Linear Algebra

Math 314 Fall 2025

Lecture 26

Today's poll code:

Z374RQ

To do list:

- Webwork 7.4 due Tuesday
- Webwork 7.5 due Friday
- Webwork 7.6 and 7.7 due Sunday
- Lab 2 due Sunday (accepted until Wednesday)

Office hours

Mondays 5–6 pm
Wednesdays 2–3 pm
in Avery 339 (Dr. Grifo)

Tuesdays 11–noon
Thursdays 1–2 pm
in Avery 337 (Kara)

Last quiz

on Friday December 5

Review problem on basis and dimension

Lab 2

Due Sunday December 7

Late work accepted with no penalty until Wednesday December 10

To be discussed in recitation December 5

Theorem (Spectral Theorem)**.**

Every symmetric matrix is orthogonally diagonalizable.

In fact, if A is a square matrix

$$A \text{ is orthogonally diagonalizable} \iff A \text{ is symmetric}$$

Moreover, the eigenvalues of a (real) symmetric matrix must be real numbers.

Note: for any matrix A, both $A^\mathsf{T}A$ and AA^T are symmetric.

Least Squares

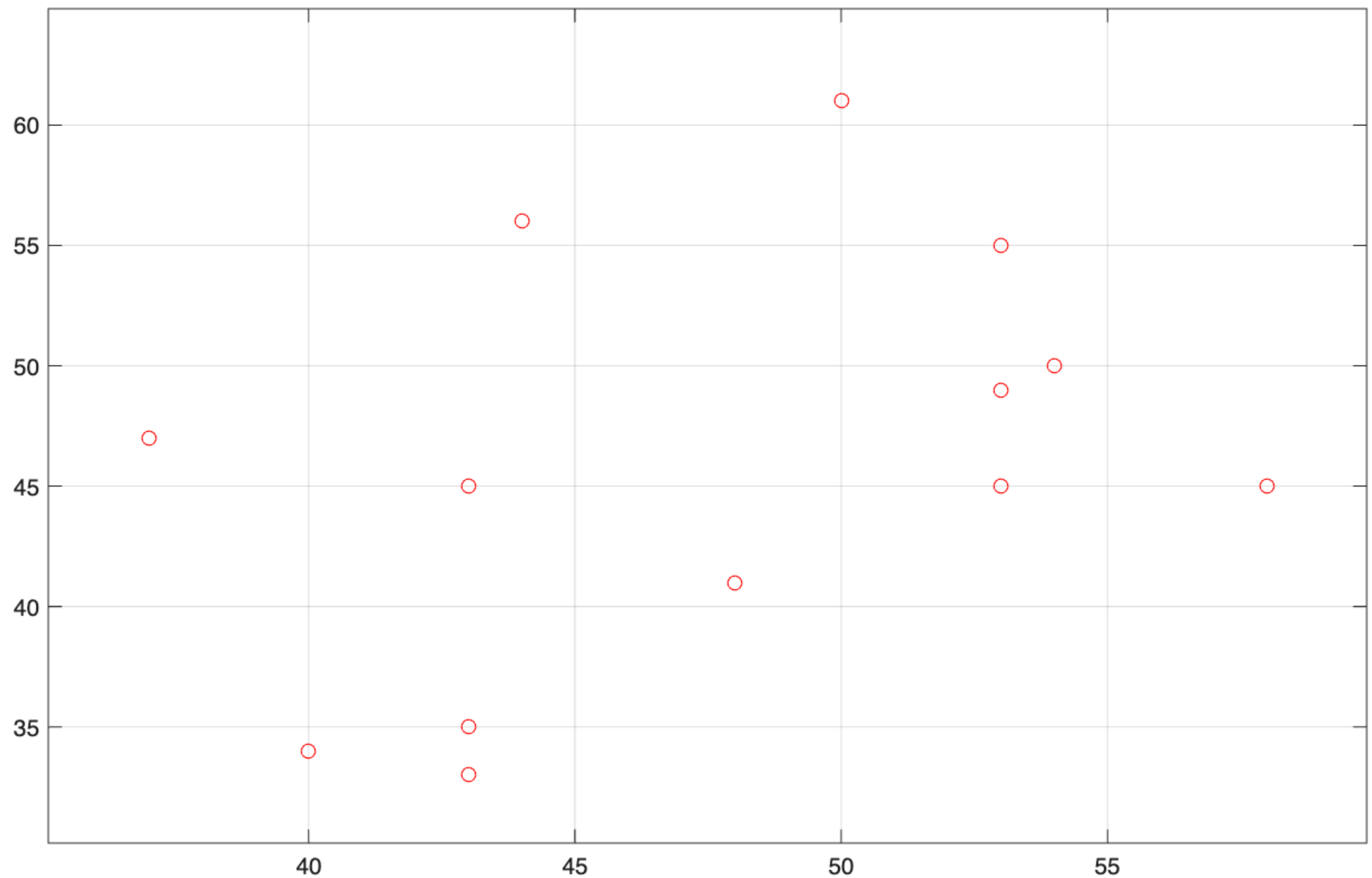
Day	2024 Avg. Temp.	2025 Avg. Temp.
March 1	43	33
2	53	45
3	53	49
4	43	45
5	40	30
6	40	34
7	43	35
8	35	38
9	37	47
10	44	56
11	58	45
12	54	50
13	53	55
14	50	61
15	48	41

