Mathematical Methods in Physics

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Review

- Series
- Convergence, absolute convergence, and uniform convergence
- Radius and disk of convergence
- Multi-valued functions
- Branch point, branch cut, and Riemann sheet

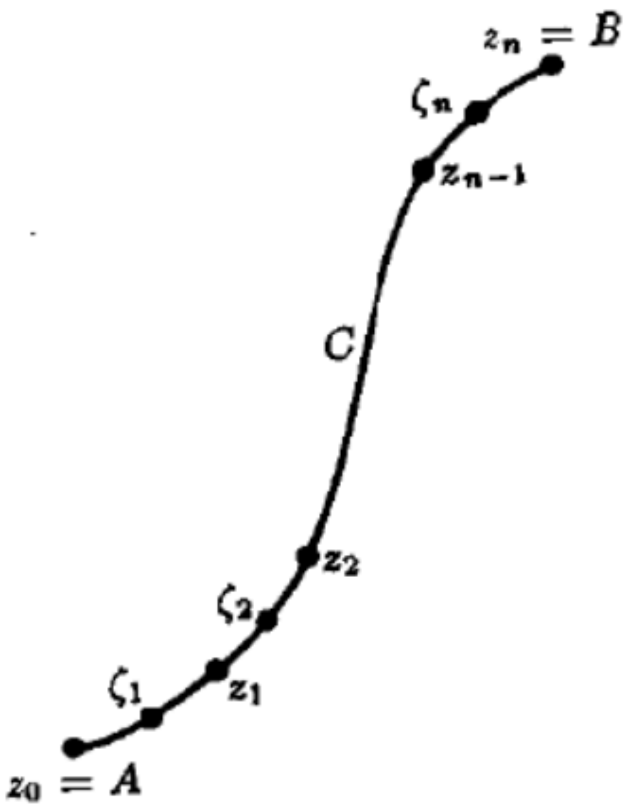
Chapter – 02: Complex integration

- In this week, we will be discussing complex integration (复数积分), Cauchy theorem (柯西定理), Cauchy integral formula (柯西积分公式).

- You should recall the knowledge about integration learned from Calculus.

- You should also recall the knowledge about limit, series, and analytic functions learned in previous weeks.

Complex integration (复变积分)



C is a curve defined in $\mathbb{C}$ that connects points A and B. Complex function $f(z)$ is defined along *C*.

$$\sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\zeta_k)\Delta z_k$$

If when $n \to \infty$, $\max|\Delta z_k| \to 0$, the limit of the above summation exists, and does not depend on the choice of ζ_k, then we call the limit of the above summation the integration of $f(z)$ along curve C.

$$\int_C f(z)dz = \lim_{\max|\Delta z_k| \to 0} \sum_{k=1}^{n} f(\zeta_k)\Delta z_k$$

Complex integration and real integration

$$\int_C f(z)dz = \int_C (u + \mathrm{i}v)(dx + idy) = \int_C (udx - vdy) + \mathrm{i} \int_C (vdx + udy)$$

If $f(z)$ is continuous along the curve *C* (assumed to be smooth), then the integral exists.

Note: here we do not require that $f(z)$ must be analytic.

Recall that **analytic** is a stronger condition than **continuous**.

Some properties

$$\int_C [f_1(z) + f_2(z) + \ldots + f_n(z)]dz = \int_C f_1(z)dz + \int_C f_2(z)dz + \ldots + \int_C f_n(z)dz$$

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots + \int_{C_n} f(z)dz \qquad C = C_1 + C_2 + \ldots + C_n$$

$$\int_{C^-} f(z)dz = -\int_C f(z)dz$$ where the direction of C^- is opposite to that of C

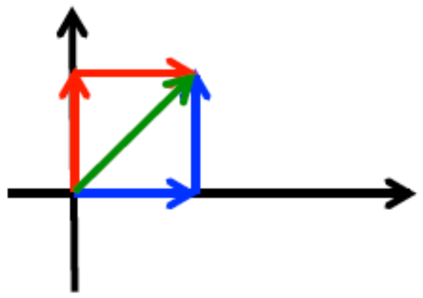
Some properties

$$\int_C af(z)dz = a\int_C f(z)dz$$ where a is a constant complex number

$$\left|\int_C f(z)dz\right| \leq \int_C |f(z)||dz|$$

$$\left|\int_C f(z)dz\right| \leq Ml$$ where M is the upper bound of $|f(z)|$ along C, and l is the length of C

Exercise



- [5.01] Find the values of $\int_C \mathrm{Re}(z)dz$ along the following paths:

(1) $0 \to 1$ along the real axis, then $1 \to 1+\mathrm{i}$ along the imaginary axis

(2) $0 \to \mathrm{i}$ along the imaginary axis, then $\mathrm{i} \to 1+\mathrm{i}$ along the real axis

(3) along the diagonal line $0 \to 1+\mathrm{i}$

Solution (1) = $\frac{1}{2}+\mathrm{i}$ Solution (2) = $\frac{1}{2}$ Solution (3) = $\frac{1}{2}(1+i)$

Now you get three different solutions, what can you conclude?

