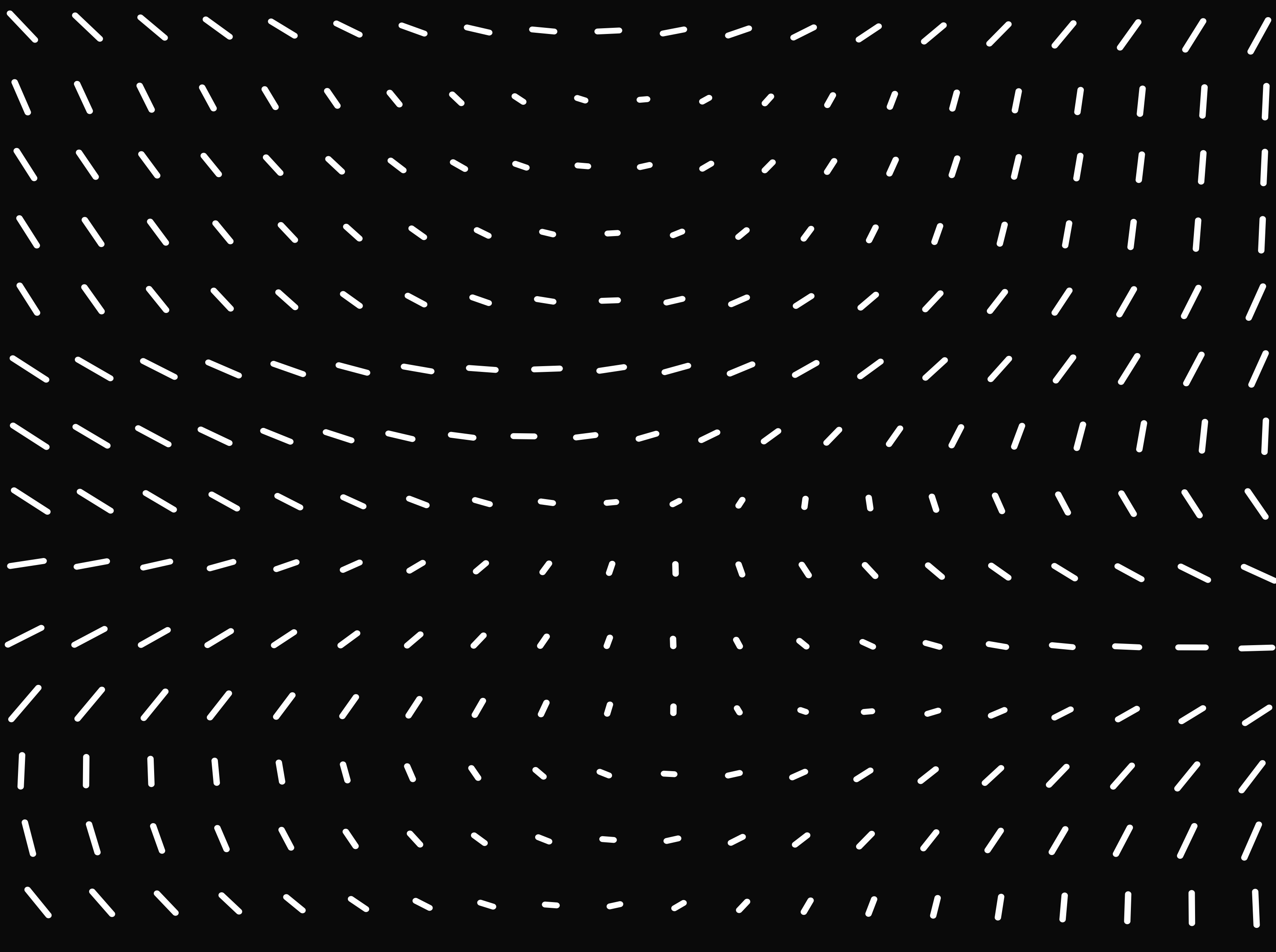
CHAMP TALK



THE CRITICAL + COCRITICAL DEGREES OF A TOTALLY ACYCLIC COMPLEX

Goals

1. **Introduce** notion of critical degree in R-Mod

2. **Present** extension to $K_{tac}(R)$

   + **Discuss** study of operations of complexes

3. **Argue** why this extension is not optimal

   + **Propose** an alternative

Setting the Stage I.

