Mathematical Methods in Physics

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Review

- Complex numbers
- Geometric representation: Cartesian and polar coordinates
- Operation of complex numbers
- Exponential representation: Euler's formula and De Moivre's theorem

Chapter – 01: Complex numbers and complex functions

- Since we have learned some basic knowledge about complex numbers, now we can define complex functions (复函数).

- Recall some elementary functions (初等函数) of a real number.

Power function (幂函数): $f(x) = x^n \quad n \text{ is an integer}$

Exponential function (指数函数): $f(x) = 10^x \quad f(x) = e^x$

Logarithmic function (对数函数): $f(x) = \log(x) \quad f(x) = \ln(x)$

Trigonometric function (三角函数): $f(x) = \sin(x),\ \cos(x),\ \tan(x),\ \sec(x)$

Inverse trigonometric function (反三角函数):

$$f(x) = \arcsin(x),\ \arccos(x),\ \arctan(x),\ \text{arcsec}(x)$$

- Note: elementary functions also include the combinations of the following functions.

Power function (幂函数)

Exponential function (指数函数)

Logarithmic function (对数函数)

Trigonometric function (三角函数)

Inverse trigonometric function (反三角函数)

- Since we have learned some basic knowledge about complex numbers, now we can define complex functions (复函数).

- Write elementary functions (初等函数) of a complex number. $z = x + iy$

Power function (幂函数): $f(z) = z^n$ $\quad n$ is an integer

Exponential function (指数函数): $f(z) = 10^z \quad f(z) = e^z$

Logarithmic function (对数函数): $f(z) = \log(z) \quad f(z) = \ln(z)$

Trigonometric function (三角函数): $f(z) = \sin(z),\ \cos(z),\ \tan(z),\ \sec(z)$

Inverse trigonometric function (反三角函数):

$$f(z) = \arcsin(z),\ \arccos(z),\ \arctan(z),\ \mathrm{arcsec}(z)$$

Similarities and differences between real functions and complex functions (for a single variable)

- There is only one variable x for real function $f(x)$; but there are technically two variables x and y for complex function $f(z) = f(x + \mathrm{i}y)$.
- Complex function $f(z) = f(x + \mathrm{i}y)$ is different from a real function $f(x, y)$, because x and y are connected for $f(z)$.
- Although $|\sin(x)| \leq 1$, $|\sin(z)|$ can be greater than 1.

Nevertheless, we assume, without providing strict proof, that many operations for real functions, such as differentiation (微分) and integration (积分), also apply to complex functions.

Power function (幂函数)

$$f(z) = z^n \qquad n \text{ is an integer}$$

$$f(z) = z^2 \qquad f(1+\mathrm{i}) = (1+\mathrm{i})^2 = 2\mathrm{i}$$

We can also extend the above expressions to polynomials.

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

And we can further introduce the ratio between two polynomials.

$$R(z) = \frac{P_n(z)}{Q_m(z)} = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m} \qquad Q_m(z) \neq 0$$

Power function

- Taylor expansion can also be applied to complex functions.

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{1}{2!}(z - z_0)^2 f''(z_0) + \ldots + \frac{1}{n!}(z - z_0)^n f^{(n)}(z_0) + \ldots$$

after rearrangement and simplification

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

Power function

- Differentiation

$$f'(z) = (z^n)' = nz^{(n-1)}$$

Be careful when $n < 0$

- Integration

$$\int f(z)dz = \int z^n dz = \frac{z^{(n+1)}}{n+1} + C$$

Be careful when $n = -1$

Exercise

[2.01] Expand $f(z) = (z+2)^3$ in the form of $P_n(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$

Exponential function (指数函数)

$$f(z) = a^z$$

$$f(z) = e^z$$

$$a = e^{\ln a}$$

$$a^z = (e^{\ln a})^z = e^{z \ln a}$$

We only need to study e^z.

Rules for exponential function

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$$

$$(e^{z_1})^{z_2} = e^{z_1 \cdot z_2}$$

$$\sqrt[z_2]{e} = e^{\frac{1}{z_2}}$$

Exponential function

- Differentiation

$$f'(z) = (e^z)' = e^z$$

The simplest forms!

