Exercise 06

Part 1

(1) $1-z^2 = \sum_{i=0}^{\infty} a_i (z-1)^i$, where $a_i = \frac{f^{(i)}(1)}{i!}$

$a_0 = 0$, $a_1 = -2$, $a_2 = -1$, $a_i = 0$ for $i \geq 3$

$\therefore 1-z^2 = -2(z-1)-(z-1)^2$

radius of convergence : $|z-1| < \infty$

(2) $\sin z = \sum_{i=0}^{\infty} a_i (z-n\pi)^i$, where $a_i = \frac{f^{(i)}(n\pi)}{i!}$

$a_i = 0$ for i even

$a_1 = \cos n\pi$ $\quad a_3 = -\frac{\cos n\pi}{3!}$, ...

$$\therefore \sin z = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{(2k+1)!} (z-n\pi)^{2k+1}$$

$(|z-n\pi| < \infty)$

(3)
$$\frac{1}{1+z+z^2} = \frac{1}{\sqrt{3}i}\left(\frac{1}{z+e^{-i\frac{\pi}{3}}} - \frac{1}{z+e^{i\frac{\pi}{3}}}\right)$$

$$= \frac{1}{\sqrt{3}i}\left(e^{i\frac{\pi}{3}}\frac{1}{1+e^{i\frac{\pi}{3}}z} - e^{-i\frac{\pi}{3}}\frac{1}{1+e^{-i\frac{\pi}{3}}z}\right)$$

$$= \frac{1}{\sqrt{3}i}\left(e^{i\frac{\pi}{3}}\left[1-e^{i\frac{\pi}{3}}z+\left(e^{i\frac{\pi}{3}}z\right)^2 - \cdots\right] - e^{-i\frac{\pi}{3}}\left[1-e^{-i\frac{\pi}{3}}z+\left(e^{-i\frac{\pi}{3}}z\right)^2 - \cdots\right]\right)$$

$$= \frac{1}{\sqrt{3}i}\sum_{n=0}^{\infty}(-z)^n\left[e^{i\frac{\pi}{3}(n+1)} - e^{-i\frac{\pi}{3}(n+1)}\right]$$

$$= \frac{2}{\sqrt{3}}\sum_{n=0}^{\infty}(-1)^n z^n \sin\left[\frac{\pi}{3}(n+1)\right]$$

radius of convergence : $|e^{\pm i\frac{\pi}{3}}z| < 1 \Rightarrow |z| < 1$

(4) Around $z=0$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad |z|<\infty$$

$$\frac{1}{z-1} = -\sum_{n=0}^{\infty} z^n, \quad |z|<1$$

$$\therefore \frac{\sin z}{z-1} = \sum_{n=0}^{\infty} a_n z^n, \quad |z|<1.$$

where $a_n = \begin{cases} \sum_{k=0}^{(n-1)/2} \frac{(-1)^{k+1}}{(2k+1)!}, & n \text{ odd} \\ a_{n-1}, & n \text{ even} \end{cases}$

Part 2.

(1) $$\ln z = \sum_{n=0}^{\infty} \frac{f^{(n)}(i)}{n!}(z-i)^n$$

$$= \frac{\pi}{2} i - \sum_{n=1}^{\infty} \frac{i^n}{n}(z-i)^n, \quad |z-i|<1$$

(2) $f(i) = \ln(z)|_{z=i} = -\frac{3}{2}\pi i$

$$\therefore \ln z = -\frac{3}{2}\pi i - \sum_{n=1}^{\infty} \frac{i^n}{n}(z-i)^n, \quad |z-i|<1$$

Part 3.

