

Linear Algebra

Math 314 Fall 2025

Today's poll code:

DF6GU3

Lecture 9

To do list:

Office hours

Mondays 5–6 pm

Wednesdays 2–3 pm

in Avery 339 (Dr. Grifo)

Tuesdays 11–noon

Thursdays 1–2 pm

in Avery 337 (Kara)

- Webwork 2.6 due Friday
- Webwork 3.1 due next Tuesday

Quiz on Friday

on Lectures 8–9

Midterm 1

in 2 weeks

Review: linear transformations

Given a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

its **standard matrix** is

$$A = [T(e_1) \quad \cdots \quad T(e_n)]$$

so

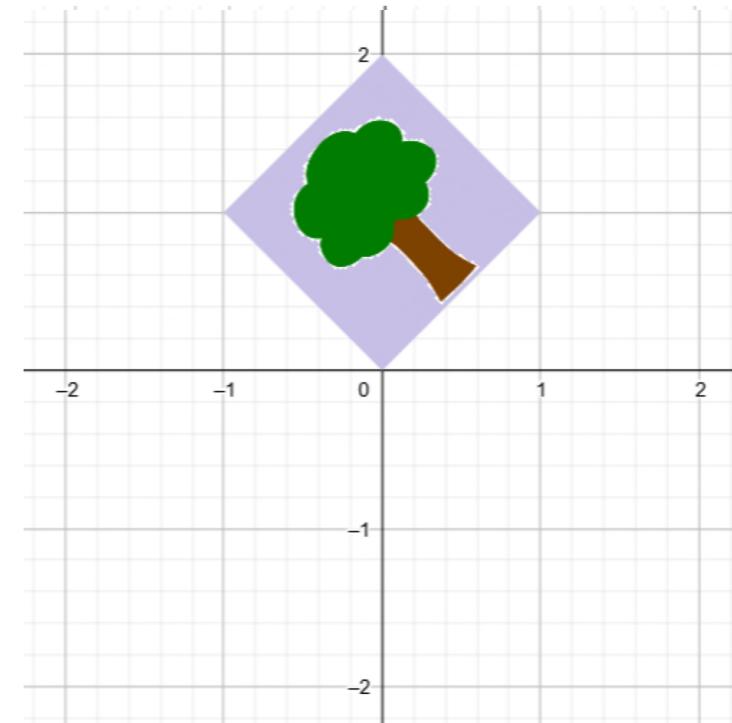
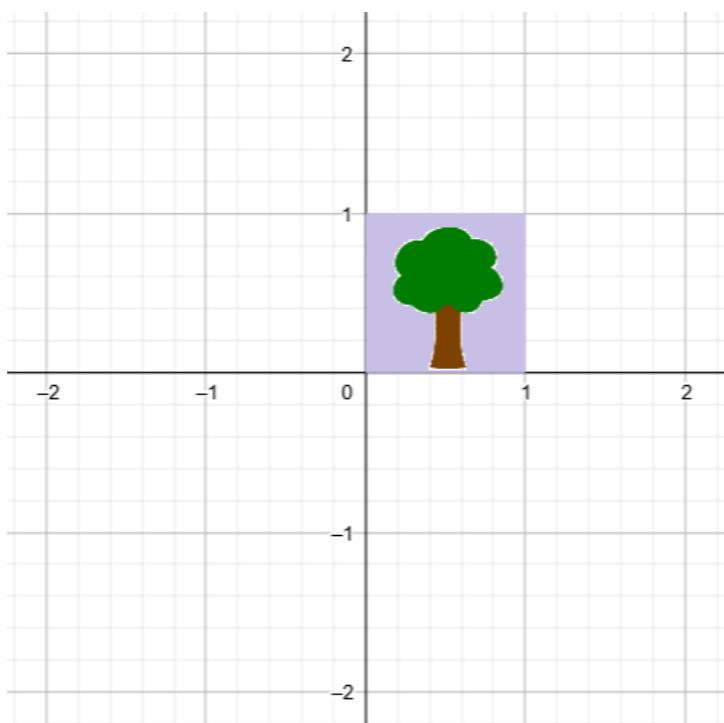
$$T(x) = Ax \text{ for all vectors } x$$

i th standard vector
in \mathbb{R}^n

$$\mathbf{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \xrightarrow{\text{position } i}$$

