Mathematical Methods in Physics

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Review

- Set and interior point
- Function of a complex variable
- Limit, derivative, and continuity
- Cauchy-Riemann relations
- Analytic functions

Chapter – 02: Functions of a complex variable

- In this week, we will be discussing complex series (复数序列), multi-valued function (多值函数), the singular point (奇点) and zero point (零点).

- For judging the convergence of series, you should recall the methods you have learned for real series and real functions.

Chapter – 1 of the textbook

- For multi-valued functions, you should recall the source(s) for causing multiple values – argument.

Review on series (级数，序列)

$$\sum_{n=0}^{+\infty} u_n = u_0 + u_1 + u_2 + \ldots + u_n + \ldots$$

We assume u_n is a complex number.

$$S_n = u_0 + u_1 + u_2 + \ldots + u_n$$

Partial summation (部分和)

If sequences $\{S_n\}$ are convergent, then the series $\sum u_n$ are convergent.

$$S = \lim_{n \to \infty} S_n \qquad \sum u_n$$

If the limit of $\{S_n\}$ does not exist, the series are said to be divergent.

Complex infinite series

Complex series: $$\sum_{n=0}^{\infty}(x_n + iy_n)$$

Partial sum: $$S_n = X_n + iY_n$$

$$X_n = \sum_{i=0}^{n} x_i,\ Y_n = \sum_{j=0}^{n} y_j$$

If S_n approaches a limit $S = X + iY$ as $n \to \infty$, we call the series convergent and call S its sum. This means that $X_n \to X$ and $Y_n \to Y$; in other words, the real and the imaginary parts of the series are each convergent series.

The sufficient and necessary condition (充分必要条件)

$\forall \varepsilon > 0$, $\exists n$ (positive integer), for any positive integer p, such that the following condition is satisfied.

$$|u_{n+1} + u_{n+2} + \ldots + u_{n+p}| < \varepsilon$$

Judge the convergence of the following series

$$\sum_{n=0}^{+\infty} u_n = u_0 + u_1 + u_2 + \ldots + u_n + \ldots$$

Especially, if we let $p = 1$, we can obtain the necessary condition for judging the convergence:

$$\lim_{n \to \infty} u_n = 0$$

Some properties of series

- If the series $\sum u_n$ are convergent, then the related sub-series (子序列) are also convergent.

  $u_0 + u_1 + u_2 + u_3 + \ldots = (u_0 + u_1) + (u_2 + u_3) + \ldots = u_0' + u_1' + \ldots$

- If $\sum_{n=0}^{\infty} |u_n|$ are convergent, then the series $\sum_{n=0}^{\infty} u_n$ are absolutely convergent (绝对收敛).

- Note: if $\sum_{n=0}^{\infty} u_n$ are convergent, $\sum_{n=0}^{\infty} |u_n|$ are not necessarily convergent.

Judging the convergence / divergence

Preliminary test. If the terms of an infinite series do *not* tend to zero (that is, if $\lim_{n\to\infty} a_n \neq 0$), the series diverges. If $\lim_{n\to\infty} a_n = 0$, we must test further.

This is *not* a test for convergence; what it does is to weed out some very badly divergent series which you then do not have to spend time testing by more complicated methods. *Note carefully:* The preliminary test can *never* tell you that a series converges. It does *not* say that series converge if $a_n \to 0$ and, in fact, often they do not. A simple example is the harmonic series (4.2); the nth term certainly tends to zero, but we shall soon show that the series $\sum_{n=1}^{\infty} 1/n$ is divergent. On the other hand, in the series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$$

the terms are tending to 1, so by the preliminary test, this series diverges and no further testing is needed.

See the necessary condition discussed on page 6.

A trick for judging the convergence

For alternating series　交错级数

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

Test for alternating series. An alternating series converges if the absolute value of the terms decreases steadily to zero, that is, if $|a_{n+1}| \leq |a_n|$ and $\lim_{n\to\infty} a_n = 0$.

Some tricks for judging the convergence

For series of positive terms

- **Comparison method**

  If $\exists N \in \mathbb{N}$, $\forall n > N$, the condition $|u_n| < v_n$ is satisfied. If $\sum_{n=0}^{\infty} v_n$ are convergent, then $\sum_{n=0}^{\infty} |u_n|$ are convergent.

- **Ratio method**

  If there exists a constant ρ (un-correlated with n), and $|u_{n+1}/u_n| < \rho < 1$, then $\sum_{n=0}^{\infty} u_n$ are absolutely convergent.

- **d'Alembert method (criterion)**

  $\overline{\lim}_{n\to\infty} |u_{n+1}/u_n| < 1 \longrightarrow \sum_{n=0}^{\infty} u_n \longrightarrow$ absolutely convergent

  $\underline{\lim}_{n\to\infty} |u_{n+1}/u_n| > 1 \longrightarrow \sum_{n=0}^{\infty} u_n \longrightarrow$ divergent

The definition of limit points of $\{z_n\}$

This is different from:
$\exists N(\varepsilon)$, such that when $n > N(\varepsilon)$

- Upper and lower limits

  If there exists z, $\forall \varepsilon > 0$ there is an infinite number of n, such that $|z_n - z| < \varepsilon$

  z is said to be a limit point (极限点) ---- Note, this is not equivalent to limit (极限).

  If the limit for a given series exists, there is only one limit point.

- A real series $\{x_n\}$ can have multiple limit points, the largest one (upper limit) is denoted $\overline{\lim}_{n\to\infty}\{x_n\}$, and the smallest one (lower limit) is denoted $\underline{\lim}_{n\to\infty}\{x_n\}$

Some tricks for judging the convergence

- **Gauss method** (when $\lim_{n\to\infty} |u_{n+1}/u_n| = 1$

  Assume that the ratio between two neighboring terms has the following form:

  $$\frac{u_n}{u_{n+1}} = 1 + \frac{\mu}{n} + O(n^{-\lambda})$$

  where $\mu = a + ib,\ \lambda > 1$. If $a > 1$, then $\sum_{n=0}^{\infty} u_n$ => absolutely convergent.

  If $a \leq 1$, $\sum_{n=0}^{\infty} |u_n|$ => divergent.

- **Cauchy method**

  $$\overline{\lim}_{n\to\infty} |u_n|^{1/n} < 1 \longrightarrow \sum_{n=0}^{\infty} u_n \longrightarrow \text{absolutely convergent}$$

  $$\overline{\lim}_{n\to\infty} |u_n|^{1/n} > 1 \longrightarrow \sum_{n=0}^{\infty} u_n \longrightarrow \text{divergent}$$

Some properties

If $\sum_{n=0}^{\infty} u_n$ is **absolutely convergent**, then you can perform the following

(1) Change the order of summation

$$u_0 + u_1 + u_2 + u_3 + u_4 + \ldots = u_0 + u_1 + u_2 + u_4 + u_3$$

(2) Divide the original series into several sub-series

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} u_{2n+1} + \sum_{n=0}^{\infty} u_{2n}$$

(3) For two series that are absolutely convergent, their product is also absolutely convergent

$$\sum_k u_k \cdot \sum_l v_l = \sum_{k,j} u_k v_l$$

Double series (二重级数)

$$a_{11} + a_{12} + \ldots + a_{1n} + \ldots$$
$$+a_{21} + a_{22} + \ldots + a_{2n} + \ldots$$
$$+\ldots$$
$$a_{m1} + a_{m2} + \ldots + a_{mn} + \ldots$$

Think about the concept of matrix you have learned from linear algebra.

$$S_{mn} = \sum_{1\leq k\leq m, 1\leq l\leq n} a_{kl}$$ Partial summation

$$\lim_{m\to\infty, n\to\infty} S_{mn} = S$$ If the limit exists $$S = \sum_{k,l=1}^{\infty} a_{kl}$$

Double series (二重级数)

A row-wise summation

$$
\begin{array}{l}
a_{11} + a_{12} + \ldots + a_{1n} + \ldots \\
+a_{21} + a_{22} + \ldots + a_{2n} + \ldots \\
\qquad\qquad\qquad\qquad\qquad + \ldots \\
a_{m1} + a_{m2} + \ldots + a_{mn} + \ldots
\end{array}
$$

$$
\begin{array}{l}
a_{11} + a_{12} + \ldots + a_{1n} + \ldots \\
+a_{21} + a_{22} + \ldots + a_{2n} + \ldots \\
\qquad\qquad\qquad\qquad\qquad + \ldots \\
a_{m1} + a_{m2} + \ldots + a_{mn} + \ldots
\end{array}
$$

A column-wise summation

$$
\begin{array}{l}
a_{11} + a_{12} + \ldots + a_{1n} + \ldots \\
+a_{21} + a_{22} + \ldots + a_{2n} + \ldots \\
\qquad\qquad\qquad\qquad\qquad + \ldots \\
a_{m1} + a_{m2} + \ldots + a_{mn} + \ldots
\end{array}
$$

A diagonal-wise summation

Unless the double series are absolutely convergent, otherwise the result could depend on the order by which the summation is made.

