Mathematical Methods in Physics

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Review

- Cauchy’s theorem
- Small and great arc lemmas
- Cauchy’s integral formula
- Higher-order derivatives of an analytic function

Chapter – 02: Taylor and Laurent series, Residue theorem, and definite integrals

- In this week, we will be discussing Taylor and Laurent series (泰勒级数和劳伦级数), residue theorem (留数定理), definite integrals using contour integration (使用回路积分计算定积分).

- You should recall the knowledge about series and integration learned from Calculus.

- You should also recall the knowledge of Cauchy's theorem and Cauchy's integral formula learned in the previous week.

Taylor series (泰勒级数)

From lecture 04, we know that a power series represents an analytic function inside its disk of convergence. Now we would like to ask how to represent an analytic function by a power series.

Theorem (Taylor expansion)

Assume $f(z)$ is analytic inside circle *C* (centered at point *a*) as well as along *C*, then for any point *z* inside *C*, $f(z)$ can be represented by a power series (expanded around *a*).

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

where $a_n = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta = \frac{f^{(n)}(a)}{n!}$

Note: the contour integration is along the positive (anti-clockwise) direction of *C*.

Proof:

Following Cauchy's integral formula, we have $f(z) = \dfrac{1}{2\pi \mathrm{i}} \oint_C \dfrac{f(\zeta)}{\zeta - z} d\zeta$

Let's do some tricks here. Recall how to get the value of a geometric series

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z - a)} = \frac{1}{\zeta - a} \sum_{n=0}^{\infty} \left(\frac{z-a}{\zeta - a} \right)^n \qquad |z - a| < |\zeta - a|$$

Mind the radius of convergence

Because $\displaystyle\sum_{n=0}^{\infty} \left(\frac{z-a}{\zeta - a} \right)^n$ is uniformly convergent in $\left| \dfrac{z-a}{\zeta - a} \right| \le r < 1$, we can exchange the order of summation and integration.

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_C \left[\sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^{n+1}} \right] f(\zeta) d\zeta$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta \right] (z-a)^n$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n, \quad \left| \frac{z-a}{\zeta - a} \right| \le r < 1 \quad \Rightarrow \quad a_n = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta = \frac{f^{(n)}(a)}{n!}$$

Some remarks

- The overall form of Taylor expansion looks similar to that for a real function. However, there are some differences. For a real function, we usually require that the series be expanded for a small region around the reference point (if the region is too large, the series may not converge). But for a complex function, the condition of analytic determines that the limit of series must exist.

Some remarks

Page 670, textbook

Theorem III (which we state without proof). If $f(z)$ is analytic in a region (R in Figure 2.3), then it has derivatives of all orders at points inside the region and can be expanded in a Taylor series about any point z_0 inside the region. The power series converges *inside* the circle about z_0 that extends to the nearest singular point (C in Figure 2.3).

Some definitions:

A *regular point* of $f(z)$ is a point at which $f(z)$ is analytic.

A *singular point* or *singularity* of $f(z)$ is a point at which $f(z)$ is not analytic. It is called an *isolated* singular point if $f(z)$ is analytic everywhere else inside some small circle about the singular point.

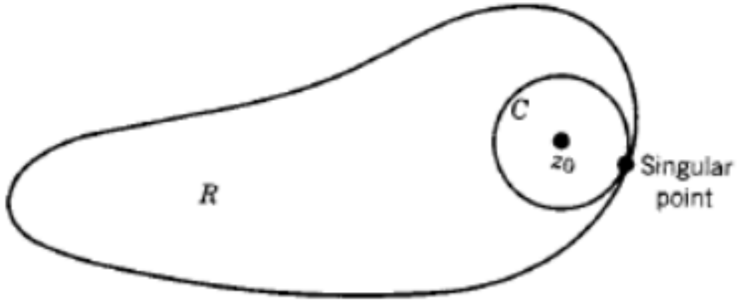


Figure 2.3

Notice again what a strong condition it is on $f(z)$ to say that it has a derivative. It is quite possible for a function of a real variable $f(x)$ to have a first derivative but not higher derivatives. But if $f(z)$ has a first derivative with respect to z, then it has derivatives of all orders, and all these derivatives are analytic functions.

Generally speaking, if b is the nearest singular point of $f(z)$ relative to point a, then the radius of convergence $R = |b - a|$

Page 671, textbook

Exercise

- [6.01] $\dfrac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}, \quad |z| < 1$

From L.H.S., we know that the nearest singular point (relative to z=0) is $\pm\mathrm{i}$.

Then we can guess that the radius of convergence is $R = |\pm\mathrm{i} - 0| = 1.$

From R.H.S., you easily verify the radius of convergence.

Some remarks

- **The uniqueness**

  For a given function, which is analytic inside a circle *C* (centered at a), its Taylor expansion is unique.

  Proof: $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$

  $$= a_0' + a_1'(z-a) + a_2'(z-a)^2 + \ldots$$

  Let $z = a$, we have $a_0 = a_0'$. Then we can compare the expansion for $f'(z)$ at $z = a$.

Conclusion: No matter how you expand the analytic function around the same point, the coefficients for its Taylor expansion must be unique.

Exercise

- [6.02] $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n} \qquad |z| < 1$$

Note: both examples do the expansion around z=0.

Apply known results to new problems.

Exercise

- [6.03] $$\frac{1}{1-3z+2z^2} = -\frac{1}{1-z} + \frac{2}{1-2z} = \sum_{n=0}^{\infty} \left(2^{n+1}-1\right) z^n, \quad |z| < \frac{1}{2}$$

Decomposition of partial fraction (部分分式分解) – you have learned this trick from calculus.

Then apply the results of [6.02].

The series are expanded around z=0, the nearest singular point is z=1/2. Therefore we get R=1/2.

Exercise

$$|z|<1 \ \cap \ |z|<1/2 = |z|<1/2$$

- [6.04] $$\frac{1}{1-3z+2z^2} = \frac{1}{1-z}\cdot\frac{1}{1-2z} = \sum_{k=0}^{\infty} z^k \cdot \sum_{l=0}^{\infty} 2^l z^l = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2^l z^{k+l}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} 2^l\right) z^n = \sum_{n=0}^{\infty}\left(2^{n+1}-1\right) z^n, \quad |z|<\frac{1}{2}$$

Recall the double series in Lecture 04 .

