Exercise – 04 (total = 100')

Due date: Mar. 19, 2022, 23:59

Note: Unless mentioned otherwise, z is treated as a complex number, while x and y are treated as real numbers.

Part – 1: Judge the convergence (2' x 10 = 20')

Judge the convergence (convergent, absolutely convergent, divergent) of the following series and **present your reasoning**.

(1) $\sum_{n=2}^{\infty} \frac{i^n}{\ln n}$

(2) $\sum_{n=1}^{\infty} \frac{i^n}{n}$

(3) $\sum_{n=1}^{\infty} \frac{i^n}{\sqrt{n}}$

(4) $\sum_{n=1}^{\infty} \frac{(1+i)^n}{2^n}$

(5) $\sum_{n=1}^{\infty} (1+i)^n$

(6) $\sum_{n=1}^{\infty} \frac{1}{(1+i)^n}$

(7) $\sum_{n=1}^{\infty} \frac{1+i}{n^2}$

(8) $\sum_{n=1}^{\infty} e^{in\pi/4}$

(9) $\sum_{n=1}^{\infty} \frac{(2+3i)^n}{n!}$

(10) $\sum_{n=1}^{\infty} \left(\frac{1+i}{2-i}\right)^n$

Part – 2: Disk of convergence (2.5' x 8 =20')

Find the disk (or radius) of convergence for the following series.

(1) $\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n} z^n$

(2) $\sum_{n=1}^{\infty} \frac{1}{2^{2n}} z^{2n}$

(3) $\sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n$

(4) $\sum_{n=1}^{\infty} \frac{z^n}{n!}$

(5) $\sum_{n=1}^{\infty} \frac{(iz)^n}{n^2}$

(6) $\sum_{n=1}^{\infty} n^{\ln n} z^n$

(7) $\sum_{n=1}^{\infty} \frac{\ln n^n}{n!} z^n$

(8) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n z^n$

Part – 3: Proof (10' x 1 = 10')

Please prove the following relation:

$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - ..., \ |z| < 1$

Note: You are not allowed to directly apply the knowledge of Taylor expansion.

Part – 4: Judgment (4' x 5 = 20')

Please judge if the following functions are multi-valued and **present your reasoning**.

(1) $\sqrt{z^2+1}$

(2) $\cos(\sqrt{z})$

(3) $\tan(\sqrt{z})$

(4) $\frac{\sin(\sqrt{z})}{\sqrt{z}}$

(5) $\sin(i \ln z)$

Part – 5: Branch point (5' x 4 = 20')

Please find all the branch points (don't forget to judge ∞) of the following functions.

(1) $\sqrt{z^2+4}$

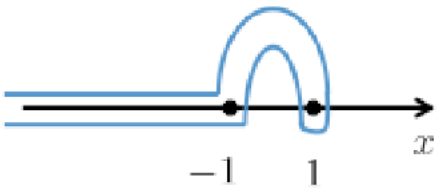
(2) $\sqrt[3]{1-z^3}$

(3) $\ln(z^2+1)$

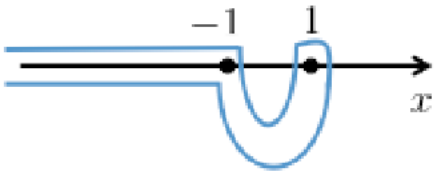
(4) $\ln[\cos(z)]$

Part – 6 Find the value (5' x 2 = 10')

Let's consider function $w=\ln(1-z^2)$, which is of course a multi-valued function. Now let's draw some branch cut and further assume $w(0)=0$, then we can uniquely determine the value of w at points not along the branch cut. Please follow this idea to figure out the value of $w(3)$ in (a) and (b). Note: the way of drawing branch cut (the blue curve) differs between (a) and (b), which may cause a different value of $w(3)$ between (a) and (b).



(a)



(b)