Mathematical Methods in Physics

Instructor: Shiqing Xu

Office: Room 609, No. 9 Innovation Park

Email: xusq3@sustech.edu.cn

Tel: 0755-88018653

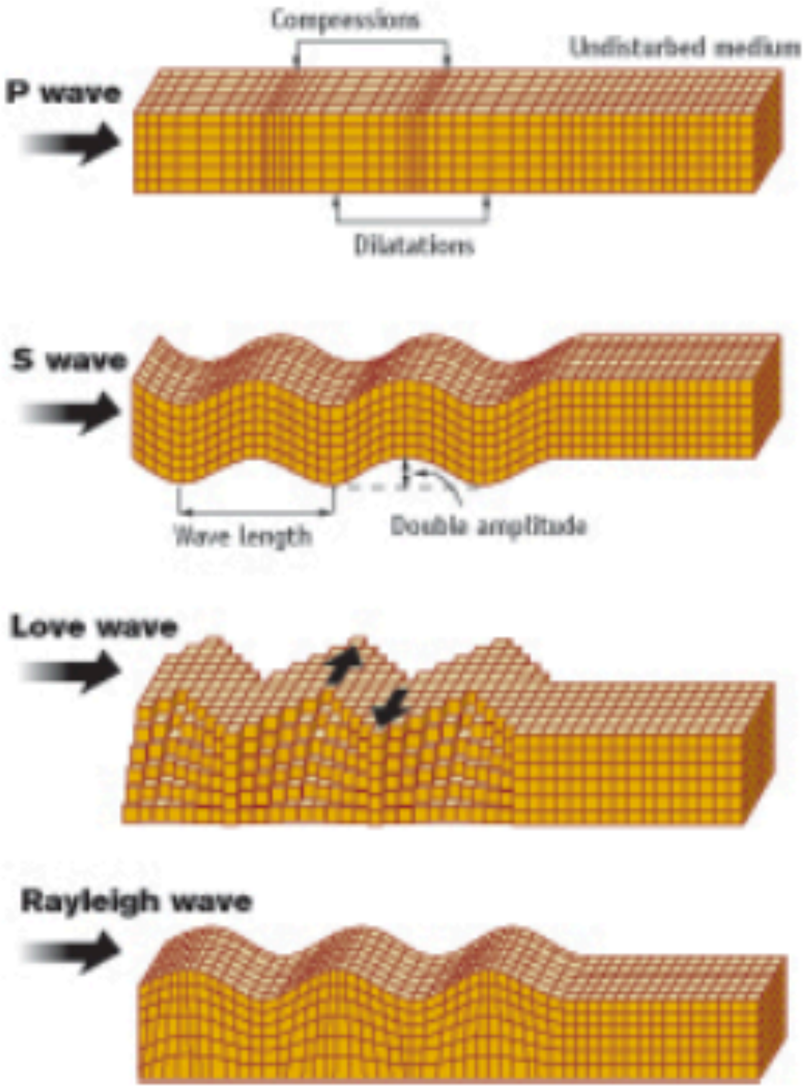
Review

- PDEs
- Some examples
- Linear operator and decomposition of linear operator
- General solutions and particular solutions

Chapter – 03: Partial Differential Equations (PDEs)

- We will continue to discuss PDEs.
- We will be focusing on some specific types of PDEs: the wave equation and the heat equation (or the diffusion equation).
- You will learn how to formulate a problem.

The wave equation



Different types of wave during an earthquake

https://www.iris.edu/gallery3/general/posters/exploring_earth/WaveIllustration

The wave equation (transverse wave)

Let's try to work out the problem associated with string vibration.

Assume the two ends of a soft string are $x = 0, \text{ and } x = l$

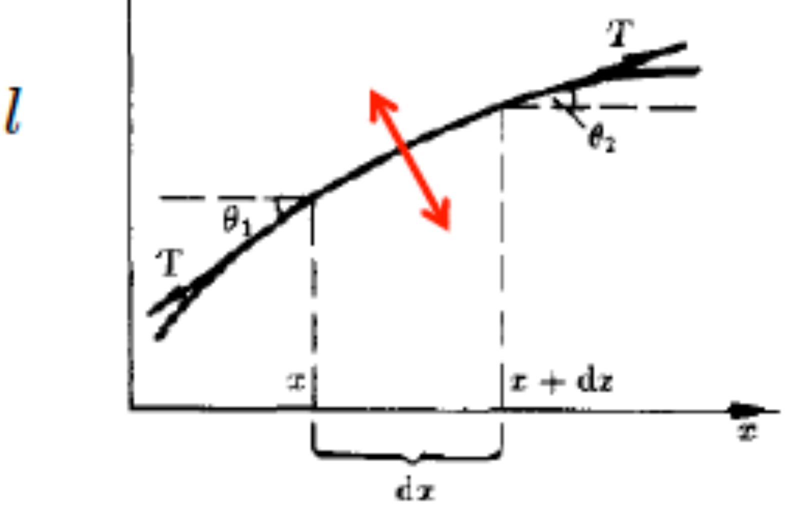
Every point along the string can be associated with a unique coordinate *x*.

u(*x*, *t*) represents the ***transverse displacement*** of point *x* at time *t*. We further assume that *u*(*x*, *t*) is very small.

$$\tan\theta(x,t) = \frac{\partial u}{\partial x}|_x$$

String is soft, so there is no stress along the normal direction.

We can also ignore gravity (elastic force is much larger than gravitational force).



Force balance

$$(T\cos\theta)_{x+\Delta x} - (T\cos\theta)_x = 0$$ Horizontal

$$(T\sin\theta)_{x+\Delta x} - (T\sin\theta)_x = \rho\Delta x\frac{\partial^2 u}{\partial t^2}$$ Vertical

The wave equation (transverse wave)

String is soft, so there is no stress along the normal direction.

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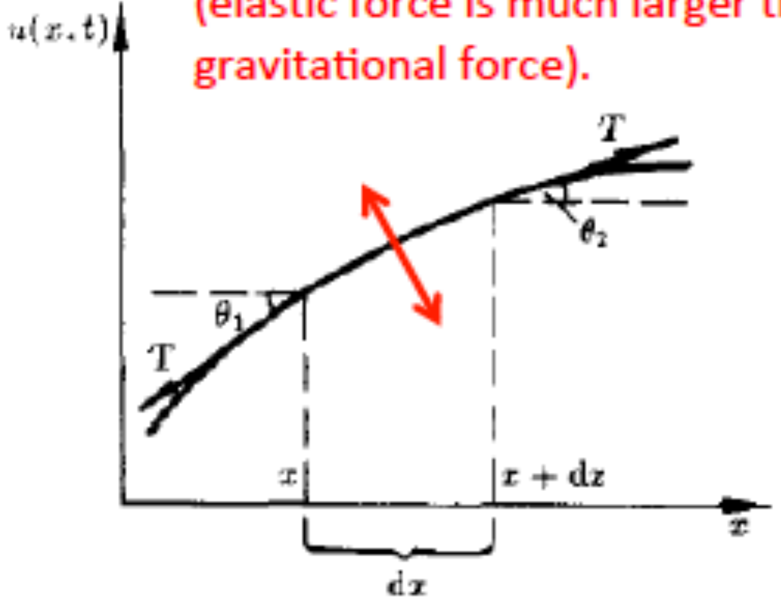


Force balance

$$(T\cos\theta)_{x+\Delta x} - (T\cos\theta)_x = 0 \qquad \text{Horizontal}$$

$$(T\sin\theta)_{x+\Delta x} - (T\sin\theta)_x = \rho\Delta x\overline{\frac{\partial^2 u}{\partial t^2}} \qquad \text{Vertical}$$

Under the assumption of small displacement $|\frac{\partial u}{\partial x}| \ll 1$

$$\left.\begin{aligned} &\cos\theta \approx 1 \\ &\sin\theta \approx \tan\theta = \frac{\partial u}{\partial x} \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} &T|_{x+\Delta x} = T|_x \\ &T\underbrace{\left(\frac{\partial u}{\partial x}|_{x+\Delta x} - \frac{\partial u}{\partial x}|_x\right)}_{\frac{\partial^2 u}{\partial x^2}|_x \cdot \cancel{\Delta x}} = \rho\cancel{\Delta} x\overline{\frac{\partial^2 u}{\partial t^2}} \end{aligned}\right.$$

$$\xrightarrow{\Delta x \to 0} \quad \boxed{T\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}}$$

