Linear Algebra

Math 314 Fall 2025

Lecture 17

Today's poll code:

PQRT8D

To do list:

- Webwork 4.4 due tomorrow
- Webwork 5.1 due Friday
- Webwork 5.2 due next Tuesday

Quiz on Friday

on determinants

Office hours

Mondays 5–6 pm
Wednesdays 2–3 pm
in Avery 339 (Dr. Grifo)

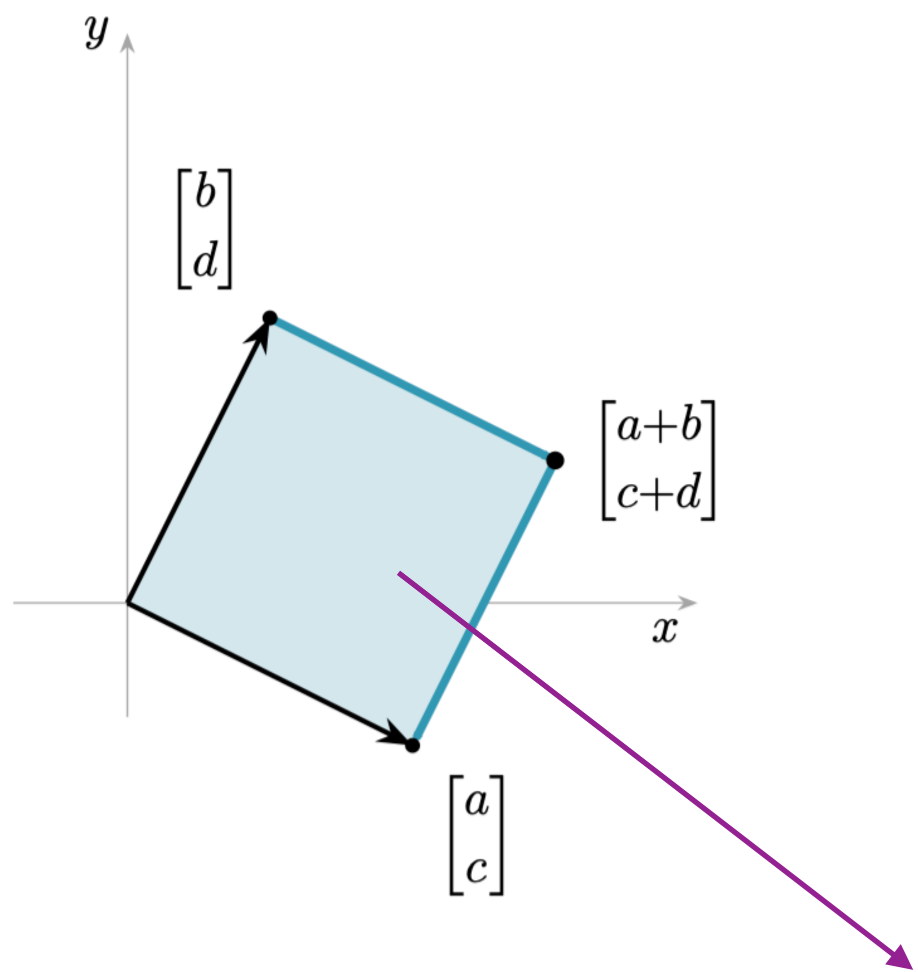
Tuesdays 11–noon
Thursdays 1–2 pm
in Avery 337 (Kara)

Midterm 2
On Monday November 10

Determinants

Determinants

$$\det(A) = \pm \begin{array}{c} \text{volume of the } n\text{-dimensional solid} \\ \text{determined by the columns of } A \end{array}$$



$$\text{area} = \pm \det \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

To compute the determinant of a square matrix:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

A_{ij} = delete row i and column j from A

expand along row 1:

$$\det(A) = a_{11}\det(A_{11}) \pm a_{12}\det(A_{12}) \pm \cdots \pm a_{1n}\det(A_{1n})$$

OR

expand along row i:

$$\det(A) = \pm a_{i1}\det(A_{i1}) \pm a_{i2}\det(A_{i2}) \pm \cdots \pm a_{in}\det(A_{in})$$