(1) Let $S(z) = \sum_{n=0}^{\infty} \frac{1}{2n+1} z^{2n+1}$

Radius of convergence: $R = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{2n+1}\right)^{\frac{1}{2n+1}}} = 1$

When $|z|<1$, $S(z)$ uniformly converges.

$$\therefore S'(z) = \left(\sum_{n=0}^{\infty} \frac{1}{2n+1} z^{2n+1}\right)'$$

$$= \sum_{n=0}^{\infty} \frac{1}{2n+1} (z^{2n+1})'$$

$$= \sum_{n=0}^{\infty} z^{2n} = \frac{1}{1-z^2}$$

$$\therefore S(z) - S(0) = \int_0^z S'(z)\,dz$$
$$= \int_0^z \frac{1}{1-z^2}\,dz$$
$$= \frac{1}{2}\ln\frac{1+z}{1-z}$$

And $S(0) = 0$

$$\therefore S(z) = \frac{1}{2}\ln\frac{1+z}{1-z}, \quad (|z|<1)$$

(2) Let $S(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}$

$$\because \lim_{n\to\infty}\left|\frac{\frac{1}{(2n+2)!}z^{2n+2}}{\frac{1}{(2n)!}z^{2n}}\right| = \lim_{n\to\infty}\frac{z^2}{(2n+1)(2n+2)} = 0$$

for any z

$\therefore R = \infty$

Note that
$$e^z = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}$$
$$e^{-z} = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!} - \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}$$

$$\therefore S(z) = \frac{e^z + e^{-z}}{2}$$

Part 4.

(1) Around $z=1$

$$\frac{1}{z^2} = 1 - 2(z-1) + 3(z-1)^2 - \cdots$$
$$= \sum_{n=0}^{\infty}(-1)^n(n+1)(z-1)^n \qquad (|z-1|<1)$$

$$\therefore \frac{1}{z^2(z-1)} = \sum_{n=0}^{\infty}(-1)^n(n+1)(z-1)^{n-1},$$
$$(0<|z-1|<1)$$

(2) $$\frac{1}{z^2(z-1)} = \frac{1}{z^3(1-\frac{1}{z})} = \frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{-3} z^n$$
$$(1<|z|<\infty)$$

(3) $\frac{(z-1)(z-2)}{(z-5)(z-6)} = 1 + \frac{20}{z-6} - \frac{12}{z-5}$

$= 1 - \frac{10}{3}\frac{1}{1-\frac{z}{6}} - \frac{12}{z}\frac{1}{1-\frac{5}{z}}$

$= 1 - \frac{10}{3}\sum_{n=0}^{\infty}\left(\frac{z}{6}\right)^n - \frac{12}{z}\sum_{n=0}^{\infty}\left(\frac{5}{z}\right)^n$

where $\begin{cases} \left|\frac{z}{6}\right| < 1 \\ \left|\frac{5}{z}\right| < 1 \end{cases} \Rightarrow 5 < |z| < 6$

(4) $\frac{(z-1)(z-2)}{(z-5)(z-6)} = 1 + \frac{20}{z-6} - \frac{12}{z-5}$

$= 1 + \frac{20}{z}\frac{1}{1-\frac{6}{z}} - \frac{12}{z}\frac{1}{1-\frac{5}{z}}$

$= 1 + \frac{20}{z}\sum_{n=0}^{\infty}\left(\frac{6}{z}\right)^n - \frac{12}{z}\sum_{n=0}^{\infty}\left(\frac{5}{z}\right)^n$

where $\begin{cases} \left|\frac{6}{z}\right| < 1 \\ \left|\frac{5}{z}\right| < 1 \end{cases} \Rightarrow 6 < |z| < \infty$

Part 5

(1) $\frac{1}{z^2+16} = \frac{1}{(z+4i)(z-4i)}$, $z = \pm 4i$ are first-order poles.

(2) $\frac{\cos(az)}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n\frac{(az)^{2n}}{(2n)!} = \sum_{n=-1}^{\infty}(-1)^{n+1}\frac{a^{2n+2}\, z^{2n}}{(2n+2)!}$

$z=0$ is a second-order poles.

By setting $z = \frac{1}{t}$, the expansion has infinite negative-power terms.

$\therefore$ $z=\infty$ is an essential singularity.

(3) $\frac{\cos(az)-\cos(bz)}{z^2} = \frac{1}{z^2}\left[\sum_{n=0}^{\infty}(-1)^n\frac{(az)^{2n}}{(2n)!} - \sum_{n=0}^{\infty}(-1)^n\frac{(bz)^{2n}}{(2n)!}\right]$

$= \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}(a^{2n}-b^{2n})z^{2n-2}$

no negative-power terms $\Rightarrow$ $z=0$ is removable singularity

Similar to (2), $z=\infty$ is an essential singularity.