Average daily temperatures in Lincoln, NE.
Data obtained from `https://lincolnweather.unl.edu/`.

Can we model the weather in Lincoln in March using this data?

How does the temperature in Lincoln in March in 2025 depend on the temperature in Lincoln in March in 2024?

Is the temperature in 2025 a linear function of the temperature in 2024?



It does not look very linear, but

can we approximate the data with a linear function?

Suppose there was a linear function

$$y = f(x) = ax + b$$

with $x =$ temperature in 2024

and $y =$ temperature in 2025

Then all of our values would satisfy

$$y = ax + b$$

So we would have real numbers a and b such that

$$\begin{bmatrix} ax_1 + b \\ \vdots \\ ax_{15} + b \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_{15} \end{bmatrix}$$

$x_i =$ temperature on March i, 2024

$y_i =$ temperature on March i, 2025

So we would have real numbers a and b such that

$$y = ax + b$$

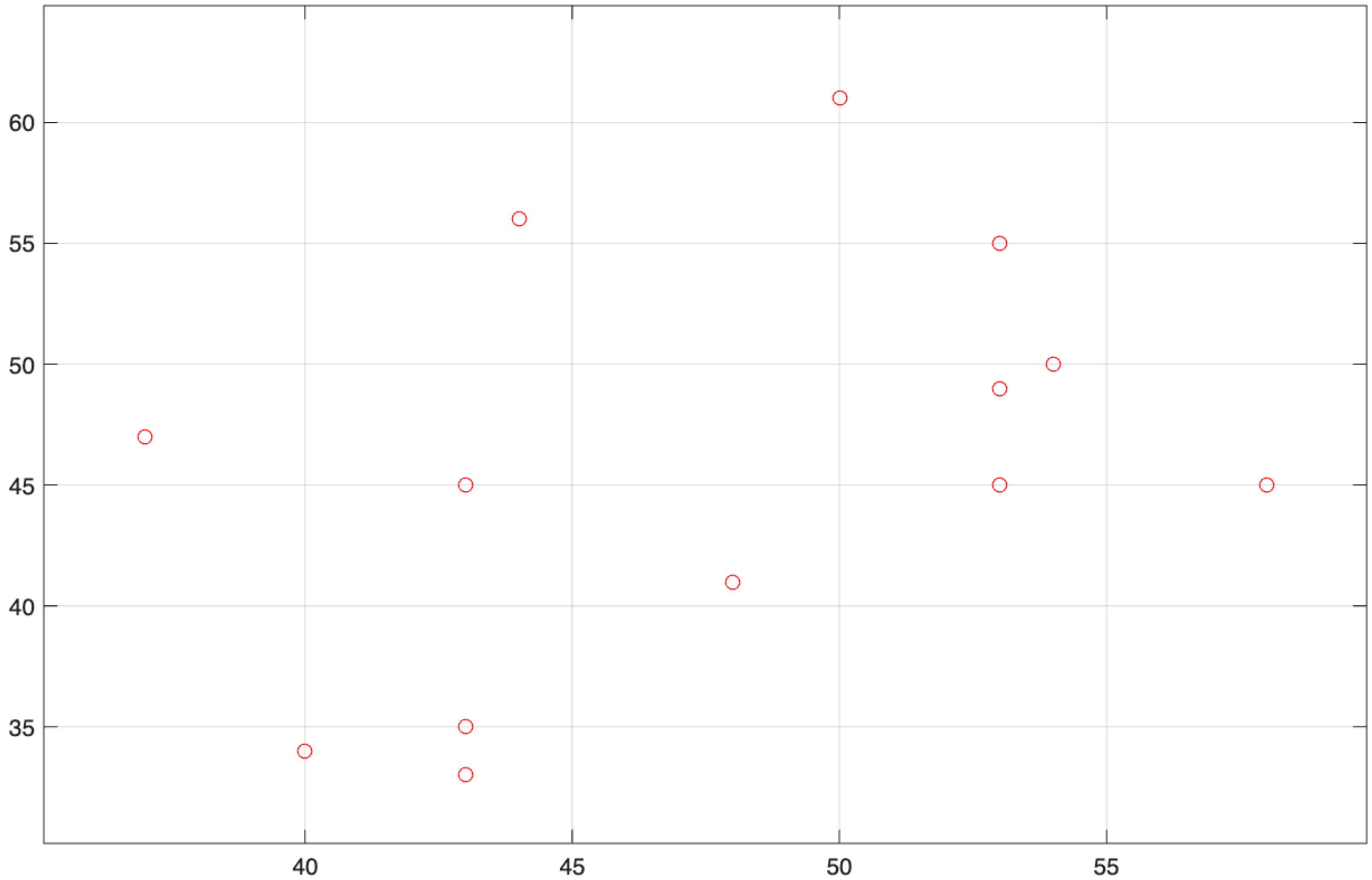
$$\begin{bmatrix} ax_1 + b \\ \vdots \\ ax_{15} + b \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_{15} \end{bmatrix}$$