Rearrange the double series, according to the power index of z^n (n=0, 1, 2, 3…)

It is easy to figure out R=1/2.

Because the two individual series are absolutely convergent in $|z|<\dfrac{1}{2}$, their product is also absolutely convergent in the common region.

Taylor expansion for a multi-valued function

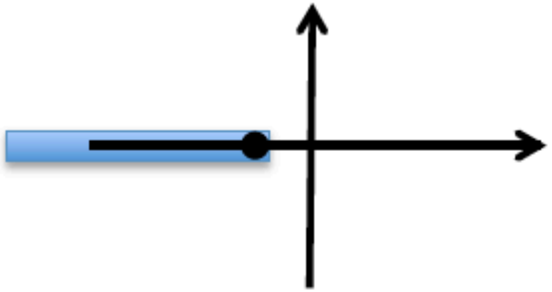
For a multi-valued function, if we set the branch cut (or assume the path of z), then the function can be treated like a single-valued function, and hence Taylor expansion can be performed.

Please also compare it to Problem 3 in Exercise04.

Example: Find the Taylor expansion of $\ln(1+z)$ around $z=0$. We assume that

$$\ln(1+z)|_{z=0} = 0.$$

$$\ln(1+z) = \ln(1+z) - \ln(1+z)|_{z=0} = \int_0^z \frac{1}{1+\zeta}d\zeta$$



$$= \int_0^z \sum_{n=0}^{\infty}(-1)^n \zeta^n d\zeta = \sum_{n=0}^{\infty}(-1)^n \int_0^z \zeta^n d\zeta = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}z^{n+1}$$

Note: (1+z)=0 (or z=-1) is a branch point. Therefore, the maximum possible radius of convergence = 1. The actual radius depends on how the branch cut is chosen. In the example above, we draw it to the negative direction of the real axis, then $|z| < 1$ is the disk of convergence.

Taylor expansion at infinity

- If $f(z)$ is analytic at ∞ , then its Taylor expansion also exists around ∞ .
- We can translate the problem into: $f(1/t)$ at $t = 0$.

$$f(1/t) = a_0 + a_1 t + a_2 t^2 + ..., \ |t| < r$$

$$f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + ..., \ |z| > \frac{1}{r}$$

Cannot contain terms such as $a_k z^k$ $(\text{where } k > 0)$, because $f(z)$ is analytic at ∞.

Zero point (零点) of an analytic function

If $f(z)$ is analytic in a region around $z = a$, and is not always equal to zero. If $f(a) = 0$, then $z = a$ is called a zero point of $f(z)$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \ |z-a| < \rho$$

If $z = a$ is zero point, then $a_0 = a_1 = ... = a_{m-1} = 0, \quad a_m \neq 0$ $(z-1)^{10}$

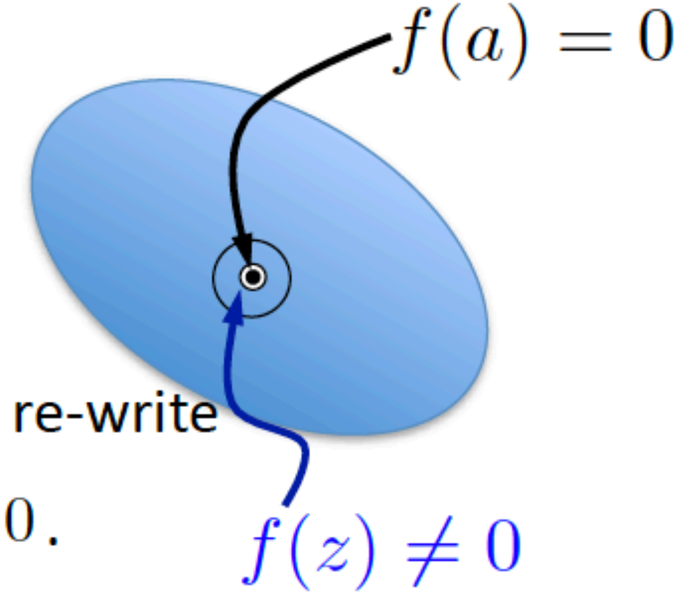
And we call $z = a$ the *m*-th order zero point of $f(z)$.

$$f(a) = f'(a) = ... = f^{(m-1)}(a) = 0, \quad f^{(m)}(a) \neq 0$$

m must be a positive integer, because $f(z)$ is analytic and is not always equal to zero.

Zero point (零点) of an analytic function

If $z = a$ is the zero point of $f(z)$, and $f(z)$ is not always equal to zero for a small region around $z = a$, then $\exists \rho > 0$, such that $f(z)$ does not have zero points in $0 < |z - a| < \rho$.



In other words, the zero point of $f(z)$ is isolated in space.

Proof: Assume that $z = a$ is an *m*-th order zero point. Then we can re-write

$f(z) = a_m (z-a)^m \phi(z)$, where $\phi(z)$ is analytic and $\phi(z) \neq 0$.

Because $\phi(z)$ is analytic (and hence continuous) at $z = a$.

$\forall \varepsilon > 0,\ \exists \rho > 0$, such that when $|z - a| < \rho$, $|\phi(z) - \phi(a)| < \varepsilon$. Let $\varepsilon = |\phi(a)|/2$,

$$|\phi(z)| > |\phi(a)| - \varepsilon = \frac{1}{2}|\phi(a)| > 0 .$$

Some remarks

- Assume $f(z)$ is analytic in $G: \ |z-a|<R$. If there is an infinite number of zero points $\{z_n\}$ in G (these zero points are not identical to one another), and $\lim_{n\to\infty} z_n = a$, but $z_n \not\equiv a$, then $f(z) \equiv 0$.

