Mathematical Methods in Physics

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Review

- PDEs: the wave equation and the heat equation
- Classification of boundary conditions
- Characteristics of solutions

Chapter – 04: PDE, separation of variables

- We will continue to discuss PDEs.

- We will introduce the method called separation of variables.

- You will learn when and why the above method works.

The wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \ t > 0 \\ u(x,t)|_{t=0} = \phi(x), \ -\infty < x < \infty \\ \dfrac{\partial u}{\partial t}|_{t=0} = \psi(x), \ -\infty < x < \infty \\ u(x,t)|_{x\to\pm\infty} \to 0 \quad \text{or is bounded} \end{cases}$$

$$u(x,t) = f(x-at) + g(x+at) = \frac{1}{2}[\phi(x-at) + \phi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi) d\xi$$

- We can only find the **traveling wave solution** under some special conditions.
- Can we find a solution of the form $u(x,t) = X(x)T(t)$? If yes, the original PDE can be reduced to ODE (about *X* and about *T*) , which will be easier to solve.

Key components for this week

- When and how can we do variable separation for solving a PDE?

- What ideas and skills can we learn from variable separation?

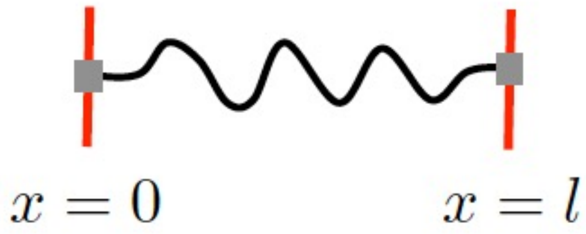
- Some practice

The wave equation – free vibration

No external force

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, \ t > 0$$ **Homogeneous PDE**

$$u|_{x=0} = 0, \quad u|_{x=l} = 0, \quad t \geq 0$$ **Homogeneous boundary condition**



$$u|_{t=0} = \phi(x), \quad \frac{\partial u}{\partial t}|_{t=0} = \psi(x), \quad 0 \leq x \leq l$$

Step 1: Variable separation $u(x,t) = X(x)T(t)$

u(x,t) is not always equal to 0

Insert into PDE, $X(x)T''(t) = a^2 X''(x)T(t)$ $\Rightarrow$ $$\frac{1}{a^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

Function of t = Function of x = some constant independent of t and x $-\lambda$

$$T''(t) + \lambda a^2 T(t) = 0,$$

$$X''(x) + \lambda X(x) = 0$$

T(t) is not always equal to 0

$$X(0)T(t) = 0, \ X(l)T(t) = 0 \ => \ X(0) = 0, \ X(l) = 0$$

The wave equation – free vibration

Step 2: Find the eigenvalues and eigenfunctions

$$X''(x) + \lambda X(x) = 0$$

$$X(0) = 0,\ X(l) = 0$$

An ODE problem

$x = 0$ $x = l$

We want to find non-zero solution of $X(x)$ (called an eigenfunction); the corresponding λ is called eigenvalue. --- Similar to linear algebra

- Search for the solution in the complex domain
- Mind if multiple roots exist

(1) $\lambda = 0$, then $X(x) = Ax + B$

Use boundary condition (B.C.) => $A = 0,\ B = 0, X(x) = 0$ <= Not an eigenfunction!

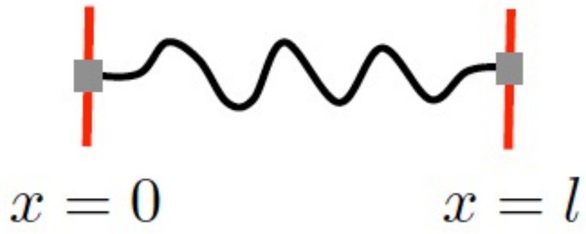
(2) $\lambda \neq 0$, then $X(x) = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$ <= independent solutions

Use B.C. => $B = 0,\ A\sin\sqrt{\lambda}l = 0$ => $\sqrt{\lambda}l = n\pi,\ \text{or},\ \lambda_n = \left(\frac{n\pi}{l}\right)^2,\ n = 1, 2, 3$

The corresponding eigenfunction is $X_n(x) = \sin\frac{n\pi}{l}x$ $\sin\frac{-n\pi}{l}x = -\sin\frac{n\pi}{l}x$

The wave equation – free vibration

Step 3: Find the particular solution, then the general solution.



Now insert $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ to the ODE of T(t)

$$T''(t) + \lambda_n a^2 T(t) = 0$$

We have $T_n(t) = C_n \sin \frac{n\pi}{l} at + D_n \cos \frac{n\pi}{l} at$

One particular solution that satisfies the PDE and B.C. (not necessarily for I.C.)

$$u_n(x,t) = \left(C_n \sin \frac{n\pi}{l} at + D_n \cos \frac{n\pi}{l} at\right) \sin \frac{n\pi}{l} x, \;\; n = 1, 2, 3$$

Due to the homogeneous condition (for PDE and B.C.)

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \sin \frac{n\pi}{l} at + D_n \cos \frac{n\pi}{l} at\right) \sin \frac{n\pi}{l} x$$

If the series is convergent and differentiable, then it also satisfies the PDE and B.C.

The rest work is to find the values for C_n and D_n from I.C.