$x_i =$ temperature on March i, 2024

$y_i =$ temperature on March i, 2025

so a and b would be solutions to the system

$$\begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_{15} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_{15} \end{bmatrix}$$



These points are not all on a line, so the system is inconsistent

But can we find an approximate solution?

A $m \times n$ matrix

b vector in $\mathbb{R}^m$

A **least squares solution** to $Ax = b$ is

a vector y in $\mathbb{R}^n$ such that

$$\|b - Ay\| \leqslant \|b - Ax\|$$

for every vector x in $\mathbb{R}^n$

In other words: a least squares solution to $Ax = b$ is a vector y that is as close to being a solution as possible

Note: if $Ax = b$ is consistent, any actual solution x to $Ax = b$ *is* a least squares solution to $Ax = b$

The least squares solution to $Ax = b$ is the

<u>best approximation</u> of b to the set of vectors of the form Ax

Theorem.

least squares solutions to $Ax = b$	$=$	solutions to $A^TAx = A^Tb$

$$\boxed{A^TAx = A^Tb}$$ normal equations for $Ax = b$

To find the least squares solution to $Ax = b$, we need to solve the normal equations $A^T Ax = A^T b$

Theorem. If the columns of A are linearly independent, then $A^{\mathsf{T}} A$ is invertible.

In that case, the least squares approximation to $Ax = b$ is

$$\dots$$

Theorem. If A is an invertible $n \times n$ matrix, then for each $b \in \mathbb{R}^n$ the equation

$$Ax = b$$

has a unique solution, which is given by $x = A^{-1}b$.

To find the least squares solution to $Ax = b$, we need to solve the normal equations $A^T Ax = A^T b$

Theorem. If the columns of A are linearly independent, then $A^\top A$ is invertible.

