Mathematical Methods in Physics

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Review

- Elementary functions and complex functions
- Power function
- Exponential function
- Trigonometric functions
- Hyperbolic functions
- Radical or *n*th root function
- Logarithmic function
- Inverse trigonometric functions

Chapter – 02: Functions of a complex variable

- In this week, we will be discussing properties of functions of a complex variable (复变函数) and analytic functions (解析函数).

- Last week we introduced the properties of several complex functions (复函数), but here we use the term “functions of a complex variable”, to emphasize that z in $f(z)$ is a variable (more precisely, a complex variable).

- For now we mainly focus on functions of a **single** complex variable.

Review on set (集合) and connectivity (连通性)

- We have already learned the relation between point (点) and set (集合) in MATH 101.

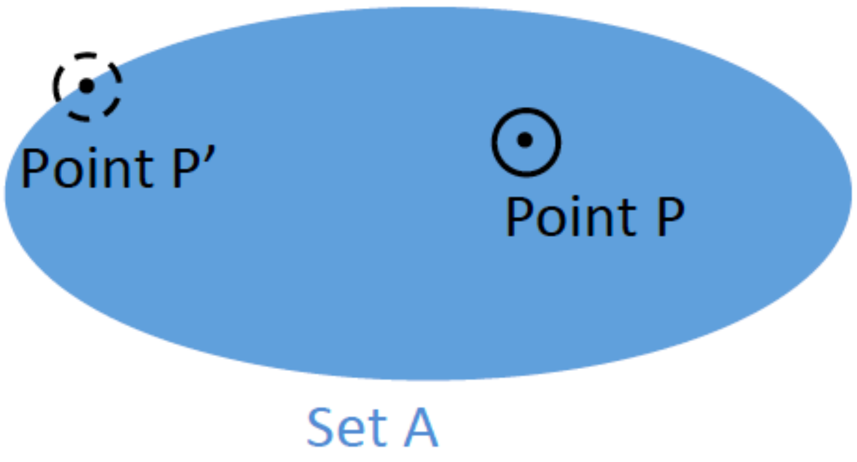
- Now let’s define interior point (内点).

Suppose there is a set A that includes many points. Point P is said to be an interior point of set A, if there exists a circle surrounding point P, such that **all the points that fall within the circle** also belongs to set A.

P is an interior point of set A.

P’ is not an interior point of set A. (Some of you may guess that P’ is at the boundary of set A).

For P’, no matter how small the circle is, there are always some points that will fall outside set A.

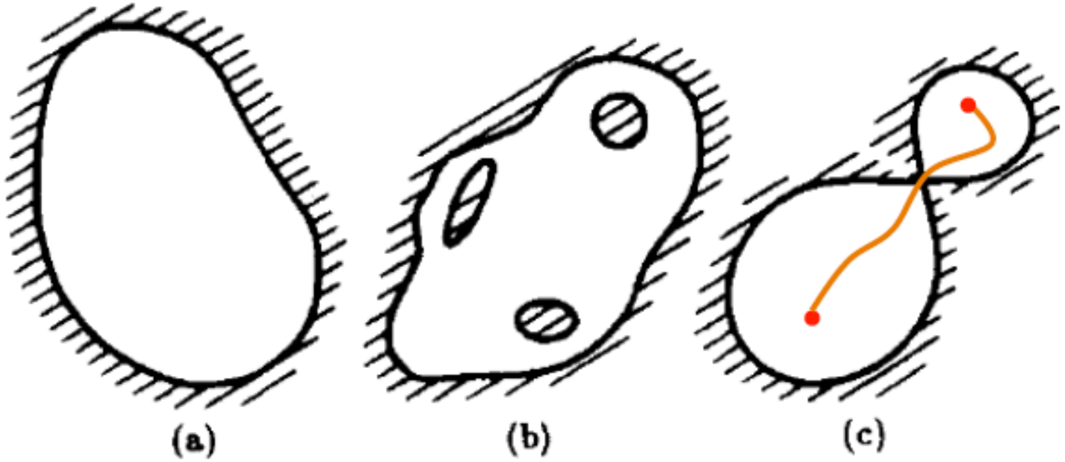


Review on set (集合) and connectivity (连通性)

- Then we can define region (区域).

- A set is said to be a region, if it satisfies the following two conditions: (1) all the points are interior points, and (2) the set is connected (具有连通性).

- About connectivity: For any two points in the set, they can be connected with a curve (or line), and all the points along the curve (or line) also belong to the same set.



For the left three examples, sets (a) and (b) are regions, but set (c) is not a region.

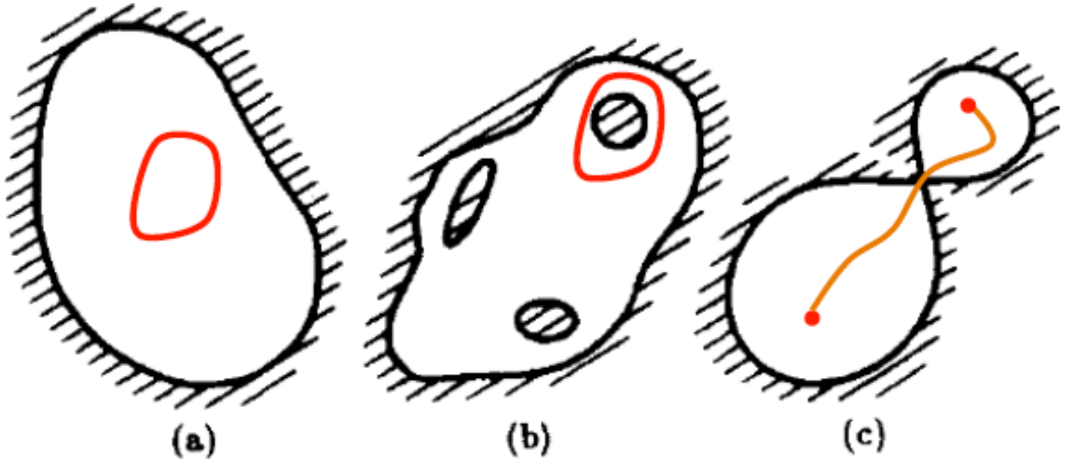
For (a)-(c), we consider the points in the white area.