- $(R, \mathfrak{m}, k)$ = local, comm Ring
- M = fin-gen R-mod w/ min free resoln F
- $\{b_i\}$ = betti sequence $\Rightarrow$ $b_i = \operatorname{rk} F_i$

Q. When $\operatorname{pd}_R M = \infty$, what patterns can occur in $\{b_i\}$?

E.g.

- $R \neq CI$: $\{b_i(k)\}$ has exponential growth
- $R = CI$: $\{b_i(k)\}$ is eventually given by a poly

$\boxed{R = CI}$ $\rightarrow$ $\{b_i\}$ has "nice" patterns at ∞

Pre-show

"Complete Intersection Dimension"
[Luchezar Avramov, Vesselin Gasharov, + Irena Peeva (c. 1997)]

Idea: $\{\text{CI-dim}_R M\}$

→ Define class of modules which admit structural similarities in F's to modules over CI's

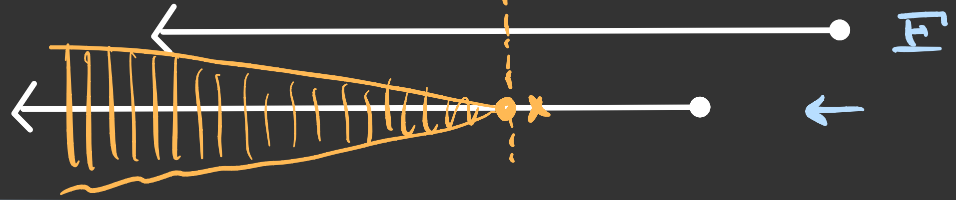
Point: $\text{CI-dim}_R M < \infty$, patterns @ ∞ do arise in $\{b_i\}$

Q. When do these patterns arise?

Defn. The critical degree of M, denoted $\text{crdeg}_R M$, is the least integer s for which F admits a chain map of degree $-q<0$ such that $\mu_{n+q}: F_{n+q} \to F_n$ is surjective for all $n > s$.

- If no such integer exists, set $\text{crdeg}_R M = \infty$.
- If $M \neq 0$ then $-1 \leq \text{crdeg}_R M \leq \infty$ and $\text{crdeg}_R 0 = -\infty$

Pictorially:



Thm. (CID) When CI-$\dim_R M < \infty$ $(M \neq 0)$:

① If $cx_R M \leq 1 \Rightarrow b_n = b_{n+1}$ for $n > s$

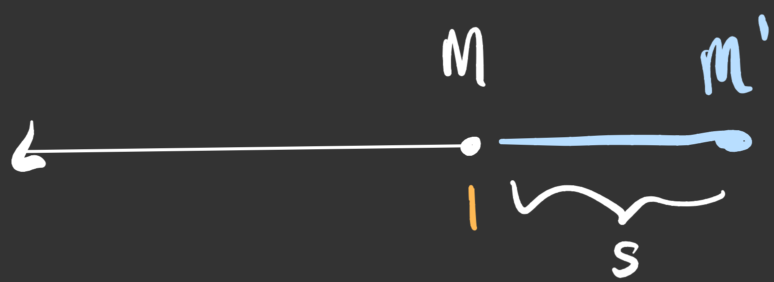
② If $cx_R M \geq 2 \Rightarrow b_n < b_{n+1}$ for $n > s$

$\Rightarrow$ After $\operatorname{crdeg}_R M$, $\{b_i\}$ has strict growth

- Authors were able to give bounds for the $\operatorname{crdeg}_R M$ in terms of $\operatorname{depth} R - \operatorname{depth}_R M$ when $cx_R M \leq 1$ and discussed case when $cx_R M = 2$ in later paper.

However, there is no bound valid for all modules of a given $cx_R M \geq 2$

Ex.



$\Rightarrow M = \Omega^{s+1} M'$

$\Rightarrow \operatorname{crdeg} M' = s$

but $cx_R M = cx_R M'$.

Setting the Stage II.

- For remainder of talk, $\mathcal{C} = R$-complex.

