Last time — Localization of a ring R at a multiplicative set W

$$W^{-1}R := \left\{ \frac{x}{w} \mid x \in R, w \in W \right\} / \sim \qquad \text{where}$$

$$\frac{x}{w} \sim \frac{x'}{w'} \iff u(xw' - x'w) = 0 \text{ for some } u \in W$$

canonical map: $\begin{array}{ccc} R & \longrightarrow & W^{-1}R \\ x & \longmapsto & \frac{x}{1} \end{array}$ (not necessarily injective)

eg, $R = \dfrac{k[x,y]}{(xy)}$ In $R_{(x)}$, $\dfrac{x}{1} = \underset{y \notin (x)}{\dfrac{yx}{y}} = \dfrac{0}{y} = \dfrac{0}{1}$

Localization at a prime — R ring, P prime

Localization of R at P is a local ring $(W = R \setminus P)$

(R_P, P_P)

with residue field $k(p) = R_P / P_P \cong (R/P)_P$

will show:

Ideals in R_P		Ideals in R
I_P	$\longleftrightarrow$	$I \subseteq P$
primes Q_P	$\longleftrightarrow$	primes $Q \subseteq P$

many properties of rings/ideals/modules are local, meaning they can be checked at all prime ideals (by localizing)

Eg, containments are local: $I \subseteq J \iff I_P \subseteq J_P \ \forall P \in \operatorname{Spec} R$

Def M R-mod

$W \subseteq R$ multiplicative set

$$W^{-1}M := \left\{ \frac{m}{w} \;\middle|\; m \in M,\ w \in W \right\} / \sim$$

$$\frac{m}{w} \sim \frac{m'}{w'} \quad \text{if} \quad u(mw' - m'w) = 0 \text{ for some } u \in W$$

this is an R_W-module via

$$\frac{m}{w} + \frac{m'}{w'} = \frac{mw' + m'w}{ww'} \qquad \frac{r}{w} \cdot \frac{m}{w'} = \frac{rm}{ww'}$$

Remark $M \xrightarrow{\alpha} N$ R-mod homomorphism

$\Longrightarrow$ induces R_W-mod homomorphism

$$W^{-1}M \xrightarrow{W^{-1}\alpha} W^{-1}N$$
$$\frac{m}{w} \longmapsto \frac{\alpha(m)}{w}$$

Lemma $\frac{m}{w} = 0 \in W^{-1}M \iff vm = 0$ for some $v \in W$

$\iff \operatorname{ann}_R(m) \cap W \neq \emptyset$

Proof $\frac{m}{w} = \frac{0}{1} \iff v(m \cdot 1 - 0 \cdot w) = 0$
for some $v \in W$

$\iff vm = 0$ for some $v \in W$

$\iff \operatorname{ann}(m) \cap W \neq \emptyset$

Remark $M \xrightarrow{\alpha} N$ injective $\Rightarrow$ $W^{-1}M \xrightarrow{W^{-1}\alpha} W^{-1}N$ injective

$$\frac{\alpha(m)}{w} = 0 \Rightarrow u\alpha(m) = 0 \text{ for some } u \in W$$
$$\Leftrightarrow \alpha(um) = 0 \text{ for some } u \in W$$
$$\alpha \text{ injective} \Rightarrow um = 0 \text{ for some } u \in W$$
$$\Leftrightarrow \frac{m}{w} = 0$$

Lemma $N_1, \dots, N_t \subseteq M$ R-mods, $W \subseteq R$ multiplicative set

$$W^{-1}(N_1 \cap \cdots \cap N_t) = W^{-1}N_1 \cap \cdots \cap W^{-1}N_t \subseteq W^{-1}M$$

Proof $\subseteq$ ✓ $\quad$ w

find common denominator $\frac{n_1}{w_1} = \frac{n_2}{w_2} = \cdots = \frac{n_t}{w_t}$

thm Localization is exact:

$0 \to A \to B \to C \to 0$ ses of R-mods

$\Rightarrow$ $0 \to W^{-1}A \to W^{-1}B \to W^{-1}C \to 0$ ses of R-mods

Corollary $W^{-1}(M/N) \cong W^{-1}M / W^{-1}N$

Prop $W \subseteq R$ multiplicatively closed set
$I \subseteq R$ ideal $\qquad p \in \operatorname{Spec}(R)$