Review on set (集合) and connectivity (连通性)

- We can further introduce the following classification (分类) of connectivity.

- Simply connected region (单连通区域): Draw a ***simply closed curve*** (e.g. a circle) inside region A, if all the points enclosed by the curve belong to region A, we call region A a "simply connected region". Note:  is not a simply closed curve.

- Multi-connected region (多连通或复连通区域): a region that does not satisfy the definition of simply connected region.



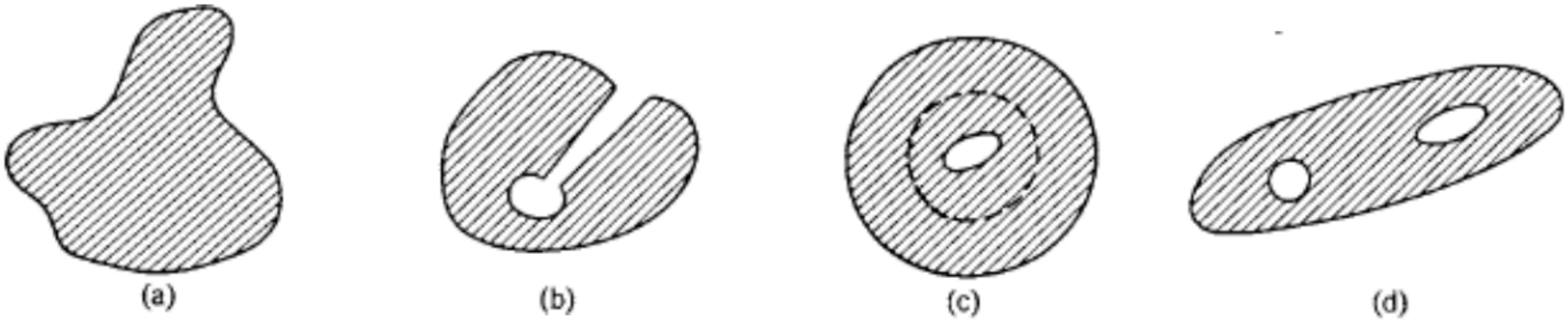
Set (a) is a simply connected region.

Set (b) is a multi-connected region.

Set (c) is not a region.

Exercise

- [3.01] Please judge the type of the following regions.



For (a)-(d), we consider the points in the hatched area.

