Exercise 05

Part 1

(1) $\int_0^{2+i} Re(z)\,dz = \int_0^{2+i} x\,(dx + i\,dy)$

(i) $0 \to 2$: $dy = 0 \Rightarrow \int_0^2 x\,dx = \frac{1}{2}x^2\Big|_0^2$
$= 2$

$2 \to 2+i$: $x = 2$, $dx = 0$, $y: 0 \to 1$

$\therefore\ i\int_0^1 2\,dy = 2i$

$\therefore\ \int_0^{2+i} Re(z)\,dz = 2 + 2i$

(ii) $z = (2+i)t$, $x = 2t$, $dz = (2+i)dt$

$\therefore\ \int_0^{2+i} Re(z)\,dz = \int_0^1 2t(2+i)\,dt$

$= 2+i$

(2) Along the unit circle, $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta$

$\sqrt{z}\big|_{z=1} = 1 \Rightarrow \theta = 0$ at $z = 1$

(i) Along the upper half of the unit circle:

$\int_C \frac{dz}{\sqrt{z}} = \int_0^{\pi} e^{-i\frac{\theta}{2}}\, i e^{i\theta}\,d\theta$

$= i\int_0^{\pi} e^{i\frac{\theta}{2}}\,d\theta$

$= 2i - 2$

(ii) Along the lower half of the unit circle:

$\int_C \frac{dz}{\sqrt{z}} = \int_0^{-\pi} e^{-i\frac{\theta}{2}}\; i e^{i\theta}\,d\theta$

$= -2i - 2$

Part 2.

When $|z|=2$, $z=2e^{i\theta}$, $dz=2ie^{i\theta}d\theta$

(1) $\oint_{|z|=2} \frac{dz}{z} = \int_0^{2\pi} \frac{1}{2} e^{-i\theta} 2ie^{i\theta} d\theta$

$= 2\pi i$

(2) $|dz| = 2d\theta$

$\oint_{|z|=2} \frac{|dz|}{z} = \int_0^{2\pi} \frac{1}{2} e^{-i\theta} 2d\theta = 0$

(3) $\oint_{|z|=2} \frac{dz}{|z|} = \frac{1}{2}\int_0^{2\pi} 2ie^{i\theta} d\theta = 0$

(4) $\oint_{|z|=2} \left|\frac{dz}{z}\right| = \int_0^{2\pi} \frac{2d\theta}{2} = 2\pi$

Part 3.

$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$

Let $f(z) = \frac{1}{2}\frac{1}{z-1}\sin\frac{\pi}{4}z$

$g(z) = \frac{1}{2}\frac{1}{z+1}\sin\frac{\pi}{4}z$

Then, $\frac{1}{z^2-1}\sin\frac{\pi}{4}z = f(z) - g(z)$.

(1) In the disk $|z|<\frac{1}{2}$, $f(z)$ and $g(z)$ are analytic.

$\therefore \oint_{|z|=\frac{1}{2}} \frac{1}{z^2-1}\sin\frac{\pi}{4}z\,dz = 0$ by Cauchy theorem.

(2) In the disk $|z-1|<1$, $g(z)$ is analytic

$$\therefore \oint_{|z-1|=1} g(z)\,dz = 0$$

And $$\oint_{|z-1|=1} f(z)\,dz = \frac{1}{2}\oint_{|z-1|=1} \frac{\sin\frac{\pi}{4}z}{z-1}\,dz$$

$$= i\pi\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}\pi i \quad \text{(by Cauchy integral formula)}$$

(3) By Cauchy integral formula.

$$\oint_{|z|=3} f(z)\,dz = \frac{1}{2}\oint_{|z|=3} \frac{\sin\frac{\pi}{4}z}{z-1}\,dz$$

$$= i\pi\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}\pi i$$

$$\oint_{|z|=3} g(z)\,dz = \frac{1}{2}\oint_{|z|=3} \frac{\sin\frac{\pi}{4}z}{z+1}\,dz$$

$$= i\pi\sin\left(-\frac{\pi}{4}\right)$$

$$= -\frac{\sqrt{2}}{2}\pi i$$

$$\therefore \oint_{|z|=3} \frac{1}{z^2-1}\sin\frac{\pi}{4}z\,dz = \oint_{|z|=3} \big(f(z)-g(z)\big)\,dz$$

$$= \sqrt{2}\pi i$$

(4) Similar to (3), the result is $\sqrt{2}\pi i$.

Part 4.

(1) $\cos z$ is analytic in the disk $|z|<2$.

