nabla^2 + k^2$

How does an ordinary differential equation (ODE) look like?

- Usually a function of **a single variable** is involved.
- The operation of **full derivative** is involved.
- There has to be an equation.

Newton's Law

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\frac{d^2\mathbf{r}}{dt^2}$$

The order of a differential equation

The *order* of a differential equation is the order of the highest derivative in the equation. Thus the equations

$$
\begin{aligned}
y' + xy^2 &= 1, \\
xy' + y &= e^x, \\
\frac{dv}{dt} &= -g, \\
L\frac{dI}{dt} + RI &= V,
\end{aligned} \tag{1.4}
$$

P 391 in the textbook

are first-order equations, while (1.3) and

$$m\frac{d^2r}{dt^2} = -kr$$

are second-order equations. A *linear* differential equation (with x as independent and y as dependent variable) is one of the form

$$a_0y + a_1y' + a_2y'' + a_3y''' + \cdots = b,$$

Linear or non-linear

are second-order equations. A *linear* differential equation (with x as independent and y as dependent variable) is one of the form

$$a_0y + a_1y' + a_2y'' + a_3y''' + \cdots = b,$$

where the a's and b are either constants or functions of x. The first equation in (1.4) is not linear because of the y^2 term; all the other equations we have mentioned so far are linear. Some other examples of nonlinear equations are:

$$
\begin{aligned}
y' &= \cot y && \text{(not linear because of the term } \cot y\text{);}\\
yy' &= 1 && \text{(not linear because of the product } yy'\text{);}\\
y'^2 &= xy && \text{(not linear because of the term } y'^2\text{).}
\end{aligned}
$$

P 391 in the textbook

What does a solution to an ODE mean?

A *solution* of a differential equation (in the variables x and y) is a relation between x and y which, if substituted into the differential equation, gives an identity.

P 391 in the textbook

Example 1. The relation

$$y = \sin x + C \tag{1.5}$$

Mind the arbitrary constant C

is a solution of the differential equation

$$y' = \cos x \tag{1.6}$$

because if we substitute (1.5) into (1.6) we get the identity $\cos x = \cos x$.

P 392 in the textbook

What does a solution to an ODE mean?

Example 2. The equation $y'' = y$ has solutions $y = e^x$ or $y = e^{-x}$ or $y = Ae^x + Be^{-x}$ as you can verify by substitution.

Two particular solutions (特解)

Mind the arbitrary constants A and B

Solution like this form is called a general solution (通解).

If we integrate $y' = f(x)$, the expression for y, namely $y = \int f(x)\,dx + C$, contains one arbitrary constant of integration. If we integrate $y'' = g(x)$ twice to get $y(x)$, then y contains two independent integration constants. We might expect that in general the solution of a differential equation of the nth order would contain n independent arbitrary constants. Note that in Example 1 above, the solution of the first-order equation $y' = \cos x$ contained one arbitrary constant C, and in Example 2 the solution $y = Ae^x + Be^{-x}$ of the second-order equation $y'' = y$ contained two arbitrary constants A and B.

> Any *linear* differential equation of order n has a solution containing n independent arbitrary constants, from which *all* solutions of the differential equation can be obtained by letting the constants have particular values. This solution is called the *general* solution of the linear differential equation.

(This may not be true for nonlinear equations; see Section 2.)

In applications, we usually want a *particular* solution, that is, one which satisfies the differential equation and some other requirements as well. Here are some examples of this.

P 392 in the textbook

What does a solution to a PDE mean?

$$\frac{\partial^2 U}{\partial x \partial y} = 2y - x$$

- Is this a linear partial differential equation? What is the order of this equation?

