Mathematical Methods in Physics

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Instruction

- This course will be taught in both English and Chinese.
- The lecture notes, either PowerPoint (PPT) or handwriting materials, will be made in English. Some key concepts and terminologies will be translated into Chinese as well. I, the instructor, will be speaking in both Chinese (major language) and English (minor language) during the class.

Why study mathematical methods

- Mathematics
- Physics
- Chemistry
- Biology
- Earth Sciences
- Engineering
- etc.

Ordinary/Partial Differential Equations,
Real/Complex Analysis, Functional Analysis, etc.

- They have their own logic system
- There are many proofs (lemma, theorem)
- Need to understand the essence

Mathematical Methods in Physics
Methods of Mathematical Physics
Methods of Theoretical Physics

- Learn some basic concepts and ideas
- Some proofs can be skipped
- Master some useful skills (need practice)

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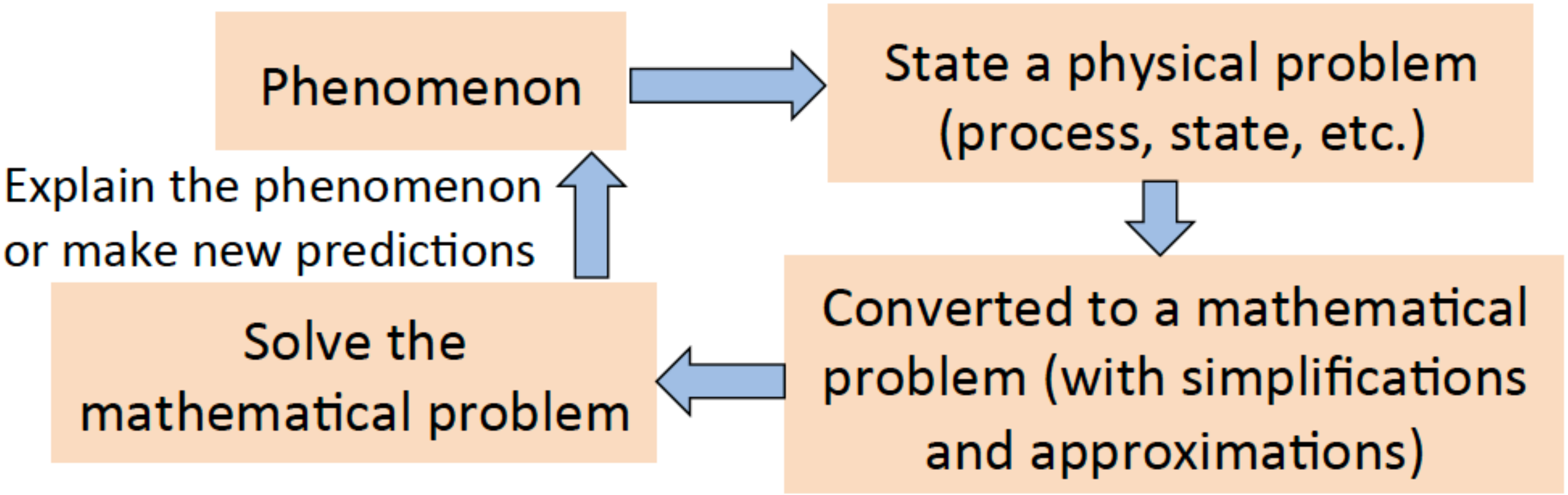

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- Apply rules for real numbers to complex numbers
- Existence and uniqueness of the solution
- Dependence on initial and boundary conditions
- Definition of infinity (无穷大或无穷远)

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Why study mathematical methods

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- Chemistry
- Biology
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- etc.

- You need to practice
- Master transferable skills
- You know how to formulate a problem and then solve it
- You know where to get help (resources)

Mathematical Methods in Physics
Methods of Mathematical Physics
Methods of Theoretical Physics

- Learn some basic concepts and ideas
- Some proofs can be skipped
- Master some useful skills (need practice)

Some examples

- Mathematics
- Physics
- Chemistry
- Biology
- Earth Sciences
- Engineering
- etc.

The Schrödinger equation – Quantum mechanics

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi(x,t) \equiv \widetilde{H}\Psi(x,t)$$

Navier-Stokes equation – Fluid dynamics

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla P}{\rho} + \nu\nabla^2\mathbf{u}$$

Wave equation – a number of scientific fields

$$\nabla^2\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}$$

What you need to know before taking this course

- Calculus → We will frequently use the knowledge on trigonometric function (三角函数), series (序列), differentiation and integration (微分和积分), Taylor expansion (泰勒展开), etc.

- General Physics → There will be some problems on mechanics (力学), thermodynamics (热力学), electromagnetics (电磁学), and optics (光学).

- Linear Algebra → We will be using the concepts of eigenvectors (特征向量) and eigenvalues (特征值).