Series of complex functions

- Suppose that $u_k(z)$ $(k=1,2,\ldots)$ is defined in region G. If for $z_0 \in G$, $\sum_{k=1}^{\infty} u_k(z_0)$ are convergent, then it is said that $\sum_{k=1}^{\infty} u_k(z)$ are convergent at $z = z_0$.

- If $\sum_{k=1}^{\infty} u_k(z)$ are convergent for every point in G, then it is said that the series are convergent in G (point-wise convergent). The related summation $S(z)$ is a single-valued function defined in G .

Exercise

- [4.01] Examine the convergence of $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + ... + z^n + ...$

$$S_n = 1 + z + z^2 + ... + z^{n-1} = \frac{1 - z^n}{1 - z}$$

$|z| < 1$ $\quad \lim_{n \to \infty} S_n = \frac{1}{1 - z}$ convergent

$|z| \geq 1$ $\quad |z^n| \geq 1$ divergent

Uniform convergence (一致收敛)

- If $\forall\varepsilon > 0,\ \exists N(\varepsilon)$ (un-correlated with z), such that for $\forall n > N(\varepsilon),\ \forall z \in G$, the following condition is satisfied:

$$|S(z) - \sum_{k=1}^{n} u_k(z)| < \varepsilon$$

Then $\sum_{k=1}^{\infty} u_k(z)$ => uniformly convergent in G .

Uniform convergence (一致收敛)

- **Continuity** $u_k(z)$ is continuous in G, and $\sum_{k=1}^{\infty} u_k(z)$ is uniformly convergent,

  then $S(z) = \sum_{k=1}^{\infty} u_k(z)$ is continuous in G.

- Exchange the order of summation and integration

$$\int_C \sum_{k=1}^{\infty} u_k(z) dz = \sum_{k=1}^{\infty} \int_C u_k(z) dz$$

pre-requisite

C is a smooth curve in *G*. $u_k(z)$ is continuous along C. Satisfy uniform convergence.

- Exchange the order of summation and differentiation

$f(z) = \sum_{k=1}^{\infty} u_k(z)$ is analytic in *G*

$$f^{(p)}(z) = \sum_{k=1}^{\infty} u_k^{(p)}(z)$$

pre-requisite

$u_k(z)$ is single-valued and analytic in *G*. Satisfy uniform convergence.

Power series (幂级数)

$$\sum_{n=0}^{\infty} c_n (z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots$$

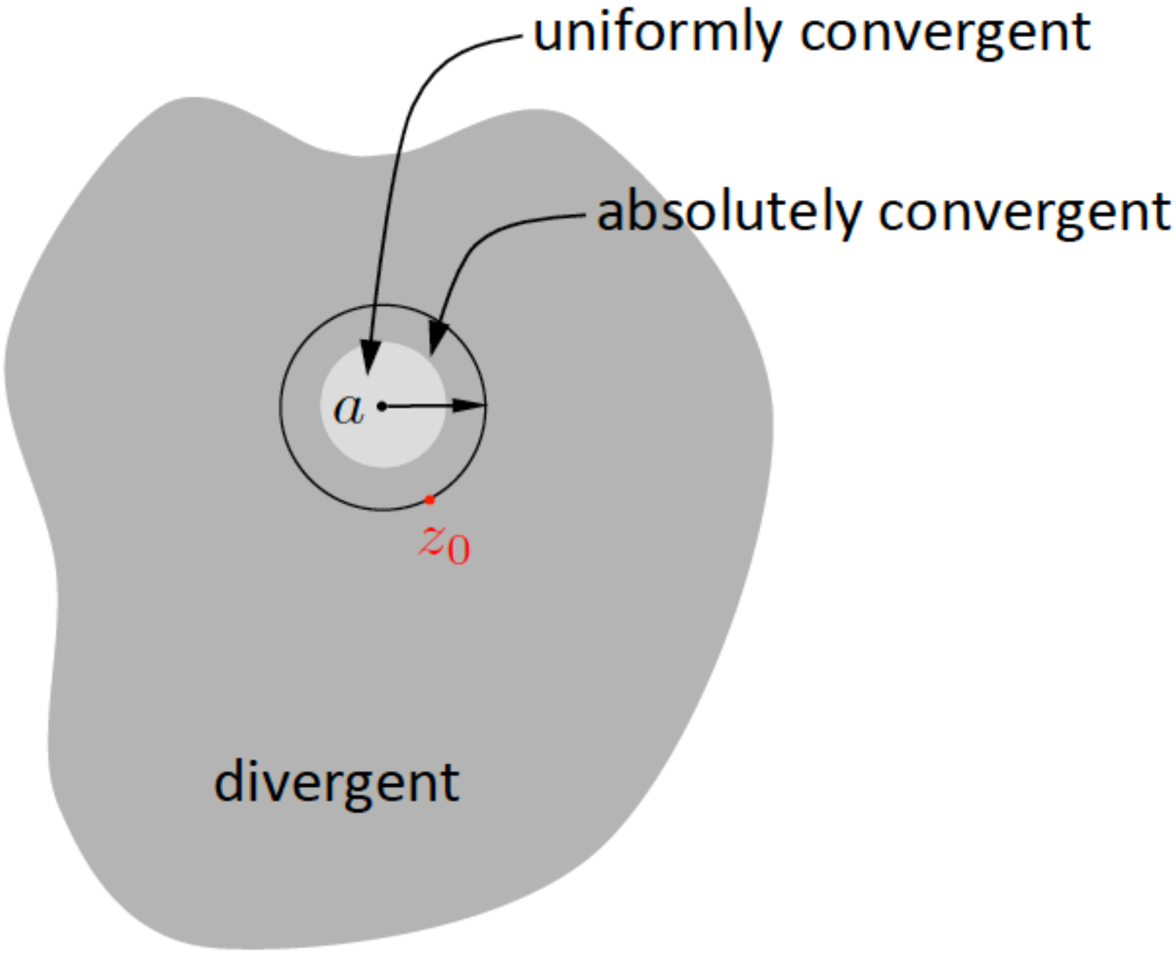
Abel theorem: If the series $\sum_{n=0}^{\infty} c_n (z-a)^n$ are convergent at $z = z_0$, then

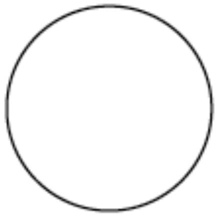
the series are absolutely convergent in a disk region (with a radius of $|z_0 - a|$)

surrounding z_0, and are uniformly convergent in the region $|z-a| \le r \ (r < |z_0 - a|)$.

Corollary: If $\sum_{n=0}^{\infty} c_n (z-a)^n$ are divergent at z_1 , then also divergent in $|z-a| > |z_1 - a|$.

Power series (幂级数)





Circle of convergence
(收敛圆)

Its interior- called the
disk of convergence

Radius of convergence
(收敛半径)

How to find radius of convergence

Cauchy-Hadamard formula (recall Cauchy method on page 12)

$$\overline{\lim}_{n\to\infty}|c_n(z-a)^n|^{1/n} < 1 \Rightarrow |z-a| < \frac{1}{\overline{\lim}_{n\to\infty}|c_n|^{1/n}} \quad \text{convergent}$$

$$\overline{\lim}_{n\to\infty}|c_n(z-a)^n|^{1/n} > 1 \Rightarrow |z-a| > \frac{1}{\overline{\lim}_{n\to\infty}|c_n|^{1/n}} \quad \text{divergent}$$

$$R = \frac{1}{\overline{\lim}_{n\to\infty}|c_n|^{1/n}} = \underline{\lim}_{n\to\infty}\left|\frac{1}{c_n}\right|^{1/n}$$

How to find radius of convergence

According to **d'Alembert criterion** (page 10)

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(z-a)^{n+1}}{c_n(z-a)^n}\right| < 1 \quad \Rightarrow \quad |z-a| < \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

