Last time   All rings are commutative and have 1

Categories $\mathcal{C}$: • objects ($\mathcal{C}$)
+
• arrows $A \xrightarrow{f} B$ with source A and target B

such that:

• Arrows $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ can be composed to make an arrow $A \xrightarrow{gf} B$
and composition is associative

• For every object A, there is a special arrow $1_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$

$$A \xrightarrow{1} A \xrightarrow{f} B = A \xrightarrow{f} B, \quad B \xrightarrow{g} A \xrightarrow{1_A} A = B \xrightarrow{g} A$$

Notation: the collection of all arrows $A \to B$ is $\mathrm{Hom}_{\mathcal{C}}(A, B)$

Examples

1) Set : objects are all sets, arrows are all functions
2) Grp : groups and group homomorphisms
3) Ab : abelian groups and homomorphisms
4) Ring : rings and ring homomorphisms
5) Top : topological spaces and continuous functions
6) R-mod : R-modules and R-module homomorphisms

also known as: Mod(R)   (vs mod(R) = fg R-modules)

Another example

X partially ordered set

we can regard X as a category

objects $x \in X$

arrows if $x \le y \Rightarrow \mathrm{Hom}_{\mathcal{C}}(x, y)$ singleton

if $x \not\le y \Rightarrow \mathrm{Hom}_{\mathcal{C}}(x,y) = \emptyset$

Special types of arrows:

- $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is an iso $\equiv$ isomorphism if there exists $g \in \mathrm{Hom}_{\mathcal{C}}(B,A)$ such that $gf = 1_A$, $fg = 1_A$

- $f \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ is <u>monic</u> or a <u>monomorphism</u> if

$$A \overset{g_1}{\underset{g_2}{\rightrightarrows}} B \xrightarrow{f} C \qquad fg_1 = fg_2 \Rightarrow g_1 = g_2$$

- $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is epi or an epimorphism if

$$A \xrightarrow{f} B \overset{g_1}{\underset{g_2}{\rightrightarrows}} C \qquad g_1 f = g_2 f \Rightarrow g_1 = g_2$$

Functors $\mathcal{C}, \mathcal{D}$ categories.

A **covariant** functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a mapping that assigns

- to each object C in $\mathcal{C}$, an object $F(C)$ in $\mathcal{D}$
- to each arrow $\begin{array}{c} A \\ f \downarrow \\ B \end{array}$ in $\mathcal{C}$, an arrow $\begin{array}{c} F(A) \\ \downarrow F(f) \\ F(B) \end{array}$ in $\mathcal{D}$

such that:

① $F(1_A) = 1_{F(A)}$

② $F(fg) = F(f)F(g)$ for all composable arrows f, g

A **Contravariant** functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ flips all arrows:

- to each object C in $\mathcal{C}$, an object $F(C)$ in $\mathcal{D}$
- to each arrow $\begin{array}{c} A \\ f \downarrow \\ B \end{array}$ in $\mathcal{C}$, an arrow $\begin{array}{c} F(A) \\ \uparrow F(f) \\ F(B) \end{array}$ in $\mathcal{D}$

such that:

① $F(1_A) = 1_{F(A)}$

② $F(fg) = F(g)\,F(f)$

$$\begin{array}{c} A \\ g \downarrow \\ B \\ f \downarrow \\ C \end{array} \quad \rightsquigarrow \quad \begin{array}{c} F(A) \\ \uparrow F(g) \\ F(B) \\ \uparrow F(f) \\ F(C) \end{array}$$

A contravariant functor $\mathcal{C} \longrightarrow \mathcal{D}$
$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$
a covariant functor $\mathcal{C}^{op} \longrightarrow \mathcal{D}$

where $\mathcal{C}^{op}$ is a category obtained from $\mathcal{C}$ by flipping all the arrows

so an arrow $\begin{array}{c} A \\ f\downarrow \\ B \end{array}$ in $\mathcal{C}$ $\equiv$ an arrow $\begin{array}{c} A \\ \uparrow f \\ B \end{array}$ in $\mathcal{C}^{op}$

Examples of Functors

① **Forgetful functor**

Grp $\longrightarrow$ Set : forgets the group structure

R-mod $\longrightarrow$ Set : forgets the R-mod structure

② **Identity Functor** $\mathcal{C} \longrightarrow \mathcal{C}$

③ **Localization** Fix ring R, multiplicative subset $W \ni 1$

$$W^{-1} : R\text{-mod} \longrightarrow W^{-1}R\text{-mod}$$
$$M \longmapsto W^{-1}M$$

$$\begin{array}{c} M \\ f\downarrow \\ N \end{array} \longmapsto \begin{array}{c} W^{-1}M \\ W^{-1}f \downarrow \\ W^{-1}N \end{array} \quad \begin{array}{c} \ni \frac{m}{w} \\ \downarrow \\ \frac{f(m)}{w} \end{array}$$

Special types of functors: $\mathcal{C}, \mathcal{D}$ locally small. $F: \mathcal{C} \to \mathcal{D}$ is

<u>faithful</u> $\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ all <u>injective</u>

<u>full</u> $\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ all <u>surjective</u>

<u>fully faithful</u> full and faithful

<u>embedding</u> fully faithfully and injective on objects

A subcategory $\mathcal{C}$ of $\mathcal{D}$ is <u>full</u> if for all objects A, B in $\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}}(A,B) = \mathrm{Hom}_{\mathcal{D}}(A,B).$$

Equivalently, the inclusion functor $\mathcal{C} \to \mathcal{D}$ is full

<u>Examples</u>

1. Ab is a full subcategory of Grp
2. The forgetful functor R-mod $\to$ Set is faithful, not full.

$F, G: \mathcal{C} \to \mathcal{D}$ functors

A <u>natural transformation</u> $\eta: F \Rightarrow G$ is a mapping

object $C \in \mathcal{C} \longmapsto \eta_C \in \mathrm{Hom}_{\mathcal{D}}(F(C), G(C))$

such that

for all arrows $\begin{array}{c} A \\ f\downarrow \\ B \end{array}$ in $\mathcal{C}$ $\rightsquigarrow$

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \end{tikzcd}$$

commutes

η is a <u>natural</u> <u>isomorphism</u> if η_A is an iso for all A.

