

# 数学物理方法

Mathematical Methods in Physics

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2022 年 4 月 19 日

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## 复变函数部分

# 第一章 复数

约定：我们认为  $z \in \mathbb{C}$ ,  $x, y \in \mathbb{R}$

## 1.1 定义以及运算

定义 1.1.1.

$$i = \sqrt{-1}$$

称之为虚数单位。通过虚数单位和『实数单位 (1)』的线性组合，可以得到任意复数的表示方式：

$$z = x + iy, z \in \mathbb{C}, x, y \in \mathbb{R}$$

$x, y$  分别称为实部和虚部，记为：

$$x = \operatorname{Re} z$$

$$y = \operatorname{Im} z$$

定义 1.1.2.

$$z^* = x - yi$$

称为  $z$  的共轭复数。容易得到，

$$z \cdot z^* = x^2 + y^2 = |z|^2 \geq 0$$

$$x = \operatorname{Re} z = \frac{z + z^*}{2}$$

$$y = \operatorname{Im} z = \frac{z - z^*}{2i}$$

注意到复数的运算与实数的运算存在许许多多的不同之处，例如

例 1.1.3.

$$\lim_{y \rightarrow 0} \frac{1}{x + yi} \neq \frac{1}{x}$$

$$\lim_{y \rightarrow 0} \frac{1}{x + yi} = \lim_{y \rightarrow 0} \frac{x - yi}{x^2 + y^2} \rightarrow$$

$$\operatorname{Re} z = \begin{cases} 0, & x = 0 \\ \frac{1}{x}, & x \neq 0 \end{cases} \quad \operatorname{Im} z = -i\pi\delta(x)$$

## 1.2 复数的几何表示

引入复平面可以容易地表示复数的几何形式：即  $z = x + yi$  在  $x$  轴 (实轴) 上的投影为  $x$ ，在  $y$  轴 (虚轴) 上的投影为  $y$ 。那么，对应向量的 (主) 辐角  $\theta$  以及模  $\rho$  便定义为：

**定义 1.2.1.**

$$\theta = \operatorname{Arg} z; \quad \rho = \sqrt{x^2 + y^2}$$

主辐角记为  $\operatorname{Arg} z \in [-\pi, \pi] = \arctan \frac{y}{x}$ ，辐角记为  $\arg z$  那么得到：

**引理 1.2.2.**

$$z = \rho(\cos \theta + i \sin \theta)$$

注意到：

$$\frac{1}{z} = \frac{1}{\rho(\cos \theta + i \sin \theta)} = \frac{1}{\rho}(\cos \theta - i \sin \theta)$$

**引理 1.2.3.** 假设  $1/z = n \in \mathbb{C}$ ,

$$\rho_n = 1/\rho_z; \quad \arg z = -\arg n$$

同样

**引理 1.2.4.** 假设

$$z = \prod_{i=1}^n z_i \rightarrow \rho_z = \prod_{i=1}^n \rho_{z_i}; \quad \arg z = \sum_{i=1}^n \arg z_i$$

$$z_i \in \mathbb{C}$$

**定理 1.2.5.** *de Moivre's* 定理：

$$z_1 = \rho_1(\cos \theta_1 + i \sin \theta_1) \quad z_2 = \rho_2(\cos \theta_2 + i \sin \theta_2)$$

$\Rightarrow$

$$z_1 \cdot z_2 = \rho_1 \rho_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

结合1.2.3和1.2.4, 我们可以得到任意个复数的乘法除法公式:

**推论 1.2.6.**

$$z = \frac{\prod_{i=1}^n a_i \in \mathbb{C}}{\prod_{i=1}^n b_i \in \mathbb{C}} \implies \rho_z = \frac{\prod_{i=1}^n \rho_{a_i}}{\prod_{i=1}^n \rho_{b_i}} \quad \arg z = \sum_{i=1}^n \arg a_i - \sum_{i=1}^n \arg b_i$$

### 1.3 复数数列

形式如下的序列称为复数数列

$$z_n = x_n + iy_n, \quad n = 1, 2, 3, 4, \dots$$

$$z_n \text{ 收敛} \Leftrightarrow x_n, y_n \text{ 收敛}$$

### 1.4 欧拉公式以及复数的指数函数形式

**定理 1.4.1.** 欧拉公式:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

证明. 由 Taylor-Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

得到

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!} = \left[ 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \dots \right] + i \left[ \theta - \frac{\theta^3}{3!} + \dots \right]$$

考虑到  $\cos \theta$  和  $\sin \theta$  的 Taylor-Series, 得到:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

□

显然如上的证明并不是一个严格的证明, 因为我们没有证明如上的展开适用于复数域, 以及在交换次序时没有事先证明它绝对收敛. 结合1.2.2得到

$$z = \rho e^{i\theta}$$

称为复数的指数函数形式。

**例 1.4.2.** 计算无穷级数:  $\cos \theta + \cos 2\theta + \cos 3\theta + \dots$

证明. 原式等价于

$$\operatorname{Re} [e^{i\theta} + e^{i2\theta} + e^{i3\theta} + \dots]$$

$$\begin{aligned} & e^{i\theta} + e^{i2\theta} + e^{i3\theta} + \dots \\ &= \lim_{n \rightarrow \infty} \frac{e^{i\theta} - e^{i(n+1)\theta}}{1 - e^{i\theta}} \end{aligned}$$

□

## 1.5 复数域上的指数函数的反函数

对于  $\forall z \in \mathbb{C}$ , 如何定义函数  $g = f(z) = e^z$  的反函数? 即定义一个函数, 使得:

$$f^{-1}(g) = z$$

这个函数称为复对数函数, 区别于  $\mathbb{R}$  上的指数函数  $\ln(x)$ 。

**定义 1.5.1.** 复对数函数:

$$\operatorname{Ln} z = \ln |z| + i \arg z + 2n\pi i$$

$$s.t. \operatorname{Ln} g = \operatorname{Ln} |z| e^{i \arg z + 2ni\pi}$$

其多值性来源于

$$g = e^z = e^{z+2ni\pi}$$



## 第二章 复变序列

对于某一复数序列  $u_n = x_n + iy_n$ , 其和前  $n$  项和  $S_n$ :

$$\sum_{n=0}^{\infty} (x_n + iy_n)$$
$$S_n = X_n + iY_n$$
$$X_n = \sum_{i=0}^n x_i, Y_n = \sum_{j=0}^n y_j$$

无穷级数收敛的充要条件:  $\forall \varepsilon > 0, \exists n > 0; n \in \mathbb{Z} \text{ s.t. } \forall p > 0$

$$|u_{n+1} + u_{n+2} + \dots + u_{n+p}| < \varepsilon$$

级数收敛的必要条件:

$$\text{Preliminary Test: } \lim_{n \rightarrow \infty} u_n = 0$$

### 2.1 级数收敛性判别法

**Test for alternating series:** An alternating series converges if the absolute value of the terms decreases steadily to zero, that is, if  $|a_{n+1}| \leq |a_n|$  and  $\lim_{n \rightarrow \infty} a_n = 0$ . **一致递减 至 0**

**Comparison Method:** If  $\exists N \in \mathbb{N}, \forall n > N$ , the condition  $|u_n| < v_n$  is satisfied. If  $\sum_{n=0}^{\infty} v_n$  are convergent, then  $\sum_{n=0}^{\infty} |u_n|$  are convergent.

**Ratio Method:** If there exists a constant  $\rho$  (**un-correlated with  $n$** ), and  $|u_{n+1}/u_n| < \rho < 1$ , then  $\sum_{n=0}^{\infty} u_n$  are absolutely convergent.

**d'Alembert Method(Criterion):** 级数的通项比值  $\left(\frac{u_{n+1}}{u_n}\right)$  的模的上极限小于 1, 则原级数绝对收敛; 级数的通项比值的模下极限大于 1, 则原级数发散。

**Gauss Method:** Assume that the ratio between two neighboring terms has the following form:  $\frac{u_n}{u_{n+1}} = 1 + \frac{\mu}{n} + O(n^{-\lambda})$  where  $\mu = a + ib, \lambda > 1$ .

If  $a > 1$ ,  $\sum_{n=0}^{\infty} u_n$  absolutely convergent.

If  $a \leq 1$ ,  $\sum_{n=0}^{\infty} |u_n|$  divergent.