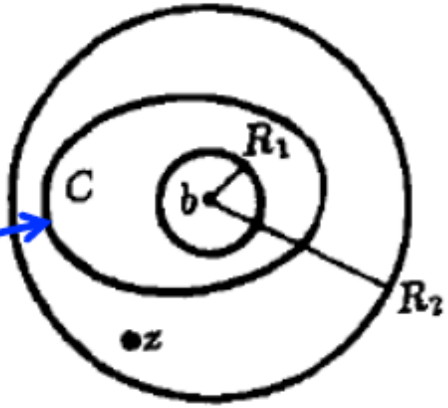
- **Uniqueness theorem**: $f_1(z)$ and $f_2(z)$ are two analytic functions defined in G. If there exists a series $\{z_n\}$ in G, such that $\forall n, f_1(z_n) = f_2(z_n)$. If one limit point of $\{z_n\}$ also falls inside G, then $f_1(z) \equiv f_2(z)$.

Laurent series (劳伦级数)

- Assume $f(z)$ is single-valued and analytic in a ring-like region $R_1 \leq |z-b| \leq R_2$, then for any z inside the ring, $f(z)$ can be expanded as a power series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-b)^n, \quad R_1 < |z-b| < R_2$$

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(\zeta)}{(\zeta - b)^{n+1}} d\zeta$$



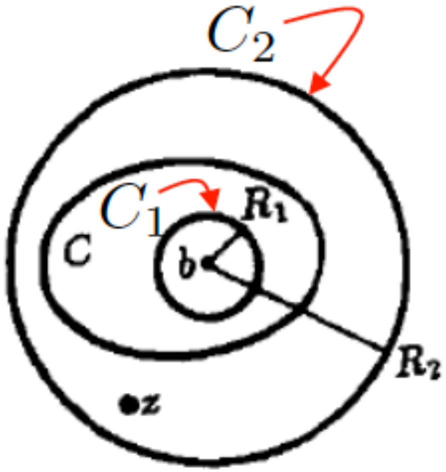
Here C is an arbitrary closed curve inside the ring.

Laurent series (劳伦级数)

Proof: Let's denote the inner and outer boundaries C_1 and C_2 .

According to Cauchy's integral formula, we have

$$f(z)=\frac{1}{2\pi\mathrm{i}}\oint_{C_2}\frac{f(\zeta)}{\zeta-z}d\zeta-\frac{1}{2\pi\mathrm{i}}\oint_{C_1}\frac{f(\zeta)}{\zeta-z}d\zeta$$



For the integration along C_2 , we can refer to the results of Taylor expansion

$$\frac{1}{2\pi\mathrm{i}}\oint_{C_2}\frac{f(\zeta)}{\zeta-z}d\zeta=\sum_{n=0}^{\infty}a_n(z-b)^n,\ \boxed{|z-b|<R_2}$$

$$a_n=\frac{1}{2\pi\mathrm{i}}\oint_{C_2}\frac{f(\zeta)}{(\zeta-b)^{n+1}}d\zeta$$

Laurent series (劳伦级数)

Proof: For the integration along C_1, we can re-express the integral relative to point b

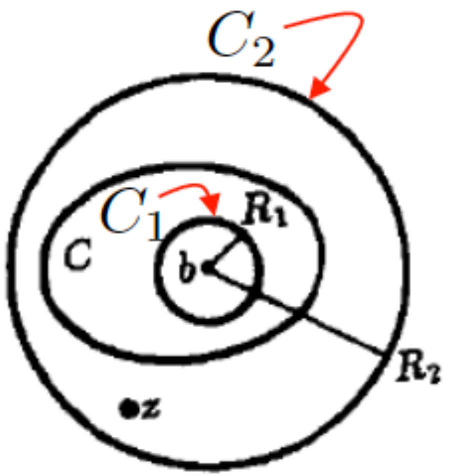
$$-\frac{1}{2\pi \mathrm{i}}\oint_{C_1}\frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi \mathrm{i}}\oint_{C_1}\frac{f(\zeta)}{(z-b)-(\zeta-b)}d\zeta$$

$$= \frac{1}{2\pi \mathrm{i}}\oint_{C_1}\frac{f(\zeta)}{z-b}\sum_{k=0}^{\infty}\left(\frac{\zeta-b}{z-b}\right)^k d\zeta$$

$$= \sum_{k=0}^{\infty}(z-b)^{-k-1}\cdot\frac{1}{2\pi \mathrm{i}}\oint_{C_1}f(\zeta)(\zeta-b)^k d\zeta$$

$$n = -(k+1) \qquad = \sum_{n=-1}^{-\infty}a_n(z-b)^n, \quad \boxed{|z-b| > R_1}$$

$$a_n = \frac{1}{2\pi \mathrm{i}}\oint_{C_1}\frac{f(\zeta)}{(\zeta-b)^{n+1}}d\zeta$$



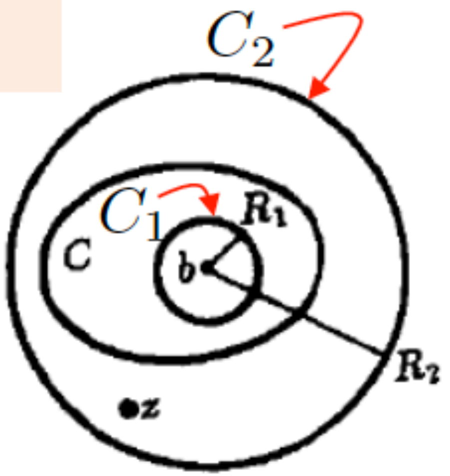
Laurent series (劳伦级数)

Proof: Combine page 19 with page 20, we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-b)^n, \quad R_1 < |z-b| < R_2$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - b)^{n+1}} d\zeta$$

Laurent expansion
Laurent series



where we have replaced the integration path from C_1 (or C_2) to an arbitrary closed curve *C*, according to the Cauchy's integral formula.

Some remarks

- For a Laurent expansion, $a_n \neq \frac{1}{n!} f^{(n)}(b)$ (different from a Taylor expansion)
- $f(z)$ is not analytic inside $\overline{C_1}$. Generally speaking, it has a singular point along C_1.

  It may or may not be analytic at $z = b$.

  If $z = b$ is the only singular point inside $\overline{C_1}$, it is also called the isolated singular point (孤立奇点) of $f(z)$. Then R_1 can be arbitrarily small.
- Unlike a Taylor expansion, Laurent expansion includes both positive and negative terms (in their power index). $\sum_{0}^{\infty}$ is absolutely convergent in $|z-b| < R_2$, while $\sum_{-1}^{-\infty}$ is absolutely convergent in $|z-b| > R_1$.

