# Exercise – 06 (total = 100')

Due date: Apr. 10, 2022, 23:59

**Note:** Unless mentioned otherwise, $z$ is treated as a complex number.

**Part – 1: Taylor series (2.5' x 4 = 10')**

Expand the following functions by Taylor series around the given point and find out the radius of convergence.

(1) $1 - z^2,\ \text{around}\ z = 1$

(2) $\sin(z),\ \text{around}\ z = n\pi$

(3) $\dfrac{1}{1 + z + z^2},\ \text{around}\ z = 0$

(4) $\dfrac{\sin(z)}{z - 1},\ \text{around}\ z = 0$

**Part – 2: Taylor series (2.5' x 2 =5')**

Expand the following functions by Taylor series around the given point and find out the radius of convergence.

(1) $\ln(z),\ \text{around}\ z = \mathrm{i},\ 0 \le \arg z < 2\pi$

(2) $\ln(z),\ \text{around}\ z = \mathrm{i},\ \ln(z)|_{z=\mathrm{i}} = -\dfrac{3}{2}\pi\mathrm{i}$

**Part – 3: Summation (2.5' x 2 = 5')**

Find the sum function of the following series and figure out the radius of convergence.

(1) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1} z^{2n+1}$

(2) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}$

**Part – 4: Laurent series (2.5' x 4 = 10')**

Expand the following functions by Laurent series in the given region.

(1) $\dfrac{1}{z^2(z-1)},\ \text{around } z=1$

(2) $\dfrac{1}{z^2(z-1)},\ \text{in } 1<|z|<\infty$

(3) $\dfrac{(z-1)(z-2)}{(z-5)(z-6)},\ \text{in } 5<|z|<6$

(4) $\dfrac{(z-1)(z-2)}{(z-5)(z-6)},\ \text{in } 6<|z|<\infty$

**Part – 5: Singular points and poles (2.5' x 4 = 10')**

Find the singular points (don't forget ∞) and judge their types. If these points are poles, also figure out their orders.

(1) $\dfrac{1}{z^2+16}$

(2) $\dfrac{\cos(az)}{z^2},\ a\neq 0$

(3) $\dfrac{\cos(az)-\cos(bz)}{z^2},\ a\neq b$

(4) $\dfrac{\sin(z)}{z^2}-\dfrac{1}{z}$

**Part -6: At infinity (2.5' x 4 = 10')**

Judge the behaviors of the following functions at infinity (analytic, singular, pole, isolated/essential singular, etc.)

(1) $\dfrac{1}{z^3}$

(2) $\dfrac{\cos(z)}{z}$

(3) $e^{-\frac{1}{z^2}}$

(4) $\sqrt{(z-5)(z-6)}$

**Part – 7: Calculate the residue (5' x 4 = 20')**

Calculate the residue at the given point.

(1) $\dfrac{1}{z-1}e^{z^2},\ \text{at } z_0=1$

(2) $\left[\dfrac{z}{1-\cos(z)}\right]^2,\ \text{at } z_0=0$

(3) $\dfrac{1}{z^2 \sin(z)}$, at $z_0 = 0$

(4) $\dfrac{e^z}{(z^2-1)^2}$, at $z_0 = 1$

**Part – 8: Calculation (5' x 2 =10')**

Figure out the values of the following integrations.

(1) $\oint_{|z-1|=1} \dfrac{1}{1+z^4} dz$

(2) $\oint_{|z|=1} \dfrac{e^z}{z^3} dz$

**Part – 9: Calculation (5' x 4 = 20')**

Figure out the values of the following integrations.

(1) $\int_0^{\pi} \dfrac{1}{1+\sin^2(\theta)} d\theta$

(2) $\int_0^{\infty} \dfrac{\cos(x)}{1+x^4} dx$

(3) v.p. $\int_{-\infty}^{\infty} \dfrac{1}{x(x-1)(x-2)} dx$

(4) $\int_0^{\infty} \dfrac{x^{s-1}}{1-x} dx,\ 0 < s < 1$