**例 2.1.1.** 使用 Gauss Method 判别级数  $S_n = \sum_{n=0}^{\infty} \frac{1}{n}$  的收敛性:

$$\frac{u_n}{u_{n+1}} = \frac{n+1}{n} = 1 + \frac{1}{n}$$

则  $a = 1$ , 原级数发散。

使用 Gauss Method 判别级数  $S_n = \sum_{n=0}^{\infty} \frac{1}{n^2}$  的收敛性:

$$\frac{u_n}{u_{n+1}} = \left(\frac{n+1}{n}\right)^2 = 1 + \frac{1}{n^2} + \frac{2}{n}$$

则  $a = 2$ , 原级数绝对收敛。

**Cauchy Method:**  $|u_n|^{1/n}$  的上极限小于 1, 原级数绝对收敛; 大于 1, 原级数发散。

## 2.2 复数序列的一致收敛

如果  $S_n$  一致收敛, 则:

- Continuity  $u_k(z)$  is continuous in  $G$ , and  $\sum_{k=1}^{\infty} u_k(z)$  is uniformly convergent, then  $S(z) = \sum_{k=1}^{\infty} u_k(z)$  is continuous in  $G$
- $\int_C \sum_{k=1}^{\infty} u_k(z) dz = \sum_{k=1}^{\infty} \int_C u_k(z) dz$
- $f(z) = \sum_{k=1}^{\infty} u_k(z)$  is analytic in  $G \rightarrow f^{(p)}(z) = \sum_{k=1}^{\infty} u_k^{(p)}(z)$

## 2.3 幂级数与阿贝尔定理

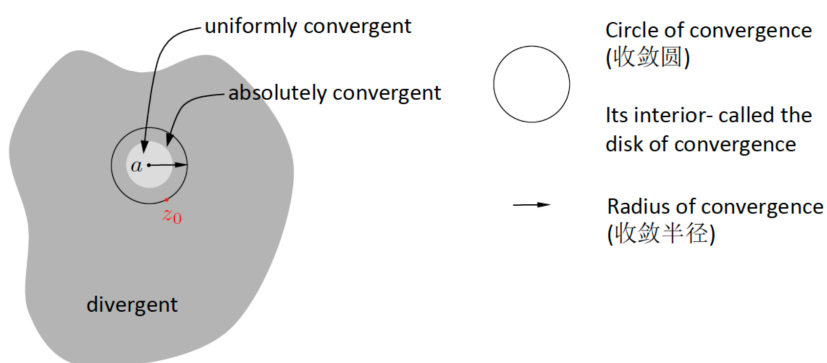
对于幂级数

$$\sum_{n=0}^{\infty} c_n(z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + \dots$$

有

**定理 2.3.1. Abel theorem:** If the series  $\sum_{n=0}^{\infty} c_n(z-a)^n$  are convergent at  $z = z_0$ , then the series are absolutely convergent in a disk region (with a radius of  $|z_0 - a|$ ) surrounding  $a$ , and are uniformly convergent in the region  $|z-a| \leq r$  ( $r < |z_0 - a|$ ).

**推论 2.3.2.** If  $\sum_{n=0}^{\infty} c_n(z-a)^n$  are divergent at  $z_1$ , then also divergent in  $|z-a| > |z_1 - a|$ .



计算幂级数的收敛半径的方法:

**方法 2.3.3. Cauchy-Hadamard Formula:**

$$R = \frac{1}{\overline{\lim}_{n \rightarrow \infty} |c_n|^{1/n}} = \lim_{n \rightarrow \infty} \left| \frac{1}{c_n} \right|^{1/n}$$

**方法 2.3.4. d'Alembert Critrion:**

$$R = \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right|$$

## 第三章 复变函数

### 3.1 复变函数的概念

**定义 3.1.1.** 复变函数是复数区域到复数区域的映射。

$$f: \mathbb{C} \rightarrow \mathbb{C}$$

$$f(z) = u(x, y) + iv(x, y) \quad z = x + iy \quad x, y \in \mathbb{R}$$

与实变函数不同，**区域**与**区间**是有显著差异的。

**定义 3.1.2.** 如果复平面上的点集  $D$  满足以下条件：

1. 开集性：不包含边界。 $\forall z_0 \in D, \exists \epsilon > 0 \quad s.t. \{z \mid |z - z_0| < \epsilon\} \subset D$
2. 连通性：任意两点之间可以用区域内的线连通。

那么点集  $D$  称为 (开) 区域。闭区域

$$\overline{D} = D + \partial D$$

$\partial D$  是  $D$  区域的边界。边界具有方向，其正方向定义为使得区域位于运动方向的左手侧的方向。

**定义 3.1.3.** 双曲函数定义为：

$$\begin{aligned} \sinh(z) &= \frac{e^z - e^{-z}}{2} & \cosh(z) &= \frac{e^z + e^{-z}}{2} & \tanh(z) &= \frac{\sinh(z)}{\cosh(z)} \\ \coth(z) &= \frac{\cosh(z)}{\sinh(z)} & \operatorname{sech}(z) &= \frac{1}{\cosh(z)} & \operatorname{csch}(z) &= \frac{1}{\sinh(z)} \end{aligned}$$

类比  $\cos z, \sin z$  可以得到：双曲函数的周期性：

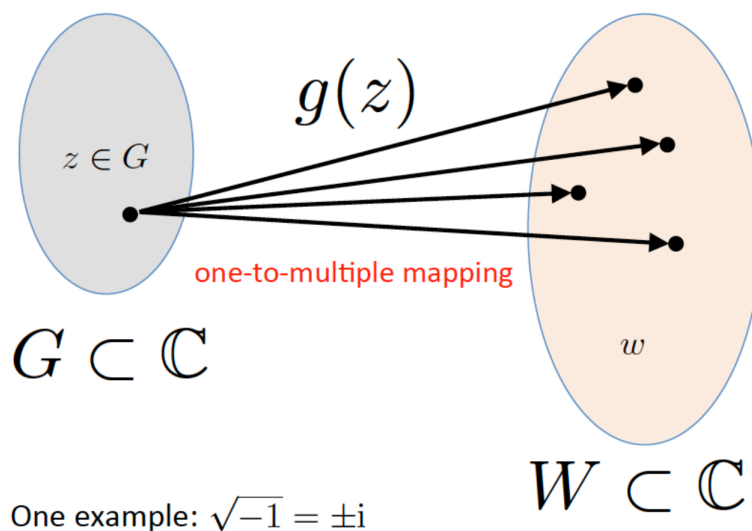
$$\sinh(z) = \sinh(z + i2n\pi) \quad \cosh(z) = \cosh(z + i2n\pi) \quad \tanh(z) = \tanh(z + in\pi) \quad n \in \mathbb{Z}$$

**定义 3.1.4.** 函数  $f^{-1}(z)$  称为函数  $f(z)$  的逆函数，如果：

$$f^{-1}(f(z)) = z$$

### 3.2 单值性与黎曼面

注意函数的多值性：



定义域  $G$   $z = x + iy$   $\xrightarrow{f(z)}$  值域  $W$   $w = f(z) = u(x, y) + iv(x, y)$

**例 3.2.1.** 根式函数:  $w = \sqrt[n]{z-a}$ 。令  $z-a = re^{i\theta}$  得到  $w$  有  $n$  个根:

$$w_1 = \sqrt[n]{r}e^{i\theta/n} \quad w_2 = \sqrt[n]{r}e^{i(\theta/n+2\pi/n)} \quad \dots \quad w_n = \sqrt[n]{r}e^{i(\theta/n+2(n-1)\pi/n)}$$

辐角的多值性

**例 3.2.2.** 对数函数:

$$w = \ln z = \ln|z| + i(\theta \pm 2n\pi)$$

模的多值性

反三角函数:

$$\begin{aligned} \arcsin(z) &= \frac{1}{i} \ln \left( iz + \sqrt{1-z^2} \right) \\ \arccos(z) &= \frac{1}{i} \ln \left( z + \sqrt{z^2-1} \right) \\ \arctan(z) &= \frac{1}{2i} \ln \frac{1+iz}{1-iz} \end{aligned}$$

**例 3.2.3.** 以  $\arcsin z$  为例:

$$\sin(w) = \frac{e^{iw} - e^{-iw}}{2i} = z$$

Multiply  $e^{iw}$  for both sides, we have

$$\begin{aligned}(e^{iw})^2 - 2iz(e^{iw}) - 1 &= 0 \\ e^{iw} &= \frac{2iz \pm \sqrt{4 - 4z^2}}{2} = iz \pm \sqrt{1 - z^2} \\ \Rightarrow w &= \frac{1}{i} \ln \left( iz \pm \sqrt{1 - z^2} \right)\end{aligned}$$

复合函数多值性的判断:

**例 3.2.4.**  $\sin \sqrt{z}$  是多值函数 (两个值), 而  $\cos \sqrt{z}$  是单值函数。

**定义 3.2.5.** 当自变量  $z$  围绕某点  $z_0$  旋转一圈 (辐角增加  $2\pi$ ) 之后, 若得到的新的函数与原函数不相等, 则  $z_0$  称为一个支点。

例如:

**例 3.2.6.**  $w = \sqrt{z}$ :

$$z' = z \cdot e^{2\pi i} \rightarrow w' = \sqrt{z} \cdot e^{\pi i} = -\sqrt{z} \neq w$$

所以  $z_1 = 0, z_2 = \infty$  是  $w$  的两个支点。

**方法 3.2.7.** 多值函数的单值化:

- 限定辐角的范围, 例如  $(0, 2\pi]$
- 规定某点  $z_0$  的值, 然后描绘途径该点到目标点  $z$  的不同路径下的  $f(z)$  的取值。

### 3.3 导数及解析函数的定义

**定义 3.3.1.**  $f(z)$  在  $z_0$  以及其邻域上有定义, 且沿任何路径  $z \rightarrow z_0$  时均有

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

则  $f(z)$  在  $z_0$  上连续。

**定义 3.3.2.** 若  $f(z)$  在其定义域上处处连续, 则称其为连续函数。

**定义 3.3.3.** 若  $f(z)$  在其  $z_0$  上连续, 且沿任何路径  $\Delta z \rightarrow 0$

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

存在且唯一, 则称  $f(z)$  在  $z_0$  可导。

**定义 3.3.4.** 若  $f(z)$  在  $z_0$  及其邻域各点均可导, 则称为在  $z_0$  解析。

**定义 3.3.5.** 若  $f(z)$  在域  $D$  上处处解析, 则称为  $D$  上的解析函数。

### 3.4 柯西-黎曼条件

若  $f(z) = u(x, y) + iv(x, y)$  其中  $u, v$  均为二元实函数, 那么  $f(z)$  可导的必要条件之一为柯西-黎曼条件:

**定义 3.4.1.** Cauchy-Riemann Condition:

$$\begin{cases} \frac{\partial u(x, y)}{\partial x} = \frac{\partial v(x, y)}{\partial y} \\ \frac{\partial v(x, y)}{\partial x} = -\frac{\partial u(x, y)}{\partial y} \end{cases}$$

极坐标的柯西黎曼条件:

$$z = re^{i\theta} \Rightarrow \Delta z = \frac{\partial z}{\partial r} \Delta r + \frac{\partial z}{\partial \theta} \Delta \theta = e^{i\theta} \Delta r + ire^{i\theta} \Delta \theta$$

(1) Along  $r$  direction ( $\Delta \theta = 0$ )

$$\lim_{\Delta r \rightarrow 0} \frac{\Delta u + i\Delta v}{\Delta r e^{i\theta}} = \frac{1}{e^{i\theta}} \left( \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right)$$

(2) Along  $\theta$  direction ( $\Delta r = 0$ )

$$\begin{aligned} \lim_{\Delta \theta \rightarrow 0} \frac{\Delta u + i\Delta v}{r \Delta \theta i e^{i\theta}} &= \frac{1}{r i e^{i\theta}} \left( \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \right) = \frac{1}{e^{i\theta}} \left( \frac{-i}{r} \frac{\partial u}{\partial \theta} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right) \\ &\Rightarrow \begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta} \end{cases} \end{aligned}$$

**推论 3.4.2.** 在某一个点,  $f(z)$  可导的充分必要条件:

1. 函数的实部和虚部均为二元可微实函数。
2. 满足柯西黎曼条件。

证明. 假设  $f(z) = u(x, y) + iv(x, y)$ , 由条件 1 得:

$$\begin{aligned} \Delta u &= \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \\ \Delta v &= \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y \\ \lim_{\Delta x \rightarrow 0, \Delta y \rightarrow 0} \epsilon_i &= 0 \quad i = 1, 2, 3, 4 \end{aligned}$$

$$\begin{aligned}
\Delta f &= \Delta u + i\Delta v = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y + \\
&\quad i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \epsilon_3\Delta x + \epsilon_4\Delta y\right) \\
&= \left(i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right)\Delta x + \left(i\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\right)\Delta y + (\epsilon_1 + i\epsilon_3)\Delta x + (\epsilon_2 + i\epsilon_4)\Delta y \\
&= \left(i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right)\Delta x + i\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)\Delta y + (\epsilon_1 + i\epsilon_3)\Delta x + (\epsilon_2 + i\epsilon_4)\Delta y
\end{aligned}$$

由条件 2 得:

$$\begin{aligned}
\Delta f &= \left(i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right)\Delta z + (\epsilon_1 + i\epsilon_3)\Delta x + (\epsilon_2 + i\epsilon_4)\Delta y \\
z &= x + iy \rightarrow \Delta z = \Delta x + i\Delta y \\
\lim_{\Delta z \rightarrow 0} \frac{\Delta f}{\Delta z} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}
\end{aligned}$$

□

**推论 3.4.3.** 在某一个区域  $G$  内,  $f(z)$  解析的充分必要条件:

1. 函数的实部和虚部均为二元可微实函数, 且其四个偏导  $\left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \frac{\partial v}{\partial y}\right)$  连续。
2. 满足柯西黎曼条件。

**注意:** 多值函数一定不可导, 不解析。

**例 3.4.4.**  $e^z$  在  $z \rightarrow \infty$  时一定不解析, 因为其在  $z \rightarrow \infty$  时是多值的。同理, 三角函数和双曲函数在  $z \rightarrow \infty$  时也是不可导的。

### 3.5 解析函数的特性

假设某个复变解析函数:  $f(z) = u(x, y) + iv(x, y)$   $u, v \in \mathbb{R}$ 。由柯西-黎曼条件得到:

**引理 3.5.1.**

$$\frac{\partial^2 u(x, y)}{\partial^2 x} + \frac{\partial^2 u(x, y)}{\partial^2 y} = 0 \quad (3.1)$$

$$\frac{\partial^2 v(x, y)}{\partial^2 x} + \frac{\partial^2 v(x, y)}{\partial^2 y} = 0 \quad (3.2)$$



3.1和3.2 是拉普拉斯方程。所以解析函数的实部和虚部均为调和函数。即：

$$\begin{aligned}\Delta u &= \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \Delta v &= \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0\end{aligned}$$

**定理 3.5.2.**

$$\frac{\partial f(z)}{\partial z^*} = 0$$

即解析函数与其自变量的共轭无关。

证明.

$$x = \frac{z + z^*}{2}, \quad y = \frac{z - z^*}{2}$$

$$\begin{aligned}\frac{\partial f(z)}{\partial z^*} &= \frac{\partial f(z)}{\partial x} \frac{\partial x}{\partial z^*} + \frac{\partial f(z)}{\partial y} \frac{\partial y}{\partial z^*} = \\ \frac{1}{2} \left[ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right] + \frac{i}{2} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] &= 0\end{aligned}$$

□

**定理 3.5.3.** 解析函数的实部和虚部的等值线的切向量相互垂直。

证明.

$$u(x, y) = C \Rightarrow du(x, y) = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$d\vec{s}$  是实部等值线的切向量, 则

$$d\vec{s} = (dx, dy) \propto \left( \frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x} \right)$$

$$v(x, y) = C' \Rightarrow dv(x, y) = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = 0$$

$d\vec{s}'$  是虚部等值线的切向量, 则

$$d\vec{s}' = (dx', dy') \propto \left( \frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x} \right)$$

$$d\vec{s} \cdot d\vec{s}' \propto \left( \frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x} \right) \cdot \left( \frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x} \right) = 0 \Rightarrow d\vec{s} \perp d\vec{s}'$$

□

### 3.6 由部分确定整个解析函数

如果已知某个解析函数的实部  $u(x, y)$  以及在某点  $z_0$  的取值, 可以确定整个解析函数:

**方法 3.6.1.** 由于柯西-黎曼条件,

$$\begin{aligned}\frac{\partial u(x, y)}{\partial x} &= \frac{\partial v(x, y)}{\partial y} \rightarrow v(x, y) = \int \frac{\partial v}{\partial x} dx + h(y) \\ \frac{\partial u(x, y)}{\partial y} &= -\frac{\partial v(x, y)}{\partial x} \rightarrow v(x, y) = -\int \frac{\partial u}{\partial y} dx + h(y) \\ &\Rightarrow \\ \frac{\partial v(x, y)}{\partial y} &= -\int \frac{\partial^2 u}{\partial y^2} dx + h'(y) = \frac{\partial u(x, y)}{\partial x} \\ &\Rightarrow \\ h'(y) &= \frac{\partial u(x, y)}{\partial x} + \int \frac{\partial^2 u(x, y)}{\partial y^2} dx \rightarrow h(y) = \int h'(y) dy + C\end{aligned}$$

**方法 3.6.2.** 利用  $C-R$  条件, 先找到解析函数的导数:

$$\begin{aligned}\frac{df}{dz} &= \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \equiv g(z) \\ &\Rightarrow \\ f(z) &= \int g(z) dz + C\end{aligned}$$

**方法 3.6.3.**

$$f(z) = u(x, y) + iv(x, y), \quad f^*(z) = u(x, y) - iv(x, y)$$

$\Rightarrow$

$$u(x, y) = \frac{f(z) + f^*(z)}{2}, \quad v(x, y) = \frac{f(z) - f^*(z)}{2i}$$

通过代数运算, 我们可以将  $u(x, y)$  写成:

$$u(x, y) = u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right) = h(z) + h^*(z) = [h(z) + iC] + [h(z) + iC]^*$$

对比系数可得:

$$f(z) = 2h(z) + 2iC$$

## 第四章 复变函数的积分

### 4.1 解析函数的积分特性

定义 4.1.1. 复变函数的积分定义为：

$$\int_L f(z)dz \equiv \lim_{n \rightarrow \infty} \sum_{j=1}^n f(\xi_j)(z_j - z_{j-1})$$

其中  $L$  为有向路径。

一些较为常用的性质：

$$\int_L f_1(z) + f_2(z)dz = \int_L f_1(z)dz + \int_L f_2(z)dz$$

$$\int_L f(z)dz = - \int_{-L} f(z)dz$$

$$\int_{L_1+L_2} f(z)dz = \int_{L_1} f(z)dz + \int_{L_2} f(z)dz$$

$$\int_C a f(z)dz = a \int_C f(z)dz \quad \text{where } a \text{ is a constant complex number}$$

$$\left| \int_C f(z)dz \right| \leq \int_C |f(z)||dz|$$

$$\left| \int_C f(z)dz \right| \leq Ml \quad \text{where } M \text{ is upper bound of } f(z)$$

定理 4.1.2. 单连通域上解析函数的柯西积分定理：假设  $C$  是某个单连通域的边界。

$$\oint_C f(z)dz = 0$$

证明. 假设将复变解析函数  $f(z)$  沿着某一单连通域做回路积分：

$$\oint_C f(z)dz = \oint_C [u(x, y) + iv(x, y)](dx + idy) \quad (4.1)$$

其中正方向定义为确保解析区域在左手边的方向。展开4.1得到：

$$\oint_C [u dx - v dy] + i \oint_C [u dy + v dx]$$

由格林公式

$$\oint_C [P dx + Q dy] = \iint_{\Sigma} \left[ -\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right] dx dy$$

得到：

$$\oint_C f(z) dz = \iint_{\Sigma} \left[ -\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right] dx dy + i \iint_{\Sigma} \left[ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right] dx dy \quad (4.2)$$

考虑 C-R 条件，

$$4.2 \equiv 0$$

□

**定理 4.1.3.** *Morera's theorem (莫列拉定理):* If  $f(z)$  is continuous in  $\bar{G}$ , if for any closed curve (contour) in  $\bar{G}$ ,  $\oint_C f(z) dz = 0$ , then  $f(z)$  is analytic in  $G$ .

