Symbolic Powers

UC Riverside Colloquium 21/02/2019

Algebra $\longleftrightarrow$ Geometry

$R = \mathbb{C}[x_1, \dots, x_d] \longmapsto \mathbb{C}^d$

$\cup\!|$

$\{f_1, \dots, f_n\} \longmapsto X = \{x \in \mathbb{C}^d : f_i(x) = 0 \ \forall i\}$ variety

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$(f_1, \dots, f_n) \longmapsto \{x \in \mathbb{C}^d : (g_1 f_1 + \dots + g_n f_n)(x) = 0 \ \ \forall g_i \in R\}$

$\|$

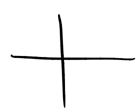
$\{g_1 f_1 + \dots + g_n f_n : g_i \in R\}$

ideal generated by $f_1, \dots, f_n$

$\{f \in R : f(x) = 0 \ \forall x \in X\} \longleftarrow X \subseteq \mathbb{C}^d$ variety

<u>Examples</u> Algebra Geometry

$(x^2 - y) \longmapsto$ 

$(xy) \longmapsto$ 

$(x^2 y) \longmapsto$

An ideal I is radical if $f^n \in I \Rightarrow f \in I$

An ideal P is prime if $fg \in P \Rightarrow f \in P$ or $g \in P$

Hilbert's Nullstellensatz

Algebra	$\longleftrightarrow$	Geometry
$\mathbb{C}[x_1, \dots, x_n]$	$\longleftrightarrow$	$\mathbb{C}^d$
radical ideals	$\longleftrightarrow$	varieties
prime ideals	$\longleftrightarrow$	irreducible varieties
maximal ideals $\mathfrak{m} = (x_1 - a_1, \dots, x_d - a_d)$	$\longleftrightarrow$	points in $\mathbb{C}^d$ $\{(a_1, \dots, a_d)\}$

Example curve (t^3, t^4, t^5) in $\mathbb{C}^3$

$\equiv$ ideal $P = (\underbrace{x^3 - yz}_{f}, \underbrace{y^2 - xz}_{g}, \underbrace{z^2 - x^2 y}_{h})$ in $\mathbb{C}[x, y, z]$

Geometry of this curve $\Leftrightarrow$ algebra of this ideal

Question How do we measure vanishing?

$I \subseteq R$ ideal $\leftrightarrow$ $X \subseteq \mathbb{C}^d$ variety

we say $f \in R$ vanishes up to order n along X / on I if

- the power series expansion of f around each $x \in X$ has no terms of order $< n$
- $g \dfrac{\partial^{a_1 + \dots + a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} (f) \in I$ for all $a_1 + \dots + a_d \leq n-1$, $g \in R$
- $f \in \bigcap_{x \in X} \mathfrak{m}_x^n = \bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \text{ max}}} \mathfrak{m}^n$ $\quad \mathfrak{m}_x$ = maximal ideal corresponding to $x \in X$

the n-th power of I is the ideal I^n generated by $f_1 \cdots f_n$, $f_i \in I$

<u>Working Example</u> Curve (t^3, t^4, t^5) in $\mathbb{C}^3$

$$P = (\underbrace{x^3 - yz}_{f}, \underbrace{y^2 - xz}_{g}, \underbrace{z^2 - x^2 y}_{h}) \text{ in } \mathbb{C}[x,y,z]$$

$$P^2 = (f^2, g^2, h^2, fg, fh, gh)$$

all vanish up to order 2 on our curve

But! there are more polynomials that vanish up to order 2

Vanishing is a local condition — localize!