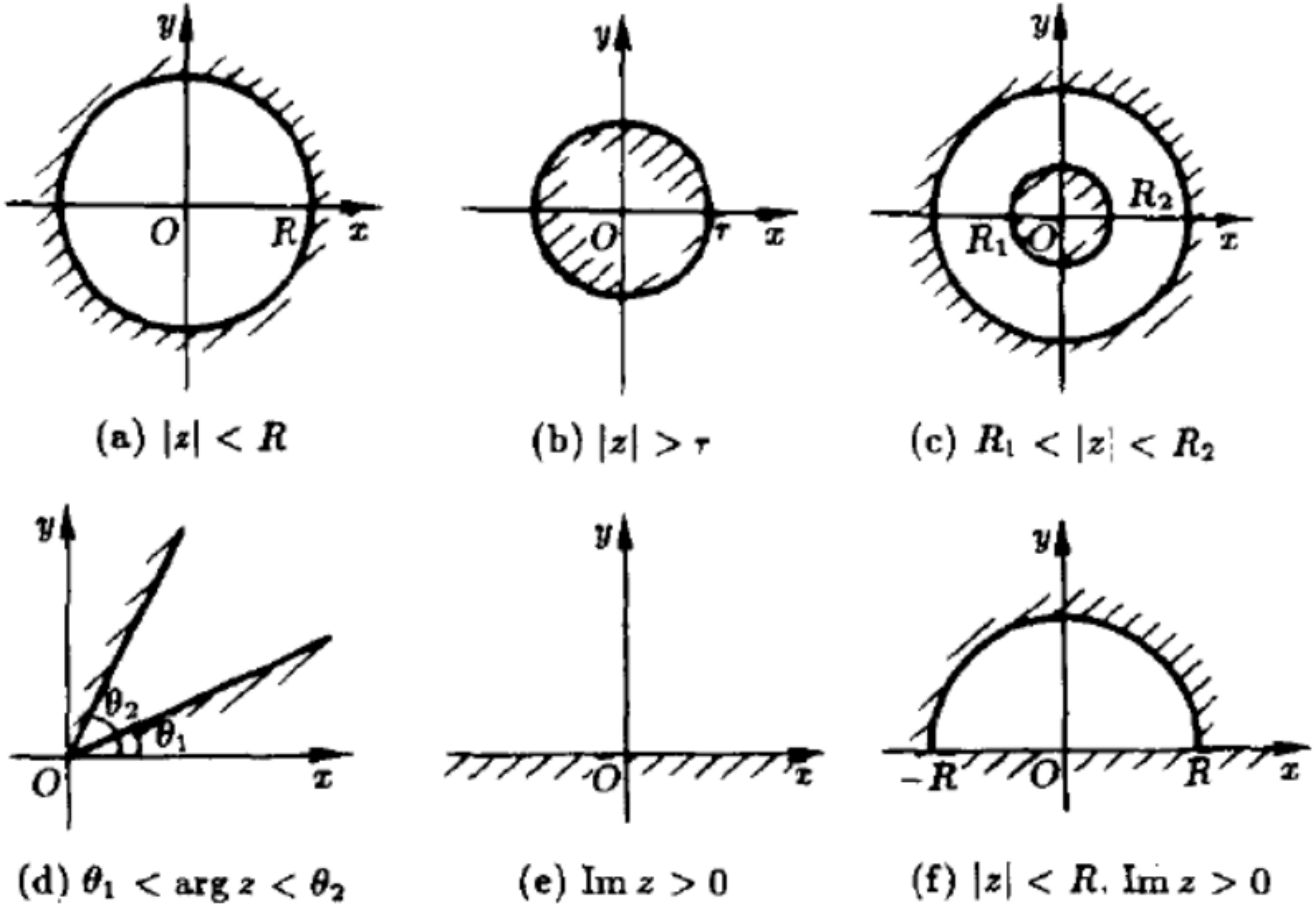
With or without bound

- For a region G, if $\exists M > 0$ (存在 $M > 0$), which satisfies the following relation $|z| < M$ for $\forall z \in G$ (任意 $z \in G$), then we call G is a bounded region (有界区域).

- Otherwise, G is said to be unbounded.

- How to understand "otherwise": you just need to find one exception.

More examples of region



(a) $|z| < R$ (b) $|z| > r$ (c) $R_1 < |z| < R_2$

(d) $\theta_1 < \arg z < \theta_2$ (e) $\operatorname{Im} z > 0$ (f) $|z| < R$, $\operatorname{Im} z > 0$

For (a)-(f), we consider the points in the white area.

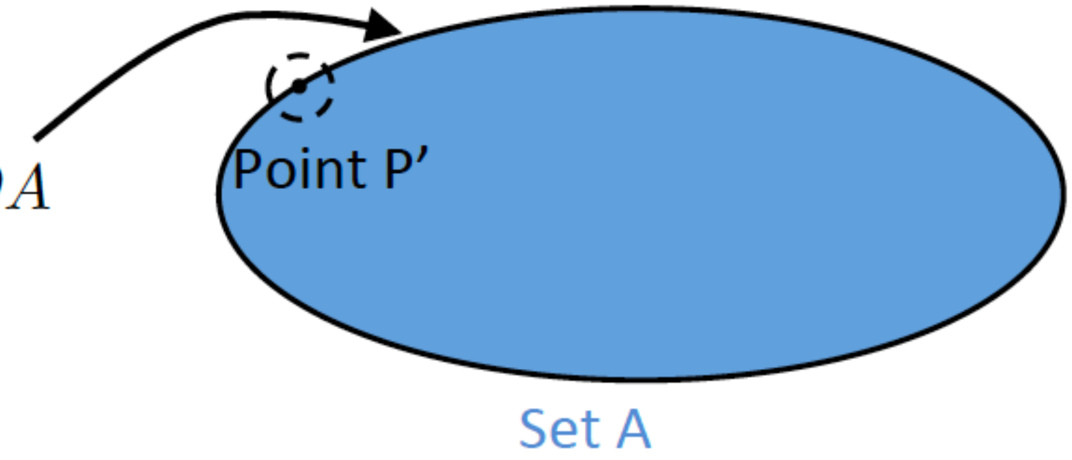
- Can you tell which of these examples are simply connected?
- Which of these examples are bounded?

Boundary and boundary point

- For region G, its boundary point $z_1 \notin G$. But if we draw a circle surrounding z_1, $\forall r > 0$, the interior of the circle ($|z - z_1| < r$) always contains some points that belong to G.

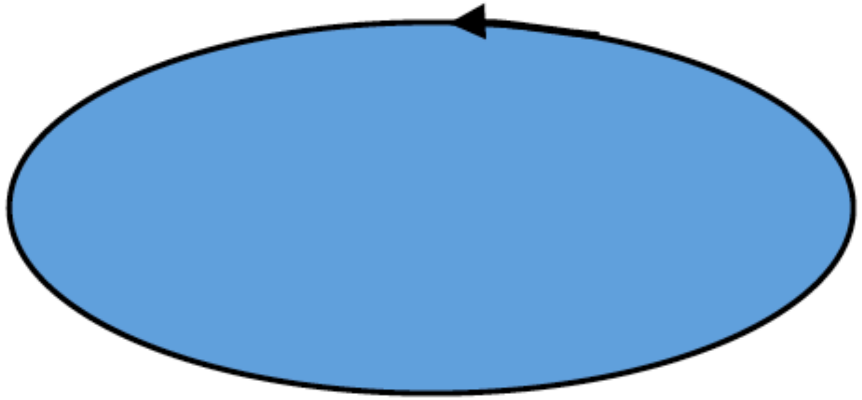
- All the boundary points together constitute the boundary of G.

Boundary of set A: ∂A

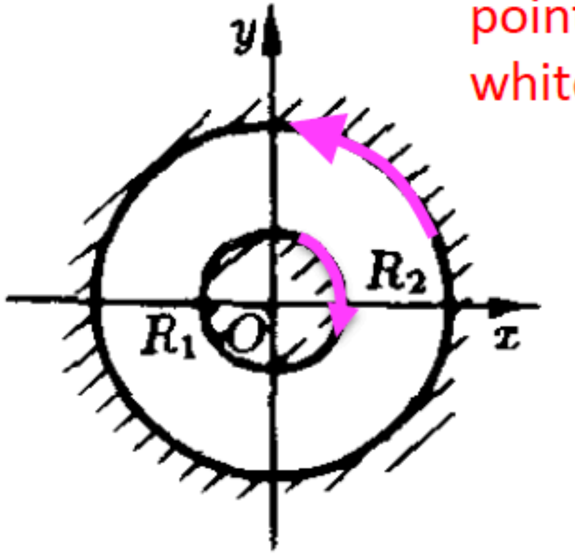


P’ is a boundary point of set A

Direction



We consider the points in the white area.