In that case, the least squares approximation to $Ax = b$ is

$$z = (A^\top A)^{-1} A^\top b$$

In this case, there is only <u>one</u> best approximation

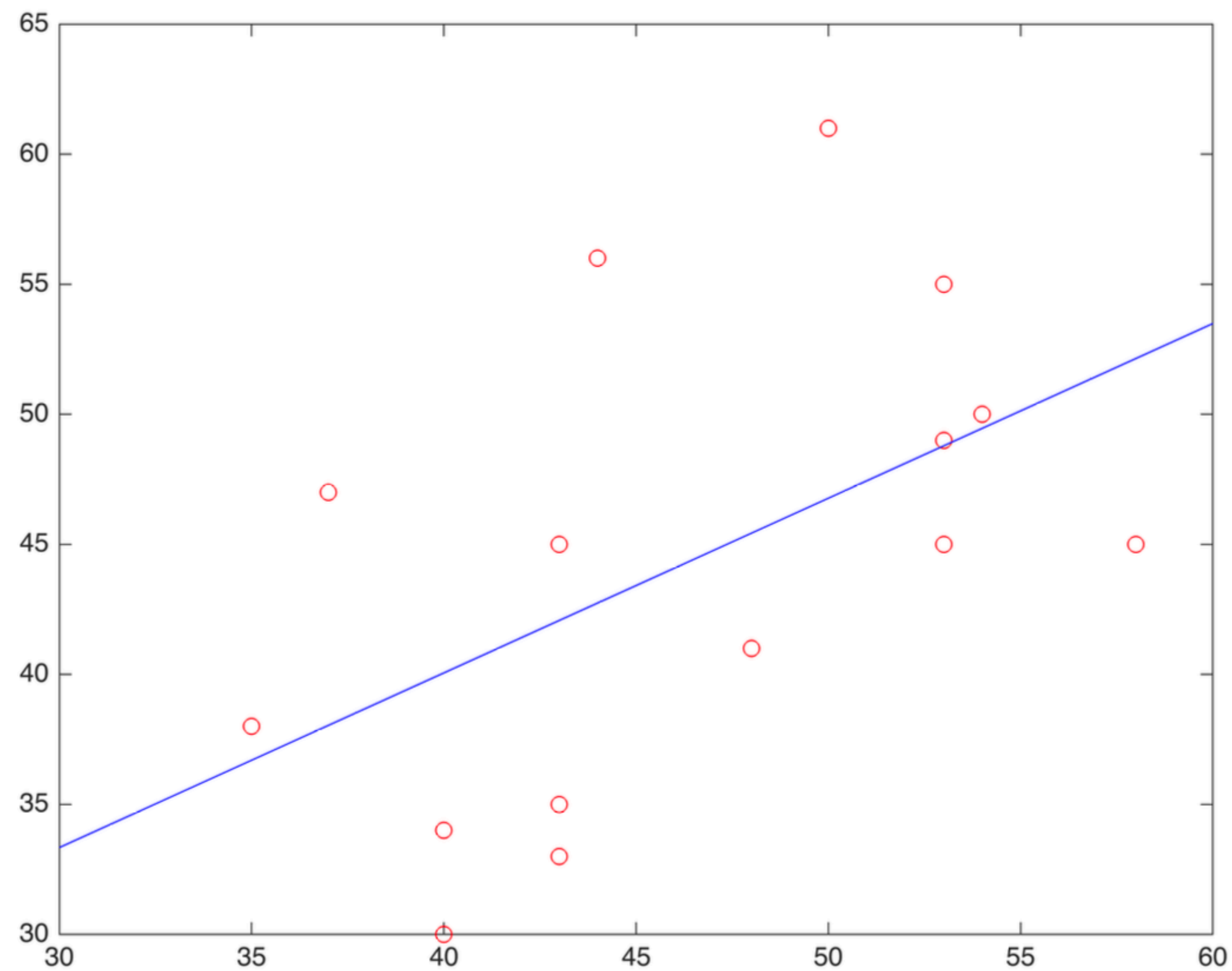
If $A^T A$ is not invertible, then there are multiple least squares solutions to $Ax = b$.

Original data

Day	2024 Avg. Temp.	2025 Avg. Temp.
March 1	43	33
2	53	45
3	53	49
4	43	45
5	40	30
6	40	34
7	43	35
8	35	38
9	37	47
10	44	56
11	58	45
12	54	50
13	53	55
14	50	61
15	48	41

Average daily temperatures in Lincoln, NE.
Data obtained from `https://lincolnweather.unl.edu/`.

Least squares approximation



Line of best fit

Today's poll code:

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$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$$[A|b] = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

How many solutions does the system $Ax = b$ have?

A. None B. 1 C. 2 D. infinitely many

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$$[A|b] = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot in the last column $\implies$ inconsistent system

so there are no solutions to $Ax = b$

Let us find a least squares solution

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

normal equations

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}^{-1} = \tfrac{1}{85-1} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

Least squares solution to $Ax = b$:

$$z = (A^T A)^{-1} b = \tfrac{1}{85-1} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 19 \\ 11 \end{bmatrix} = \tfrac{1}{84} \begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

If $A^\intercal A$ is invertible, then

there is a unique least squares solution to $Ax = b$

given by

$$z = (A^\intercal A)^{-1} A^\intercal b$$

If $A^T A$ is not invertible, then

there are multiple least squares solutions to $Ax = b$.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

$$[A|b] = \begin{bmatrix} 1 & 1 & 0 & 0 & -3 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$[A|b] = \begin{bmatrix} 1 & 1 & 0 & 0 & -3 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

How many solutions does the system $Ax = b$ have?

A. None B. 1 C. 2 D. infinitely many

Today's poll code:

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$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Is $A^{\mathsf{T}}A$ invertible?

A. Yes

B. No

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \\ 6 \end{bmatrix}$$

$$[A^{\mathsf{T}} A \mid A^{\mathsf{T}} b] = \begin{bmatrix} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & -5 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Today's poll code:

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$$[A^\mathsf{T}A \mid A^\mathsf{T}b] = \begin{bmatrix} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & -5 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

How many least squares solutions does $Ax = b$ have?