- Integration

$$\int f(z)dz = \int e^z dz = e^z + C$$

Exercise

[2.02] Use previously studied rules to verify that $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$

Solution – 1: $e^{z_1} \cdot e^{z_2} = e^{x_1+\mathrm{i}y_1} \cdot e^{x_2+\mathrm{i}y_2}$

$$= e^{x_1+x_2} \cdot e^{\mathrm{i}(y_1+y_2)} = e^{z_1+z_2}$$

Exercise

[2.02] Use previously studied rules to verify that $e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$

Solution – 2: $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$

$$e^{z_1} \cdot e^{z_2} = \left(1 + z_1 + \frac{z_1^2}{2!} + \frac{z_1^3}{3!} + \ldots\right) \cdot \left(1 + z_2 + \frac{z_2^2}{2!} + \frac{z_2^3}{3!} + \ldots\right)$$

$$= 1 + (z_1 + z_2) + \left(\frac{z_1^2}{2!} + \frac{z_2^2}{2!} + z_1 \cdot z_2\right) + \left(\frac{z_1^3}{3!} + \frac{z_2^3}{3!} + \frac{z_1 \cdot z_2^2}{2!} + \frac{z_1^2 \cdot z_2}{2!}\right) + \ldots$$

$$= 1 + (z_1 + z_2) + \frac{(z_1 + z_2)^2}{2!} + \frac{(z_1 + z_2)^3}{3!} + \ldots = e^{z_1 + z_2}$$

Hint: You may need to recall binomial expansion (二项展开式) $(a+b)^n = \sum_{r=0}^{n} C_n^r a^{n-r} b^r = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} a^{n-r} b^r$

Exponential function

- A complex exponential function can be periodic (具有周期性).

$$e^z \cdot e^{2n\pi \mathrm{i}} = e^z \cdot 1 = e^z, \text{ where } n = \text{integer}$$

- A real exponential function does not have the above property.

- Examine the asymptotic behavior (渐近行为) of e^z

when $z = x \to +\infty$ $\qquad |e^z| \to +\infty$

$z = x \to -\infty$ $\qquad |e^z| \to 0$

$z = iy \text{ and } y \to +\infty$ $\qquad |e^z| = 1$

e^z does not converge when $z \to \infty$

Trigonometric function (三角函数)

Since we already know how to work with the exponential function, we can re-express trigonometric functions using the exponential function.

$$\cos(z) = \frac{e^{\mathrm{i}z} + e^{-\mathrm{i}z}}{2}$$

$$\sin(z) = \frac{e^{\mathrm{i}z} - e^{-\mathrm{i}z}}{2\mathrm{i}}$$

Note: both e^{z} and $e^{\mathrm{i}z}$ are complex exponential functions.

Trigonometric function

- Differentiation

$$(\cos z)' = -\sin z \qquad (\sin z)' = \cos z$$

$$\cos(z) = \frac{e^{\mathrm{i}z} + e^{-\mathrm{i}z}}{2}$$

$$\sin(z) = \frac{e^{\mathrm{i}z} - e^{-\mathrm{i}z}}{2\mathrm{i}}$$

- Integration

$$\int \sin z dz = -\cos z + C \qquad \int \cos z dz = \sin z + C$$

Exercise

[2.03] Use previously studied rules to verify that $(\cos z)' = -\sin z$

$$(\cos(z))' = \left(\frac{e^{iz} + e^{-iz}}{2}\right)' = \frac{ie^{iz} - ie^{-iz}}{2}$$

$$= -\frac{e^{iz} - e^{-iz}}{2i} = -\sin(z)$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Trigonometric function

- Similar to the complex exponential function, trigonometric functions are also periodic. Note: their periods are not exactly the same (mind the difference i).

$$e^z = e^{(z \pm \mathrm{i}2n\pi)} \qquad \text{period} = 2n\pi\mathrm{i}$$

$$\sin(z) = \sin(z \pm 2n\pi), \text{ and } \cos(z) = \cos(z \pm 2n\pi) \qquad \text{period} = 2n\pi$$

$$\cos(z) = \frac{e^{\mathrm{i}z} + e^{-\mathrm{i}z}}{2} \qquad \sin(z) = \frac{e^{\mathrm{i}z} - e^{-\mathrm{i}z}}{2\mathrm{i}}$$

Trigonometric function

- Unlike real trigonometric functions, the modulus of a complex trigonometric function can be greater than 1.

$$\cos(z = \mathrm{i}) = \frac{e^{-1} + e^{1}}{2} \approx 1.543$$

- Also think about the asymptotic behavior of trigonometric function, when $z \to \infty$

$$\lim_{z=\mathrm{i}y\to+\mathrm{i}\infty} \cos(z) = \frac{e^{(-\infty)} + e^{(+\infty)}}{2}$$

Unbounded (无界的)

$$\lim_{z=x\to+\infty} \cos(z) = \frac{e^{\mathrm{i}x} + e^{-\mathrm{i}x}}{2}$$

Bounded (有界的)

Ordinary real function

Trigonometric function

- We can use $\cos(z)$ and $\sin(z)$ to further define other types of complex trigonometric function.

$$\tan(z) = \frac{\sin(z)}{\cos(z)} \qquad \cot(z) = \frac{\cos(z)}{\sin(z)} \qquad \sec(z) = \frac{1}{\cos(z)} \qquad \csc(z) = \frac{1}{\sin(z)}$$

- Many well-known relations for real trigonometric functions also apply to complex trigonometric functions.