Cauchy theorem

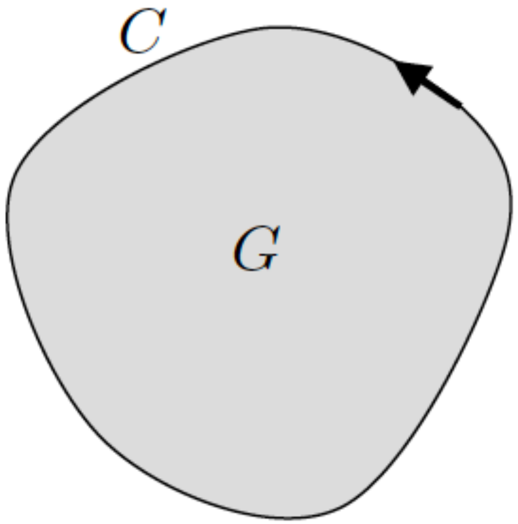
If $f(z)$ is analytic in region $\overline{G}$, then along its boundary C, we have: $\boxed{\oint_C f(z)dz = 0}$

Proof

- First for simply connected region

$$\oint_C f(z)dz = \oint_C (udx - vdy) + \mathrm{i} \oint_C (vdx + udy)$$

Green formula
&
Cauchy-Riemann relations

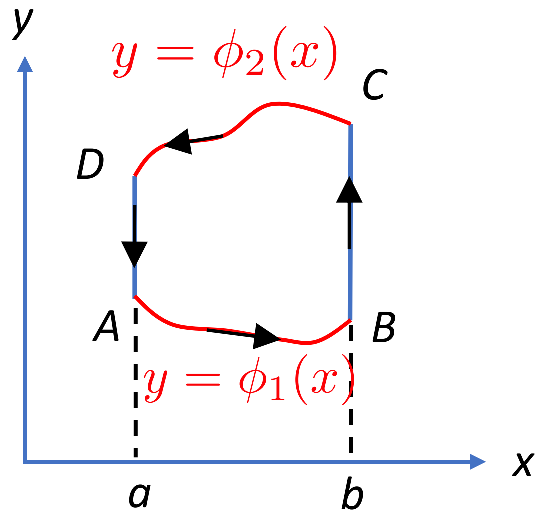


$$\oint_C [P(x,y)dx + Q(x,y)dy] = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

$$\oint_C (udx - vdy) = -\iint_S \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dxdy = 0$$

$$\oint_C (vdx + udy) = \iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy = 0$$

Green formula



$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_L Pdx + Qdy$$

Along the positive direction of *L*

Let's first consider: $\iint_D \left(-\frac{\partial P}{\partial y}\right) dxdy = \oint_L Pdx$

$$\mathrm{LHS} = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \left(-\frac{\partial P}{\partial y}\right) dydx = \int_a^b \{P[x, \phi_1(x)] - P[x, \phi_2(x)]\}\, dx$$

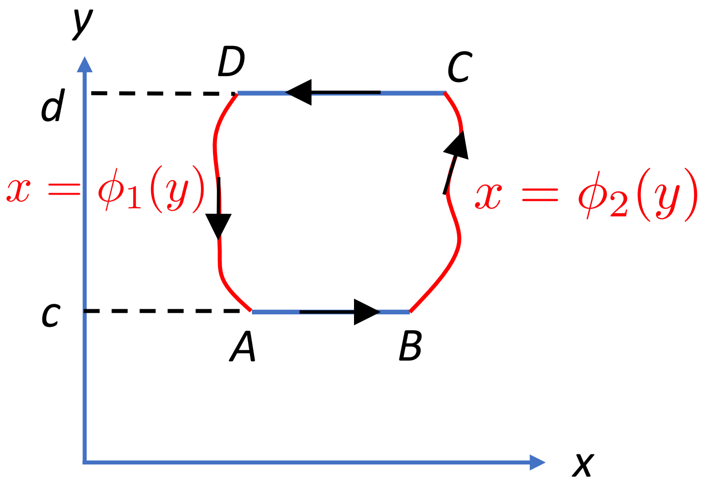
$$\mathrm{RHS} = \int_{A\to B} Pdx + \int_{B\to C} Pdx + \int_{C\to D} Pdx + \int_{D\to A} Pdx$$

$$= \int_a^b P[x, \phi_1(x)]dx + 0 - \int_a^b P[x, \phi_2(x)]dx + 0 = \mathrm{LHS}$$

