Commutative Algebra Seminar 04/24/20

Combinatorial Structure of Squarefree Monomial Ideals

Setting: $R = k[x_1, \dots, x_n]$

$I = (u_1, \dots, u_m)$ squarefree monomial ideal

↳ if $x_j \mid u_i$, then $x_j^2 \nmid u_i$

Assume that $\{u_1, \dots, u_m\}$ is a minimal generating set for I

↳ no subset would generate I

Outline:

I. Combinatorial structure of squarefree monomial ideals

II. Combinatorial structure vs free resolutions

III. Combinatorial structure vs Cohen-Macaulay property

References:

[Herzog-Hibi]: "Monomial ideals", GTM 260, Springer

Section 1.5 → Simplicial complexes

Section 5.1 → Reduced simplicial homology

Section 7.1 → Taylor resolution

Chapter 8 → Hochster / Eagon-Reiner Formulas and graded Betti numbers

Chapter 9 → Graphs & edge ideals

[Villarreal]: "Rees algebras of edge ideals," Communications in Algebra, 1995

[Francisco-Mermin-Schweig]: "A survey on Stanley-Reisner Theory" → on Chris Francisco's webpage

[Ha-Van Tuyl]: "Resolutions of squarefree monomial ideals via facet ideals: a survey", arxiv: math/0604301v2, 2007

I. Squarefree monomial ideals

First scenario: u_i squarefree monomials of degree 2

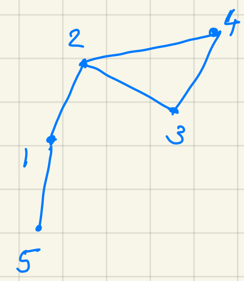
$$\{x_1, \dots, x_n\} \longleftrightarrow \{1, \dots, n\} = [n]$$

We can construct a graph $\mathcal{G}$ associated with I:

$$V(\mathcal{G}) = [n]$$

$$\{i, j\} \in E(\mathcal{G}) \iff x_i x_j \text{ is a minimal monomial generator of } I$$

Ex: $I = (\underbrace{x_1 x_2,\ x_2 x_3,\ x_3 x_4,\ x_2 x_4,\ x_1 x_5}_{\text{minimal generating set}})$



Notice: I squarefree $\Rightarrow$ G has no loops nor multiple edges.

A graph $\mathcal{G}$ with this property is called a simple graph.

Now, let $\mathcal{G}$ be a simple graph on n vertices.

Let $I(G) = \langle x_i x_j : \{i, j\} \in E(\mathcal{G}) \rangle$ edge ideal of G

Hence:

$$\left\{ \begin{array}{c} \text{simple graphs} \\ \mathcal{G} \text{ on } [n] \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{squarefree monomial ideals} \\ I(\mathcal{G}), \text{ generated in degree } 2 \end{array} \right\}$$

Second scenario : u_i squarefree monomials of degrees $d_i \geq 2$

$$\{x_1, \dots, x_n\} \longleftrightarrow \{1, \dots, n\} = [n]$$

We can construct a simplicial complex Δ associated with I:

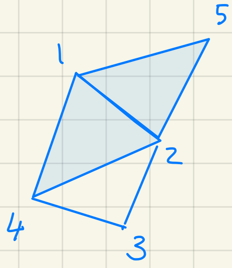
$$F = \{i_1, \dots, i_k\} \in \mathcal{F}(\Delta) \iff x_{i_1} \cdots x_{i_k} \text{ is a minimal monomial generator of } I$$

↳ facets of Δ = faces of maximal dimension

Recall: $\Delta \subseteq \mathcal{P}([n])$ is a simplicial complex if for each $F \in \Delta$ every $F' \subseteq F$ is in Δ as well.