The wave equation (transverse wave)

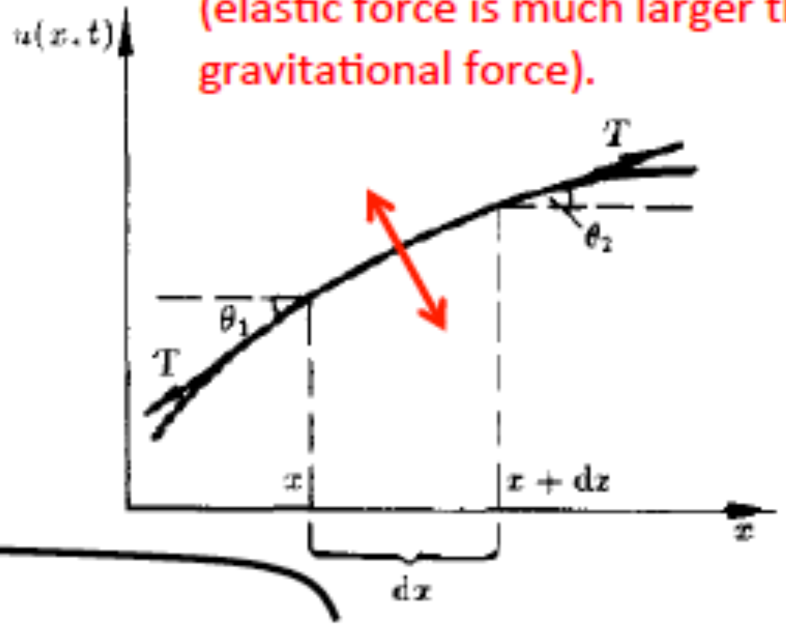
String is soft, so there is no stress along the normal direction.

We can also ignore gravity (elastic force is much larger than gravitational force).

Force balance

$$(T\cos\theta)_{x+\Delta x} - (T\cos\theta)_x = 0 \qquad \text{Horizontal}$$

$$(T\sin\theta)_{x+\Delta x} - (T\sin\theta)_x = \rho\Delta x\overline{\frac{\partial^2 u}{\partial t^2}} \qquad \text{Vertical}$$



Under the assumption of small displacement $|\frac{\partial u}{\partial x}| \ll 1$

$$T\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2} \Rightarrow \frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0$$

The wave equation (homogeneous PDE)
Free vibration, **no external force**

$$a = \sqrt{T/\rho}$$

Wave propagation velocity

u(*x*,*t*) is along the transverse (vertical) direction, while propagation is along the *x* direction. **Transverse wave**

The wave equation (transverse wave)

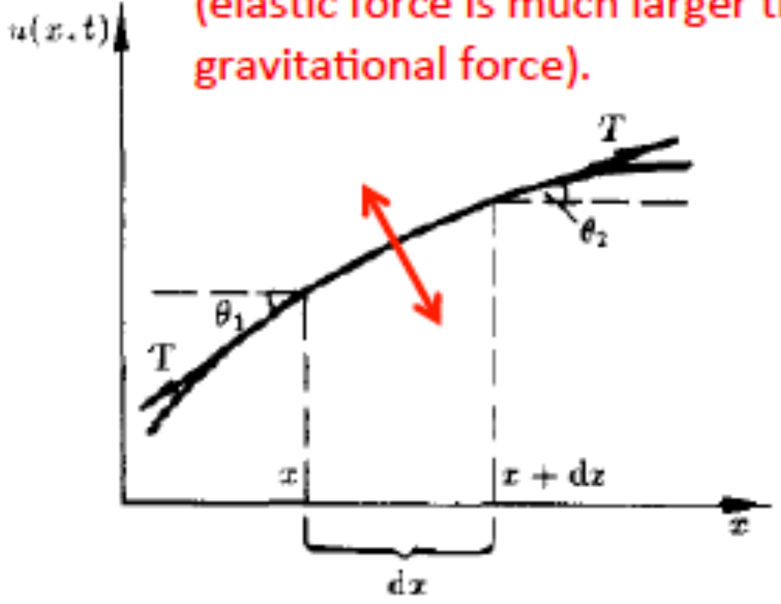
String is soft, so there is no stress along the normal direction.

We can also ignore gravity (elastic force is much larger than gravitational force).

If there is an external force, say f (for a unit length of the string)

$$\Delta s - \Delta x = \sqrt{\Delta u^2 + \Delta x^2} - \Delta x$$

$$\approx \left[\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} - 1\right] \Delta x = O\left(\left(\frac{\partial u}{\partial x}\right)^2\right)$$



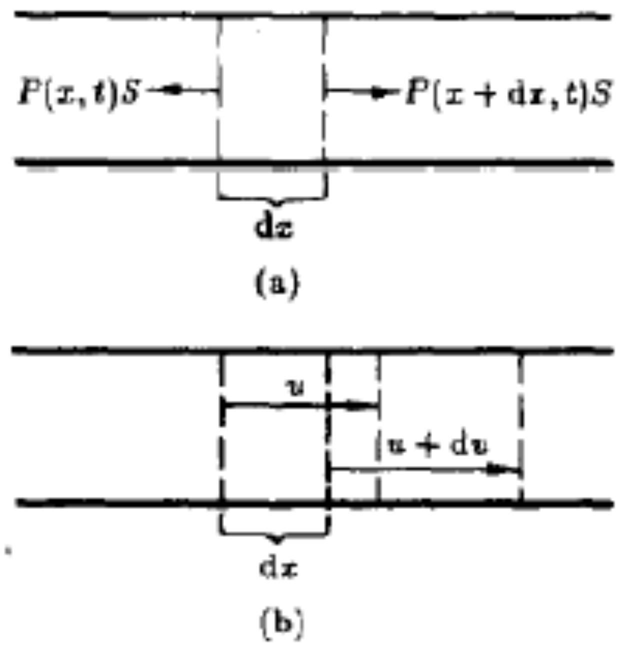
$$\Delta s \approx \Delta x$$

$$\rho \Delta x \overline{\frac{\partial^2 u}{\partial t^2}} = T\left[\frac{\partial u}{\partial x}|_{x+\Delta x} - \frac{\partial u}{\partial x}|_{x}\right] + f\Delta x$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = \frac{f}{\rho}}$$

The wave equation (inhomogeneous PDE) with external force

The wave equation (longitudinal wave)



Let's consider the vibration inside a 1D bar.

Similar to the case analyzed before, we assume every point inside the bar is associated with a unique coordinate *x*.

Displacement *u*(*x*,*t*) is **along the *x* direction.**

Wave propagates along the *x* direction

$$\rho \cancel{S \Delta x} \overline{\frac{\partial^2 u}{\partial t^2}} = \underbrace{[P(x+\Delta x, t) - P(x,t)]}_{\frac{\partial P}{\partial x}\cancel{\Delta x}} \cancel{S}$$

Force balance along x direction

$$\boxed{P|_x = E\frac{\partial u}{\partial x}}$$

Hooke's law, *E*=Young's modulus

$$\Delta x \to 0$$

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial P}{\partial x}$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0}$$

$$a = \sqrt{E/\rho}$$

Wave propagation velocity

The wave equation (in 3D)