To compute the determinant of a square matrix:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

A_{ij} = delete row i and column j from A

expand along column 1:

$$\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) \pm \cdots + (-1)^{n+1}a_{n1}\det(A_{n1})$$

OR

expand along column j:

$$\det(A) = (-1)^{1+j}a_{1j}\det(A_{1j}) + \cdots + (-1)^{n+j}a_{nj}\det(A_{nj})$$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Expand along row 1:

$$\begin{aligned}
\det A &= (+)\,1 \cdot \det\left(\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix}\right) \;-\; 5 \cdot \det\left(\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\right) \;+\; 0 \cdot \det\left(\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}\right) \\
&= 1 \cdot \big(4 \cdot 0 - (-1)(-2)\big) \;-\; 5 \cdot \big(2 \cdot 0 - (-1) \cdot 0\big) \;+\; 0 \cdot \big(2 \cdot (-2) - 4 \cdot 0\big) \\
&= 1 \cdot (0 - 2) \;-\; 5 \cdot (0 - 0) \;+\; 0 \cdot (-4 - 0) \\
&= -2.
\end{aligned}$$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Expand along row 2:

$$\begin{aligned}
\det A &= (-)\,2 \cdot \det\left(\begin{bmatrix} 5 & 0 \\ -2 & 0 \end{bmatrix}\right) \;+\; 4 \cdot \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \;-\; (-1) \cdot \det\left(\begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix}\right) \\
&= -2 \cdot \big(5 \cdot 0 - 0 \cdot (-2)\big) \;+\; 4 \cdot \big(1 \cdot 0 - 0 \cdot 0\big) \;+\; 1 \cdot \big(1 \cdot (-2) - 5 \cdot 0\big) \\
&= -2 \cdot (0 - 0) \;+\; 4 \cdot (0 - 0) \;+\; (-2 - 0) \\
&= -2.
\end{aligned}$$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Expand along column 3:

$$\begin{aligned}
\det A &= (+)\,0 \cdot \det\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} \;-\; (-1) \cdot \det\begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix} \;+\; 0 \cdot \det\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \\
&= 0 \cdot \big(2 \cdot (-2) - 4 \cdot 0\big) \;+\; 1 \cdot \big(1 \cdot (-2) - 5 \cdot 0\big) \;+\; 0 \cdot \big(1 \cdot 4 - 5 \cdot 2\big) \\
&= 0 \cdot (-4 - 0) \;+\; (-2 - 0) \;+\; 0 \cdot (4 - 10) \\
&= -2.
\end{aligned}$$

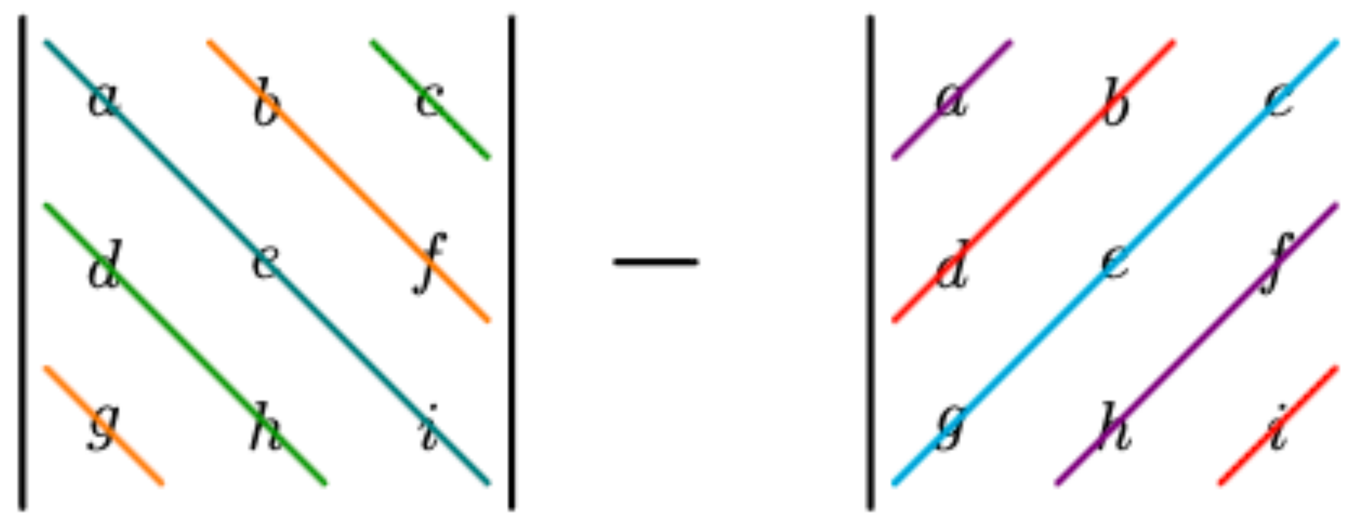
An alternative formula for 3×3 matrices:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$(\,\mathrm{aei} + \mathrm{bfg} + \mathrm{cdh}\,) - (\,\mathrm{ceg} + \mathrm{bdi} + \mathrm{afh}\,).$$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$