Mind the two arbitrary functions *A(x)* and *B(y)*

- One particular solution is $U(x, y) = xy^2 - \frac{1}{2}x^2 y$

- The general solution is $U(x, y) = xy^2 - \frac{1}{2}x^2 y + A(x) + B(y)$

- **In general, the general solution to a *n*-th order linear PDE contains n arbitrary functions.**

Some remarks

- For now we mainly focus on linear PDEs, which only contain linear operations of unknown functions.

$$\hat{L}[u] = f \qquad\qquad \hat{L}: \text{ linear operator} \quad \text{线性算符}$$

- In the above, *u* is the unknown function to be solved, whereas *f* is a known function.

- In general when $f \neq 0$, the equation is said to be inhomogeneous (非齐次的). If $f \equiv 0$, then the equation is homogeneous (齐次的).

Some remarks

Suppose $\hat{L}$ is a linear operator, we have:

$$\hat{L}[c_1 u_1 + c_2 u_2] = c_1 \hat{L}[u_1] + c_2 \hat{L}[u_2], \ (c_1, c_2 = \text{constant})$$

If both u_1 and u_2 are solutions to a homogeneous equation $\hat{L}[u] = 0$, then we have

$$\hat{L}[c_1 u_1 + c_2 u_2] = 0$$

If both u_1 and u_2 are solutions to an inhomogeneous equation $\hat{L}[u] = f$, then we have

$$\hat{L}[u_1 - u_2] = 0$$

Because $\boxed{u_1} = \boxed{(u_1 - u_2)} + \boxed{u_2}$, this means the summation of a solution to a homogeneous equation and a solution to an inhomogeneous equation is still a solution to the original inhomogeneous equation.

Some remarks

If u_1 and u_2 are solutions to the following inhomogeneous equations,

$$\hat{L}[u_1] = f_1,\ \hat{L}[u_2] = f_2$$

then their linear combination is a solution to the following inhomogeneous equation:

$$\hat{L}[c_1 u_1 + c_2 u_2] = c_1 f_1 + c_2 f_2$$

Now we focus on linear PDEs of two variables

$$A_0 \frac{\partial^n u}{\partial x^n} + A_1 \frac{\partial^n u}{\partial x^{n-1} \partial y} + \ldots + A_n \frac{\partial^n u}{\partial y^n} + B_0 \frac{\partial^{n-1} u}{\partial x^{n-1}} + \ldots + M \frac{\partial u}{\partial x} + N \frac{\partial u}{\partial y} + Pu = f(x, y)$$

Denote $\hat{D}_x \equiv \partial / \partial x, \;\; \hat{D}_y \equiv \partial / \partial y$

$$\hat{L}(\hat{D}_x, \hat{D}_y) u = (\underline{A_0} \hat{D}_x^n + \underline{A_1} \hat{D}_x^{n-1} \hat{D}_y + \ldots + \underline{A_n} \hat{D}_y^n + \underline{B_0} \hat{D}_x^{n-1} + \ldots + \underline{M} \hat{D}_x + \underline{N} \hat{D}_y + \underline{P}) u = f(x, y)$$

Known functions of *x* and *y*, called the coefficients of the equation

Now we focus on linear PDEs of two variables

$$A_0 \frac{\partial^n u}{\partial x^n} + A_1 \frac{\partial^n u}{\partial x^{n-1} \partial y} + \dots + A_n \frac{\partial^n u}{\partial y^n} + B_0 \frac{\partial^{n-1} u}{\partial x^{n-1}} + \dots + M \frac{\partial u}{\partial x} + N \frac{\partial u}{\partial y} + Pu = f(x, y)$$

Denote $\hat{D}_x \equiv \partial/\partial x, \ \hat{D}_y \equiv \partial/\partial y$

$$\hat{L}(\hat{D}_x, \hat{D}_y)u = (\underline{A_0}\hat{D}_x^n + \underline{A_1}\hat{D}_x^{n-1}\hat{D}_y + \dots + \underline{A_n}\hat{D}_y^n + \underline{B_0}\hat{D}_x^{n-1} + \dots + \underline{M}\hat{D}_x + \underline{N}\hat{D}_y + \underline{P})u = f(x, y)$$

Known functions of *x* and *y*, called the coefficients of the equation

We can further simplify the problem by requiring that all the coefficients are constant.