$$\boxed{\frac{\partial^2 u}{\partial t^2} - a^2 \nabla^2 u = 0}$$

$$\nabla^2 u = \nabla \cdot (\nabla u)$$

Inner product

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Laplace operator

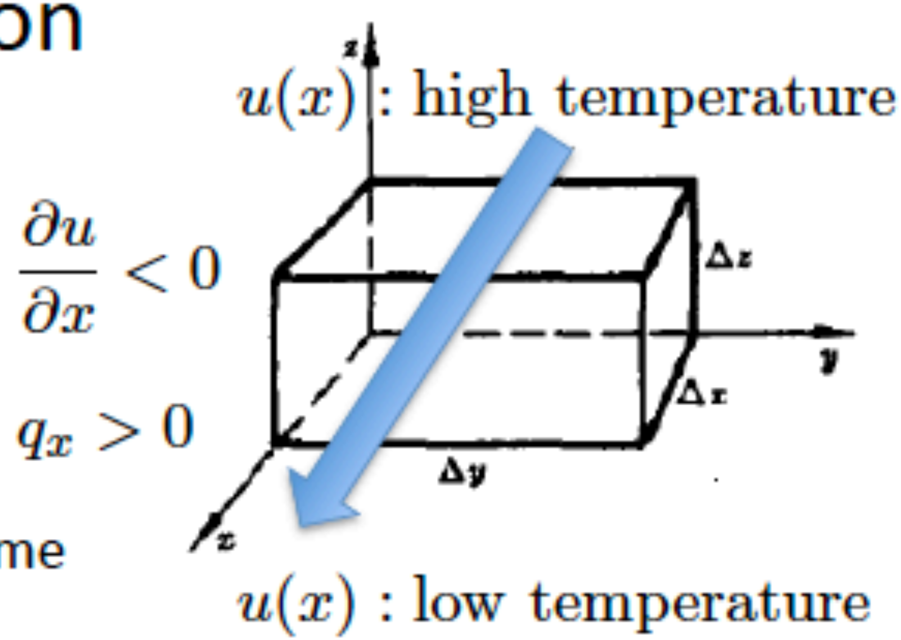
The heat equation

Let's consider another type of problem: heat transfer

Fourier's law

$$q_x = -k\frac{\partial u}{\partial x}$$ Heat flux along *x* direction

Heat per unit area of $\Delta y \Delta z$ per unit time



k: thermal conductivity; *u*: temperature

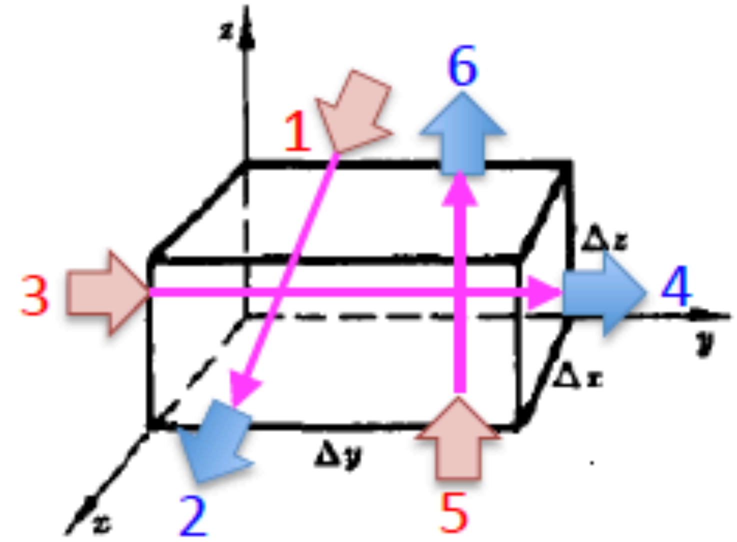
The negative sign implies the heat flux is from high temperature to low temperature

$$q_x = -k\frac{\partial u}{\partial x},\ q_y = -k\frac{\partial u}{\partial y},\ q_z = -k\frac{\partial u}{\partial z}$$ in 3D

K: 3 x 3 matrix

For isotropic $\boxed{\mathbf{q} = -k\nabla u}$ vector For anisotropic $\boxed{\mathbf{q} = -\mathbf{K}\cdot\nabla u}$

The heat equation



The total heat flux transferred ***into*** the cuboid over time Δt

1-2: $(q_x|_x - q_x|_{x+\Delta x})\Delta y \Delta z \Delta t = -\frac{\partial q_x}{\partial x}\Delta x \Delta y \Delta z \Delta t$

3-4: $(q_y|_y - q_y|_{y+\Delta y})\Delta x \Delta z \Delta t = -\frac{\partial q_y}{\partial y}\Delta y \Delta x \Delta z \Delta t$

5-6: $(q_z|_z - q_z|_{z+\Delta z})\Delta x \Delta y \Delta t = -\frac{\partial q_z}{\partial z}\Delta z \Delta x \Delta y \Delta t$

Consider the conservation of total energy

Mass density

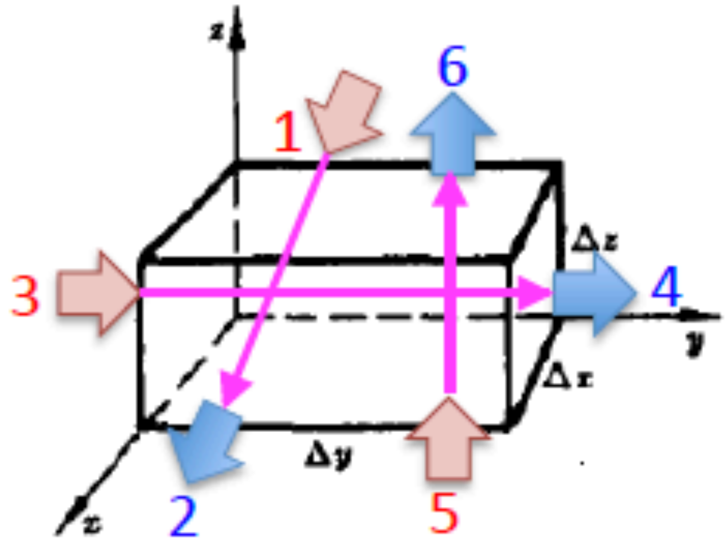
Temperature change

$$-\nabla \cdot \mathbf{q} \Delta x \Delta y \Delta z \Delta t + \underline{F(x,y,z,t)} \Delta x \Delta y \Delta z \Delta t = \rho \Delta x \Delta y \Delta z \cdot c \cdot \Delta u$$

Heat production (***source term***)
per unit volume per unit time

Heat capacity

The heat equation



Consider the conservation of total energy

$$-\nabla \cdot \mathbf{q}\Delta x\Delta y\Delta z\Delta t + F(x,y,z,t)\Delta x\Delta y\Delta z\Delta t = \rho\Delta x\Delta y\Delta z \cdot c \cdot \Delta u$$

$$\frac{\partial(\rho c u)}{\partial t} + \nabla \cdot \mathbf{q} = F(x,y,z,t)$$

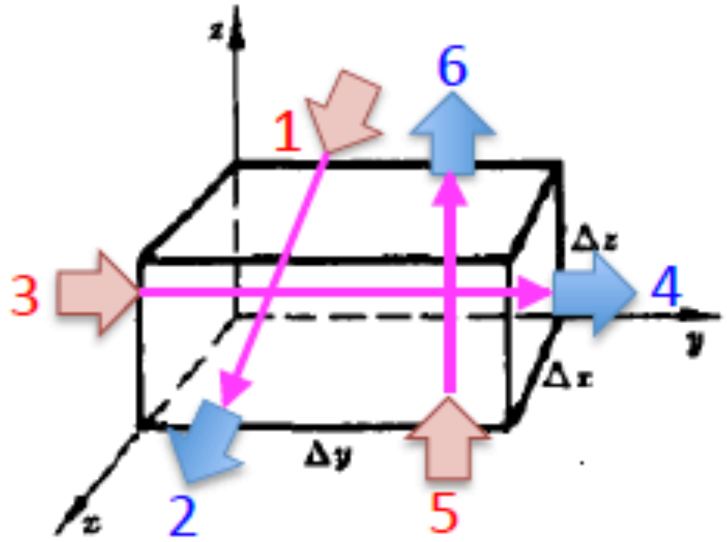
For isotropic medium: $\dfrac{\partial u}{\partial t} - \kappa\nabla^2 u = \dfrac{F}{\rho c} = f$

$\kappa = k/(\rho c)$

For anisotropic medium: $\dfrac{\partial(\rho c u)}{\partial t} - \nabla \cdot (\mathbf{K} \cdot \nabla u) = F(x,y,z,t)$

The heat equation

$$\boxed{\frac{\partial u}{\partial t} - \kappa \nabla^2 u = \frac{F}{\rho c} = f \quad \text{Inhomogeneous heat equation}}$$



If there is ***no source/sink term*** inside the cuboid, $F = 0$

$$\boxed{\frac{\partial u}{\partial t} - \kappa \nabla^2 u = 0 \quad \text{Homogeneous heat equation}}$$

Like the heat problem, the flux of fluid (or molecules) follows a similar law: **Fick's law**

$$\boxed{\frac{\partial u}{\partial t} - D \nabla^2 u = f(x, y, z, t)}$$

u: density/concentration (浓度) of molecules

D: diffusivity

The diffusion equation

Problem in steady state

Note: both the wave equation and the heat equation depend on time.