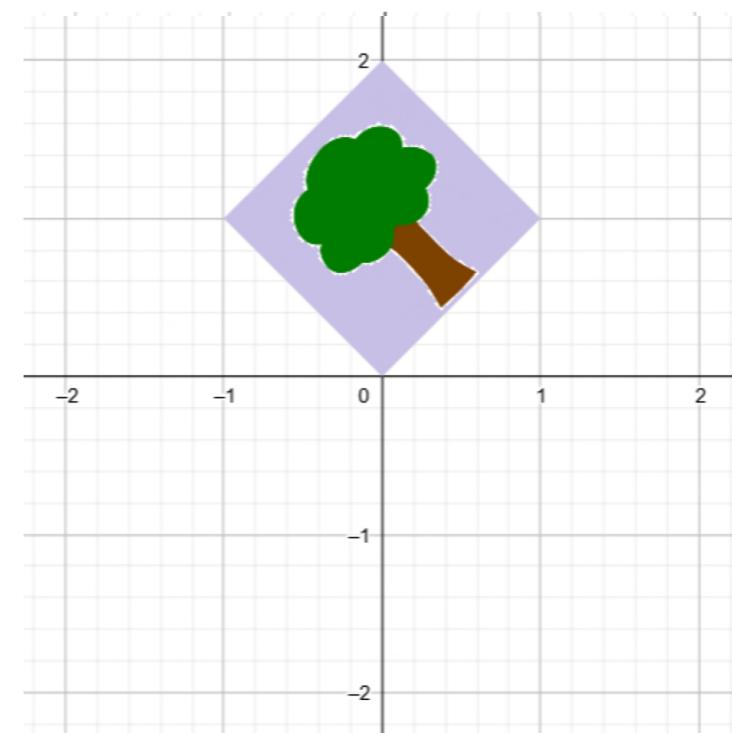
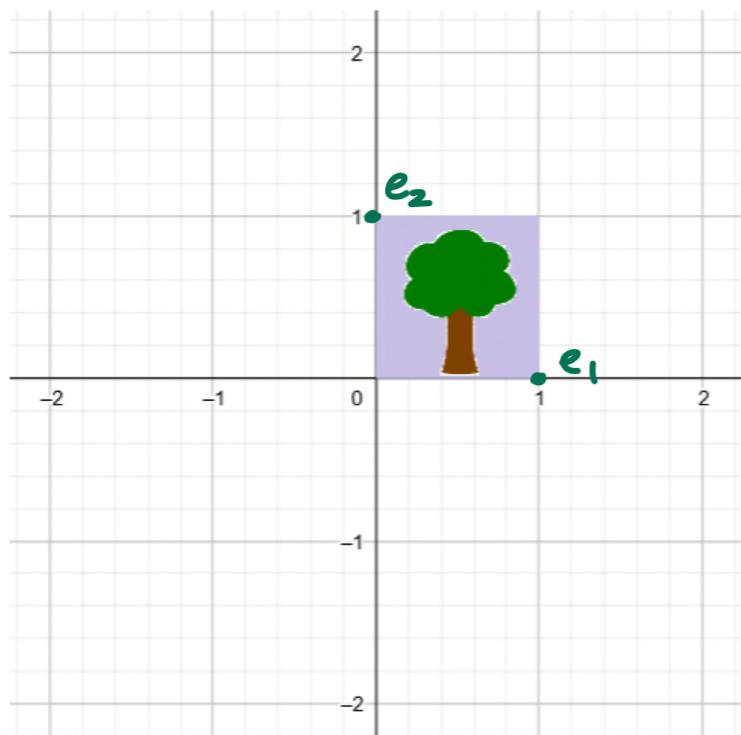
Review: example from last Wednesday's class

Consider the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that does the following:



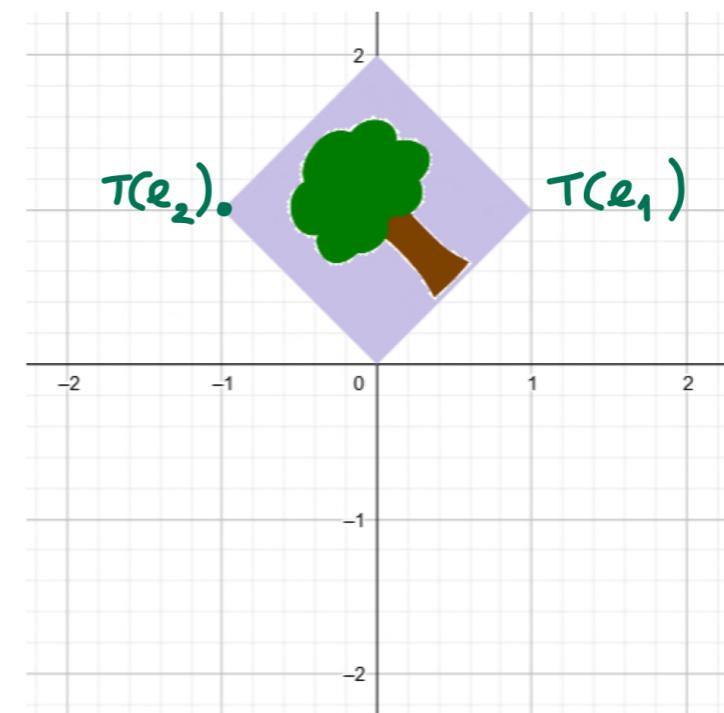
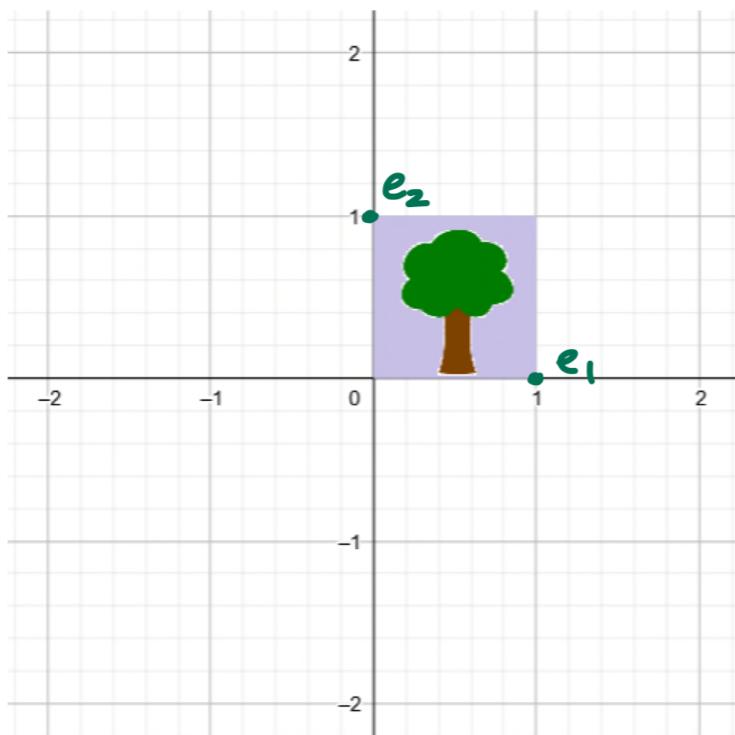
Review: example from last Wednesday's class

Consider the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that does the following:



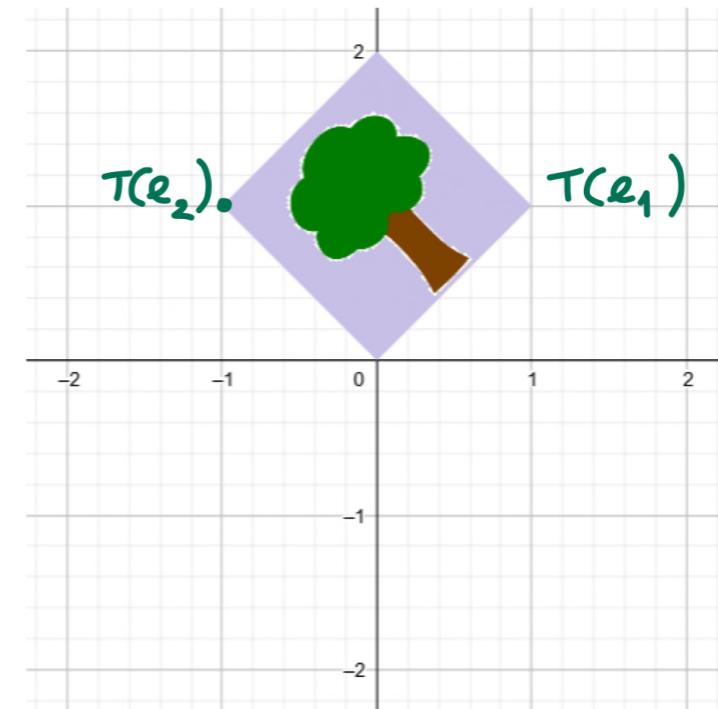
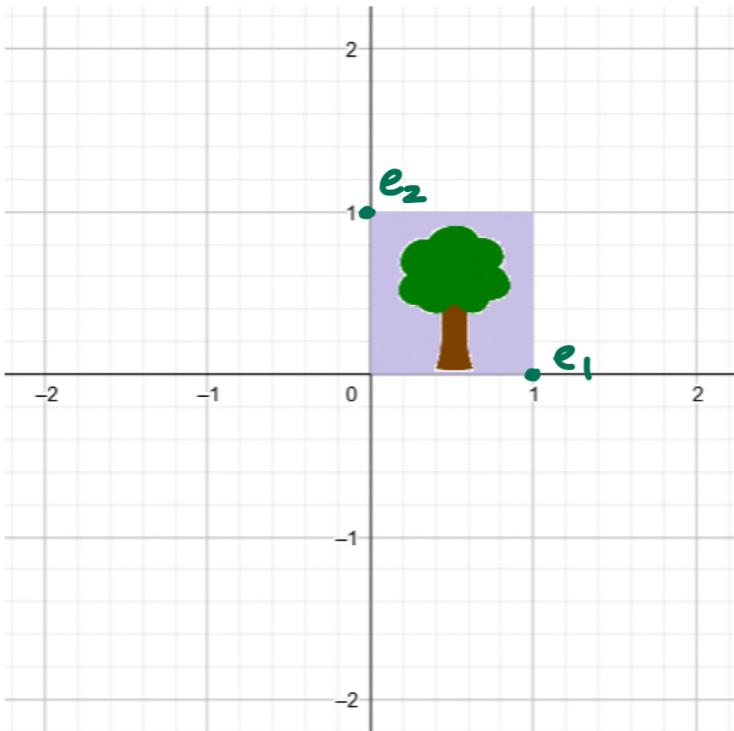
Review: example from last Wednesday's class

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Review: example from last Wednesday's class