Some remarks

- If $f(z)$ is not analytic at infinity, but is analytic (and single-valued) in a region around infinity, then we can do Laurent expansion for $f(z)$ around ∞.

  Variable substitution $1/t = z$

  $$f(1/t) = \sum_{n=-\infty}^{\infty} a_n t^n,\ 0 < |t| < r \qquad f(z) = \sum_{n=-\infty}^{\infty} a_n z^{-n},\ 1/r < |z| < \infty$$

- The expansion of a Laurent series for a given region (with respect to the same reference point) is unique.

Exercise

- [6.05] Find the Laurent expansion of $\dfrac{1}{z(z-1)}$ when (i) $0<|z|<1$ and (ii) $|z|>1$

You should refer to the previous exercises.

convergent in $0<|z|<1$

(i) $0<|z|<1$ $\quad \dfrac{1}{z(z-1)} = -\dfrac{1}{z}\dfrac{1}{1-z} = -\dfrac{1}{z}\displaystyle\sum_{n=0}^{\infty} z^n = -\sum_{n=-1}^{\infty} z^n$

And we can confirm that $\dfrac{1}{z(z-1)}$ has singular point at $z=0$.

When changing the region, the result changes.

(ii) $|z|>1$ $\quad \dfrac{1}{z(z-1)} = \dfrac{1}{z^2}\dfrac{1}{1-\frac{1}{z}} = \dfrac{1}{z^2}\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=-2}^{-\infty} z^n$

convergent in $|z|>1$

And we can confirm that $\dfrac{1}{z(z-1)}$ has singular point along $|z|=1$.

Isolated singular point

For a single-valued function (or one branch of a multi-valued function), if $f(z)$ is not analytic at b, but there $\boxed{\text{exists } r > 0}$, such that $f(z)$ has derivative everywhere in $0 < |z - b| < r$, then b is called an isolated singular point (孤立奇点) of $f(z)$.

Otherwise, if $\boxed{\forall r > 0}$, $f(z)$ has singular point(s) in $0 < |z - \beta| < r$, then β is called a non-isolated singular point of $f(z)$.

Isolated singular point

If b is an isolated singular point of $f(z)$, then there exists $R > 0$, and $f(z)$ can be expanded as a Laurent series in $0 < |z-b| < R$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-b)^n, \ 0 < |z-b| < R$$

(1) If the series does not contain negative term, then b is said to be a removable singular point (可去奇点).

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}, \ |z| < \infty$$

(2) The number of negative terms is finite, then b is said to be a pole (极点).

(3) The number of negative terms is infinite, then b is said to be an essential singular point (本性奇点).

Pole and zero point

- The relation between pole and zero point

$$f(z) = a_{-m}(z-b)^{-m} + a_{-m+1}(z-b)^{-m+1} + \dots + a_0 + a_1(z-b) + \dots$$

$$= (z-b)^{-m}[a_{-m} + a_{-m+1}(z-b) + \dots]$$

$$= (z-b)^{-m}\phi(z), \quad 0 < |z-b| < R$$

$\phi(z)$ is analytic for a region around $z = b$. If $\phi(b) = a_{-m} \neq 0$, then b is said to be the *m*-th order pole of $f(z)$.

$$\frac{1}{f(z)} = (z-b)^m \frac{1}{\phi(z)}$$

b is the *m*-th order zero point

Essential singular point

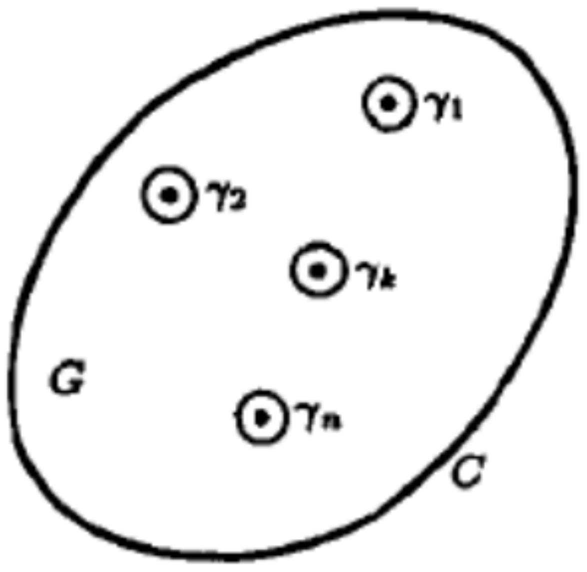
- If b is an essential singular point of $f(z)$, then when $z \to b$, the limit of $f(z)$ does not exist – it depends on the path of $z \to b$.

- It is straightforward to show that $z = 0$ is an essential singular point of

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z} \right)^n, \ 0 < |z| < \infty$$

- Similarly, using $1/t = z$, we can verify that $z = \infty$ is an essential singular point of e^z (we have discussed this example before).

Residue theorem (留数定理)



Assume that G is a bounded region, and its boundary *C* is a smooth, simply closed curve. If except for a finite number of isolated singular points b_k, $k = 1, 2, 3..., n$, $f(z)$ is single-valued and analytic in G, and is continuous in $\overline{G}$ (including along *C*). Then we have the following relation:

$$\oint_C f(z)dz = 2\pi \mathrm{i} \sum_{k=1}^{n} \boxed{\mathrm{res}\, f(b_k)}$$

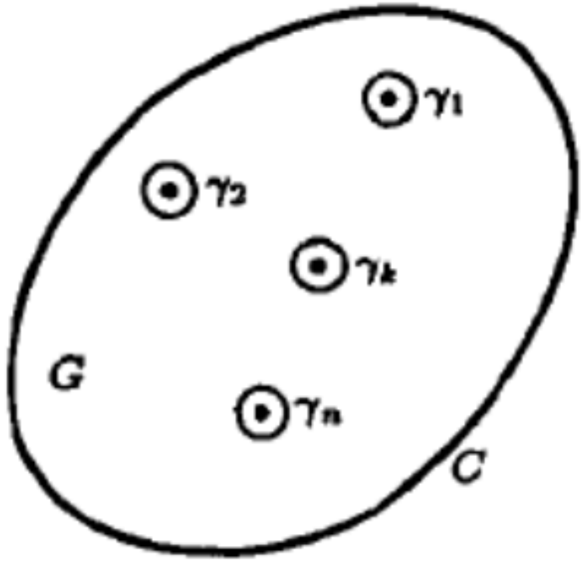
called the residue (留数) of $f(z)$ at b_k

$$f(z) = \sum_{l=-\infty}^{\infty} a_l^{(k)} (z - b_k)^l, \; 0 < |z - b_k| < r$$

along the positive direction of *C*

which is equal to the coefficient of $(z - b_k)^{-1}$, $\boxed{a_{-1}^{(k)}}$

Residue theorem (留数定理)