(4) $\frac{\sin z}{z^2}-\frac{1}{z} = \frac{1}{z^2}\left[\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\right]-\frac{1}{z}$

$= \sum_{n=1}^{\infty}(-1)^n\frac{z^{2n-1}}{(2n+1)!}$

no negative-power terms $\Rightarrow$ $z=0$ is removable singularity

Similar to (2), $z=\infty$ is an essential singularity.

Part 6.

(1) Let $t=\frac{1}{z}$.

$\frac{1}{z^3} \Rightarrow t^3$, which is analytic at $t=0$

$\therefore$ $\frac{1}{z^3}$ is analytic at $z=\infty$.

(2) Similar to part 5 (2), $z=\infty$ is an essential singularity.

(3) Let $t=\frac{1}{z}$, then $e^{-\frac{1}{z^2}} \Rightarrow e^{-t^2}$

e^{-t^2} is analytic around $t=0$

$\therefore$ e^{-1/z^2} is analytic at $z=\infty$.

(4) $\sqrt{(z-5)(z-6)} \xrightarrow{t=\frac{1}{z}} \sqrt{(\frac{1}{t}-5)(\frac{1}{t}-6)}$

$= \frac{\sqrt{(1-5t)(1-6t)}}{t}$

$\therefore$ $t=0$ is a simple pole

$\therefore z=\infty$ is also a simple pole of $\sqrt{(z-5)(z-6)}$

Part 7.

(1) Let $Q(z)=z-1$.

$z=1$ is a first-order zero point of $Q(z)$

$\therefore$ res $f(1) = \frac{e^1}{1} = e$

(2) $\left(\frac{z}{1-\cos z}\right)^2 = \frac{z^2}{\left(\frac{1}{2}z^2 - \frac{1}{4!}z^4 + \cdots\right)^2}$

$= \frac{1}{\left(\frac{1}{2}z - \frac{1}{4!}z^3 + \cdots\right)^2}$

$\because z^2 f(z)\Big|_{z=0} = 4$, $z^3 f(z)\Big|_{z=0} = 0$

$\therefore z=0$ is a pole of order 2 of $f(z)$.

res $f(0) = \left(z^2 f(z)\right)'\Big|_{z=0} = 0$

(3) $\frac{1}{z^2 \sin z} = \frac{1}{z^2 \sum\limits_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}}$

By similar reasoning to (2), $z=0$ is a pole of order 3.

$\therefore$ res $f(0) = \frac{1}{2!} \frac{d^2}{dz^2} \frac{z^3}{z^2 \sin z}\Big|_{z=0} = \frac{1}{6}$

(4) $\frac{e^z}{(z^2-1)^2} = \frac{e^z}{(z+1)^2 (z-1)^2}$

$\therefore z=1$ is a pole of order 2

res $f(1) = \frac{d}{dz}(z-1)^2 \frac{e^z}{(z^2-1)^2}\Bigg|_{z=1} = 0$

Part 8.

(1) There are two simple poles $z_1 = e^{i\frac{\pi}{4}}$, $z_2 = e^{i\frac{3}{4}\pi}$ along the integration path

res $f(e^{i\frac{\pi}{4}}) = (z - e^{i\frac{\pi}{4}}) \frac{1}{1+z^4}\Big|_{z=e^{i\frac{\pi}{4}}} = \frac{1}{4e^{i\frac{3}{4}\pi}}$

res $f(e^{i\frac{3}{4}\pi}) = \frac{1}{4e^{\frac{9}{4}\pi i}}$

$$\therefore \oint_{|z-1|=1} \frac{1}{1+z^4} dz = 2\pi i\left[\operatorname{res} f(e^{i\frac{\pi}{4}}) + \operatorname{res} f(e^{i\frac{7}{4}\pi})\right]$$
$$= -\frac{\sqrt{2}}{2}\pi i$$

(2) Apply the higher-order formula for derivatives:

$$\oint_{|z|=1} \frac{e^z}{z^3} dz = \frac{2\pi i}{2!} e^z\Big|_{z=0} = \pi i$$

Part 9.