**定理 4.1.4.** 复连通域上解析函数的柯西积分定理：假设  $C$  是一个复连通域的边界，而填上这个复连通域中的  $C_1, C_2, \dots, C_N$  所围成的区域可以将该域变为单连通域。那么：

$$\oint_C f(z) dz = \sum_{n=1}^N \oint_{C_n} f(z) dz$$

**例 4.1.5.** Find the value of  $\oint_C z^n dz$ , where  $n$  is an integer,  $C$  is a simply closed curve in  $\mathbb{C}$ .

- If  $n$  is non-negative,  $z^n$  is analytic, then  $\oint_C z^n dz = 0$ .
- If  $n$  is negative, and if the contour does not enclose  $z = 0$ , then  $z^n$  is analytic inside the region bounded by  $C$ , and again we have  $\oint_C z^n dz = 0$ .
- If  $n$  is negative, and if the contour encloses  $z = 0$ . We can draw a simple circle around  $z = 0$ , and apply the Cauchy theorem for a multi-connected

$$\oint_C z^n dz = \oint_{|z|=\varepsilon} z^n dz = \int_0^{2\pi} \varepsilon^{n+1} e^{i(n+1)\theta} i d\theta = \begin{cases} 2\pi i, n = -1; \\ 0, n = -2, -3, -4, \dots \end{cases}$$

**推论 4.1.6.** 函数  $f(z)$  在  $\Sigma_G$  内解析，如果  $\Sigma_C \subset \Sigma_G$ ，其线积分  $\int_C f(z) dz$  与路径无关。

**引理 4.1.7.** 小圆弧引理 *Small Arc Lemma*:

If  $f(z)$  is continuous in a small region around  $z = a$ , and satisfies the following relation: when  $\theta_1 \leq \arg(z - a) \leq \theta_2, |z - a| \rightarrow 0, (z - a)f(z)$  uniformly approaches  $k$ . Then we have

$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = ik(\theta_2 - \theta_1)$$

**引理 4.1.8.** 大圆弧引理 *Great Arc Lemma*:

If  $f(z)$  is continuous in a region around  $z = \infty$ , and satisfies the following relation: when  $\theta_1 \leq \arg(z) \leq \theta_2, z \rightarrow \infty, zf(z)$  uniformly approaches  $K$ . Then we have

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = iK(\theta_2 - \theta_1)$$

## 4.2 柯西积分公式

**定理 4.2.1.** 柯西积分公式: 假设  $C$  包围了  $f(z)$  的单连通解析区域,  $z_0$  为区域内一点, 则

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

证明. 不妨用一个小圆将  $z_0$  包围, 其边界设为  $C_r: \forall z \in \Sigma_{C_r} \quad z = z_0 + re^{i\theta}$ , 则由4.1.4,

$$\begin{aligned} \oint_C \frac{f(z)}{z - z_0} dz &= \oint_{C_r} \frac{f(z)}{z - z_0} dz = \\ \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta &= i \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \end{aligned} \quad (4.3)$$

再不妨令  $r \rightarrow 0$ , 那么4.3化为

$$i \int_0^{2\pi} f(z_0) d\theta = 2\pi i f(z_0)$$

即:

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) \quad (4.4)$$

4.4 同样可以使用小圆弧定理导出:

As  $z \rightarrow z_0, (z - z_0) \cdot f(z)/(z - z_0) \rightarrow f(z_0)$ . And when  $r \rightarrow 0$ ,

$$\oint_{C_r} \frac{f(z)}{z - z_0} dz = 2i\pi^1 f(z_0)$$

□

<sup>1</sup>为什么不是  $2ni\pi$ ? 因为该函数被单值化了

**定理 4.2.2.** Cauchy Integration Equation for Unbounded Region:

If  $f(z)$  is a single-valued analytic function defined along and beyond a simply closed curve  $C$  (including the infinity), then we have the following relation

$$\frac{1}{2\pi i} \left[ \oint_{C_R} \frac{f(z)}{z-a} dz + \oint_C \frac{f(z)}{z-a} dz \right] = f(a)$$

where the integration is done along the positive direction of  $C$  (which is clockwise) and the positive direction of  $C_R$  (anti-clockwise).

由4.2.2, 使用大圆弧引理:

$$\text{当 } K = 0, \text{ 即 } \lim_{z \rightarrow \infty} f(z) = 0 : f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

退化到了4.2.1

**引理 4.2.3.** 令4.4中的  $f(z) = 1$ , 推出公式:

$$\frac{1}{2\pi i} \oint_C \frac{1}{z-z_0} dz = \begin{cases} 1 & z_0 \in \Sigma_C \\ 0 & z_0 \notin \Sigma_C \end{cases}$$

可以利用4.4计算解析函数的导数:

**引理 4.2.4.**

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi-z} d\xi \rightarrow \\ f'(z) &= \frac{1}{2\pi i} \frac{d}{dz} \oint_C \frac{f(\xi)}{\xi-z} d\xi = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi-z)^2} d\xi \rightarrow \\ f''(z) &= \frac{2!}{2\pi i} \oint_C \frac{f(\xi)}{(\xi-z)^3} d\xi \\ &\dots \\ f^{(n)}(z) &= \frac{n!}{2\pi i} \oint_C \frac{f(\xi)}{(\xi-z)^{n+1}} d\xi \end{aligned}$$

这说明解析函数是任意阶可导的。

**定义 4.2.5.** 柯西型积分 *Cauchy-type Integral*:

If  $\phi(\zeta)$  is a continuous function defined along curve  $C$  (piece-wisely smooth), the

$$f(z) = \frac{1}{2\pi i} \int_C \frac{\phi(\zeta)}{\zeta-z} d\zeta, \quad z \notin C$$

is an analytic function defined outside  $C$ . And we have

$$f^{(p)}(z) = \frac{p!}{2\pi i} \int_C \frac{\phi(\zeta)}{(\zeta-z)^{p+1}} d\zeta, \quad z \notin C$$

**方法 4.2.6.** *Integral that Contains a Parameter*

1.  $f(t, z)$  is a continuous function of  $t$  and  $z, t \in [a, b], z \in \bar{G}, \bar{G}$  is bounded
2. For any value  $t \in [a, b], f(t, z)$  is a single-valued analytic function defined in  $\bar{G}$ . Then  $F(z) = \int_a^b f(t, z)dt$  is analytic in  $G$ , and

$$F'(z) = \int_a^b \frac{\partial f(t, z)}{\partial z} dt, z \in G$$

**4.3 最大模定理**

**定理 4.3.1.** 最大模定理：设  $f(z)$  在闭区域上解析，则其模  $|f(z)|$  的最大值只能出现在该区域的边界上，除非  $f(z)$  是一个常函数。

证明.

$$\begin{aligned} f^n(z) &= \frac{1}{2\pi i} \oint \frac{f^n(\xi)}{\xi - z} d\xi \\ |f(z)|^n &= |[f(z)]^n| = \left| \frac{1}{2\pi i} \oint \frac{f^n(\xi)}{\xi - z} d\xi \right| \\ &\leq \frac{1}{2\pi} \oint \frac{|f(\xi)|^n}{|\xi - z|} |d\xi| \leq \frac{M^n}{2\pi d} \oint_C |d\xi| = \frac{M^n}{2\pi d} l \end{aligned}$$

$d$  为  $z$  至边界的最短距离,  $\forall z: |z - \xi| \geq d$

$M$  为  $|f(\xi)|$  的最大值,  $\forall z: |f(\xi)| \leq M, \xi \in C$

即

$$|f(z)| \leq M \left[ \frac{l}{2\pi d} \right]^{1/n} \rightarrow |f(z)| \leq \lim_{n \rightarrow \infty} M \left[ \frac{l}{2\pi d} \right]^{1/n} = M$$

即  $f(z), z \in \overline{\Sigma_C}$  的最大值便是  $f(\xi), \xi \in C$  的最大值。

□

## 第五章 复变函数的级数

### 5.1 复变函数在其解析圆域上的泰勒级数展开

$f(z)$  在  $z_0$  为圆心的圆域内解析, 则对于任意一圆域内点  $z$ , 有

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi = \frac{f^{(n)}(z_0)}{n!}$$

证明.