Scenario 1: the homogeneous case

with respect to the operator

- If $\hat{L}(\hat{D}_x, \hat{D}_y)$ is homogeneous about $\hat{D}_x$ and $\hat{D}_y$

$$(\underbrace{A_0\hat{D}_x^n}_{n^{th}\text{ order}} + A_1\underbrace{\hat{D}_x^{n-1}\hat{D}_y}_{n^{th}\text{ order}} + \dots + A_n\underbrace{\hat{D}_y^n}_{n^{th}\text{ order}})u = 0$$

类比多项式分解 / 多项式展开

then we can **decompose the linear operator** – similar to polynomial decomposition (*a*+*b*)^n, but now replacing *a* with $\hat{D}_x$, and *b* with $\hat{D}_y$.

$$\hat{L}(\hat{D}_x, \hat{D}_y) = A_0(\hat{D}_x - a_1\hat{D}_y)(\hat{D}_x - a_2\hat{D}_y)\dots(\hat{D}_x - a_n\hat{D}_y)$$

We assume different components can exchange their orders

$e.g.\ \hat{D}_x\hat{D}_y = \hat{D}_y\hat{D}_x$

But this may not be true for some special cases.

You can verify that the coefficient for the highest order $\hat{D}_x^n$ is A_0

If we assume that a trial solution can be written as $u = \phi(y + ax)$

note that

Chain rule

$$\hat{D}_x^k = a^k\phi^{(k)}(y + ax),\ \hat{D}_y^k = \phi^{(k)}(y + ax),\ \hat{D}_x^r\hat{D}_y^s = a^r\phi^{(r+s)}(y + ax)$$

Scenario 1: the homogeneous case

- After plugging-in and re-arrangement, we have

$$(A_0 a^n + A_1 a^{n-1} + ... + A_n)\phi^{(n)}(y + ax) = 0$$

- Assume the algebraic equation (known as auxiliary equation)

$$(A_0 a^n + A_1 a^{n-1} + ... + A_n) = 0$$

has the solutions a_1, a_2, ..., a_n (no multiple roots), then the general solution to the original equation is

$$u = \phi_1(y + a_1 x) + \phi_2(y + a_2 x) + ... + \phi_n(y + a_n x)$$

$\phi_i \ (i = 1, 2, ..., n)$ are independent, *n*-th order differentiable, arbitrary functions.

Exercise

- [7.01] Find the general solution to $\dfrac{\partial^2 u}{\partial x^2} - m^2 \dfrac{\partial^2 u}{\partial y^2} = 0,\ m = \text{constant}$

Solution: $\hat{L}(\hat{D}_x, \hat{D}_y) = (\hat{D}_x - m\hat{D}_y)(\hat{D}_x + m\hat{D}_y)$

$$a^2 - m^2 = 0,\ a = \pm m \Longrightarrow u = \phi_1(y + mx) + \phi_2(y - mx)$$

If we have multiple roots, e.g. when $m = 0$ $\Longleftrightarrow$ $\dfrac{\partial^2 u}{\partial x^2} = 0$

$$\hat{L}(\hat{D}_x, \hat{D}_y) = (\hat{D}_x - 0)^2 \Longrightarrow u = x\phi_1(y) + \phi_2(y)$$

Two different cases

Generally, if *a* is n multiple root, i.e. $\hat{L}(\hat{D}_x, \hat{D}_y)u = (\hat{D}_x - a\hat{D}_y)^n u = 0$

then the general solution is

$$u = x^{n-1}\phi_1(y + ax) + x^{n-2}\phi_2(y + ax) + \ldots + x\phi_{n-1}(y + ax) + \phi_n(y + ax)$$