(c) $R_1 < |z| < R_2$

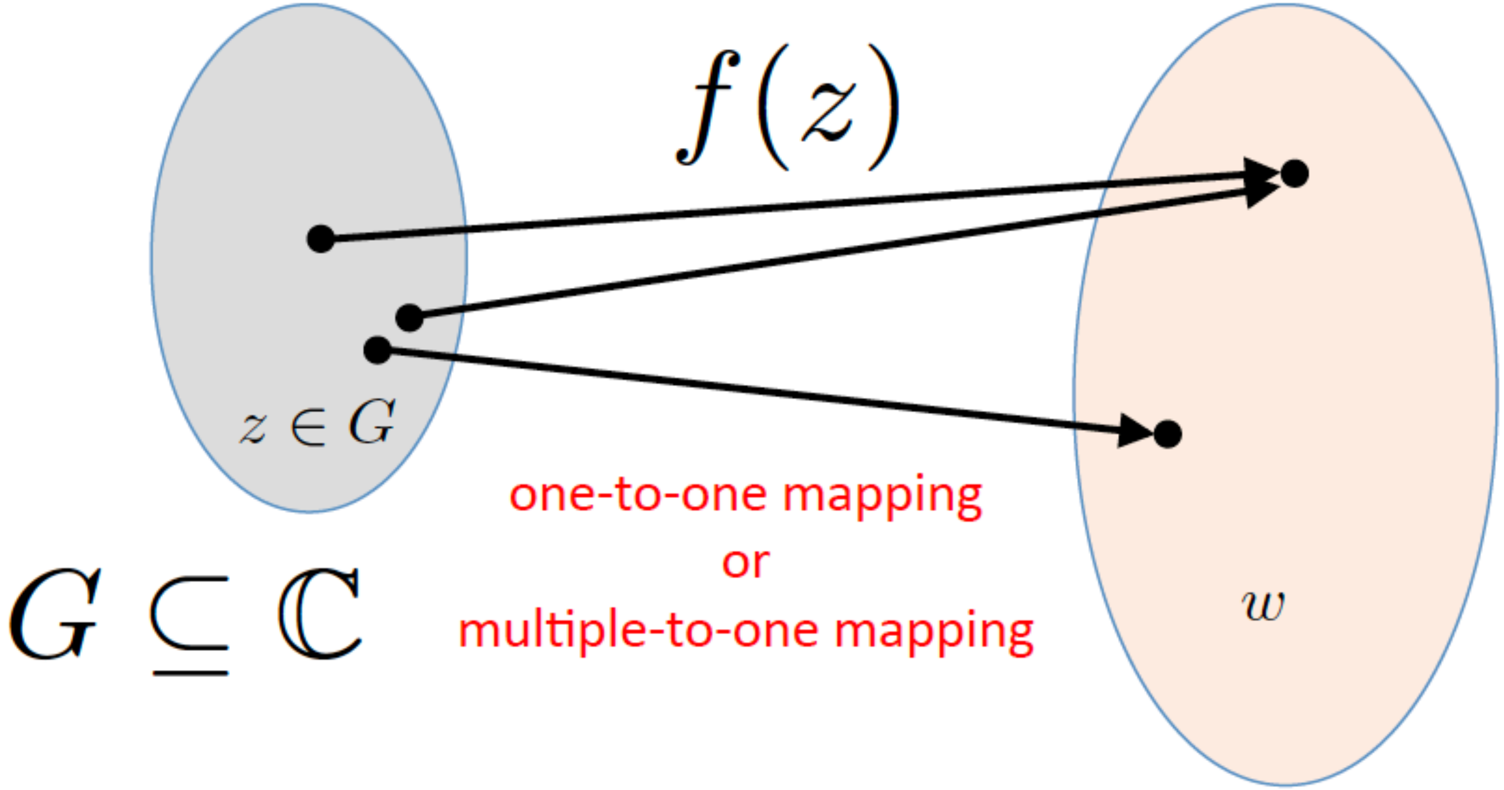
Let's move along the boundary of a region. If the region keeps on the left-hand side of the moving direction, we define that moving direction as the positive direction.

For the above example, the positive direction is the anti-clockwise direction.

What is the positive direction along R_1?