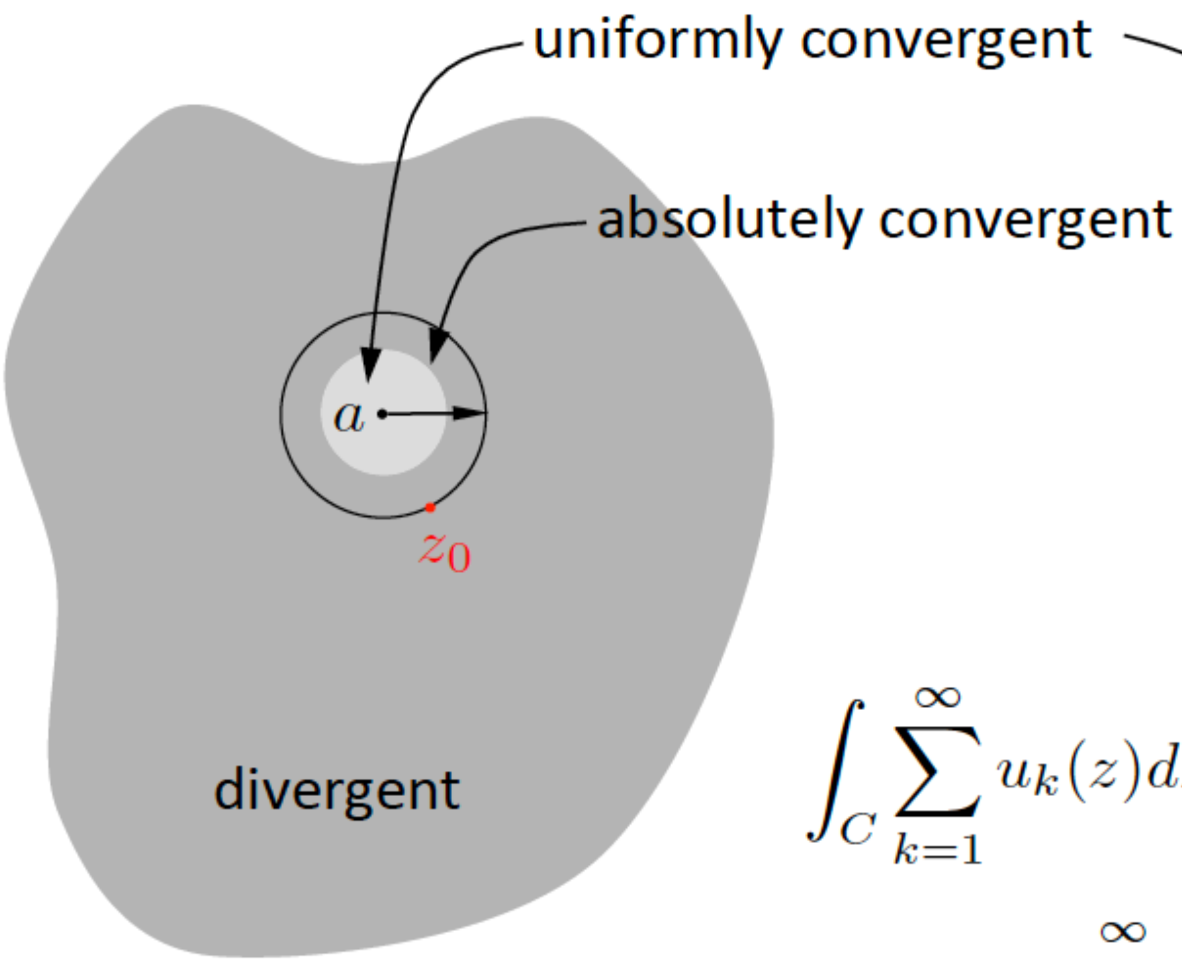
convergent

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(z-a)^{n+1}}{c_n(z-a)^n}\right| > 1 \quad \Rightarrow \quad |z-a| > \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

divergent

$$\boxed{R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|}$$

Power series (幂级数)



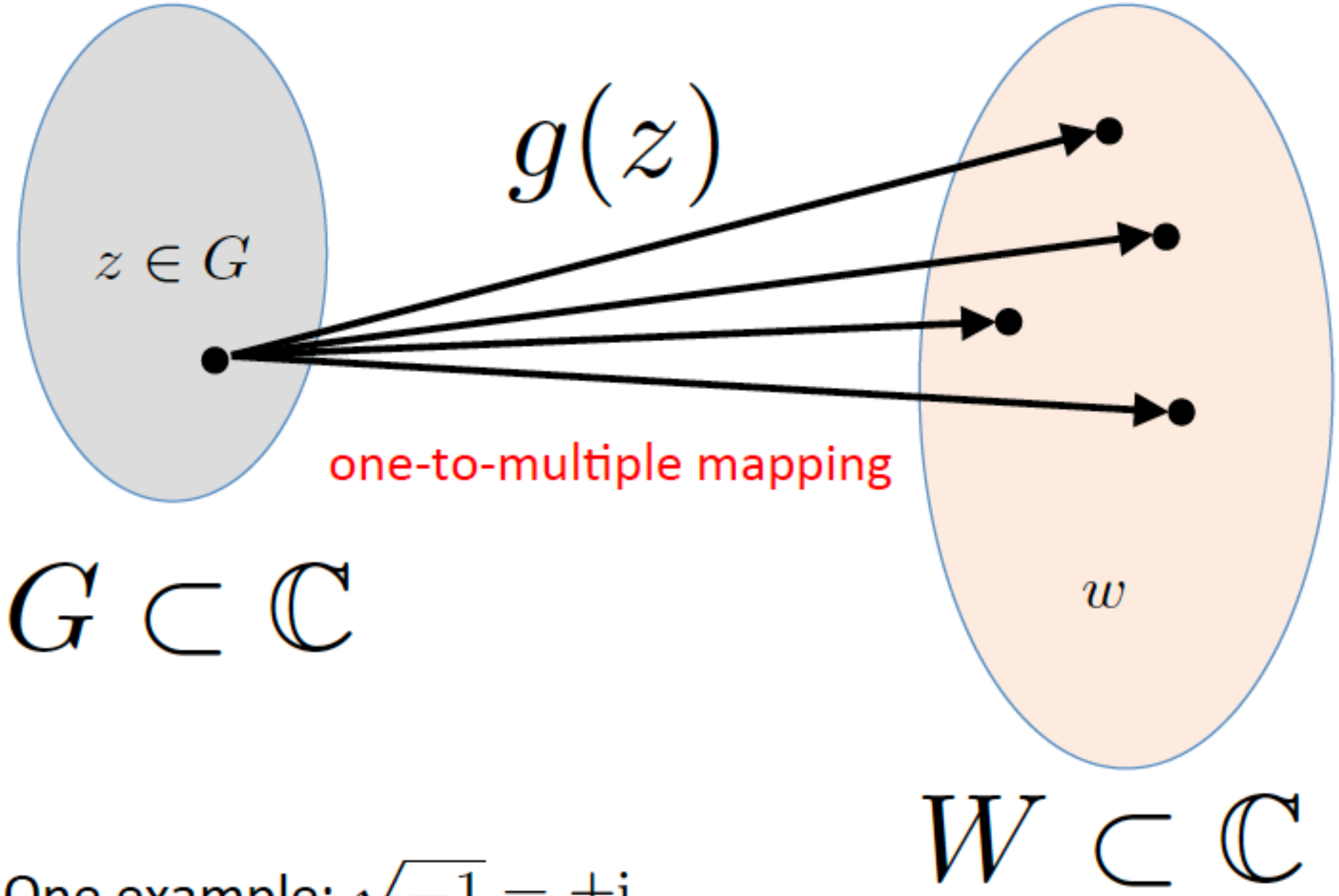
$u_k(z) = c_k(z-a)^k$ is an analytic function defined in $\mathbb{C}$. The series are also uniformly convergent inside the radius of convergence.

$$\int_C \sum_{k=1}^{\infty} u_k(z)dz = \sum_{k=1}^{\infty} \int_C u_k(z)dz$$

$$f^{(p)}(z) = \sum_{k=1}^{\infty} u_k^{(p)}(z)$$

Exchange the order of operations

Multi-valued functions and branch cut



One example: $\sqrt{-1} = \pm \mathrm{i}$

Radical or *n*th root function (根式函数)

Let's recall the geometric representation of a complex number.