Green formula

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_L Pdx + Qdy$$

Along the positive direction of *L*



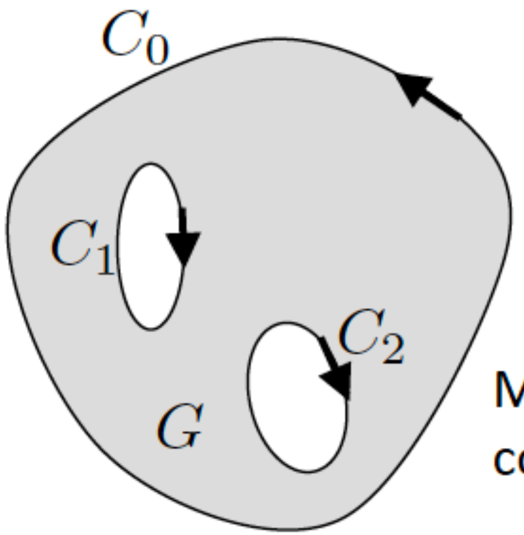
Then let's consider: $\displaystyle \iint_D \frac{\partial Q}{\partial x} dxdy = \oint_L Qdy$

$$\text{LHS} = \int_c^d \int_{\phi_1(y)}^{\phi_2(y)} \frac{\partial Q}{\partial x} dxdy = \int_c^d \{Q[\phi_2(y), y] - Q[\phi_1(y), y]\}\, dy$$

$$\text{RHS} = \int_{A\to B} Qdy + \int_{B\to C} Qdy + \int_{C\to D} Qdy + \int_{D\to A} Qdy$$

$$= 0 + \int_c^d Q[\phi_2(y), y]dy + 0 - \int_c^d Q[\phi_1(y), y]dy = \text{LHS}$$

Cauchy theorem



- Then for multi-connected region

$f(z)$ is analytic

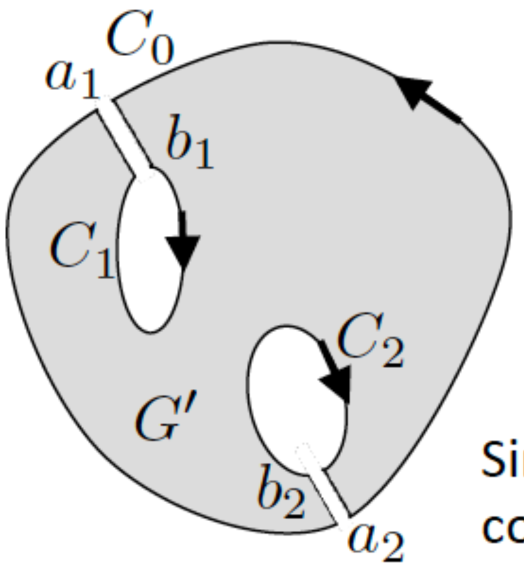
$$\oint_C f(z)dz = 0$$

Multi-connected

$$\int_{a_1}^{b_1} f(z)dz + \int_{b_1}^{a_1} f(z)dz = 0$$

$$\int_{a_2}^{b_2} f(z)dz + \int_{b_2}^{a_2} f(z)dz = 0$$

$$\Rightarrow \oint_{C_0} f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz = 0$$



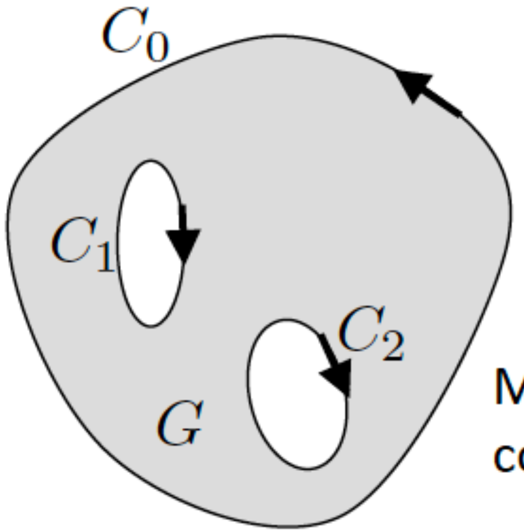
Simply connected

$$\oint_{C_0} f(z)dz + \int_{a_1}^{b_1} f(z)dz + \oint_{C_1} f(z)dz + \int_{b_1}^{a_1} f(z)dz + \int_{a_2}^{b_2} f(z)dz + \oint_{C_2} f(z)dz + \int_{b_2}^{a_2} f(z)dz = 0$$