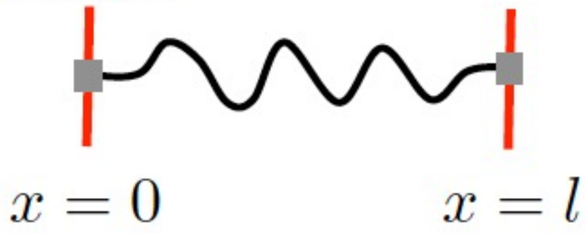
$$\sum_{n=1}^{\infty} D_n \sin \frac{n\pi}{l} x = \phi(x)$$

$$\sum_{n=1}^{\infty} C_n \frac{n\pi a}{l} \sin \frac{n\pi}{l} x = \psi(x)$$

The wave equation – free vibration

Step 4: Use the orthogonality of eigenfunctions to find the coefficients.

$$\int_0^l X_n(x)X_m(x)dx = 0,\ n \neq m$$



Proof:

$$\left.\begin{aligned} &X_n''(x) + \lambda_n X_n(x) = 0, \\ &X_n(0) = 0,\ X_n(l) = 0 \\ &X_m''(x) + \lambda_m X_m(x) = 0, \\ &X_m(0) = 0,\ X_m(l) = 0 \end{aligned}\right\} \boxed{\int_0^l [X_n''(x)X_m(x) - X_m''(x)X_n(x)]dx} + (\lambda_n - \lambda_m)\int_0^l X_n(x)X_m(x)dx = 0$$

d(A*B) = dA*B + A*dB

$$\int_0^l [X_n''(x)X_m(x) - X_m''(x)X_n(x)]dx = [X_n'(x)X_m(x) - X_m'(x)X_n(x)]_{x=0}^{x=l}$$

$$- \int_0^l [X_n'(x)X_m'(x) - X_m'(x)X_n'(x)]dx = 0$$

➡ When $\lambda_n \neq \lambda_m$, $\int_0^l X_n(x)X_m(x)dx = 0$

The wave equation – free vibration

Step 4: Use the orthogonality of eigenfunctions to find the coefficients.

$$\int_0^l \phi(x) \sin \frac{m\pi}{l} x dx = \int_0^l \sum_{n=1}^{\infty} D_n \sin \frac{n\pi}{l} x \sin \frac{m\pi}{l} x dx$$

$$= \sum_{n=1}^{\infty} D_n \int_0^l \sin \frac{n\pi}{l} x \sin \frac{m\pi}{l} x dx = D_m \int_0^l \sin^2 \frac{m\pi}{l} x dx = D_m \frac{l}{2}$$

$$\frac{1 - \cos \frac{2m\pi}{l} x}{2}$$

You've learned this in Calculus!

$$D_n = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi}{l} x dx$$

Similarly, we have $C_n = \dfrac{2}{n\pi a} \displaystyle\int_0^l \psi(x) \sin \frac{n\pi}{l} x dx$

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \sin \frac{n\pi}{l} at + D_n \cos \frac{n\pi}{l} at \right) \sin \frac{n\pi}{l} x$$

Some remarks

- We still need to verify if the summation $u(x,t) = \sum_{n=1}^{\infty} \left(C_n \sin \frac{n\pi}{l} at + D_n \cos \frac{n\pi}{l} at \right) \sin \frac{n\pi}{l} x$ satisfies the PDE – i.e. whether we can do differentiation term by term.
- We also need to verify if the summation u(x,t) satisfies the B.C. – i.e. whether u(x,t) is continuous.
- We further need to verify if it is valid to exchange the order of summation and integration (page 10).

To answer the above questions, we need to examine the convergence of the summation (the series).

Some remarks

To successfully do variable separation, the followings conditions are critical:

- The eigenvalue problem has nontrivial solution (i.e. solution is not always equal to zero). $X''(x) + \lambda X(x) = 0$
- The solution can be expanded by using the superposition of eigenfunctions – the set of eigenfunctions is complete.
- Different eigenfunctions are orthogonal to each other in the defined space.

$$\int_0^l X_n(x) X_m(x) dx = 0, \ n \neq m$$

Some remarks

Let's check the total energy of the string. $u(x,t) = \sum_{n=1}^{\infty} \left(C_n \sin \frac{n\pi}{l} at + D_n \cos \frac{n\pi}{l} at \right) \sin \frac{n\pi}{l} x$

$$E(t) = \frac{1}{2} \int_0^l \rho \left(\frac{\partial u}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^l T \left(\frac{\partial u}{\partial x} \right)^2 dx$$

Kinetic energy Strain energy

$$E(t) = \frac{m\pi^2 a^2}{4l^2} \sum_{n=1}^{\infty} n^2 \left[|C_n|^2 + |D_n|^2 \right] \qquad m = \rho \cdot l$$

The total energy does not depend on time (conservation)!

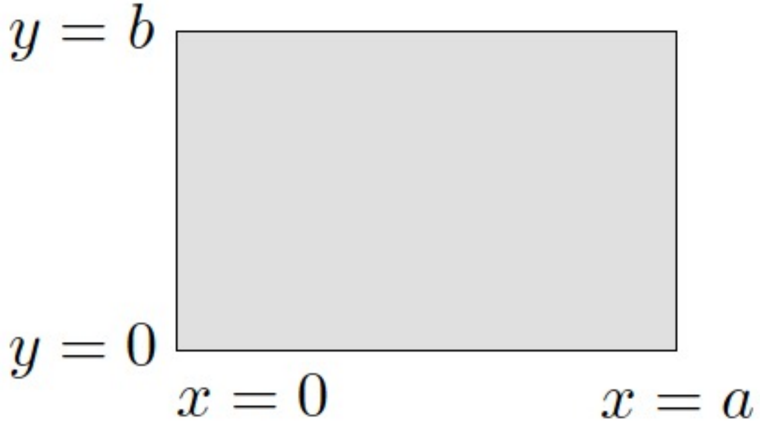
The Laplace equation – rectangular region

e.g. the heat equation in steady state

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a,\ 0 < y < b$$

$$\boxed{u|_{x=0} = 0, \quad \frac{\partial u}{\partial x}|_{x=a} = 0, \quad 0 \le y \le b}$$

$$u|_{y=0} = f(x), \quad \frac{\partial u}{\partial y}|_{y=b} = 0, \quad 0 \le x \le a$$