Exercise

[2.04] Prove the following relation for complex trigonometric functions.

$$\cos^2(z) + \sin^2(z) = 1$$

$$\text{Solution}: \ \text{L.H.S} = \left(\frac{e^{\mathrm{i}z} + e^{-\mathrm{i}z}}{2}\right)^2 + \left(\frac{e^{\mathrm{i}z} - e^{-\mathrm{i}z}}{2\mathrm{i}}\right)^2$$

$$= \frac{e^{2\mathrm{i}z} + e^{-2\mathrm{i}z} + 2}{4} - \frac{e^{2\mathrm{i}z} + e^{-2\mathrm{i}z} - 2}{4} = 1$$

L.H.S. = Left hand side
R.H.S. = Right hand side

- Attention: here $\cos^2(z)$ is no longer a non-negative real number, but a complex number.
- Recall that $|\cos(z)|$ can be unbounded, the above relation => $|\sin(z)|$ can be unbounded, too.

Exercise

[2.05] Prove the following relation for complex trigonometric functions.

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$$

$$\text{Solution:}\quad \text{L.H.S.} = \frac{e^{\mathrm{i}(z_1+z_2)} - e^{-\mathrm{i}(z_1+z_2)}}{2\mathrm{i}} = \frac{e^{\mathrm{i}z_1}e^{\mathrm{i}z_2} - e^{-\mathrm{i}z_1}e^{-\mathrm{i}z_2}}{2\mathrm{i}}$$

$$= \frac{(e^{\mathrm{i}z_1}e^{\mathrm{i}z_2} \boxed{+ e^{-\mathrm{i}z_1}e^{\mathrm{i}z_2}}) \boxed{- (e^{-\mathrm{i}z_1}e^{\mathrm{i}z_2}} + e^{-\mathrm{i}z_1}e^{-\mathrm{i}z_2})}{2\mathrm{i}}$$

$$= \frac{2\cos(z_1)\cdot(\cos(z_2) + \mathrm{i}\sin(z_2)) - 2\cos(z_2)\cdot(\cos(z_1) - \mathrm{i}\sin(z_1))}{2\mathrm{i}}$$

$$= \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2) = \text{R.H.S.}$$

Question: Can you find other ways to prove the above relation?

Exercise

[2.06] Solve the equation $\cos(z) = \mathrm{i}$

$$\text{Solution}: \ \text{L.H.S} = \frac{e^{\mathrm{i}z} + e^{-\mathrm{i}z}}{2} = \mathrm{i}$$

$e^{\mathrm{i}z} \neq 0$ multiply $e^{\mathrm{i}z}$ on both sides

recall that $\sqrt{-1} = \pm\mathrm{i}$

Then we have $e^{2\mathrm{i}z} - 2\mathrm{i}e^{\mathrm{i}z} + 1 = 0$

Let $e^{iz} = z_1$ we have $z_1^2 - 2\mathrm{i}z_1 + 1 = 0$

$$z_1 = \frac{2\mathrm{i} \pm \sqrt{-4-4}}{2} = (1 \pm \sqrt{2})\mathrm{i}$$

$=> e^{-y}e^{\mathrm{i}x} = (1 \pm \sqrt{2})\mathrm{i}$

Situation-1: $x = 2n\pi + \pi/2, \ y = -\ln(1 + \sqrt{2})$

Situation-2: $x = 2n\pi - \pi/2, \ y = -\ln(\sqrt{2} - 1)$

Hyperbolic function (双曲函数)

After mastering exponential and trigonometric functions, we can naturally introduce hyperbolic function(s).

$$\sinh(z) = \frac{e^z - e^{-z}}{2} \qquad \cosh(z) = \frac{e^z + e^{-z}}{2} \qquad \tanh(z) = \frac{\sinh(z)}{\cosh(z)}$$

$$\coth(z) = \frac{\cosh(z)}{\sinh(z)} \qquad \mathrm{sech}(z) = \frac{1}{\cosh(z)} \qquad \mathrm{csch}(z) = \frac{1}{\sinh(z)}$$

- **Periodicity:** $\sinh(z) = \sinh(z \pm \mathrm{i}2n\pi)$ $\quad \tanh(z) = \tanh(z \pm \mathrm{i}n\pi)$
- **Differentiation:** $(\sinh(z))' = \cosh(z)$