Cauchy theorem

- Then for multi-connected region

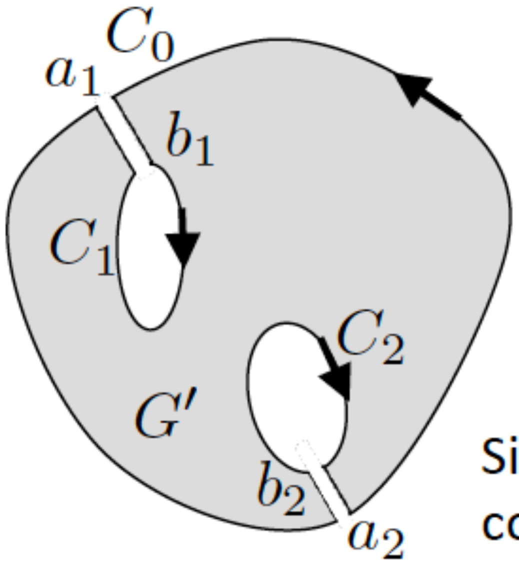
$$\boxed{\oint_C f(z)dz = 0}$$



Multi-connected

$$\oint_{C_0} f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz = 0$$

equivalent to



Simply connected

$$\oint_{C_0} f(z)dz = \oint_{C_1^-} f(z)dz + \oint_{C_2^-} f(z)dz$$

Here C_i^- keeps the same direction as C_0
(i.e. clockwise or anti-clockwise)

Attention

- Condition for Cauchy theorem: the region has to be finite, but cannot include ∞.

- Or, you have to find all the singular points. Again, recall that a complex function cannot be analytic at singular points.

Exercise

- [5.02] Find the value of $\oint_C z^n dz$, where *n* is an integer, *C* is a simply closed curve in $\mathbb{C}$.

  1. If *n* is non-negative, z^n is analytic, then $\oint_C z^n dz = 0$.

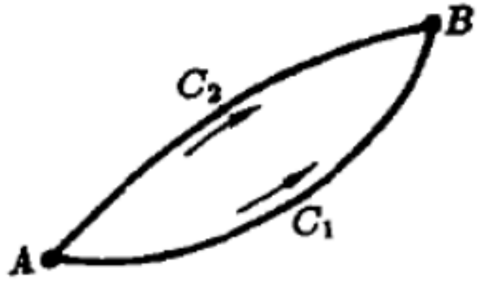
  2. If *n* is negative, and if the contour does not enclose $z = 0$, then z^n is analytic inside the region bounded by *C*, and again we have $\oint_C z^n dz = 0$.

  3. If *n* is negative, and if the contour encloses $z = 0$. We can draw a simple circle around $z = 0$, and apply the Cauchy theorem for a multi-connected region.

$$\oint_C z^n dz = \oint_{|z|=\varepsilon} z^n dz = \int_0^{2\pi} \varepsilon^{n+1} e^{\mathrm{i}(n+1)\theta} \mathrm{i} d\theta = \begin{cases} 2\pi\mathrm{i}, \ n = -1; \\ 0, \ n = -2, -3, -4, \ldots; \end{cases}$$

Corollary

- If $f(z)$ is analytic in a bounded and simply connected region G, then $\int_C f(z)dz$ does not depend on the path of integration, where $C \subset G$.



- Questions: can you prove the above statement?

- If we fix the starting point as z_0, and make the ending point a variable, then we can define an indefinite integral $F(z)$, which is a single-valued function in G.

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$$

A useful theorem

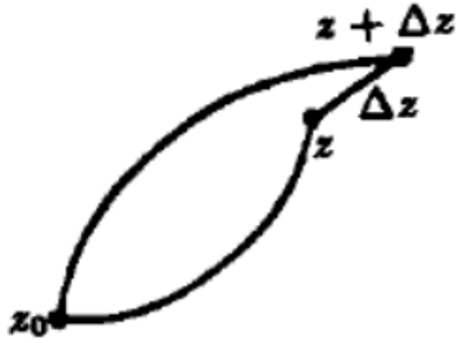
- If $f(z)$ is analytic in a bounded and simply connected region G, then its indefinite integral $F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$ is also analytic in G , and satisfies the following relation:

$$F'(z) = \frac{d}{dz}\int_{z_0}^{z} f(\zeta)d\zeta = f(z) \qquad \text{for } z \in G$$

- Proof: $$\frac{\Delta F}{\Delta z} = \frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta)d\zeta$$

$$\left|\frac{\Delta F}{\Delta z} - f(z)\right| = \left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(\zeta) - f(z)]d\zeta\right|$$

$$\leq \boxed{\left|\frac{1}{\Delta z}\right|\int_{z}^{z+\Delta z}|f(\zeta) - f(z)|\cdot|d\zeta|}$$