If the problem does not depend on time, it is said to be in steady state: $\frac{\partial u}{\partial t} = 0$

For example, after a long enough time, equilibrium in temperature can be reached.

Poisson equation

$$\nabla^2 u = -\frac{f}{\kappa}$$

Laplace equation

$$\nabla^2 u = 0$$

Classification

- The wave equation (***hyperbolic equation***): $$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0$$

  2nd order in time and space

- The heat equation (***parabolic equation***): $$\frac{\partial u}{\partial t} - \kappa \nabla^2 u = 0$$

  1st order in time, 2nd order in space

- Poisson equation (***elliptical equation***): $$\nabla^2 u = -\frac{f}{\kappa}$$

  0th order in time, 2nd order in space

Some remarks

- Differential equation alone is not enough for fully determining the solution, other conditions are also needed, such as ***the initial condition*** and/or ***the boundary condition***.

$$\frac{d^2u}{dt^2} = a,\ u|_{t=0} = u_0,\ \frac{du}{dt}|_{t=0} = v_0$$ Newton's second law

$$u = u_0 + v_0 t + \frac{1}{2}at^2$$

- Also note that differential equation only applies to the interior of the considered region. The behavior along the boundary has to be determined separately.

Some remarks

- Differential equation alone is not enough for fully determining the solution, other conditions are also needed, such as ***the initial condition*** and/or ***the boundary condition***.

The wave equation is 2nd order in time

$$u|_{t=0} = \phi(x,y,z),\ \frac{\partial u}{\partial t}|_{t=0} = \psi(x,y,z),\ (x,y,z) \in \overline{V}$$

The heat equation is 1st order in time

$$u|_{t=0} = \phi(x,y,z),\ (x,y,z) \in \overline{V}$$

Some remarks

- Differential equation alone is not enough for fully determining the solution, other conditions are also needed, such as ***the initial condition*** and/or ***the boundary condition***.

The wave equation (transverse wave)

$$u|_{x=0} = 0,\ u|_{x=l} = 0,\ t \geq 0$$

Fixed at the two ends

Some remarks

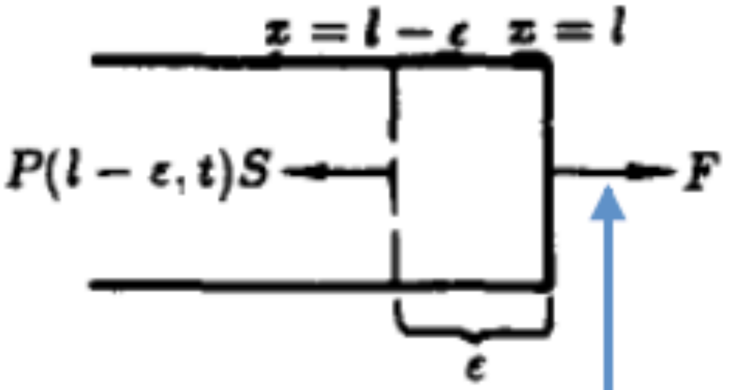
- Differential equation alone is not enough for fully determining the solution, other conditions are also needed, such as ***the initial condition*** and/or ***the boundary condition***.

The wave equation (longitudinal wave)

- Fixed at one end

$$\boxed{u|_{x=0} = 0}$$

- Force balance at the other end



$$\rho\varepsilon S\overline{\frac{\partial^2 u}{\partial t^2}} = F(t)S - P(l - \varepsilon, t)S \xrightarrow{\varepsilon \to 0} \boxed{\frac{\partial u}{\partial x}|_{x=l} = \frac{1}{E}F(t)}$$

$$P|_x = E\frac{\partial u}{\partial x}$$

Traction at the boundary

Some remarks

- Differential equation alone is not enough for fully determining the solution, other conditions are also needed, such as ***the initial condition*** and/or ***the boundary condition***.

The wave equation (longitudinal wave)

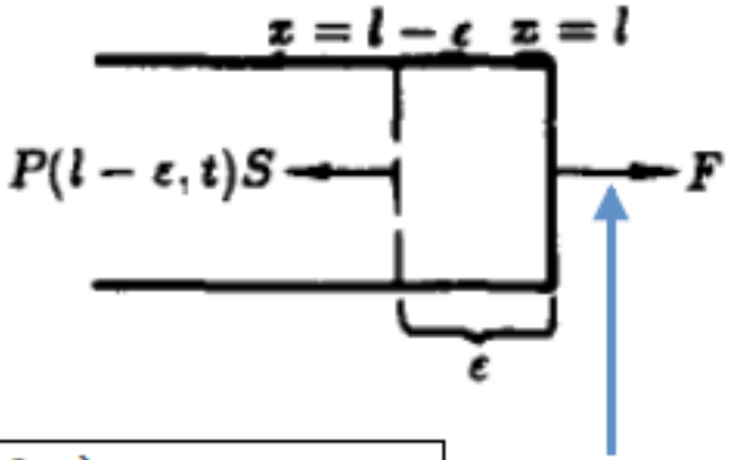
- If the traction is zero

$$\boxed{\frac{\partial u}{\partial x}|_{x=l} = 0}$$

- If the traction is given by a spring

$$F(t)S = -k\left[u(l,t) - u_0\right] \Rightarrow \boxed{\left(ku + ES\frac{\partial u}{\partial x}\right)|_{x=l} = ku_0}$$

u_0: the equilibrium position of the spring



Traction at the boundary

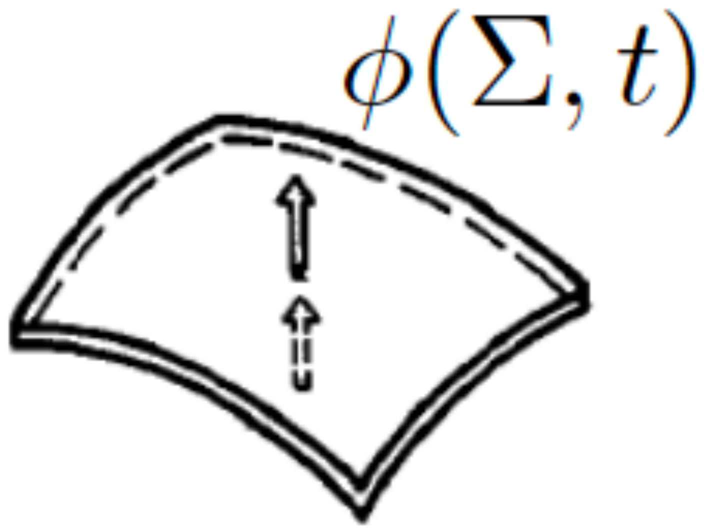
Some remarks

- Differential equation alone is not enough for fully determining the solution, other conditions are also needed, such as ***the initial condition*** and/or ***the boundary condition***.

The heat equation

- If the temperature is known at the boundary

$$\boxed{u|_\Sigma = \phi(\Sigma, t)}$$



Some remarks

- Differential equation alone is not enough for fully determining the solution, other conditions are also needed, such as ***the initial condition*** and/or ***the boundary condition***.

The heat equation

- If the heat flux is known at the boundary

$$\boxed{-k\frac{\partial u}{\partial n}|_{\Sigma} = \psi(\Sigma, t)}$$

$$\frac{\partial}{\partial n} = \mathbf{n} \cdot \nabla$$

$$\psi(\Sigma, t)$$

$$\text{thickness} = \varepsilon \to 0$$

$$m \to 0$$

No heat production or loss inside the thin layer, no flux along other boundaries

$$q_n = -k\frac{\partial u}{\partial n}|_{\Sigma - \varepsilon}$$

Some remarks

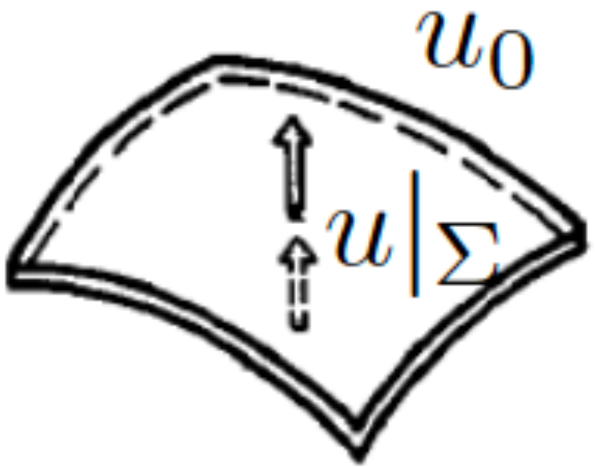
- Differential equation alone is not enough for fully determining the solution, other conditions are also needed, such as ***the initial condition*** and/or ***the boundary condition***.