$w = \rho e^{i\phi}$ $\quad z - a = re^{i\theta}$ $\quad$ (we assume $r \geq 0, \rho \geq 0$)

$$w = \sqrt{z-a} <=> \rho^2 = r, \text{ and } 2\phi = \theta \pm 2n\pi$$

$$\rho = \sqrt{r}$$

$$\boxed{\phi = \theta/2 \pm n\pi}$$

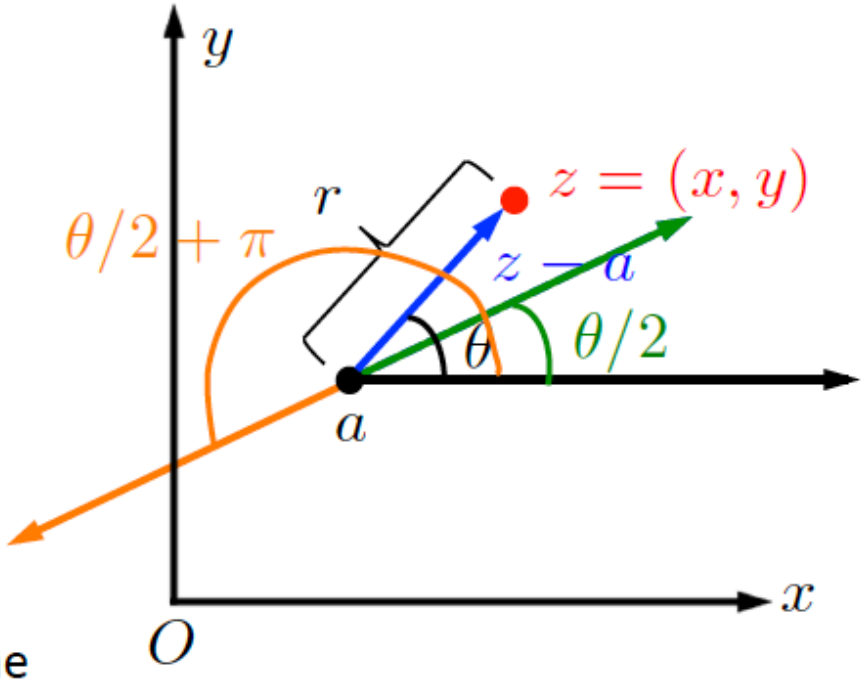
The source of multiple values

$$\left.\begin{aligned} w &= \sqrt{r}e^{i\theta/2} \\ w &= \sqrt{r}e^{i(\theta/2+\pi)} = -\sqrt{r}e^{i\theta/2} \end{aligned}\right\} \text{Two solutions}$$

$$w = \sqrt{r}e^{i(\theta/2+2\pi)} = \sqrt{r}e^{i\theta/2}$$

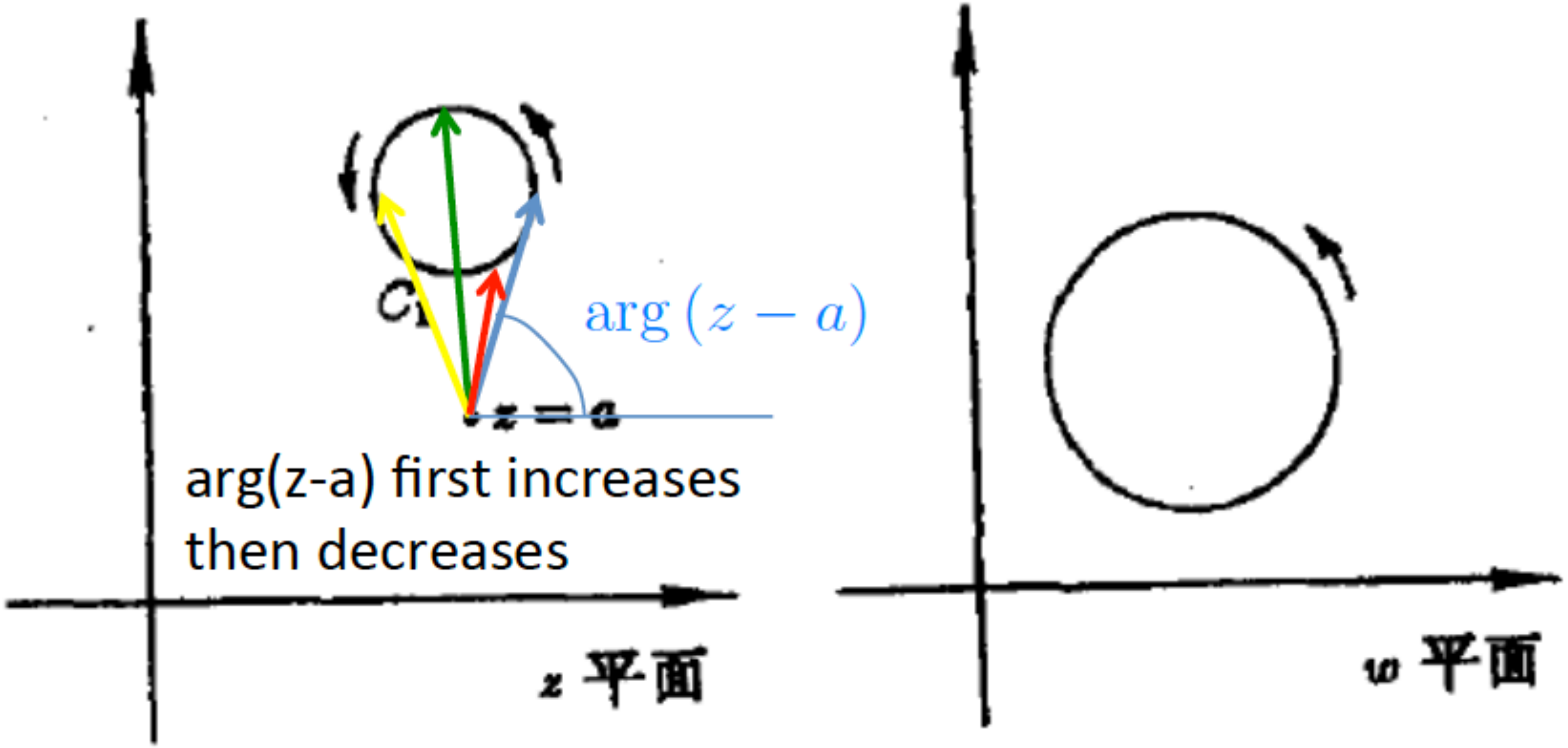
$$w = \sqrt{r}e^{i(\theta/2+3\pi)} = -\sqrt{r}e^{i\theta/2}$$

