

Symmetric powers Algebra (Day 1)

BRIDGES Day 1
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Theorem (Fundamental theorem of Arithmetic)

Every positive integer n can be written as a product

$$n = p_1^{a_1} \cdots p_k^{a_k}$$

of primes p_i (with $a_i \geq 1$) which is unique up to the order of the factors.

How about in other rings?

For us, rings are always commutative with 1.

Example $R = \mathbb{Z}[\sqrt{-5}] \cong \mathbb{Z}[x]/(x^2 + 5)$

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

These are distinct products in medecules.

What's wrong?

We are focusing on elements when we should be focusing on ideals!

Def: An ideal I is prime if $xy \in I \Rightarrow x \in I$ or $y \in I$

Def: An ideal I is primary if

$$xy \in I \Rightarrow x \in I \text{ or } y^n \in I \text{ for some } n$$

Def: the radical of I is $\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some } n\}$

Def An ideal I is primary if

$$xy \in I \Rightarrow x \in I \text{ or } y \in \sqrt{I}$$

Notes : • I prime $\Rightarrow I = \sqrt{I}$

$$\bullet I \text{ primary} \Rightarrow \sqrt{I} \text{ prime}$$

Def Q is I -primary if Q is primary and $\sqrt{Q} = I$.

Warning \sqrt{I} prime $\not\Rightarrow I$ primary

Fact: I any ideal

$$\sqrt{I} = \bigcap_{\substack{Q \supseteq I \\ Q \text{ prime}}} Q = \bigcap_{\substack{Q \supseteq I \\ \text{minimal} \\ \text{wrt } \supseteq I}} Q$$

Exercises

① Q_1, Q_2 I -primary $\Rightarrow Q_1 \cap Q_2$ I -primary

② \sqrt{I} maximal $\Rightarrow I$ primary

Def A primary decomposition of I is a collection of primary ideals Q_i

$$I = Q_1 \cap \dots \cap Q_k$$

A primary decomposition is redundant if

- no Q_i can be deleted
- $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$

$\left(\begin{array}{l} \text{each } Q_i \text{ is called} \\ \text{a (primary) component} \\ \text{of } I \end{array} \right)$

Note Any primary decomposition can be made redundant by

- deleting unnecessary components
- intersecting components with the same radical

Theorem (Starken, 1905, Noether, 1921)

Every ideal in a Noetherian ring has a primary decomposition

* Noetherian rings = Commutative algebrists favorite rings

R is Noetherian if every ideal in R is finitely generated

(big example: $R = k[x_1, \dots, x_d]/I$ k field)

Examples

a) $I = \sqrt{I} = \bigcap Q$

fact above
 $Q \supseteq I$
prime

In fact, $\text{Min}(I) := \{Q \text{ prime in } R \mid Q \supseteq I\}$

is a finite set (in any Noetherian ring).

so $I = \underbrace{P_1 \cap \dots \cap P_k}_{\text{primary components of } I} \quad P_i \text{ minimal primes}$

b) $I = (xy, xz, yz) = (x, y) \cap (y, z) \cap (x, z) \subseteq k[x, y, z]$

c) Primary decompositions are not unique \therefore

$$I = (x^2, xy) \subseteq k[x, y]$$

$$= (x) \cap (x^2, xy, y^2) = (x) \cap (x^2, xy, y^n)$$

these are different primary decompositions. But what do they have in common?

$$(x) \cap (x^2, xy, y^2) = (x) \cap (x^2, xy, y^n)$$

\downarrow radical

$$(x), (x, y)$$

$$(x), (x, y)$$

Theorem (Uniqueness of primary decompositions)

R Noetherian ring

I ideal in R

$$I = Q_1 \cap \dots \cap Q_k \text{ irredundant primary decomposition}$$

① $\{\sqrt{Q_1}, \dots, \sqrt{Q_k}\}$ is unique and does not depend on our choice of primary decomposition

In fact, $\{\sqrt{Q_1}, \dots, \sqrt{Q_k}\} = \text{Ass}(R/I) = \text{associated primes of } I$

where a prime P is associated to I if

$$P = \{f \in R \mid fa \in I\} \quad \text{for some } a \in R$$

(2) the components Q_i coming from minimal primes of I are unique and given by the following formula:

$P \in \text{Min}(I) \subseteq \text{Ass}(R/I) \Rightarrow$ the P -primary Component of I is

$$\begin{aligned} IR_P \cap R &= \left\{ f \in R \mid s f \in I, s \notin P \right\} \\ &= \left(\begin{array}{l} \left\{ f \in R \mid \frac{s}{s} \cdot \frac{f}{1} \in IR_P \right\} \\ = \left\{ f \in R \mid \frac{f}{1} \in IR_P \right\} \end{array} \right) \end{aligned}$$

Most important facts about associated primes and primary decompositions:

→ Every (proper) ideal has associated primes

→ $\text{Min}(I) \subseteq \text{Ass}(R/I)$

→ associated primes that are not minimal are called embedded

→ primary decompositions are computationally difficult to find

What does it really mean to be an associated prime?

when M is an R -module (abelian group with a scalar product by elements in R)

$\text{ann}_M(m) := \{ r \in R \mid rm = 0 \}$ annihilator of m

$\mathfrak{P} \in \text{Ass}(R/I)$ $\Leftrightarrow \mathfrak{P} = \text{ann}(a + I)$ for some $a \in R$

\Leftrightarrow there exists an inclusion $R/\mathfrak{P} \hookrightarrow R/I$

Symbolic powers

let I be an ideal in R

$I^n := (f_1 \cdots f_n \mid f_i \in I)$ n th power of I

Example $(x, y)^2 = (x^2, xy, y^2)$

\mathfrak{P} prime ideal $\rightsquigarrow \mathfrak{P}^n$ is not necessarily primary!

Example $R = \frac{k[x, y, z]}{(xy - z^2)}$ $\mathfrak{P} = (x, z) \text{ (in } R\text{)}$

\mathfrak{P}^2 is not primary! Because

$$xy = z^2 \in \mathfrak{P}^2, \text{ but } x \notin \mathfrak{P}^2, y \notin \sqrt{\mathfrak{P}^2} = \mathfrak{P}$$

but \mathfrak{P}^n has a primary decomposition

What primes are going to appear?

$$\text{Min}(\mathfrak{P}^n) = \{\mathfrak{P}\} \text{ since } \mathfrak{P}^n \subseteq Q \stackrel{Q \text{ prime}}{\Rightarrow} \mathfrak{P} \subseteq Q$$

so an redundant primary decomposition of \mathfrak{P}^n looks like

$$\mathfrak{P}^n = \underbrace{Q_1}_{\substack{\mathfrak{P}-\text{primary} \\ \text{component}}} \cap \underbrace{Q_2 \cap \dots \cap Q_k}_{\substack{\text{embedded} \\ \text{components}}}$$

and we know $Q_{\mathfrak{P}} = \{f \in R \mid sf \in \mathfrak{P}^n \text{ for some } s \notin \mathfrak{P}\}$

Definition \mathfrak{P} prime ideal

the n -th symbolic power of \mathfrak{P} is given by

$$\begin{aligned} \mathfrak{P}^{(n)} &:= \{f \in R \mid sf \in \mathfrak{P}^n \text{ for some } s \notin \mathfrak{P} \\ &= \mathfrak{P}\text{-primary Component of } \mathfrak{P}^n \end{aligned}$$

exercise \uparrow = smallest \mathfrak{P} -primary ideal containing \mathfrak{P}

Note: $\mathfrak{P}^n \subseteq \mathfrak{P}^{(n)}$ always! But in general, $\mathfrak{P}^n \subsetneq \mathfrak{P}^{(n)}$

Example $R = \frac{k[x, y, z]}{(xy - z^2)}$ $\mathfrak{P} = (x, z)$ (in R)

$$xy = z^2 \in \mathfrak{P}^2 \quad \Rightarrow \quad x \in \mathfrak{P}^{(2)} \quad \begin{array}{l} \text{but } x \notin \mathfrak{P}^2 (!) \\ \text{so } \mathfrak{P}^2 \subsetneq \mathfrak{P}^{(2)} \end{array}$$

General definition $I = \sqrt{I} = P_1 \cap \dots \cap P_k$

$$I^{(n)} = P_1^{(n)} \cap \dots \cap P_k^{(n)}$$

In fact: this is actually the same we would get if we took
the minimal components in an redundant primary decomposition of I^n