What you will learn after taking this course

- Complex number (复数), function of a complex variable (复变函数)
- Analytic functions (解析函数), series (序列), residual theorem (留数定理)
- Special functions (特殊函数)
- Ordinary and partial differential equations (常和偏微分方程)
- ~~Sturm-Liouville equation (斯图姆—刘维尔型方程)~~
- ~~Green's function (格林函数)~~
- Basic concepts and ideas, skills, where to get resources – believe me, these are what you will keep in your mind for a long time

How should you take this course

- Preview the course materials for the coming week (on blackboard)
- Attend class (or online course) on time
- Take your own notes, do not just rely on the course materials
- Ask questions if you don't understand – don't be shy!
- Consult with your classmates, instructor (me), or TA for help
- Do homework, and if necessary extra exercises Very important!
- Do NOT cheat!
- Make summary after each chapter

A brief overview

- Credit value: 4
- Credit hours: 64
- Class time: Tue (16:20 - 18:10), Thu (10:20 - 12:10)
- Class place: 1st Teaching Building 402
- Assessment: Homework (30%), Midterm (30%), Final exam (40%)

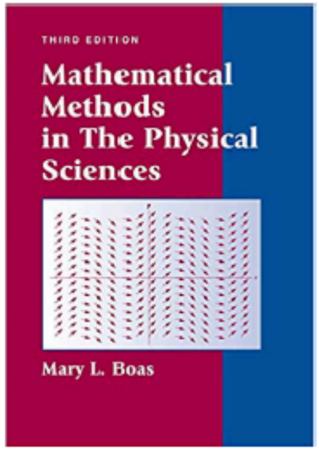
A brief overview of syllabus

- Chapter – 01: Complex numbers and hyperbolic functions (2 weeks)
- Chapter – 02: Complex variables (4 weeks)
- Chapter – 03: Partial differential equations: general and particular solutions (2 weeks)
- Midterm (1 week)
- Chapter – 04: Partial differential equations: separation of variables (2 weeks)
- Chapter – 05: Series solutions of ordinary differential equations (2 weeks)
- Chapter – 06: Special functions (3 weeks)
- Final

Changes could be made in the future.

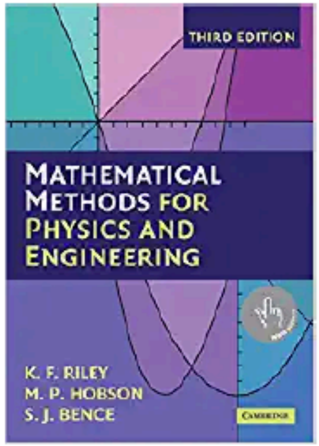
Textbook and references

- Mathematical Methods in the Physical Sciences, by Mary L. Boas. The third Edition; Wiley

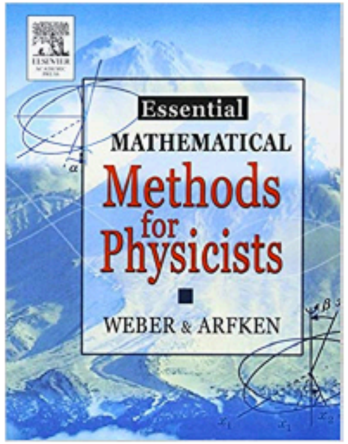


- 数学物理方法（第三版）吴崇试 高春媛，北京大学出版社

- Mathematical Methods for Physics and Engineering, by K. F. Riley, M. P. Hobson and S. J. Bence. The third Edition; Cambridge University Press

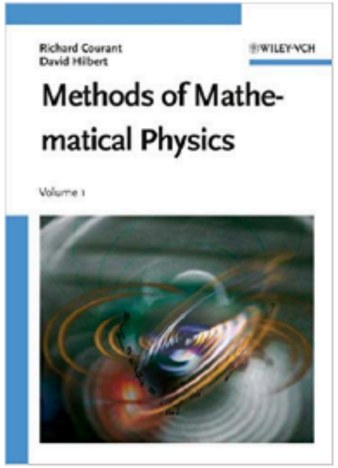


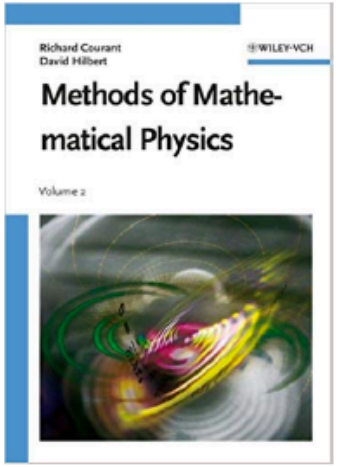
- Essential Mathematical Methods for Physicists, by Hans J. Weber and George B. Arfken. Academic Press



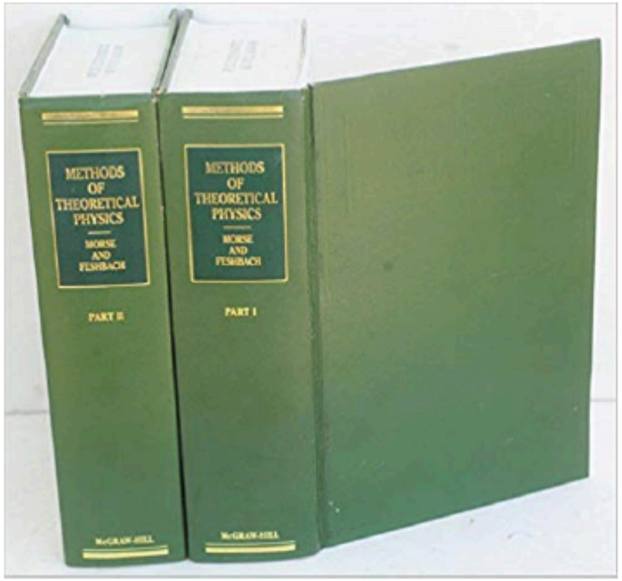
Advanced readings

- Methods of Mathematical Physics. Vol. 1 & 2, by Richard Courant and David Hilbert. Wiley-VCH





- Methods of Theoretical Physics. Part 1 & 2, by Herman Morse, and Philip M. Feshbach. McGraw Hill



Any questions?