We are looking for the solution of the form $u(x, y) = X(x)Y(y)$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \Rightarrow \boxed{X''(x) + \lambda X(x) = 0, \qquad X(0) = 0,\ X'(a) = 0}$$

$$Y''(y) - \lambda Y(y) = 0$$

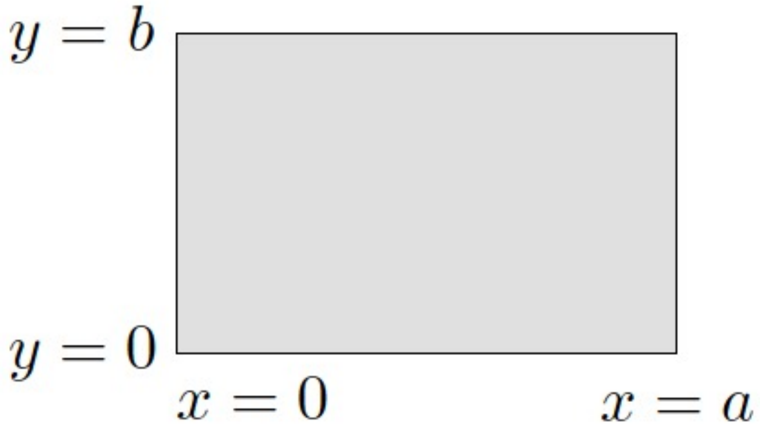
The eigenvalue problem

The Laplace equation – rectangular region

e.g. the heat equation in steady state

$$\boxed{X''(x) + \lambda X(x) = 0, \qquad X(0) = 0,\ X'(a) = 0}$$

$$Y''(y) - \lambda Y(y) = 0$$



$$X(x) = A_0 x + B_0,\ \lambda = 0 \qquad\qquad A_0 = B_0 = 0$$

$$X(x) = A \sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x,\ \lambda \neq 0$$

$$B = 0,\ \cos\sqrt{\lambda}a = 0 \quad \Rightarrow \quad \begin{cases} \lambda_n = \left(\dfrac{2n+1}{2a}\pi\right)^2, \ n = 0, 1, 2, \dots \\ X_n(x) = \sin\dfrac{2n+1}{2a}\pi x \end{cases}$$

Insert the value of λ_n into the ODE for *Y*(*y*), we have:

$$Y_n(y) = C_n \sinh\frac{2n+1}{2a}\pi y + D_n \cosh\frac{2n+1}{2a}\pi y$$

<= Note: the choice of independent solutions is not unique

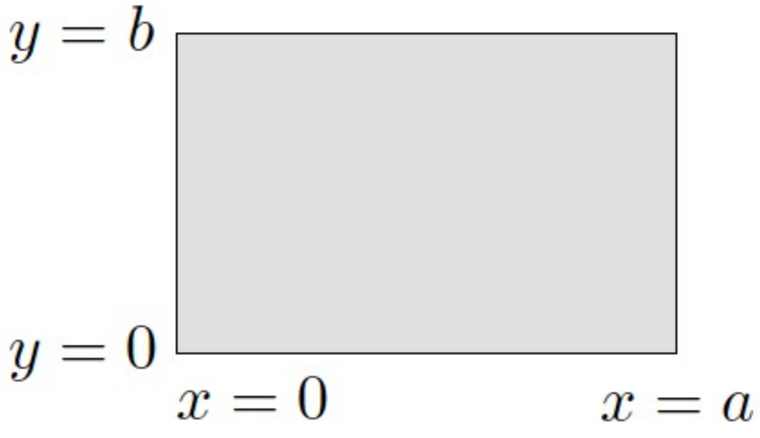
The Laplace equation – rectangular region

e.g. the heat equation in steady state

Then we obtain the “eigen” solution that satisfies the PDE

and B.C. at x=0 and x=a.

$$u_n(x,y) = \left(C_n \sinh \frac{2n+1}{2a}\pi y + D_n \cosh \frac{2n+1}{2a}\pi y \right) \sin \frac{2n+1}{2a}\pi x$$



By superposition, we have:

$$u(x,y) = \sum_{n=0}^{\infty} \left(C_n \sinh \frac{2n+1}{2a}\pi y + D_n \cosh \frac{2n+1}{2a}\pi y \right) \sin \frac{2n+1}{2a}\pi x$$

Mind the first term

By inserting the superposed form into B.C. at y=0 and y=b:

$$u|_{y=0} = \sum_{n=0}^{\infty} D_n \sin \frac{2n+1}{2a}\pi x = f(x)$$

$$\frac{\partial u}{\partial y}|_{y=b} = \sum_{n=0}^{\infty} \frac{2n+1}{2a}\pi \left(C_n \cosh \frac{2n+1}{2a}\pi b + D_n \sinh \frac{2n+1}{2a}\pi b \right) \sin \frac{2n+1}{2a}\pi x = 0$$