A. None B. 1 C. 2 D. infinitely many

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

$$[A^\mathsf{T} A \mid A^\mathsf{T} b] = \begin{bmatrix} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & -5 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Least squares solutions to $Ax = b$:

$$z = x_4 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -5 \\ -2 \\ 0 \end{bmatrix}$$

Singular Value Decomposition (SVD)

A $m \times n$ matrix

$$(A^\top A)^\top = A^\top (A^\top)^\top = A^\top A$$

$$\Downarrow$$

$A^\top A$ is always symmetric

$$\Downarrow$$

$A^\top A$ $n \times n$ orthogonally diagonalizable matrix

$A^{\mathsf{T}}A$ is always symmetric

$\{v_1, \ldots, v_n\}$ orthonormal basis for $\mathbb{R}^n$ of eigenvectors of $A^{\mathsf{T}}A$

$\underbrace{\lambda_1 \geqslant \cdots \geqslant}_{\text{(maybe repeated)}} \lambda_n$ corresponding eigenvalues of $A^{\mathsf{T}}A$

$$\begin{aligned}
\|Av_i\|^2 &= (Av_i)^T(Av_i) \\
&= v_i^T((A^TA)v_i) \\
&= v_i^T(\lambda_i v_i) \quad \longleftarrow \quad v_i \text{ is an eigenvector associated to } \lambda_i \\
&= \lambda_i(v_i^T v_i) \\
&= \lambda_i(v_i \bullet v_i) \\
&= \lambda_i
\end{aligned}$$

Consequences: $\lambda_i \geqslant 0$ and $\sqrt{\lambda_i} = \|Av_i\|$

A $m \times n$ matrix

$A^\mathsf{T} A$ is always symmetric

$\{v_1, \dots, v_n\}$ orthonormal basis for $\mathbb{R}^n$ of eigenvectors of $A^\mathsf{T} A$

$\underbrace{\lambda_1 \geqslant \cdots \geqslant}_{\text{(maybe repeated)}} \lambda_n$ corresponding eigenvalues of $A^\mathsf{T} A$

$$\lambda_i \geqslant 0 \qquad \text{and} \qquad \sqrt{\lambda_i} = \|Av_i\|$$

The **singular values** of A are

$$\sigma_1 = \sqrt{\lambda_1}, \dots, \sigma_n = \sqrt{\lambda_n}$$

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

$$A^{\mathsf{T}}A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

$$\det(A^{\mathsf{T}}A - \lambda I) = -\lambda^3 + 450\lambda^2 - 32400\lambda = -\lambda(\lambda - 90)(\lambda - 360)$$

Singular values of A:

$$\sigma_1 = \sqrt{360} = 6\sqrt{10} \qquad \sigma_2 = \sqrt{90} = 3\sqrt{10} \qquad \sigma_3 = 0$$

Theorem.

A $m \times n$ matrix

$\{v_1, \ldots, v_n\}$ basis for $\mathbb{R}^n$ of eigenvectors of $A^\mathsf{T} A$

corresponding to eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$

$$\begin{aligned} r &= \text{number of nonzero } \lambda_i \\ &= \text{number of nonzero singular values of } A \end{aligned}$$

so $\lambda_1 \geqslant \cdots \geqslant \lambda_r > 0$ and $\lambda_{r+1} = \cdots = \lambda_n = 0$

Then $\{Av_1, \ldots, Av_r\}$ is an orthogonal basis for $\mathrm{col}(A)$.

In particular, $\mathrm{rank}(A) = r$.

A $m \times n$ matrix

$A^\mathsf{T} A$ is always symmetric

$\{v_1, \dots, v_n\}$ orthonormal basis for $\mathbb{R}^n$ of eigenvectors of $A^\mathsf{T} A$

$\underbrace{\lambda_1 \geqslant \cdots \geqslant}_{\text{(maybe repeated)}} \lambda_n$ corresponding eigenvalues of $A^\mathsf{T} A$

singular values of A: $\sigma_1 = \sqrt{\lambda_1}, \dots, \sigma_n = \sqrt{\lambda_n}$

$$\boxed{r = \operatorname{rank}(A) = \text{number of nonzero singular values of } A}$$

$\{Av_1, \dots, Av_r\}$ orthogonal basis for $\operatorname{col}(A)$

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \qquad A^\mathsf{T} A = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

$$\det(A^\mathsf{T} A - \lambda I) = -\lambda^3 + 450\lambda^2 - 32400\lambda = -\lambda(\lambda - 90)(\lambda - 360)$$

Singular values of A:

$$\sigma_1 = \sqrt{360} = 6\sqrt{10} \qquad \sigma_2 = \sqrt{90} = 3\sqrt{10} \qquad \sigma_3 = 0$$

What is the rank of A?

