Symbolic Powers and the Containment Problem
KU Algebra Seminar (November 30, 2017)

R regular ring
I radical ideal
$I^{(n)} = \bigcap_{Q \in \mathrm{Ass}(I)} (I^n R_Q \cap R)$

h = big height of I = $\max\{\mathrm{ht}\, Q : Q \in \mathrm{Ass}(I)\}$

Q prime ideal
the n-th symbolic power of Q is given by

$$Q^{(n)} = Q^n R_Q \cap R$$
$$= \{ r \in R : sr \in Q^n, \text{ for some } s \notin Q \}$$
$$= \text{smallest } Q\text{-primary ideal containing } Q^n$$

I radical ideal
the n-th symbolic power of I is given by

$$I^{(n)} = \bigcap_{Q \in \mathrm{Ass}(I)} (I^n R_Q \cap R)$$

Properties:

1) $I^n \subseteq I^{(n)}$
2) $I^{(n+1)} \subseteq I^{(n)}$
3) If I is generated by a regular sequence, then $I^{(n)} = I^n$.

the converse is false.

the n-th symbolic power of I is the set of functions that vanish up to order n along the corresponding variety.

Example $\quad I = (x,y) \cap (x,z) \cap (y,z) = (xy, xz, yz)$

$$I^{(2)} = (x,y)^2 \cap (y,z)^2 \cap (x,z)^2 \ni xyz$$

elements in I^2 have degree ≥ 4, but $\deg(xyz) = 3 \implies xyz \notin I^2$

$$I^2 \subsetneq I^{(2)} \qquad \text{but} \qquad I^{(3)} \subseteq I^2$$

Containment Problem $\quad$ When is $I^{(a)} \subseteq I^b$?

Theorem (Ein – Lazarsfeld – Smith, 2001, Hochster – Huneke, 2002, Ma – Schwede, 2017)

$$I^{(hn)} \subseteq I^n \qquad \forall n \geq 1$$

Example $\quad I = (x,y) \cap (y,z) \cap (x,z) \qquad h = 2$

theorem says $I^{(2n)} \subseteq I^n$ for all n, so $I^{(4)} \subseteq I^2$.
But actually we can do better: $\quad I^{(3)} \subseteq I^2$.

Question (Huneke, 2000) If Q is a prime of height 2, does $Q^{(3)} \subseteq Q^2$?

Conjecture (Harbourne, ≤ 2006) $\quad I^{(hn-h+1)} \subseteq I^n \qquad \forall n \geq 2$.

Example (Dumnicki, Szemberg, Tutaj-Gasińska, 2013)

$$I = \left(x(y^{\overset{n}{\cancel{3}}} - z^{\overset{n}{\cancel{3}}}),\ y(z^{\overset{n}{\cancel{3}}} - x^{\overset{n}{\cancel{3}}}),\ z(x^{\overset{n}{\cancel{3}}} - y^{\overset{n}{\cancel{3}}})\right) \subseteq \overset{k}{\cancel{\mathbb{C}}}[x,y,z]$$

$h = 2$, but $\quad I^{(3)} \not\subseteq I^2$.
this corresponds to some special configuration of 12 points in $\mathbb{P}^2$.

Harbourne's Conjecture does hold for:

- General sets of points in $\mathbb{P}^2$ (Harbourne–Huneke) and $\mathbb{P}^3$ (Dumnicki)
- Squarefree monomial ideals.

Hochster and Huneke's proof of $I^{(hn)} \subseteq I^n$ uses basic characteristic p techniques (and reduction to characteristic $p>0$). So maybe we could study this question in characteristic $p>0$.

F: Frobenius map $F(x) = x^p$ $\qquad$ $I^{[q]} = (f^q : f \in I)$ $\qquad$ $q = p^e$

We will ask that R/I be F-pure.

<u>Fedder's Criterion</u> (1983) $\quad$ (R, m) regular local ring of char $p>0$, I radical ideal.

$$R/I \text{ is } F\text{-pure} \iff (I^{[q]} : I) \not\subseteq m^{[q]} \quad \text{for all } q = p^e \gg 0$$

<u>Theorem</u> (–, Huneke) $\quad$ If R/I is F-pure, then $I^{(hn-h+1)} \subseteq I^n$ $\forall n \geq 1$.

<u>Sketch of proof</u>: $\quad$ <u>Step 1</u>: Reduce to the local case.

<u>Step 2</u> $\quad I \subseteq J \iff (J : I) = R \iff (J : I) \not\subseteq m$

<u>Step 3</u> $\quad (I^{[q]} : I) \subseteq (I^n : I^{(hn-h+1)})^{[q]}$ $\quad$ for all $q = p^e \gg 0$

If $I^{(hn-h+1)} \not\subseteq I^n$, then $(I^{[q]} : I) \subseteq m^{[q]} \implies R/I$ <u>not</u> F-pure

Is this best possible for the class of F-pure rings? Yes – even for monomial ideals.