$$\begin{aligned} \det(A) &= (1 \cdot 5 \cdot 0 + 5 \cdot (-1) \cdot 0 + 2 \cdot (-2) \cdot 0) - (0 \cdot 4 \cdot 0 + 5 \cdot 2 \cdot 0 + (-1) \cdot (-2) \cdot 1) \\ &= 2. \end{aligned}$$

Our new way of computing the determinant matches the old one:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = +a\det([d]) - b\det([c]) = ad - bc$$

Properties of determinants

Theorem. A square matrix

$$\det(A) \neq 0$$

$$\Updownarrow$$

$$A \text{ is invertible}$$

A square matrix

A is **upper triangular** if all entries below the main diagonal are zero

A is **lower triangular** if all entries above the main diagonal are zero

A is **triangular** if it is upper or lower triangular

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & 4 \end{bmatrix}$$

upper triangular

lower triangular

Theorem.

If A is a triangular matrix, then

$$\det(A) = \text{product of the main diagonal entries}$$

$$A = \begin{bmatrix} -1 & -1 & 7 & 5 \\ 0 & -5 & 42 & 2 \\ 0 & 0 & 2 & 13 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

A upper triangular

$$\Downarrow$$

$$\det(A) = (-1) \cdot (-5) \cdot 2 \cdot 5 = 50$$

Theorem.

If A is a triangular matrix, then

$$\det(A) = \text{product of the main diagonal entries}$$

Corollary. $\det(I) = 1$

identity
matrix

Today’s poll code:

PQRT8D

Today's poll code:

PQRT8D

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

A. 1　　B. 0　　C. -1

D. None of the above

Today's poll code:

PQRT8D

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

A. 1

B. 0

C. -1

D. None of the above

Theorem.

A square matrix

1. $\det(A^{\mathsf{T}}) = \det(A)$.
2. $\det(AB) = \det(A)\det(B)$.
3. $\det(cA) = c^n \det(A)$.

Warning! No formula for $\det(A + B)$

Theorem.

A square matrix

1. $\det(A^\mathsf{T}) = \det(A)$.
2. $\det(AB) = \det(A)\det(B)$.
3. $\det(cA) = c^n \det(A)$.

Consequence: if A and B are invertible, then AB is invertible.

Why?

Theorem.

A square matrix

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3. $\det(cA) = c^n \det(A)$.

Consequence: if A and B are invertible, then AB is invertible.

Why?
$$\begin{matrix} \det(A) \neq 0 \\ \det(B) \neq 0 \end{matrix} \implies \det(AB) = \det(A)\,\det(B) \neq 0$$

Theorem.

A square matrix

1. $\det(A^{\mathsf{T}}) = \det(A)$.
2. $\det(AB) = \det(A)\det(B)$.
3. $\det(cA) = c^n \det(A)$.

Consequence: $$\det(A^{-1}) = \frac{1}{\det(A)}$$

Why?

Theorem.

A square matrix

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2. $\det(AB) = \det(A)\det(B)$.
3. $\det(cA) = c^n \det(A)$.