Then apply the continuity of $f(z)$ - see the next page

A useful theorem

- Following the last page

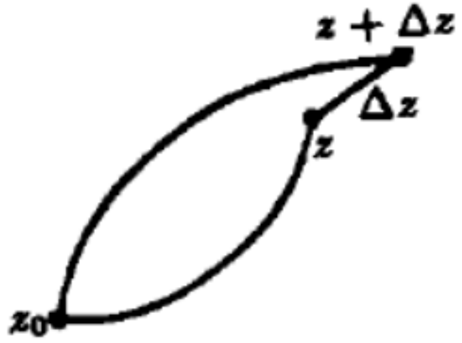
Because $f(z)$ is continuous, we have $\forall \varepsilon > 0,\ \exists \delta > 0$, such that when $|\zeta - z| < \delta$,

the following relation holds: $|f(\zeta) - f(z)| < \varepsilon$

Plug the above relation back to the relation inside the red box

on page 17, we have

$$\left| \frac{\Delta F}{\Delta z} - f(z) \right| \leq \frac{1}{|\Delta z|} \cdot \varepsilon \cdot |\Delta z| = \varepsilon$$



$$F'(z) = \lim_{\Delta z \to 0} \frac{\Delta F}{\Delta z} = f(z)$$

- If $\Phi'(z) = f(z)$, then $\Phi(z)$ is called the primitive function (原函数) of $f(z)$.

- If $\Phi_1'(z) = f(z) \quad \Phi_2'(z) = f(z)$

  Then we have $\Phi_1(z) = \Phi_2(z) + C$

- Similar to real functions, we have

  $$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta = \Phi(z) + C$$

Exercise

- [5.03] Find the value of $\int_a^b z^n dz,\ \text{where } n \text{ is an integer}$

Solution: (1) when *n* is non-negative, $F(z) = \int^z z^n dz$ is analytic in a finite region in $\mathbb{C}$. Then we have $\int_a^b z^n dz = \frac{1}{n+1}(b^{n+1} - a^{n+1})$

(2) when *n* is negative but not equal to -1, z^n is analytic in a finite region not including 0. Then we also have $\int_a^b z^n dz = \frac{1}{n+1}(b^{n+1} - a^{n+1})$

(3) when *n* = -1, z^{-1} is analytic in a finite region not including 0. We have

$$\int_a^b z^{-1} dz = \ln b - \ln a$$

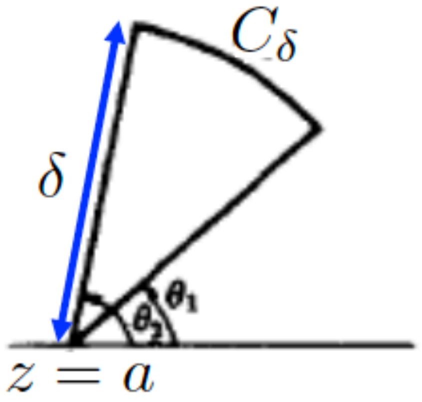
Pay attention to $\ln z$. If we fix the Riemann sheet (or branch), then its value can be uniquely determined.

Two useful lemmas

- **Small arc lemma (小圆弧引理)**

  If $f(z)$ is continuous in a small region around $z=a$, and satisfies the following relation: when $\theta_1 \le \arg(z-a) \le \theta_2$, $|z-a| \to 0$, $(z-a)f(z)$ uniformly approaches k. Then we have

$$\lim_{\delta\to 0}\int_{C_\delta} f(z)dz = ik(\theta_2-\theta_1)$$



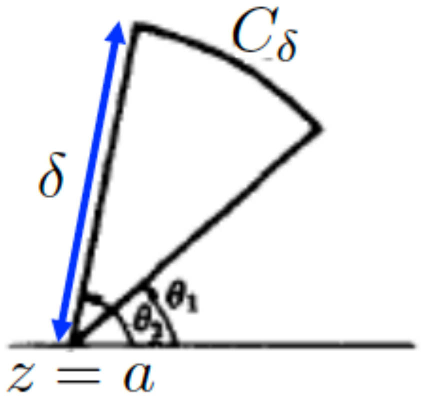
Two useful lemmas

- **Small arc lemma (小圆弧引理)**

Proof: Recall the results on page 20. $\int_{C_\delta} \frac{dz}{z-a} = \mathrm{i}(\theta_2 - \theta_1)$

$$\left| \int_{C_\delta} f(z)dz - \mathrm{i}k(\theta_2 - \theta_1) \right| = \left| \int_{C_\delta} \left[f(z) - \frac{k}{z-a} \right] dz \right|$$

$$\leq \int_{C_\delta} |(z-a)f(z) - k| \frac{|dz|}{|z-a|}$$



Then apply the property of **uniform convergence** around $z-a$, when $|z-a| \to 0$.

$\forall \varepsilon > 0$, $\exists r(\varepsilon) > 0$ **(independent of *z*)**, such that when $|z-a| = \delta < r$, the relation $\boxed{|(z-a)f(z) - k| < \varepsilon}$ holds.

