Symbolic powers and the (stable) containment problem
Neuchâtel – Fribourg – Bern Seminar 15/05/2020

R regular (great working example: $R = k[x, y, z]$)
$I = \sqrt{I} = P_1 \cap \dots \cap P_k$
$h =$ big height of $I = \max_i \{ \operatorname{ht} P_i \}$

The n-th symbolic power of I:

$$I^{(n)} = \bigcap_i \left(I^n R_{P_i} \cap R \right)$$
$$= \{ r \in R : sr \in I^n, \text{ some } s \notin \cup P_i \}$$

theorem (Zariski–Nagata) $R = k[x_1, \dots, x_d]$, k perfect field

$$I^{(n)} = \{ f \in R : f \text{ vanishes to order } n \text{ along the variety defined by } I \}$$
$$= \left\{ f \in R : \frac{\partial^{a_1 + \dots + a_d} f}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} \in I \text{ for all } a_1 + \dots + a_d \leq n-1 \right\}$$

Facts
1) $I^{(n+1)} \subseteq I^{(n)}$
2) $I^n \subseteq I^{(n)}$
3) If $I = $ (regular sequence), $I^n = I^{(n)}$ for all n

Example $P = \ker\left(k[\overset{3}{x}, \overset{4}{y}, \overset{5}{z}] \to k[t^3, t^4, t^5]\right)$ prime

$$= (\underbrace{x^3 - yz}_{\substack{f \\ \deg 9}}, \underbrace{y^2 - xz}_{\substack{g \\ \deg 8}}, \underbrace{z^2 - x^2y}_{\substack{h \\ \deg 10}})$$

Claim: $P^{(2)} \supsetneq P^2$

$$P^2 \ni \underbrace{f^2 - gh}_{\deg 18} = \underbrace{x}_{\notin P} q \implies q \in P^{(2)}$$

$$\deg 18 = \deg 3 + \deg 15 \longrightarrow q \notin P^2 \leftarrow \deg \geq 16$$

$$P^{(3)} \subseteq P^{(2)} \subsetneq P^2$$

Note the ideals P defining (t^a, t^b, t^c) have very interesting symbolic powers. Eg

$\bigoplus_{n \geq 0} P^{(n)} t^n \subseteq R[t]$ is not always fg

(sometimes it is, and generated in degree $\leq 1, 2, 3, 4, \ldots$)

Big Questions

1) Give generators for $I^{(n)}$
2) When is $I^{(n)} = I^n$?
3) What degrees does $I^{(n)}$ live in?

__Containment Problem__ When is $I^{(a)} \subseteq I^b$?

__Theorem__ (Ein-Lazarsfeld-Smith 2001, Hochster-Huneke 2002, Ma-Schwede 2018)

$$I^{(hn)} \subseteq I^n \text{ for all } n \geq 1$$

__Example__ Our prime P defining (t^3, t^4, t^5) has $h=2$ so $P^{(4)} \subseteq P^2$. But actually $P^{(3)} \subseteq P^2$.

__Question__ P prime of height 2 in a RLR. Is $P^{(3)} \subseteq P^2$?

__Conjecture__ (Harbourne) $I^{(hn-h+1)} \subseteq I^n$ for $n \geq 1$

__Fact__ In char p, $I^{(hq-h+1)} \subseteq I^{[q]} \subseteq I^q$ for $q = p^e$.

Counterexample (Dumnicki – Szemberg – Tutaj-Gasińska, 2013
Harbourne – Seceleanu, 2015)

$$I = (x(y^n - z^n), y(z^n - x^n), z(x^n - y^n)) \subseteq k[x,y,z]$$

char $k \neq 2$, $n \geqslant 3$ (n^2+3 points in $\mathbb{P}^2$) $h=2$

$$I^{(3)} \not\subseteq I^2$$

But actually, $I^{(2n-1)} \subseteq I^n$ for $n \geqslant 3$.

Harbourne's Conjecture is satisfied when:

- I general points in $\mathbb{P}^2$ (Bocci–Harbourne) and $\mathbb{P}^3$ (Dumnicki)
- I squarefree monomial ideal

$\cap$

- R/I F-pure / of dense F-pure type (G-Huneke)

eg:

F-reg $\begin{cases} \cdot\ I = I_t(X), \quad X\ n\times m \text{ generic matrix} \\ \cdot\ R/I \cong \text{Veronese} \\ \cdot\ R/I \text{ ring of invariants of a linearly reductive group} \end{cases}$

R/I F-regular $\Rightarrow$ can replace h by $h-1$

When $h=2$, get $I^{(n)} = I^n$ for all $n \geqslant 1$

<u>Stable Harbourne Conjecture</u> $I^{(hn-h+1)} \subseteq I^n$ for $n \gg 0$.

<u>Question</u> If $I^{(hk-h+1)} \subseteq I^k$ for some k,
does that imply $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$?

<u>Note</u> If yes, then Harbourne stable holds in char p.

<u>theorem</u> (G) If $I^{(hk-h)} \subseteq I^k$ for some k
then $I^{(hn-h)} \subseteq I^n$ for all $n \gg 0$

<u>Note</u> No known examples that fail the hypothesis.

<u>Resurgence</u> (Bocci - Harbourne)

$$\rho(I) = \sup\left\{\frac{a}{b} : I^{(a)} \not\subseteq I^b\right\}$$

$$1 \leq \rho(I) \leq h$$

<u>Note</u>: If $\rho(I) < h$, then Harbourne stable holds.

Why?

$$\frac{hn-c}{n} > \rho(I) \implies I^{(hn-c)} \subseteq I^n$$

$$\Updownarrow$$

$$n > \frac{c}{h-\rho} \neq 0$$

Assume $h \geq 2$.

<u>Question</u> Can $\rho(I) = h$?

I has <u>expected</u> resurgence if $\rho(I) < h$.

<u>Theorem</u> (G–Huneke–Mukundan)

(R,m) RLR / polynomial ring with I homogeneous

Assume $I^{(n)} = I^n : m^\infty$.

(eg, $I = P$ prime of height $d-1$, or ideal of points)

If $I^{(hn-h+1)} \subseteq m I^n$ for some n, then $\rho(I) < h$.

Applications :

1) char 0 $\qquad I^{(n)} = I^n : m^\infty$

I Homogeneous ideal generated in degree $a < h$

2) P defining (t^a, t^b, t^c)

$$P^{(3)} \subseteq m P^2 \qquad \text{(Knödel–Schenzel–Zonzarov)}$$

3) R/I Gorenstein, $I^{(n)} = I^n : m^\infty$