(1) $\displaystyle\int_0^{\pi} \frac{1}{1+\sin^2\theta} d\theta = \int_0^{\pi} \frac{1}{3-\cos(2\theta)} d(2\theta)$

$$\xrightarrow{\text{Let } \gamma = 2\theta} \int_0^{2\pi} \frac{1}{3-\cos\gamma} d\gamma$$

Let $z = e^{i\gamma}$, $\cos\gamma = \dfrac{z^2+1}{2z}$, $d\gamma = \dfrac{dz}{iz}$

$$\therefore \text{integral} = \oint_{|z|=1} \frac{1}{3-\frac{z^2+1}{2z}} \frac{dz}{iz}$$
$$= -2i \oint_{|z|=1} \frac{1}{6z-z^2-1} dz$$

There is one simple pole $z = 3-2\sqrt{2}$ inside the circle $|z|=1$

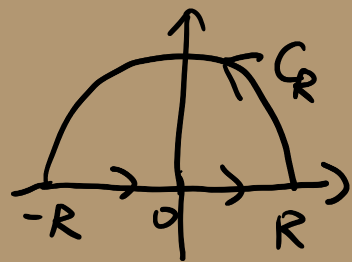
$$\operatorname{res} f(3-2\sqrt{2}) = \frac{z-3+2\sqrt{2}}{6z-z^2-1}\Big|_{z=3-2\sqrt{2}} = \frac{\sqrt{2}}{8}$$

$$\therefore \text{Integral} = -2i\cdot(2\pi i)\operatorname{res} f(3-2\sqrt{2}) = \frac{\sqrt{2}}{2}\pi$$

(2) First, look at

$$\oint_C \frac{e^{iz}}{1+z^4} dz = \int_{-R}^{R} \frac{e^{ix}}{1+x^4} dx + \int_{C_R} \frac{e^{iz}}{1+z^4} dz$$

$$\because \lim_{z\to\infty} \frac{1}{1+z^4} = 0$$



$\therefore$ by Jordan's lemma,

$$\int_{C_R} \frac{1}{1+z^4} e^{iz} dz = 0$$

For the L.H.S., $\oint_C \frac{1}{1+z^4} e^{iz} dz$

$$= 2\pi i \left[\operatorname{res} f(e^{i\frac{\pi}{4}}) + \operatorname{res} f(e^{i\frac{3}{4}\pi}) \right]$$

$$= 2\pi i \left(\left.\frac{e^{iz}}{4z^3}\right|_{z=e^{i\frac{\pi}{4}}} + \left.\frac{e^{iz}}{4z^3}\right|_{z=e^{i\frac{3}{4}\pi}} \right)$$

$$= \frac{\pi i}{2} e^{-\frac{\sqrt{2}}{2}} \left[e^{i(\frac{\sqrt{2}}{2} - \frac{3}{4}\pi)} + e^{-i(\frac{\sqrt{2}}{2} + \frac{\pi}{4})} \right]$$

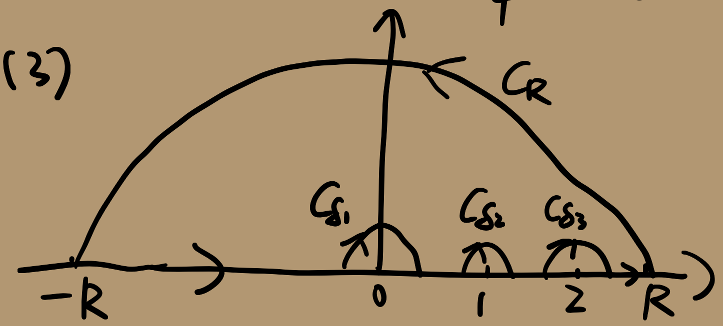
$$\therefore \int_0^{\infty} \frac{\cos x}{1+x^4} dx = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\cos x}{1+x^4} dx$$

$$= \frac{1}{2} \operatorname{Re} \left[\int_{-\infty}^{+\infty} \frac{\cos z}{1+z^4} dz \right]$$

$$= \lim_{R\to\infty} \frac{1}{2} \operatorname{Re} \left[\oint_C \frac{e^{z}}{1+z^4} dz \right]$$

$$= \frac{\sqrt{2}}{4} \pi e^{-\frac{\sqrt{2}}{2}} \left(\cos\frac{\sqrt{2}}{2} + \sin\frac{\sqrt{2}}{2} \right)$$