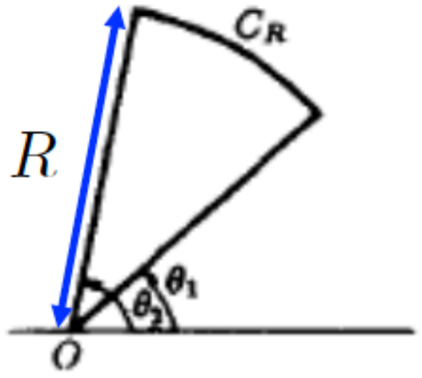
$$\left| \int_{C_\delta} f(z)dz - \mathrm{i}k(\theta_2 - \theta_1) \right| \leq \varepsilon(\theta_2 - \theta_1) \quad \Rightarrow \quad \lim_{\delta \to 0} \int_{C_\delta} f(z)dz = \mathrm{i}k(\theta_2 - \theta_1)$$

Two useful lemmas

- **Great arc lemma (大圆弧引理)**

  If $f(z)$ is continuous in a region around $z=\infty$, and satisfies the following relation: when $\theta_1 \leq \arg(z) \leq \theta_2$, $z \to \infty$, $zf(z)$ uniformly approaches K. Then we have

$$\lim_{R\to\infty}\int_{C_R} f(z)dz = \mathrm{i}K(\theta_2-\theta_1)$$



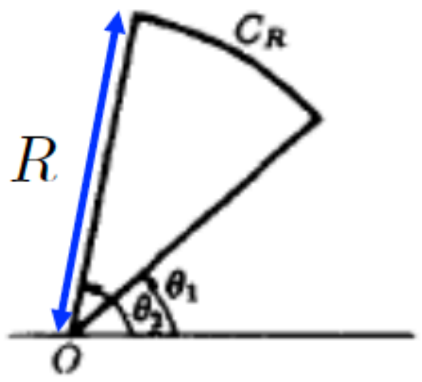
Two useful lemmas

- **Great arc lemma (大圆弧引理)**

Proof: Similar to that for the small arc lemma.

$$\left|\int_{C_R} f(z)dz - \mathrm{i}K(\theta_2-\theta_1)\right| = \left|\int_{C_R}[f(z)-\frac{K}{z}]dz\right|$$
$$\leq \int_{C_R}|zf(z)-K|\cdot\frac{|dz|}{|z|}$$

Then apply the property of uniform convergence around $z=\infty$, when $R\to\infty$.



Please do it by yourself (the rest part of the proof), following the procedure on page 22.

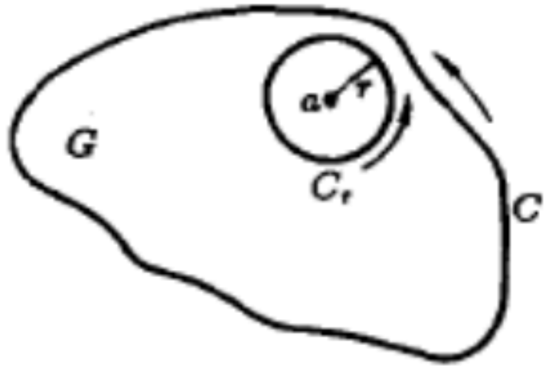
Cauchy integral formula (柯西积分公式)

For bounded region

- If $f(z)$ is a single-valued analytic function defined in $\overline{G}$ (which is bounded), and the boundary of $\overline{G}$ is a smooth curve C, a is a point inside G. Then we have:

$$f(a) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{z-a} dz$$

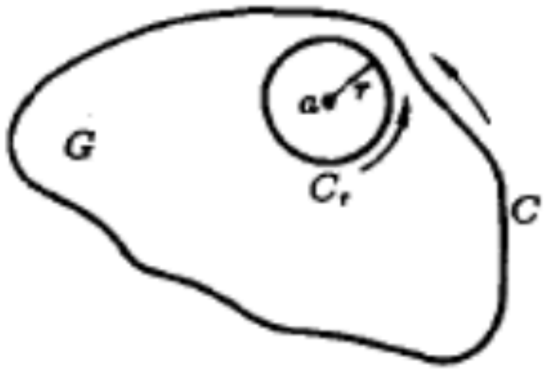
where the integration is done along the positive direction of C.



Cauchy integral formula (柯西积分公式)

For bounded region

Proof: We can draw a small circle around $z = a$. The radius of the circle is small enough, such that the enclosed disk falls within G.



According to the results on page 13:

$$\oint_C \frac{f(z)}{z-a}dz = \oint_{|z-a|=r} \frac{f(z)}{z-a}da$$

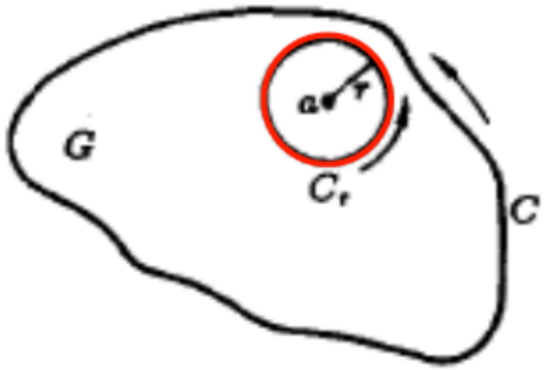
Let $r \to 0$, we know that $\lim_{z\to a}(z-a)\frac{f(z)}{z-a} = f(a)$. Apply the small arc lemma (page 21)

$$\lim_{z\to a}\oint_{C_r}\frac{f(z)}{z-a}dz = 2\pi\mathrm{i}f(a) \Longrightarrow f(a) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(z)}{z-a}dz$$