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \qquad A^{\mathsf{T}}A = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

$$\det(A^{\mathsf{T}}A - \lambda I) = -\lambda^3 + 450\lambda^2 - 32400\lambda = -\lambda(\lambda - 90)(\lambda - 360)$$

Singular values of A:

$$\sigma_1 = \sqrt{360} = 6\sqrt{10} \qquad \sigma_2 = \sqrt{90} = 3\sqrt{10} \qquad \sigma_3 = 0$$

$$\boxed{\mathrm{rank}(A) = 2}$$

Orthonormal basis of eigenvectors for $A^{\mathsf{T}}A$:

$$v_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

Orthogonal basis for $\mathrm{col}(A)$: $\{Av_1, Av_2\}$

Theorem (Singular Value Decomposition).

A $m \times n$ matrix $\qquad \mathrm{rank}(A) = r$

We can always decompose A as a product

$$\boxed{A = U\Sigma V^T} \quad \text{SVD of } A$$

where:

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \qquad m \times n \text{ diagonal in blocks}$$

$D =$ diagonal $r \times r$ matrix with singular values of A

U orthogonal $m \times m$ matrix

V is an orthogonal $n \times n$ matrix

Theorem (Singular Value Decomposition).

A $m \times n$ matrix $\qquad$ $\operatorname{rank}(A) = r$

We can always decompose A as a product

$$\boxed{A = U\Sigma V^T} \quad \text{SVD of } A$$

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \quad m \times n \text{ diagonal in blocks}$$

D = diagonal $r \times r$ matrix with singular values of A

$$\underset{m \times m}{U = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}} \qquad \underset{n \times n}{V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}$$

$\{v_1, \ldots, v_n\}$ orthonormal basis for $\mathbb{R}^n$ of eigenvectors of $A^\mathsf{T} A$

$\{u_1, \ldots, u_m\}$ orthonormal basis for $\mathbb{R}^m$

u_i = normalization of Av_i for $i \leqslant r$

Traditionally, we write

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

with

$$\sigma_1 \geqslant \cdots \geqslant \sigma_n$$

so that there is only one

singular value decomposition of A

Recipe for finding a SVD for A

Recipe to find the singular value decomposition of A:

Step 1: Find the eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ of $A^\mathsf{T} A$.

Singular values of A: $$\begin{gathered}\sigma_1 = \sqrt{\lambda_1} \geqslant \cdots \geqslant \sqrt{\lambda_r} > 0 \\ \text{and } \sigma_{r+1} = \cdots = \sigma_n = 0\end{gathered}$$

Step 2: Find an orthonormal basis $\{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$, with v_i eigenvector of $A^\mathsf{T} A$ corresponding to λ_i

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After we find a basis of eigenvectors of $A^T A$,
we might need to use Gram-Schmidt on some of the eigenspaces

For each space with dimension $\geqslant 2$,

check if the basis is orthogonal.

If not, use Graham–Schmidt.

Recipe to find the singular value decomposition of A:

Step 1: Find the eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ of $A^\mathsf{T} A$.

Singular values of A: $\sigma_1 = \sqrt{\lambda_1} \geqslant \cdots \geqslant \sqrt{\lambda_r} > 0$ and $\sigma_{r+1} = \cdots = \sigma_n = 0$

Step 2: Find an orthonormal basis $\{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$, with v_i eigenvector of $A^\mathsf{T} A$ corresponding to λ_i

We know $\{Av_1, \ldots, Av_r\}$ is an orthogonal basis for $\operatorname{col}(A)$.

Step 3: Normalize each Av_i to obtain $u_1, \ldots u_r$.

$$u_i = \frac{Av_i}{\|Av_i\|} = \frac{Av_i}{\sigma_i}$$

We know $\{u_1, \ldots, u_r\}$ is an orthonormal basis for $\operatorname{col}(A)$.

Step 4: Extend $\{u_1, \ldots, u_r\}$ to an orthonormal basis $\{u_1, \ldots, u_m\}$ for $\mathbb{R}^m$.

A $m \times n$ matrix $\qquad \operatorname{rank}(A) = r$

SVD of A $\qquad \boxed{A = U\Sigma V^T}$

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \qquad m \times n \text{ diagonal in blocks}$$

$D =$ diagonal $r \times r$ matrix with singular values of A

$$\underset{m \times m}{U = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}} \qquad \underset{n \times n}{V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}$$

$\{v_1, \ldots, v_n\}$ orthonormal basis for $\mathbb{R}^n$ of eigenvectors of $A^\mathsf{T} A$

$\{u_1, \ldots, u_m\}$ orthonormal basis for $\mathbb{R}^m$

$u_i =$ normalization of Av_i for $i \leqslant r$

Example of a singular value decomposition

Let us find a singular value decomposition of

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

Recipe to find the singular value decomposition of A:

Step 1: Find the eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ of $A^\mathsf{T} A$.