  $(\cosh(z))' = \sinh(z)$

Exercise

[2.07] Please verify that $(\tanh(z))' = \mathrm{sech}^2(z)$

$$(\tanh(z))' = \left(\frac{e^z - e^{-z}}{e^z + e^{-z}}\right)' = \frac{(e^z - e^{-z})' \cdot (e^z + e^{-z}) - (e^z - e^{-z}) \cdot (e^z + e^{-z})'}{(e^z + e^{-z})^2}$$

$$= \frac{(e^{2z} + e^{-2z} + 2) - (e^{2z} + e^{-2z} - 2)}{(e^z + e^{-z})^2} = \frac{4}{(e^z + e^{-z})^2} = \left(\frac{2}{e^z + e^{-z}}\right)^2 = \mathrm{sech}^2(z)$$

Hint: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Exercise

[2.08] Please verify that $\cosh^2(z) - \sinh^2(z) = 1$

$$\text{L.H.S.} = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2$$

$$= \frac{(e^{2z} + e^{-2z} + 2) - (e^{2z} + e^{-2z} - 2)}{4}$$

$$= 1 = \text{R.H.S.}$$

What about the inverse functions (逆函数)?

$$f(z) = z^n \qquad g(z) = \sqrt[n]{z}$$

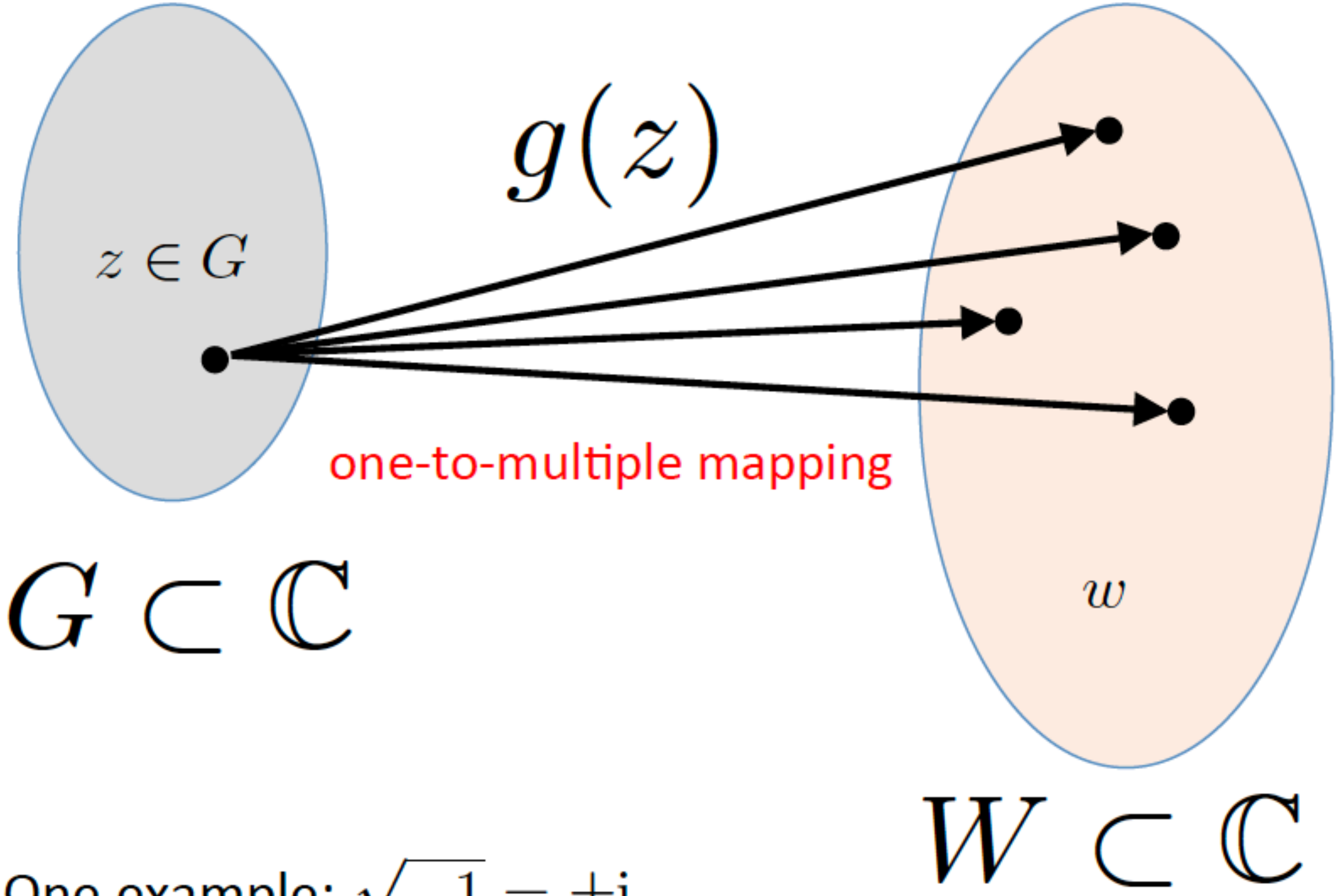
$$f(z) = e^z \qquad g(z) = \ln(z)$$

$$f(z) = \sin(z),\ \cos(z),\ \tan(z),\ \sec(z) \qquad g(z) = \arcsin(z), \arccos(z),\ \arctan(z),\ \operatorname{arcsec}(z)$$

We already know that trigonometric and hyperbolic functions can be expressed using combinations of several exponential functions. Therefore, to understand the inverse functions, it is better to first investigate $\sqrt[n]{z}$ and $\ln(z)$.