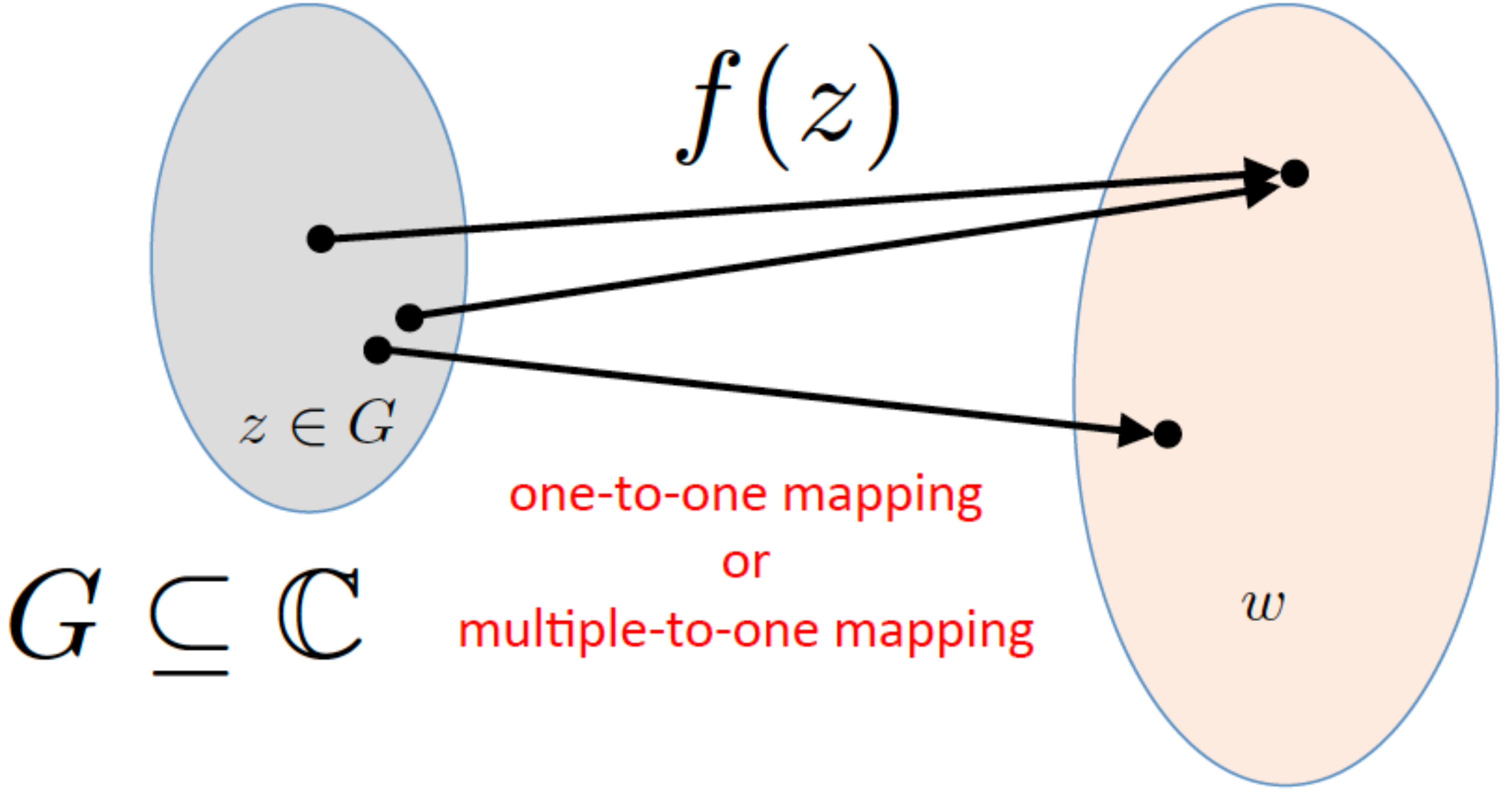
What about along R_2?

Definition: function of a complex variable



For every complex number in region G, there exists a unique corresponding complex number w. We use f to denote the relation between z and w, and called f a function of a complex variable defined in region G.

Definition: function of a complex variable



According to the definition, both z and w are complex numbers. Therefore, we can write:

$$z = x + \mathrm{i}y \xrightarrow{f(z)} w = f(z) = u(x, y) + \mathrm{i}v(x, y)$$

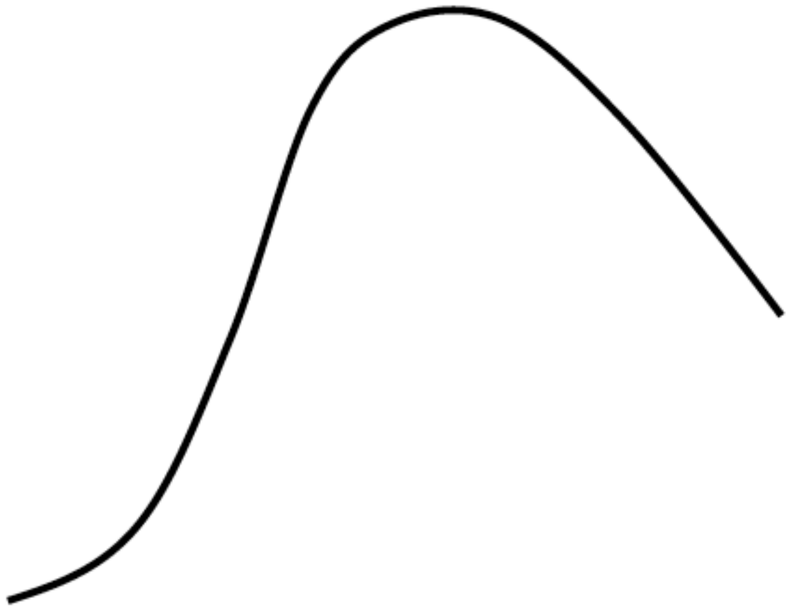
Limit (极限) and continuity (连续性)

- Assume $f(z)$ can be defined in a small region around z_0. If there exists a complex number A, $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$, such that when $0 < |z - z_0| < \delta$, the relation $0 < |f(z) - A| < \varepsilon$ always holds. We have: $\lim_{z \to z_0} f(z) = A$

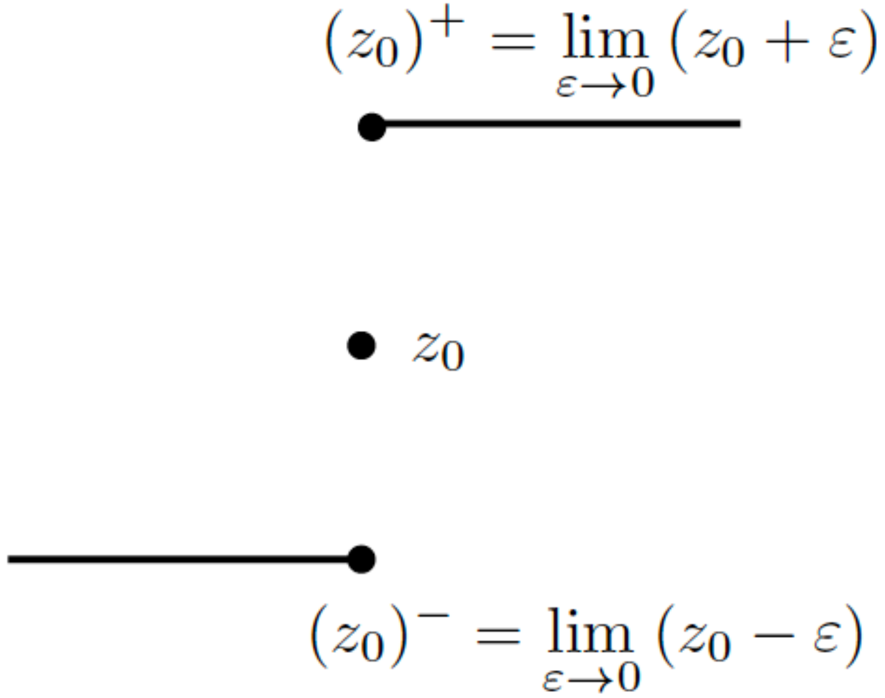
- If $f(z)$ is also defined at z_0, and that $\lim_{z \to z_0} f(z) = f(z_0)$, then $f(z)$ is continuous at $z = z_0$.