The heat equation

- If the heat flux across the boundary follows the cooling law

$$\boxed{-k\frac{\partial u}{\partial n}|_\Sigma = H(u|_\Sigma - u_0)}$$

$$\text{thickness} = \varepsilon \to 0$$
$$m \to 0$$



$$q_n = -k\frac{\partial u}{\partial n}|_{\Sigma-\varepsilon}$$

Some remarks

- Differential equation alone is not enough for fully determining the solution, other conditions are also needed, such as ***the initial condition*** and/or ***the boundary condition***.

Classification of boundary conditions

- First kind (Dirichlet): $u|_{\Sigma} = \phi(\Sigma, t)$
- Second kind (Neumann): $\frac{\partial u}{\partial n}|_{\Sigma} = \psi(\Sigma, t)$
- Third kind (mixed): $\left(A \cdot u + B \cdot \frac{\partial u}{\partial n}\right)|_{\Sigma} = F(\Sigma, t)$

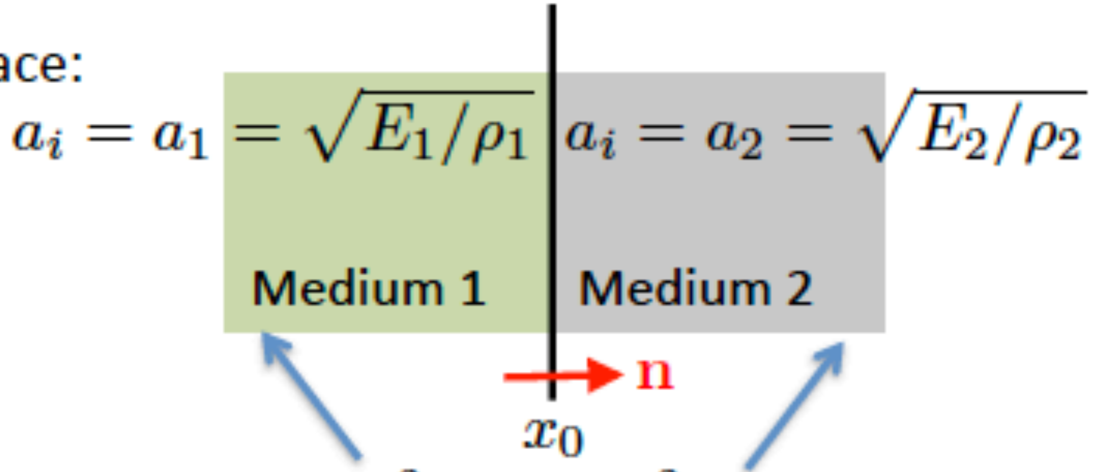
Some remarks

- So far discussion is made for a bounded region.
- Sometimes we need to discuss the behavior at the infinity.
- Sometimes there could be a discontinuity (or jump) across an interface ***inside the region***. It is difficult to apply PDE directly to such discontinuity; instead, we often require the following conditions across the interface:
  - Continuity in displacement (along ***n*** direction)

$$u_1(x,t)|_{x=x_0-0} = u_2(x,t)|_{x=x_0+0}$$

  - Continuity in stress (along ***n*** direction)

$$\left[E_1 \frac{\partial u_1(x,t)}{\partial x}\right]|_{x=x_0-0} = \left[E_2 \frac{\partial u_2(x,t)}{\partial_x}\right]|_{x=x_0+0}$$

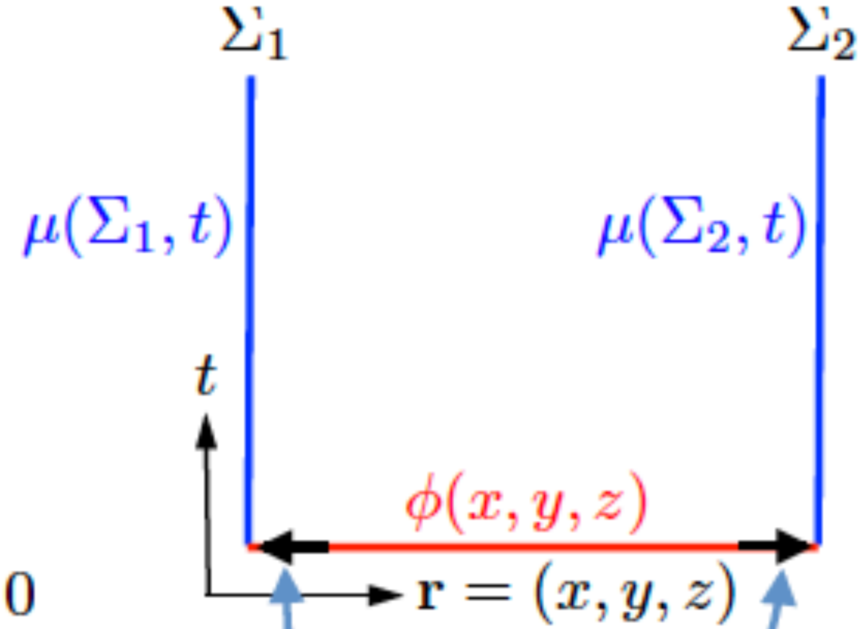


$$\frac{\partial^2 u}{\partial t^2} - a_i^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Well posedness (适定性)

- Let's consider the following problem, where PDE, initial and boundary conditions are given: $$\frac{\partial u}{\partial t} - \kappa \nabla^2 u = f(x,y,z,t),\ (x,y,z,t) \in V,\ t > 0$$
$$u|_{\Sigma} = \mu(\Sigma, t),\quad t \geq 0$$
$$u|_{t=0} = \phi(x,y,z),\quad (x,y,z) \in \overline{V}$$

$\overline{V} = V + \Sigma$



We assume that $f(x,y,z,t),\ \mu(\Sigma,t),\ \phi(x,y,z)$ are continuous, then the solution should satisfy:

(1) u is continuous over $(x,y,z) \in V,\ t > 0$

(2) $\partial u/\partial t,\ \partial^2 u/\partial x_i^2$ are continuous over $(x,y,z) \in V,\ t > 0$

(3) u satisfies the PDE over $(x,y,z) \in V,\ t > 0$

(4) u satisfies the boundary condition for $t > 0,$ and $t \to 0^+$

(5) u satisfies the initial condition for $(x,y,z) \in V,$ and $\boxed{(x,y,z) \to \Sigma^-}$

Approach the boundary from inside

Well posedness (适定性)

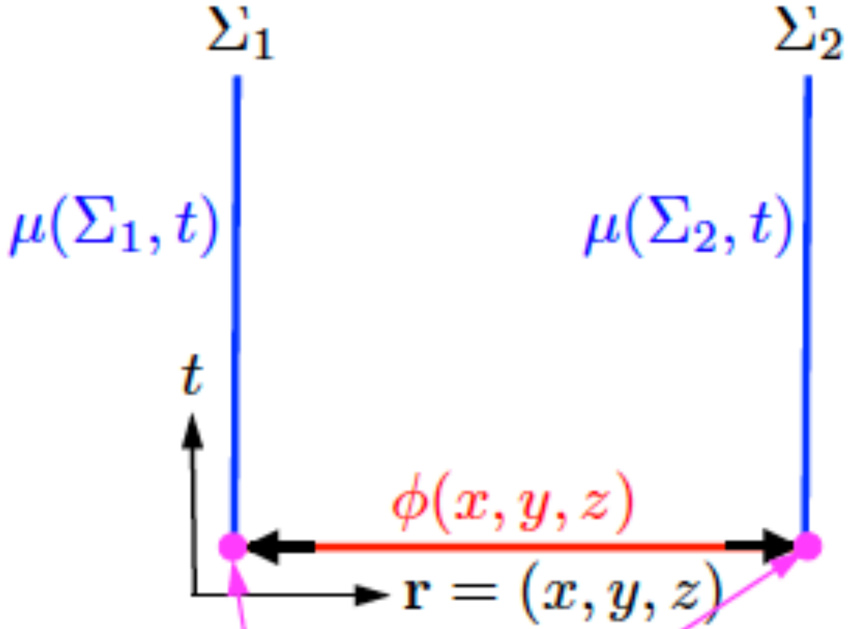
- Let's consider the following problem, where PDE, initial and boundary conditions are given:

$$\frac{\partial u}{\partial t} - \kappa \nabla^2 u = f(x,y,z,t), \; (x,y,z,t) \in V, \; t > 0$$

$$u|_{\Sigma} = \mu(\Sigma, t), \quad t \geq 0$$

$$u|_{t=0} = \phi(x,y,z), \quad (x,y,z) \in \overline{V}$$

$$\overline{V} = V + \Sigma$$



We want to know:

- Whether the solution exists
- Whether there exists a unique solution
- Whether the solution is stable

Well posedness

well-posed or ill-posed?

