Exercise 02

Part 1.

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Part 2.

(1) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$$(\sin z)' = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

(2) $L.H.S. = \dfrac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i}$

$$= \frac{\left(e^{iz_1}e^{iz_2} + e^{-iz_1}e^{iz_2}\right) - \left(e^{-iz_1}e^{iz_2} + e^{-iz_1}e^{-iz_2}\right)}{2i}$$

$$= \frac{\cos z_1 \, e^{iz_2} - e^{-iz_1} \cos z_2}{i}$$

$$= \frac{\cos z_1 \left(\cos z_2 + i\sin z_2\right) - \left(\cos z_1 - i\sin z_2\right)\cos z_2}{i}$$

$= R.H.S.$

(3) $L.H.S. = \dfrac{e^{i(z_1 - z_2)} + e^{i(z_2 - z_1)}}{2}$

$$= \frac{\left(e^{iz_1}e^{-iz_2} + e^{iz_1}e^{iz_2}\right) - \left(e^{iz_1}e^{iz_2} - e^{iz_2}e^{-iz_1}\right)}{2}$$

$$= e^{iz_1}\cos z_2 - i e^{iz_2}\sin z_1$$

$$= (\cos z_1 + i\sin z_1)\cos z_2 - i(\cos z_2 + i\sin z_1)\sin z_1$$

$$= \text{R.H.S.}$$

(4) $\text{L.H.S.} = \dfrac{\sin(z_1+z_2)}{\cos(z_1+z_2)}$

$$= \frac{\sin z_1\cos z_2 + \cos z_1\sin z_2}{\cos z_1\cos z_2 - \sin z_1\sin z_2}$$

$$= \frac{\dfrac{\sin z_1}{\cos z_1} + \dfrac{\sin z_2}{\cos z_2}}{1 - \dfrac{\sin z_1}{\cos z_1}\dfrac{\sin z_2}{\cos z_2}}$$

$$= \text{R.H.S.}$$

(5) $\text{R.H.S.} = -i\,\dfrac{e^{i(iz)} - e^{-i(iz)}}{2i}$

$$= \frac{e^{z} - e^{-z}}{2}$$

$$= \text{L.H.S.}$$

(6) $\text{R.H.S.} = \dfrac{e^{i(iz)} + e^{-i(iz)}}{2}$

$$= \frac{e^{z} + e^{-z}}{2}$$

$$= \text{L.H.S.}$$

(7) $L.H.S. = \dfrac{\sinh z}{\cosh z}$

$$= -i\,\frac{\sin(iz)}{\cos(iz)}$$

$= R.H.S.$

(8) $L.H.S. = 1 - \left(\dfrac{e^{z} - e^{-z}}{e^{z} + e^{-z}}\right)^2$

$$= \frac{4}{(e^{z} + e^{-z})^2}$$

$= R.H.S.$

(9) $L.H.S. = -i\sin(iz_1 + iz_2)$

$$= -i\sin(iz_1)\cos(iz_2) - i\cos(iz_1)\sin(iz_2)$$

$= R.H.S.$

(10)

$$|\cos(x+iy)| = \left|\frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2}\right|$$

$$\leq \frac{e^{-y} + e^{y}}{2} = \cosh(y)$$

$$|\cos(x+iy)| \geq \frac{\left|\,|e^{-ix}e^{y}| - |e^{ix}e^{-y}|\,\right|}{2}$$

$$= \left|\frac{e^{y} - e^{-y}}{2}\right| \geq |\sinh(y)|$$

Part 3.

(1) $\cos z = 4 \Rightarrow \dfrac{e^{iz}+e^{-iz}}{2} = 4$

$$\Rightarrow e^{i2z} - 8e^{iz} + 1 = 0$$

$$\Rightarrow e^{iz} = e^{ix}e^{-y} = \frac{8 \pm \sqrt{8^2-4}}{2}$$

$$= 4 \pm \sqrt{15}$$

$\Rightarrow y = -\ln(4 \pm \sqrt{15})$ $\Rightarrow z = 2n\pi - i\ln(4 \pm \sqrt{15})$

$x = 2n\pi, \quad n \in \mathbb{Z}$

(2) $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i} = \dfrac{3}{4} + \dfrac{i}{4}$

Let $a = e^{iz}$, then we have,

$$a^2 + \left(\tfrac{1}{2} - \tfrac{3}{2}i\right)a - 1 = 0$$

$\Rightarrow a = i-1$ or $a = \frac{1}{2} + \frac{1}{2}i$

If $e^{iz} = e^{ix}e^{-y} = (\cos x + i\sin x)e^{-y}$

$$= i-1$$

then, $\begin{cases} x = \frac{3}{4}\pi + 2n\pi, \quad n \in \mathbb{Z} \\ y = -\ln\sqrt{2} \end{cases}$

If $e^{iz} = \frac{1}{2} + \frac{1}{2}i$, then $\begin{cases} x = \frac{\pi}{4} + 2n\pi \\ y = -\ln\frac{\sqrt{2}}{2} \end{cases}$

Thus, $z = \frac{3}{4}\pi + 2n\pi - \ln\sqrt{2}\, i$ or $z = \frac{\pi}{4} + 2n\pi - \ln\frac{\sqrt{2}}{2}\, i$

(3) Let $a = \cosh(z)$

then $2a^2 - 3a + 1 = 0$

$\Rightarrow \quad a = \frac{1}{2}$ or $a = 1$

If $a = \frac{1}{2}$, $\quad \dfrac{e^z + e^{-z}}{2} = \frac{1}{2}$

$$\Rightarrow e^{2z} - e^z + 1 = 0$$

$$\Rightarrow e^z = \frac{1 \pm \sqrt{3}\, i}{2}$$

$$= e^x e^{iy} = e^x(\cos y + i\sin y)$$

$$\Rightarrow \begin{cases} x = 0 \\ y = \pm\frac{\pi}{3} + 2n\pi, \quad n \in \mathbb{Z} \end{cases}$$

If $a = 1$, $\quad \dfrac{e^z + e^{-z}}{2} = 1$

$$\Rightarrow e^{2z} - 2e^z + 1 = 0$$

$$\Rightarrow e^z = 1$$

$$\Rightarrow x = 0, \quad y = 2n\pi$$

Thus, $z = 2n\pi i$ or $z = \left(\pm\frac{\pi}{3} + 2n\pi\right) i$

Part 4.

(1) Let $W = \arccos z$.

$$z = \cos w = \frac{e^{iw} + e^{-iw}}{2}$$

$$\Rightarrow e^{iw} = z \pm \sqrt{z^2 - 1} = z + \sqrt{z^2 - 1}$$

$$\Rightarrow w = \frac{1}{i} \ln\left(z + \sqrt{z^2 - 1}\right)$$

(2) Let $z = \tan w$

$$= \frac{e^{iw} - e^{-iw}}{i\left(e^{iw} + e^{-iw}\right)}$$

$$\Rightarrow e^{iw} = \sqrt{\frac{1+iz}{1-iz}}$$

$$\Rightarrow w = \frac{1}{2i} \ln \frac{1+iz}{1-iz}$$