Assume that G is a bounded region, and its boundary C is a smooth, simply closed curve. If except for a finite number of isolated singular points b_k, $k = 1, 2, 3..., n$, $f(z)$ is single-valued and analytic in G, and is continuous in $\overline{G}$ (including along C). Then we have the following relation:

$$\oint_C f(z)dz = 2\pi\mathrm{i}\sum_{k=1}^{n} \mathrm{res} f(b_k)$$

Laurent expansion + known result of $\oint_C z^n dz$ $n = -1$

Proof: $\underbrace{\oint_C f(z)dz = \sum_{k=1}^{n}\oint_{\gamma_k}}_{\text{Cauchy's theorem}} f(z)dz = 2\pi\mathrm{i}\sum_{k=1}^{n} a_{-1}^{(k)} = 2\pi\mathrm{i}\sum_{k=1}^{n} \mathrm{res} f(b_k)$

How to compute residue

- Find the coefficient for the term $(z-b)^{-1}$.

  Let's assume b is an *m*-th order pole of $f(z)$.

$$f(z) = a_{-m}(z-b)^{-m} + a_{-m+1}(z-b)^{-m+1} \ldots + \ldots a_0 + a_1(z-b) + \ldots,\ \ 0 < |z-b| < r$$

$$(z-b)^m \cdot f(z) =$$

$$a_{-m} + a_{-m+1}(z-b) \ldots + \ldots + \boxed{a_{-1}(z-b)^{m-1}} + a_0(z-b)^m + a_1(z-b)^{m+1} + \ldots$$

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-b)^m f(z)|_{z=b}$$

How to compute residue

- The general form of $f(z)$ is $\dfrac{P(z)}{Q(z)}$. If $P(z)$ and $Q(z)$ are analytic around b, and

  $P(b) \neq 0$, $z = b$ is the first-order zero point of $Q(z)$.

  $$a_{-1} = \lim_{z \to b}(z-b)f(z) = \lim_{z \to b}(z-b)\frac{P(z)}{Q(z)} = \frac{P(b)}{Q'(b)}$$

  L'Hôpital's rule

Exercise

- [6.06] Find the residue of $1/(z^2+1)^3$ at its isolated singular points.

$$\frac{1}{(z^2+1)^3} = \frac{1}{(z+\mathrm{i})^3(z-\mathrm{i})^3}$$

$\pm\mathrm{i}$ 3rd-order pole

$$\mathrm{res} f(\pm\mathrm{i}) = \frac{1}{2!}\frac{d^2}{dz^2}\left[(z\mp\mathrm{i})^3\cdot\frac{1}{(z^2+1)^3}\right]\Big|_{z=\pm\mathrm{i}} = \mp\frac{3}{16}\mathrm{i}$$

Some tricks

$$f(z) = \frac{1}{(z-1)^2(z-2)(z-3)} = \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-2} + \frac{D}{z-3}$$

You can perform decomposition of partial fraction to figure out values for A-D.

Let's try a different approach.

$$(z-1)f(z) = \frac{1}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + B + \frac{C(z-1)}{z-2} + \frac{D(z-1)}{z-3}$$

Then $A = \mathrm{res}(z-1)f(z)|_{z=1} = \frac{1}{2}$

$$B = \mathrm{res} f(z)|_{z=1} = \frac{3}{4}$$

$$C = \mathrm{res} f(z)|_{z=2} = -1$$

$$D = \mathrm{res} f(z)|_{z=3} = \frac{1}{4}$$

Residue at infinity

If ∞ is not a non-isolated singular point of $f(z)$, then we can define

$$\mathrm{res} f(\infty) = \frac{1}{2\pi \mathrm{i}} \oint_{C'} f(z) dz$$

Along the clockwise direction, around ∞

Inside C′, $f(z)$ is analytic except for ∞

$$\mathrm{res} f(\infty) = \frac{1}{2\pi \mathrm{i}} \oint_{C'} f(z) dz = -\frac{1}{2\pi \mathrm{i}} \oint_{C} \frac{f(1/t)}{t^2} dt \qquad z = 1/t,\ dz = -1/t^2 dt$$

$$= -\frac{f(1/t)}{t^2}$$ coefficient of term t^{-1} around 0

$$= -f(1/t)$$ coefficient of term t^{1} around 0

$$= -f(z)$$ coefficient of term z^{-1} around ∞

Note z^{-1} is analytic at ∞

$f(z)$ may be analytic at ∞, but $\mathrm{res} f(\infty) \neq 0$.

∞ can be an isolated singular point of $f(z)$, but $\mathrm{res} f(\infty)$ could be 0.

Application of Residue Theorem: For computing integration

$$I = \int_0^{2\pi} R(\sin\theta,\ \cos\theta)d\theta$$

$R(,)$ is continuous in $[0, 2\pi]$

$$z = e^{i\theta} \qquad \frac{1}{z} = e^{-i\theta}$$

Variable substitution

$$\Updownarrow$$

$$I = \oint_{|z|=1} R\left(\frac{z^2-1}{2iz},\ \frac{z^2+1}{2z}\right)\frac{dz}{iz}$$

$R(,)$ does not have any singular point along the unit circle $|z| = 1$; it may have singular points inside the circle