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0) - (z - z_0)} d\xi \\ &= \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z_0} \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} d\xi \end{aligned}$$

由于

$$\frac{z - z_0}{\xi - z_0} \leq 1, \quad \frac{1}{1 - t} = \sum_{n=0}^{\infty} t^n, \quad |t| < 1 \quad (5.1)$$

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z_0} \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} d\xi &= \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z_0} \sum_{n=0}^{\infty} \left[ \frac{z - z_0}{\xi - z_0} \right]^n d\xi = \\ \sum_{n=0}^{\infty} \left[ \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right] (z - z_0)^n &= f(z) \end{aligned}$$

□

复变函数在其解析圆域上的泰勒级数展开的收敛半径为:

$$R = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \text{or} \quad R = |z_0 - z_1| \quad z_1 \text{ 是离 } z_0 \text{ 最近的奇点}$$



**例 5.1.1.** 计算  $f(z) = \frac{1}{1-z^2}$  的泰勒展开, 求出收敛半径。

By equation 5.1,

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} t^n \Big|_{t=z^2} = \sum_{n=0}^{\infty} z^{2n}$$

**引理 5.1.2.** 对于给定的  $f(z)$ 、 $z_0$ , 其泰勒展开 (系数) 唯一。

## 5.2 利用泰勒级数讨论最大模定理

**定义 5.2.1.** Kronecker- $\delta$  符号:

$$\delta_{mn} \equiv \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta} d\theta = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

假设最大模定理不成立, 即:

$$\exists z_0 \in \Sigma, z_0 \notin \partial\Sigma \text{ s.t. } |f(z_0)| = \max |f(z)|$$

那么以  $z_0$  为中心做泰勒展开:

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \rightarrow a_0 = f(z_0)$$

由于

$$z - z_0 = r e^{i\theta}$$

$$\begin{aligned} |a_0|^2 &= \frac{1}{2\pi} \int_0^{2\pi} |a_0|^2 d\theta = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|^2 d\theta \geq \frac{1}{2\pi} \int_0^{2\pi} f^*(z) \cdot f(z) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{m=1}^{\infty} a_m^* [(z - z_0)^*]^m \cdot \sum_{n=1}^{\infty} a_n (z - z_0)^n d\theta \\ &= \sum_{m,n=0}^{\infty} a_m^* a_n r^{m+n} \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta} d\theta \\ &= \sum_{m,n=0}^{\infty} a_m^* a_n r^{m+n} \delta_{mn} = \sum_{n=0}^{\infty} a_n^* a_n r^{2n} = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \\ &= |a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \end{aligned} \tag{5.2}$$

考虑到

$$\sum_{n=1}^{\infty} |a_n|^2 r^{2n} \geq 0 \Rightarrow |a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \geq |a_0|^2$$

若想要5.2成立, 那么

$$\sum_{n=1}^{\infty} |a_n|^2 r^{2n} = 0 \rightarrow a_n = 0 \rightarrow f(z) = \text{constant}.$$

**定理 5.2.2.** 刘维尔定理: 在全复平面内解析且有界的复变函数必为常函数。

证明. 以  $z_0 = 0$  为中心做泰勒展开:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi^{n+1}} d\xi$$

由于

$$\xi \in C \rightarrow \xi = re^{i\theta} \rightarrow d\xi = ire^{i\theta} d\theta$$

则  $|a_n|$  可以化为

$$|a_n| \leq \frac{1}{2\pi} \oint_C \frac{|f(\xi)|}{|\xi^{n+1}|} |d\xi| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{M}{r^n} d\theta = \frac{M}{r^n}$$

由于  $f(z)$  在整个复平面上解析, 即其泰勒展开的收敛半径  $R = \infty$ , 那么

$$|a_n| \leq \lim_{r \rightarrow \infty} \frac{M}{r^n} = 0 \rightarrow \forall n \neq 0: a_n = 0$$

□

### 5.3 解析函数的零点及其孤立性

**定义 5.3.1.**  $f(z)$  在  $z_0$  点有  $f(z_0) = 0$ , 且在以  $z_0$  为圆心的圆域内的泰勒级数展开式最低幂次 (最小的使得  $a_n \neq 0$  的  $n$ ) 为  $k$  次, 则称  $z_0$  为  $f(z)$  的  $k$ -阶零点。

由定义得到, 若  $z_0$  是  $f(z)$  的  $k$ -阶零点, 则  $\forall k > n > 0: f^{(n)}(z_0) = 0$

**定理 5.3.2.** 零点的孤立性: 假设  $z_0$  为  $f(z)$  的一个零点, 则

$$\exists r > 0 \text{ s.t. } \forall z \in \{z \mid |z - z_0| < r\}, f(z) \neq 0$$

即零点不能构成区域。

证明. 假设  $z_0$  为  $f(z)$  的一个  $k$ -阶零点:

$$f(z) = \sum_{n=k}^{\infty} a_n (z - z_0)^n = (z - z_0)^k \sum_{m=0}^{\infty} a_{m+k} (z - z_0)^m = (z - z_0)^k \varphi(z)$$

$$\varphi(z_0) \equiv a_k \neq 0$$

由于函数解析，函数必定连续，则

$$\forall \epsilon > 0: \exists z \neq z_0 \text{ s.t. } |\varphi(z_0) - \varphi(z)| < \epsilon$$

令  $\epsilon = |\varphi(z_0)|/2$ :

$$|\varphi(z_0)| - |\varphi(z)| < |\varphi(z_0) - \varphi(z)| < |\varphi(z_0)|/2$$

$$|\varphi(z)| > |\varphi(z_0)|/2 > 0$$

$$f(z) = (z - z_0)^k \varphi(z) \neq 0$$

即总可以在  $z_0$  为中心找到一圆域使得在该圆域内除圆心  $z_0$  外的所有点  $z$  满足  $f(z) \neq 0$  □

## 5.4 解析环域上的洛朗级数展开

$f(z)$  在以  $z_0$  为圆心的环域内解析，则对于该环域内任何一点  $z$ ，有

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad a_n = \frac{1}{2\pi i} \oint f(\xi) (\xi - z_0)^{-n-1} d\xi$$

证明. 将环域的外环和内环建立一微小链接，使得  $L = C_1 + C_2 + \partial L - \partial L = C_1 + C_2$  为一单连通区域的边界，

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_L \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i} \oint_{C_2} \frac{f(\xi)}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \oint_{C_2} \frac{f(\xi)}{\xi - z} d\xi \end{aligned}$$

回想起证明泰勒级数时的过程，不妨将  $\xi - z_0$  设为  $r$ ， $z - z_0$  设为  $R$ ，不难发现：对于  $C_1$ ， $r > R$ ，对于  $C_2$ ， $r < R$ 。

$$\begin{aligned} \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{\xi - z} d\xi &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{r - R} d\xi \\ &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{1 - R/r} d\xi \\ &= \frac{1}{2\pi i} \oint_{C_1} \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^n \frac{f(\xi)}{\xi - z_0} d\xi \\ &= \frac{1}{2\pi i} \oint_{C_1} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^n \frac{f(\xi)}{\xi - z_0} d\xi \end{aligned}$$

同理可得

$$\begin{aligned}
 -\frac{1}{2\pi i} \oint_{C_2} \frac{f(\xi)}{\xi - z} d\xi &= -\frac{1}{2\pi i} \oint_{C_2} \frac{f(\xi)}{r - R} d\xi \\
 &= \frac{1}{2\pi i} \oint_{C_2} \frac{f(\xi)}{1 - r/R} d\xi \\
 &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_{C_2} \left(\frac{r}{R}\right)^n \frac{f(\xi)}{z - z_0} d\xi \\
 &= \frac{1}{2\pi i} \oint_{C_2} \sum_{n=0}^{\infty} \left(\frac{\xi - z_0}{z - z_0}\right)^n \frac{f(\xi)}{z - z_0} d\xi
 \end{aligned}$$

代入  $f(z)$  中得到

$$\begin{aligned}
 f(z) &= \frac{1}{2\pi i} \oint_{C_1} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^n \frac{f(\xi)}{\xi - z_0} d\xi + \frac{1}{2\pi i} \oint_{C_2} \sum_{n=0}^{\infty} \left(\frac{\xi - z_0}{z - z_0}\right)^n \frac{f(\xi)}{z - z_0} d\xi \\
 &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left[ \oint \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right] (z - z_0)^n \\
 &\quad + \frac{1}{2\pi i} \sum_{n=-\infty}^{-1} \left[ \oint \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right] (z - z_0)^n \\
 &= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \left[ \oint \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right] (z - z_0)^n
 \end{aligned}$$

□

洛朗级数的收敛半径:

$$R_1 < |z - z_0| < R_2, \quad R_1 = \lim_{n \rightarrow -\infty} \left| \frac{a_{n-1}}{a_n} \right| \quad R_2 = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

或者可以认为

$R_1$  := 以  $z_0$  为圆心的包含考察点  $z$  的最大解析环域的内径

$R_2$  := 以  $z_0$  为圆心的包含考察点  $z$  的最大解析环域的外径

**例 5.4.1.** Find the Laurent expansion of  $\frac{1}{z(z-1)}$  when (i)  $0 < |z| < 1$  and (ii)  $|z| > 1$  You should refer to the previous exercises. convergent in  $0 < |z| < 1$

1.

$$0 < |z| < 1: \quad \frac{1}{z(z-1)} = -\frac{1}{z} \frac{1}{1-z} = -\frac{1}{z} \sum_{n=0}^{\infty} z^n = -\sum_{n=-1}^{\infty} z^n$$

And we can confirm that  $\frac{1}{z(z-1)}$  has singular point at  $z = 0$ .