Repeating the above ones



Let's take a closer look

Case – I: The closed curve C_1 does not enclose point **a**



$$w = \sqrt{z-a}$$

$$|w| = \sqrt{r}$$

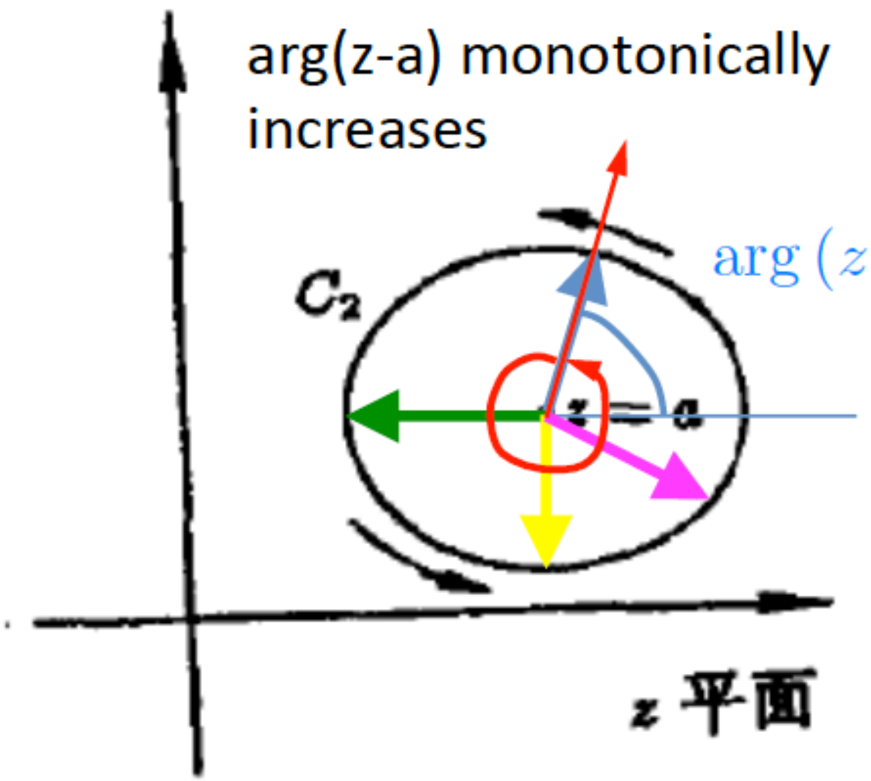
$$\arg(w) = \frac{1}{2}\arg(z-a)$$

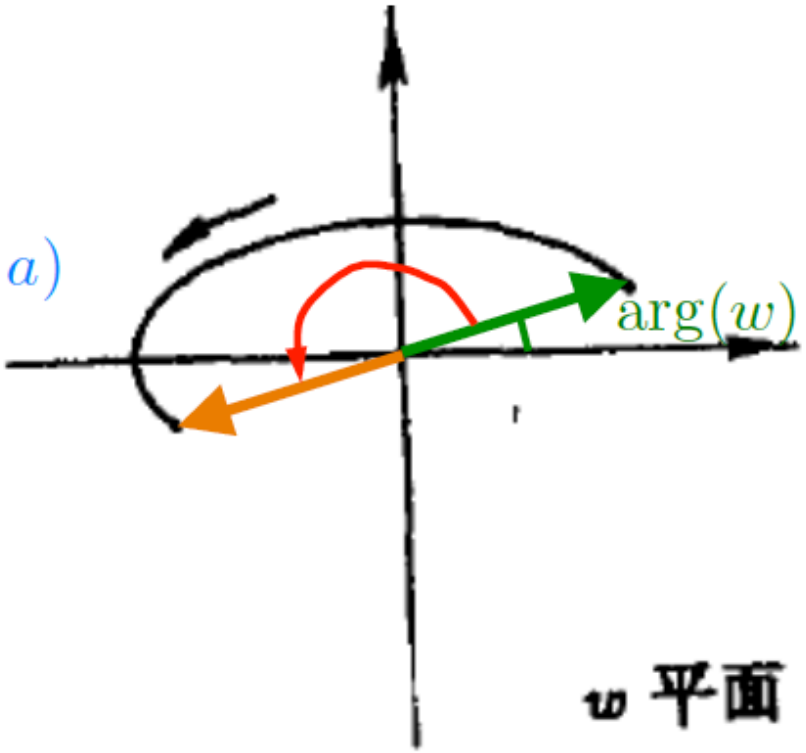
After one round, arg (z-a) returns to its original value, so does arg(w) = ½ arg (z-a)

Let's take a closer look

Case – II: The closed curve C_2 encloses point **a**

arg(z-a) monotonically increases





$$w = \sqrt{z-a}$$

$$|w| = \sqrt{r}$$

$$\arg(w) = \frac{1}{2}\arg(z-a)$$

After one round, arg (z-a) increases by 2π, so arg(w) = ½ arg (z-a) increases by π.

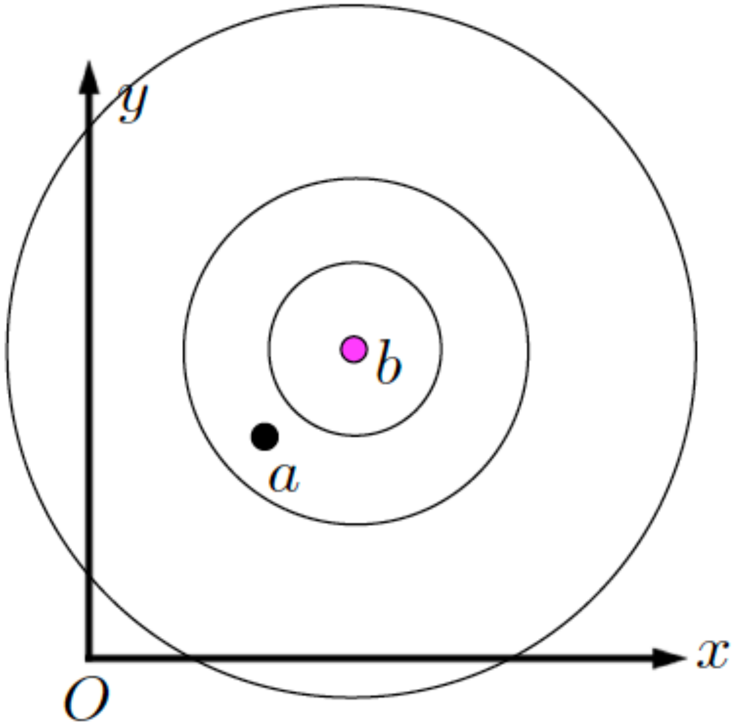
Branch point (分支点)

- Assume that $w = f(z)$ can be defined for a small region around z_0 (but excluding z_0), if $\exists r > 0$, when z returns to its original point along a simply closed curve such as a circle that encloses z_0, w does not return to its original position (including when the radius of the circle approaches zero); then z_0 is said to be the branch point of $w = f(z)$.
- $z = a$ is a branch point of $w = \sqrt{z-a}$.

Exercise

- [4.02] Judge if $z = b$ is a branch point of $w = \sqrt{z-a}$.

  How about $z = \infty$?



Exercise

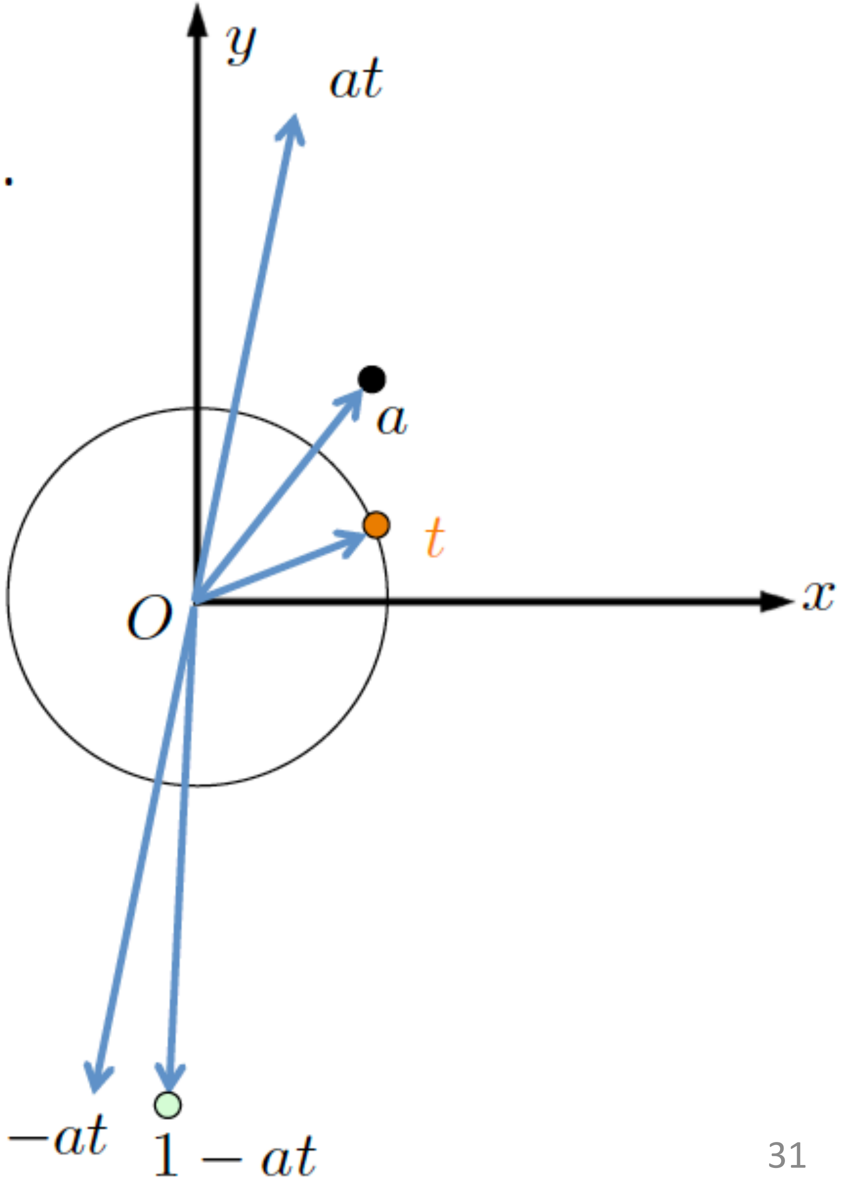
- [4.02] Judge if $z = b$ is a branch point of $w = \sqrt{z-a}$.

How about $z = \infty$?