The Laplace equation – rectangular region

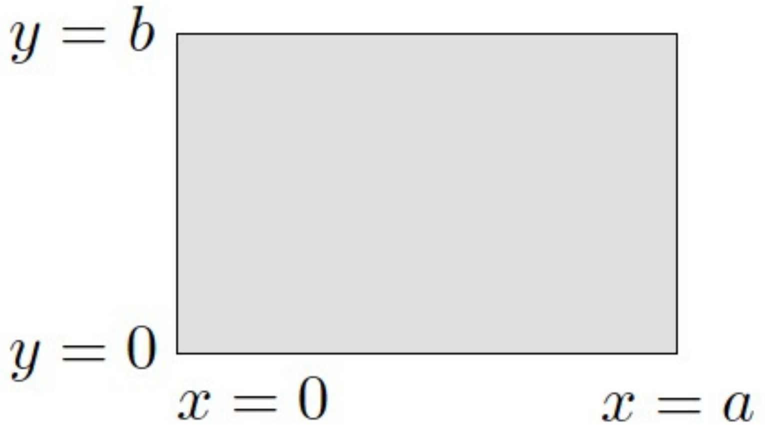
e.g. the heat equation in steady state

Again we can use the orthogonality of eigenfunctions.

$$\int_0^a \sin\frac{2n+1}{2a}\pi x \sin\frac{2m+1}{2a}\pi x dx = \frac{a}{2}\delta_{nm}$$

a/2: Normalization factor

Kronecker Delta $\begin{matrix} 0, & m \neq n \\ 1, & m = n \end{matrix}$



$$u|_{y=0} = \sum_{n=0}^{\infty} D_n \sin\frac{2n+1}{2a}\pi x = f(x)$$

$$\frac{\partial u}{\partial y}|_{y=b} = \sum_{n=0}^{\infty} \frac{2n+1}{2a}\pi\left(C_n \cosh\frac{2n+1}{2a}\pi b + D_n \sinh\frac{2n+1}{2a}\pi b\right)\sin\frac{2n+1}{2a}\pi x = 0$$

Find the coefficients

$$D_n = \frac{2}{a}\int_0^a f(x)\sin\frac{2n+1}{2a}\pi x dx$$

$$C_n = -D_n \tanh\frac{2n+1}{2a}\pi b$$

$$\Rightarrow u(x,y) = \sum_{n=0}^{\infty}\left(C_n \sinh\frac{2n+1}{2a}\pi y + D_n \cosh\frac{2n+1}{2a}\pi y\right)\sin\frac{2n+1}{2a}\pi x$$

With more than two variables

So far we have only considered the cases with two variables (pages 6 and 14). But sometimes the PDE problem could involve more variables.

$$\frac{\partial u}{\partial t} - \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \quad 0 < x < a,\ 0 < y < b,\ t > 0$$ PDE

$$\frac{\partial u}{\partial x}|_{x=0} = 0,\ \frac{\partial u}{\partial x}|_{x=a} = 0, \quad 0 \le y \le b,\ t \ge 0$$

B.C.

$$\frac{\partial u}{\partial y}|_{y=0} = 0,\ \frac{\partial u}{\partial y}|_{y=b} = 0, \quad 0 \le x \le a,\ t \ge 0$$

$$u|_{t=0} = \phi(x, y), \quad 0 \le x \le a,\ 0 \le y \le b$$ I.C.

With more than two variables

$$u(x,y,t) = X(x)Y(y)T(t) \xrightarrow{\text{Into PDE}} \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} - \frac{1}{\kappa}\frac{T'(t)}{T(t)} = 0$$

$$X''(x) + \mu X(x) = 0,$$
$$Y''(y) + \nu Y(y) = 0, \qquad \underline{\mu + \nu = \lambda}$$
$$T'(t) + \lambda\kappa T(t) = 0$$

Use the homogeneous B.C. for X and Y: $\begin{array}{l} X'(0) = 0,\ X'(a) = 0, \\ Y'(0) = 0,\ Y'(b) = 0. \end{array}$

Then we can solve the eigenvalue problems. We use X for illustration.

(1) $\mu = 0$ => $X_0 = 1\ (\neq 0)$, is an eigenfunction.

(2) $\mu \neq 0$ => $\mu_n = \left(\frac{n\pi}{a}\right)^2$, $n = 1, 2, 3$, $X_n(x) = \cos\frac{n\pi}{a}x$

Taken together, we have $\mu_n = \left(\frac{n\pi}{a}\right)^2$, $n = 0, 1, 2, 3...$ $X_n(x) = \cos\frac{n\pi}{a}x$

With more than two variables

Similarly, we can work out the eigenvalue problem for Y:

$$\nu_m = \left(\frac{m\pi}{b}\right)^2, \ m = 0, 1, 2, 3...$$

$$Y_m(y) = \cos\frac{m\pi}{b}y$$

$$X''(x) + \mu X(x) = 0,$$
$$Y''(y) + \nu Y(y) = 0,$$
$$T'(t) + \lambda\kappa T(t) = 0$$

$$\underline{\mu + \nu = \lambda}$$

Use the expressions of X and Y, we can solve for T:

$$T_{nm}(t) = A_{nm}e^{-\lambda_{nm}\kappa t}, \ n = 0, 1, 2, 3..., \ m = 0, 1, 2, 3...$$

$$\lambda_{nm} = \mu_n + \nu_m = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

Then we can obtain the superposed solution:

$$u(x, y, t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}\cos\frac{n\pi}{a}x\cos\frac{m\pi}{b}y \cdot \exp\left\{-\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]\kappa t\right\}$$

With more than two variables

Then we need to figure out the coefficients by using the I.C.

$$u(x,y,t)|_{t=0} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}\cos\frac{n\pi}{a}x\cos\frac{m\pi}{b}y = \phi(x,y)$$