Exercise

- [6.07] Compute $I = \int_0^{\pi} \frac{1}{1+\varepsilon\cos\theta} d\theta, |\varepsilon| < 1$

$$I = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1}{1+\varepsilon\cos\theta} d\theta = \frac{1}{2}\oint_{|z|=1} \frac{1}{1+\varepsilon\frac{z^2+1}{2z}} \frac{dz}{\mathrm{i}z}$$

$$= \frac{1}{2}\oint_{|z|=1} \frac{2}{\varepsilon z^2 + 2z + \varepsilon} \frac{dz}{\mathrm{i}} = \pi \sum_{|z|<1} \operatorname{res} \frac{2}{\varepsilon z^2 + 2z + \varepsilon}$$

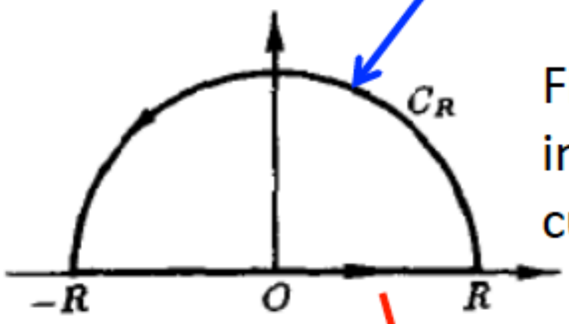
Only one root satisfies the condition

$$= \pi \frac{2}{2\varepsilon z + 2}\Big|_{z=(-1+\sqrt{1-\varepsilon^2})/\varepsilon} = \frac{\pi}{\sqrt{1-\varepsilon^2}}$$

Application of Residue Theorem: Infinite Integration

$$I = \int_{-\infty}^{\infty} f(x)dx$$

Judge the behavior along the great arc



Find the residue(s) inside the closed curve

$$I = \lim_{R_1 \to \infty, R_2 \to \infty} \int_{-R_1}^{R_2} f(x)dx$$

$$\text{v.p.} \int_{-\infty}^{\infty} f(x)dx = I = \lim_{R \to \infty} \int_{-R}^{R} f(x)dx$$

The basic idea is to expand the region of integration.

For now we assume that there are no singular points along the closed curve.

Exercise

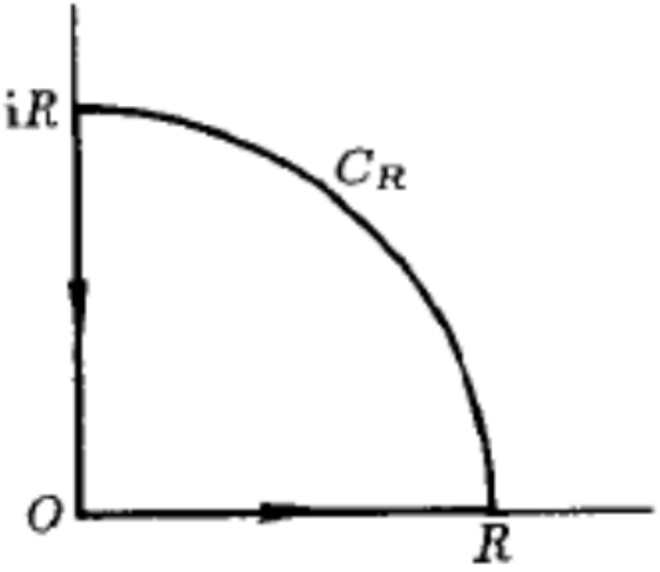
- [6.08] Compute $\displaystyle\int_0^\infty \frac{dx}{1+x^4}$

$$\oint_C \frac{1}{1+z^4}dz = \int_0^R \frac{1}{1+x^4}dx + \underbrace{\boxed{\int_{C_R} \frac{1}{1+z^4}dz}}_{} + \int_R^0 \frac{\mathrm{i}dy}{1+(\mathrm{i}y)^4}$$

$= 0$

$$= (1-\mathrm{i})\int_0^R \frac{dx}{1+x^4} + \int_{c_R} \frac{dz}{1+z^4}$$

$$= 2\pi\mathrm{i}\,\mathrm{res}\frac{1}{1+z^4}\Big|_{z=e^{\mathrm{i}\pi/4}} = \frac{\pi}{2}\frac{1-\mathrm{i}}{\sqrt{2}}$$



There is only one singular point inside the first quadrant

Let $R \to \infty$

$$\lim_{R\to\infty} \frac{z}{1+z^4} = 0$$

$$\int_0^\infty \frac{1}{1+x^4}dx = \frac{\sqrt{2}}{4}\pi$$

Application of Residue Theorem: Infinite Integration

$$I = \int_{-\infty}^{\infty} f(x)\cos px\,dx,\ \ I = \int_{-\infty}^{\infty} f(x)\sin px\,dx \qquad (p > 0)$$

We do not take the substitution $f(z)\cos pz$ or $f(z)\sin pz$, because $z = \infty$ is an essential singular point.

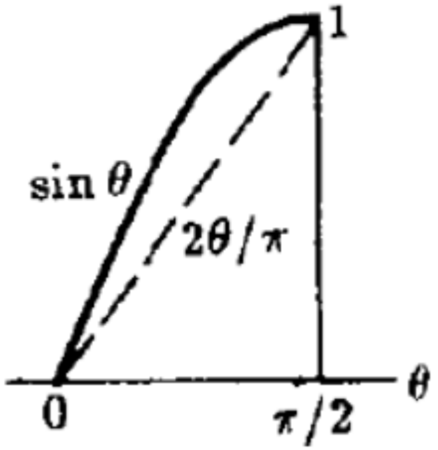
Instead, we take $f(z)e^{\mathrm{i}pz}$

$$\oint_C f(z)e^{\mathrm{i}pz}dz = \int_{-R}^{R} f(x)(\cos px + \mathrm{i}\sin px)dx + \int_{C_R} f(z)e^{\mathrm{i}pz}dz$$

Jordan's lemma

Assume that in $0 \le \arg z \le \pi$, when $|z| \to \infty$, $Q(z)$ uniformly converges to 0, then

$$\lim_{R\to\infty}\int_{C_R} Q(z)e^{\mathrm{i}pz}dz = 0$$



where $p > 0$, C_R is an arc centered at origin, in the upper half space.

Proof: Along the arc, we have $z = Re^{\mathrm{i}\theta}$.