It is important to note that these functions can be multi-valued (多值的).

Multi-valued functions (多值函数)



One example: $\sqrt{-1} = \pm \mathrm{i}$

Radical or *n*th root function (根式函数)

- For a given variable z (complex number), as long as w satisfies the relation

  $w^2 = z - a, \text{here} \quad a = \text{a constant complex number}$

  We call w the square root of $z - a$.

- In general, $\sqrt[n]{z-a}$ is called " $z - a$ radical n ", or "the *n*th root of $z - a$".

Radical or *n*th root function (根式函数)

Let's recall the geometric representation of a complex number.

$w = \rho e^{i\phi}$ $z - a = r e^{i\theta}$ (we assume $r \geq 0, \rho \geq 0$)

$w = \sqrt{z-a} <=> \rho^2 = r, \text{ and } 2\phi = \theta \pm 2n\pi$

$$\rho = \sqrt{r}$$

$$\boxed{\phi = \theta/2 \pm n\pi}$$

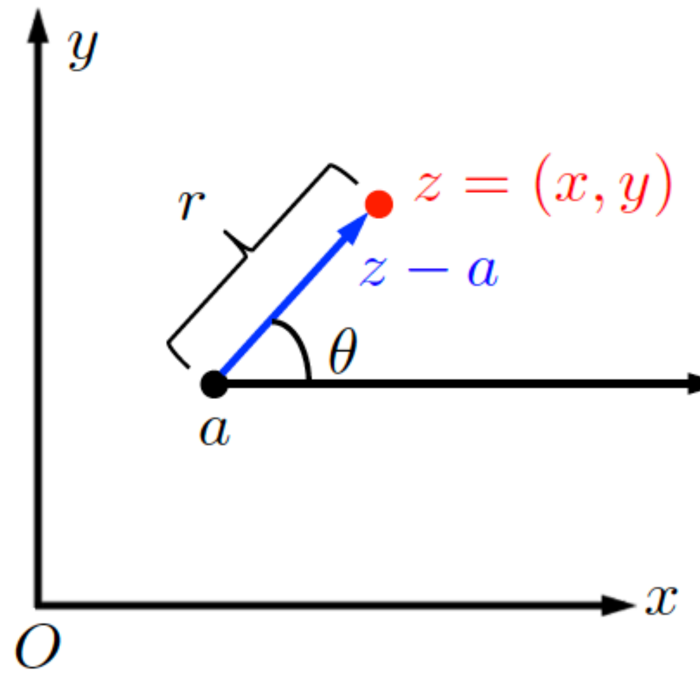
The source of multiple values

$$\left.\begin{aligned} w &= \sqrt{r}e^{i\theta/2} \\ w &= \sqrt{r}e^{i(\theta/2+\pi)} = -\sqrt{r}e^{i\theta/2} \end{aligned}\right\} \text{Two solutions}$$

$$w = \sqrt{r}e^{i(\theta/2+2\pi)} = \sqrt{r}e^{i\theta/2}$$

$$w = \sqrt{r}e^{i(\theta/2+3\pi)} = -\sqrt{r}e^{i\theta/2}$$

Repeating the above ones



Radical or *n*th root function (根式函数)

Let’s recall the geometric representation of a complex number.

$w = \rho e^{\mathrm{i}\phi} \qquad z - a = re^{\mathrm{i}\theta}$ (we assume $r \geq 0, \rho \geq 0$)