As before, we can use the orthogonality of eigenfunctions

$$\int_0^a X_n(x)X_{n'}(x)dx = \frac{a}{2}(1+\delta_{n0})\delta_{nn'}$$

$$\int_0^b Y_m(y)Y_{m'}(y)dy = \frac{b}{2}(1+\delta_{m0})\delta_{mm'}$$

$$\left(\cos\frac{n\pi x}{a}\right)^2 = \frac{1+\cos\frac{2n\pi x}{a}}{2}$$

Mind the special case when n = 0 and m=0

$$A_{nm} = \frac{4}{ab}\frac{1}{(1+\delta_{n0})(1+\delta_{m0})}\int_0^a\int_0^b \phi(x,y)\cos\frac{n\pi}{a}x\cos\frac{m\pi}{b}ydxdy$$

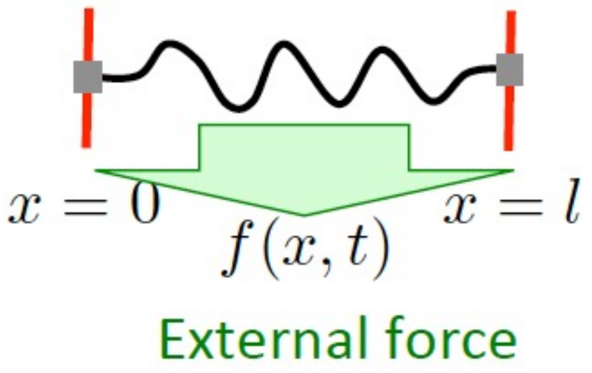
Inhomogeneous cases: PDE

- So far we have mainly focused on problems with homogeneous conditions (for PDE and for B.C.). Such conditions ensure that we can do variable separation, and can set up the corresponding eigenvalue problem.
- How about problems with inhomogeneous conditions?

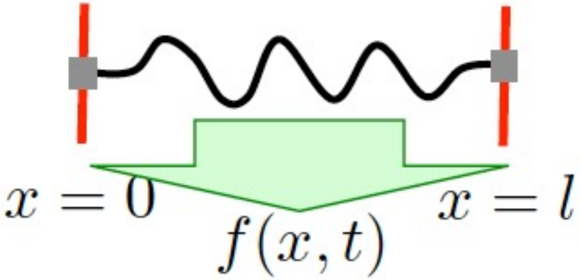
$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad 0 < x < l,\ t > 0$$

$$u|_{x=0} = 0, \quad u|_{x=l} = 0, \quad t \geq 0$$

$$u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0, \quad 0 \leq x \leq l$$



Inhomogeneous cases: PDE



We can try to homogenize the PDE.

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad 0 < x < l,\ t > 0$$

$$u|_{x=0} = 0, \quad u|_{x=l} = 0, \quad t \geq 0$$

$$u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0, \quad 0 \leq x \leq l$$

$$u(x,t) = v(x,t) + w(x,t)$$

$$\frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2} = f(x,t), \quad 0 < x < l,\ t > 0$$

$$v(x,t)|_{x=0} = 0, \quad v(x,t)|_{x=l} = 0$$

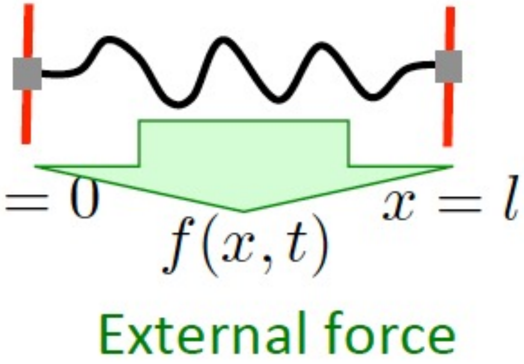
$$\frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^2 w}{\partial x^2} = 0, \quad 0 < x < l,\ t > 0$$

$$w|_{x=0} = 0, \quad w|_{x=l} = 0, \quad t \geq 0$$

$$w|_{t=0} = -v|_{t=0}, \quad \frac{\partial w}{\partial t}|_{t=0} = -\frac{\partial v}{\partial t}|_{t=0}, \quad 0 \leq x \leq l$$

Inhomogeneous cases: PDE

We already know how to solve for w.



$$u(x,t)=v(x,t)+w(x,t) \quad \frac{\partial^2 w}{\partial t^2}-a^2\frac{\partial^2 w}{\partial x^2}=0, \quad 0<x<l,\ t>0$$

$$w|_{x=0}=0, \quad w|_{x=l}=0, \quad t\geq 0$$

$$w|_{t=0}=-v|_{t=0}, \quad \frac{\partial w}{\partial t}|_{t=0}=-\frac{\partial v}{\partial t}|_{t=0}, \quad 0\leq x\leq l$$

$$u(x,t)=v(x,t)+\sum_{n=1}^{\infty}\left(C_n\sin\frac{n\pi}{l}at+D_n\cos\frac{n\pi}{l}at\right)\sin\frac{n\pi}{l}x$$

Use the I.C. and the orthogonality of eigenfunctions to figure out coefficients

$$\sum_{n=1}^{\infty}D_n\sin\frac{n\pi}{l}x=-v(x,t)|_{t=0} \qquad D_n=-\frac{2}{l}\int_0^l v(x,0)\sin\frac{n\pi}{l}xdx$$

$$\sum_{n=1}^{\infty}C_n\frac{n\pi a}{l}\sin\frac{n\pi}{l}x=-\frac{\partial v(x,t)}{\partial t}|_{t=0} \qquad C_n=-\frac{2}{n\pi a}\int_0^l\frac{\partial v(x,t)}{\partial t}|_{t=0}\sin\frac{n\pi}{l}xdx$$