The condition of uniform convergence implies that

$\forall\varepsilon > 0, \exists M(\varepsilon) > 0$ (not related to z), when $|z| = R > M$

and $0 \le \arg z \le \pi$, $|Q(z)| < \varepsilon$

$$\sin\theta \ge 2\theta/\pi$$

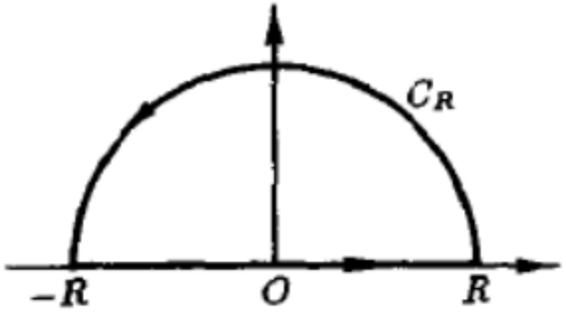
$$|\int_{C_R} Q(z)e^{\mathrm{i}pz}dz| = |\int_0^\pi Q(Re^{\mathrm{i}\theta})e^{\mathrm{i}pR(\cos\theta+\mathrm{i}\sin\theta)}Re^{\mathrm{i}\theta}\mathrm{i}d\theta|$$

$$\le \int_0^\pi |Q(Re^{\mathrm{i}\theta})|e^{-pR\sin\theta}Rd\theta < \varepsilon R\int_0^\pi e^{-pR\sin\theta}d\theta = 2\varepsilon R\int_0^{\pi/2} e^{-pR\sin\theta}d\theta$$

$$< 2\varepsilon R\int_0^{\pi/2} e^{-2pR\theta/\pi}d\theta = \frac{\varepsilon\pi}{p}(1 - e^{-pR})$$

Exercise

- [6.09] Compute $\displaystyle\int_0^\infty \frac{x\sin x}{x^2+a^2}dx, a>0$



$$\oint_C \frac{ze^{\mathrm{i}z}}{z^2+a^2}dz = \int_{-R}^{R} \frac{xe^{\mathrm{i}x}}{x^2+a^2}dx + \underbrace{\boxed{\int_{C_R} \frac{ze^{\mathrm{i}z}}{z^2+a^2}dz}}_{=0}$$

$$= 2\pi\mathrm{i}\ \mathrm{res}\frac{ze^{\mathrm{i}z}}{z^2+a^2}\Big|_{z=a\mathrm{i}} = \pi\mathrm{i}e^{-a}$$

Apply Jordan's lemma

$$\lim_{R\to\infty}\frac{z}{z^2+a^2} = 0$$

$$\int_0^\infty \boxed{\frac{x\sin x}{x^2+a^2}}dx = \frac{\pi}{2}e^{-a}$$

An even function with respect to *x*

A complementary lemma

Assume that $Q(z)$ only has a limited number of singular points. In the lower half space, when $|z| \to \infty$, $Q(z)$ converges to 0 uniformly. Then

$$\lim_{R\to\infty} \int_{C_R} Q(z)e^{-\mathrm{i}pz}dz = 2\pi\mathrm{i} \cdot \sum_{\text{whole space}} \mathrm{res}\,\{Q(z)e^{-\mathrm{i}pz}\}$$

where $p > 0$, C_R is a half circle centered at origin, in the upper half space.

With singular point(s) along the path

We have to get around those singular points, and apply the limit of improper integral.

$$\int_a^b f(x)dx = \lim_{\delta_1 \to 0} \int_a^{c-\delta_1} f(x)dx + \lim_{\delta_2 \to 0} \int_{c+\delta_2}^b f(x)dx$$

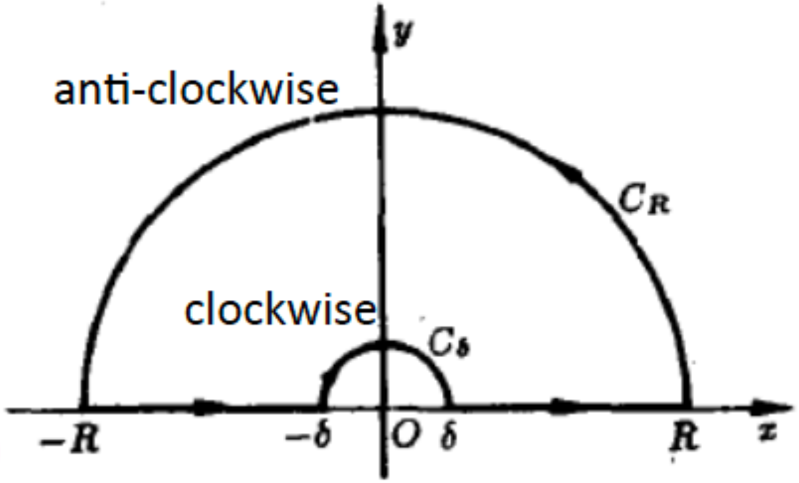
$$\text{v.p.} \int_a^b f(x)dx = \lim_{\delta \to 0} \left[\int_a^{c-\delta} f(x)dx + \int_{c+\delta}^b f(x)dx \right]$$

Usually, the singular point is of 1st-order pole, then we can apply the small/great arc lemma.

$$\lim_{|z-a| \to 0} (z-a)f(z) \to k \quad \text{or} \quad \lim_{z \to \infty} zf(z) \to K$$

If the order is much higher, then we may not apply the small/great arc lemma.

Exercise



[6.10] Compute $\text{v.p.} \displaystyle\int_{-\infty}^{\infty} \frac{dx}{x(1+x+x^2)}$

$-\pi \mathrm{i}$

small arc lemma
(mind the direction)

$$\oint_C \frac{dz}{z(1+z+z^2)} = \int_{-R}^{-\delta} \frac{dx}{x(1+x+x^2)} + \boxed{\int_{C_\delta} \frac{dz}{z(1+z+z^2)}} + \int_{\delta}^{R} \frac{dx}{x(1+x+x^2)}$$

$\lim\limits_{\delta \to 0}$

$$+ \boxed{\int_{C_R} \frac{dz}{z(1+z+z^2)}}$$

= 0 (great arc lemma)