Cauchy integral formula (柯西积分公式)

For bounded region

In particular, we choose *C* as a closed circle around $z = a$



$$\boxed{f(a)} = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{z-a} dz$$

$$= \boxed{\frac{1}{2\pi} \int_0^{2\pi} f(a + r e^{\mathrm{i}\theta}) d\theta}$$ (after variable substitution)

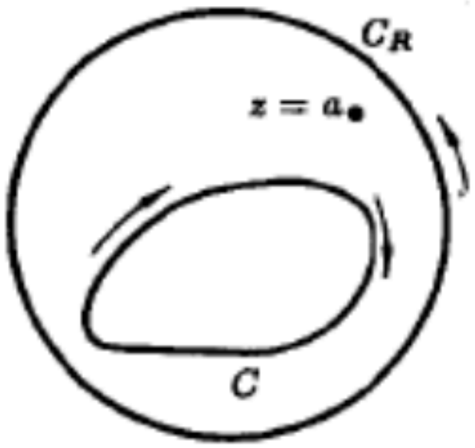
Mean-value theorem: the value at a point equals to the average along the circle that encloses that point.

Cauchy integral formula (柯西积分公式)

For unbounded region

- If $f(z)$ is a single-valued analytic function defined along and beyond a simply closed curve C (including the infinity), then we have the following relation

$$\frac{1}{2\pi \mathrm{i}}\left[\oint_{C_R}\frac{f(z)}{z-a}dz+\oint_{C}\frac{f(z)}{z-a}dz\right]=f(a)$$



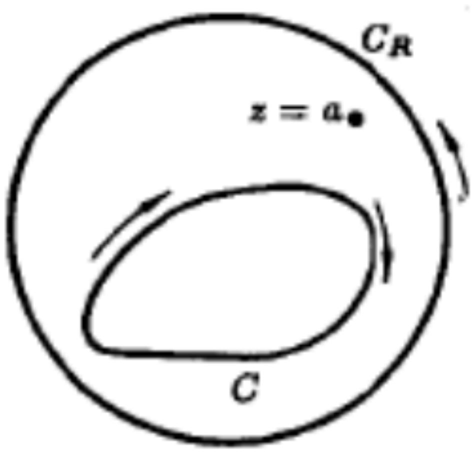
where the integration is done along the positive direction of C (which is clockwise) and the positive direction of C_R (anti-clockwise).

Cauchy integral formula (柯西积分公式)

For unbounded region

If $K=0$, then we can formally introduce the Cauchy integral formula for unbounded region:

If $f(z)$ is a analytic function defined along and beyond a simply closed curve C, and satisfies the condition $\lim_{z\to\infty} f(z)=0$, then the integral formula is given by:



$$f(a)=\frac{1}{2\pi \mathrm{i}}\oint_C \frac{f(z)}{z-a}dz$$

where a is a point enclosed by C along the clockwise direction.

Higher-order derivatives of an analytic function

- Based on the Cauchy integral formula, we can deduce a very important conclusion:

If $f(z)$ is analytic in a bounded region $\overline{G}$, then any higher-order derivative of $f(z)$ exists, which is given by

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

Then you see again, why analytic function is so special!

l: length of C

δ: shortest distance to C

- **Proof**

  Let's first work with $f'(z)$ - again we follow the definition

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi \mathrm{i}}\frac{1}{h}\oint_C \left[\frac{f(\zeta)}{\zeta - z - h} - \frac{f(\zeta)}{\zeta - z}\right]$$

$$= \frac{1}{2\pi \mathrm{i}}\oint_C \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} d\zeta$$

Then we let $h \to 0$, the L.H.S is $f'(z)$, while the R.H.S is $\frac{1}{2\pi \mathrm{i}}\oint_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$ if we can exchange the order of limit and integration.

