Exercise 04

Part 1

(1) ① $\sum_{n=2}^{\infty} \frac{i^n}{\ln n} = \left(-\frac{1}{\ln 2} + \frac{1}{\ln 4} - \cdots\right) + i\left(-\frac{1}{\ln 3} + \frac{1}{\ln 5} - \cdots\right)$

$= \sum_n a_n + i \sum_n b_n$

Here, $\{a_n\}$ and $\{b_n\}$ are alternating series

and $|a_{n+1}| < |a_n|$, $\lim_{n \to \infty} a_n = 0$

$|b_{n+1}| < |b_n|$, $\lim_{n \to \infty} b_n = 0$

$\therefore$ They converge and so does the original series.

② It is not absolutely convergent because

$\left|\frac{i^n}{\ln n}\right| > \frac{1}{n}$ and $\sum_n \frac{1}{n}$ diverges.

(2) $\sum_{n=1}^{\infty} \frac{i^n}{n} = \left(-\frac{1}{2} + \frac{1}{4} - \cdots\right) + i\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$

Real part and imaginary part are alternating series and satisfying $|u_{n+1}| < |u_n|$, $\lim_{n \to \infty} u_n = 0$

$\therefore$ convergent.

But it's not absolutely convergent because

$\left|\frac{i^n}{n}\right| = \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(3) $\sum_{n=1}^{\infty} \frac{i^n}{\sqrt{n}} = \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} - \cdots\right) + i\left(1 - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \cdots\right)$

Real part and imaginary part are alternating series and satisfying $|u_{n+1}| < |u_n|$, $\lim_{n \to \infty} u_n = 0$

$\therefore$ convergent.

But it's not absolutely convergent because

$\left|\frac{i^n}{\sqrt{n}}\right| > \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(4) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{1+i}{2}\right| = \frac{\sqrt{2}}{2} < 1$

$\therefore$ absolutely convergent

(5) $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \sqrt{2} > 1$,

$\therefore$ divergent.

(6) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{1}{1+i}\right| = \frac{1}{\sqrt{2}} < 1$

$\therefore$ absolutely convergent

(7) $\frac{u_n}{u_{n+1}} = 1 + \frac{2}{n} + \frac{1}{n^2}$

According to Gauss method, it is absolutely convergent

(8) $\lim_{n\to\infty} e^{in\frac{\pi}{4}}$ does not exist

$\therefore$ divergent.

(9) $\overline{\lim_{n\to\infty}}\left|\frac{u_{n+1}}{u_n}\right| = \frac{\sqrt{13}}{n+1} < 1$

$\therefore$ absolutely convergent

(10) $\left|\frac{u_{n+1}}{u_n}\right| = \sqrt{\frac{2}{5}} < 1$

$\therefore$ absolutely convergent

Part 2.

We use R to denote radius of convergence.

(1) $C_n = \frac{(n!)^2}{n^n}$

$R = \lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\frac{1}{n+1} = 0$

(2) $\sum_{n=1}^{\infty} \frac{z^{2n}}{2^{2n}} = \sum_{n=1}^{\infty} \left[\left(\frac{z}{2}\right)^2\right]^n$

$\left|\frac{z}{2}\right|^2 < 1 \Rightarrow |z| < 2 \quad \therefore R = 2$

(3) $\left|\frac{z}{2}\right| < 1 \Rightarrow |z| < 2 \quad \therefore R = 2$

(4) $C_n = \frac{1}{n!}$

$\lim_{n\to\infty} \left|\frac{C_n}{C_{n+1}}\right| = \lim_{n\to\infty} (n+1) = \infty$

$\therefore R = \infty$

(5) $C_n = \frac{i^n}{n^2}$ $\quad \lim_{n\to\infty} \left|\frac{C_n}{C_{n+1}}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2$

$= 1$

$\therefore R = 1$

(6) $C_n = n^{\ln n}$

$R = \lim_{n\to\infty} \left|\frac{1}{C_n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} n^{-\frac{\ln n}{n}}$

$= \lim_{n\to\infty} \left(e^{\ln n}\right)^{-\frac{\ln n}{n}}$

$= \lim_{n\to\infty} e^{-\frac{(\ln n)^2}{n}}$

$= 1$

(7) $C_n = \frac{\ln n^n}{n!}$

$R = \lim_{n\to\infty} \left|\frac{C_n}{C_{n+1}}\right| = \lim_{n\to\infty} \frac{n\ln n}{n!} \frac{(n+1)!}{(n+1)\ln(n+1)}$

$= \lim_{n\to\infty} n \frac{\ln n}{\ln(n+1)}$

$= \infty$

(8) $c_n = \left(1-\frac{1}{n}\right)^n$

$$R = \lim_{n\to\infty} \left|\frac{1}{c_n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{n}}$$

$$= 1$$

Part 3.

Let $S(z) = \sum_{n=1}^{\infty} c_n z^n$, where $c_n = -\frac{1}{n}$

radius of convergence: $R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$

$$= \lim_{n\to\infty} \frac{n+1}{n}$$

$$= 1$$

For $|z|<1$, $S(z)$ is uniformly convergent.

$$\therefore \frac{dS(z)}{dz} = -\sum_{n=1}^{\infty} \frac{1}{n} \frac{d(z^n)}{dz} = -\sum_{n=0}^{\infty} z^n$$

$$= -\frac{1}{1-z}$$

Then, $\int_0^z \frac{dS(z')}{dz'} dz'$

$$= -\int_0^z \frac{1}{1-z'} dz'$$

$$= \ln(1-z')\Big|_0^z$$

$$= S(z) - S(0)$$

$\therefore S(0) = \ln 1 = 0$, $S(z) = \ln(1-z)$.

