A note on homology:

$H_n : Ch(R) \longrightarrow R\text{-mod}$ is an additive functor

But is <u>not</u> exact!

$$\underset{\text{exact}}{A \to B \to C} \implies \underset{\text{is a } \underline{\text{Complex}}}{H_n(A) \to H_n(B) \to H_n(C)}$$

but <u>not</u> necessarily exact!

Examples:

$$\begin{array}{cccc}
 & 0 \longrightarrow A & \xrightarrow{f} & B \\
2 & 0 & \xrightarrow[0]{} & 0 \\
 & \downarrow & & \downarrow \\
1 & 0 & \xrightarrow[0]{} & \mathbb{Z} \\
 & {\scriptstyle 0}\downarrow & & \| \\
0 & \mathbb{Z} & = & \mathbb{Z} \\
 & \downarrow & & \downarrow \\
-1 & 0 & \xrightarrow[0]{} & 0
\end{array}
\quad \overset{H_0}{\rightsquigarrow} \quad
\begin{array}{c}
0 \to H_0(A) \xrightarrow{H_0(f)} H_0(B) \\
0 \longrightarrow \mathbb{Z} \xrightarrow{0} 0
\end{array}$$

not exact!

$$\begin{array}{c|ccc}
 & B & \xrightarrow{g} & C \longrightarrow 0 \\
2 & 0 & \xrightarrow[0]{} & 0 \\
 & \downarrow & & \downarrow \\
1 & \mathbb{Z} & = & \mathbb{Z} \\
 & \| & & \downarrow \\
0 & \mathbb{Z} & \xrightarrow{0} & 0 \\
 & \downarrow & & \downarrow \\
-1 & 0 & \xrightarrow[0]{} & 0
\end{array}
\quad \overset{H_1}{\rightsquigarrow} \quad
\begin{array}{ccccc}
H_1(B) & \xrightarrow{H_2(g)} & H_1(C) & \longrightarrow & 0 \\
0 & \longrightarrow & \underbrace{\mathbb{Z}}_{\text{not exact}} & \longrightarrow & 0
\end{array}$$

What's really going on:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

$$\rightsquigarrow$$

induces a LES in homology

the connecting homomorphism is not 0 (!)
(in some degrees)

$$H_{n+1}(C) \xrightarrow[0]{} \boxed{H_n(A) \rightarrow H_n(B) \rightarrow H_n(C)} \xrightarrow[0]{} H_{n-1}(A)$$

usually not 0 usually not 0

Previously, on Homological Algebra:

→ P projective $\Leftrightarrow$ $\mathrm{Hom}_R(P,-)$ is exact $\Leftrightarrow$

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dotted, "\exists"'] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

$\Uparrow$

P free

- Every R-module M is a quotient of a (free $\Rightarrow$) projective module

→ E injective $\Leftrightarrow$ $\mathrm{Hom}_R(-,E)$ is exact $\Leftrightarrow$

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & I \arrow[u] \arrow[r] & R \arrow[ul, dotted, "\exists"'] \end{tikzcd}$$

ideal

- Every R-module M embeds into some injective module.

Projective Resolutions

<u>Slogan</u>: Approximate M by projectives

A projective resolution of M is a complex

$$\cdots \to \overset{2}{P_2} \to \overset{1}{P_1} \to \overset{0}{P_0} \to 0 \to \cdots$$

with $H_i(P_\bullet) = 0$ for $i > 0$ and $H_0(P_\bullet) = M$.

$\Leftrightarrow$ an exact complex $\cdots \to \overset{2}{P_2} \to \overset{1}{P_1} \to \overset{0}{P_0} \to M \to 0$

A free resolution of M is a projective resolution where all the P_i are free.

We sometimes write $P_\bullet \twoheadrightarrow M$ or $P_\bullet \to M$

<u>Remark</u> projective resolution

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "0"'] & P_1 \arrow[r] \arrow[d, "0"'] & P_0 \arrow[r] \arrow[d] & 0 \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

quasi iso

Every module has a free resolution:

Step 1 Find a surjection from a free module $P_0 \xrightarrow{\pi_0} M$

Step 2 Look at the kernel of π_0.
Find a free module surjecting onto $K_0 := \ker \pi_0$

$$
\begin{tikzcd}
P_1 \arrow[rr, dashed, "i_0 \circ \pi_1"] \arrow[dr, "\pi_1"'] & & P_0 \arrow[r, "\pi_0"] & M \arrow[r] & 0 \\
& K_0 \arrow[ur, "i_0"'] \arrow[d] & & & \\
0 \arrow[ur] & 0 & & &
\end{tikzcd}
$$

Note $\ker \pi_1 \underset{\substack{\downarrow \\ i_0 \text{ injective}}}{=} \ker (i_0 \circ \pi_1)$

$$
\begin{tikzcd}
0 \arrow[dr] & & 0 & & & & \\
& K_1 \arrow[ur] \arrow[dr] & & & & & \\
P_2 \arrow[ur] \arrow[rr, dashed] & & P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[r] & M \arrow[r] & 0 \\
& & & K_0 \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

Repeat. If $K_i = 0$ at some point, stop

$$\operatorname{pdim}_R(M) := \inf \{ d \mid \exists P_\bullet \to M \text{ with } P_i = 0 \text{ for } i > d \}$$

Minimal Free resolution

Setup:

(R, m) Noetherian local ring

or

$\mathbb{N}$-graded k-algebra, $R_0 = k$, $m = \bigoplus_{n \geq 1} R_n$

$\left(\text{so } R = \frac{k[x_1, \dots, x_d]}{I}, \ I \text{ homogeneous}, \ m = (x_1, \dots, x_d)\right)$

M fg (graded) R-module

Note: We can find a free resolution of M where all the P_i are fg

Recall $\mu(M) :=$ minimal # of generators of $M = \dim_k (M/mM)$

Can find a surjection $R^{\mu(M)} \twoheadrightarrow M = Rf_1 + \cdots + Rf_n$

$(r_1, \dots, r_n) \longmapsto r_1 f_1 + \cdots + r_n f_n$

In the graded case we can take all the maps to be graded

$R(-f_1) \oplus \cdots \oplus R(-f_n) \twoheadrightarrow M = Rf_1 + \cdots + Rf_n$ $\qquad \deg(f_i) = d_i$

is a degree graded R-module map

$(R(-s))_t = R_{t-s}$ so R_0 lives in degree s

A minimal free resolution of M is one where each $P_i \cong R^{n_i}$ has n_i the smallest possible. In the graded case, we also ask for the maps in the resolution to be degree preserving. So

$\mu(P_0) = \mu(M), \ \mu(P_1) = \mu(K_0), \ \mu(P_{i+1}) = \mu(K_i)$

Will show: Minimal free resolutions are unique!

<u>Betti numbers</u> $\beta_i(M) :=$ rank of F_i in a minimal free resolution

<u>Graded betti numbers</u>: $\beta_{ij}(M) :=$ # copies of $R(-j)$ in homological degree i

<u>Betti table</u> has $\beta_{ij}(M)$ in position $(i, i+j)$

<u>Example</u> $R = k[x,y,z]$ $M = R/(xy, xz, yz)$

$$0 \to R^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & x \end{pmatrix}} R^3 \xrightarrow{(xy\ xz\ yz)} R \to M$$

$\beta_1(M) = 3$ $\beta_2(M) = 2$ $\beta_0(M) = 1$

<u>Graded resolution</u>:

$$0 \to R(-3)^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & x \end{pmatrix}} R(-2)^3 \xrightarrow{(xy\ xz\ yz)} R \to M$$

degree 1
$1+2=3$

degree 2

$$\beta(M) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & & \\ 1 & & 3 & 2 \\ 2 & & & \end{array}$$

β_{12} β_{23}

$\beta_{12}(M) = 3$

$\beta_{23}(M) = 2$

<u>Example</u> $R = k[x,y]$ $M = R/(x^2, xy, y^3)$

$$0 \to \begin{matrix} R(-3) \\ \oplus \\ R(-4) \end{matrix} \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y^2 \\ 0 & x \end{pmatrix}} \begin{matrix} R(-2)^2 \\ \oplus \\ R(-3) \end{matrix} \xrightarrow{(x^2\ xy\ y^3)} R \to M$$

Note: $\begin{pmatrix} 0 \\ y^2 \\ x \end{pmatrix}$ lands in deg 2, deg 2, deg 3 so deg $2+2=4$, deg $1+3=4$ ✓

Equivalently, $F_\bullet$ is a minimal free resolution of M if it is a <u>minimal</u> complex, meaning

$$\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to 0$$

$\operatorname{im} \partial_i \subseteq \mathfrak{m} F_{i-1}$.

$\Leftrightarrow$ all the entries in the matrices are in $\mathfrak{m}$.

<u>Proof</u>: Suppose $\operatorname{im} \partial_n \subseteq \mathfrak{m}\, \partial_{n-1}$ for all n, but F_n is the free module on non-minimal generators $f_1, \dots, f_s$ generators for $K_{n-1} := \ker \partial_{n-1}$

$$F_n \longrightarrow M$$
$$e_i \longmapsto f_i$$

$f_1, \dots, f_s$ linearly dependent in $M/\mathfrak{m}M$

$\Rightarrow r_1 f_1 + \dots + r_u f_s = 0$ for some $r_i \in R$, not all in $\mathfrak{m}$

Assume wlog that r_1 is invertible, and multiply by its inverse

$\Rightarrow e_s - r_1 e_1 - \dots - r_{s-1} e_{s-1} \in \ker \partial_n = \operatorname{im} \partial_{n-1} \subseteq \mathfrak{m} F_n$

$\Rightarrow e_1, \dots, e_s$ are linearly dependent! $\lightning$

Suppose $\mu(F_n) = \mu(K_n)$ for all n, where $K_{n-1} = \ker \partial_{n-1}$, but $\operatorname{im} \partial_{n+1} \not\subseteq \mathfrak{m} F_n$ for some n.

$F_n = \bigoplus_{i=1}^{s} R e_i$, $\{\partial_n(e_i)\}$ minimal generating set for K_{n-1}

$\exists\, r_1 e_1 + \dots + r_s e_s \notin \mathfrak{m} F_n$, $\in \ker \partial_n \xRightarrow[r_1 \notin \mathfrak{m}]{\text{WLOG}} e_1 - c_2 e_2 - \dots - c_s e_s \in \ker \partial_n$

$\Rightarrow \partial_n(e_1) = c_2 \partial_n(e_2) + \dots + c_s \partial_n(e_s)$ $\lightning$