Linear Algebra

Math 314 Fall 2025

Lecture 16

Today's poll code:

P8LAVE

To do list:

- Webwork 4.2 due today
- Webwork 4.3 due Friday
- Webwork 4.4 due next Tuesday
- Webwork 5.1 due next Friday

Quiz on Friday

on bases and dimension

Office hours

Mondays 5–6 pm
Wednesdays 2–3 pm
in Avery 339 (Dr. Grifo)

Tuesdays 11–noon
Thursdays 1–2 pm
in Avery 337 (Kara)

Midterm 2
On Monday November 10

Quick recap

$$W = \operatorname{span}(\{v_1, \ldots, v_n\}) \text{ subspace of } \mathbb{R}^m$$

How to find a basis for W?

Find a basis for the column space of

$$A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

To find a basis for the column space of A:

Step 1: Find the RREF of A.

Step 2: Collect the pivot columns of A.

Warning: Make sure to use the pivot columns of A, not of its reduced echelon form!

To find a basis for the null space of A:

<u>Step 1</u>: Find the general solution for $Ax = 0$.

<u>Step 2</u>: Write the solution in parametric vector form.

Use one vector for each free variable.

<u>Step 2</u>: The vectors we used form a basis for $\mathrm{Nul}(A)$.

$$\dim(\mathrm{Nul}(A)) = \# \text{ free variables}$$

$$\dim(\mathrm{col}(A)) = \#\text{ of pivots in } A = \mathrm{rank}(A)$$

$$\dim(\mathrm{Nul}(A)) = \#\text{ free variables } = \text{nullity of } A$$

Theorem (Rank–Nullity Theorem)**.**

For any $m \times n$ matrix A,

$$\mathrm{rank}(A) + \dim(\mathrm{Nul}(A)) = n.$$

Trace

A $n \times n$ square matrix

The **trace** of A is

$$\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$$

the sum of the entries in the main diagonal

Example: $\quad \mathrm{tr}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 1 + 4 = 5$

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The set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$$

A. is a basis for $\mathbb{R}^3$

B. is linearly independent, but does not span $\mathbb{R}^3$

C. spans $\mathbb{R}^3$, but is not linearly independent

D. None of the above

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$$W = \operatorname{span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7 \end{bmatrix} \right\} \right)$$

What is the dimension of W?

Bases and coordinates

A **basis*** for a vector space V is

a spanning set of linearly independent vectors.

The **dimension** of a vector space is

the number of vectors in a basis.

Notation: $\dim(V)$

* The plural of basis is bases.

V vector space

$\mathcal{B} = \{b_1, \ldots, b_n\}$ basis for V

Any vector $v \in V$ can be written uniquely as

$$v = v_1 b_1 + \cdots + v_n b_n$$

$\{b_1, \ldots, b_n\}$ basis for V $\implies$ The system $v_1 b_1 + \cdots + v_n b_n = v$ has exactly one solution

V vector space

$\mathcal{B} = \{b_1, \ldots, b_n\}$ basis for V

Any vector $v \in V$ can be written uniquely as

$$v = v_1 b_1 + \cdots + v_n b_n$$

The **coordinates** of v relative to $\mathcal{B}$ are the unique scalars $v_1, \ldots, v_n$

$$[v]_{\mathcal{B}} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Example: $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^2$

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } \mathbb{R}^2$$

$$v = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow [v]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$C = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } \mathbb{R}^2$$

$$v = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow [v]_C = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Example: $v = 3 + t^2$ in $\mathbb{P}_2$

$\mathcal{B} = \{1, t, t^2\}$ is a basis for $\mathbb{P}_2$

$$[v]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$\mathcal{C} = \{1 + t, t, t^2\}$ is a basis for $\mathbb{P}_2$

$$[v]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$$

Linear transformations

V vector space $\qquad$ W vector space

A **linear transformation** from V to W is a function

$$T\colon V \to W$$

such that

$$T(u+v) = T(u) + T(v) \text{ for all } u \text{ and } v \text{ in } V$$

and

$$T(cv) = cT(v) \text{ for all } v \text{ in } V \text{ and all scalars } c$$

V vector space $\qquad\qquad$ W vector space

Example: The **identity function** $\mathrm{id}_V : V \to V$

$$\mathrm{id}_V(v) = v$$

is a linear transformation

Example: The **zero function** $Z : V \to W$

$$Z(v) = \underset{\substack{\text{zero vector} \\ \text{in } W}}{0} \quad \text{for all } v \in V$$

is a linear transformation

Example:

$$T : \mathbb{P}_2 \to \mathbb{R}^3 \quad \text{given by} \quad T(a_0 + a_1 t + a_2 t^2) = \begin{bmatrix} a_1 \\ a_2 \\ a_1 + a_2 \end{bmatrix}$$

is a linear transformation:

$$\begin{aligned} T(p+q) &= T((a_0 + a_1 t + a_2 t^2) + (b_0 + b_1 t + b_2 t^2)) \\ &= T((a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2) \\ &= \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ (a_1 + b_1) + (a_2 + b_2) \end{bmatrix} \\ &= \begin{bmatrix} a_1 \\ a_2 \\ a_1 + a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ a_2 + b_2 \end{bmatrix} \\ &= T(a_0 + a_1 t + a_2 t^2) + T(b_0 + b_1 t + b_2 t^2) \\ &= T(p) + T(q) \end{aligned}$$

$$\begin{aligned} T(cp) &= T((ca_0) + (ca_1)t + (ca_2)t^2) \\ &= \begin{bmatrix} ca_1 \\ ca_2 \\ ca_1 + ca_2 \end{bmatrix} \\ &= c \begin{bmatrix} a_1 \\ a_2 \\ a_1 + a_2 \end{bmatrix} \\ &= cT(a_0 + a_1 t + a_2 t^2) \\ &= cT(p(t)) \end{aligned}$$

$$D\colon \mathbb{P}_2 \to \mathbb{P}_2$$

is a linear transformation:

$$D(p(t)) = \tfrac{d}{dt}\,(p(t))$$

$$D(p(t) + q(t)) = \tfrac{d}{dt}\,(p(t) + q(t)) = \tfrac{d}{dt}p(t) + \tfrac{d}{dt}q(t) = D(p(t)) + D(q(t))$$

and

$$D(c\,p(t)) = \tfrac{d}{dt}(c\,p(t)) = c\tfrac{d}{dt}p(t) = cD(p(t))$$

V vector space $\qquad \mathcal{B} = \{b_1, \ldots, b_n\}$ basis for V

W vector space $\qquad \mathcal{C} = \{c_1, \ldots, c_m\}$ basis for W

$T\colon V \to W$ linear transformation

The matrix representing T in the bases $\mathcal{B}$ and $\mathcal{C}$ is

$$A_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} [T(b_1)]_{\mathcal{C}} & \cdots & [T(b_n)]_{\mathcal{C}} \end{bmatrix}$$

standard matrix for $T\colon \mathbb{R}^n \to \mathbb{R}^m$

$$\|$$

matrix representing $T\colon \mathbb{R}^n \to \mathbb{R}^m$

in the standard matrices of $\mathbb{R}^n$ and $\mathbb{R}^m$

$$D\colon \mathbb{P}_2 \to \mathbb{P}_2$$

is a linear transformation

$$D(p(t)) = \tfrac{d}{dt}\,(p(t))$$

$\mathcal{B} = \{1, t, t^2\}$ is a basis for $\mathbb{P}_2$

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$\mathcal{B} = \{1, t, t^2\}$ is a basis for $\mathbb{P}_2$

$$D(1) = 0 \implies [D(1)]_{\mathcal{B}} = [0]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D(t) = 1 \implies [D(t)]_{\mathcal{B}} = [1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$D(t^2) = 2t \implies [D(t^2)]_{\mathcal{B}} = [2t]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

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$$D\colon \mathbb{P}_2 \to \mathbb{P}_2 \qquad D(p(t)) = \tfrac{d}{dt}\,(p(t))$$

$\mathcal{B} = \{1, t, t^2\}$ is a basis for $\mathbb{P}_2$ $\qquad A_{\mathcal{B}\leftarrow\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$p = 3 + 7t - t^2$ $\qquad$ What is $D(p)$?

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$$p = 3 + 7t - t^2 \qquad \text{What is } D(p)?$$

1 Find the coordinates of p wrt $\mathcal{B}$ $\longrightarrow$ $[p]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix}$

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$$[D(p)]_{\mathcal{B}} = A_{\mathcal{B}\leftarrow\mathcal{B}}[p]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix}$$

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3 so $D(p) = 7 - 2t$

Determinants

Determinants

$$\det(A) = \pm \begin{array}{c} \text{volume of the } n\text{-dimensional solid} \\ \text{determined by the columns of } A \end{array}$$

To compute the determinant of a square matrix:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$A_{ij} = \text{delete row } i \text{ and column } j \text{ from } A$$

expand along row 1:

$$\det(A) = a_{11}\det(A_{11}) \pm a_{12}\det(A_{12}) \pm \cdots \pm a_{1n}\det(A_{1n})$$

OR

expand along row i:

$$\det(A) = \pm a_{i1}\det(A_{i1}) \pm a_{i2}\det(A_{i2}) \pm \cdots \pm a_{in}\det(A_{in})$$

To compute the determinant of a square matrix:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$A_{ij} =$ delete row i and column j from A

expand along column 1:

$$\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) \pm \cdots + (-1)^{n+1} a_{n1}\det(A_{n1})$$