- Remark: to judge the continuity of a function at point z_0, one has to investigate the behavior of that function around z_0 and at z_0.

Some examples



Continuous



Not continuous at z_0, but continuous at other points

Derivative (导数)

- Suppose $f(z)$ is a single-valued function defined in G, if it satisfies the following:

  The limit $\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$ exists. Then we say $f(z)$ is derivable (可导) at z.

  We use $f'(z)$ to denote the above limit when it exists.

- Remark: The above statement implies that for an arbitrary path $\Delta z \to 0$, the ratio

  $\frac{f(z+\Delta z)-f(z)}{\Delta z}$ exists and **approaches the same value.**

Example 1. Show that $(d/dz)(z^2) = 2z$. By (2.1) we have

$$\frac{d}{dz}(z^2) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z}$$
$$= \lim_{\Delta z \to 0} (2z + \Delta z) = 2z.$$

We see that the result is independent of *how* Δz tends to zero; thus z^2 is an analytic function. By the same method it follows that $(d/dz)(z^n) = nz^{n-1}$ if n is a positive integer (Problem 30).

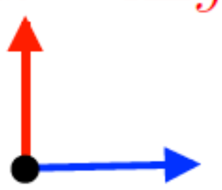
p 668 in the Textbook

Derivative (导数)

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$f(z) = f(x, y) = u(x, y) + \mathrm{i}v(x, y)$$

$$\Delta z = \mathrm{i}\Delta y, \ \ (\Delta x = 0)$$



$$\Delta z = \Delta x, \ \ (\Delta y = 0)$$

$$\begin{aligned} z &= (x, y) \\ &= x + \mathrm{i}y \end{aligned}$$

(1) Along x direction

$$\lim_{\Delta x \to 0} \frac{\Delta u + \mathrm{i}\Delta v}{\Delta x} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}$$

(2) Along y direction

$$\lim_{\Delta y \to 0} \frac{\Delta u + \mathrm{i}\Delta v}{\mathrm{i}\Delta y} = -\mathrm{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

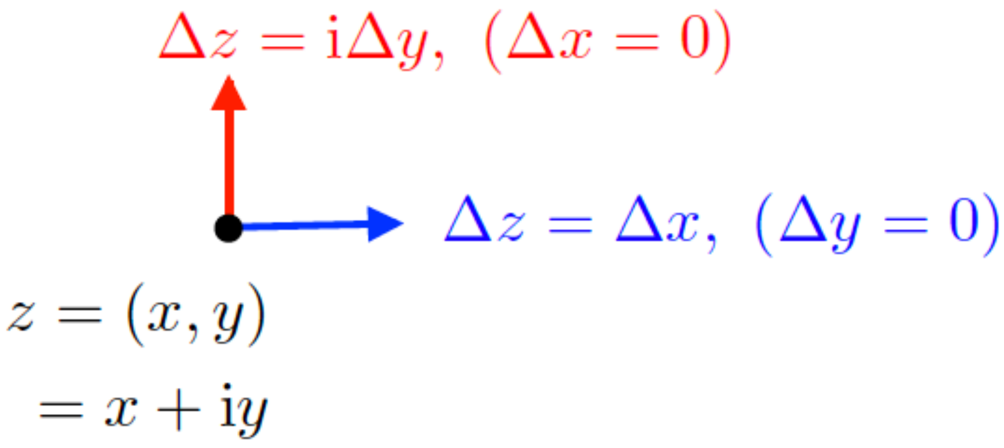
The Cauchy-Riemann relations

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Derivative (导数)

Some remarks: $f(z) = f(x,y) = u(x,y) + \mathrm{i}v(x,y)$

- The CR relations only provide a necessary condition (必要条件) for the existence of $f'(z)$, because the CR relations only guarantee that the limit along two particular paths exists.



- If both u and v are differentiable (可微) at z , and satisfy the CR relations, then $f(z)$ is derivable at z .

The Cauchy-Riemann relations

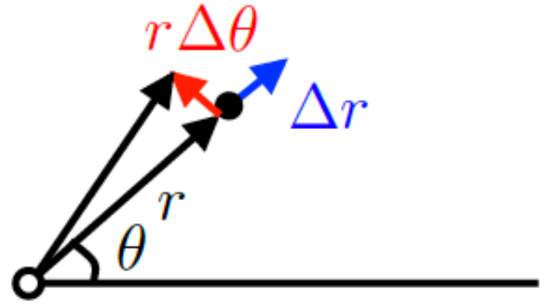
$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Exercise

- [3.02] Please derive the Cauchy-Riemann relations in a polar coordinate system.

$$z = re^{i\theta}$$

$$\Delta z = e^{i\theta}\Delta r + ire^{i\theta}\Delta\theta$$



(1) Along r direction $(\Delta\theta = 0)$

$$\lim_{\Delta r\to 0}\frac{\Delta u + i\Delta v}{\Delta r e^{i\theta}} = \frac{1}{e^{i\theta}}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

(2) Along θ direction $(\Delta r = 0)$

$$\lim_{\Delta\theta\to 0}\frac{\Delta u + i\Delta v}{r\Delta\theta i e^{i\theta}} = \frac{1}{rie^{i\theta}}\left(\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right) = \frac{1}{e^{i\theta}}\left(\frac{-i}{r}\frac{\partial u}{\partial\theta} + \frac{1}{r}\frac{\partial v}{\partial\theta}\right)$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta}}$$

Any other methods?