Some remarks

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad 0 < x < l,\ t > 0$$

$$u|_{x=0} = 0, \quad u|_{x=l} = 0, \quad t \geq 0$$

$$u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0, \quad 0 \leq x \leq l$$

$$\frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^2 w}{\partial x^2} = 0, \quad 0 < x < l,\ t > 0$$

$$\boxed{w|_{x=0} = 0, \quad w|_{x=l} = 0, \quad t \geq 0}$$

$$w|_{t=0} = -v|_{t=0}, \quad \frac{\partial w}{\partial t}|_{t=0} = -\frac{\partial v}{\partial t}|_{t=0}, \quad 0 \leq x \leq l$$

$$u(x,t) = v(x,t) + w(x,t)$$

- Find ***v*** and ***w***, such that ***w*** satisfies the homogeneous PDE and the homogeneous B.C.
- I.C. does not matter much. It is mainly used to figure out the values of coefficients.

Example: forced vibration of a string

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = \underline{A_0 \sin \omega t}, \quad 0 < x < l,\ t > 0$$

$$u|_{x=0} = 0, \quad u|_{x=l} = 0, \quad t \geq 0$$

$$u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0, \quad 0 \leq x \leq l$$

We can simply follow the steps mentioned on pages 23 and 24.

$$u(x,t) = v(x,t) + w(x,t)$$

Let's choose ***v*** of the form $v(x,t) = f(x) \sin \omega t$

To keep the B.C. homogeneous, f(x) has to satisfy the following conditions:

$$-\omega^2 f(x) - a^2 f''(x) = A_0$$

$$f(0) = 0, \quad f(l) = 0$$

$$\Rightarrow$$

$$f(x) = -\frac{A_0}{\omega^2} + A \sin \frac{\omega}{a} x + B \cos \frac{\omega}{a} x$$

$$f(x) = -\frac{A_0}{\omega^2} \left[\left(1 - \cos \frac{\omega}{a} x\right) - \tan \frac{\omega l}{2a} \sin \frac{\omega}{a} x \right] = -\frac{A_0}{\omega^2} \left[1 - \frac{\cos(\omega(x - l/2)/a)}{\cos(\omega l/2a)} \right]$$

Example: forced vibration of a string

Refer to page 24, we have:

$$D_n = 0$$

$$C_n = -\frac{2\omega}{n\pi a}\int_0^l f(x)\sin\frac{n\pi}{l}xdx = -\frac{2A_0\omega l^3}{\pi^2 a}\frac{1-(-1)^n}{n^2}\frac{1}{(n\pi a)^2-(\omega l)^2}$$

When n = an even number, $C_n = 0$.

Some tricks related to trigonometric function
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$$w(x,t) = -\frac{4A_0\omega l^3}{\pi^2 a}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{1}{[(2n+1)\pi a]^2-(\omega l)^2}\sin\frac{2n+1}{l}\pi x\sin\frac{2n+1}{l}\pi at$$

$$u(x,t) = -\frac{A_0}{\omega^2}\left[1-\frac{\cos(\omega(x-l/2)/a)}{\cos(\omega l/2a)}\right]\sin\omega t$$

$$-\frac{4A_0\omega l^3}{\pi^2 a}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{1}{[(2n+1)\pi a]^2-(\omega l)^2}\sin\frac{2n+1}{l}\pi x\sin\frac{2n+1}{l}\pi at$$

Example: forced vibration of a string

$$u(x,t) = -\frac{A_0}{\omega^2}\left[1 - \frac{\cos(\omega(x-l/2)/a)}{\underline{\cos(\omega l/2a)}}\right]\sin\omega t$$

$$-\frac{4A_0\omega l^3}{\pi^2 a}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{1}{\underline{[(2n+1)\pi a]^2-(\omega l)^2}}\sin\frac{2n+1}{l}\pi x\sin\frac{2n+1}{l}\pi at$$

What if $\omega = (2k+1)\pi a/l,\ k = \text{integer}$?

Resonance: denominator = 0

$$f(x) = -\frac{A_0}{\omega^2}\left[\left(1-\cos\frac{\omega}{a}x\right) - \tan\frac{\omega l}{2a}\sin\frac{\omega}{a}x\right] = -\frac{A_0}{\omega^2}\left[1 - \frac{\cos(\omega(x-l/2)/a)}{\underline{\cos(\omega l/2a)}}\right]$$

Let's expand f(x) as a series of eigenfunctions:

$$f(x) = -\sum_{n=1}^{\infty}\frac{n\pi a}{\omega l}C_n\sin\frac{n\pi}{l}x$$

Example: forced vibration of a string

$$u(x,t) = \frac{4A_0 l^2}{\pi^2 a} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \frac{1}{[(2n+1)\pi a]^2 - \underline{(\omega l)^2}} \sin\frac{2n+1}{l}\pi x \cdot \left[(2n+1)\pi a \underline{\sin\omega t} - \underline{(\omega l)} \sin\frac{2n+1}{l}\pi a t\right]$$

$\omega = (2k+1)\pi a/l,\ \ k = \text{integer}$ L'Hopital’s rule

Summation without $n = k$

$$u(x,t) = \left.\begin{aligned}&\frac{4A_0 l^2}{\pi^2 a} {\sum_{n=0}^{\infty}}' \frac{1}{(2n+1)^2} \frac{1}{[(2n+1)\pi a]^2 - (\omega l)^2} \sin\frac{2n+1}{l}\pi x \\ &\cdot \left[(2n+1)\pi a \sin\omega t - (\omega l) \sin\frac{2n+1}{l}\pi a t\right]\end{aligned}\right\} n \neq k$$

$$-\frac{2A_0 l}{\pi^2 a} \frac{1}{(2k+1)^2} \sin\frac{2k+1}{l}\pi x \left[t\cos\frac{2k+1}{l}\pi a t - \frac{l}{(2k+1)\pi a} \sin\frac{2k+1}{l}\pi a t\right] \qquad n = k$$

What if $f(x,t)$ is very complex?