Consequence: $$\det(A^{-1}) = \frac{1}{\det(A)}$$

Why?

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$$

Effect of Elementary Row Operations on the determinant:

$$A \xrightarrow[\text{add a multiple of a row of } A \text{ to another row}]{\text{Replace}} B \qquad \det(B) = \det(A)$$

$$A \xrightarrow[\text{switch two rows of } A]{\text{Swap}} B \qquad \det(B) = -\det(A)$$

$$A \xrightarrow[\text{multiply a row of } A \text{ by } c]{\text{Rescale}} B \qquad \det(B) = c\det(A)$$

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$$

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$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2} R_1} \begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} = B$$

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$$

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$$\det(B) = \frac{1}{2} \det(A) \implies \det(A) = 2 \det(B)$$

$$B = \begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \qquad \det(A) = 2\det(B)$$

$$B = \begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \qquad \det(A) = 2\det(B)$$

$$B = \begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - 3R_1 \qquad R_4 \to R_4 - R_1} \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 4R_2} \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix} \xrightarrow{R_4 \to R_4 - \frac{1}{2} R_3} \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = C$$

$$B = \begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \qquad \det(A) = 2\det(B)$$

$$B = \begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - 3R_1 \qquad R_4 \to R_4 - R_1} \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

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$$\det(C) = \det(B)$$

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 2\det(B) \qquad \det(C) = \det(B)$$

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 2\det(B) \qquad \det(C) = \det(B)$$

$$\det(C) = 1 \cdot 3 \cdot (-6) \cdot 1 = -18$$

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 2\det(B) \qquad \det(C) = \det(B)$$

$$\det(C) = 1 \cdot 3 \cdot (-6) \cdot 1 = -18$$

$$\det(A) = 2\det(B) = 2\det(C) = 2 \cdot (-18) = -36$$

Theorem.

$$A \sim E$$

E in echelon form

(does not have to be in RREF!)

r = number of row switches from A to E

no rescaling from A to E

$$\det(A) = \begin{cases} (-1)^r \cdot \begin{array}{c}\text{product of the entries in} \\ \text{the pivot positions of } E\end{array} & \text{if } A \text{ is invertible} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem.

A square matrix

1. If A has a column of zeroes, then $\det(A) = 0$.
2. If A has a row of zeroes, then $\det(A) = 0$.
3. If one of the columns of A is a scalar multiple of another, then $\det(A) = 0$.
4. If one of the rows of A is a scalar multiple of another, then $\det(A) = 0$.

Theorem.

The following are equivalent for an $n \times n$ matrix A:

1. A is invertible.
2. There exists B such that $BA = I$.
3. There exists B such that $AB = I$.
4. We have $A \sim I$.
5. The matrix A has rank n.
6. The equation $Ax = 0$ has only the trivial solution.
7. The columns of A form a linearly independent set.
8. The rows of A form a linearly independent set.
9. The linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x) = Ax$ is injective.
10. The linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x) = Ax$ is surjective.
11. The equation $Ax = b$ has at least one solution for each b.
12. The transpose A^{T} is invertible.
13. The determinant of A is nonzero: $\det(A) \neq 0$.

Today's poll code:

PQRT8D

$$\det\left(\begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix}\right)$$

A. 1 B. 0 C. -1

D. None of the above

Today's poll code:

PQRT8D

$$\text{If } \det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = 1 \text{ then}$$

$$\det\left(\begin{bmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{bmatrix}\right) \text{ is}$$

A. 1 B. 2 C. -2 D. -1

Today's poll code:

PQRT8D

$$\text{If } \det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = 1 \text{ then}$$

$$\det\left(\begin{bmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{bmatrix}\right) \text{ is}$$

A. 1 B. 2 C. -2 D. -1

Today's poll code:

PQRT8D

$$\text{If } \det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = 1 \text{ then}$$

$$\det\left(\begin{bmatrix} g & h & i \\ d+2a & e+2b & f+2c \\ a & b & c \end{bmatrix}\right) \text{ is}$$

