A stable version of Harbourne's Conjecture

R regular ring (eg, $R = k[x,y,z]$, k a field)
I radical ideal (can also assume prime – still interesting)
h = big height of I = $\max\{ \operatorname{ht} Q : Q \in \operatorname{Min}(R/I) \}$

the n-th symbolic power of I is given by

$$I^{(n)} = \bigcap_{Q \in \operatorname{Min}(R/I)} (I^n R_Q \cap R)$$

<u>theorem</u> (Zariski-Nagata) $R = \mathbb{C}[x_1, \dots, x_d]$

$$I^{(n)} = \bigcap_{\substack{m \supseteq I \\ m \text{ max}}} m^n = \left\{ f \in R : \frac{\partial^{a_1 + \dots + a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} (f) \in I \right\}$$

<u>Note</u>: there are versions of this over any field, or over $\mathbb{Z}$, or a DVR

<u>Facts</u>
1) $I^n \subseteq I^{(n)}$
2) $I^{(n+1)} \subseteq I^{(n)}$
3) $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$
4) If I = (regular sequence), then $I^n = I^{(n)}$ $\forall n \geqslant 1$
In general, $I^{(n)} \neq I^n$.

Example $P = \ker\left(k[x,y,z] \longrightarrow k[\overset{x}{t^3},\overset{y}{t^4},\overset{z}{t^5}]\right)$ prime

$\deg x = 3$
$\deg y = 4$
$\deg z = 5$

$$= (\underbrace{x^3-yz}_{\substack{f \\ \deg 9}},\ \underbrace{y^2-xz}_{\substack{g \\ \deg 8}},\ \underbrace{z^2-x^2y}_{\substack{h \\ \deg 10}})$$

$$P^2 \subsetneq P^{(2)} = P^2 R_P \cap R = \{ r \in R : sr \in P^2,\ s \notin P \}$$

$$\underbrace{f^2-gh}_{\in P^2} = \underbrace{x}_{\notin P}\, q \Rightarrow q \in P^{(2)},\ q \notin P^2$$

$\deg 18 = \deg 3 + \deg 15$ In P^2, elements have $\deg \geq 16$

But actually, $P^{(3)} \subseteq P^2$.

Containment Problem When is $I^{(a)} \subseteq I^b$?

theorem (Ein–Lazarsfeld–Smith 2001, Hochster–Huneke 2002, Ma–Schwede 2017)

$$I^{(hn)} \subseteq I^n \quad \text{for all } n \geq 1$$

$$\Longrightarrow \quad I^{((\dim R)n)} \subseteq I^n \quad \text{for all } n \geq 1$$

Example $P \sim (t^3, t^4, t^5)$ $h = 2$ $P^{(2n)} \subseteq P^n\ \forall n \Rightarrow P^{(4)} \subseteq P^2$.

Question (Huneke, 2000) P prime of ht 2 in RLR. Is $P^{(3)} \subseteq P^2$?

Conjecture (Harbourne, 2008) $I^{(hn-h+1)} \subseteq I^n$ for all $n \geq 1$

Note In char $p>0$, $I^{(hq-h+1)} \subseteq I^{[q]} \subseteq I^q$ $\quad q=p^e$

Example (Dumnicki, Szemberg, Tutaj-Gasińska, 2013)

$\exists$ radical (not prime) ideal in $\mathbb{C}[x,y,z]$, $h=2$ st

$I^{(2n-1)} \nsubseteq I^n$ for $n=2$. $\quad (I^{(3)} \nsubseteq I^2)$

Harbourne's Conjecture does hold for:

- General points in $\mathbb{P}^2$ (Bocci–Harbourne) and $\mathbb{P}^3$ (Dumnicki)
- In char p, if R/I is F-pure (G-Huneke)

$\downarrow$

only need R F-finite Gorenstein, pdim $(I) < \infty$ (G-Ma-Schwede)

Stable Harbourne $I^{(hn-h+1)} \subseteq I^n$ for $n \gg 0$.

Question $I^{(hn-h+1)} \subseteq I^n$ for some $n \overset{?}{\Longrightarrow}$ all $n \gg 0$?

Remark If yes, then we are done in char p.

Remark the answer is yes provided I verifies:

$$I^{(n+h)} \subseteq I I^{(n)} \quad \text{for all } n \geqslant 1$$

False in general, but

thm If R/I is F-pure, $I^{(n+h)} \subseteq I I^{(n)}$ for all $n \geqslant 1$

theorem If $I^{(hk-h)} \subseteq I^k$ for some n then

+ Schwede in mixed char

$$I^{(hn-h)} \subseteq I^n \quad \text{for all } n \gg 0$$

No counterexamples to $I^{(hn-C)} \subseteq I^n$ for $n \gg 0$, C fixed.

Resurgence (Bocci–Harbourne) $\rho(I) = \sup\left\{ \frac{a}{b} : I^{(a)} \not\subseteq I^b \right\}$

$$1 \leq \rho(I) \leq h$$

Remark If $\rho(I) < h$, then Stable Harbourne holds.

$$\frac{hn-C}{n} > \rho(I) \implies I^{(hn-C)} \subseteq I^n$$

C fixed

$$\Updownarrow$$

$$n > \frac{C}{h-\rho(I)}$$

<u>Question</u> Can $\rho(I) = h$?

<u>theorem</u> (G–Huneke–Mukundan)
(R,m) RLR containing a field
Assume $I^{(n)} = I^n : m^\infty = \bigcup_{k\geq 1} I^n : m^k$
(eg R/I Cohen-Macaulay, $\dim(R/I) = 1$)

If $I^{(hn-h+1)} \subseteq m I^n$ for some n, then $\rho(I) < h$.
(and stable Harbourne holds)

<u>Applications</u>

1) I homogeneous ideal, generated in degree $a < h$, char 0

2) Space monomial curves $P \sim (t^a, t^b, t^c)$, char $\neq 3$

$P^{(3)} \subseteq m P^2$ (using Knödel–Schenzel–Zonsarow implicit generators for $P^{(3)}$)

$\Downarrow$

$\rho(P) < 2$
and
stable Harbourne

3) Future work: R graded, char p, R/I Gorenstein