Symbolic powers and differential operators

(joint with Alessandro De Stefani and Jack Jeffries)

Algebra		Geometry
$R = \mathbb{C}[x_1, \dots, x_d]$	$\longleftrightarrow$	$\mathbb{P}^n_{\mathbb{C}}$
$I \subseteq R$ ideal	$\longleftrightarrow$	$X = \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in I\}$
Q prime ideal	$\longleftrightarrow$	X is an irreducible variety
m maximal ideal	$\longleftrightarrow$	$X = \{x\}$

the n-th symbolic power of Q is

$$Q^{(n)} = \{f \in R : f \text{ vanishes up to order } n \text{ along } X\}$$

What does that mean, precisely?

1) m maximal ideal $\iff X = \{x\}$

$$m^{(n)} = m^n = (f_1 \cdots f_n : f_i \in m)$$

Given a general ideal, the n-th power of I is given by $I^n = (f_1 \cdots f_n : f_i \in I)$

2) a prime ideal

$$Q = \bigcap_{x \in X} m_x = \bigcap_{\substack{m \supseteq Q \\ m \text{ maximal}}} m$$

$$Q^{(n)} = \bigcap_{x \in X} m_x^n = \bigcap_{\substack{m \supseteq Q \\ m \text{ maximal}}} m^n$$

$\boxed{m_x := \text{maximal ideal corresponding to the point } x}$

But when X is infinite, we will be taking an infinite intersection
Can we write this in a finite way?
Maybe we need to describe $Q^{(n)}$ in a more algebraic manner.

Vanishing at a point is a local property — in Commutative Algebra, "locally" means localizing

Q defines X, so all we care about is what happens in $R_Q = \{ \frac{f}{g} : g \notin Q \}$

$$Q^{(n)} = Q^n R_Q \cap R = \{ f : gf \in Q^n, g \notin Q \}$$
$= $ elements that "locally" live in Q^n

Warning! This not the same as Q^n (!)

Example $P = \ker(\mathbb{C}[x,y,z] \to \mathbb{C}[t^3, t^4, t^5])$

$$P = (\underbrace{x^3 - yz}_{\substack{f \\ \deg 9}}, \underbrace{y^2 - xz}_{\substack{g \\ \deg 8}}, \underbrace{z^2 - x^2 y}_{\substack{h \\ \deg 10}})$$

$\deg x = 3$
$\deg y = 4$
$\deg z = 5$

$P^2 = (f^2, g^2, h^2, fg, gh, fh)$ $\quad$ deg of an element in $P^2 \geq 16$

But $\underbrace{f^2 - gh}_{\deg 18} = \underbrace{x}_{\deg 3} q \in P^2$, $x \notin P$ $\Rightarrow$ $\underbrace{q}_{\deg 18-3=15} \in P^{(2)}$

$\Rightarrow q \notin P^2$, $q \in P^{(2)}$ $\quad$ So $P^2 \subsetneq P^{(2)}$

$$Q^{(n)} = Q^n R_Q \cap R = \bigcap_{\substack{m \supseteq Q \\ m \text{ maximal}}} m^n$$

$\downarrow$

Classical Zariski-Nagata

this can also be described via differential operators:

$$D_{R|\mathbb{C}} = \mathbb{C}\langle x_1, \dots, x_n, \partial/\partial x_1, \dots, \partial/\partial x_n \rangle \quad \text{ring of differential operators}$$

more generally, if R is an A-algebra, we define $D_{R|A} \subseteq \mathrm{Hom}_A(R,R)$ by:

$$D^0_{R|A} = \mathrm{Hom}_R(R,R) = \{\cdot r,\ r \in R\} \quad (\cdot r = \text{multiplication by } r)$$

$$D^n_{R|A} = \{\delta \in \mathrm{Hom}_A(R,R) : [\delta, r] = \delta \cdot r - r \cdot \delta \in D^{n-1}_{R|A} \ \forall r \in D^0_{R|A}\}$$

the n-th differential power of the prime ideal Q is given by

$$Q^{\langle n \rangle} = \{f \in R : \delta(f) \in Q \text{ for all } \delta \in D^{n-1}_{R|A}\}$$

<u>theorem</u> (Zariski-Nagata for differential operators)

$Q^{\langle n \rangle} = Q^n \quad \forall n \geq 1$, as long as A is a perfect field.

this has been known by experts for a while.
For a proof of the statement as stated, see a survey on symbolic powers by Dao, De Stefani, —, Huneke, Núñez Betancourt.
We will follow the steps in this proof later in the talk.

Can we extend this result to polynomial rings over things that aren't fields?

$R = A[x_1, \dots, x_d]$, $A = \mathbb{Z}, \mathbb{Z}_p$, a DVR with uniformizer p (prime integer)

<u>Example</u> $R = \mathbb{Z}[x]$, $Q = (2)$ principal ideal $\Rightarrow$ $Q^n = Q^{(n)}$ $\forall n \geq 1$.

