A Feder-type criterion over Gorenstein rings

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(joint work with Linquan Ma and Karl Schwede)

R Cohen-Macaulay ring

I radical ideal

$h =$ big height of I

the n-th symbolic power of I is

$$I^{(n)} = \bigcap_{P \in \mathrm{Min}(I)} (I^n R_P \cap R)$$

<u>Facts</u>

1) $I^n \subseteq I^{(n)}$
2) $I^{(n+1)} \subseteq I^{(n)}$
3) If $I = (\text{reg seq})$, then $I^n = I^{(n)}$ for all $n \geq 1$

In general, $I^n \neq I^{(n)}$.

<u>Example</u> $I = (xy, xz, yz) = (x,y) \cap (x,z) \cap (y,z) \subseteq k[x,y,z]$

$$I^{(2)} = (x,y)^2 \cap (x,z)^2 \cap (y,z)^2 \ni \underbrace{xyz}_{\deg 3} \notin I^2$$

Elements in I^2 have degree ≥ 4

so $I^2 \subsetneq I^{(2)}$. But $I^{(3)} \subseteq I^2$

<u>Containment Problem</u> When is $I^{(a)} \subseteq I^b$?

Theorem (Ein-Lazarsfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)

R regular ring, excellent in mixed characteristic
I radical ideal of big height h
then
$$I^{(hn)} \subseteq I^n \quad \text{for all } n \geq 1.$$

When $h=2$, $I^{(4)} \subseteq I^2$.

Question (Huneke, 2000) P prime of ht 2 in a RLR, is $P^{(3)} \subseteq P^2$?

Conjecture (Harbourne, 2008) I radical ideal of big height h
$$I^{(hn-h+1)} \subseteq I^n \quad \text{for all } n \geq 1.$$

Fact In char p, $I^{(hq-h+1)} \subseteq I^{[q]}$ for all $q=p^e$

Example (Dumnicki-Szemberg-Tutaj-Gasińska, 2013, Harbourne-Seceleanu, 2015)
$$I = (x(y^a-z^a), y(z^a-y^a), z(x^a-y^a)) \subseteq k[x,y,z]$$
$a \geq 3$, $h=2$ but $I^{(3)} \not\subseteq I^2$. char $k \neq 2$

Harbourne's Conjecture holds for:
- General points in $\mathbb{P}^2$ (Harbourne-Huneke) and $\mathbb{P}^3$ (Dumnicki)
- Certain cases of (t^a, t^b, t^c) over a field of char $\neq 3$ (G-Huneke-Mukundan)
- Squarefree monomial ideals

Theorem (G-Huneke, 2017) Let R be a regular ring of char p.

1) If R/I is F-pure, then I verifies Harbourne's Conjecture.

2) If R/I is strongly F-regular, then I verifies Harbourne's Conjecture with h replaced by $h-1$, if $h \geq 2$.

When $h=2$, $I^{(n)} = I^n$ for all $n \geq 1$

Theorem (G-Huneke) Let R be a regular ring, R/I F-pure. If $I^{(hk-C)} \subseteq I^n$ for some k, then $I^{(hn-C)} \subseteq I^n$ for all $n \geq k$.

$\Rightarrow$ From now on, all rings have char $p>0$, $q=p^e$

F is the Frobenius map $x \mapsto x^p$

$F_*^e R \equiv R$ with the R-module structure induced by F^e

Theorem (Kunz, 1969) R regular $\Leftrightarrow$ F is flat.

R is F-finite if $F_*(R)$ is a fg R-module.
eg, if $k=k^p$, $R = k[x_1, \dots, x_d]/J$.

Will assume: R F-finite

A ring R is F-pure $\equiv$ F-split if F^e splits.

$$\begin{tikzcd} R \arrow[r, "F^e"] & F_*^e(R) \arrow[l, bend left, dotted] \end{tikzcd}$$

$\exists \phi$ splitting

Fedder's Criterion (1983) Let (R, m) be a regular local ring, I ideal in R

$$R/I \text{ F-pure} \iff (I^{[q]} : I) \not\subseteq m^{[q]} \text{ for all/some/large } q$$

Glassbrenner's Criterion (1996) Let (R, m) be an F-finite RLR, I ideal in R.