Singular values of A: $\sigma_1 = \sqrt{\lambda_1} \geqslant \cdots \geqslant \sqrt{\lambda_r} > 0$ and $\sigma_{r+1} = \cdots = \sigma_n = 0$

Step 2: Find an orthonormal basis $\{v_1, \dots, v_n\}$ for $\mathbb{R}^n$, with v_i eigenvector of $A^\mathsf{T} A$ corresponding to λ_i

We know $\{Av_1, \dots, Av_r\}$ is an orthogonal basis for $\text{col}(A)$.

Step 3: Normalize each Av_i to obtain $u_1, \dots u_r$.

$$u_i = \frac{Av_i}{\|Av_i\|} = \frac{Av_i}{\sigma_i}$$

We know $\{u_1, \dots, u_r\}$ is an orthonormal basis for $\text{col}(A)$.

Step 4: Extend $\{u_1, \dots, u_r\}$ to an orthonormal basis $\{u_1, \dots, u_m\}$ for $\mathbb{R}^m$.

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \qquad A^{\mathsf{T}}A = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

Singular values of A:

$$\sigma_1 = \sqrt{360} = 6\sqrt{10} \qquad \sigma_2 = \sqrt{90} = 3\sqrt{10} \qquad \sigma_3 = 0$$

$$\boxed{\operatorname{rank}(A) = 2}$$

$$D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} D & 0 \end{bmatrix} = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

Recipe to find the singular value decomposition of A:

Step 1: Find the eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ of $A^\mathsf{T} A$.

Singular values of A: $\sigma_1 = \sqrt{\lambda_1} \geqslant \cdots \geqslant \sqrt{\lambda_r} > 0$
and $\sigma_{r+1} = \cdots = \sigma_n = 0$

$$\boxed{\text{Step 2: Find an orthonormal basis } \{v_1, \ldots, v_n\} \text{ for } \mathbb{R}^n\text{, with } v_i \text{ eigenvector of } A^\mathsf{T} A \text{ corresponding to } \lambda_i}$$

We know $\{Av_1, \ldots, Av_r\}$ is an orthogonal basis for $\operatorname{col}(A)$.

Step 3: Normalize each Av_i to obtain $u_1, \ldots u_r$.

$$u_i = \frac{Av_i}{\|Av_i\|} = \frac{Av_i}{\sigma_i}$$

We know $\{u_1, \ldots, u_r\}$ is an orthonormal basis for $\operatorname{col}(A)$.

Step 4: Extend $\{u_1, \ldots, u_r\}$ to an orthonormal basis $\{u_1, \ldots, u_m\}$ for $\mathbb{R}^m$.

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \qquad A^\top A = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

Singular values of A:

$$\sigma_1 = \sqrt{360} = 6\sqrt{10} \qquad \sigma_2 = \sqrt{90} = 3\sqrt{10} \qquad \sigma_3 = 0$$

Orthonormal basis of eigenvectors for $A^\top A$:

$$v_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \qquad A^{\mathsf{T}}A = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

Singular values of A:

$$\sigma_1 = \sqrt{360} = 6\sqrt{10} \qquad \sigma_2 = \sqrt{90} = 3\sqrt{10} \qquad \sigma_3 = 0$$

$$\boxed{\operatorname{rank}(A) = 2}$$

$$D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} D & 0 \end{bmatrix} = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Recipe to find the singular value decomposition of A:

Step 1: Find the eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ of $A^\mathsf{T} A$.

Singular values of A: $\sigma_1 = \sqrt{\lambda_1} \geqslant \cdots \geqslant \sqrt{\lambda_r} > 0$ and $\sigma_{r+1} = \cdots = \sigma_n = 0$

Step 2: Find an orthonormal basis $\{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$, with v_i eigenvector of $A^\mathsf{T} A$ corresponding to λ_i

$\boxed{\text{We know } \{Av_1, \ldots, Av_r\} \text{ is an orthogonal basis for } \operatorname{col}(A).}$

Step 3: Normalize each Av_i to obtain $u_1, \ldots u_r$.

$$u_i = \frac{Av_i}{\|Av_i\|} = \frac{Av_i}{\sigma_i}$$

We know $\{u_1, \ldots, u_r\}$ is an orthonormal basis for $\operatorname{col}(A)$.

Step 4: Extend $\{u_1, \ldots, u_r\}$ to an orthonormal basis $\{u_1, \ldots, u_m\}$ for $\mathbb{R}^m$.

Recipe to find the singular value decomposition of A:

Step 1: Find the eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ of $A^\mathsf{T} A$.