(3)



$z = 0, 1, 2$ are 3 simple poles of $\frac{1}{z(z-1)(z-2)}$

First, let $f(z) = \frac{1}{z(z-1)(z-2)}$

$$\oint_C f(z)dz = \int_{-R}^{-\delta_1} f(z)dz + \int_{C_{\delta_1}} f(z)dz + \int_{\delta_1}^{1-\delta_2} f(z)dz + \int_{C_{\delta_2}} f(z)dz$$
$$+ \int_{1+\delta_2}^{2-\delta_3} f(z)dz + \int_{C_{\delta_3}} f(z)dz + \int_{2+\delta_3}^{R} f(z)dz + \int_{C_R} f(z)dz$$

$\because \lim_{z\to\infty} zf(z) = 0$, $\therefore \int_{C_R} f(z)dz \to 0$ as $R \to \infty$

by great arc lemma.

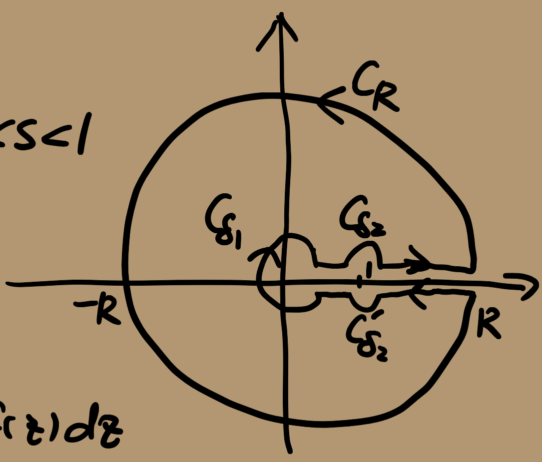
$\because \lim_{z\to 0} zf(z) = \frac{1}{2}$, $\therefore \int_{C_{\delta_1}} f(z)dz = -\frac{\pi}{2} i$ by small arc lemma.

Similarly, $\int_{C_{\delta_2}} f(z)dz = \pi i$, $\int_{C_{\delta_3}} f(z)dz = -\frac{\pi}{2} i$

$\because f(z)$ is analytic inside the enclosed region

$\oint_C f(z)dz = 0$ by Cauchy's theorem.

$\therefore$ v.p. $\int_{-\infty}^{+\infty} \frac{1}{x(x-1)(x-2)}dx = 0$



(4) $f(z) := \frac{z^{s-1}}{1-z}$, $g(z) := \frac{(ze^{i2\pi})^{s-1}}{1-z}$, $0<s<1$

$$\oint_C f(z)dz = \int_{1+\delta_2}^{R} f(z)dz + \int_{C_R} f(z)dz + \int_{R}^{1+\delta_2} g(z)dz$$

$$\int_{C'_{\delta_2}} g(z)\,dz + \int_{1-\delta_2}^{\delta_1} g(z)\,dz + \int_{C_\delta} f(z)dz$$

$$+ \int_{\delta}^{1-\delta} f(z)dz + \int_{C_{\delta_2}} f(z)dz$$

Great arc Lemma: $\int_{C_R} \frac{z^{s-1}}{1-z}dz = 0$ since $\lim_{z\to\infty} zf(z)=0$

Small arc Lemma: $\int_{C_\delta} f(z)dz = 0$ since $\lim_{z\to 0} zf(z)=0$

$\int_{C_{\delta_2}} f(z)dz = \pi i$ $\int_{C'_{\delta_2}} g(z)dz = \pi i e^{2\pi i s}$

Cauchy's theorem: $\oint_C f(z)\,dz = 0$

$$\therefore \int_0^\infty \frac{x^{s-1}}{1-x}dx = -\frac{\pi i + \pi i e^{2\pi i s}}{1-e^{2\pi i s}}$$

$$= \pi \cot(\pi s)$$