$$w = \sqrt{z-a} <=> \rho^2 = r, \text{ and } 2\phi = \theta \pm 2n\pi$$

$$\rho = \sqrt{r}$$

$$\boxed{\phi = \theta/2 \pm n\pi}$$

The source of multiple values

$$w = \sqrt{r}e^{\mathrm{i}\theta/2}$$

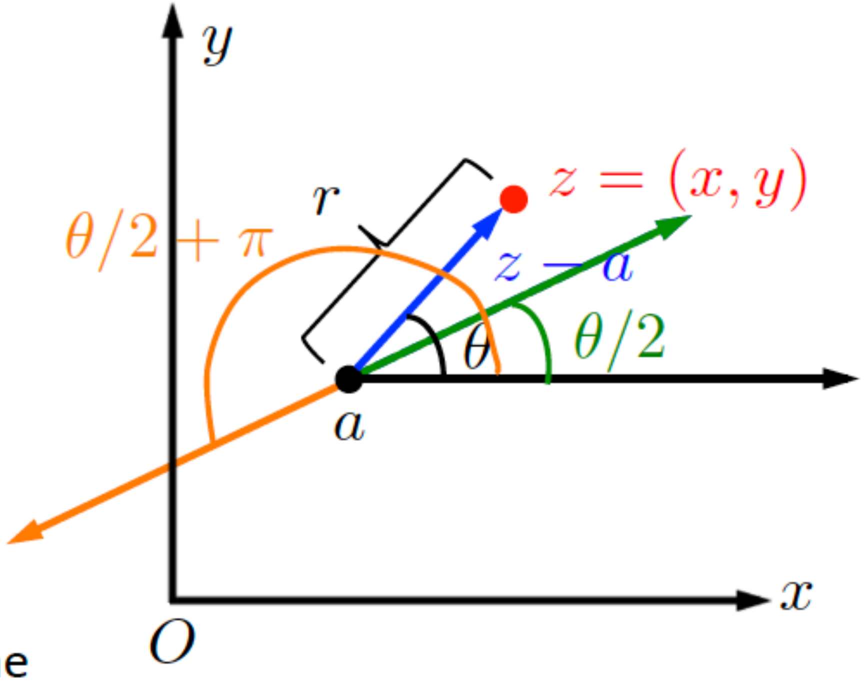
$$w = \sqrt{r}e^{\mathrm{i}(\theta/2+\pi)} = -\sqrt{r}e^{\mathrm{i}\theta/2}$$

Two solutions

$$w = \sqrt{r}e^{\mathrm{i}(\theta/2+2\pi)} = \sqrt{r}e^{\mathrm{i}\theta/2}$$

$$w = \sqrt{r}e^{\mathrm{i}(\theta/2+3\pi)} = -\sqrt{r}e^{\mathrm{i}\theta/2}$$

Repeating the above ones



Radical or *n*th root function (根式函数)

Let's recall the geometric representation of a complex number.

$w = \rho e^{\mathrm{i}\phi}$ $\quad z - a = r e^{\mathrm{i}\theta}$ $\quad$ (we assume $r \geq 0, \rho \geq 0$)

$$w = \sqrt{z-a} <=> \rho^2 = r, \text{ and } 2\phi = \theta \pm 2n\pi$$

$$\rho = \sqrt{r}$$

$$\boxed{\phi = \theta/2 \pm n\pi}$$

→ The source of multiple values

$$\left.\begin{aligned} w &= \sqrt{r} e^{\mathrm{i}\theta/2} \\ w &= \sqrt{r} e^{\mathrm{i}(\theta/2+\pi)} = -\sqrt{r} e^{\mathrm{i}\theta/2} \end{aligned}\right\} \text{Two solutions}$$

$$|w| = \sqrt{r}$$

$$\arg(w) = \frac{1}{2}\arg(z-a)$$

Note ***arg*** itself is multi-valued due to the periodicity.

Radical or *n*th root function (根式函数)

What about $w = \sqrt[3]{z-a}$?

$w = \rho e^{\mathrm{i}\phi}$ $\quad z - a = re^{\mathrm{i}\theta}$ (we assume $r \geq 0, \rho \geq 0$)

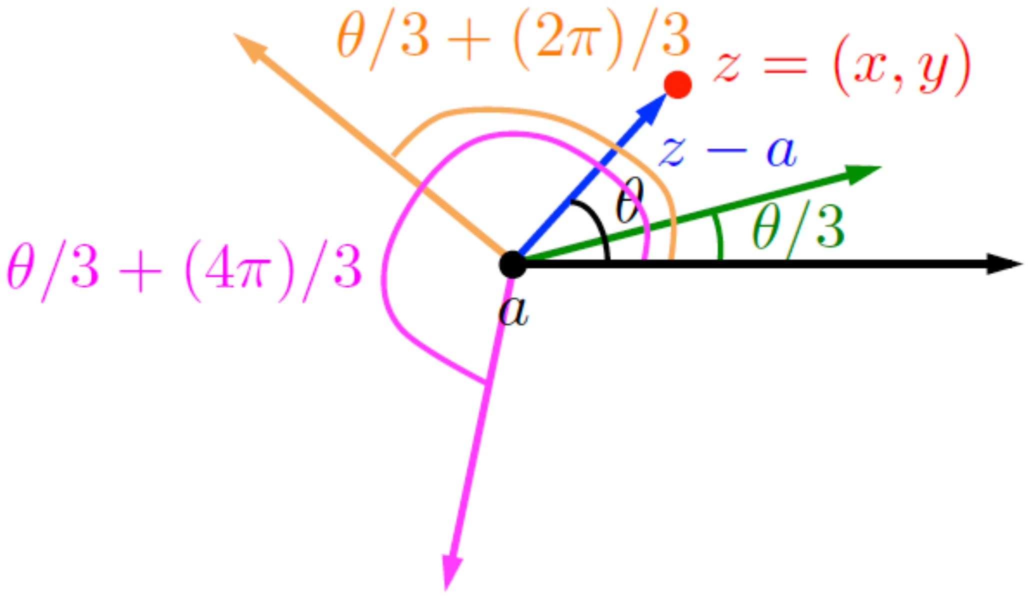
$$\boxed{\begin{aligned} &\rho = \sqrt[3]{r} \\ &\phi = \theta/3 \pm (2n\pi)/3 \end{aligned}}$$