$Nat(F, G) :=$ natural transformations from F to G

can construct a functor category $\mathcal{D}^{\mathcal{C}}$ with:

- objects all functors $\mathcal{C} \to \mathcal{D}$
- arrows all natural transformations between such functors

<u>Example</u> $Id: Grp \to Grp$, $ab: Grp \to Grp$

$$G \longmapsto G^{ab} = G/[G,G]$$

the mapping $\pi: Id \Rightarrow ab$ is a natural transformation:

$$G \longmapsto \begin{array}{c} G \\ \downarrow \pi_G \\ G^{ab} \end{array} \quad \text{quotient map}$$

$$\begin{tikzcd} G \arrow[r, "\pi_G"] \arrow[d, "f"'] & G^{ab} \arrow[d, dashed, "f^{ab}"] \\ H \arrow[r, "\pi_H"'] & H^{ab} \end{tikzcd}$$

for all $G \xrightarrow{f} H$ group homomorphism.

Hom functors $\quad \mathcal{C}$ locally small category

the covariant functor

$$\mathrm{Hom}_{\mathcal{C}}(A,-) : \mathcal{C} \longrightarrow \mathrm{Set}$$
$$B \longmapsto \mathrm{Hom}_{\mathcal{C}}(A,B)$$

$$\begin{tikzcd} B \arrow[d, "f"'] & \longmapsto & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d] & \ni & g \arrow[d, mapsto] \\ C & & \mathrm{Hom}_{\mathcal{C}}(A,C) & \ni & f\circ g \end{tikzcd}$$

$$\mathrm{Hom}_{\mathcal{C}}(A,f) =: f^*$$

$$f^*(g) = fg \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[dr, dashed] & B \arrow[d, "f"] \\ & C \end{tikzcd}$$

the contravariant

$$\mathrm{Hom}_{\mathcal{C}}(-,B) : \mathcal{C} \longrightarrow \mathrm{Set}$$
$$A \longmapsto \mathrm{Hom}_{\mathcal{C}}(A,B)$$

$$\begin{tikzcd} A \arrow[d, "f"'] & \longmapsto & \mathrm{Hom}_{\mathcal{C}}(A,B) & \ni & \\ C & & \mathrm{Hom}_{\mathcal{C}}(C,B) \arrow[u] & \ni & g \arrow[u, mapsto] \end{tikzcd}$$

$$\mathrm{Hom}_{\mathcal{C}}(f,B) =: f_*$$

$$f_*(g) = gf \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, dashed] & C \arrow[dl, "g"] \\ B & \end{tikzcd}$$

Yoneda Lemma $\mathscr{C}$ locally small category

$F : \mathscr{C} \longrightarrow \mathrm{Set}$ covariant functor

Fix an object A in $\mathscr{C}$

there is a bijection

$$\mathrm{Nat}\big(\mathrm{Hom}_{\mathscr{C}}(A,-), F\big) \xrightarrow[\gamma]{\cong} F(A)$$

this correspondance is natural in both A and F.

Proof $\varphi : \mathrm{Hom}_{\mathscr{C}}(A,-) \Rightarrow F$

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(A,A) &\xrightarrow{\varphi_A} F(A) \\ 1_A &\longmapsto \varphi_A(1_A) = u \end{aligned}$$

Set $\gamma(\varphi) := \varphi_A(1_A) = u$. Why is γ injective/surjective?

Fix an arrow $A \xrightarrow{f} X$. φ natural transformation gives

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(A,A) \arrow[rrr, "{\mathrm{Hom}_{\mathscr{C}}(A,f)}"] \arrow[ddd, "\varphi_A"'] & \ni 1_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d] & \mathrm{Hom}_{\mathscr{C}}(A,X) \arrow[ddd, "\varphi_X"] \\
& u \arrow[r, mapsto] & \boxed{} & \\
& & & \\
F(A) \arrow[rrr, "F(f)"'] & & & F(X)
\end{tikzcd}$$

$$\boxed{} = \varphi_X \circ \mathrm{Hom}_{\mathscr{C}}(A,f)(1_A) = \varphi_X(f)$$

$$= F(f) \circ \varphi_A(1_A) = F(f)(u)$$

so:

- φ is <u>completely determined</u> by $\varphi_A(1_A)$:

$$\varphi_X(f) = F(f)(\varphi_A(1_A))$$

so the values of φ are set for any X, f from $\varphi_A(1_A)$

$\therefore$ different choices of $u = \varphi_A(1_A) \Rightarrow$ different φ

$\Rightarrow \gamma$ is injective

$\therefore$ given any $u \in F(A)$, setting $\varphi_X(f) = F(f)(u)$ gives a natural transformation φ with $\gamma(\varphi) = u$.

$\Rightarrow \gamma$ is surjective