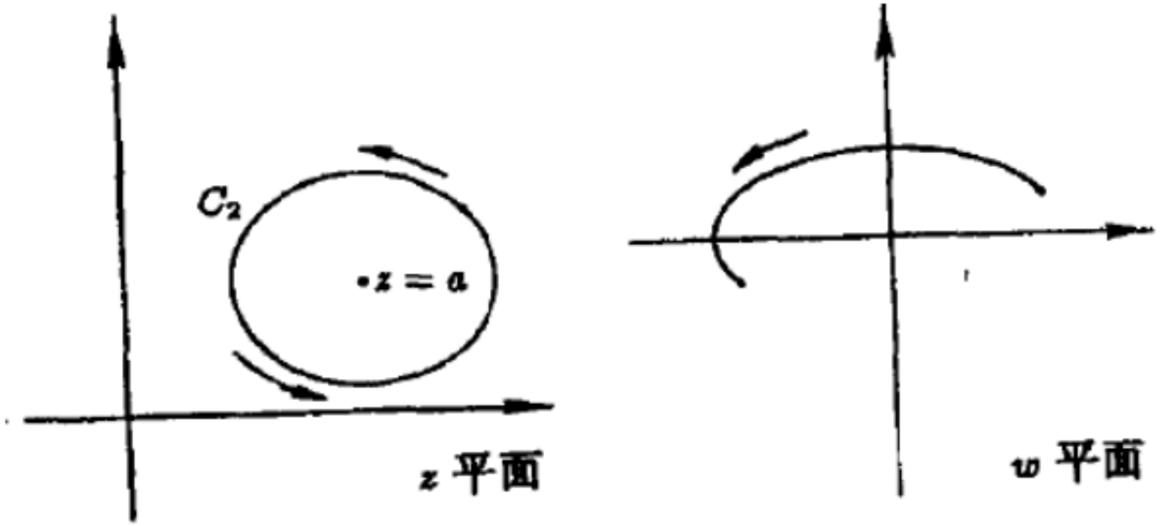
If we let $t = 1/z$

$$w = \sqrt{\frac{1}{t} - a} = \sqrt{\frac{1-at}{t}}$$

$\boxed{z = \infty \text{ is also a branch point for } w = \sqrt{z-a}}$



How to make multi-valued function single-valued?



$$w = \sqrt{z-a}$$

$$|w| = \sqrt{r}$$

$$\arg(w) = \frac{1}{2}\arg(z-a)$$

We can artificially determine the range of arg (z-a), then the range of arg (w) is set, too.

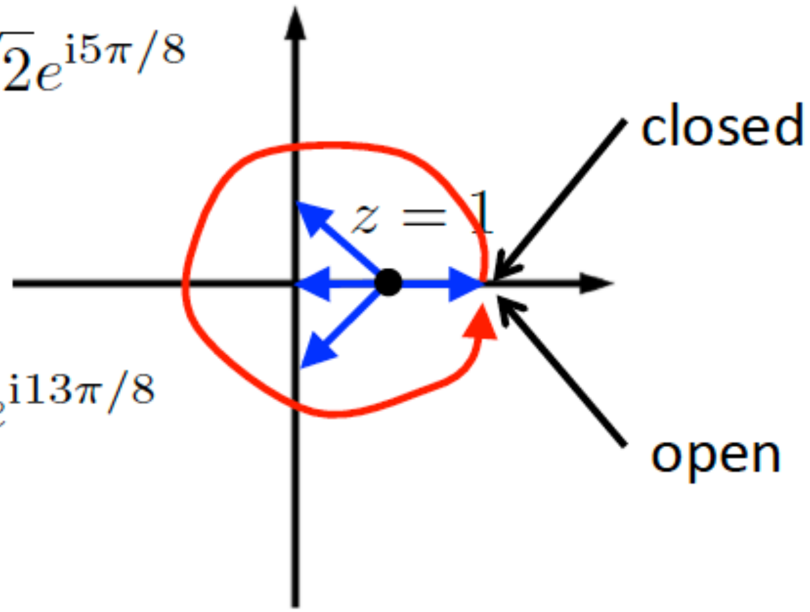
Example

- Assume that $w=\sqrt{z-1}$ and $0 \leq \arg(z-1) < 2\pi$, find the values for

$$w(2)=1,\ w(\mathrm{i})=\sqrt[4]{2}e^{\mathrm{i}3\pi/8},\ w(0)=\mathrm{i},\ w(-\mathrm{i})=\sqrt[4]{2}e^{\mathrm{i}5\pi/8}$$

- What if we set $2\pi \leq \arg(z-1) < 4\pi$?

$$w(2)=-1,\ w(\mathrm{i})=\sqrt[4]{2}e^{\mathrm{i}11\pi/8},\ w(0)=-\mathrm{i},\ w(-\mathrm{i})=\sqrt[4]{2}e^{\mathrm{i}13\pi/8}$$



Note: the red curve is NOT a closed curve

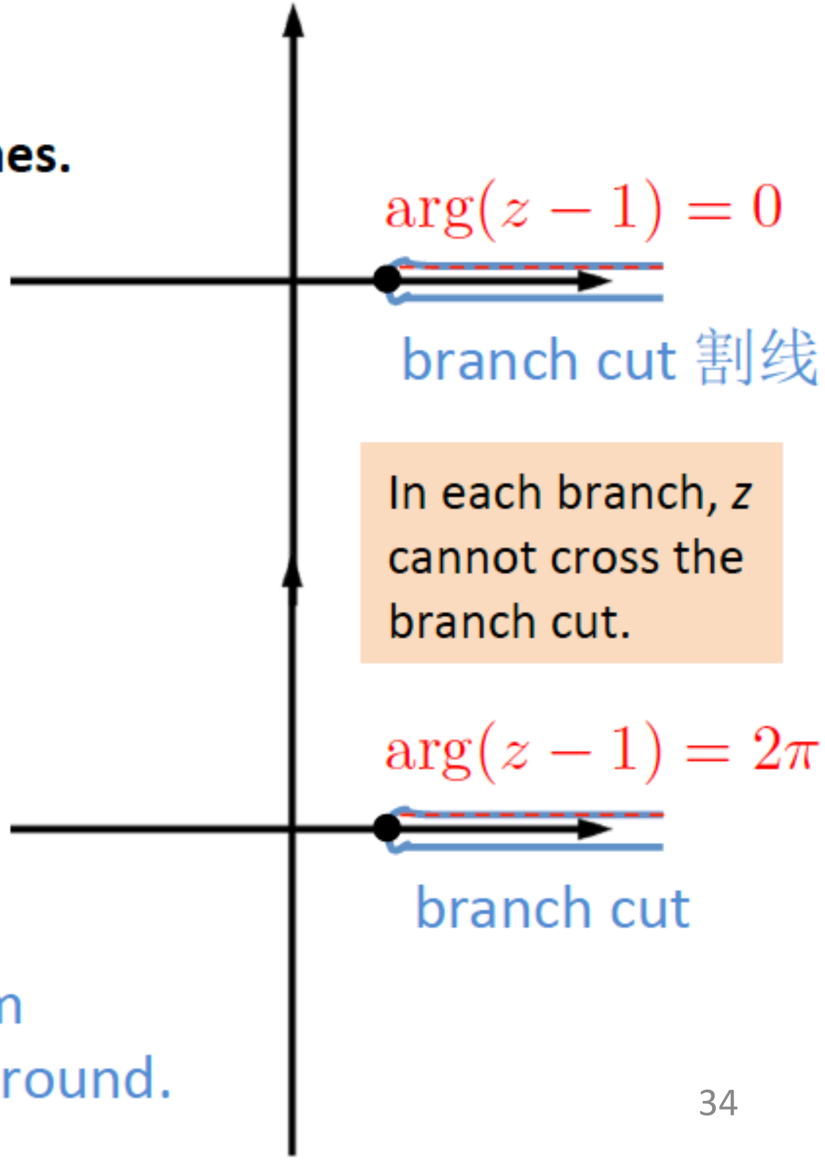
We can define the branch cut

$w = \sqrt{z-1}$

The entire multi-valued function *w* can be represented as the summation of two branches.

$0 \leq \arg(z-1) < 2\pi$	*w* is single-valued in **branch I**
$0 \leq \arg(w) < \pi$	connected

$2\pi \overbrace{\leq \arg(z-1) <}^{\text{no overlap}} 4\pi$	*w* is single-valued in **branch II**
$\pi \leq \arg(w) < 2\pi$	



The purpose of introducing branch cut is to prevent *z* from rotating around its branch point(s) (1 and infinity) by one round.

About singular point (奇点)

Some definitions:

A *regular point* of $f(z)$ is a point at which $f(z)$ is analytic.

A *singular point* or *singularity* of $f(z)$ is a point at which $f(z)$ is not analytic. It is called an *isolated* singular point if $f(z)$ is analytic everywhere else inside some small circle about the singular point.