Definitions: ① $\mathcal{C}$ is a totally acyclic complex if:

- C_i = projective (R = local $\Rightarrow$ C_i = free)
- $H(\mathcal{C}) = 0 = H(\mathcal{C}^*)$ where $\mathcal{C}^* = \mathrm{Hom}_R(\mathcal{C}, R)$

② We say $\mathcal{C}$ is minimal if: $\partial(\mathcal{C}) \subseteq \mathfrak{m}\mathcal{C}$

Fact: Every R-complex can be written as

$$\mathcal{C} = \bar{\mathcal{C}} \oplus T.$$

$\bar{\mathcal{C}} \to$ minimal

$T \to$ trivial/contractible: $1^{\mathcal{C}} \sim 0$

- Define: $K_{tac}(R)$ = category with

[Objects] $\longrightarrow$ [T.A. Complexes]

[Morphisms] $\longrightarrow$ [Homotopy eq. classes of R-comp chain maps]

Connection to R-Mod + Free Resolutions

Defn. A complete resolution of M is a diagram

$$\mathcal{V} \xrightarrow{\rho} P \xrightarrow{\pi} M \quad \text{where } \mathcal{V} \in K_{tac}(R),$$

P : projective resolution of M, and ρ_n is bijective for $n \gg 0$.

Fact: ① Any fin-gen M has a comp resoln:

$$\mathcal{G} \xrightarrow{\rho} F \to M$$

② Every $\mathcal{G} \in K_{tac}(R)$ $\exists$ a diagram:

$$\mathcal{G} \to \mathcal{G}^{\geq n} \to \operatorname{Im} \partial_n^{\mathcal{G}}$$

Rmk. Abuse notation + call $\mathcal{V}$ = complete resolution.

Point:

$$\begin{tikzcd} \mathcal{G} \arrow[r, bend right] & M \arrow[l, bend right] \end{tikzcd}$$

$$F \longrightarrow \mathcal{G}$$

$$\{b_i\} \longrightarrow \{b_i\} : \text{"complete" } \cancel{\text{resoln}} \text{ betti seq.}$$

Structure of $K_{tac}(R)$

⇒ Triangulated Category

Main Structural Components:

① Shifts

⇒ Σ's "suspension endofunctor"

$$\mathcal{C} \longrightarrow \Sigma\mathcal{C} : \text{complex w/ } (\Sigma\mathcal{C})_n = \mathcal{C}_{n-1}$$

$$\Sigma^a\mathcal{C} \qquad \text{and } \partial_n^{\Sigma\mathcal{C}} = -\partial_{n-1}^{\mathcal{C}}$$

② Class of Distinguished Triangles

$$[\text{Dist } \Delta\text{'s}] \xrightarrow{\text{Analogue}} [\text{SES}]$$

Act I. The Definitions + Basic Properties

Defn. 1 • A totally acyclic complex C has <u>critical degree relative to an endomorphism</u> $\mu: C \to \Sigma^q C$ (or μ-critical degree) $s = \operatorname{crdeg}_R^\mu C$ if:

$$s = \inf\{n \mid \bar{\mu}_{i+q}: \bar{C}_{i+q} \to \bar{C}_i \text{ is surjective } \forall\, i > n\}$$

where $\bar{\mu}$ is the induced endomorphism on the minimal subcomplex $\bar{C}$.

→ If no such integer exists and $s \neq -\infty$ then set $\operatorname{crdeg}_R^\mu C = \infty$.

• C has <u>cocritical degree relative</u> to $\mu: C \to \Sigma^q C$ (or μ-cocritical degree) $t = \operatorname{cocrdeg}_R^\mu C$ if:

$$t = \sup\{n \mid \bar{\mu}_i: \bar{C}_i \to \bar{C}_{i-q} \text{ is split injective } \forall\, i < n\}$$

→ If no such integer exists and $t \neq \infty$ then set $\operatorname{cocrdeg}_R^\mu C = -\infty$.

Main Defn. The Critical + Cocritical Degrees

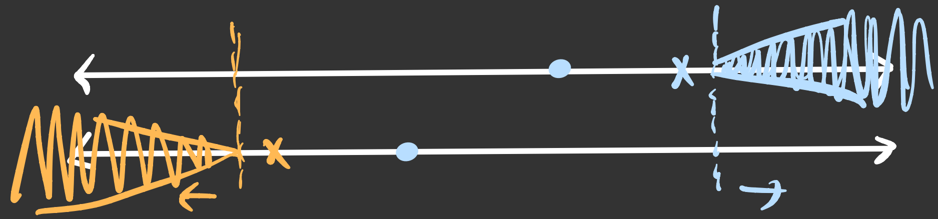
- Define the critical degree of C as:

$$\operatorname{crdeg}_R C = \inf\{\operatorname{crdeg}_R^\mu C \mid \mu = \text{negative degree chain endomorphism}\}$$