Chapter – 01: Complex numbers (复数)

- Recall **real numbers** (实数)

Comparison: $x > y, x < y, x = y$

Operation (addition, subtraction, multiplication, and division): $x + y, x - y, x \cdot y, x/y$

- Let's expand real numbers to **ordered pairs of real numbers** (有序实数对)

Define: $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, where x_i and y_i are real numbers

If $z_1 = z_2$, then $(x_1, y_1) = (x_2, y_2) \Rightarrow x_1 = x_2,\ y_1 = y_2$

In general it is meaningless to say $z_1 > z_2$ or $z_1 < z_2$ (We will come back to it later)

mind the negative sign

Operation: $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$ $\quad z_1 \cdot z_2 = (x_1 \cdot x_2 - y_1 \cdot y_2, x_1 \cdot y_2 + x_2 \cdot y_1)$

If $z = (x, y) = x \cdot (1, 0) + y \cdot (0, 1)$ obeys the above rules, we call z a complex number.

$$z = (x, y) = x \cdot (1, 0) + y \cdot (0, 1)$$

x is called the **real part** (实部) of z, and y is called the **imaginary part** (虚部) of z

$$x = \mathrm{Re}\ z \qquad\qquad y = \mathrm{Im}\ z$$

We use $\mathbb{C}$ to denote the set of complex numbers.

Recall that a complex number is an ordered pair of real numbers. It means complex numbers include real numbers.

e.g. $z = (x, 0) = x \cdot (1, 0) = x$

From above example, we see that $(1, 0)$ is a very special complex number, which is identical to the "unit" of real numbers 1.

Then we can have another very special case by setting the real part as zero:

$z = (0, y) = y \cdot (0, 1) = \mathrm{i}y$ $\qquad z$

Complex numbers like the above example are called **pure imaginary numbers** (**纯虚数**).

We use i to denote the unit of pure imaginary numbers: $\mathrm{i} = (0, 1)$

Now we can simplify the writing of a complex number:

$z = (x, y) = x \cdot (1, 0) + y \cdot (0, 1) = x + \mathrm{i}y$

Exercise

[1.01] What are the real and imaginary parts of $15 + 9\mathrm{i}$?

[1.02] The imaginary part of a complex number is a pure imaginary number, true or false?

[1.03] Real numbers are a subset of complex numbers, true or false?

[1.04] $(0, -5) = -5$, true or false?

[1.05] What is the result of $(-1, 3) \cdot (-2, 9)$?

Some properties of i

$$\mathrm{i} \cdot \mathrm{i} = (0, 1) \cdot (0, 1) = (0 - 1, 0) = -1$$

$$\mathrm{i}^2 = -1$$

i plays a central role in complex numbers and complex analysis

Now we see that i is the square root of a negative real number -1.

From above, we can simplify the performance of multiplication:

$$z_1 \cdot z_2 = (x_1 + \mathrm{i}y_1) \cdot (x_2 + \mathrm{i}y_2) = (x_1 \cdot x_2 - y_1 \cdot y_2) + \mathrm{i}(x_1 \cdot y_2 + x_2 \cdot y_1)$$

Please do it by yourself, following the multiplication rule for multinomials (多项式乘法)!

But also note that i is not the only square root of -1.

$$(-\mathrm{i})^2 = (-1)^2 \cdot (\mathrm{i})^2 = -1$$

Exercise

[1.06] Write $(0, -5)$ in the form of $x + \mathrm{i}y$

[1.07] i is the only square root of -1, true or false?

[1.08] Re-express $(-1, 3) \cdot (-2, 9)$ in the form of $x + \mathrm{i}y$, and then calculate the result

Subtraction for complex numbers

Similar to the positive and negative real numbers, every complex number has an opposite counterpart (相反数):

$$z = x + \mathrm{i}y,\ \ (-z) = (-x) + (-\mathrm{i}y),\ \text{ and } z + (-z) = 0$$

Then we can define the operation of subtraction for complex numbers:

$$z_1 - z_2 = z_1 + (-z_2)$$

We will come to the remaining operation of division for complex numbers later!

Exercise

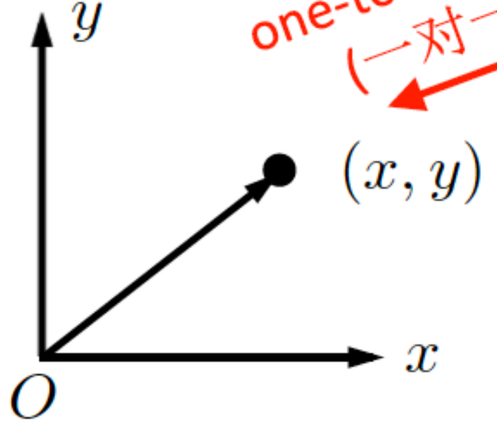
[1.09] $-(9,-3)=(-9,3)$, true or false?

[1.10] Each complex number has multiple opposite numbers, true or false?

The geometric representation of complex numbers

- Recall what you have learned from the geometry course.

A complex number can be uniquely determined once its real part x and imaginary part y are known.



one-to-one mapping
(一对一映射)

$$z = (x, y) = x + \mathrm{i}y$$

- Therefore, we can express a complex number in a graph – the geometric representation.