1) $W^{-1}I \cap R = \{ r \in R \mid wr \in I \text{ for some } w \in W \}$
2) $W \cap p = \emptyset \Rightarrow W^{-1}p = p\,W^{-1}R$ is prime
3) $\operatorname{Spec}(W^{-1}R) \longrightarrow \operatorname{Spec}(R)$ is injective
image $= \{ p \in \operatorname{Spec}(R) \mid p \cap W = \emptyset \}$

<u>Proof</u> 1) $W^{-1}(R/I) \cong \frac{W^{-1}R}{W^{-1}I} \Rightarrow \ker\left(R \longrightarrow W^{-1}(R/I)\right) = R \cap W^{-1}I$

2) $W \cap p = \emptyset \Rightarrow$ no element in W kills $\bar{1} \in R/p$

$\Rightarrow W^{-1}(R/p) \neq 0$, so a domain (localization of a domain is a domain)

$\Rightarrow W^{-1}R/W^{-1}p \neq 0$ domain $\Rightarrow W^{-1}p$ prime

3) <u>Claim</u>

$$\begin{array}{ccc} \operatorname{Spec}(W^{-1}R) & & \{p \in \operatorname{Spec}(R) \mid p \cap W = \emptyset\} \\ q & \longmapsto & q \cap R \\ W^{-1}p & \longleftarrow\!\shortmid & p \quad \text{(ok by 2)} \end{array}$$

are inverse maps.

- $\mathfrak{a} \subseteq W^{-1}R$ ideal: $\mathfrak{a} = \left(\left\{\frac{a_i}{w_i}\right\}\right) = \left(\left\{\frac{a_i}{1}\right\}\right)$

unit multiple of $\frac{a_i}{w_i}$

$\therefore \mathfrak{a} \cap R = (\{a_i\}) \Rightarrow (\mathfrak{a} \cap R)W^{-1}R = \mathfrak{a}$

$W \cap p = \emptyset \Rightarrow W^{-1}p \cap R = \{x \in R \mid wx \in p \text{ for some } w \in W\}$

$\qquad = \{x \in R \mid x \in p\} = p$

p prime

<u>Corollary</u> $R \longrightarrow R_p$ induces the following map on Spectra:

$\{q \in \operatorname{Spec}(R) \mid q \subseteq p\} \hookrightarrow \operatorname{Spec}(R)$

<u>Determinantal Trick</u> $A \in M_{n\times n}(R)$, $v \in R^{\oplus n}$, $x \in R$

If $Av = xv$, then $\det(x I_{n\times n} - A)\, v = 0$

<u>Nakayama's Lemma</u> (Nakayama - Azumaya - Krull)

<u>NAK1</u> R ring
I ideal
M fg R-module

If $IM = M$, then:

1. there is $r \in 1 + I$ st $rM = 0$
2. there is $a \in I$ st $am = m$ for all M

<u>Proof</u> $M = Rm_1 + \cdots + Rm_s$

1. $m_i = a_{i1} m_1 + \cdots + a_{is} m_s \in IM$ $\qquad a_{ij} \in I$

$A = [a_{ij}]$, $v = \begin{pmatrix} m_1 \\ \vdots \\ m_s \end{pmatrix}$

$$Av = v \Rightarrow \underbrace{\det(1 \cdot I_{s\times s} - A)}_{\in R}\, v = 0$$

$$\det(I_{s\times s} - \underset{0 \bmod I}{A}) = \det(I_{s\times s}) = 1 \pmod{I}$$

so $r = \det(I_{s\times s} - A) \in 1 + I$ kills M.

2. take $a = 1 - r \in I$. For all $m \in M$:

$$am = (1-r)m = m - \underbrace{rm}_{=0} = m$$

NAK 2 $(R, \mathfrak{m})$ local ring
M fg R-mod

If $M = \mathfrak{m}M$, then $M = 0$

Proof $M = \mathfrak{m}M \Rightarrow rM = 0$ for some $r \in \underbrace{1+\mathfrak{m}}_{\notin \mathfrak{m} \Rightarrow \text{unit}}$

$\Rightarrow \quad 1 \cdot M = 0$

$\Leftrightarrow \quad M = 0$

Note In fact, $M = 0 \Leftrightarrow M = \mathfrak{m}M$.