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad 0 < x < l,\ t > 0$$

$$u|_{x=0} = 0, \quad u|_{x=l} = 0, \quad t \geq 0$$

$$u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0, \quad 0 \leq x \leq l$$

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

$$X_n''(x) + \lambda_n X_n(x) = 0$$

$$X_n(0) = 0, \quad X_n(l) = 0$$

$$f(x,t) = \sum_{n=1}^{\infty} g_n(t) X_n(x)$$

As long as the set $\{X_n(x)\}$ is complete

$$\sum_{n=1}^{\infty} T''(t) X_n(x) - a^2 \sum_{n=1}^{\infty} T_n(t) X_n''(x) = \sum_{n=1}^{\infty} g_n(t) X_n(x)$$

Use the orthogonality of eigenfunctions

$$T_n''(t) + \lambda_n a^2 T_n(t) = g_n(t)$$

What if $f(x,t)$ is very complex?

$$T_n''(t) + \lambda_n a^2 T_n(t) = g_n(t)$$

According to the I.C.

$$u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0, \quad 0 \le x \le l$$

$$\left.\begin{aligned} \sum_{n=1}^{\infty} T_n(0) X_n(x) = 0, \\ \sum_{n=1}^{\infty} T_n'(0) X_n(x) = 0. \end{aligned}\right\} T_n(0) = 0,\ T'(0) = 0$$

ODE with I.C.

$$T_n(t) = \frac{l}{n\pi a} \int_0^t g_n(\tau) \sin \frac{n\pi}{l} a(t-\tau) d\tau$$

You can verify the above solution by inserting it into the ODE and the I.C.

Be careful when performing the differentiation with respect to *t*

Apply this method to the problem (page 26)

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = A_0 \sin \omega t, \quad 0 < x < l,\ t > 0$$

$$u|_{x=0} = 0, \quad u|_{x=l} = 0, \quad t \geq 0$$

$$u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0, \quad 0 \leq x \leq l$$

$$T_n''(t) + \lambda_n a^2 T_n(t) = g_n(t)$$

$$T_n(t) = \frac{l}{n\pi a} \int_0^t g_n(\tau) \sin \frac{n\pi}{l} a(t-\tau) d\tau$$

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi}{l} x$$

$$A_0 \sin \omega t = \frac{2A_0}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin \frac{n\pi}{l} x \sin \omega t$$

$$T_n''(t) + \left(\frac{n\pi}{l} a\right)^2 T_n(t) = \frac{2A_0}{\pi} \frac{1-(-1)^n}{n} \sin \omega t$$

$$T_n(0) = 0,\ T'(0) = 0$$

$$T_n(t) = \frac{2A_0 l^2}{n\pi} \frac{1-(-1)^n}{(n\pi a)^2 - (\omega l)^2} \sin \omega t - \frac{2A_0 \omega l^3}{n^2 \pi^2 a} \frac{1-(-1)^n}{(n\pi a)^2 - (\omega l)^2} \sin \frac{n\pi}{l} a t$$

Inhomogeneous cases: B.C.

Sometimes, the B.C. could be inhomogeneous. If we still want to apply the tricks learned before, we can homogenize the B.C.

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, t > 0$$

$$u|_{x=0} = \mu(t), \quad u|_{x=l} = \nu(t), \; t \geq 0$$

$$u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0, \; 0 \leq x \leq l$$

$$u(x,t) = v(x,t) + w(x,t)$$

$$v(x,t)|_{x=0} = \mu(t), \quad v(x,t)|_{x=l} = \nu(t)$$

$$\frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^2 w}{\partial x^2} = -\left(\frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2}\right)$$

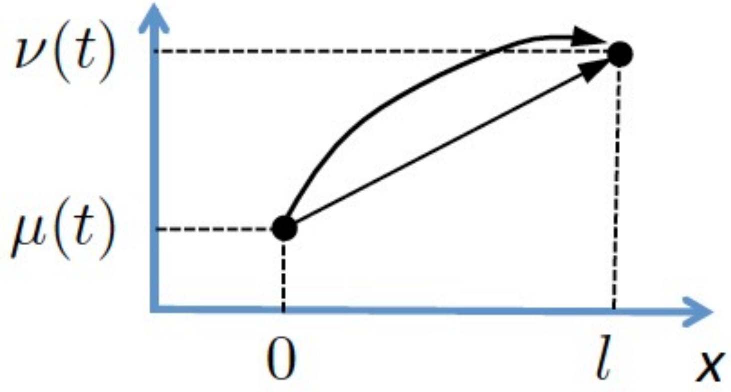
$$\underline{w|_{x=0} = 0, \; w|_{x=l} = 0}$$

$$w|_{t=0} = -v|_{t=0}, \; \frac{\partial w}{\partial t}|_{t=0} = -\frac{\partial v}{\partial t}|_{t=0}$$