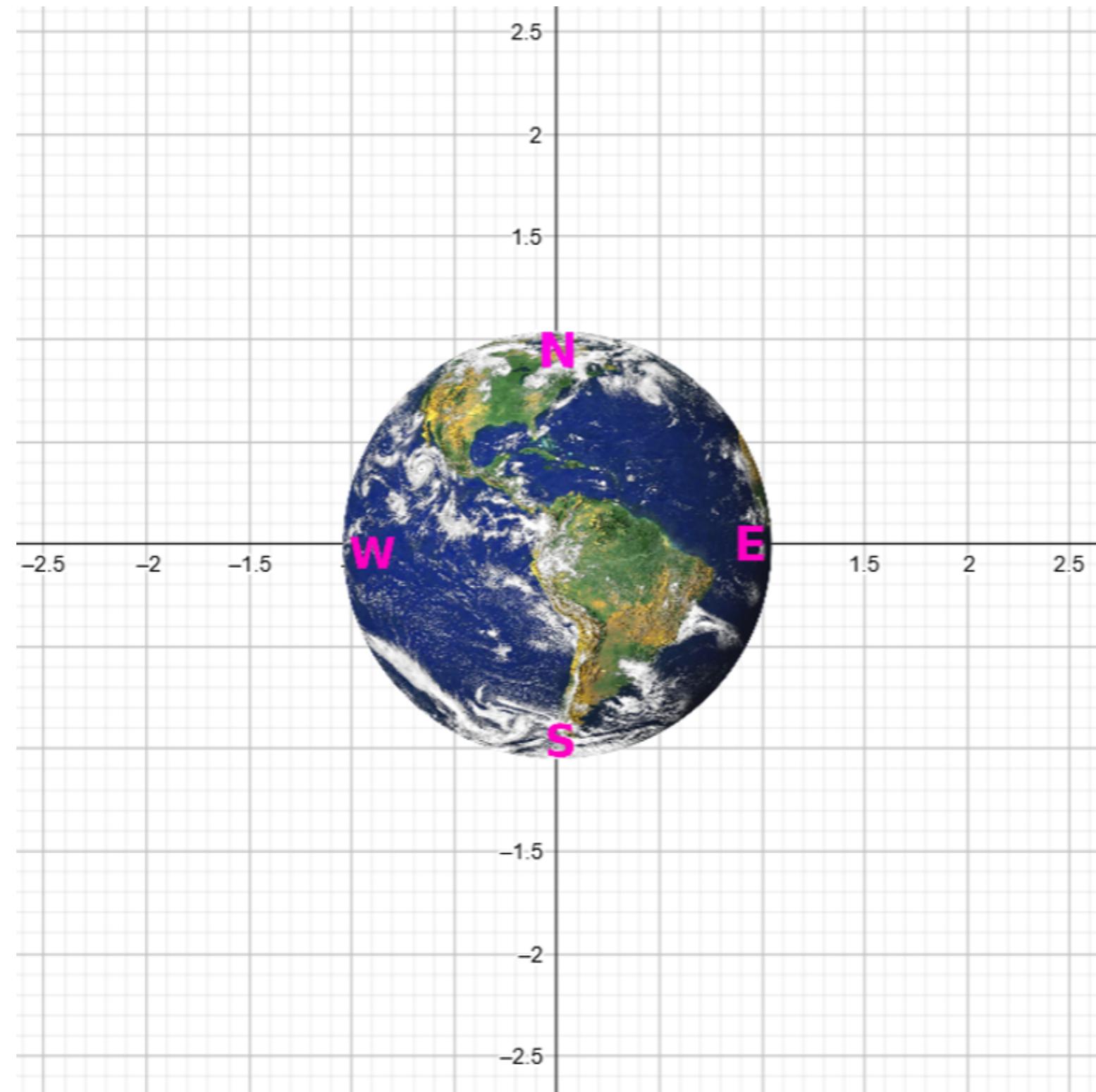
Consider the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that does the following:



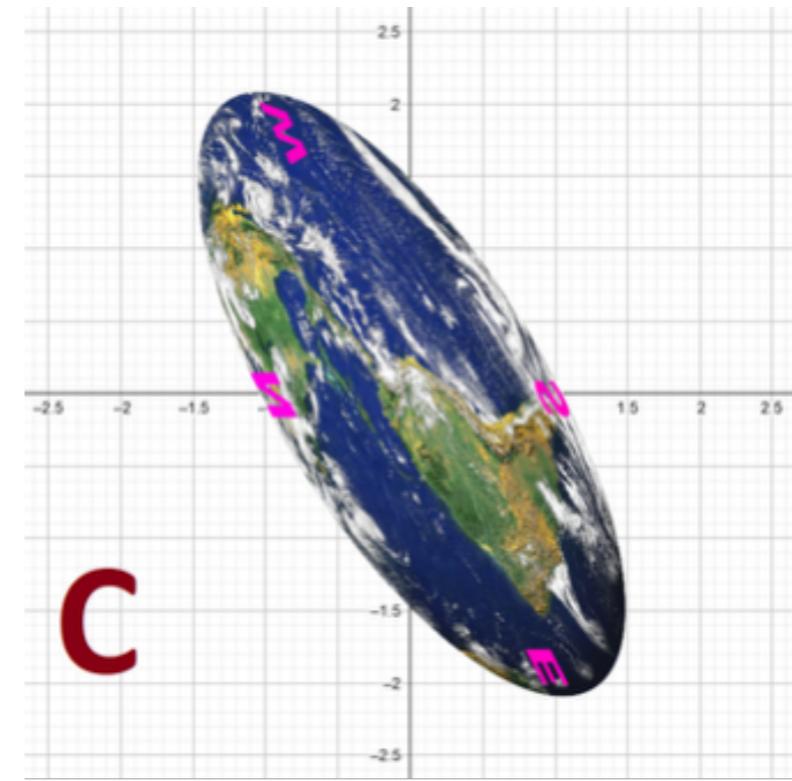
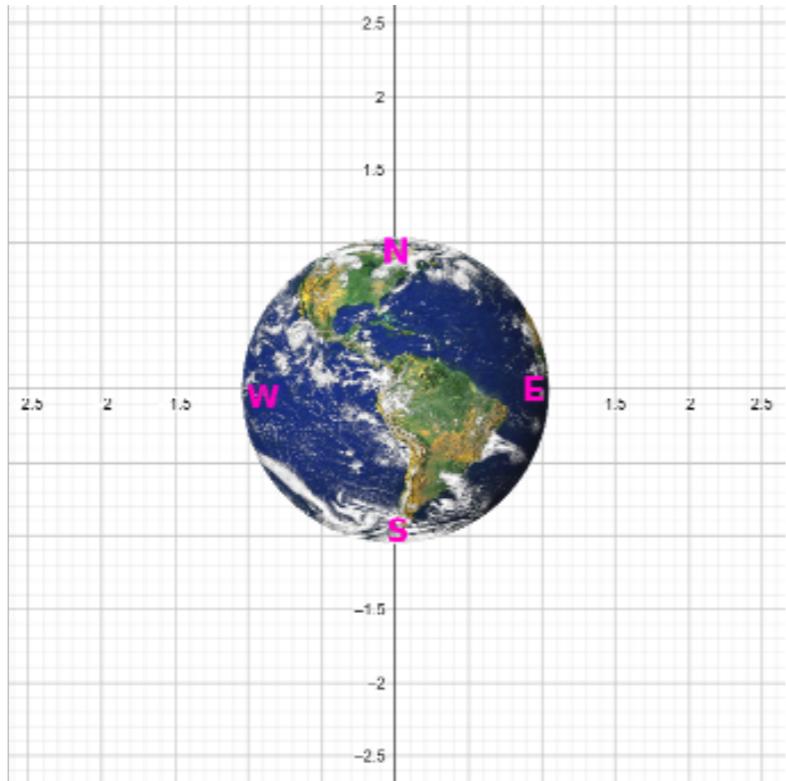
Note how we explicitly marked the images of e_1 and e_2 . This is sufficient for us to find the standard matrix, and thus to completely describe the linear transformation: the matrix is

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Let's look at an example involving the earth:

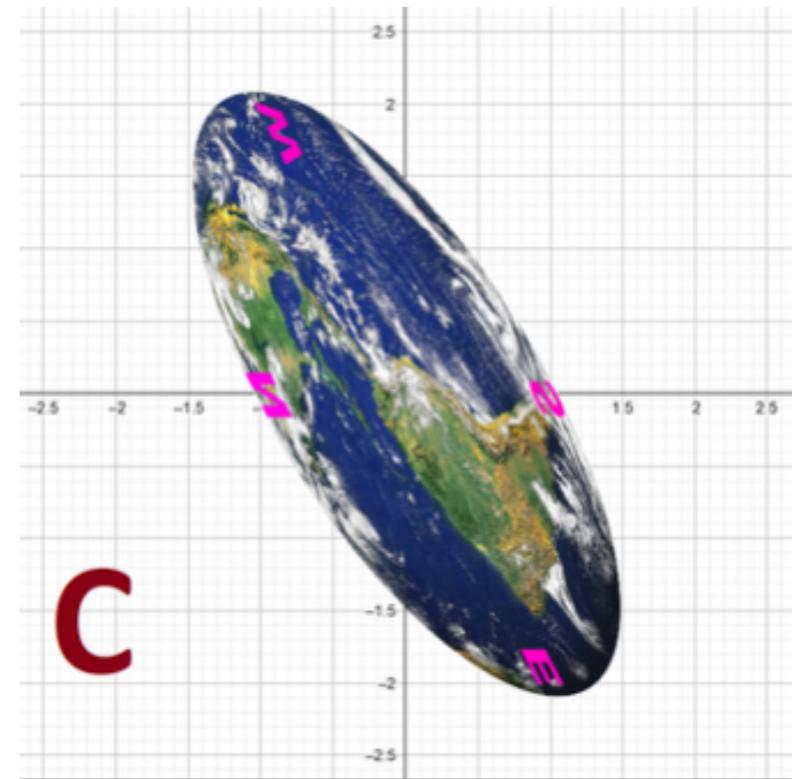
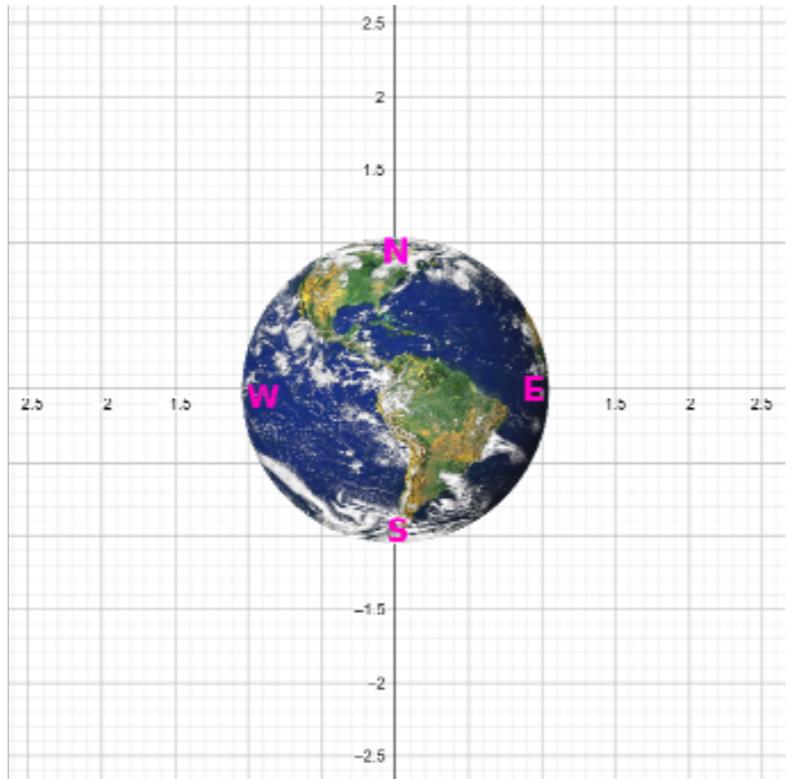


Consider the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that does the following:



The standard matrix for T is

Consider the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that does the following:



The standard matrix for T is $\begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$

Midterm 1

Midterm 1

October 6

Material for Midterm 1: Chapters 1, 2, 3

Study materials:

- Class notes (see canvas)
- Textbook
- Slides from lecture
- Poll questions
- Quizzes
- Webwork questions
- Study guide (quick summary)

Office hours

Mondays 5–6 pm and Wednesdays 2–3 pm
in Avery 339 (Dr. Grifo)

Tuesdays 11–noon and Thursdays 1–2 pm
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Special office hours next week

Monday 5–6 pm and Wednesday 2–3 pm
Thursday 2:30 to 3:30 pm
Friday 2:30 to 3:30 pm
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Tuesday 11–noon and Thursday 1–2 pm
in Avery 337 (Kara)

Chapter 3

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (i, j)\text{th entry of } A = a_{ij}$$

multiplying a matrix by a scalar: $cA = [ca_{ij}]$

$$3 \cdot \begin{bmatrix} -1 & 1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -3 & 15 \end{bmatrix}$$

always defined

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (i, j)\text{th entry of } A = a_{ij}$$

The sum $A + B$ is defined if

A and B have the same size $\begin{pmatrix} \# \text{ rows of } A = \# \text{ rows of } B \\ \# \text{ columns of } A = \# \text{ columns of } B \end{pmatrix}$

sum of matrices:

$$A + B := [a_{ij} + b_{ij}]$$

$$\begin{bmatrix} 3 & -1 \\ 2 & -11 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3+1 & -1+3 \\ 2+4 & -11+5 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 6 & -6 \end{bmatrix}$$

product of matrices: The product AB is defined if

$$\# \text{ of columns of } A = \# \text{ of rows of } B$$

$$\# \text{ of rows of } AB = \# \text{ of rows of } A$$

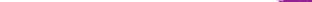
$$\# \text{ of columns of } AB = \# \text{ of columns of } B$$

In AB :

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

The diagram illustrates the calculation of the (i, j) th entry of the product matrix AB . It shows a row vector a_i (labeled "row i ") being multiplied by a column vector b_j (labeled "column j "). The result is a scalar value represented by a red dotted line with a circled plus sign at the intersection point, labeled "row i " and "column j ".

