Exercise – 03 (total = 100’)

Due date: Mar. 13, 2022, 23:59

Note: Unless mentioned otherwise, z is treated as a complex number, while x and y are treated as real numbers. We use z^* or $\overline{z}$ to denote the complex conjugate of z.

Part – 1: True or False (3’ x 5 = 15’)

(1) If a complex function $f(z)$ has derivative at $z = z_0$, then it is also continuous at $z = z_0$.

(2) If a complex function $f(z)$ is analytic at $z = z_0$, then it also has derivative at $z = z_0$.

(3) A boundary point P of region G also belongs to G, i.e. $P \in G$.

(4) The complex function e^{iz} is not differentiable at $z \longrightarrow \infty$.

(5) If $f(z)$ is an analytic function about the complex variable z, so is $\overline{f(z)}$.

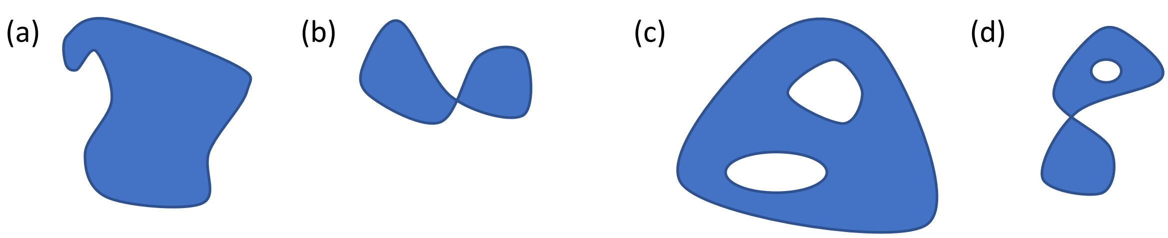
Part – 2: Graph (5’ x 3 = 15’)

Please answer the following questions related to the graphs.
Note: we consider the points in the blue area.

(1) Which of the following graphs is a region?

(2) Which of the following graphs is a simply connected region?

(3) Which of the following graphs is a multi-connected region?



Part – 3: Proof (5’ x 5 = 25’)

$f(z) = u(x, y) + i \cdot v(x, y)$ or $u(r, \theta) + i \cdot v(r, \theta)$.

(1) Prove that the Cauchy-Riemann relations (hereafter referred to as the C-R relations) in a Cartesian coordinate system are equivalent to

$$i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$$

(2) Utilize the relations $z = z(r,\theta) = re^{i\theta}$ and $f(z) = f(r,\theta) = u(r,\theta) + i \cdot v(r,\theta)$ to show that the C-R relations in a polar coordinate system are given by:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \text{and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Note: you should try a new method other than the geometry-based one discussed in the class.

(3) Suppose there is a complex function $f(z) = f(x+iy) = u(x,y) + i \cdot v(x,y)$. Now apply variable substitution $z = x + iy, \overline{z} = x - iy$, or $x = (z + \overline{z})/2\,, y = (z - \overline{z})/2i$, to show that the C-R relations are equivalent to

$$\frac{\partial f}{\partial \overline{z}} = 0$$

That is, f cannot be an explicit function of variable $\overline{z}$.

(4) If $f(z)$ is an analytic function defined in region G, and further satisfies the condition $f'(z) = 0$, then show that $f(z)$ is a constant in G.

(5) If both $f(z)$ and $\overline{f(z)}$ are analytic in region G, then show that $f(z)$ is a constant in G.

Part – 4: Verification (5' x 2 = 10')

Please follow the definition of derivative to verify the following relations.

(1) $\frac{d}{dz}(z^4) = 4z^3$

(2) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}, g(z) \neq 0$

Part – 5 Judgement (5' x 3 = 15')

Please judge whether the following functions are analytic in a bounded region (not including ∞) and present your reason.

(1) $|z|^2$

(2) e^{iz}

(3) $\frac{y - ix}{x^2 + y^2}$

Part – 6 Find the expression (5' x 4 = 20')

Suppose $z = x + iy$. If the real part of the analytic function $f(z) = u(x, y) + i \cdot v(x, y)$, i.e., $u(x, y)$, is given by the following expression, please find the full explicit form of $f(z)$.

Note: (1) you should express the final answer in terms of z, not x and y.

(2) don't forget the constant term.

(1) $u(x, y) = \frac{x}{x^2+y^2}$

(2) $u(x, y) = x^2 - y^2 + x$

(3) $u(x, y) = e^y \cos(x)$

(4) $u(x, y) = \cos(x) \cosh(y)$