How to choose *v*(x,t)

$$u(x,t) = v(x,t) + w(x,t)$$

$$v(x,t)|_{x=0} = \mu(t), \quad v(x,t)|_{x=l} = \nu(t)$$



- Line: $v(x,t) = A(t)x + B(t)$
  $$B(t) = \mu(t),\ A(t) = \frac{1}{l}\left[\nu(t) - \mu(t)\right]$$
- Parabola: $v(x,t) = A(t)x^2 + B(t)$
  $$A(t) = \frac{1}{l^2}\left[\nu(t) - \mu(t)\right],\ B(t) = \mu(t)$$

$$v(x,t) = A(t)(l-x)^2 + B(t)x^2$$
$$A(t) = \frac{1}{l^2}\mu(t),\ B(t) = \frac{1}{l^2}\nu(t)$$

Example

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, t > 0$$

$$u|_{x=0} = A \sin \omega t, \quad u|_{x=l} = 0, \quad t \geq 0$$

$$u|_{t=0} = 0, \quad 0 \leq u \leq l$$

$$u(x,t) = v(x,t) + w(x,t)$$

$$v(x,t) = A\left(1 - \frac{x}{l}\right) \sin \omega t$$

$$\frac{\partial w}{\partial t} - \kappa \frac{\partial^2 w}{\partial x^2} = -A\omega\left(1 - \frac{x}{l}\right) \cos \omega t, \quad 0 < x < l, t > 0$$

$$w|_{x=0} = 0, \quad w|_{x=l} = 0, \quad t \geq 0$$

$$w|_{t=0} = 0, \quad 0 \leq x \leq l$$

Example

We can expand all unknown functions as a series of eigenfunctions

$$w(x,t) = \sum_{n=1}^{\infty} T(t) \sin \frac{n\pi}{l} x, \quad 1 - \frac{x}{l} = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{l} x$$

Do the integration by parts

$$T_n'(t) + \kappa \left(\frac{n\pi}{l}\right)^2 T_n(t) = -\frac{2A\omega}{n\pi} \cos \omega t$$

See page 401 of the Textbook

$$T_n(0) = 0$$

$$T_n(t) = \frac{2A\omega l^2}{\kappa^2 (n\pi)^4 + \omega^2 l^4} \frac{1}{n\pi} \left[\kappa (n\pi)^2 e^{-(n\pi/l)^2 \kappa t} - \kappa (n\pi)^2 \cos \omega t - \omega l^2 \sin \omega t \right]$$

How to solve 1st order ODE?

3. LINEAR FIRST-ORDER EQUATIONS

A first-order equation contains y' but no higher derivatives. A *linear* first-order equation means one which can be written in the form

$$y' + Py = Q, \tag{3.1}$$

where P and Q are functions of x. To see how to solve (3.1), let us first consider the simpler equation when $Q = 0$. The equation

$$\left.\begin{aligned} ye^I &= \int Qe^I\,dx + c, \qquad \text{or} \\ y &= e^{-I}\int Qe^I\,dx + ce^{-I}, \end{aligned}\right\} \quad \text{where} \quad I = \int P\,dx. \tag{3.9}$$

See page 401 of the Textbook

Another example

For the example on page 35, the PDE for *w* is not homogeneous, although the B.C. for *w* is homogeneous. Sometimes we can set both PDE and B.C. as homogeneous for *w*.

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, \ t > 0$$

$$u|_{x=0} = 0, \quad \frac{\partial u}{\partial x}|_{x=l} = A \sin \omega t, \quad t \geq 0$$

$$u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0, \quad 0 \leq x \leq l$$

Again we consider:

$$u(x,t) = v(x,t) + w(x,t)$$

$$v(x,t) = f(x) \sin \omega t$$

$$f''(x) + \left(\frac{\omega}{a}\right)^2 f(x) = 0$$

$$f(0) = 0, \quad f'(l) = A$$

It is easy to find the solution for f(x): $f(x) = \frac{Aa}{\omega} \frac{1}{\cos(\omega l/a)} \sin \frac{\omega}{a} x$

Another example

Then w satisfies the following conditions: $u(x,t) = v(x,t) + w(x,t)$

$$\frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^2 w}{\partial x^2} = 0, \quad 0 < x < l, \ t > 0$$

$$w|_{x=0} = 0, \ \frac{\partial w}{\partial x}|_{x=l} = 0, \quad t \geq 0$$

$$w|_{t=0} = 0, \quad \frac{\partial w}{\partial t}|_{t=0} = -\frac{Aa}{\cos(\omega l/a)} \sin \frac{\omega}{a} x, \quad 0 \leq x \leq l$$

Following the procedure around page 6, we can find the solution for w:

$$w(x,t) = \sum_{n=1}^{\infty} \left(C_n \sin \frac{2n+1}{2l} \pi a t + D_n \cos \frac{2n+1}{2l} \pi a t \right) \sin \frac{2n+1}{2l} \pi x$$

$$D_n = 0$$

$$C_n = -\frac{4A}{\pi \cos(\omega l/a)} \frac{1}{2n+1} \int_0^l \sin \frac{\omega}{a} x \sin \frac{2n+1}{2l} \pi x dx$$

Some remarks

- For the inhomogeneous cases, how to choose the decomposition ***u* = *v* + *w*** is not unique. We can either choose ***v*** such that ***w*** satisfies homogeneous B.C. (page 35), or choose ***v*** such that ***w*** satisfies homogeneous PDE and homogeneous B.C. (pages 38-39). The detailed choice could depend on whether ***v*** is easy to solve.
- So far the eigenvalue problem has been made for the homogeneous B.C.. What about the I.C.?

$$T''(t) + \lambda a^2 T(t) = 0$$
$$T(0) = 0,\ T'(0) = 0$$

- Cannot find nontrivial solutions
- The I.C. only gives the information at one point in time (t=0), while B.C. can give the information at different points in space (x=0 and x=l).