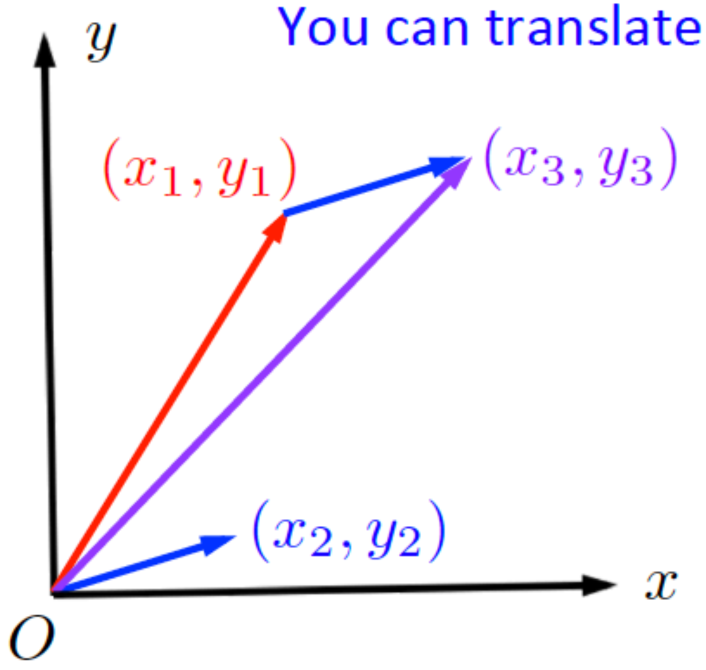
- Unlike the usual case, here the two axes x and y have quite different properties (see previous slides).

The geometric representation of complex numbers

- Recall the operation of addition for complex numbers:

$z_1 = x_1 + \mathrm{i}y_1,\ \text{and}\ z_2 = x_2 + \mathrm{i}y_2$ $\qquad z_3 = z_1 + z_2 = (x_1 + x_2) + \mathrm{i}(y_1 + y_2)$

You can translate (平移) the vector for (x_2, y_2)

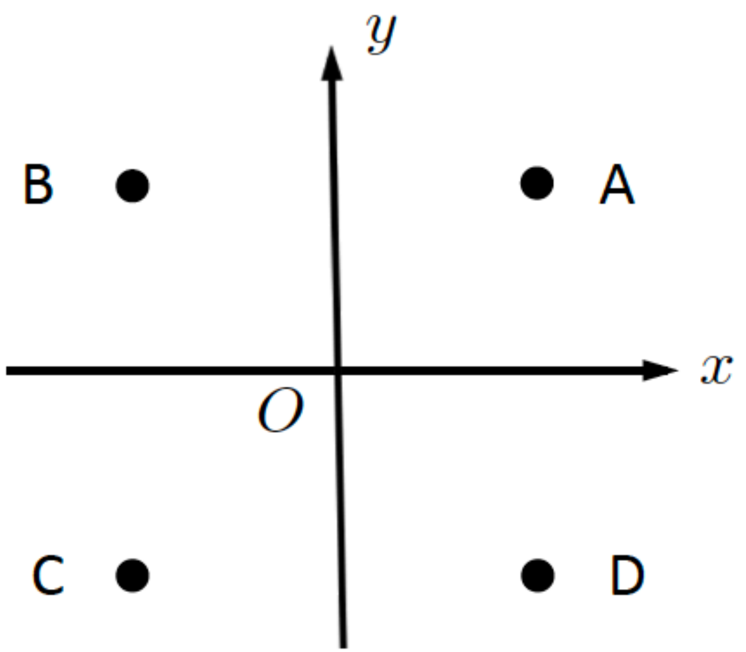


- So the basic knowledge you learned from General Physics can be applied here: how to get the sum of two vectors.

- Following the same principle, you should be able to figure out how to do subtraction between two vectors (see p 27).

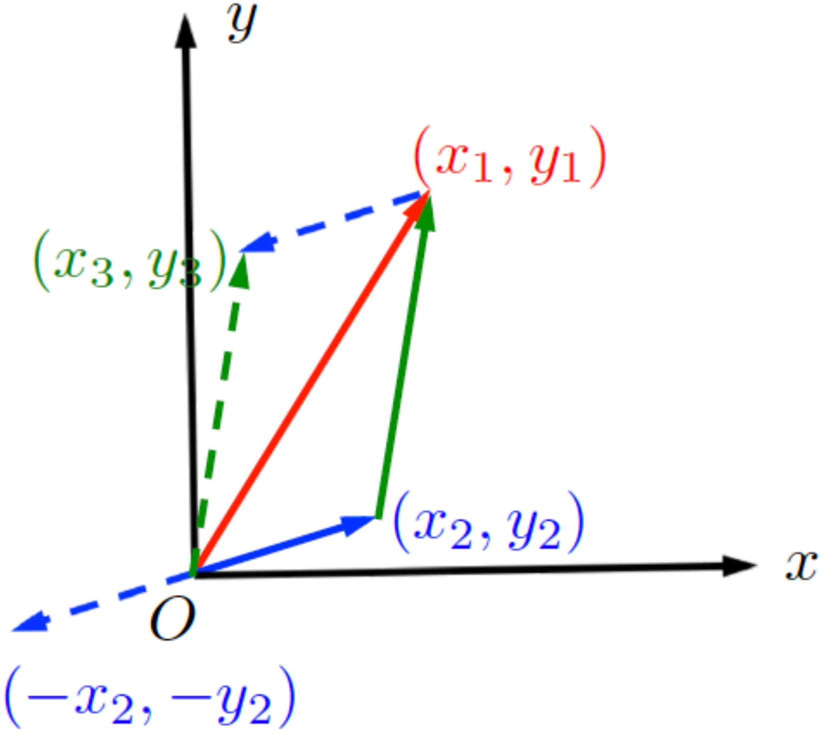
Exercise

[1.11] Which point below (A, B, C, and D) most likely represents the complex number $-3+3\mathrm{i}$?