The dependence of solution on parameters related to PDE and initial/boundary conditions

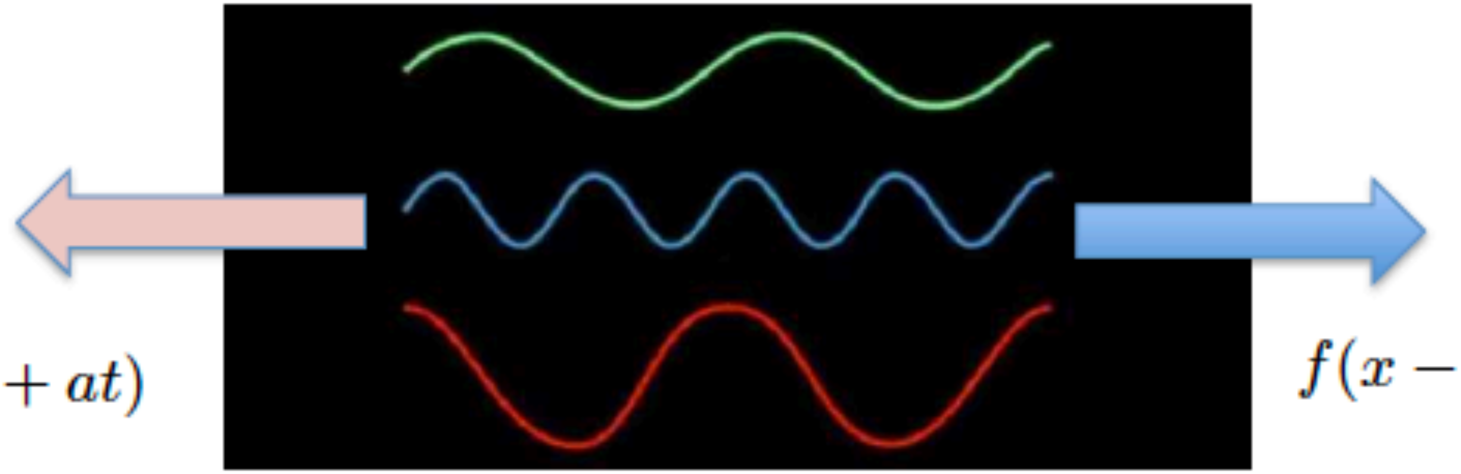
$$\kappa, \; f, \; \mu, \; \phi$$

$$\boxed{\mu(\Sigma, t)|_{t=0} = \phi(x,y,z)|_{\Sigma}}$$

Traveling wave solution to the wave equation

- **Traveling wave (行波)**

  The wave propagates

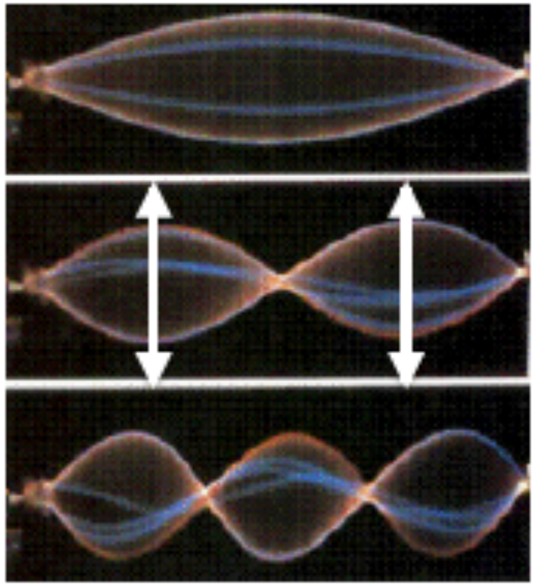


$g(x+at)$ $f(x-at)$

- **Standing wave (驻波)**

  The wave does not “propagate”

  (superposition of two propagating waves in opposite directions)



Traveling wave solution to the wave equation

- **Traveling wave (行波)**

  The wave propagates

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \ t > 0 \\ u(x,t)|_{t=0} = \phi(x), \ -\infty < x < \infty \\ \dfrac{\partial u}{\partial t}|_{t=0} = \psi(x), \ -\infty < x < \infty \\ \boxed{u(x,t)|_{x\to\pm\infty} \to 0 \quad \text{or is bounded}} \end{cases}$$

$$u(x,t) = f(x - at) + g(x + at)$$

Apply the initial condition

$$\begin{cases} f(x) + g(x) = \phi(x) \\ a[f'(x) - g'(x)] = -\psi(x) \end{cases} \Rightarrow f(x) - g(x) = -(1/a)\int_0^x \psi(\xi)d\xi + C$$

$$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2a}\int_0^x \psi(\xi)d\xi + \frac{C}{2}, \ g(x) = \frac{1}{2}\phi(x) + \frac{1}{2a}\int_0^x \psi(\xi)d\xi - \frac{C}{2}$$

Traveling wave solution to the wave equation

- **Traveling wave (行波)**

  The wave propagates

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty,\ t > 0 \\ u(x,t)|_{t=0} = \phi(x), \ -\infty < x < \infty \\ \dfrac{\partial u}{\partial t}|_{t=0} = \psi(x), \ -\infty < x < \infty \\ \boxed{u(x,t)|_{x\to\pm\infty} \to 0 \quad \text{or is bounded}} \end{cases}$$

$$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2a}\int_0^x \psi(\xi)d\xi + \frac{C}{2}$$

$$g(x) = \frac{1}{2}\phi(x) + \frac{1}{2a}\int_0^x \psi(\xi)d\xi - \frac{C}{2}$$

Traveling wave solution or d'Alembert solution

$$\boxed{u(x,t) = f(x-at) + g(x+at) = \frac{1}{2}[\phi(x-at) + \phi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)d\xi}$$

Traveling wave solution to the wave equation

$$u(x,t)|_{t=0} = \phi(x), \ -\infty < x < \infty$$

$$\frac{\partial u}{\partial t}|_{t=0} = \psi(x), \ -\infty < x < \infty$$

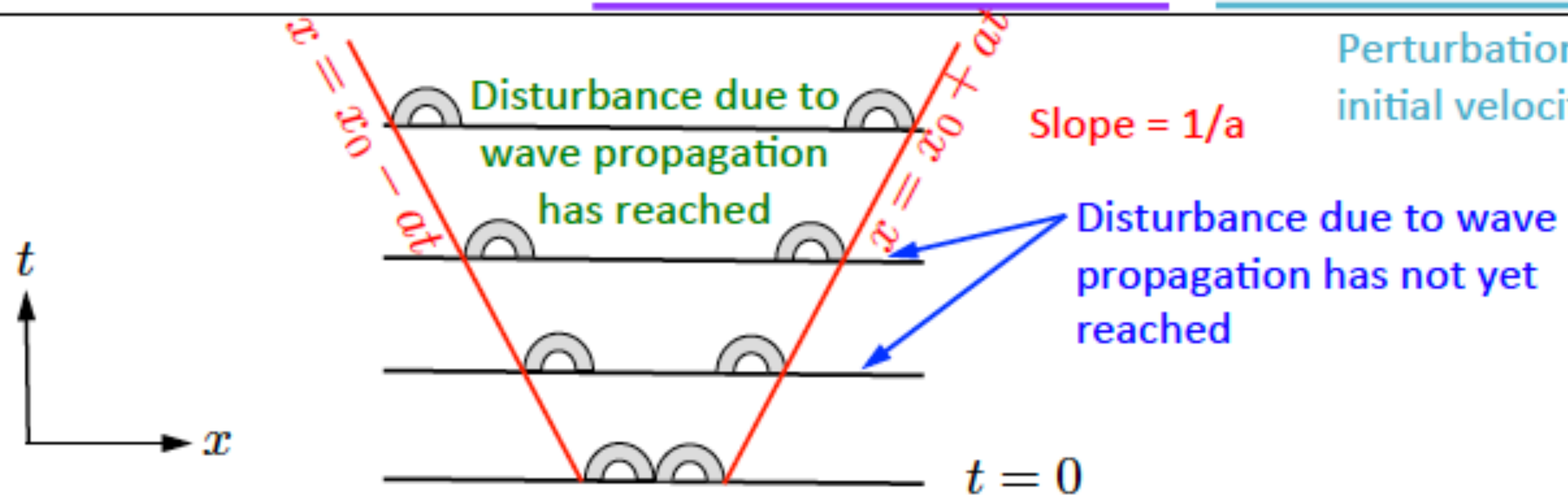
- **Traveling wave (行波)**

  The wave propagates

Perturbation due to initial displacement

$$\boxed{u(x,t) = f(x-at) + g(x+at) = \underbrace{\frac{1}{2}[\phi(x-at) + \phi(x+at)]} + \underbrace{\frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)d\xi}}$$