OR

expand along column j:

$$\det(A) = (-1)^{1+j} a_{1j}\det(A_{1j}) + \cdots + (-1)^{n+j} a_{nj}\det(A_{nj})$$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Expand along row 1:

$$\begin{aligned}
\det A &= (+)\, 1 \cdot \det\left(\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix}\right) \; - \; 5 \cdot \det\left(\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\right) \; + \; 0 \cdot \det\left(\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}\right) \\
&= 1 \cdot \big(4 \cdot 0 - (-1)(-2)\big) \; - \; 5 \cdot \big(2 \cdot 0 - (-1) \cdot 0\big) \; + \; 0 \cdot \big(2 \cdot (-2) - 4 \cdot 0\big) \\
&= 1 \cdot (0 - 2) \; - \; 5 \cdot (0 - 0) \; + \; 0 \cdot (-4 - 0) \\
&= -2.
\end{aligned}$$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Expand along row 2:

$$\begin{aligned}
\det A &= (-)\,2 \cdot \det\left(\begin{bmatrix} 5 & 0 \\ -2 & 0 \end{bmatrix}\right) \;+\; 4 \cdot \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \;-\; (-1) \cdot \det\left(\begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix}\right) \\
&= -2 \cdot \big(5 \cdot 0 - 0 \cdot (-2)\big) \;+\; 4 \cdot \big(1 \cdot 0 - 0 \cdot 0\big) \;+\; 1 \cdot \big(1 \cdot (-2) - 5 \cdot 0\big) \\
&= -2 \cdot (0 - 0) \;+\; 4 \cdot (0 - 0) \;+\; (-2 - 0) \\
&= -2.
\end{aligned}$$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Expand along column 3:

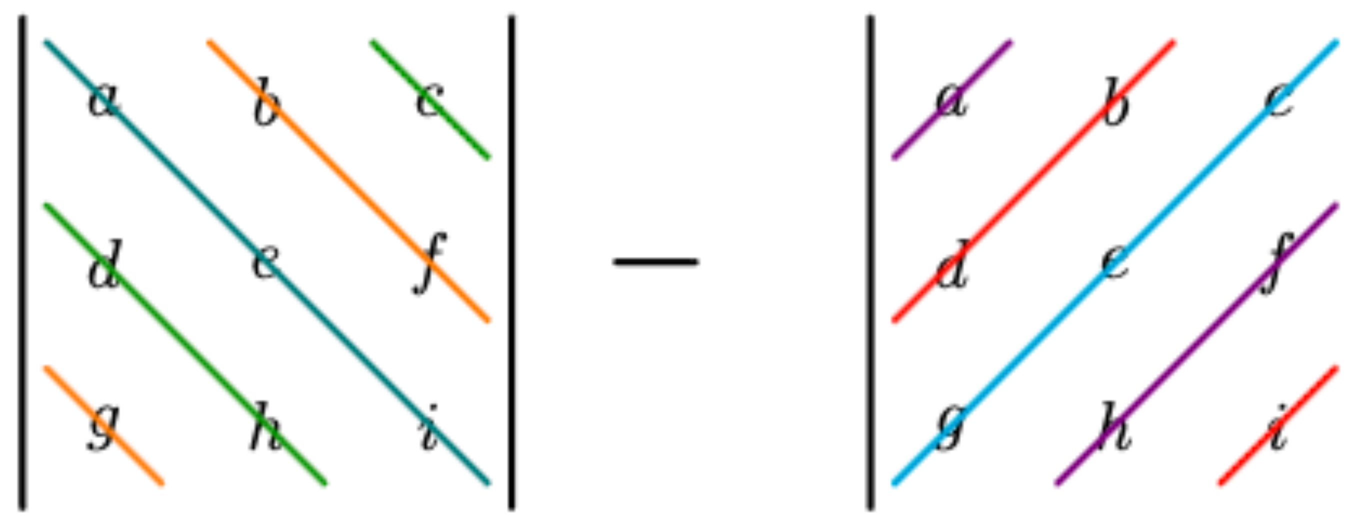
$$\begin{aligned} \det A &= (+)\,0 \cdot \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} \;-\; (-1) \cdot \det \begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix} \;+\; 0 \cdot \det \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \\ &= 0 \cdot \big(2 \cdot (-2) - 4 \cdot 0\big) \;+\; 1 \cdot \big(1 \cdot (-2) - 5 \cdot 0\big) \;+\; 0 \cdot \big(1 \cdot 4 - 5 \cdot 2\big) \\ &= 0 \cdot (-4 - 0) \;+\; (-2 - 0) \;+\; 0 \cdot (4 - 10) \\ &= -2. \end{aligned}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$(\ \mathrm{aei} + \mathrm{bfg} + \mathrm{cdh}\ ) - (\ \mathrm{ceg} + \mathrm{bdi} + \mathrm{afh}\ ).$$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$



$$\begin{aligned} \det(A) &= (1 \cdot 5 \cdot 0 + 5 \cdot (-1) \cdot 0 + 2 \cdot (-2) \cdot 0) - (0 \cdot 4 \cdot 0 + 5 \cdot 2 \cdot 0 + (-1) \cdot (-2) \cdot 1) \\ &= 2. \end{aligned}$$

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