The geometric representation of complex numbers

$$z_1 = x_1 + \mathrm{i}y_1, \text{ and } z_2 = x_2 + \mathrm{i}y_2 \qquad z_3 = z_1 - z_2 = (x_1 - x_2) + \mathrm{i}(y_1 - y_2)$$



- For performing subtraction, you can directly do it following the rule for vector subtraction (see the solid green arrow in the left figure).

- Alternatively, you can first define the opposite of z_2, and then perform a regular addition of z_1 and $-z_2$ (see the dashed green arrow in the left figure).

- Note both methods will give you the same answer! Why? Because the two green vectors are identical after an operation of translation.

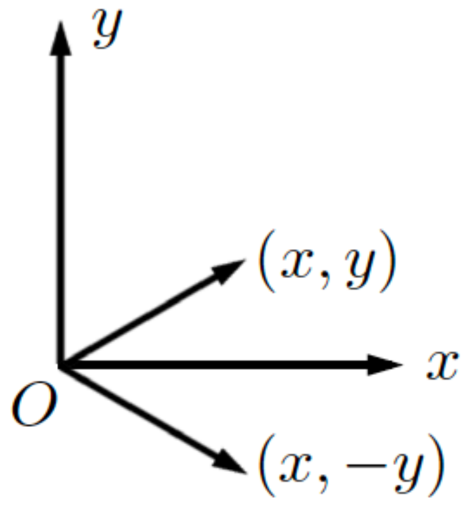
Complex conjugate and division for complex numbers

Previously we have introduced the concept of opposite (相反数), now we will introduce another concept called conjugate (共轭), or more specifically, complex conjugate (复共轭)

$z^* = x - \mathrm{i}y$ is the complex conjugate of $z = x + \mathrm{i}y$; conversely, $z = x + \mathrm{i}y$ is the complex conjugate of $z^* = x - \mathrm{i}y$.

Graphically, z and z^* are symmetric with respect to the real axis x.

Sometimes z^* is also written as $\overline{z}$.



Complex conjugate and division for complex numbers

You should master the following properties related to conjugate (possible exam problems!)

$$z = x + \mathrm{i}y \qquad z^* = x - \mathrm{i}y$$

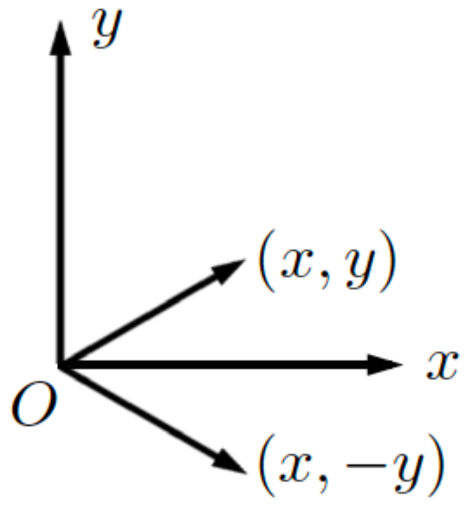
$(z^*)^* = z$

$z^* = z$, if and only if z is a real number.

$z + z^* = 2x$ is a real number.

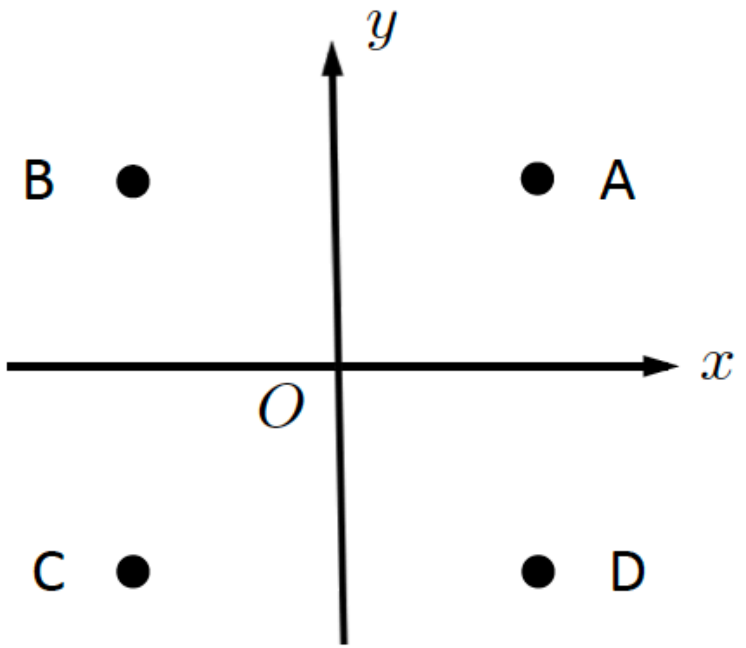
$z - z^* = 2\mathrm{i}y$ is a pure imaginary number.

$z \cdot z^* = x^2 + y^2 \geq 0$



Exercise

[1.12] Which point below (A, B, C, and D) most likely represents the complex number $(-3+3\mathrm{i})^*$?



Exercise

[1.13] Suppose z is a complex number, $z \cdot z^*$ can be a negative real number, true or false?

[1.14] What is the opposite of $5 - 7\mathrm{i}$, and what is the conjugate of $5 - 7\mathrm{i}$?