Perturbation due to initial velocity



Traveling wave solution to the wave equation

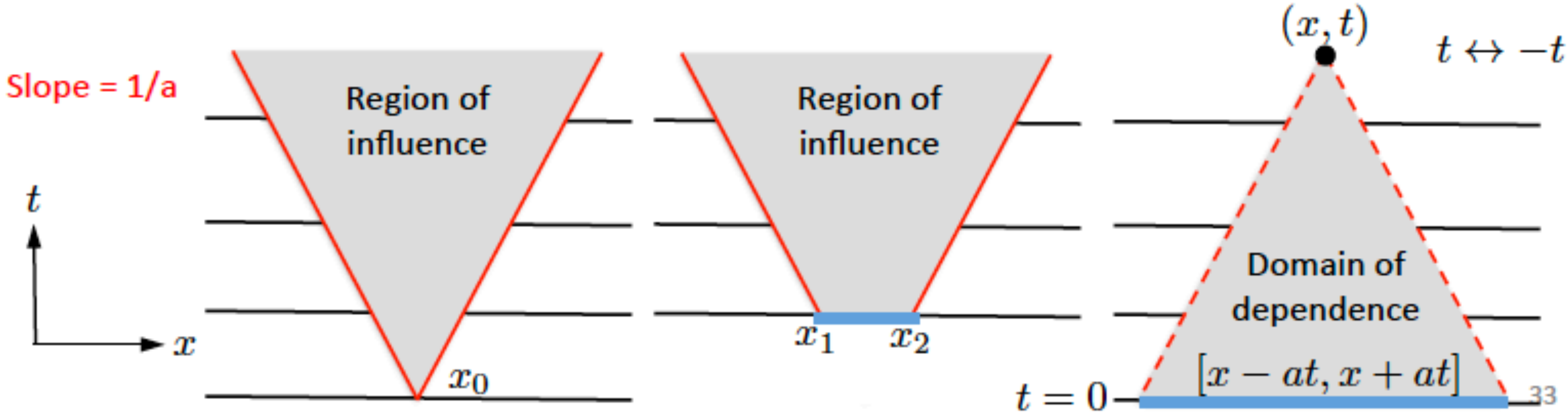
- **Traveling wave (行波)**

  The wave propagates

$$u(x,t)|_{t=0} = \phi(x),\ -\infty < x < \infty$$

$$\frac{\partial u}{\partial t}|_{t=0} = \psi(x),\ -\infty < x < \infty$$

$$\boxed{u(x,t) = f(x-at) + g(x+at) = \frac{1}{2}[\phi(x-at) + \phi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)d\xi}$$



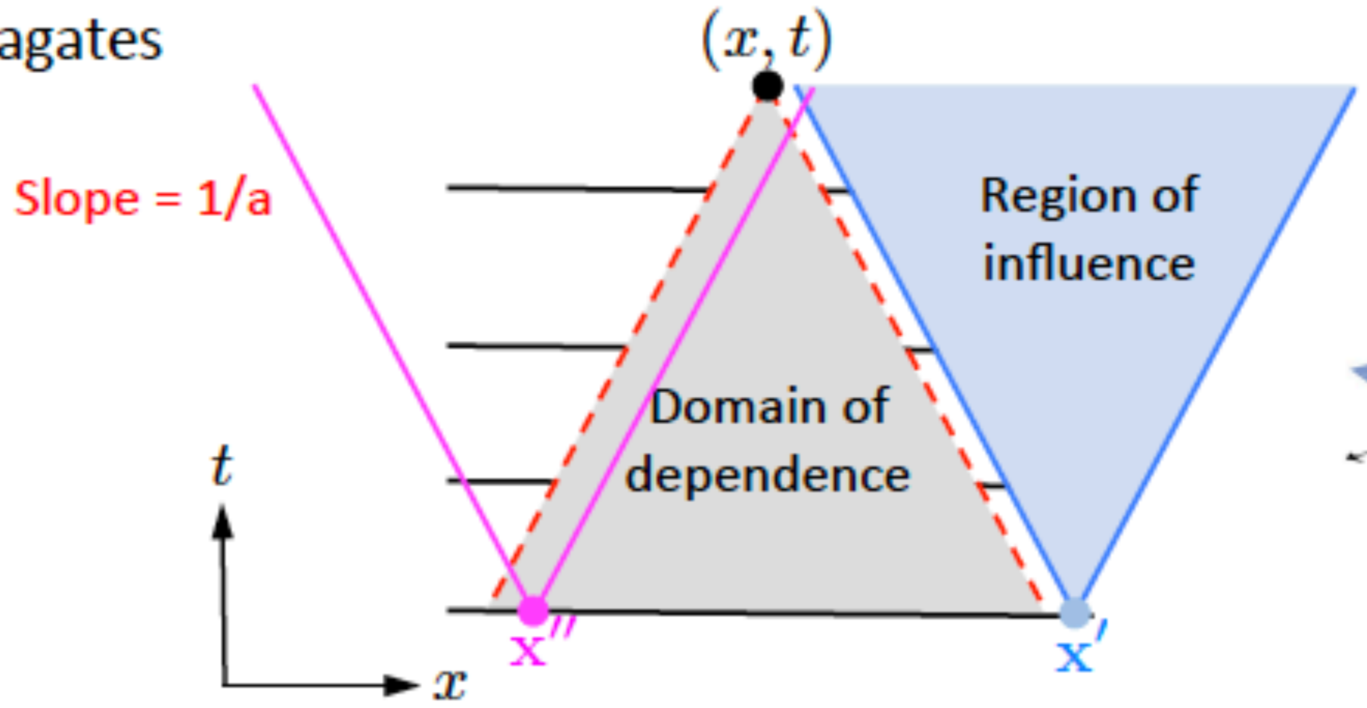
Traveling wave solution to the wave equation

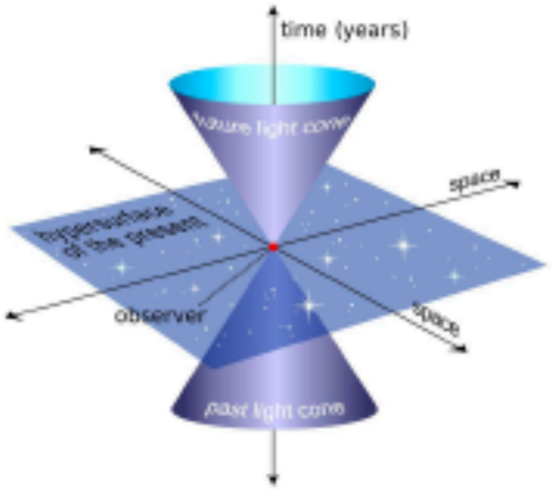
- **Traveling wave (行波)**

  The wave propagates

Relativity: event horizon
light cone

The propagation velocity is finite.





x'' can influence **(x,t)**, while x' cannot. All the points (or regions) that can influence **(x,t)** constitute the domain of dependence of **(x,t)**.