Let's examine $\oint_C \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} d\zeta - \oint_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta = h\oint_C \frac{f(\zeta) d\zeta}{(\zeta - z - h)(\zeta - z)^2}$

We know $f(z)$ is analytic (therefore continuous), then it is also bounded (e.g. by *M*).

$$\left|\oint_C \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} d\zeta - \oint_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta\right| \le h\frac{Ml}{\delta^2(\delta - h)} \to 0$$

l : length of C

δ: shortest distance to C

- **Proof**

Then let's assume the conclusion holds for

$$f^{(k)}(z) = \frac{k!}{2\pi \mathrm{i}} \oint_C \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

and further examine if it also holds for $f^{(k+1)}(z)$

Again, we start with the definition

$$\frac{f^{(k)}(z+h) - f^{(k)}(z)}{h} = \frac{k!}{2\pi \mathrm{i}} \frac{1}{h} \oint_C \left[\frac{f(\zeta)}{(\zeta - z - h)^{k+1}} - \frac{f(\zeta)}{(\zeta - z)^{k+1}} \right] d\zeta$$

$$= \frac{k!}{2\pi \mathrm{i}} \frac{1}{h} \oint_C f(\zeta) \frac{(\zeta - z)^{k+1} - (\zeta - z - h)^{k+1}}{(\zeta - z)^{k+1} (\zeta - z - h)^{k+1}} d\zeta$$

Apply differentiation

$(k+1)!$

$$= \frac{k!}{2\pi \mathrm{i}} \oint_C \frac{f(\zeta)}{(\zeta - z)^{k+1} (\zeta - z - h)^{k+1}} \left[(k+1)(\zeta - z - h)^k + O(h) \right] d\zeta$$

$(\zeta - z)^{k+2}$

$$h \to 0$$

$$\left| \frac{f^{(k)}(z+h) - f^{(k)}(z)}{h} - \frac{(k+1)!}{2\pi \mathrm{i}} \oint_C \frac{f(\zeta)}{(\zeta - z)^{k+2}} d\zeta \right| = O(h)$$

Some remarks

- **Morera's theorem** (莫列拉定理): If $f(z)$ is continuous in $\overline{G}$, if for any closed curve (contour) in $\overline{G}$, $\oint_C f(z)dz = 0$, then $f(z)$ is analytic in G.

- **Maximum-modulus principle** (最大模原理): If $f(z)$ is analytic in $\overline{G}$, then the maximum of $|f(z)|$ must be reached at the boundary of $\overline{G}$.

- **Liouville's theorem** (刘维尔定理): If $f(z)$ is analytic in $\mathbb{C}$. And $|f(z)|$ is bounded when $z \to \infty$, then $f(z)$ must be a constant.

Some remarks

- **Cauchy-type integral**

  If $\phi(\zeta)$ is a continuous function defined along curve *C* (piece-wisely smooth) , then

$$f(z)=\frac{1}{2\pi\mathrm{i}}\int_C\frac{\phi(\zeta)}{(\zeta-z)}d\zeta,\quad z\notin C$$

is an analytic function defined outside *C*.

geometric decay

And we have

$$f^{(p)}(z)=\frac{p!}{2\pi\mathrm{i}}\int_C\frac{\phi(\zeta)}{(\zeta-z)^{p+1}}d\zeta,\quad z\notin C$$

In mechanics and physics, you will frequently encounter formulas like the type mentioned here.

Imagine the curve *C* is some sort of "source" term, and you want to know the "field", which corresponds to a convolution ("summation") over the entire source, at any point outside the source region.

Some remarks

- **Integral that contains a parameter**

  (1) $f(t,z)$ is a continuous function of t and z , $t \in [a,b],\ z \in \overline{G},\ \overline{G}$ is bounded

  (2) For any value $t \in [a,b]$, $f(t,z)$ is a single-valued analytic function defined in $\overline{G}$.

  Then $F(z) = \int_a^b f(t,z)dt$ is analytic in G, and

  $$F'(z) = \int_a^b \frac{\partial f(t,z)}{\partial z} dt,\ z \in G$$

Some remarks

- **Integral that contains a parameter**

$$F(z)=\int_a^b f(t,z)dt \qquad F'(z)=\int_a^b \frac{\partial f(t,z)}{\partial z}dt,\ \ z\in G$$

- **Proof**

Since $f(t,z)$ is analytic in $\overline{G}$, then $f(t,z)=\dfrac{1}{2\pi \mathrm{i}}\displaystyle\oint_C \frac{f(t,\zeta)}{\zeta-z}d\zeta$

Cauchy-type integral

$$F(z)=\int_z^b \frac{dt}{2\pi \mathrm{i}}\oint_C \frac{f(t,\zeta)}{\zeta-z}d\zeta=\boxed{\frac{1}{2\pi \mathrm{i}}\oint_C \frac{1}{\zeta-z}\left[\int_a^b f(t,\zeta)dt\right]d\zeta}$$

We can exchange the order because $f(t,z)$ is continuous

$\displaystyle\int_a^b f(t,z)dt$ is continuous in $\mathbb{C}$, then $F(z)$ is analytic.

$$F'(z)=\frac{1}{2\pi \mathrm{i}}\oint_C \frac{1}{(\zeta-z)^2}\left[\int_a^b f(t,\zeta)dt\right]d\zeta=\int_a^b\left[\frac{1}{2\pi \mathrm{i}}\oint_C \frac{f(t,\zeta)}{(\zeta-z)^2}d\zeta\right]dt=\int_a^b \frac{\partial f(t,z)}{\partial z}dt$$