Page 670 in the textbook

Theorem III (which we state without proof). If $f(z)$ is analytic in a region (R in Figure 2.3), then it has derivatives of all orders at points inside the region and can be expanded in a Taylor series about any point z_0 inside the region. The power series converges *inside* the circle about z_0 that extends to the nearest singular point (C in Figure 2.3).

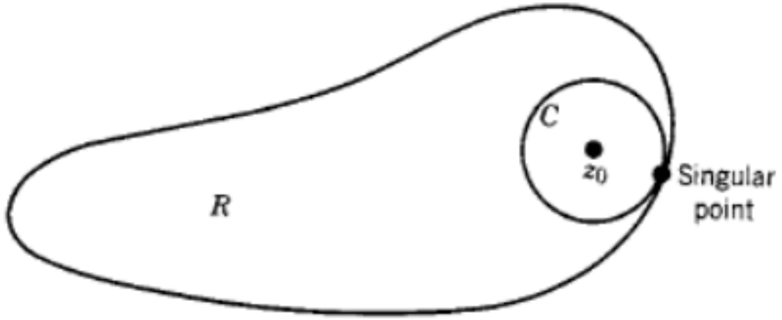


Figure 2.3

Page 671 in the textbook

About singular point (奇点)

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Page 670 in the textbook

Apparently, if $f(z)$ is not defined at z_0, or does not have derivative at z_0, or has derivative but is not analytic at z_0, z_0 is said to be a singular point.

According to the above definitions, branch point and point along the branch cut are singular points, because these points are shared by different branches.

About zero point (零点)

The values of x for which $\sin x = 0$ (called the zeros for $\sin x$)

Page 591 in the textbook

The above concept can be extended to functions of a complex variable.

Why studying zeros or zero points? => Sometimes we need to find zeros of a denominator, such as Q(z).

$$\int \frac{P(z)}{Q(z)} dz$$

About poles (极点)

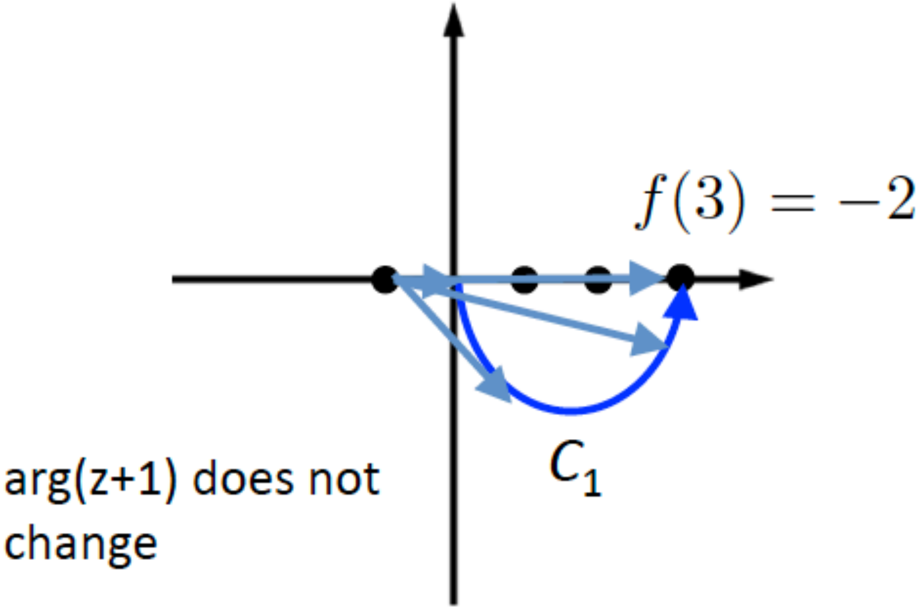
Most of the functions we shall consider will be analytic except for poles—such functions are called *meromorphic* functions. If $f(z)$ has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$. A three-dimensional graph with $|f(z)|$ plotted vertically over a horizontal complex plane would look like a tapered pole near $z = z_0$. We can often see that a function has a pole and find the order of the pole without finding the Laurent series.

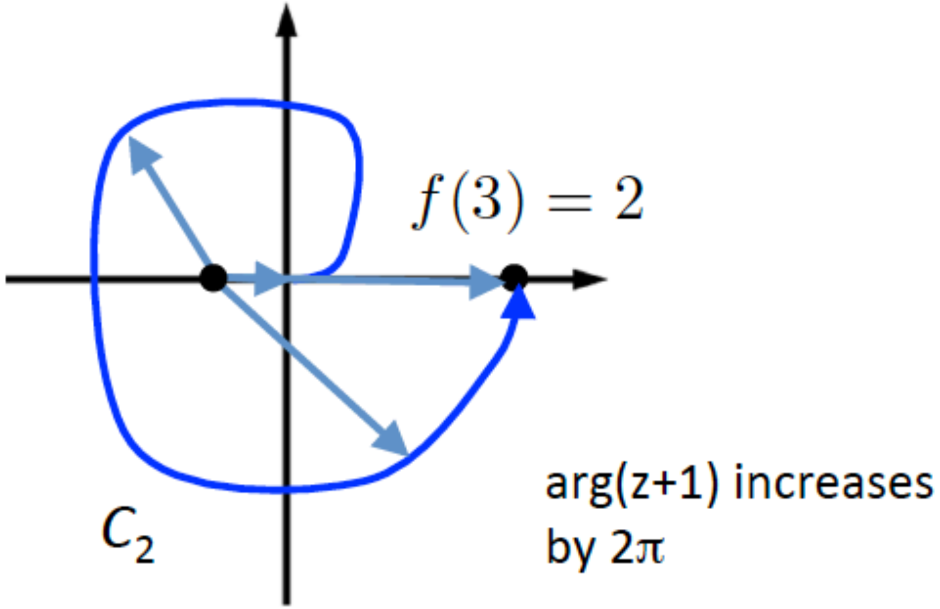
Page 681 in the textbook

Alternative approaches

We can pre-define the value of $w = f(z)$ at $z = z_0$ (outside the branch point) and the evolutionary path towards other points. This allows z to change its position continuously.

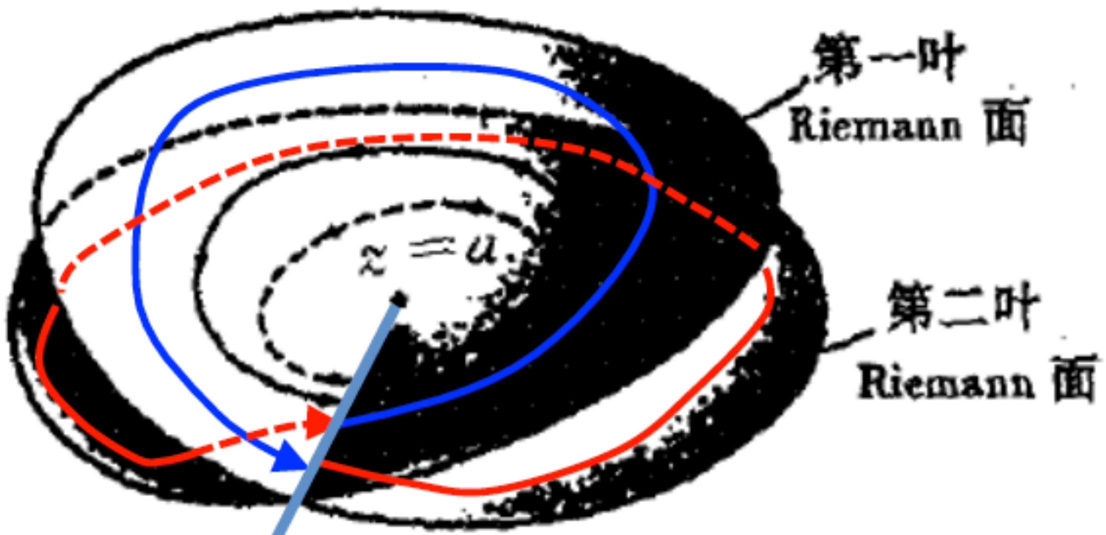
Assume that $f(z) = \sqrt{z+1}$, $f(0) = -1$, find the values of $f(3)$ along C_1 and C_2.