P prime

R_P is a ring with elements $\frac{f}{g}$ where $f, g \in R$, $g \notin P$

the n-th symbolic power of P

"elements that live in P^n locally"

$$P^{(n)} = \{ f \in R : sf \in P^n, s \notin P \}$$
$$= \{ f \in R : \frac{sf}{s} = \frac{f}{1} \in P^n \text{ in } R_P \}$$

<u>theorem</u> (Zariski–Nagata)

P prime ideal in $R = \mathbb{C}[x_1, \ldots, x_d]$

$$P^{(n)} = \{ f \in R : f \text{ vanishes up to order } n \text{ at } P \}$$

Working Example curve (t^3, t^4, t^5) in $\mathbb{C}^3$

$$\underline{P} = (\underbrace{x^3 - yz}_{\substack{f \\ \deg 9}},\ \underbrace{y^2 - xz}_{\substack{g \\ \deg 8}},\ \underbrace{z^2 - x^2 y}_{\substack{h \\ \deg 10}}) \text{ in } \mathbb{C}[x, y, z]$$

$\deg x = 3$, $\deg y = 4$, $\deg z = 5$

$$\underbrace{f^2 - gh}_{\substack{\in \underline{P}^2 \\ \deg 18}} = \boxed{\underbrace{x}_{\substack{\notin \underline{P} \\ \deg 3}}}\ \underbrace{q}_{\deg 15} \implies q \in \underline{P}^{(2)},\ q \notin \underline{P}^2$$

$$\therefore \underline{P}^2 \subsetneq \underline{P}^{(2)}$$

Elements in $\underline{P}^2$ have degree ≥ 16

Can we do this over other rings?

$k[x_1, \ldots, x_d]$

k field

- $\bigcap_{\substack{m \supseteq \underline{P} \\ m \text{ max}}} m^n$

  $\parallel$

- $\underline{P}^{(n)}$

  $\parallel$

- Grothendieck's k-linear differential operators

$\mathbb{Z}[x_1, \ldots, x_d]$

eg, $\underline{P} = (3, x)$ in $\mathbb{Z}[x]$

- $\bigcap_{\substack{m \supseteq \underline{P} \\ m \text{ max}}} m^n$

  $\parallel$

- $\underline{P}^{(n)}$

- What does "d/dp" mean?

theorem (De Stefani - G - Jeffries)

Using Grothendieck's differential operator's and Buium's p-derivations, we can describe vanishing up to order n, and this coincides with symbolic powers.

Questions

1) When is $I^n = I^{(n)}$?
2) Give generators for $I^{(n)}$.
3) What degrees does $I^{(n)}$ live in?
4) Compare I^n to $I^{(n)}$.

Containment Problem When is $I^{(a)} \subseteq I^b$?

theorem (Ein - Lazersfeld - Smith 2001, Hochster - Huneke 2002, Ma - Schwede 2017)

$R = k[x_1, \dots, x_d]$, k field or $\mathbb{Z}$

P prime of codim h

then $P^{(hn)} \subseteq P^n$ for all $n \geq 1$

Working Example curve (t^3, t^4, t^5) has codim 2

$P^{(2n)} \subseteq P^n$, in particular $P^{(4)} \subseteq P^2$.

Question (Huneke, 2000) P prime of codim 2. Is $P^{(3)} \subseteq P^2$?

theorem (-) $P \longleftrightarrow (t^a, t^b, t^c)$ in char $\neq 3$. then $P^{(3)} \subseteq P^2$.

Conjecture (Harbourne, 2008) $I^{(hn-h+1)} \subseteq I^n$ for all $n \geq 1$

More generally, Harbourne's Conjecture is about radical ideals of big height h
Unfortunately, the Conjecture is false in this full generality
However, the Conjecture holds under mild conditions on the ideal

The Conjecture holds for general points in $\mathbb{P}^2$ (Harbourne - Huneke) and $\mathbb{P}^3$ (Dumnicki)

theorem (G - Huneke - Mukundan) Harbourne's Conjecture holds for (t^a, t^b, t^c)

theorem (G - Huneke) $R = k[x_1, \dots, x_d]$, char $k = p$, I radical ideal

R/I F-pure $\Rightarrow$ I verifies Harbourne's Conjecture

R/I strongly F-regular $\Rightarrow$ I verifies Harbourne's Conjecture with $h-1$ instead of h

(G - Ma - Schwede)
More general version when R is an F-finite Gorenstein ring of char p.