Scenario 2: the inhomogeneous case

with respect to the operator, not to the R.H.S. of the PDE

- Let's consider the simple case of 1st order

$$\underbrace{(\hat{D}_x}_{\text{1st order}} - \underbrace{a\hat{D}_y}_{\text{1st order}} - \underbrace{b)}_{\text{0th order}}u = 0$$

- We already know the general solution when *b*=0: $u = \phi(y + ax)$
- Then we can assume the solution for the inhomogeneous case has the form

$$u(x, y) = f(x)\phi(y + ax)$$

$$(\hat{D}_x - a\hat{D}_y - b)[f(x)\phi(y + ax)] = f(x)\underbrace{\boxed{(\hat{D}_x - a\hat{D}_y)\phi(y + ax)}}_{=0} + \phi(y + ax)(\hat{D}_x - b)f(x) = 0$$

$$f'(x) - bf(x) = 0 \Longrightarrow f(x) = Ce^{bx}$$

$u = Ce^{bx}\phi(y + ax) = e^{bx}\phi(y + ax)$ The constant *C* can be merged into $\phi(y + ax)$

Exercise

- [7.02] Find the general solution to $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial x \partial y} - 2\dfrac{\partial^2 u}{\partial y^2} + 2\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$

- Solution: Again, let's try to decompose the operator – to reduce from 2D to 1D

$$(\hat{D}_x^2 - \hat{D}_x\hat{D}_y - 2\hat{D}_y^2 + 2\hat{D}_x + 2\hat{D}_y)u = (\hat{D}_x + \hat{D}_y)(\hat{D}_x - 2\hat{D}_y + 2)u = 0$$

Following the procedures on pages 18-20, we have

$$u = \phi(y - x) + e^{-2x}\psi(y + 2x)$$

Remark: If multiple roots (for inhomogeneous case) exist, e.g. $(\hat{D}_x - a\hat{D}_y - b)^2 u = 0$

Then the general solution takes the form

$$u = xe^{bx}\phi(y + ax) + e^{bx}\psi(y + ax)$$

General solutions to linear inhomogeneous PDEs

with respect to the R.H.S. of the PDE

$$\hat{L}(\hat{D}_x, \hat{D}_y)u = \boxed{f(x,y)}$$

The general solution (linear inhomogeneous PDE) = one special solution (linear inhomogeneous PDE) + general solution (linear homogeneous PDE)

How to obtain one special solution to the inhomogeneous case?

$$u_0 = \frac{f(x,y)}{\hat{L}(\hat{D}_x, \hat{D}_y)}$$

(1) If $f(x,y) = e^{ax+by}$, then $\dfrac{1}{\boxed{\hat{L}(\hat{D}_x, \hat{D}_y)}} e^{ax+by} = \dfrac{1}{\boxed{L(a,b)}} e^{ax+by}, \ L(a,b) \neq 0$

operator

polynomial

Easy to verify

$$\hat{D}_x e^{ax+by} = a e^{ax+by}, \quad \hat{D}_y e^{ax+by} = b e^{ax+by}$$

General solutions to linear inhomogeneous PDEs

with respect to the R.H.S. of the PDE

How to obtain one special solution to the inhomogeneous case? – One is enough!

(1) If $f(x,y) = e^{ax+by}$, then $\frac{1}{\hat{L}(\hat{D}_x, \hat{D}_y)} e^{ax+by} = \cancel{\frac{1}{L(a,b)} e^{ax+by}, \; L(a,b) \neq 0}$

But if $L(a,b) = 0$, without losing generality, we can assume

$\hat{L}(\hat{D}_x, \hat{D}_y) = b\hat{D}_x - a\hat{D}_y, \quad [L(a,b) = b \cdot a - a \cdot b = 0]$

$(b\hat{D}_x - a\hat{D}_y)u = e^{ax+by}$ Plug in

we assume the special solution as $u_0(x,y) = f(x,y)e^{ax+by}$

$(b\hat{D}_x - a\hat{D}_y)f(x,y) = 1$ Let's further assume $f(x,y) = \alpha x + \beta y + \gamma \Rightarrow b\alpha - a\beta = 1$