By Cauchy integral formula,

$$\oint_{|z|=2} \frac{\cos z}{z} dz = 2\pi i \cos 0 = 2\pi i$$

(2) $$\oint_{|z|=2} \frac{z^2-1}{z^2+1} dz$$

$$= \frac{1}{2i} \oint_{|z|=2} \left(\frac{z^2-1}{z-i} - \frac{z^2-1}{z+i} \right) dz$$

$$= \pi \left[(i^2-1) - \left((-i)^2 - 1 \right) \right]$$

$$= 0$$

(3) $$\oint_{|z|=2} \frac{\sin(e^z)}{z} dz = 2\pi i \sin(e^0)$$

$$= 2\pi \sin(1)\, i$$

(4) $$\oint_{|z|=2} \frac{e^z}{\cosh(z)} dz = \oint_{|z|=2} \frac{2e^z}{e^z + e^{-z}} dz$$

Let $t = e^z$, $dt = e^z dz \Rightarrow dz = \frac{dt}{t}$

$$\therefore \text{Integral} = \oint_{C_t} \frac{2t}{t^2+1} dt$$

$$= -i \oint_{C_t} \left(\frac{t}{t-i} - \frac{t}{t+i} \right) dt$$

The integral path C_t:

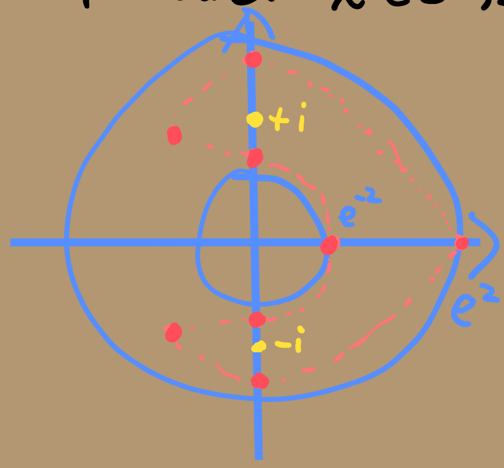
When $|z|=2$, we have $x^2+y^2=4$ and $x \in [-2,2]$

$|t| = |e^{x+iy}| = e^x \in [e^{-2}, e^2]$

With the constraint $x^2+y^2=2$,

one can see that the path

will pass the following prints:



$\left(0,\ e^{\pm\sqrt{16-\pi^2}/2}\right)$, $\left(e^{\pm 2}, 0\right)$, $\left(0,\ -e^{\pm\sqrt{16-\pi^2}/2}\right)$, $\left(\cos(\pm 2),\ \sin(\pm 2)\right)$, which are the red points in the figure.

$$\therefore \text{ Integral} = -i \oint_{C_t} \left(\frac{t}{t-i} - \frac{t}{t+i}\right) dt$$

$$= -i\, 2\pi i\, [i - (-i)]$$

$$= 4\pi i$$

(5) $$f^{(n)}(0) = \frac{n!}{2\pi i} \oint_{|z|=2} \frac{f(z)}{z^{n+1}}\, dz$$

where $f(z) = \sin z$ and $n = 1$

$\therefore$ The integral $= 2\pi i \cos(0) = 2\pi i$

(6) $$\oint_{|z|=2} \frac{|z|\, e^z}{z^2}\, dz = \oint_{|z|=2} \frac{2e^z}{z^2}\, dz$$

By higher-order Cauchy integral formula,

it equals $2 \cdot 2\pi i\, e^0 = 4\pi i$

(7) $$\oint_{|z|=2} \frac{\sin z}{z^4}\, dz = \frac{2\pi i}{3!} \left(\sin z\right)^{(3)}\Big/_{z=0}$$

$$= -\frac{1}{3}\pi i$$

(8) $$\oint_{|z|=2} \frac{dz}{z^2(z^2+16)} = \frac{1}{16} \oint_{|z|=2} \left(\frac{1}{z^2} - \frac{1}{z^2+16}\right) dz$$

$$= \frac{1}{16} \oint_{|z|=2} \frac{1}{z^2}\, dz$$

since $\frac{1}{z^2+16}$ is analytic in the region $|z|<2$.

With the higher-order formula, we have,

$$\text{integral} = \frac{1}{16} 2\pi i \cdot 0 = 0$$

Part 5.

(1) $$\oint_{|z|=1} \frac{e^z}{z^3} dz = \frac{2\pi i}{2!} e^0 = \pi i$$

(2) $F(z) = \int_{z_0}^{z} e^{\xi} \left(\frac{1}{\xi} + \frac{a}{\xi^3} \right) d\xi$ is single-valued

$\Rightarrow$ when $z = |z| e^{i\theta} \to z = |z| e^{i(\theta + 2\pi)}$,

$F(z)$ is unchanged.

$$\therefore \int_{z_0}^{|z|e^{i\theta}} e^{\xi} \left(\frac{1}{\xi} + \frac{a}{\xi^3} \right) d\xi = \int_{z^0}^{|z|e^{i(\theta+2\pi)}} e^{\xi} \left(\frac{1}{\xi} + \frac{a}{\xi^3} \right) d\xi$$

$$\therefore \int_{|z|e^{i\theta}}^{|z|e^{i(\theta+2\pi)}} e^{\xi} \left(\frac{1}{\xi} + \frac{a}{\xi^3} \right) d\xi$$

$$\Rightarrow 2\pi i e^0 + \frac{2\pi i}{2!} a e^0 = 0$$

$$\Rightarrow a = -2$$