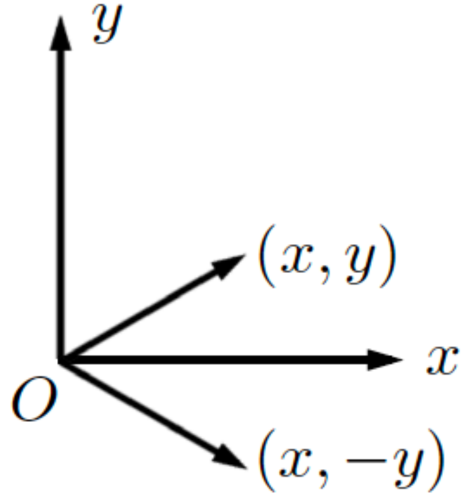
[1.15] $(-(-z)^*) = z$, true or false?

Complex conjugate and division for complex numbers

Since you have become familiar with the concept of conjugate, now we can discuss how to perform the division between two complex numbers.

$$\frac{z_1}{z_2} = \frac{x_1 + \mathrm{i}y_1}{x_2 + \mathrm{i}y_2} = \frac{(x_1 + \mathrm{i}y_1)\boxed{(x_2 - \mathrm{i}y_2)}}{(x_2 + \mathrm{i}y_2)\boxed{(x_2 - \mathrm{i}y_2)}} \quad \longleftarrow z_2^*$$

$$= \frac{(x_1x_2 + y_1y_2) + \mathrm{i}(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$



- Similar to the case for real numbers, we require that the denominator (分母) cannot be zero: $z_2 \neq 0$.
- $z_2 \neq 0$ also means $z_2^* \neq 0$.

Exercise

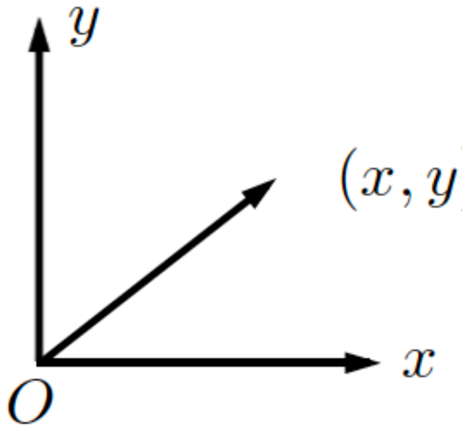
[1.16] Calculate the result of $\dfrac{1+i}{1-i}$

[1.17] Calculate the result of $\left(\dfrac{1+i}{1-i}\right)^2$

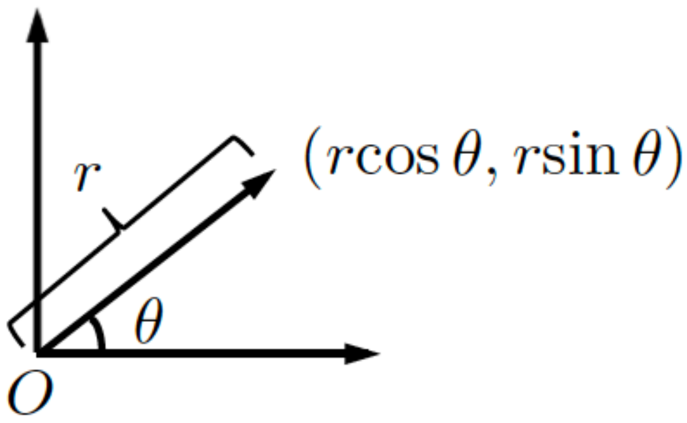
[1.18] Calculate the result of $\left(\dfrac{(1+i)^*}{(1-i)^*}\right)^2$

The geometric representation of complex numbers

- Recall **again** what you have learned from the geometry course.



Cartesian coordinate (笛卡尔坐标系）



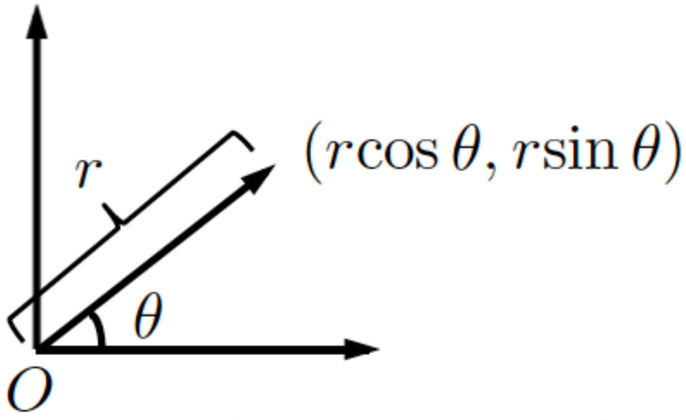
Polar coordinate (极坐标系）

$$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta)$$

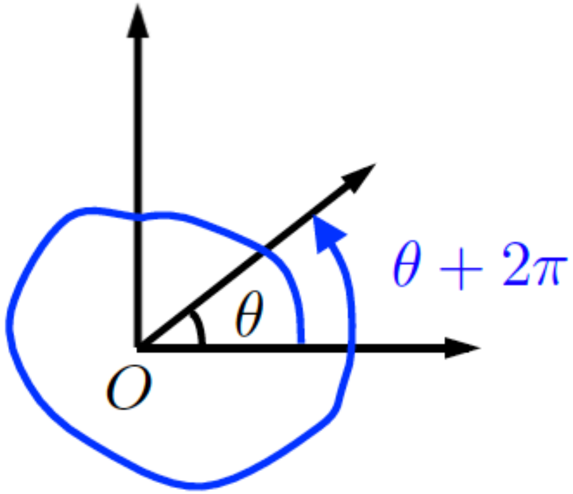
$r = |z| = \sqrt{x^2 + y^2}$: called modulus (模) of z

$\theta = \arg z$: called angle or argument (辅角) of z

Complex numbers in a polar coordinate system



$$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta)$$



$\theta = \mathrm{Arg}\ z$

if $\theta \in [-\pi, \pi]$

called principal value (主值)

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \arg z$$

- r is non-negative, and is used to represent the length of z.
- When $r = 0$, θ can be any value.
- Due to the periodicity of triangular functions, the value of θ is not unique. θ $<==>$ $\theta \pm 2n\pi$

$z_1 = z_2 : |z_1| = |z_2|$ and $\arg z_1 = \arg z_2 \pm 2n\pi$

About the unit of angle

While you may be more familiar with the unit of degrees, e.g. 30°, the common practice for representing complex numbers in a polar coordinate system is to use the unit of rad, i.e., $\pi/6$.