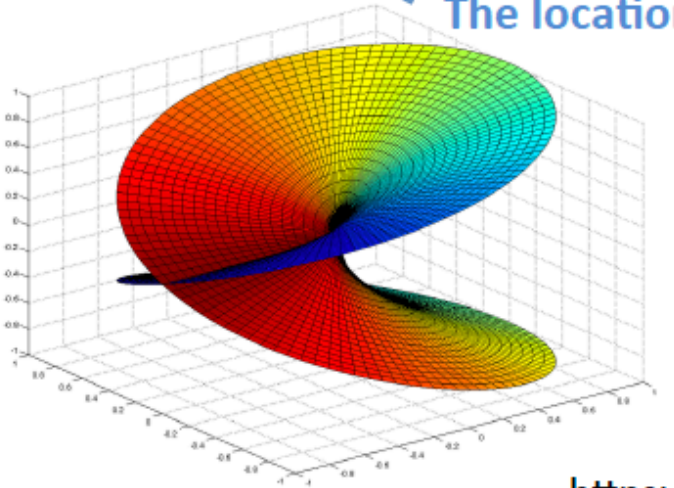


Alternative approaches

Perspective view



The location of the branch cut (page 34)
It is not needed here.

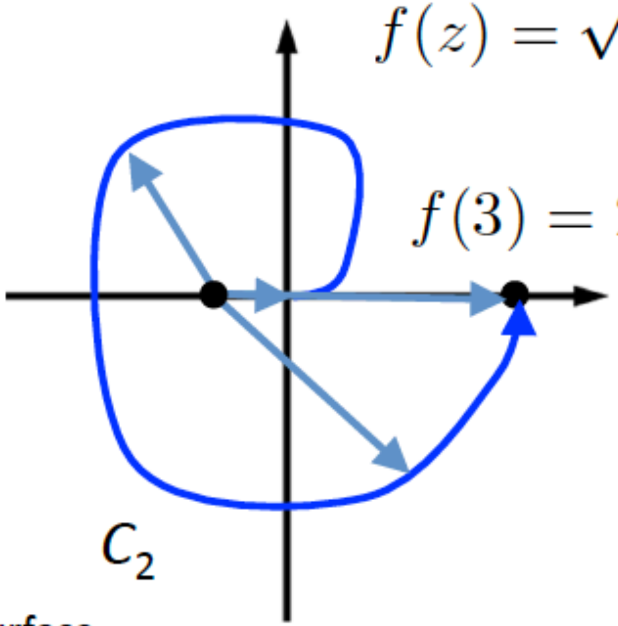


https://en.wikipedia.org/wiki/Riemann_surface

Connect the end of the first Riemann surface (blue arrow) to the beginning of the second Riemann surface.

Connect the end of the second Riemann surface (red arrow) to the beginning of the first Riemann surface.

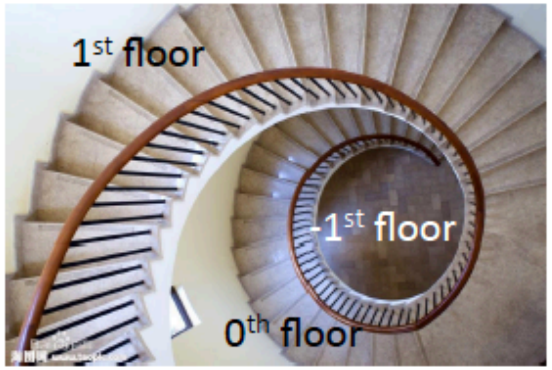
$$f(z)=\sqrt{z+1},\, f(0)=-1$$



Logarithmic function (对数函数)

- For a given variable z (complex number), as long as w satisfies the relation

$e^w = z$



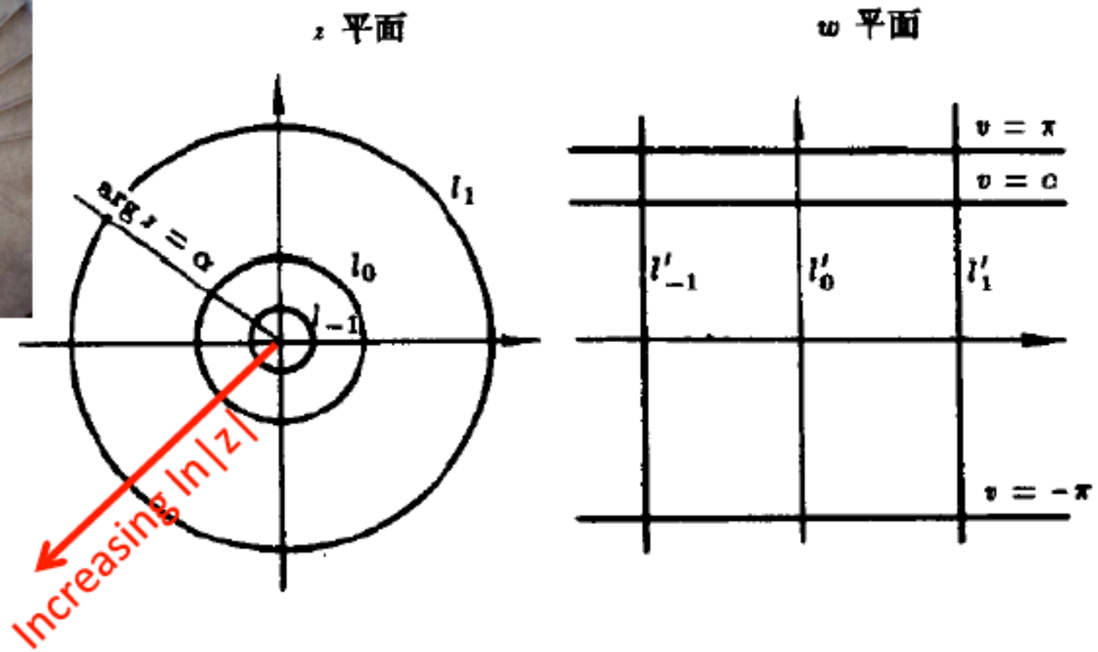
We call $w = \ln z$.

$w = u + \mathrm{i}v,\;$ and $z = re^{\mathrm{i}\theta}$

$e^u e^{\mathrm{i}v} = re^{\mathrm{i}\theta}$

$=> u = \ln|z|,\;$ and $v = \theta \pm 2n\pi$

$$=> w = \ln z = \ln|z| + \boxed{\mathrm{i}(\theta \pm 2n\pi)}$$



The source of multiple values, again due to the argument.

There is infinite number of solutions.

Logarithmic function (对数函数)

- For a given variable z (complex number), as long as w satisfies the relation

$$e^w = z$$

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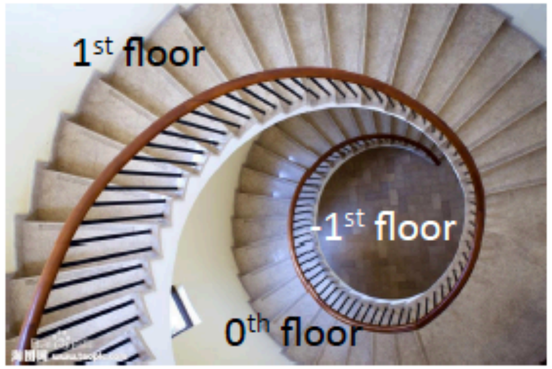

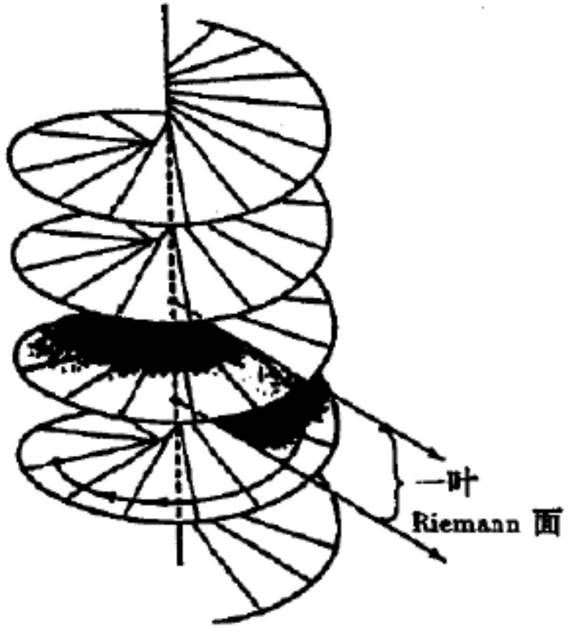

0 and ∞ are the branch points of ln(z).

ln(z) has an infinite number of Riemann surfaces (see left figure).

$$w = u + \mathrm{i}v, \text{ and } z = re^{\mathrm{i}\theta}$$

$$e^u e^{\mathrm{i}v} = re^{\mathrm{i}\theta}$$

$$=> u = \ln|z|, \text{ and } v = \theta \pm 2n\pi$$

$$\boxed{=> w = \ln z = \ln|z| + \boxed{\mathrm{i}(\theta \pm 2n\pi)}}$$

The source of multiple values, again due to the argument.

There is infinite number of solutions.