- If we take $\beta = \gamma = 0, \; \alpha = 1/b \quad u_0 = (x/b) \cdot e^{ax+by}$
- If we take $\alpha = \gamma = 0, \; \beta = -1/a \quad u_0 = (-y/a) \cdot e^{ax+by}$

General solutions to linear inhomogeneous PDEs

with respect to the R.H.S. of the PDE

How to obtain one special solution to the inhomogeneous case? – One is enough!

(2) If $f(x,y) = e^{\mathrm{i}(ax+by)}$, then

$$\frac{1}{\hat{L}(\hat{D}_x, \hat{D}_y)} e^{\mathrm{i}(ax+by)} = \frac{1}{L(\mathrm{i}a, \mathrm{i}b)} e^{\mathrm{i}(ax+by)}$$

When *a* & *b* are real numbers, and when the coefficients of the PDE are real numbers

$$\frac{1}{\hat{L}(\hat{D}_x, \hat{D}_y)} \sin(ax+by) = \mathrm{Im}\left[\frac{1}{L(\mathrm{i}a, \mathrm{i}b)} e^{\mathrm{i}(ax+by)}\right]$$

$$\frac{1}{\hat{L}(\hat{D}_x, \hat{D}_y)} \cos(ax+by) = \mathrm{Re}\left[\frac{1}{L(\mathrm{i}a, \mathrm{i}b)} e^{\mathrm{i}(ax+by)}\right]$$

General solutions to linear inhomogeneous PDEs

with respect to the R.H.S. of the PDE

How to obtain one special solution to the inhomogeneous case? – One is enough!

(3) If $f(x,y) = e^{ax+by}g(x,y)$, then

$$\hat{L}(\hat{D}_x,\hat{D}_y)$$

$$\frac{1}{\hat{L}(\hat{D}_x,\hat{D}_y)} e^{ax+by} g(x,y) = e^{ax+by} \frac{1}{\hat{L}(\hat{D}_x+a,\hat{D}_y+b)} g(x,y)$$

Easy to verify the above, because

$$\left.\begin{aligned} \hat{D}_x\left[e^{ax+by}g(x,y)\right] &= e^{ax+by}(\hat{D}_x+a)g(x,y) \\ \hat{D}_y\left[e^{ax+by}g(x,y)\right] &= e^{ax+by}(\hat{D}_y+b)g(x,y) \end{aligned}\right\} \quad \hat{L}(\hat{D}_x,\hat{D}_y)\left[e^{ax+by}g(x,y)\right] = e^{ax+by}\hat{L}(\hat{D}_x+a,\hat{D}_y+b)g(x,y)$$

$$\boxed{\begin{aligned} &\underline{\hat{L}(\hat{D}_x,\hat{D}_y)}\left[e^{ax+by}\frac{1}{\hat{L}(\hat{D}_x+a,\hat{D}_y+b)}g(x,y)\right] \\ &= e^{ax+by}\hat{L}(\hat{D}_x+a,\hat{D}_y+b)\left[\frac{1}{\hat{L}(\hat{D}_x+a,\hat{D}_y+b)}g(x,y)\right] = e^{ax+by}g(x,y) \end{aligned}}$$

General solutions to linear inhomogeneous PDEs

with respect to the R.H.S. of the PDE

How to obtain one special solution to the inhomogeneous case? – One is enough!

(4) If $f(x,y) = x^m y^n$, we can expand the operator $\dfrac{1}{\hat{L}(\hat{D}_x, \hat{D}_y)}$ as a power series of $\hat{D}_x,\ \hat{D}_y$, and then find one special solution.