2.

$$|z| > 1: \quad \frac{1}{z(z-1)} = \frac{1}{z^2} \frac{1}{1 - \frac{1}{z}} = \frac{1}{z^2} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{n=-2}^{\infty} z^n$$

And we can confirm that  $\frac{1}{z(z-1)}$  has singular point along  $|z| = 1$ .

## 第六章 留数定理

### 6.1 留数定理

**定义 6.1.1.** 孤立奇点: 若  $f(z)$  在  $z_0$  处不解析, 但在其邻域内全都可导 ( $\exists r > 0$  s.t.  $\forall z: 0 < |z - z_0| < r, f'(z)$  exists.), 那么  $z_0$  是  $f(z)$  的一个孤立奇点。若  $z_0$  是一个孤立奇点, 那么我们可以在其邻域内对  $f(z)$  做洛朗展开, 若:

1. 若其洛朗级数展开不包含负数项,  $z_0$  称为一个可去奇点 (*removable singular point*)
2. 若其洛朗级数展开包含有限个负数项,  $z_0$  称为一个极点 (*pole*)
3. 若

$$\begin{aligned} f(z) &= a_{-m}(z-b)^{-m} + a_{-m+1}(z-b)^{-m+1} + \dots + a_0 + a_1(z-b) + \dots \\ &= (z-b)^{-m} [a_{-m} + a_{-m+1}(z-b) + \dots] \\ &= (z-b)^{-m} \phi(z), \quad 0 < |z-b| < R \end{aligned}$$

$\phi(z)$  is analytic for a region around  $z = b$ . If  $\phi(b) = a_{-m} \neq 0$ , then  $b$  is said to be the  $m$ -th order pole of  $f(z)$ .

$$\frac{1}{f(z)} = (z-b)^m \frac{1}{\phi(z)}$$

4. 若其洛朗级数展开包含无限个负数项,  $z_0$  称为一个本性奇点 (*essential singular point*)

如果  $z_0$  是  $f$  的一个本质奇点, 那么  $\lim_{z \rightarrow z_0} f(z)$  不存在。

**例 6.1.2.** 判断  $z_0 = 0$  是  $f(z) = e^{1/z}$  的何种奇点。

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{(-n)!} z^n$$

则,  $z_0 = 0$  是其本质奇点, 且不难验证, 其趋于本质奇点的极限不存在。

**定理 6.1.3.** Assume that  $G$  is a bounded region, and its boundary  $C$  is a smooth, simply closed curve. If except for a finite number of isolated singular points  $b_k, k = 1, 2, 3, \dots, n$ ,  $f(z)$  is single-valued and analytic in  $G$ , and is continuous in  $\bar{G}$  (including along  $C$ ). Then we have the following relation:

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^n \text{res } f(b_k)$$

$\text{res } f(b_k)$  called the residue (留数) of  $f(z)$  at  $b_k$

$$f(z) = \sum_{l=-\infty}^{\infty} a_l^{(k)} (z - b_k)^l, 0 < |z - b_k| < r, \text{res } f(b_k) := a_{-1}^{(k)}$$

证明.

$$\oint_C f(z)dz = \sum_{k=1}^n \oint_{\gamma_k} f(z)dz$$

Recalling that result of

$$\oint_C z^n dz = \oint_{|z|=\varepsilon} z^n dz = \int_0^{2\pi} \varepsilon^{n+1} e^{i(n+1)\theta} i d\theta = \begin{cases} 2\pi i, n = -1; \\ 0, n = -2, -3, -4, \dots \end{cases}$$

we concluded before, and we can say that:

$$O.E. = 2\pi i \sum_{k=1}^n a_{-1}^{(k)} = 2\pi i \sum_{k=1}^n \text{res } f(b_k)$$

□

**方法 6.1.4.** Computation Method of Residue:

$$a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-b)^m f(z) \Big|_{z=b}$$

$$a_{-1} = \frac{1}{(m-1)!} \lim_{z \rightarrow a} [(z-a)^m f(z)]^{m-1}$$

The general form of  $f(z)$  is  $\frac{P(z)}{Q(z)}$ . If  $P(z)$  and  $Q(z)$  are analytic around  $b$ , and  $P(b) \neq 0, z = b$  is the first-order zero point of  $Q(z)$ .

$$a_{-1} = \lim_{z \rightarrow b} (z-b)f(z) = \lim_{z \rightarrow b} (z-b) \frac{P(z)}{Q(z)} = \frac{P(b)}{Q'(b)}$$

$$f(z) = \frac{1}{(z-1)^2(z-2)(z-3)} = \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-2} + \frac{D}{z-3}$$

You can perform decomposition of partial fraction to figure out values for  $A-D$ .

Let's try a different approach.

$$(z-1)f(z) = \frac{1}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + B + \frac{C(z-1)}{z-2} + \frac{D(z-1)}{z-3}$$

Then

$$A = \operatorname{res}(z-1)f(z)|_{z=1} = \frac{1}{2}$$

$$B = \operatorname{res} f(z)|_{z=1} = \frac{3}{4}$$

$$C = \operatorname{res} f(z)|_{z=2} = -1$$

$$D = \operatorname{res} f(z)|_{z=3} = \frac{1}{4}$$

**例 6.1.5.** 如果  $\infty$  不是一个  $f(z)$  的非孤立奇点, 那么我们可以定义  $f$  在  $\infty$  处的留数:

$$\operatorname{res} f(\infty) = \frac{1}{2\pi i} \oint C f(z) dz \quad \text{顺时针积分可以包含 } \infty$$

$$\begin{aligned} \operatorname{res} f(\infty) &= \frac{1}{2\pi i} \oint C f(z) dz = -\frac{1}{2\pi i} \oint_C \frac{f(1/t)}{t^2} dt \\ &= -\frac{f(1/t)}{t^2} \quad \text{coefficient of term } t^{-1} \text{ around } 0 \\ &= -f(1/t) \quad \text{coefficient of term } t^1 \text{ around } 0 \\ &= -f(z) \quad \text{coefficient of term } z^{-1} \text{ around } \infty \text{ Note } z^{-1} \text{ is analytic at } \infty \end{aligned}$$

也就是说,  $f$  有可能在  $\infty$  处不解析的同时, 在  $\infty$  处有非零留数。

同样,  $\infty$  可能是  $f$  的一个孤立奇点, 但是留数为 0。

## 6.2 留数定理的应用: 无穷积分

**方法 6.2.1.** 使用留数定理计算积分

$$\int_0^{2\pi} R(\sin \theta, \cos \theta) d\theta \Leftrightarrow I = \oint_{|z|=1} R\left(\frac{z^2-1}{2iz}, \frac{z^2+1}{2z}\right) \frac{dz}{iz}$$

**例 6.2.2.** Compute

$$I = \int_0^\pi \frac{1}{1 + \varepsilon \cos \theta} d\theta, |\varepsilon| < 1$$

$$\begin{aligned} I &= \frac{1}{2} \int_{-\pi}^\pi \frac{1}{1 + \varepsilon \cos \theta} d\theta = \frac{1}{2} \oint_{|z|=1} \frac{1}{1 + \varepsilon \frac{z^2+1}{2z}} \frac{dz}{iz} \\ &= \frac{1}{2} \oint_{|z|=1} \frac{2}{\varepsilon z^2 + 2z + \varepsilon} \frac{dz}{i} = \pi \sum_{|z|<1} \operatorname{res} \frac{2}{\varepsilon z^2 + 2z + \varepsilon} \\ &= \pi \frac{2}{2\varepsilon z + 2} \Big|_{z=(-1+\sqrt{1-\varepsilon^2})/\varepsilon} = \frac{\pi}{\sqrt{1-\varepsilon^2}} \end{aligned}$$



**定义 6.2.3.** 当我们在计算形如

$$I = \int_{-\infty}^{\infty} f(x) dx$$

的广义积分时, 如果发现最终结果发散, 我们常常可以定义一个主值积分:

$$I = \lim_{R_1, R_2 \rightarrow \infty} \int_{-R_1}^{R_2} f(x) dx$$

, 其主值积分定义为:

$$v.p. \ I = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$$

**例 6.2.4.** Compute

$$\int_0^{\infty} \frac{dx}{1+x^4}$$

$$\begin{aligned} \oint_C \frac{1}{1+z^4} dz &= \int_0^R \frac{1}{1+x^4} dx + \int_{C_R} \frac{1}{1+z^4} dz + \int_R^0 \frac{idy}{1+(iy)^4} \\ &= (1-i) \int_0^R \frac{dx}{1+x^4} + \int_{C_R} \frac{dz}{1+z^4} \\ &= 2\pi i \operatorname{res} \frac{1}{1+z^4} \Big|_{z=e^{i\pi/4}} = \frac{\pi}{2} \frac{1-i}{\sqrt{2}} \end{aligned}$$

The red integral is calculated by Large Arc Lemma: Let  $R \rightarrow \infty$

$$\lim_{R \rightarrow \infty} \frac{z}{1+z^4} = 0$$

Thus,

$$O.E. = \frac{\sqrt{2}}{4} \pi$$

**方法 6.2.5.** 计算

$$I = \int_{-\infty}^{\infty} f(x) \cos px dx, I = \int_{-\infty}^{\infty} f(x) \sin px dx$$

我们使用:

$$\oint_C f(z) e^{ipz} dz = \int_{-R}^R f(x) (\cos px + i \sin px) dx + \int_{C_R} f(z) e^{ipz} dz$$

**引理 6.2.6.** *Jordan's Lemma:*

Assume that in  $0 \leq \arg z \leq \pi$ , when  $|z| \rightarrow \infty$ ,  $Q(z)$  uniformly converges to 0. then

$$\lim_{R \rightarrow \infty} \int_{C_R} Q(z) e^{ipz} dz = 0$$

where  $p > 0$ ,  $C_R$  is an arc centered at origin, in the upper half space.