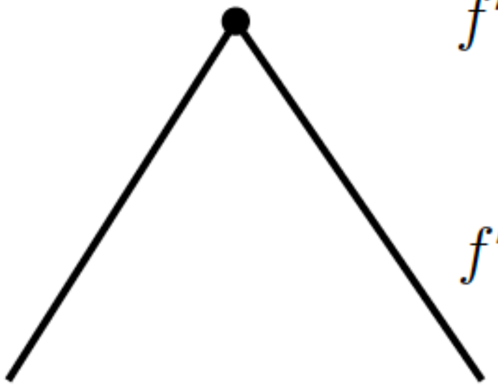
The relation between derivative and continuity

- If $f(z)$ is derivable at z, then $f(z)$ is continuous at z.

- If $f(z)$ is continuous at z, $f(z)$ may or may not be derivable at z.

$z_0 = x_0$

$f'(z)$ does not exists at z_0



$f'(z = x) = f'(x) > 0$

$f'(z = x) = f'(x) < 0$

Derivative and differentiable

- If **the total change** of $w = f(z)$ around z can be written as

  $$\Delta w = f(z + \Delta z) - f(z) = A(z)\Delta z + \rho(\Delta z)$$

  higher-order term(s)

  and satisfies the condition: $$\lim_{\Delta z \to 0} \frac{\rho(\Delta z)}{\Delta z} = 0$$

  Then it is said that $f(z)$ is differentiable at z .

- The linear part of the total change $dw = A(z)\Delta z$ is called the differentiation (微分) of w at z .

Derivative ⇔ Differentiable

Analytic function (解析函数)

- If $f(z)$ is derivable for every point $z \in G$, $f(z)$ is said to be an analytic function defined in G.   e.g. z^n
- Note: If $f(z)$ is analytic at z_0, then $f(z)$ is derivable everywhere for a small region around z_0 and at z_0. $\Rightarrow$ Analytic is a stronger condition than derivable.

Definition: A function $f(z)$ is *analytic* (or *regular* or *holomorphic* or *monogenic*) in a region* of the complex plane if it has a (unique) derivative at every point of the region. The statement "$f(z)$ is analytic *at a point* $z = a$" means that $f(z)$ has a derivative at every point inside some small circle about $z = a$.

*Isolated points and curves are not regions; a region must be two-dimensional.

p 667 in the Textbook

Analytic function (解析函数)

- If $f(z)$ is derivable for every point $z \in G$, $f(z)$ is said to be an analytic function defined in G. e.g. z^n
- Note: If $f(z)$ is analytic at z_0, then $f(z)$ is derivable everywhere for a small region around z_0 and at z_0. $\Rightarrow$ Analytic is a stronger condition than derivable.
- According to the above definition, if $f(z)$ is analytic (解析的), then the Cauchy-Riemann relations must hold.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f(z) = f(x, y) = \underbrace{u(x, y) + iv(x, y)}_{\text{inter-connected}}$$

$f(z)$ is different from an arbitrary function of two real variables

How to judge if a function is analytic

$f(z) = u(x,y) + iv(x,y)$ is analytic $\Longleftrightarrow$ CR relations must be satisfied.

AND

$u(x,y)$ and $v(x,y)$ are differentiable in G.

$\Updownarrow$

CR relations must be satisfied.

AND

$\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$ exist, and are continuous in G.

p 670 in the Textbook

Theorem II (which we state without proof). If $u(x,y)$ and $v(x,y)$ and their partial derivatives with respect to x and y are continuous and satisfy the Cauchy-Riemann conditions in a region, then $f(z)$ is analytic at all points inside the region (not necessarily on the boundary).

Exercise

- [3.03] Find the region where $f(z) = z\,\mathrm{Im}z$ has a derivative, and is analytic.

$$f(z) = (x + \mathrm{i}y)y = xy + \mathrm{i}y^2$$

$$\frac{\partial u}{\partial x} = y \qquad \frac{\partial v}{\partial y} = 2y$$

$$\frac{\partial u}{\partial y} = x \qquad \frac{\partial v}{\partial x} = 0$$

CR relations require that $x = 0, \ y = 0$

All four partial derivatives are continuous.

↓

$f'(z)|_{z=0} = 0$

But, if $z \neq 0$, CR relations do not satisfy => not analytic

The behavior at infinity

- Sometimes we need to investigate the behavior of a function at infinity

Variable substitution: $z = 1/t$

$f(z \to \infty)$ $\qquad$ $f(1/t, t \to 0)$

Limit

Continuity

Derivative

Analytic

Exercise

- [3.04] Please verify that the following functions are analytic (except for $z = \infty$)

$$z^n \qquad e^z \qquad \cos z \qquad \cosh z$$

$$\sin z \qquad \sinh z$$

Analytic function (解析函数)

$$f(z) = f(x,y) = u(x,y) + \mathrm{i}v(x,y) \quad \Longleftrightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The Cauchy-Riemann relations imply that once you know $u(x,y)$ or $v(x,y)$, you can figure out the other.

Recall the integration of a function of multiple variables.

$$dv(x,y) = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Full derivative

$$v(x,y) = \int dv(x,y) = \int -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

Exercise

- [3.05] Judge if the function $f(z) = f(x, y) = x - \mathrm{i}y$ is an analytic function.

  Solution-1: $u(x, y) = x \quad v(x, y) = -y$

  $$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = -1$$ do not satisfy the Cauchy-Riemann relations

  Solution-2: $f(z) = f(x, y) = z^*$

  It is an explicit function of z^*, rather than z.

