Exercise – 03 (total = 100’)

Due date: Mar. 13, 2022, 23:59

Note: Unless mentioned otherwise, z is treated as a complex number, while x and y are treated as real numbers. We use z^* or $\overline{z}$ to denote the complex conjugate of z.

Part – 1: True or False (3’ x 5 = 15’)

T (1) If a complex function $f(z)$ has derivative at $z = z_0$, then it is also continuous at $z = z_0$.

T (2) If a complex function $f(z)$ is analytic at $z = z_0$, then it also has derivative at $z = z_0$.

F (3) A boundary point P of region G also belongs to G, i.e. $P \in G$.

T (4) The complex function e^{iz} is not differentiable at $z \longrightarrow \infty$.

F (5) If $f(z)$ is an analytic function about the complex variable z, so is $\overline{f(z)}$.

Part – 2: Graph (5’ x 3 = 15’)

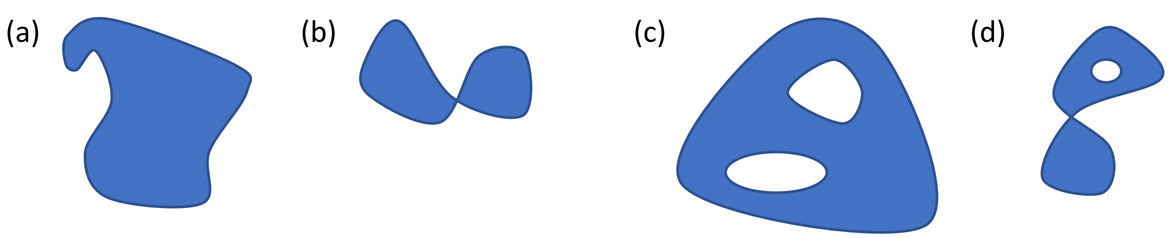
Please answer the following questions related to the graphs.

Note: we consider the points in the blue area.

(1) Which of the following graphs is a region? (a) (c)

(2) Which of the following graphs is a simply connected region? (a)

(3) Which of the following graphs is a multi-connected region? (c)



Part – 3: Proof (5’ x 5 = 25’)

$f(z) = u(x, y) + i \cdot v(x, y)$ or $u(r, \theta) + i \cdot v(r, \theta)$.

(1) Prove that the Cauchy-Riemann relations (hereafter referred to as the C-R relations) in a Cartesian coordinate system are equivalent to

$$i \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$$

(2) Utilize the relations $z = z(r,\theta) = re^{i\theta}$ and $f(z) = f(r,\theta) = u(r,\theta) + i \cdot v(r,\theta)$ to show that the C-R relations in a polar coordinate system are given by:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \text{and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Note: you should try a new method other than the geometry-based one discussed in the class.

(3) Suppose there is a complex function $f(z) = f(x+iy) = u(x,y) + i \cdot v(x,y)$. Now apply variable substitution $z = x + iy, \bar{z} = x - iy$, or $x = (z+\bar{z})/2\,, y = (z-\bar{z})/2i$, to show that the C-R relations are equivalent to

$$\frac{\partial f}{\partial \bar{z}} = 0$$

That is, f cannot be an explicit function of variable $\bar{z}$.

(4) If $f(z)$ is an analytic function defined in region G, and further satisfies the condition $f'(z) = 0$, then show that $f(z)$ is a constant in G.

(5) If both $f(z)$ and $\overline{f(z)}$ are analytic in region G, then show that $f(z)$ is a constant in G.

Part – 4: Verification (5' x 2 = 10')

Please follow the definition of derivative to verify the following relations.

(1) $\frac{d}{dz}(z^4) = 4z^3$

(2) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z)-f(z)g'(z)}{g^2(z)}, g(z) \neq 0$

Part – 5 Judgement (5' x 3 = 15')

Please judge whether the following functions are analytic in a bounded region (not including ∞) and present your reason.

(1) $|z|^2$

(2) e^{iz}

(3) $\frac{y-ix}{x^2+y^2}$

Part 3.

(1) $$i\frac{\partial f}{\partial x} = i\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}$$
$$= i\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}$$
$$= \frac{\partial f}{\partial y}$$

(2) $z = re^{i\theta} = x + iy$, where $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

① $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$
$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$$
$$= -\frac{\partial v}{\partial x} r\sin\theta + \frac{\partial v}{\partial y} r\cos\theta$$
$$= \frac{\partial u}{\partial y} r\sin\theta + \frac{\partial u}{\partial x} r\cos\theta \quad \text{(C-R relation)}$$
$$= r\frac{\partial u}{\partial r} \quad \Rightarrow \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

② $$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}$$
$$= \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= -\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta$$

$$= -\frac{\partial v}{\partial y} r\sin\theta - \frac{\partial v}{\partial x} r\cos\theta$$

$$= -r\frac{\partial v}{\partial r} \Rightarrow \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

(3) First, C-R relations are equivalent to

$$i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$$

Then,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}$$

$$= \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2}\frac{\partial f}{\partial x}$$

$$= 0$$

(4) $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = f'(z) = 0 = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$\Rightarrow \frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = 0 = -\frac{\partial u}{\partial y}$$

$\therefore$ $u(x,y) = C_1$, $v(x,y) = C_2$

$f(z) = u + iv = C_1 + iC_2$, a constant.