证明. Along the arc, we have  $z = Re^{i\theta}$ . The condition of uniform convergence implies that

$$\forall \varepsilon > 0, \exists M(\varepsilon) > 0 \text{ (not related to } z\text{)}$$

when  $|z| = R > M$  and  $0 \leq \arg z \leq \pi, |Q(z)| < \varepsilon$

$$\begin{aligned} \left| \int_{C_R} Q(z) e^{ipz} dz \right| &= \left| \int_0^\pi Q(Re^{i\theta}) e^{ipR(\cos\theta + i\sin\theta)} Re^{i\theta} i d\theta \right| \\ &\leq \int_0^\pi |Q(Re^{i\theta})| e^{-pR\sin\theta} R d\theta \\ &< \varepsilon R \int_0^\pi e^{-pR\sin\theta} d\theta = 2\varepsilon R \int_0^{\pi/2} e^{-pR\sin\theta} d\theta \\ &< 2\varepsilon R \int_0^{\pi/2} e^{-2pR\theta/\pi} d\theta = \frac{\varepsilon\pi}{p} (1 - e^{-pR}) \rightarrow 0 \end{aligned}$$

□

**例 6.2.7.** Compute

$$\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx, a > 0$$

We choose one integral path:

$$(0, 0) \rightarrow (R, 0) \rightarrow (0, R) \rightarrow (-R, 0) \rightarrow (0, 0)$$

and let  $R \rightarrow \infty$

$$\begin{aligned} \oint_C \frac{ze^{iz}}{z^2 + a^2} dz &= \int_{-R}^R \frac{xe^{ix}}{x^2 + a^2} dx + \int_{C_R} \frac{ze^{iz}}{z^2 + a^2} dz \\ &= 2\pi i \operatorname{res} \frac{ze^{iz}}{z^2 + a^2} \Big|_{z=ai} \\ &= \pi i e^{-a} \end{aligned}$$

And the **red integral** is calculated by Jordan lemma:

$$\lim_{R \rightarrow \infty} \frac{z}{z^2 + a^2} = 0$$

Thus,

$$O.E. = \frac{\pi}{2} e^{-a}$$

**引理 6.2.8.** A Complementary Lemma of Jordan Lemma:

Assume that  $Q(z)$  only has a limited number of singular points. In the lower half space, when  $|z| \rightarrow \infty, Q(z)$  converges to 0 uniformly. Then

$$\lim_{R \rightarrow \infty} \int_{C_R} Q(z) e^{-ipz} dz = 2\pi i \cdot \sum_{\text{whole space}} \operatorname{res} \{Q(z) e^{-ipz}\}$$

where  $p > 0, C_R$  is a half circle centered at origin, in the upper half space.

### 6.3 路径上含有奇点的积分

**定义 6.3.1.** 对于路径上有奇点的积分，我们同样定义主值积分：

$$\int_a^b f(x)dx = \lim_{\delta_1 \rightarrow 0} \int_a^{c-\delta_1} f(x)dx + \lim_{\delta_2 \rightarrow 0} \int_{c+\delta_2}^b f(x)dx$$

$$v.p. \int_a^b f(x)dx = \lim_{\delta \rightarrow 0} \left[ \int_a^{c-\delta} f(x)dx + \int_{c+\delta}^b f(x)dx \right]$$

对于一阶极点，我们可以使用 Large&Small Arc Lemma 轻松搞定，然而，考虑如下积分：

**例 6.3.2.** Compute

$$v.p. \int_{-\infty}^{\infty} \frac{dx}{x(1+x+x^2)}$$

这个积分，我们尝试如下积分路径：

$$(\delta, 0) \rightarrow (R, 0) \rightarrow (0, R) \rightarrow (-R, 0) \rightarrow (-\delta, 0) \rightarrow (0, \delta) \rightarrow (\delta, 0)$$

We let  $R \rightarrow \infty, \delta \rightarrow 0$ .

$$\oint_C \frac{dz}{z(1+z+z^2)} = \int_{-R}^{-\delta} \frac{dx}{x(1+x+x^2)} + \int_{C_\delta} \frac{dz}{z(1+z+z^2)} + \int_{\delta}^R \frac{dx}{x(1+x+x^2)} + \int_{C_R} \frac{dz}{z(1+z+z^2)}$$

Red Integral:

$$R = v.p. \int_{-\infty}^{\infty} \frac{dx}{x(1+x+x^2)}$$

Blue Integral:

$$\int_{C_\delta} \text{clockwise} = - \int_{C_\delta} \text{anti-clockwise}$$

so,

$$\lim_{z \rightarrow 0} z \cdot \frac{1}{z(1+z+z^2)} = 1 \Rightarrow B = -i\pi \text{ (by small arc lemma)}$$

Violet Integral:

$$\lim_{z \rightarrow \infty} z \cdot \frac{1}{z(1+z+z^2)} = 0 \Rightarrow V = 0 \text{ (by large arc lemma)}$$

$$L.H.S = 2\pi i \operatorname{res} \left. \frac{1}{z(1+z+z^2)} \right|_{z=e^{i2\pi/3}} = -\frac{\pi}{\sqrt{3}} - i\pi$$

Thus, we have:

$$O.E. = -\frac{\pi}{\sqrt{3}}$$

## 6.4 包含割线的积分

**方法 6.4.1.**  $z^{s-1}$  是一个多值函数，其割线为  $(0 \rightarrow (\infty, 0))$ ,  $Q(z)$  是一个单值函数。计算：

$$\int_0^\infty x^{s-1} Q(x) dx$$

我们使用如下路径：

$$\begin{aligned} \delta \rightarrow 0, R \rightarrow \infty : & (\delta, 0^-) \rightarrow (0, -\delta) \rightarrow (-\delta, 0) \rightarrow (0, \delta) \rightarrow \\ & (\delta, 0^+) \rightarrow (R, 0^+) \rightarrow (0, R) \rightarrow (-R, 0) \rightarrow (0, -R) \rightarrow (R, 0^-) \rightarrow (\delta, 0^-) \end{aligned}$$

$$\begin{aligned} \oint_C z^{s-1} Q(z) dz &= \int_{C_\delta} z^{s-1} Q(z) dz + \int_\delta^R x^{s-1} Q(x) dx \\ &+ \int_{C_R} z^{s-1} Q(z) dz + \int_R^\delta (xe^{i2\pi})^{s-1} Q(x) dx \end{aligned}$$

**例 6.4.2.** Compute

$$\int_0^\infty \frac{x^{a-1}}{x + e^{i\phi}} dx, 0 < a < 1, -\pi < \phi < \pi$$

$$\begin{aligned} \oint_C \frac{z^{a-1}}{z + e^{i\phi}} dz &= \int_\delta^R \frac{x^{a-1}}{x + e^{i\phi}} dx + \int_{C_R} \frac{z^{a-1}}{z + e^{i\phi}} dz \\ &+ \int_R^\delta \frac{(xe^{i2\pi})^{a-1}}{x + e^{i\phi}} dx + \int_{C_\delta} \frac{z^{a-1}}{z + e^{i\phi}} dz \end{aligned}$$

**Red Integral** gives:

$$(1 - e^{i2\pi a}) \int_\delta^R \frac{x^{a-1}}{x + e^{i\phi}} dx$$

**Cyan Integral** gives:

$$\lim_{z \rightarrow 0} \frac{z^a}{z + e^{i\phi}} = 0 \Rightarrow \text{Origin.Integral.} = 0$$

**Blue Integral** gives:

$$\lim_{z \rightarrow \infty} \frac{z^a}{z + e^{i\phi}} = 0 \Rightarrow \text{Origin.Integral.} = 0$$

$$L.H.S = 2\pi i \sum \text{res} \frac{z^{a-1}}{z + e^{i\phi}} z = e^{i(\phi+\pi)} \quad (0 < \phi + \pi < 2\pi)$$

Thus,

$$\int_0^\infty \frac{x^{a-1}}{x + e^{i\phi}} dx = -\frac{2\pi i}{1 - e^{i2\pi a}} e^{i\pi a} e^{i\phi(a-1)} = \frac{\pi}{\sin \pi a} e^{i\phi(a-1)}$$

## 微分方程部分

## 第七章 偏微分方程

### 7.1 Some Basic Assumptions

**定义 7.1.1.** *Order of differential equation:* Equals to the highest order of the equation.

**定义 7.1.2.** *Linear:*

$$a_0y + a_1y' + a_2y'' + a_3y''' + \cdots = b$$

Where  $a_i$  and  $b_i$  are not function of  $y$ .

$$y' = \cot y \quad (\text{not linear because of the term } \cot y)$$

$$y' = 1 \quad (\text{not linear because of the product } yy')$$

$$y'^2 = xy \quad (\text{not linear because of the term } y'^2).$$

**定义 7.1.3.** *A solution of an PDE:* A solution of a differential equation (in the variables  $x$  and  $y$ ) is a relation between  $x$  and  $y$  which, if substituted into the differential equation, gives an identity.