Unless mentioned otherwise, I recommend using rad as the unit for $\theta = \arg z$.

Exercise

[1.19] Calculate the result of $|1-i|$

[1.20] $|1-2\mathrm{i}| = |1+2\mathrm{i}|$, true or false?

[1.21] What is the principal argument value of $1+\sqrt{3}\mathrm{i}$?

[1.22] What is the principal argument value of $-\sqrt{3}-\mathrm{i}$?

Complex equations

Let's recall we have learned before:

$z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$

$z_1 = z_2$, if and only if $\underline{x_1 = x_2}$ and $\underline{y_1 = y_2}$ Note: there are actually two equations

Using the above relations, we can work with complex equations.

Exercise

[1.23] Find the values of x and y, which satisfy the following equation.

$$(x + \mathrm{i}y)^2 = 2\mathrm{i}$$

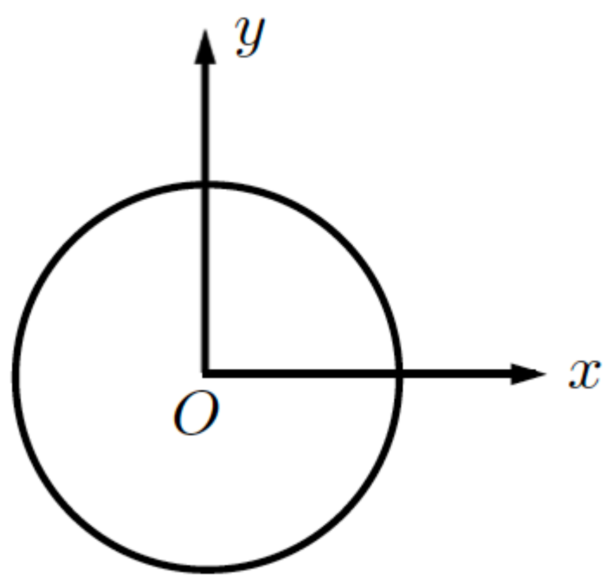
Hint: Expand both the left-hand side and the right-hand side, and equalize their real parts and imaginary parts, respectively.

Draw graphs

We can represent the solution(s) to a complex equation in a graph

$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta)$

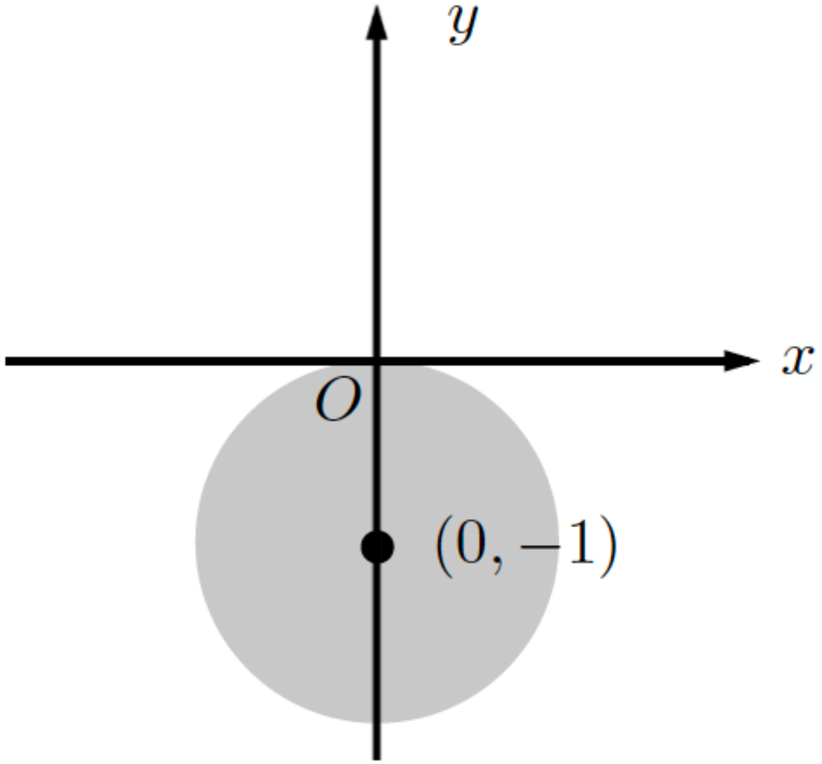
$|z| = 2 \;\; <=> \;\; x^2 + y^2 = 4, \;\text{or}\; r = 2$



Exercise

[1.24] Draw the graph that satisfies the following relation:

$$|z + \mathrm{i}| \leq 1$$



Complex numbers in a polar coordinate system

Now let's do some math under the polar coordinate system.