($F \in \Delta$: faces of Δ)

Ex: $I = (\underbrace{x_1 x_2 x_4, x_1 x_2 x_5, x_2 x_3, x_3 x_4}_{\text{minimal generating set}})$



2-dim. facets $\{\{1,2,4\}, \{1,2,5\}\}$

1-dim. facets $\{\{2,3\}, \{3,4\}\}$

Now, let Δ be a simplicial complex on $[n]$

faces: $F \in \Delta$ $\qquad x_F = \prod_{\{i_1, \dots, i_k\} \in F} x_{i_1} \cdots x_{i_k}$ monomial in $k[x_1, \dots, x_n]$

facets: $F \in \mathcal{F}(\Delta) = \{F \in \Delta : F \text{ maximal wrt } \subseteq\}$

$$I(\Delta) = \langle x_F \mid F \in \mathcal{F}(\Delta) \rangle \qquad \text{facet ideal of } \Delta$$

Hence:

$$\left\{ \begin{matrix} \text{simplicial complexes} \\ \Delta \subseteq [n] \end{matrix} \right\} \xleftrightarrow{1-1} \left\{ \begin{matrix} \text{squarefree monomial} \\ \text{ideals } I(\Delta) \subseteq k[x_1, \dots, x_n] \end{matrix} \right\}$$

Let $\mathcal{N}(\Delta) = \{ F \in \mathcal{P}([n]) \setminus \Delta : F \text{ minimal wrt } \subseteq \}$ $\longrightarrow$ minimal non-faces of Δ

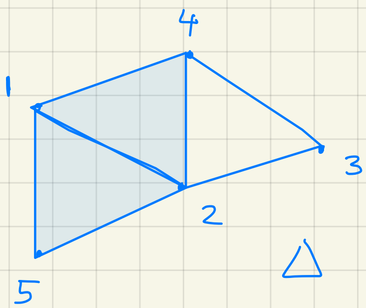
$I_\Delta = \langle x_F : F \in \mathcal{N}(\Delta) \rangle$ Stanley-Reisner ideal of Δ

Consider the following auxiliary simplicial complexes

- $\overline{\Delta}$ so that $\mathcal{F}(\overline{\Delta}) = \{ [n] \setminus F : F \in \mathcal{F}(\Delta) \}$ $\quad$ $[n] \setminus F$ = complement of the face F
- $\Delta^\vee$ so that $\mathcal{F}(\Delta^\vee) = \{ [n] \setminus F : F \in \mathcal{N}(\Delta) \}$

$\longrightarrow$ Alexander dual of Δ : $\Delta^{\vee\vee} = \Delta$

<u>Ex</u>: $I = (x_1 x_2 x_4, \; x_1 x_2 x_5, \; x_2 x_3, \; x_3 x_4) = I(\Delta)$

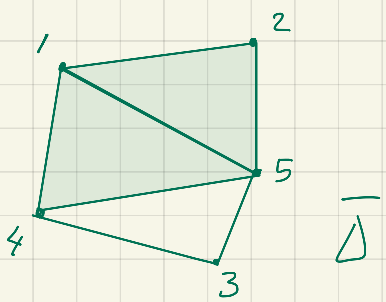


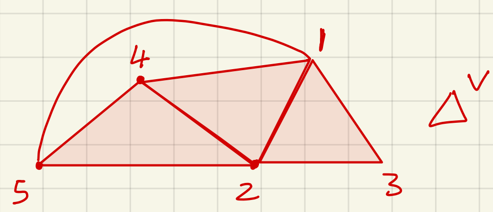
$\mathcal{F}(\Delta) = \{ \{1,2,4\}, \{1,2,5\}, \{2,3\}, \{3,4\} \}$ facets

$\mathcal{N}(\Delta) = \{ \{1,3\}, \{3,5\}, \{4,5\}, \{2,3,4\} \}$ minimal non-faces

$\mathcal{F}(\overline{\Delta}) = \{ \{3,5\}, \{3,4\}, \{1,4,5\}, \{1,2,5\} \}$

$\mathcal{F}(\Delta^\vee) = \{ \{2,4,5\}, \{1,2,4\}, \{1,2,3\}, \{1,5\} \}$





<u>Fact</u>: $I_{\Delta^\vee} = I(\overline{\Delta})$

Hence:

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \text{simplicial complexes} \\ \Delta \subseteq [n] \end{array} \right\} & \overset{1-1}{\longleftrightarrow} & \left\{ \begin{array}{c} \text{squarefree monomial} \\ \text{ideals } I(\Delta) \subseteq k[x_1, \dots, x_n] \end{array} \right\} \\
\updownarrow {\scriptstyle 1-1} & & \updownarrow \\
\left\{ \begin{array}{c} \text{simplicial complexes} \\ \overline{\Delta} \subseteq [n] \end{array} \right\} & \overset{1-1}{\longleftrightarrow} & \left\{ \begin{array}{c} \text{squarefree monomial ideals} \\ I_{\Delta^\vee} = I(\overline{\Delta}) \subseteq k[x_1, \dots, x_n] \end{array} \right\}
\end{array}$$

II. Free resolutions

Recall: $R = k[X_1, \dots, X_n]$, M a finitely generated graded R-module

A graded free resolution of M is a complex of free graded R-modules and homogeneous R-linear maps

$$\mathbb{F}: \dots F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} \dots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \twoheadrightarrow M$$

∂_i = matrices with coefficients in $k[X_1 \dots X_n]$

where

$$F_i = \bigoplus_j \underbrace{R^{\beta_{ij}}(-j)}_{\text{degree shift}}$$

$\beta_{ij} \to$ # of generators of F_i having degree j

β_{ij} = Graded Betti numbers of I $\to$ can be used in order to calculate the Hilbert function $H(R/I)$ and the projective dimension $\operatorname{projdim}(R/I)$

Fact: Every homogeneous ideal I admits a minimal graded free resolution

$\hookrightarrow$ = shortest possible $\iff$ $\forall i$ $\ker \partial_i \subseteq (X_1, \dots, X_n) F_i$

[Taylor, 1968]: Assume that $I = (u_1, \dots, u_m)$ monomial ideal.

Let $\underline{u} = (u_1, \dots, u_m)$, F = free module of basis $\{e_1, \dots, e_m\}$.

$$\mathbb{T}(\underline{u}): 0 \to \Lambda^m F \xrightarrow{\partial_m} \Lambda^{m-1} F \longrightarrow \dots \to \Lambda^1 F \xrightarrow{\partial_1} \Lambda^0 F = R \twoheadrightarrow R/I$$

where

$$\partial_k (e_{i_1} \wedge \dots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^{N_k} \frac{\gcd(u_{i_1}, \dots, u_{i_k})}{\gcd(u_{i_1}, \dots, u_{i_{j-1}}, u_{i_{j+1}}, \dots, u_{i_k})} e_{i_1} \wedge \dots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \dots \wedge e_k$$

$\to$ fully determined by the generators of I

Then, $\mathbb{T}(\underline{u})$ is a free resolution for R/I, called the Taylor resolution.

Issue: not a minimal free resolution $\to$ may be redundant

Highly depends on the generators of I

<u>My motivation</u>: $\mathcal{R}(I) = R \oplus It \oplus I^2t^2 \oplus I^3t^3 \oplus \cdots \subseteq R[t]$

Rees algebra of I

Let $R = k[X_1, \ldots, X_n]$, $I = (u_1, \ldots, u_m)$, $S = R[T_1, \ldots, T_m]$

Then, $\mathcal{R}(I) \cong S/J \longrightarrow$ defining ideal of $\mathcal{R}(I)$

[Taylor, 1968]: $I = (u_1, \ldots, u_m)$ <u>monomial</u> ideal, assume that $\{u_1, \ldots, u_m\}$ is a minimal monomial generating set.