Part 4

(1) Let $t = z^2+1 = re^{i(\theta+2k\pi)}$

Then, $\sqrt{z^2+1} = \sqrt{t} = \sqrt{r}\, e^{i\left(\frac{\theta}{2}+k\pi\right)}$

when k is odd, $\sqrt{z^2+1} = -\sqrt{r}\, e^{i\frac{\theta}{2}}$

when k is even, $\sqrt{z^2+1} = \sqrt{r}\, e^{i\frac{\theta}{2}}$

$\therefore$ multi-valued.

(2) Suppose $z = re^{i(\theta+2k\pi)}$. Then, $\sqrt{z} = \sqrt{r}\, e^{i\left(\frac{\theta}{2}+k\pi\right)}$

$= \pm\sqrt{r}\, e^{i\frac{\theta}{2}}$

$$\cos\sqrt{z} = \cos\left(\pm\sqrt{r}\, e^{i\frac{\theta}{2}}\right)$$
$$= \cos\left(\sqrt{r}\, e^{i\frac{\theta}{2}}\right)$$

$\therefore$ single-valued.

(3) Similar to (2), $\tan\sqrt{z} = \tan\left(\pm\sqrt{r}\, e^{i\frac{\theta}{2}}\right)$

$= \pm\tan\left(\sqrt{r}\, e^{i\frac{\theta}{2}}\right)$

$\therefore$ multi-valued.

(4) $$\frac{\sin\sqrt{z}}{\sqrt{z}} = \frac{\sin\left(\pm\sqrt{r}\, e^{i\frac{\theta}{2}}\right)}{\pm\sqrt{r}\, e^{i\theta/2}} = \frac{\pm\sin\left(\sqrt{r}\, e^{i\frac{\theta}{2}}\right)}{\pm\sqrt{r}\, e^{i\theta/2}}$$
$$= \frac{\sin\left(\sqrt{r}\, e^{i\frac{\theta}{2}}\right)}{\sqrt{r}\, e^{i\frac{\theta}{2}}}$$

$\therefore$ single-valued.

(5) $z = re^{i(\theta+2k\pi)} \Rightarrow \ln z = \ln r + i(\theta+2k\pi)$

$$\sin(i\ln z) = \sin(i\ln r - \theta - 2k\pi)$$
$$= \sin(i\ln r - \theta)$$

$\therefore$ single-valued.

Part 5.

(1) $w=0$ is the branch point of $\sqrt{w}$.

$z^2+4=0 \Rightarrow z=2i$ or $z=-2i$

$\therefore$ $z=2i$ and $z=-2i$ are branch points for $\sqrt{z^2+4}$.

Moreover, let $z+2i = r_1 e^{i(\theta_1+2k_1\pi)}$

$z-2i = r_2 e^{i(\theta_2+2k_2\pi)}$

$\Rightarrow f(z) = \sqrt{z^2+4} = \sqrt{r_1 r_2}\; e^{i\left(\frac{\theta_1+\theta_2}{2}\right)+(k_1+k_2)\pi}$

Encircling $z=2i$ will increase the argument of $f(z)$ by π, and encircling $z=-2i$ will increase it by π.

Thus, encircling both points will increase the argument by 2π, leaving the value of $f(z)$ unchanged.

This implies that $z=\infty$ is not a branch point.

(2) $1-z^3=0 \Rightarrow z=1$, $z=e^{i\frac{2}{3}\pi}$ or $e^{i\frac{4}{3}\pi}$.

These are branches points of $\sqrt[3]{1-z^3}$.

Moreover, traversing a large enough circle that encloses all these three points will increase the argument by 2π, and $f(z)=\sqrt[3]{1-z^3}$ will return to its original value.

$\therefore$ $z=\infty$ is not a branch point.

(3) $z^2+1=0 \Rightarrow z=+i$ or $z=-i$.

which are branch points of $\ln(z^2+1)$

Let $z+i = r_1 e^{i(\theta_1+2k_1\pi)}$, $z-i = r_2 e^{i(\theta_2+2k_2\pi)}$

Then, $\ln(z^2+1) = \ln r_1 + \ln r_2 + i(\theta_1+\theta_2+2k_1\pi+2k_2\pi)$

Encircling both $z=i$ and $z=-i$ will change

$f(z) = \ln(z^2+1)$ by $4\pi i$

$\therefore$ $z = \infty$ is a branch point.

(4) $\cos z = 0 \Rightarrow z = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

which are branch points of $\ln(\cos z)$

By similar reasoning of (3), $z = \infty$ is also a branch point.

Part 6.

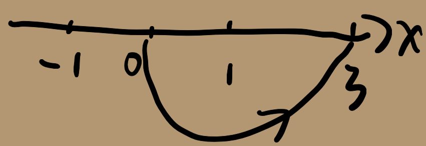
Let $1+z = r_1 e^{i\theta_1}$ $\quad 1-z = r_2 e^{i\theta_2}$

$w(z) = \ln(1-z^2) = \ln r_1 + \ln r_2 + i(\theta_1 + \theta_2)$

$w(0) = 0 \Rightarrow \theta_1 = 0$ and $\theta_2 = 0$ at $z=0$

when $z=3$, $r_1 = 4$, $r_2 = 2$, $\therefore \ln r_1 + \ln r_2 = \ln 8$

(a)



along the path

$\theta_1 : 0 \to 0$

$\theta_2 : 0 \to \pi$

$\therefore w(3) = \ln 8 + i\pi$

(b)

along the path.

$\theta_1 : 0 \to 0$

$\theta_2 : 0 \to -\pi$

$\therefore w(3) = \ln 8 - i\pi$