$\lim\limits_{R \to \infty}$

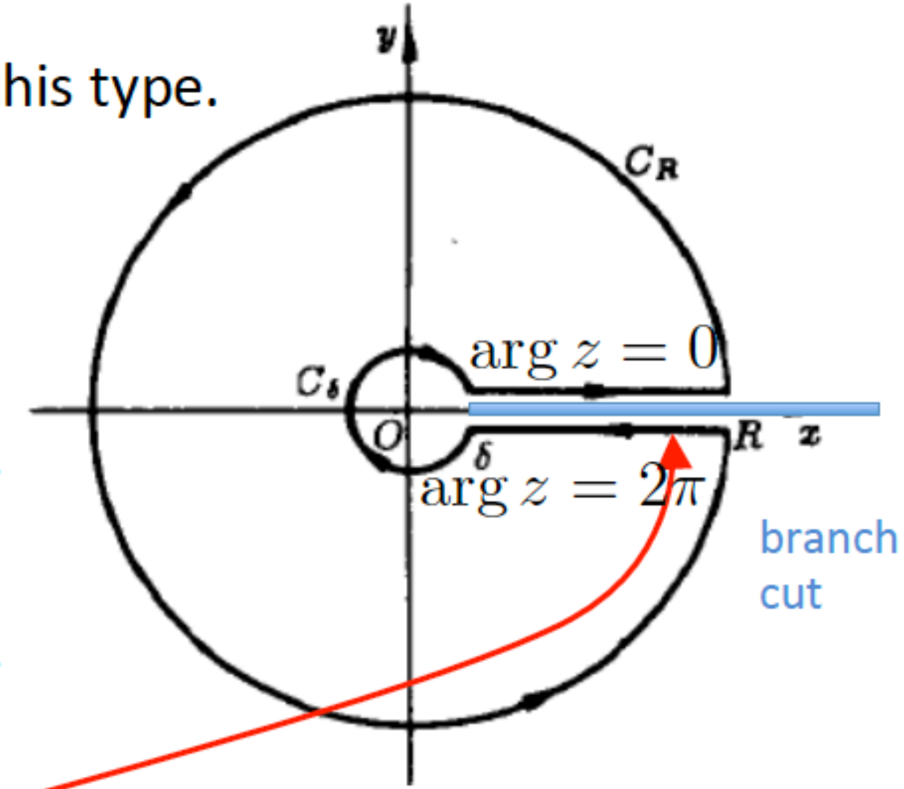
$$= 2\pi \mathrm{i} \ \mathrm{res} \frac{1}{z(1+z+z^2)}\Big|_{z=e^{\mathrm{i}2\pi/3}} = -\frac{\pi}{\sqrt{3}} - \mathrm{i}\pi$$

$$\text{v.p.} \int_{-\infty}^{\infty} \frac{dx}{x(1+x+x^2)} = -\frac{\pi}{\sqrt{3}}$$

Integration involving a multi-valued function

$$\int_0^\infty x^{s-1} Q(x)\,dx$$

There are many transforms of this type.



Assume *s* is a real number. When it is not an integer, then z^{s-1} can be multi-valued.

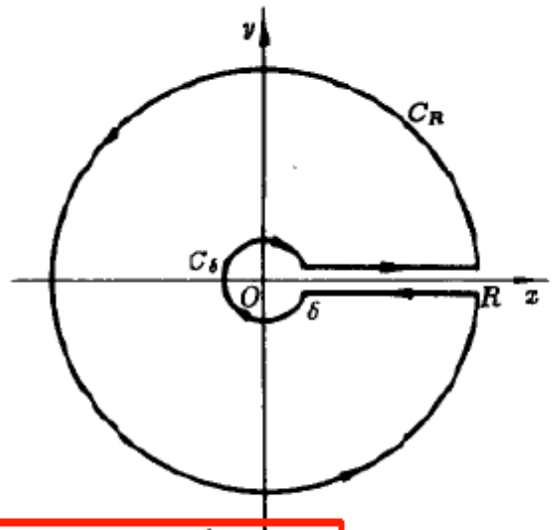
$$\oint_C z^{s-1} Q(z)\,dz = \int_{C_\delta} z^{s-1} Q(z)\,dz + \int_\delta^R x^{s-1} Q(x)\,dx$$

$$+ \int_{C_R} z^{s-1} Q(z)\,dz + \int_R^\delta (xe^{i2\pi})^{s-1} Q(x)\,dx$$

Recall that 0 and ∞ are branch points.

We assume $0 \le \arg z \le 2\pi$.

Exercise



- [6.11] Compute $\displaystyle\int_0^{\infty} \frac{x^{a-1}}{x+e^{\mathrm{i}\phi}}dx, 0<a<1, -\pi<\phi<\pi$

$$\oint_C \frac{z^{a-1}}{z+e^{\mathrm{i}\phi}}dz = \int_\delta^R \frac{x^{a-1}}{x+e^{\mathrm{i}\phi}}dx + \boxed{\int_{C_R} \frac{z^{a-1}}{z+e^{\mathrm{i}\phi}}dz} + \int_R^\delta \frac{(xe^{\mathrm{i}2\pi})^{a-1}}{x+e^{\mathrm{i}\phi}}dx + \boxed{\int_{C_\delta} \frac{z^{a-1}}{z+e^{\mathrm{i}\phi}}dz}$$

= 0 (great arc lemma)

= 0 (small arc lemma)

$$\boxed{\begin{array}{l} = 2\pi\mathrm{i} \sum \mathrm{res} \dfrac{z^{a-1}}{z+e^{\mathrm{i}\phi}} \\ z = e^{\mathrm{i}(\phi+\pi)} \\ 0<\phi+\pi<2\pi \end{array}}$$

$$-2\pi\mathrm{i}e^{\mathrm{i}\pi a}e^{\mathrm{i}\phi(a-1)} \longleftrightarrow (1-e^{\mathrm{i}2\pi a})\int_\delta^R \frac{x^{a-1}}{x+e^{\mathrm{i}\phi}}dx$$

$$\int_0^{\infty} \frac{x^{a-1}}{x+e^{\mathrm{i}\phi}}dx = -\frac{2\pi\mathrm{i}}{1-e^{\mathrm{i}2\pi a}}e^{\mathrm{i}\pi a}e^{\mathrm{i}\phi(a-1)} = \frac{\pi}{\sin\pi a}e^{\mathrm{i}\phi(a-1)}$$