(5) $\overline{f(z)} = u - iv$.

$f(z)$ analytic $\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$

$\overline{f(z)}$ analytic $\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \end{cases}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0$

$\Rightarrow u(x, y)$ and $v(x, y)$ are constant in G.

$\Rightarrow f(z)$ is constant in G.

Part 4.

(1) $$\frac{d}{dz}(z^4) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^4 - z^4}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^4 + 4z^3\Delta z + 6z^2\Delta z^2 + 4z\Delta z^3 - z^4}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(4z^3 + 6z^2\Delta z + 4z\Delta z^2\right)$$

$$= 4z^3$$

(2) $$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \lim_{\Delta z \to 0} \frac{\frac{f(z+\Delta z)}{g(z+\Delta z)} - \frac{f(z)}{g(z)}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{f(z+\Delta z)\,g(z) - f(z)\,g(z+\Delta z)}{g(z)\,g(z+\Delta z)\,\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[\frac{g(z)}{g(z)g(z+\Delta z)} \frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{f(z)}{g(z)g(z+\Delta z)} \frac{g(z+\Delta z) - g(z)}{\Delta z} \right]$$

$$= \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}$$

Part 5.

(1) $f(z) = |z|^2 = x^2 + y^2$

$u(x,y) = x^2+y^2$, $v(x,y) = 0$

C-R relations:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} 2x = 0 \\ 2y = 0 \end{cases} \Rightarrow x = y = 0$$

$\therefore$ $f(z)$ only has derivative at $(0,0)$, which is not a region. Hence, it's not analytic

(2) $f(z) = e^{iz} = e^{ix}e^{-y} = \cos x\, e^{-y} + i \sin x\, e^{-y}$

$\Rightarrow u(x,y) = \cos x\, e^{-y}$ $\quad v(x,y) = \sin x\, e^{-y}$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} -\sin x\, e^{-y} = -\sin x\, e^{-y} \\ -\cos x\, e^{-y} = -\cos x\, e^{-y} \end{cases}$$

which always hold. in a bounded region

Moreover, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ are continuous

$\therefore$ $f(z)$ is analytic in a bounded region.

(3) $u(x,y) = \frac{y}{x^2+y^2}$ $\qquad$ $v(x,y) = -\frac{x}{x^2+y^2}$

$\frac{\partial u}{\partial x} = -\frac{2xy}{(x^2+y^2)^2}$ $\qquad$ $\frac{\partial v}{\partial y} = \frac{2xy}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}$ $\qquad$ $\frac{\partial v}{\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\because \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$ $\qquad$ $\therefore$ $f(z)$ is not analytic.

Another method:

$f(z) = -\frac{i}{z^*}$, an explicit function of z^*

$\therefore$ it's not analytic

Part – 6 Find the expression (5' x 4 = 20')

Suppose $z = x + iy$. If the real part of the analytic function $f(z) = u(x,y) + i \cdot v(x,y)$, i.e., $u(x,y)$, is given by the following expression, please find the full explicit form of $f(z)$.

Note: (1) you should express the final answer in terms of z, not x and y.

(2) don't forget the constant term.

(1) $u(x,y) = \frac{x}{x^2+y^2}$

(2) $u(x,y) = x^2 - y^2 + x$

(3) $u(x,y) = e^y \cos(x)$

(4) $u(x,y) = \cos(x) \cosh(y)$

Part 6.

(1) $v(x,y) = \int -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$

$= \int \frac{2xy}{(x^2+y^2)^2} dx + \frac{y^2-x^2}{(x^2+y^2)^2} dy$

$= \int d\left(-\frac{y}{x^2+y^2}\right)$

$= -\frac{y}{x^2+y^2} + C$

$\therefore f(z) = \frac{x-iy}{x^2+y^2} + iC = \frac{\bar{z}}{|z|^2} + iC$

$= \frac{1}{z} + iC$

(2) $v(x,y) = \int -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$

$= \int 2y\, dx + (2x+1) dy$

$= \int d(2xy + y) = 2xy + y + C$

$$\therefore f(z) = x^2 - y^2 + x + i(2xy + y + c)$$
$$= z^2 + z + ic$$

(3) $$v(x,y) = \int -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$
$$= \int -e^y \cos x\, dx - \sin x e^y dy$$
$$= \int d(-e^y \sin x)$$
$$= -e^y \sin x + C$$
$$\therefore f(z) = e^y \cos x - i e^y \sin x + ic$$
$$= e^{-iz} + ic$$

(4) $$v(x,y) = \int -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$
$$= \int -\cos x \sinh y\, dx - \sin x \cosh y\, dy$$
$$= \int d(-\sin x \sinh y)$$
$$= -\sin x \sinh y + C$$
$$\therefore f(z) = \cos x \cosh y - i \sin x \sinh y + ic$$
$$= \cos x \cos(iy) - \sin x \sin(iy) + ic$$
$$= \cos(x + iy) + ic$$
$$= \cos(z) + ic$$