R/I strongly F-regular $\iff c(I^{[q]} : I) \not\subseteq m^{[q]}$ for all large q, $c \notin$ min prime of I

Sketch: R/I is F-pure/strongly F-regular $\Rightarrow I^{(a)} \subseteq I^b$

Step 0 Reduce to the local case (R, m)

Step 1 $I^{(a)} \subseteq I^b \iff (I^b : I^{(a)}) = R \iff (I^b : I^{(a)}) \not\subseteq m$

Step 2 If $I \subseteq J$, then $I^{[q]} \subseteq J^{[q]}$.

Show that $(I^b : I^{(a)})^{[q]} \not\subseteq m^{[q]}$ for $q \gg 0$

by showing

$$\underbrace{J_q}_{} \subseteq (I^b : I^{(a)})^{[q]} \quad \text{for } q \gg 0$$

witness to R/I F-pure/strongly F-regular.

tools we need to remove regularity assumption:

- Non-regular version of Feder/Glassbrenner Criteria
- Corresponding containment involving our colon ideal and our witness

Gorenstein version of Fedder/Glassbrenner's Criteria:

Back to the regular case:

- R/I F-pure $\iff \exists \phi \in \mathrm{Hom}_R(F^e_*(R/I), R/I)$ $1 \in \mathrm{im}\,\phi$
- understand $\mathrm{Hom}_R(F^e_*(R/I), R/I)$

$$\begin{tikzcd} \underline{\underline{\text{free}}}\ F^e_*(R) \arrow[r, dashed, "\exists \tilde{\phi}"] \arrow[d] & R \arrow[d, two heads] \\ F^e_*(R/I) \arrow[r, "\phi"'] & R/I \end{tikzcd}$$

Every $\phi \in \mathrm{Hom}_{R/I}(F^e_*(R/I), R/I)$ lifts to $\tilde{\phi}$ in $\mathrm{Hom}_R(F^e_* R, R)$.

- R Gorenstein $\Rightarrow$ $\mathrm{Hom}_R(F^e_*(R), R)$ is cyclic, generated by Φ_e

Every element looks like $\Phi_e(F^e_* x \cdot \text{—})$, $x \in R$

Fedder's Criterion says:

$$R/I \text{ F-pure} \Longrightarrow \Phi_e(F^e_*(\underbrace{I^{[q]} : I}_{\substack{x\in R \text{ corresponding} \\ \text{to a map on } R \\ \text{that descends to } R/I}})) = R$$

R Gorenstein $\Rightarrow$ $F^e_* R$ no longer free/projective

Our criterion (G-Ra-Schwede)
Let R be a Gorenstein F-finite ring of char $p>0$, $\mathrm{pd}(I)<\infty$

1) Every $\phi \in \mathrm{Hom}_{R/I}(F_*^e(R/I), R/I)$ lifts to $\tilde{\phi} \in \mathrm{Hom}_R(F_*^e R, R)$.

2) If R/I is F-pure, then $\Phi_e(F_*^e(I_e(I):I)) = R$ for all e.

3) If R/I is strongly F-regular, then $\Phi_e(F_*^e(c(I_e(I):I))) = R$ for all $c \notin$ min prime of I, $e \gg 0$.

Theorem (G-Ra-Schwede) Let R be an F finite Gorenstein ring, $\mathrm{pd}(I)<\infty$
If R/I is F-pure, then I verifies Harbourne's Conjecture.

Theorem (G-Ra-Schwede) R F-finite Gorenstein ring, $\mathrm{pd}(I^{(n)})<\infty$ for all n.
If R/I is strongly F-regular, then I verifies Harbourne's Conjecture with h replaced by $h-1$.

How about infinite projective dimension?

Theorem (G-Ra-Schwede) R Gorenstein F-finite ring $/ k=k^p$
J = Jacobian ideal (R/k)

If R/I is F-pure, then $J^n I^{(hn-h+1)} \subseteq I^n$ for all $n \geq 1$

If R/I is strongly F-regular, then $J^{2n} I^{((h-1)(n-1)+1)} \subseteq I^n$ for all $n \geq 1$.

If R/I is strongly F-regular and $pd(I) < \infty$, we can show

$$J^n I^{(h-1)(n-1)+1} \subseteq I^n \quad \text{for all } n \geq 1$$

But we expect $I^{((h-1)(n-1)+1)} \subseteq I^n$ might hold, and

$$J^n I^{((h-1)(n-1)+1)} \subseteq I^n$$

for the infinite projective dimension case.