$$z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$$

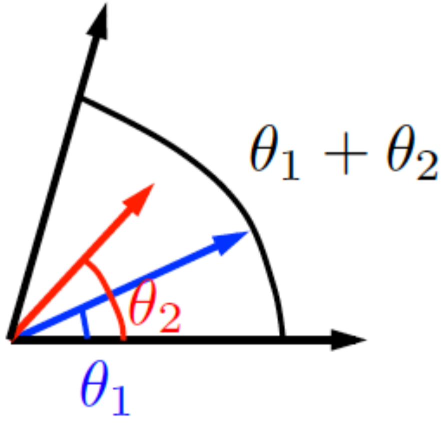
Recall some trigonometric formulas (三角公式)

- Multiplication

$$\begin{aligned} z_1 \cdot z_2 &= r_1 r_2(\cos\theta_1 + \mathrm{i}\sin\theta_1)(\cos\theta_2 + \mathrm{i}\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{i}(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)] \\ &= r_1 r_2[\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2)] \end{aligned}$$

Graphically, it means the final modulus = $r_1 r_2$ (模相乘)

the final argument = $\theta_1 + \theta_2$ (辅角相加)



Complex numbers in a polar coordinate system

Now let's do some math under the polar coordinate system.

$$z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$$

Recall some trigonometric formulas (三角公式)

- Division

$$\frac{z_1}{z_2} = \frac{z_1 \cdot z_2^*}{z_2 \cdot z_2^*} = \frac{r_1 r_2[\cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 - \theta_2)]}{r_2^2}$$
$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 - \theta_2)]$$

Graphically, it means the final modulus = $\dfrac{r_1}{r_2}$ (模相除)

the final argument = $\theta_1 - \theta_2$ (辅角相减)

As usual we require that $z_2 \neq 0$.

Exercise

[1.25] Suppose $z = r(\cos\theta + \mathrm{i}\sin\theta)$, what is the graphical representation for $z \cdot \mathrm{i}$?

[1.26] Similarly, what is the graphic representation for $z \cdot (-\mathrm{i})$?

Hint: $\pm\mathrm{i} = \cos(\pm\pi/2) + \mathrm{i}\sin(\pm\pi/2)$

Complex infinite series (复无穷序列)

Similar to the series for real numbers, we can define the series for complex numbers.

$$z_n = x_n + \mathrm{i}y_n,\ n = 1, 2, 3...$$

Note the above definition is equivalent to two series for real numbers $\{x_n\}$ and $\{y_n\}$

We can further define the limit of $\{z_n\}$, if it exists: $\lim_{n\to\infty} z_n = z$

$$\forall\varepsilon,\ \exists N(\varepsilon),\ \text{such that}\ \forall \text{positive integer}\ p,$$
$$|z_{N+p} - z_N| < \varepsilon$$

How to judge if $\{z_n\}$ are convergent (收敛的)

If the limit does not exist, we call $\{z_n\}$ divergent (发散的).

Exercise

[1.27] Judge the convergence of $z_n = (-1)^{n+1}\dfrac{n}{n+1}$

[1.28] Judge the convergence of $z_n = \mathrm{i}^n$

The exponential representation of complex numbers

- Recall the Taylor expansion you have learned from the calculus course, and let's expand function $f(x)$ around $x = a$.

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \dots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \dots$$

- Let's do the Taylor expansion for triangular functions around $\theta = 0$.

$$\left.\begin{aligned} \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \\ \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \end{aligned}\right\}$$

$$\begin{aligned} \boxed{e^{i\theta}} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} \dots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots) \\ &\boxed{= \cos\theta + i\sin\theta} \end{aligned}$$

Euler's formula (欧拉公式)

The exponential representation of complex numbers

- With Euler's formula, the operation of complex numbers can be further simplified, by referring to the rules for exponential functions.

$$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta) = re^{\mathrm{i}\theta}$$

$$z_1 \cdot z_2 = r_1 e^{\mathrm{i}\theta_1} \cdot r_2 e^{\mathrm{i}\theta_2} = r_1 r_2 e^{\mathrm{i}(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{\mathrm{i}\theta_1}}{r_2 e^{\mathrm{i}\theta_2}} = \frac{r_1}{r_2} e^{\mathrm{i}(\theta_1-\theta_2)} \qquad (z_2 \neq 0)$$

Please verify by yourself that the above results are the same as those on pages 42 and 43.

The exponential representation of complex numbers

- With Euler's formula, we can derive more useful formulas, by referring to the rules for power functions and exponential functions.

$$z^n = (re^{\mathrm{i}\theta})^n = r^n \cdot e^{\mathrm{i}n\theta}$$

Let $r = 1$ De Moivre's theorem (棣莫弗定理)

$$\boxed{(e^{\mathrm{i}\theta})^n = (\cos\theta + \mathrm{i}\sin\theta)^n = e^{\mathrm{i}n\theta} = \cos n\theta + \mathrm{i}\sin n\theta}$$

- What about the n-th root?

$$z^{1/n} = (re^{\mathrm{i}\theta})^{1/n} = r^{1/n}e^{\mathrm{i}\theta/n} = \sqrt[n]{r}(\cos\frac{\theta}{n} + \mathrm{i}\sin\frac{\theta}{n})$$

Extra care should be taken for computing the n-th root. We will come back to this point later.

Exercise

[1.29] Please write the real and imaginary parts, modulus, and argument of e^{-z}

[1.30] Simplify the expression of $\cos\theta + \cos 2\theta + ... + \cos n\theta$

Hint: You may want to use the following relations

$$\cos n\theta = \frac{e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta}}{2}$$

$$\sum_{k=1}^{n} e^{\mathrm{i}k\theta} = \frac{(1 - e^{\mathrm{i}n\theta})e^{\mathrm{i}\theta}}{1 - e^{\mathrm{i}\theta}}$$

Please review the geometric series