**定义 7.1.4.** *General solutions and particular solutions:*

$$\frac{\partial^2 U}{\partial x \partial y} = 2y - x$$

One particular solution is  $U(x, y) = xy^2 - \frac{1}{2}x^2y$

The general solution is  $U(x, y) = xy^2 - \frac{1}{2}x^2y + A(x) + B(y)$

**定义 7.1.5.**

$$\hat{L}[u] = f \quad \hat{L} : \text{linear operator}$$

If  $f = 0$ , we say that the equation is **homogeneous**.

If  $f \neq 0$ , we say that the equation is **inhomogeneous**.

Suppose  $\hat{L}$  is a linear operator, we have:

$$\hat{L}[c_1 u_1 + c_2 u_2] = c_1 \hat{L}[u_1] + c_2 \hat{L}[u_2], \quad (c_1, c_2 = \text{constant})$$

If both  $u_1$  and  $u_2$  are solutions to a homogeneous equation  $\hat{L}[u] = 0$ , then we have

$$\hat{L}[c_1 u_1 + c_2 u_2] = 0$$

If both  $u_1$  and  $u_2$  are solutions to an inhomogeneous equation  $\hat{L}[u] = f$ , then we have

$$\hat{L}[u_1 - u_2] = 0$$

Because

$$u_1 = (u_1 - u_2) + u_2$$

this means the summation of a solution to a homogeneous equation and a solution to an inhomogeneous equation is still a solution to the original inhomogeneous equation.

## 7.2 偏微分方程的解法——几种情况

对于如下偏微分方程:

$$A_0 \frac{\partial^n u}{\partial x^n} + A_1 \frac{\partial^n u}{\partial x^{n-1} \partial y} + \dots + A_n \frac{\partial^n u}{\partial y^n} + B_0 \frac{\partial^{n-1} u}{\partial x^{n-1}} + \dots + M \frac{\partial u}{\partial x} + N \frac{\partial u}{\partial y} + Pu = f(x, y)$$

如果引入记号:

$$\text{Denote } \hat{D}_x \equiv \partial/\partial x, \hat{D}_y \equiv \partial/\partial y$$

那么原式可以表示为:

$$\begin{aligned} \hat{L}(\hat{D}_x, \hat{D}_y)u &= (A_0 \hat{D}_x^n + A_1 \hat{D}_x^{n-1} \hat{D}_y + \dots + A_n \hat{D}_y^n + \\ &\quad B_0 \hat{D}_x^{n-1} + \dots + M \hat{D}_x + N \hat{D}_y + P)u = f(x, y) \end{aligned}$$

### 7.2.1 Homogeneous Case

齐次情况时, 原式写成:

$$(A_0 \hat{D}_x^n + A_1 \hat{D}_x^{n-1} \hat{D}_y + \dots + A_n \hat{D}_y^n)u = 0$$

对左侧做多项式分解, 可以得到:

$$\hat{L}(\hat{D}_x, \hat{D}_y) = A_0 (\hat{D}_x - a_1 \hat{D}_y) (\hat{D}_x - a_2 \hat{D}_y) \dots (\hat{D}_x - a_n \hat{D}_y)$$

如果我们假定原方程有一个 Trial Solution, 记为:

$$u = \phi(y + ax)$$

注意到, 由链式法则

$$\hat{D}_x^k = a^k \phi^{(k)}(y + ax), \hat{D}_y^k = \phi^{(k)}(y + ax), \hat{D}_x^r \hat{D}_y^s = a^r \phi^{(r+s)}(y + ax)$$

将上述等式代入原方程, 得到

$$\begin{aligned} (A_0 a^n + A_1 a^{n-1} + \dots + A_n) \phi^{(n)}(y + ax) &= 0 \Rightarrow \\ (A_0 a^n + A_1 a^{n-1} + \dots + A_n) &= 0 \end{aligned} \quad (7.1)$$

7.1 的解记为

$$a_1, a_2, a_3, \dots, a_n \quad (\text{Assume No Multiple Roots})$$

那么原式的解:

$$u = \phi_1(y + a_1 x) + \phi_2(y + a_2 x) + \dots + \phi_n(y + a_n x)$$

对于重根: Generally, if  $a$  is  $n$  multiple root, i.e.

$$\hat{L}(\hat{D}_x, \hat{D}_y) u = (\hat{D}_x - a \hat{D}_y)^n u = 0$$

then the general solution is

$$u = x^{n-1} \phi_1(y + ax) + x^{n-2} \phi_2(y + ax) + \dots + x \phi_{n-1}(y + ax) + \phi_n(y + ax)$$

## 7.2.2 Inhomogeneous Case——1-st Order

原式:

$$(\hat{D}_x - a \hat{D}_y - b)u = 0$$

我们假定原式有如下形式的解:

$$u = f(x) \phi(y + ax)$$

$$\begin{aligned} (\hat{D}_x - a \hat{D}_y - b) [f(x) \phi(y + ax)] &= \\ f(x) \underbrace{(\hat{D}_x - a \hat{D}_y) \phi(y + ax)}_{=0} + \phi(y + ax) (\hat{D}_x - b) f(x) &= 0 \Rightarrow \\ f' - b f = 0 \Rightarrow f = C e^{bx} \Rightarrow u = C e^{bx} \phi(y + ax) &= e^{bx} \phi(y + ax) \end{aligned}$$



### 7.2.3 inhomogeneous PDEs—Exponential

inhomogeneous PDE 的解可以写成通解与特解之和的形式。

$$1. f = e^{ax+by}, \quad \hat{L}(\hat{D}_x, \hat{D}_y) = \hat{D}_y + \hat{D}_x:$$

If so, then, it is easy to verify:

$$\hat{L}^{-1}(\hat{D}_x, \hat{D}_y) e^{ax+by} = \frac{1}{L(a, b)} e^{ax+by}, \quad L(a, b) \neq 0$$

where  $L(a, b)$  is polynomial of  $a$  and  $b$ .  $L(a, b)$  is calculated by this in this case:

$$\hat{D}_x f = af; \quad \hat{D}_y f = bf; \quad L(a, b) = a + b$$

So, a particular solution of this PDE is:

$$u_0 = \frac{f(x, y)}{\hat{L}(\hat{D}_x, \hat{D}_y)} = \frac{f(x, y)}{L(a, b)} = \frac{f(x, y)}{a + b}$$

2. If  $L(a, b) = 0$ : Without losing generality, we can assume:

$$\hat{L}(\hat{D}_x, \hat{D}_y) = b\hat{D}_x - a\hat{D}_y, \quad [L(a, b) = b \cdot a - a \cdot b = 0]$$

$$(b\hat{D}_x - a\hat{D}_y) u = e^{ax+by}$$

we assume the special solution as

$$u_0(x, y) = f(x, y) e^{ax+by}$$

$$(b\hat{D}_x - a\hat{D}_y) f(x, y) = 1$$

Let's further assume

$$f(x, y) = \alpha x + \beta y + \gamma \Rightarrow b\alpha - a\beta = 1$$

- If we take  $\beta = \gamma = 0, \alpha = 1/b$   $u_0 = (x/b) \cdot e^{ax+by}$
- If we take  $\alpha = \gamma = 0, \beta = -1/a$   $u_0 = (-y/a) \cdot e^{ax+by}$

3. If  $f = e^{ax+by} g(x, y)$

$$\frac{1}{\hat{L}(\hat{D}_x, \hat{D}_y)} e^{ax+by} g(x, y) = e^{ax+by} \frac{1}{\hat{L}(\hat{D}_x + a, \hat{D}_y + b)} g(x, y)$$

## 7.2.4 inhomogeneous PDEs—Summation

1. If  $f = x^m y^n$ :  $(\hat{D}_x^2 - 2\hat{D}_x \hat{D}_y + \hat{D}_y^2) u = 12xy$

$$\begin{aligned} u_0 &= \frac{12}{\hat{D}_x^2 - 2\hat{D}_x \hat{D}_y + \hat{D}_y^2} xy = \frac{12}{\hat{D}_x^2} \left(1 - \frac{\hat{D}_y}{\hat{D}_x}\right)^{-2} xy \\ &= \frac{12}{\hat{D}_x^2} \left(1 + 2\frac{\hat{D}_y}{\hat{D}_x} + \dots\right) xy = \frac{12}{\hat{D}_x^2} \left(xy + \frac{2}{\hat{D}_x} x\right) \\ &= 12 \left(y \frac{1}{\hat{D}_x^2} x + \frac{2}{\hat{D}_x^3} x\right) = 12 \left(\frac{1}{6} x^3 y + \frac{1}{12} x^4\right) \end{aligned}$$

As Red Part equals for

$$\hat{D}_y xy = 0 \text{ if } n \leq 2.$$

2. If  $f = f(ax + by)$ :  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12(x + y)$

$$u_0 = \frac{12}{\hat{D}_x^2 + \hat{D}_y^2} (x + y) = \frac{12}{(1^2 + 1^2)} \left[ \frac{1}{3!} (x + y)^3 \right] = (x + y)^3$$

3. If  $f = f(ax + by)$ ,  $L(a, b) = 0$ :  $(\hat{D}_x^2 - 6\hat{D}_x \hat{D}_y + 9\hat{D}_y^2) u = 6x + 2y$

$$u_0 = \frac{1}{(\hat{D}_x - 3\hat{D}_y)^2} (6x + 2y) = \frac{2}{(\hat{D}_x - 3\hat{D}_y)^2} (3x + y) = x^2(y + 3x)$$