- Define the cocritical degree of C as:

$$\operatorname{cocrdeg}_R C = \sup\{\operatorname{cocrdeg}_R^\mu C \mid \mu = \text{negative degree chain endomorphism}\}$$

Pictorially



Note: $M = \operatorname{Im} \partial_0^{\mathcal{G}} \Rightarrow \operatorname{crdeg}_R \mathcal{G} = \operatorname{crdeg}_R M$

Q. Does Defn 1 make sense in $K_{\mathrm{tac}}(R)$

Prop. If $\mathcal{G} \simeq D \Rightarrow \operatorname{crdeg} \mathcal{G} = \operatorname{crdeg} D$
and $\operatorname{cocrdeg} \mathcal{G} = \operatorname{cocrdeg} D$

Pf. (Idea) $\mathcal{G} \simeq D \Rightarrow \overline{\mathcal{G}} \cong \overline{D}$

$$\left\{\begin{matrix}\text{endos} \\ \text{on } \overline{\mathcal{G}}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{endos} \\ \text{on } \overline{D}\end{matrix}\right\}$$

Remarks. ① For any $M \Rightarrow$

$\operatorname{crdeg}_R^M \mathcal{G} \geq \operatorname{crdeg}_R \mathcal{G}$

$\operatorname{cocrdeg}_R^M \mathcal{G} \leq \operatorname{cocrdeg}_R \mathcal{G}$

② If $\operatorname{crdeg}_R \mathcal{G} = s \Rightarrow \exists$ some M s.t. $\operatorname{crdeg}_R^M \mathcal{G} = s$.

(Likewise for cocrdeg)

Act II. Operations on Complexes in $K_{tac}(R)$

THREE BASIC OPERATIONS:

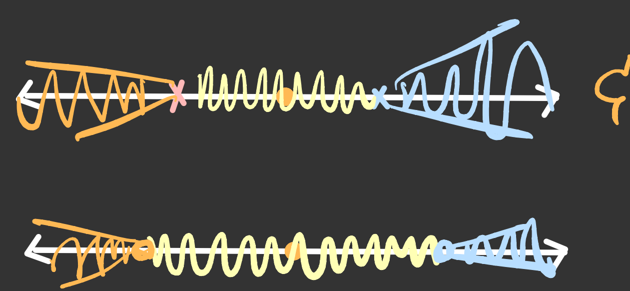
① Σ's $\Rightarrow$ e.g. $\operatorname{crdeg} \varsigma + 1 = \operatorname{crdeg} \Sigma\varsigma$

② Summands
$$\begin{tikzcd} A\oplus B \arrow[r, bend right, "?"'] & A, B \arrow[l, "\checkmark"'] \end{tikzcd}$$

③ Cones ?

Additional Operations:

④ $- \otimes_R B$ $\searrow$ bdd complex



⑤ $\otimes_R^{\bowtie}$ Pinch-tensor

The Mapping Cone and Triangles in $K_{tac}(R)$

Recall:

Given R-complexes $C = \{C_i, \partial_i^C\}$, $D = \{D_i, \partial_i^D\}$, and any chain map $f: C \to D$, the mapping cone of f is the R-complex with:

$$M(f)_n = C_{n-1} \oplus D_n \quad \text{and} \quad \partial_n^{M(f)} = \begin{pmatrix} -\partial_{n-1}^C & 0 \\ f_{n-1} & \partial_n^D \end{pmatrix}$$

[Class of Distinguished Δ's in $K_{tac}(R)$]

Defn. A sequence of objects and morphisms in $K_{tac}(R)$ of the form:

$$C \xrightarrow{f} D \to Mf \to \Sigma C$$

is called a Standard Triangle

⇒ A distinguished triangle is:

Any $\Delta \cong$ STD Δ

Ideal Situation: $f : \mathcal{G} \to \mathcal{D}$, $\operatorname{crdeg} \mathcal{G}$, $\operatorname{crdeg} \mathcal{D}$

$$\Rightarrow \operatorname{crdeg} M(f)$$

Problem ①

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[r] \arrow[d, "r"] & Mf \arrow[r] \arrow[d, dashed] & \Sigma \mathcal{G} \arrow[d] \\
\Sigma^q \mathcal{G} \arrow[r, "?"] & \Sigma^r \mathcal{D} \arrow[r] & M(?) \arrow[r] & \Sigma^{q+1} \mathcal{G}
\end{tikzcd}$$