A. 1 B. 2 C. -2 D. -1

Inverse matrices

A square matrix

cofactor matrix of A $\longrightarrow$ $C_{ij} = (-1)^{i+j}\det(A_{ij})$

adjoint of A $\longrightarrow$ $\mathrm{adj}(A) = C^{\mathsf{T}}$

Example: $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathrm{cof}(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \qquad \mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -d & a \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$$

$$\begin{aligned} C_{11} &= \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} \\ &= (-1)(-2) - (1)(4) \\ &= -2 \end{aligned} \qquad \begin{aligned} C_{12} &= -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} \\ &= -\big((1)(-2) - 1 \cdot 1\big) \\ &= -(-3) = 3 \end{aligned} \qquad \begin{aligned} C_{13} &= \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} \\ &= (1)(4) - (-1)(1) \\ &= 5 \end{aligned}$$

$$\begin{aligned} C_{21} &= -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} \\ &= -\big((1)(-2) - (3)(4)\big) \\ &= -(-14) = 14 \end{aligned} \qquad \begin{aligned} C_{22} &= \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} \\ &= (2)(-2) - (3)(1) \\ &= -4 - 3 = -7 \end{aligned} \qquad \begin{aligned} C_{23} &= -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} \\ &= -\big((2)(4) - (1)(1)\big) \\ &= -(8 - 1) = -7 \end{aligned}$$

$$\begin{aligned} C_{31} &= \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} \\ &= (1)(1) - (3)(-1) \\ &= 1 + 3 = 4 \end{aligned} \qquad \begin{aligned} C_{32} &= -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \\ &= -\big((2)(1) - (3)(1)\big) \\ &= -(2 - 3) = 1 \end{aligned} \qquad \begin{aligned} C_{33} &= \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ &= (2)(-1) - (1)(1) \\ &= -2 - 1 = -3. \end{aligned}$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = (-1)(-2) - (1)(4) = -2$$

$$C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -\big((1)(-2) - 1 \cdot 1\big) = -(-3) = 3$$

$$C_{13} = \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = (1)(4) - (-1)(1) = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = -\big((1)(-2) - (3)(4)\big) = -(-14) = 14$$

$$C_{22} = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = (2)(-2) - (3)(1) = -4 - 3 = -7$$

$$C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -\big((2)(4) - (1)(1)\big) = -(8 - 1) = -7$$

$$C_{31} = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = (1)(1) - (3)(-1) = 1 + 3 = 4$$

$$C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -\big((2)(1) - (3)(1)\big) = -(2 - 3) = 1$$

$$C_{33} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = (2)(-1) - (1)(1) = -2 - 1 = -3.$$

$$\operatorname{cof}(A) = \begin{bmatrix} -2 & 3 & 5 \\ 14 & -7 & -7 \\ 4 & 1 & -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$$

$$\operatorname{cof}(A) = \begin{bmatrix} -2 & 3 & 5 \\ 14 & -7 & -7 \\ 4 & 1 & -3 \end{bmatrix}$$

$$\operatorname{adj}(A) = (\operatorname{cof} A)^{\mathsf{T}} = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

Theorem.

A square matrix

If A is invertible, then

$$\boxed{A^{-1} = \frac{1}{\det A}\ \mathrm{adj}(A).}$$

Thus

$$A \cdot \mathrm{adj}(A) = \mathrm{adj}(A) \cdot A = \det(A) \cdot I.$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} \qquad \operatorname{adj}(A) = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

$$A \cdot \operatorname{adj} A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} \qquad \operatorname{adj}(A) = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

$$A \cdot \operatorname{adj} A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

$$\boxed{A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \det(A) \cdot I.}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} \qquad \operatorname{adj}(A) = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

$$A \cdot \operatorname{adj} A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

$$\boxed{A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \det(A) \cdot I.} \implies \det A = 14$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} \qquad \operatorname{adj}(A) = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

$$\det A = 14$$

$$\boxed{A^{-1} = \frac{1}{\det A} \operatorname{adj}(A).}$$

$$\implies A^{-1} = \tfrac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} & 1 & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{2} & \frac{1}{14} \\ \frac{5}{14} & -\frac{1}{2} & -\frac{3}{14} \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{\det A}\ \mathrm{adj}(A).}$$

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