$$w = \sqrt[3]{r}e^{\mathrm{i}\theta/3}$$

$$w = \sqrt[3]{r}e^{\mathrm{i}(\theta/3+2\pi/3)} = \sqrt[3]{r}e^{\mathrm{i}\theta/3} \cdot e^{\mathrm{i}2\pi/3}$$

$$w = \sqrt[3]{r}e^{\mathrm{i}(\theta/3+4\pi/3)} = \sqrt[3]{r}e^{\mathrm{i}\theta/3} \cdot e^{\mathrm{i}4\pi/3}$$

Three solutions



Radical or *n*th root function (根式函数)

What about $w = \sqrt[n]{z-a}$?

$w = \rho e^{\mathrm{i}\phi}$ $\qquad z - a = re^{\mathrm{i}\theta}$ (we assume $r \geq 0, \rho \geq 0$)

$$\boxed{\begin{aligned} &\rho = \sqrt[n]{r} \\ &\phi = \theta/n \pm (2k\pi)/n \end{aligned}}$$

$$\left.\begin{aligned} w &= \sqrt[n]{r}e^{\mathrm{i}\theta/n} \\ w &= \sqrt[n]{r}e^{\mathrm{i}(\theta/n+2\pi/n)} \end{aligned}\right\} \; e^{\mathrm{i}(\theta/n+2\pi/n)} = e^{\mathrm{i}\theta/n} \cdot e^{\mathrm{i}2\pi/n}$$

rotate by $2\pi/n$

...

$$w = \sqrt[n]{r}e^{\mathrm{i}(\theta/n+2(n-1)\pi/n)}$$

De Moivre's theorem (棣莫弗定理)

n solutions, equally partitioned (by angle) along the circle $\sqrt[n]{r}$

Example 3. Find and plot all values of $\sqrt[4]{-64}$. From Figure 10.2 (or by visualizing a plot of -64), we see that the polar coordinates of -64 are $r = 64$, $\theta = \pi + 2k\pi$

p 65 in the Textbook

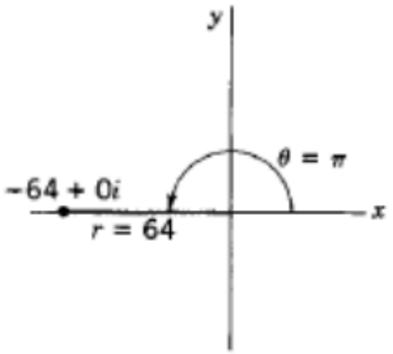


Figure 10.2

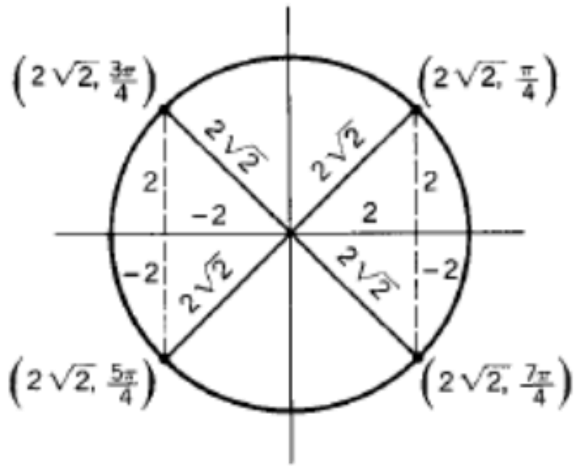


Figure 10.3

(where $k = 0, 1, 2, \cdots$). Then since $z^{1/4} = r^{1/4}e^{i\theta/4}$, the polar coordinates of $\sqrt[4]{-64}$ are

$$r = \sqrt[4]{64} = 2\sqrt{2},$$
$$\theta = \frac{\pi}{4}, \frac{\pi + 2\pi}{4}, \frac{\pi + 4\pi}{4}, \frac{\pi + 6\pi}{4}, \cdots = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