For each $s \in \{1, \ldots, m\}$, let $\mathcal{I}_s = \{\alpha = (i_1, \ldots, i_s) \mid 1 \le i_1 \le \cdots \le i_s \le n\}$

For $\alpha \in \mathcal{I}_s$, let $u_\alpha = u_{i_1} \cdots u_{i_s}$ and $T_\alpha = T_{i_1} \cdots T_{i_s}$

$$J_s = \left\{ \underbrace{\frac{u_\beta}{\gcd(u_\alpha, u_\beta)} T_\alpha - \frac{u_\alpha}{\gcd(u_\alpha, u_\beta)} T_\beta}_{} \;\middle|\; \alpha, \beta \in \mathcal{I}_s \right\}$$

$\longrightarrow$ can become of pretty high degree

Then, the defining ideal of $\mathcal{R}(I)$ is

$$J = \underbrace{SJ_1}_{\text{generators of } T\text{-degree } 1} + \underbrace{S\left(\bigcup_{i=2}^{\infty} J_i\right)}_{}$$

$\longrightarrow$ generators of T-degree ≥ 2 (this is a <u>finite</u> union in fact)

[Villarreal, 1995]:

Let $I = I(G)$ be the edge ideal of a simple graph G.

Then, the defining ideal of $\mathcal{R}(I)$ is

$$J = SJ_1 + S\left(\bigcup_{i=2}^{\infty} J_i\right)$$

where

$$J_s = \{ \underbrace{T_\alpha - T_\beta}_{} \mid u_\alpha = u_\beta \text{ for some } \alpha, \beta \in \mathcal{I}_s \}$$

$\hookrightarrow$ T-degree 1, <u>X</u>-degree 0

for $s=1$ this is called a <u>toric ideal</u>

Rees algebras of squarefree monomial ideals of degree > 2 are much less well-understood. (Open problem in its <u>greatest</u> generality)

Graded Betti numbers of squarefree monomial ideals

<u>Hochster's Formula</u> (1977): Let Δ be a simplicial complex on $[n]$.

For a monomial u in $k[x_1,\dots,x_n]$ let $U = \{j \in [n] : x_j \mid u\}$ and let Δ_U be the <u>restriction</u> of Δ to U.

Then, for all $i,j \geq 0$

$$\beta_{ij}(I_\Delta) = \sum_{\substack{u \text{ squarefree} \\ \text{monomial, } \deg u = j}} \dim_k \underbrace{\tilde{H}^{j-i-2}}(\Delta_U, k).$$

$\hookrightarrow$ reduced simplicial cohomology

- difficult to compute
- depends on char k

<u>Idea of the proof:</u>

- $I_\Delta = (u_1,\dots,u_m)$, $\underbrace{u_i = x_{i_1}^{\alpha_1}\cdots x_{i_n}^{\alpha_n}}$ $\underline{\alpha} = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n$

$\hookrightarrow$ $\mathbb{Z}^n$-homogeneous monomial

$\Rightarrow$ I_Δ admits a $\mathbb{Z}^n$-graded minimal free resolution

$$\mathbb{F}: \quad \cdots F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} \cdots \to F_1 \xrightarrow{\partial_1} F_0 \twoheadrightarrow I_\Delta$$

$$F_i = \bigoplus_j R^{\beta_{i,\underline{\alpha}}}(-\underline{\alpha})$$

$\beta_{i,\underline{\alpha}}$ → multigraded Betti numbers

Then: $$\beta_{i,\underline{\alpha}} = \dim_k \underbrace{\left[\operatorname{Tor}_i^R(k, I_\Delta)\right]_{\underline{\alpha}}}$$

i: homological degree

$\underline{\alpha}$: multidegree of generators of F_i

$$\cong \left[H_i(\mathbb{F}_\cdot \otimes_R k)\right]_{\underline{\alpha}}$$

$\mathbb{F}_\cdot \otimes_R k$: complex of $\mathbb{Z}^n$-graded <u>k-vector spaces</u>

- Let $\underline{x} = (x_1,\dots,x_n)$, $\mathbb{K}_\cdot(\underline{x}; I_\Delta)$ = Koszul complex with coefficients in I_Δ