$$\begin{cases} q = r \\ 1^{st} \text{ sq commutes} \end{cases}$$

Q. If we get $h : Mf \to \Sigma^q Mf$, Can we now say something?

Problem ② $M(f)$ almost never minimal

→ In general, can't say what $M(f)$ looks like

→ Don't know $\bar{h}$ is

Act III. An Alternative Approach

Q. What if we can "bypass" knowing what $\overline{M(f)}$ is?

Idea: Rely more on the Δ'ed structure of $K_{fac}(R)$

Fact: STD $\Delta \xrightarrow{\text{Induce}}$ l.e.s. of $\mathrm{Hom}_{\mathcal{T}}$'s for $\mathcal{T} \in \mathbb{T}$

Alternative Definition:

→ Denote K = min comp resoln of k

→ Given an endomorphism $\mu : G \to \Sigma^q G$ there exists a standard triangle:

$$G \xrightarrow{\mu} \Sigma^q G \to M(\mu) \to \Sigma G$$

→ Thus, consider the long exact sequence of abelian groups:

$$\cdots \to \mathrm{Hom}(M(\mu), \Sigma^i K) \to \mathrm{Hom}(\Sigma^q G, \Sigma^i K) \to \mathrm{Hom}(G, \Sigma^i K) \to \mathrm{Hom}(M(\mu), \Sigma^{i+1} K) \to \cdots$$

$\boxed{\textbf{Defn. 2}}$ The μ-critical degree of the complex G is defined as:

$$\mathrm{crdeg}_R^{\mu} G := \inf \{ n \mid \mathrm{Hom}(\Sigma^q G, \Sigma^{i+q} K) \xhookrightarrow{\text{injective}} \mathrm{Hom}(G, \Sigma^{i+q} K) \text{ for all } i > n \}$$

• If no such integer exists and $\mathrm{crdeg}_R^{\mu} G \neq -\infty$, then set $\mathrm{crdeg}_R^{\mu} G = \infty$.

The μ-cocritical degree of G is defined as:

$$\mathrm{cocrdeg}_R^{\mu} G := \sup \{ n \mid \mathrm{Hom}(\Sigma^q G, \Sigma^i K) \xrightarrow{\text{surjective}} \mathrm{Hom}(G, \Sigma^i K) \text{ for all } i < n \}$$

• If no such integer exists and $\mathrm{cocrdeg}_R^{\mu} G \neq \infty$, then set $\mathrm{cocrdeg}_R^{\mu} G = -\infty$.

Note: Main Defn is the same

Q. Does Defn. 2 help?

Intuition:

① $\operatorname{Hom}_{\mathbb{K}}(G, \Sigma^i \mathbb{K}) \cong \operatorname{Hom}_{\mathbb{K}}(\bar{G}, \Sigma^i \mathbb{K})$
since $G \simeq \bar{G}$.

Assume $G = \min$

② $\operatorname{Hom}_{\mathbb{K}}(G, \Sigma^i \mathbb{K}) \cong \widehat{\operatorname{Ext}}^i_R(\operatorname{Im}\partial_0^c, k)$

Take comp res of $\operatorname{Im}\partial_0^c$: G

$H^i(\operatorname{Hom}_R(\underset{\min}{G}, \underline{k}))$

$\cong \operatorname{Hom}_R(G_i, k)$

$$\boxed{\operatorname{Hom}_{\mathbb{K}}(G, \Sigma^i \mathbb{K}) \cong \operatorname{Hom}_R(G_i, k)}$$

The Point: Recall $b_i = \dim_k \mathrm{Ext}^i_R(M,k)$

$$\begin{array}{ccc} \downarrow & & \\ F & & G \\ \{b_i\} & \rightsquigarrow & \{\hat{b}_i\} \end{array}$$

Q. Defn 1 = Defn 2 ?

Proof. (Idea)

$$\begin{tikzcd}
\mathrm{Hom}(\Sigma^q G, \Sigma^i K) \arrow[r, hook, "inj"] \arrow[d, "\cong"] & \mathrm{Hom}(G, \Sigma^i K) \arrow[d, "\cong"] \\
\mathrm{Hom}_R(C_{i-q}, k) \arrow[r, hook, "inj"] & \mathrm{Hom}_R(C_i, k)
\end{tikzcd}$$

Conclusion

Defn 2 > Defn 1

1. Defn 2 can be used for "operations" that preserve minimality

2. Defn 2 utilizes the Δ'd structure $K_{tac}(R)$

Thank You!