So for all $n \geq 1$, $Q^{(n)} = Q^n = (2^n)$.
However, given any $\delta \in D_{R|\mathbb{Z}}$ (of any order!)

$$\delta(2) = \underbrace{2 \cdot \delta(1)}_{\delta \text{ is } \mathbb{Z}\text{-linear}} \in (2) = Q$$

So $2 \in Q^{\langle n \rangle}$ for any $n \geq 1$. In fact, $Q^{\langle n \rangle} = (2)$ $\forall n \geq 1$.

However, if Q is a prime that contains no integers, then everything is ok:

<u>theorem</u> (De Stefani, –, Jeffries)

If $Q \cap A = 0$, then $Q^{\langle n \rangle} = Q^n$ $\forall n \geq 1$.

And when $Q \cap A \neq 0$, say it contains a prime integer $p \in \mathbb{Z}$, what can we do?

We need to find a "differential operator"-type thing that decreases p-adic order:

$p^n \in (p)^{(n)}$ $\rightsquigarrow$ need $\delta(p^n) \in (p)$ for all δ of order $n-1$
or
$\delta(p^n) \in (p^{n-1})$ for δ of order 1

$p^n \notin Q^{(n+1)}$ $\rightsquigarrow$ need $\delta(p^n) \notin Q$ for some δ of order n
or
$\delta(p^n) \notin (p^n)$ for some δ of order 1.

Definition (Buium) Let S be a ring, $p \in S$ be an integer with $ps = 0 \Rightarrow s = 0$

A p-derivation on S is a map $\delta : S \to S$ such that:

1) $\delta(1) = 0$

2) $\delta(xy) = \delta(x) y^p + x^p \delta(y) + pxy \quad \forall x, y \in S$

3) $\delta(x+y) = \delta(x) + \delta(y) + C_p(x,y) \quad \forall x, y \in S$

where $C_p(x,y) = \dfrac{x^p + y^p - (x+y)^p}{p}$. (very much $\neq 0$ in char 0)

Note: Having a p-derivation δ is the same as having a lift ϕ of the Frobenius map $S/pS \to S/pS$:

Note that finding a lift of Frobenius ϕ is choosing a map δ that makes

$$\phi(x) = x^p + p\delta(x) \text{ a ring homomorphism.}$$

Given a p-derivation δ, $\phi(x) = x^p + p\delta(x)$ is a lift of Frobenius

Given a lift ϕ, $\delta(x) = \dfrac{\phi(x) - x^p}{p}$ is a p-derivation.

Note p-derivations don't always exist, but they do over the rings we are talking about.

When $R = \mathbb{Z}$, there is only one p-derivation:
the Fermat difference operator $\quad \delta(n) = \dfrac{n - n^p}{p}$.

Good news: p-derivations do decrease p-adic order:

$$\underbrace{\delta(p^n)}_{p\text{-adic order } n} = p^{n-1} - p^{np-1} \notin (p^n) \quad \to \quad p\text{-adic order } n-1$$

Ex: $R = \mathbb{Z}[x]$, $Q = (2)$ $\qquad \delta(2) = 1 - 2^{2-1} = -1 \notin (2) = Q$, $2 \notin Q^{(2)}$

the n-th mixed differential power of Q is

$$Q^{\langle n \rangle_{mix}} = \{ f : \delta^s \circ \partial (f) \in Q, \ \partial \in D^t_{R|A}, \ s+t \leq n-1 \}$$
$$= \bigcap_{a+b \leq n+1} \left(I^{\langle a \rangle_\delta} \right)^{\langle b \rangle}$$

A priori, this definition depends on the choice of δ – but it turns out it doesn't.

theorem (De Stefani, –, Jeffries)

If $Q \cap A \neq 0$, then $Q^{(n)} = Q^{\langle n \rangle_{mix}} = \bigcap_{\substack{m \supseteq Q \\ m \text{ max}}} m^n$

More generally, this holds for any smooth A-algebra R such that
- R has a p-derivation
- $A/pA \to R/pR$ is a separable extension.

Sketch of proof : (similar to Dao, DeStefani, –, Huneke, Núñez Betancourt)

- $Q^{\langle n \rangle_{mix}}$ is a Q-primary ideal
- $Q^n \subseteq Q^{\langle n \rangle_{mix}}$
- $\Rightarrow \quad Q^{(n)} \subseteq Q^{\langle n \rangle_{mix}}$.
- $Q^{\langle n \rangle_{mix}} R_Q = (QR_Q)^{\langle n \rangle_{mix}}$
- $m^{\langle n \rangle_{mix}} = m^n$ over a local ring (R, m)
- Since $Q^{\langle n \rangle_{mix}}$ and $Q^{(n)}$ are Q-primary, it is enough to show equality after localizing at Q.