Let $u = \underline{x}^{\underline{\alpha}}$, then

$$\left[H_i(\mathbb{F}_\cdot \otimes_R k)\right]_{\underline{\alpha}} \cong \left[H_i(\underline{x}; I_\Delta)\right]_{\underline{\alpha}} \cong \begin{cases} 0 & u \text{ not squarefree} \\ \tilde{H}^{j-i-2}(\Delta_U, k) & u \text{ squarefree of degree } j \end{cases}$$

∎

Reduced simplicial cohomology and homology

$\{x_1, \ldots, x_n\}$ variables in $k[x_1, \ldots, x_n]$ $\longleftrightarrow$ $[n] = \{1, \ldots, n\}$ $\longleftrightarrow$ $\{e_1, \ldots, e_n\}$ basis of a k-vector space V

x_F = monomial in $k[x_1, \ldots, x_n]$ $\longleftrightarrow$ $F = \{j_1 < \cdots < j_i\} \subseteq [n]$ $\longleftrightarrow$ e_F base element of $E = \Lambda V$

$I_\Delta = \langle x_F \mid F \notin \Delta \rangle$ Stanley-Reisner ideal $\longleftrightarrow$ $\Delta \subseteq [n]$ simplicial complex $\longleftrightarrow$ $J_\Delta = \langle e_F \mid F \notin \Delta \rangle$

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

$k[\Delta] = k[x_1, \ldots, x_n] / I_\Delta$ Stanley-Reisner ring $\qquad$ $k\{\Delta\} = E / J_\Delta$ exterior face ring

Let $e = e_1 + \cdots + e_n \in V = E_1$

$$(k\{\Delta\}, e): \cdots \xrightarrow{e} (k\{\Delta\})_{i-1} \xrightarrow{e} (k\{\Delta\})_i \to \cdots \quad \text{(co-complex)}$$

- $\tilde{H}^i(\Delta, k) := H^{i+1}(k\{\Delta\}, e)$ $\quad$ i^{th} reduced simplicial cohomology
- Apply $-^* := \mathrm{Hom}_k(-, k)$

$$(k\{\Delta\}, e)^*: \cdots \xrightarrow{e} ((k\{\Delta\})_i, k) \xrightarrow{e} ((k\{\Delta\})_{i+1}, k) \to \cdots \quad \text{(complex)}$$

$\Longrightarrow$ $\tilde{H}_i(\Delta, k) := H_{i+1}((k\{\Delta\}, e)^*)$ $\quad$ i^{th} reduced simplicial homology

<u>Alexander duality</u>: There is a functorial isomorphism

$$\tilde{H}^{i-2}(\Delta^\vee; k) \cong \tilde{H}_{n-i-1}(\Delta, k)$$

Def: Let Δ be a simplicial complex, $F \in \Delta$ a face.

The link of F in Δ is the simplicial complex

$$\mathrm{Link}_\Delta F := \{ G \in \Delta \,:\, G \cup F \in \Delta \text{ and } G \cap F = \emptyset \}$$

Eagon-Reiner Formula (1998):

Let Δ be a simplicial complex on $[n]$. Then, for all $i,j \geq 0$

$$\beta_{ij}(I_\Delta) = \sum_{F \in \Delta^\vee,\ |F| = n-j} \dim_k \underbrace{\tilde{H}_{i-1}(\mathrm{Link}_{\Delta^\vee} F, k)}_{}.$$

much easier to calculate than the simplicial cohomologies appearing in Hochster's formula

Question: How do properties of Δ affect the calculation of graded Betti numbers β_{ij}?

III. Cohen-Macaulay property

Recall: (from last week)

R Noetherian local ring or standard graded k-algebra

R is called Cohen-Macaulay if $\dim R = \operatorname{depth} R$

Def: A simplicial complex Δ is Cohen-Macaulay if the Stanley-Reisner ring $k[\Delta] = k[x_1, .., x_n]/_{I_\Delta}$ is Cohen-Macaulay

Def: $I = (f_1, .., f_s)$ homogeneous ideal, $\deg f_i = d \ \forall i$

I is said to have a linear resolution if $\beta_{ij} = 0$ for all $j \neq i + d$ $\longrightarrow$ so, the Betti table has a lot of 0's

Theorem [Eagon-Reiner]: Let Δ be a simplicial complex on $[n]$.