$\Leftarrow$ prop

$\Rightarrow \quad 0 = \mathfrak{m}0$

NAK 3 $(R, \mathfrak{m})$ local ring

$N \subseteq M$ R-mods

If $M = N + \mathfrak{m}M$, then $M = N$.

Proof $M = N + \mathfrak{m}M$

$$M/N = \frac{N + \mathfrak{m}M}{N} = \mathfrak{m}(M/N)$$

$\xRightarrow{\text{NAK 2}} M/N = 0 \Rightarrow M = N$

Note $M = N + \mathfrak{m}M \Leftrightarrow M = N$

Prop $(R, \mathfrak{m})$ local ring

M fg R-module

$m_1, \dots, m_s$ generate $M \iff \overline{m_1}, \dots, \overline{m_s}$ generate $M/\mathfrak{m}M$

Proof $(\Longrightarrow)$ obvious

$(\Longleftarrow)$ $N = Rm_1 + \cdots + Rm_s \subseteq M$

$$\begin{aligned} M/N = 0 &\iff \mathfrak{m}\left(M/N\right) = M/N \\ &\iff M/N = \frac{\mathfrak{m}M + N}{N} \\ &\iff M = \mathfrak{m}M + N \\ &\iff M/\mathfrak{m}M = \frac{\mathfrak{m}M+N}{\mathfrak{m}M} \\ &\iff \overline{N} \text{ generates } M/\mathfrak{m}M \end{aligned}$$

Remark $R/\mathfrak{m}$ is a field, and $M/\mathfrak{m}M$ is a vector space over $R/\mathfrak{m}$

Def $\{m_1, \dots, m_s\} \subseteq M$ is a minimal generating set for M if $\{\overline{m_1}, \dots, \overline{m_s}\}$ are a basis for the $R/\mathfrak{m}$-vector space $M/\mathfrak{m}M$

Remark these follow from facts about vector spaces:

- All minimal generating sets for M have the same number of elements.
- Every set of generators contains a minimal generating set.
- Every element in M but not in $\mathfrak{m}M$ is part of a minimal generating set.

Minimal number of generators

$\mu(M) := \dim_{R/\mathfrak{m}}(M/\mathfrak{m}M)$

$=$ number of generators in a minimal generating set

Graded NAK

G-NAK 1 R $\mathbb{N}$-graded

M $\mathbb{Z}$-graded R-mod

$M_{<a} = 0$

If $M = R_+ M$, then $M = 0$

Proof $\underbrace{M}_{\text{degrees} \geqslant a} = \underbrace{R_+ M}_{\text{degrees} \geqslant a+1} \Longrightarrow M = 0$

Remark this includes all fg $\mathbb{Z}$-graded R-modules

If M is fg, there is a finite generating set of homogeneous elements (take homogeneous components of any generating set)

Set $a :=$ min degree of a generator in a given generating set

$M \subseteq \underbrace{R}_{\text{degrees} \geqslant 0} M_{\geqslant a} \subseteq M_{\geqslant a} \Longrightarrow M_{<a} = 0$

G-NAK 2 $\quad$ R $\mathbb{N}$-graded

R_0 field

M $\mathbb{Z}$-graded R-mod

$M_{<a} = 0$

A set of elements generates M

$\Updownarrow$

images span M/R_+M over R_0 $\qquad$ $R/R_+ \cong R_0$ field

Notes:

→ In the graded setting, we can use NAK to show some modules are finitely generated, since it gives us a concrete way to find minimal generating sets. However, in the local setting, we can use NAK only if M is already fg.

→ If k is a field, $R = \bigoplus_{i \geq 0} R_i$, $R_0 = k$, I homogeneous ideal
$\Rightarrow$ I has a minimal generating set by homogeneous elements and this is unique up to k-linear combinations.

Def M fg $\mathbb{Z}$-graded module over $R = \bigoplus_{i \geq 0} R_i$, $R_0 = k$ field

$$\mu(M) := \dim_{R/R_+} (M/R_+M)$$