Exercise

You can plug in to verify

- [7.03] Find the general solution to $(\hat{D}_x^2 - 2\hat{D}_x\hat{D}_y + \hat{D}_y^2)u = 12xy$
- Solution:

One special solution is $u_0 = \dfrac{12}{\hat{D}_x^2 - 2\hat{D}_x\hat{D}_y + \hat{D}_y^2}xy = \dfrac{12}{\hat{D}_x^2}\left(1 - \dfrac{\hat{D}_y}{\hat{D}_x}\right)^{-2} xy$

$$= \frac{12}{\hat{D}_x^2}\left(1 + 2\frac{\hat{D}_y}{\hat{D}_x} + \ldots\right)xy = \frac{12}{\hat{D}_x^2}\left(xy + \frac{2}{\hat{D}_x}x\right)$$

$$= 12\left(y\frac{1}{\hat{D}_x^2}x + \frac{2}{\hat{D}_x^3}x\right) = 12\left(\frac{1}{6}x^3y + \frac{1}{12}x^4\right)$$

$$\boxed{\hat{D}_y^n xy = 0, \text{if } n \geq 2}$$

$\dfrac{1}{\hat{D}_x^2}x = \dfrac{1}{6}x^3$ Multiply by $\hat{D}_x^2$

$\dfrac{1}{\hat{D}_x^3}x = \dfrac{1}{24}x^4$ Multiply by $\hat{D}_x^3$

$$u(x,y) = x\phi(x+y) + \psi(x+y) + x^4 + 2x^3y$$

General solutions to linear inhomogeneous PDEs

with respect to the R.H.S. of the PDE

How to obtain one special solution to the inhomogeneous case? – One is enough!

(5) If $f(x, y) = f(ax + by)$, and if $\hat{L}(\hat{D}_x, \hat{D}_y)$ is homogeneous about $\hat{D}_x, \ \hat{D}_y$ (of order n), then

$$\hat{D}_x^r g(ax + by) = a^r g^{(r)}(ax + by), \ \ \hat{D}_y^s g(ax + by) = b^s g^{(s)}(ax + by)$$

$$\hat{L}(\hat{D}_x, \hat{D}_y) g(ax + by) = L(a, b) g^{(n)}(ax + by)$$

Easy to verify

When $L(a, b) \neq 0$, we have

Multiply by $\hat{L}(\hat{D}_x, \hat{D}_y)$

$$\frac{1}{\hat{L}(\hat{D}_x, \hat{D}_y)} g^{(n)}(ax + by) = \frac{1}{L(a, b)} g(ax + by)$$

Exercise

- [7.04] Find the solution to $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 12(x+y)$ $\qquad \text{Note} : n = 2$
- Solution: $u_0 = \dfrac{12}{\hat{D}_x^2 + \hat{D}_y^2}(x+y) = \dfrac{12}{(1^2+1^2)}\left[\dfrac{1}{3!}(x+y)^3\right] = (x+y)^3$

  The general solution is $u(x,y) = (x+y)^3 + \phi(x+\mathrm{i}y) + \phi(x-\mathrm{i}y)$

Refer to page 19, $m = \pm\mathrm{i}$

General solutions to linear inhomogeneous PDEs

with respect to the R.H.S. of the PDE

How to obtain one special solution to the inhomogeneous case? – One is enough!

(5) If $f(x,y) = f(ax+by)$, and if $\hat{L}(\hat{D}_x, \hat{D}_y)$ is homogeneous about $\hat{D}_x,\ \hat{D}_y$ (of order n), then

$$\hat{D}_x^r g(ax+by) = a^r g^{(r)}(ax+by),\ \ \hat{D}_y^s g(ax+by) = b^s g^{(s)}(ax+by)$$

$$\hat{L}(\hat{D}_x, \hat{D}_y) g(ax+by) = L(a,b) g^{(n)}(ax+by)$$

When $L(a,b) = 0$, we have to find other ways.