I_Δ has a linear resolution $\iff$ $\Delta^\vee$ is Cohen-Macaulay

$\hookrightarrow$ This is really a property of $k[x_1 .. x_n]/_{I_{\Delta^\vee}}$

key ingredient in the proof:

Reisner Theorem: A simplicial complex Δ is Cohen-Macaulay

$\iff \tilde{H}_i(\operatorname{link}_\Delta F; k) = 0$ for all $F \in \Delta$ and all $i < \dim(\operatorname{link}_\Delta F)$

$\hookrightarrow$ homological condition on $\Delta^\vee$

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Question: Is there a way to detect Cohen-Macaulayness from combinatorial properties of $\Delta^\vee$?

Def: A homogeneous ideal I is said to have linear quotients if there exists a system of homogeneous generators $f_1, \dots, f_m$ so that the colon ideals

$$(f_1, \dots, f_{i-1}) : f_i = \{ g \in R \mid g f_i \in (f_1, \dots, f_{i-1}) \}$$

are generated by linear forms for all i.

Theorem: $I = (u_1, \dots, u_m)$ homogeneous ideal, $\deg u_i = d \ \forall i$.

If I has linear quotients, then $k[x_1, \dots, x_n]/I$ is CM of dimension $n-d$.

Recall (from Ethan's talk):

- A simplicial complex Δ is pure if all of its facets have the same dimension.
- A simplicial complex Δ is shellable if it is pure and there exists an ordering of its facets $F_1, \dots, F_s$ so that the set

$$\{ G \subseteq F_t \mid G \not\subseteq F_e \ \forall e < t \}$$

contains a unique minimal element.

Theorem: Let Δ be a simplicial complex. Then,

$\Delta^\vee$ is $\underline{\text{shellable}}$ $\iff$ I has linear quotients w.r.t. a monomial system of generators

combinatorial property of $\Delta^\vee$

Corollary: $\Delta^\vee$ is shellable $\Rightarrow$ $\Delta^\vee$ is Cohen-Macaulay

The case of edge ideals

Let $\mathcal{G}$ be a simple graph. Then:

- $\mathcal{G}$ = simplicial complex with facets $E(\mathcal{G})$
- $I(\mathcal{G})$ = facet ideal of the simplicial complex $\mathcal{G}$
  = Stanley-Reisner ideal of the clique complex $\Delta(\mathcal{G}^c)$

Def:

- A clique of $\mathcal{G}$ is a subset $C \subseteq [n]$ so that for any $i \neq j$ in C $\{i,j\} \in E(\mathcal{G})$.
- The clique complex of $\mathcal{G}$ is the simplicial complex $\Delta(\mathcal{G})$ so that $F = \{i_1, \ldots, i_k\} \in \Delta(\mathcal{G}) \iff \mathcal{G}_F$ is a complete graph
  → induced subgraph
- The complement of $\mathcal{G}$ is the graph $\mathcal{G}^c$ so that

$$V(\mathcal{G}^c) = V(\mathcal{G})$$
$$\{i,j\} \in E(\mathcal{G}^c) \iff \{i,j\} \notin E(\mathcal{G})$$

Def:

- A cycle $\mathcal{G}$ has a chord if there exist two non-consecutive vertices i, j so that $\{i,j\} \in E(\mathcal{G})$.
- A graph $\mathcal{G}$ is chordal if every cycle of length ≥ 4 has a chord

Theorem [Fröberg]: Let $\mathcal{G}$ be a simple graph with edge ideal $I(\mathcal{G})$. Then, $I(\mathcal{G})$ has a linear resolution $\iff$ $\mathcal{G}^c$ is chordal

↓ combinatorial property of $\mathcal{G}^c$

Theorem: Every chordal graph is shellable.