Solution to the heat equation

- Now let's consider the heat equation in 1D.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \kappa \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u|_{t=0} = \phi(x) \\ u|_{x \to \pm\infty} \text{ is bounded} \end{cases}$$

Fourier transformation

$$u(x,t) = \int_{-\infty}^{\infty} A(k) e^{-\kappa k^2 t} e^{\mathrm{i}kx} dk$$

$$u(x,t=0) = \phi(x) = \int_{-\infty}^{\infty} A(k) e^{\mathrm{i}kx} dk$$

$$A(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x) e^{-\mathrm{i}kx} dx$$

Exchange the order of integration

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\kappa k^2 t} dk \int_{-\infty}^{\infty} e^{\mathrm{i}kx} \phi(x') e^{-\mathrm{i}kx'} dx'$$

Solution to the heat equation

- Now let's consider the heat equation in 1D.

$$\boxed{\int_0^{\infty} e^{-t^2} dt = \frac{1}{2}\sqrt{\pi}}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \kappa \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u|_{t=0} = \phi(x) \\ u|_{x\to\pm\infty} \text{ is bounded} \end{cases}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\kappa k^2 t} dk \int_{-\infty}^{\infty} e^{\mathrm{i}kx}\phi(x')e^{-\mathrm{i}kx'} dx'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left[\sqrt{\kappa t}k - \frac{\mathrm{i}(x-x')}{2\sqrt{\kappa t}}\right]^2} \cdot e^{-\frac{(x-x')^2}{4\kappa t}} dk \int_{-\infty}^{\infty} \phi(x') dx'$$

$$= \frac{1}{2\pi} \cdot \frac{\sqrt{\pi}}{\sqrt{\kappa t}} \int_{-\infty}^{\infty} \phi(x') \cdot e^{-\frac{(x-x')^2}{4\kappa t}} dx'$$

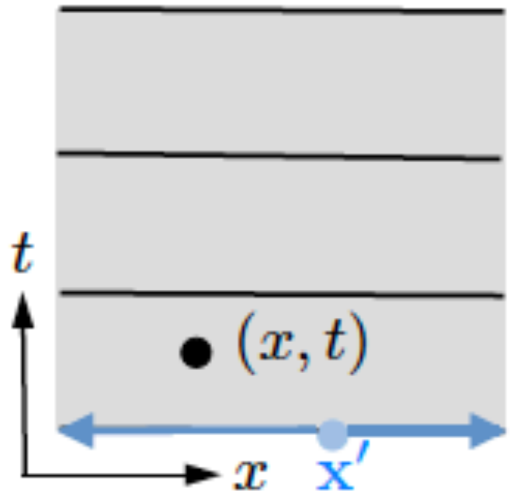
$$= \sqrt{\frac{1}{4\pi\kappa t}} \int_{-\infty}^{\infty} \phi(x') \cdot e^{-\frac{(x-x')^2}{4\kappa t}} dx'$$

Solution to the heat equation

- Now let's consider the heat equation in 1D.

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \kappa \dfrac{\partial^2 u}{\partial x^2} = 0 \\
u|_{t=0} = \phi(x) \\
u|_{x \to \pm\infty} \text{ is bounded}
\end{cases}
$$

$$u(x,t) = \sqrt{\frac{1}{4\pi\kappa t}} \int_{-\infty}^{\infty} \phi(x') \cdot e^{-\frac{(x-x')^2}{4\kappa t}} dx'$$



The initial state at point x' $\phi(x')$ can influence the state at (x,t) with a weighting factor of $e^{-\frac{(x-x')^2}{4\kappa t}}$.

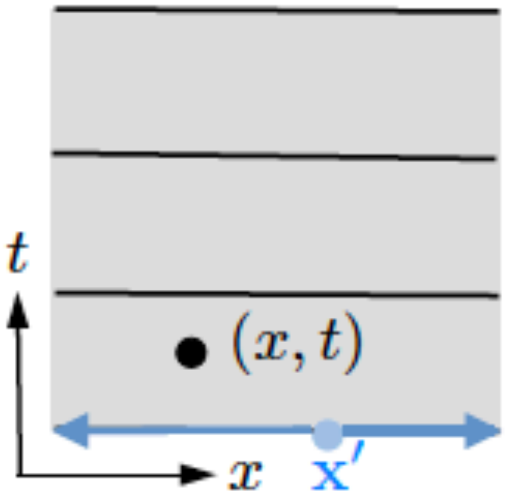
The integration is from $-\infty$ to ∞, implying that the domain of dependence is the entire x axis. Conversely, every point of the initial state can influence the entire domain – (**the "propagation velocity" of heat transfer is infinite**).

Solution to the heat equation

- Now let's consider the heat equation in 1D.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \kappa \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u\big|_{t=0} = \phi(x) \\ u\big|_{x \to \pm\infty} \text{ is bounded} \end{cases}$$

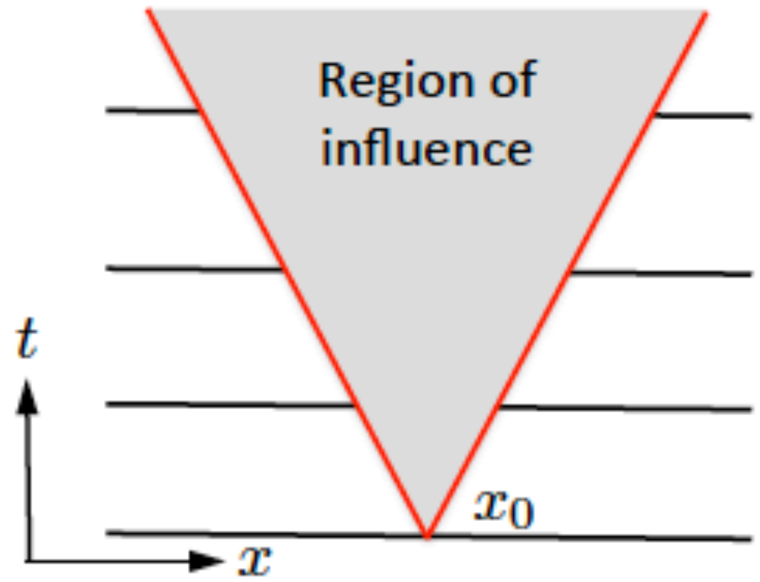
$$u(x,t) = \sqrt{\frac{1}{4\pi\kappa t}} \int_{-\infty}^{\infty} \phi(x') \cdot e^{-\frac{(x-x')^2}{4\kappa t}} dx'$$



The "propagation velocity" of heat transfer is infinite.

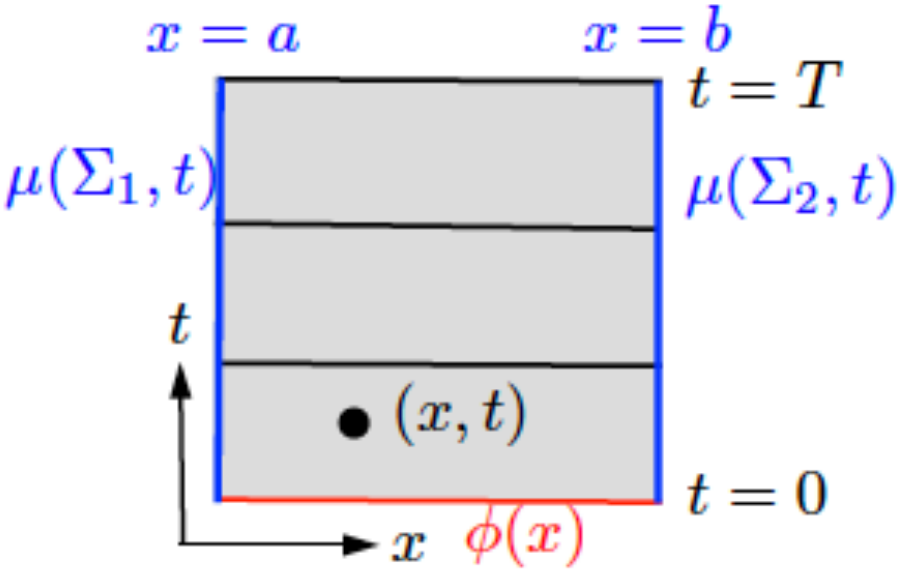
The above statement is mathematically correct, but physically incorrect. We have made some simplified assumptions when formulating the mathematical problem.

The wave equation vs. the heat equation



u(x,t) can reach its maximum/minimum inside the region.

Finite propagation velocity.



If u(x,t) is continuous in the rectangular region $a \leq x \leq b,\ 0 \leq t \leq T$, then it must reach the maximum/minimum at the initial state or at its boundary.

Infinite propagation velocity.