- Consider the 1st order case

$$(\hat{D}_x - \alpha \hat{D}_y)u = \phi(x)\psi(y+\alpha x) \qquad a=\alpha,\ b=1\ \Rightarrow\ L(a,b)=\alpha-\alpha\cdot 1=0$$

We can assume that the solution also takes the form of

$$u(x,y) = f(x)\psi(y+\alpha x) \qquad f'(x)=\phi(x)$$

After substitution, we have

$$\frac{1}{\hat{D}_x - \alpha \hat{D}_y} f'(x)\psi(y+\alpha x) = f(x)\psi(y+\alpha x)$$

Repeat the same procedure for (k-1) times

$$\frac{1}{(\hat{D}_x - \alpha \hat{D}_y)^k} f^{(k)}(x)\psi(y+\alpha x) = f(x)\psi(y+\alpha x)$$

Exercise

- [7.05] Find the solution to $(\hat{D}_x^2 - 6\hat{D}_x\hat{D}_y + 9\hat{D}_y^2)u = 6x + 2y$
- Solution:

$$L(a,b) = (6)^2 - 6\cdot 6\cdot 2 + 9\cdot(2)^2 = 0$$

Cannot apply the results on page 28; instead, let's try the results on pages 30-31.

$$u_0 = \frac{1}{(\hat{D}_x - 3\hat{D}_y)^2}(6x+2y) = \frac{2}{(\hat{D}_x - 3\hat{D}_y)^2}(3x+y) = x^2(y+3x)$$

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$$f''(x) = 1, \;\; => f(x) = x^2/2$$

The general solution is $u = x^2(y+3x) + \underbrace{x\phi(y+3x) + \psi(y+3x)}_{\text{Page 19}}$

What if the coefficients are not constant

$$x^m y^n \frac{\partial^{m+n} u}{\partial x^m \partial y^n}$$

variable

$$x = e^t,\ y = e^s \quad \Longleftrightarrow \quad t = \ln x,\ s = \ln y$$

$$\hat{D}_t = \frac{\partial}{\partial t} = x\frac{\partial}{\partial x},\ \hat{D}_s = \frac{\partial}{\partial s} = y\frac{\partial}{\partial y}$$

$$x^2 \frac{\partial^2}{\partial x^2} = \hat{D}_t(\hat{D}_t - 1) \qquad y^2 \frac{\partial^2}{\partial y^2} = \hat{D}_s(\hat{D}_s - 1)$$

$$x^3 \frac{\partial^3}{\partial x^3} = \hat{D}_t(\hat{D}_t - 1)(\hat{D}_t - 2) \qquad y^3 \frac{\partial^3}{\partial y^3} = \hat{D}_s(\hat{D}_s - 1)(\hat{D}_s - 2)$$

Now coefficients become constant

$$x^m y^n \frac{\partial^{m+n}}{\partial x^m \partial y^n} = \hat{D}_t(\hat{D}_t - 1)...(\hat{D}_t - m + 1)\hat{D}_s(\hat{D}_s - 1)...(\hat{D}_s - n + 1)$$

Exercise

- [7.06] Find the solution to $x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$
- Solution: Make variable substitution $x = e^t,\ y = e^s$

$$\hat{D}_t = \frac{\partial}{\partial t} = x\frac{\partial}{\partial x},\ \hat{D}_s = \frac{\partial}{\partial s} = y\frac{\partial}{\partial y}$$

$$[\hat{D}_t(\hat{D}_t - 1) - \hat{D}_s(\hat{D}_s - 1) + \hat{D}_t - \hat{D}_s)u = 0$$

$$(\hat{D}_t^2 - \hat{D}_s^2)u = 0$$

$$u = \phi_1(t+s) + \phi_2(t-s) = \phi_1(\ln x + \ln y) + \phi_2(\ln x - \ln y)$$
$$= \phi_1(\ln xy) + \phi_2(\ln \frac{x}{y}) = \phi(xy) + \psi(\frac{x}{y})$$