  Therefore, it is not an analytic function.

Exercise

- [3.06] Suppose $u(x,y)=x^2-y^2$, find the explicit form of $f(z)=f(x,y)$.

$$v(x,y)=\int dv(x,y)=\int -\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$$

$$\boxed{v(x,y)=\int 2ydx+2xdy=\int d(2xy)=2xy+C}$$

* Make sure you remember how to get the full derivative from partial derivatives – chain rules.

$$f(x,y)=x^2-y^2+2xy\mathrm{i}+\mathrm{i}C=(x+\mathrm{i}y)^2+\mathrm{i}C=z^2+\mathrm{i}C$$

Exercise

- [3.07] Suppose $v(x,y)=e^x\sin y$, find the explicit form of $f(z)=f(x,y)$.

$$u(x,y)=\int du(x,y)=\int \frac{\partial v}{\partial y}dx-\frac{\partial v}{\partial x}dy$$

$$u(x,y)=\int e^x\cos y dx-e^x\sin y dy=\int d(e^x\cos y)=e^x\cos y+C$$

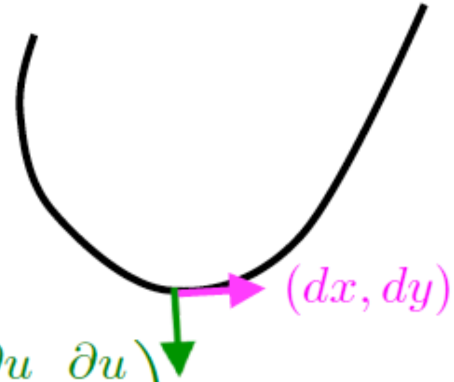
$$\begin{aligned}f(x,y)=e^x\cos y+\mathrm{i}e^x\sin y+C&=e^x(\cos y+\mathrm{i}\sin y)+C\\&=e^{x+\mathrm{i}y}+C=e^z+C\end{aligned}$$

Properties of an analytic function

- Suppose $f(z) = u(x,y) + iv(x,y)$ is an analytic function.

$$u(x,y) = C \quad \Rightarrow \quad du(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

Isoline (等值线)



$$(dx, dy) \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Inner product
(矢量点乘，内积)

The arrow (a vector) represents a unit increment of the curve, along its tangential direction

$$d\vec{s} = (dx, dy) \propto \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right)$$

A unit increment of the curve (vector) along *u* = C

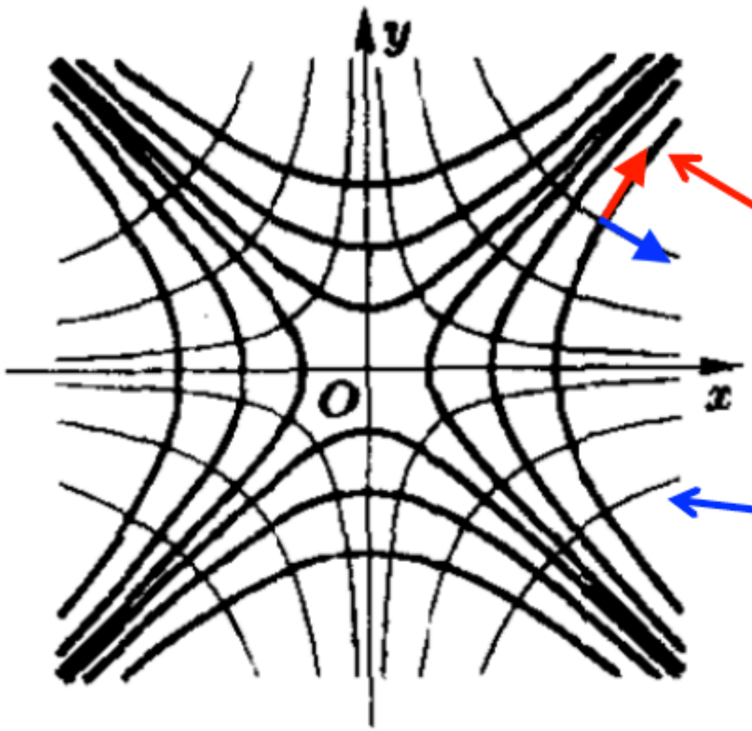
A unit increment of the curve (vector) along *v* = C′

Similarly, $v(x,y) = C' \quad \Rightarrow \quad d\vec{s'} = (dx', dy') \propto \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}\right)$

$$d\vec{s} \cdot d\vec{s'} \propto \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right) \cdot \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}\right) = 0 \quad \Longrightarrow \quad d\vec{s} \perp d\vec{s'}$$

Properties of an analytic function

$$d\vec{s} \cdot d\vec{s'} \propto \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right) \cdot \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}\right) = 0 \Longrightarrow d\vec{s} \perp d\vec{s'}$$



$$w = z^2 = (x + \mathrm{i}y)^2$$

$$u(x, y) = x^2 - y^2$$

$$v(x, y) = 2xy$$

The isolines of u(x,y) are perpendicular to the isolines of v(x,y).

Another way to understand the connection between u(x,y) and v(x,y).

Properties of an analytic function

- Suppose $f(z) = u(x,y) + \mathrm{i}v(x,y)$ is an analytic function.

  So far we have shown some inter-connections between u(x,y) and v(x,y).

  What about u(x,y) and v(x,y) themselves?

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial x \partial y}$$

Laplace's equation

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Delta v = \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Functions that satisfy $\Delta G = \nabla^2 G = 0$ are called harmonic functions (调和函数).

Properties of an analytic function

Laplace's equation

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Delta v = \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Electric potential
given at a boundary

$$\Phi = \Phi_0$$

No electric charge

$$\begin{cases} \nabla^2 \Phi = 0 \\ \Phi = \Phi_0 \text{ at } \partial\Omega \end{cases}$$

Solve for the distribution of Φ