Example $\quad I = \bigcap_{i \neq j} (x_i, x_j) = (x_1 \cdots \hat{x}_i \cdots x_v : 1 \le i \le v) \subseteq k[x_1, \dots, x_v]$

R/I F-pure, $h=2 \Rightarrow I^{(2n-1)} \subseteq I^n$ for all $n \geq 1$. Can we get $I^{(2n-2)} \subseteq I^n$?

Note: Squarefree monomial ideals always define F-pure rings.

Take $n < v$.

$$I^{(2n-2)} = \bigcap_{i \neq j} (x_i, x_j)^{2n-2} \ni (x_1 \cdots x_v)^{n-1}$$

$(x_1 \cdots x_v)^{n-1}$ has degree $(n-1)v = nv - v < nv - n = n(v-1) \le$ elements in I^n have degree

$$(x_1 \cdots x_v)^{n-1} \in I^{(2n-2)} \not\subseteq I^n \quad \text{for } n < v$$

Note: But for $n \geq v$, the containment holds.

Can we obtain tighter containments if we ask for more than F-pure?

Glassbrenner's Criterion (1996) $\quad (R, m)$ F-finite RLR, $I \subseteq R$ radical ideal

let $c \neq 0$ be an element not in any minimal prime of I.

$$R/I \text{ strongly F-regular} \iff c(I^{[q]} : I) \not\subseteq m^{[q]}.$$

Theorem (–, Huneke) $\quad R/I$ strongly F-regular $\Rightarrow I^{((h-1)(n-1)+1)} \subseteq I^n \quad \forall n \geq 1$

Sketch $\quad$ After reducing to the local case, the key lemma is the following:

$$\underbrace{(I^d : I^{(d)})}_{\substack{\text{always contains} \\ c \notin \text{any minimal} \\ \text{prime of } I}} (I^{[q]} : I) \subseteq (I\, I^{(d-h+1)} : I^{(d)})^{[q]} \quad \forall d \geq h-1,\ q = p^e$$

Corollary (–, Huneke) R/I strongly F-regular and $h=2 \Rightarrow I^{(n)} = I^n \quad \forall n \geq 1$

Note these conditions do not imply I is generated by a regular sequence.

Moral Nice conditions on I imply Harbourne's Conjecture.

But how bad are the counterexamples, really?

Example (Fermat configurations) $I = (x(y^e - z^e), y(z^e - x^e), z(x^e - y^e))$

Harbourne's Conjecture would say $I^{(2n-1)} \subseteq I^n$ for all $n \geq 1$.

this fails for $n=2$.
But holds for $n \geq 3$.

(Dumnicki, Harbourne, Nagel, Seceleanu, Szemberg, Tutaj-Gasińska)

Conjecture $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$.

How would we prove that? Maybe the following holds:

Question If $I^{(hn-h+1)} \subseteq I^n$ holds for some value of $n=k$, does that imply $I^{(hn-h+1)} \subseteq I^n$ holds for all $n \gg 0$?

Theorem (–) If $I^{(hn-h)} \subseteq I^n$ for some value of $n=k$, then $I^{(hn-h)} \subseteq I^n$ holds for all $n \geq hk$.

Question Must $I^{(hn-h)} \subseteq I^n$ always hold for some value of n? (maybe depending on I)

This holds for the known counterexamples of Harbourne's Conjecture!

<u>Resurgence</u> (Bocci–Harbourne, 2011) $\rho(I) = \sup\left\{\frac{a}{b} : I^{(a)} \not\subseteq I^{b}\right\}$

$$\underset{\substack{\downarrow \\ I^{(a)} \subseteq I^{b} \Rightarrow a \geq b}}{1} \leq \rho(I) \leq \underset{\substack{\downarrow \\ \text{ELS-HH-MS}}}{h}$$

<u>Observation</u> If $\rho(I) < h$, then for any $C > 0$

$$\frac{hn - C}{n} > \rho(I) \iff n > \frac{C}{h - \rho(I)} \quad \text{implies} \quad I^{(hn-C)} \subseteq I^{n}$$

<u>Open Question</u> Is there an ideal I with $\rho(I) = h$?

What else can we say about Harbourne's Conjecture?

It is still open for prime ideals!

In fact, whether or not $Q^{(3)} \subseteq Q^{2}$ holds for all primes Q of height 2 is an open question, even in dimension 3.

<u>Space Monomial Curves</u> $P = P(a,b,c) = \ker\left(k[x,y,z] \longrightarrow [t^{a}, t^{b}, t^{c}]\right)$, $a \leq b \leq c$

<u>theorem</u> (–) If $\operatorname{char} k \neq 3$, then $P^{(3)} \subseteq P^{2}$.
If $\operatorname{char} k \neq 2, 5$, then $P^{(5)} \subseteq P^{2}$.

Does $P^{(2n-1)} \subseteq P^{n}$ for all $n \geq 1$? True for some classes, open in general

<u>theorem</u> (–) If $\operatorname{char} k \neq 2$ and $a = 3$ or 4, then $P^{(4)} \subseteq P^{3}$

$\Rightarrow P^{(2n-2)} \subseteq P^{n}$ for all $n \geq 6$.

However, $P^{(4)} \not\subseteq P^{3}$ for $a = 9$, $b = 11$, $c = 14$