Singular values of A: $\sigma_1 = \sqrt{\lambda_1} \geqslant \cdots \geqslant \sqrt{\lambda_r} > 0$
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Singular values of A:

$$\sigma_1 = \sqrt{360} = 6\sqrt{10} \qquad \sigma_2 = \sqrt{90} = 3\sqrt{10} \qquad \boxed{\operatorname{rank}(A) = 2}$$

$$v_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, v_2 = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$Av_1 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix} \qquad u_1 = \begin{bmatrix} \frac{18}{6\sqrt{10}} \\ \frac{6}{6\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \end{bmatrix} \qquad u_2 = \begin{bmatrix} \frac{3}{3\sqrt{10}} \\ -\frac{9}{3\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \end{bmatrix}$$

Recipe to find the singular value decomposition of A:

Step 1: Find the eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ of $A^\mathsf{T} A$.

Singular values of A: $\sigma_1 = \sqrt{\lambda_1} \geqslant \cdots \geqslant \sqrt{\lambda_r} > 0$ and $\sigma_{r+1} = \cdots = \sigma_n = 0$

Step 2: Find an orthonormal basis $\{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$, with v_i eigenvector of $A^\mathsf{T} A$ corresponding to λ_i

We know $\{Av_1, \ldots, Av_r\}$ is an orthogonal basis for col(A).

Step 3: Normalize each Av_i to obtain $u_1, \ldots u_r$.

$$u_i = \frac{Av_i}{\|Av_i\|} = \frac{Av_i}{\sigma_i}$$

We know $\{u_1, \ldots, u_r\}$ is an orthonormal basis for col(A).

Step 4: Extend $\{u_1, \ldots, u_r\}$ to an orthonormal basis $\{u_1, \ldots, u_m\}$ for $\mathbb{R}^m$.

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Singular values of A:

$$\sigma_1 = \sqrt{360} = 6\sqrt{10} \qquad \sigma_2 = \sqrt{90} = 3\sqrt{10} \qquad \boxed{\operatorname{rank}(A) = 2}$$

$$u_1 = \begin{bmatrix} \frac{18}{6\sqrt{10}} \\ \frac{6}{6\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} \qquad u_2 = \begin{bmatrix} \frac{3}{3\sqrt{10}} \\ -\frac{9}{3\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \end{bmatrix}$$

Next: extend to an orthogonal basis of $\mathbb{R}^2$

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \qquad A^{\mathsf{T}}A = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

Singular values of A:

$$\sigma_1 = \sqrt{360} = 6\sqrt{10} \qquad \sigma_2 = \sqrt{90} = 3\sqrt{10} \qquad \boxed{\operatorname{rank}(A) = 2}$$

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Next: extend to an orthogonal basis of $\mathbb{R}^2$

already have two vectors, so we are done!

$$U = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \qquad A^{\mathsf{T}}A = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

Singular values of A:

$$\sigma_1 = \sqrt{360} = 6\sqrt{10} \qquad \sigma_2 = \sqrt{90} = 3\sqrt{10} \qquad \sigma_3 = 0$$

SVD of A $\quad \boxed{A = U\Sigma V^T}$

where:

$$D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} D & 0 \end{bmatrix} = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \qquad U = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \qquad A^{\mathsf{T}}A = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

Singular values of A:

$$\sigma_1 = \sqrt{360} = 6\sqrt{10} \qquad \sigma_2 = \sqrt{90} = 3\sqrt{10} \qquad \sigma_3 = 0$$

SVD of A

$$A = \underbrace{\begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}}_{V}$$

Done!

Step 4: Extend $\{u_1, \ldots, u_r\}$ to an orthonormal basis $\{u_1, \ldots, u_m\}$ for $\mathbb{R}^m$.

$\{u_1, \ldots, u_r\}$ is an orthogonal set

To complete it to an orthonormal basis for $\mathbb{R}^m$, need to find an orthonormal basis for $(\operatorname{span}(\{u_1, \ldots, u_r\}))^\perp$

How?

Find a basis for the null space of $\begin{bmatrix} u_1{}^\mathsf{T} \\ \vdots \\ u_r{}^\mathsf{T} \end{bmatrix}$

Is that basis orthogonal? If not, use Graham–Schmidt.

Normalize. The resulting vectors are $u_{r+1}, \ldots, u_m$.

If A itself is symmetric

then

singular value decomposition of A $=$ orthogonal diagonalization of A

To do list:

Quiz on Friday

on vector spaces

dimension/basis

- Webwork 7.4 due Tuesday
- Webwork 7.5 due Friday
- Webwork 7.6 and 7.7 due Sunday
- Lab 2 due Sunday (accepted until Wednesday)

Office hours

Mondays 5–6 pm and Wednesdays 2–3 pm
in Avery 339 (Dr Grifo)

Tuesdays 11–noon and Thursdays 1–2 pm
in Avery 337 (Kara)