We plot these points in Figure 10.3. Observe that they are all on a circle of radius $2\sqrt{2}$, equally spaced $2\pi/4 = \pi/2$ apart. Starting with $\theta = \pi/4$, we add $\pi/2$ repeatedly, and find exactly 4 fourth roots. We can read the values of $\sqrt[4]{-64}$ in rectangular form from Figure 10.3:

$$\sqrt[4]{-64} = \pm 2 \pm 2i \quad \text{(all four combinations of } \pm \text{ signs)}$$

or we can calculate them as in Example 2, or we can solve the equation $z^4 = -64$ by computer.

Summary In each of the preceding examples, our steps in finding $\sqrt[n]{re^{i\theta}}$ were:

(a) Find the polar coordinates of the roots: Take the nth root of r and divide $\theta + 2k\pi$ by n.

(b) Make a sketch: Draw a circle of radius $\sqrt[n]{r}$, plot the root with angle θ/n, and then plot the rest of the n roots around the circle equally spaced $2\pi/n$ apart. Note that we have now essentially solved the problem. From the sketch you can see the approximate rectangular coordinates of the roots and check your answers in (c). Since this sketch is quick and easy to do, it is worthwhile even if you use a computer to do part (c).

(c) Find the $x + iy$ coordinates of the roots by one of the methods in the examples. If you are using a computer, you may want to make a computer plot of the roots which should be a perfected copy of your sketch in (b).

p 66 in the Textbook

We will come back to multi-valued functions again later. Now you just need to know radical or *n*th root function can be multi-valued, and the source for making it multi-valued.

Logarithmic function (对数函数)

- For a given variable z (complex number), as long as w satisfies the relation

  $$e^w = z$$

  We call $w = \ln z$.

  $$w = u + \mathrm{i}v, \text{ and } z = re^{\mathrm{i}\theta}$$

  $$e^u e^{\mathrm{i}v} = re^{\mathrm{i}\theta}$$

  $$=> u = \ln|z|, \text{ and } v = \theta \pm 2n\pi$$

  $$\boxed{=> w = \ln z = \ln|z| + \boxed{\mathrm{i}(\theta \pm 2n\pi)}}$$

  The source of multiple values, again due to the argument.

  There is infinite number of solutions.

Inverse trigonometric function (反三角函数)

- We can follow the similar procedure as done before to define inverse trigonometric function.
- For a given variable z (complex number), as long as w satisfies the relation

$$\sin w = \frac{e^{\mathrm{i}w} - e^{-\mathrm{i}w}}{2\mathrm{i}} = z$$

We call $w = \arcsin(z)$.

$$\arcsin(z) = \frac{1}{\mathrm{i}} \ln\left(\mathrm{i}z + \sqrt{1 - z^2}\right) \qquad \arccos(z) = \frac{1}{\mathrm{i}} \ln\left(z + \sqrt{z^2 - 1}\right)$$

$$\arctan(z) = \frac{1}{2\mathrm{i}} \ln \frac{1 + \mathrm{i}z}{1 - \mathrm{i}z}$$

Note: we have just demonstrated that radical (*n*th root) and logarithmic functions can be multi-valued. Therefore, inverse trigonometric functions can also be multi-valued.

Exercise

[2.09] Please derive $\arcsin(z) = \frac{1}{\mathrm{i}} \ln\left(\mathrm{i}z + \sqrt{1-z^2}\right)$.

$$\sin(w) = \frac{e^{\mathrm{i}w} - e^{-\mathrm{i}w}}{2\mathrm{i}} = z$$

Multiply $e^{\mathrm{i}w}$ for both sides, we have

$$(e^{\mathrm{i}w})^2 - 2\mathrm{i}z(e^{\mathrm{i}w}) - 1 = 0$$

$$e^{\mathrm{i}w} = \frac{2\mathrm{i}z \pm \sqrt{4-4z^2}}{2} = \mathrm{i}z + \sqrt{1-z^2}$$

$$=> w = \frac{1}{\mathrm{i}} \ln\left(\mathrm{i}z + \sqrt{1-z^2}\right)$$

We have used the fact that $\sqrt{1-z^2}$ is multi-valued.

That is: $\pm\sqrt{1-z^2} \simeq \sqrt{1-z^2}$

Power function* (幂函数*)

- Let’s relax the condition for defining a complex power function

  $z^{\alpha} = (e^{\ln(z)})^{\alpha} = e^{\alpha \ln(z)}$ where α is an arbitrary complex number (not necessarily an integer)

  Recall that $\ln z = \ln |z| + \mathrm{i}\ \arg z$

  Therefore, z^{α} can be multi-valued.