A m × n matrix

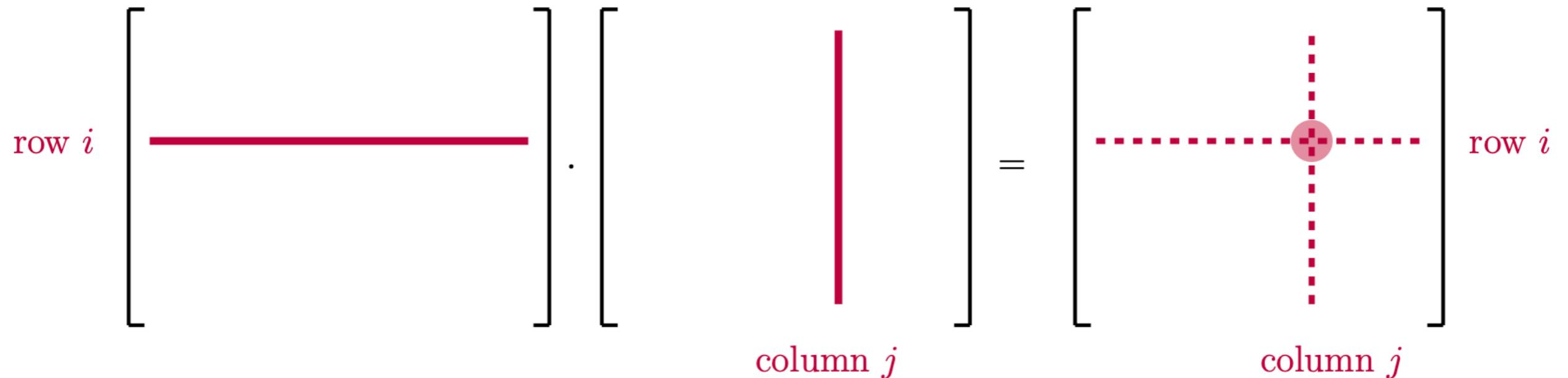
 AB $m \times p$ matrix

B $n \times p$ matrix

Informally: $(m \times n) (n \times p) = m \times p$

In AB :

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$



A $m \times n$ matrix

B $n \times p$ matrix

\longrightarrow AB $m \times p$ matrix

Informally: $(m \times \cancel{n}) (\cancel{n} \times p) = m \times p$

In AB :

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$\begin{matrix} \text{row } i \\ \left[\begin{array}{c} \text{---} \\ \cdot \\ \end{array} \right] \end{matrix} \cdot \begin{matrix} \text{column } j \\ \left[\begin{array}{c} | \\ \cdot \\ | \end{array} \right] \end{matrix} = \begin{matrix} \text{row } i \\ \left[\begin{array}{c} \dots \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \end{array} \right] \end{matrix}$$

column j

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (i, j)\text{th entry of } A = a_{ij}$$

product of matrices: The product AB is defined if

$$\# \text{ of columns of } A = \# \text{ of rows of } B$$

$$AB = [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p]$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ell} \end{bmatrix}$$

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

product of matrices: The product AB is defined if

$$\# \text{ of columns of } A = \# \text{ of rows of } B$$

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

i th row of A and j th column of B

eg:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ell} \end{bmatrix}$$

$$(1, 1)\text{th entry} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}$$

product of matrices: The product AB is defined if

$$\# \text{ of columns of } A = \# \text{ of rows of } B$$

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

i th row of A and j th column of B

eg:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$
$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ell} \end{bmatrix}$$

$$(m, 2)\text{th entry} = a_{m1}b_{12} + a_{m2}b_{22} + \cdots + a_{mn}b_{n2}$$

Example:

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + 3 \times 1 & 2 \times 3 + 3 \times (-2) & 2 \times 6 + 3 \times 3 \\ 1 \times 4 - 5 \times 1 & 1 \times 3 + (-5) \times (-2) & 1 \times 6 + (-5) \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

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Example:

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$

is undefined!

The diagram shows two matrices. The first matrix has three columns, and the second matrix has two columns. Cyan circles highlight the first two columns of the first matrix and the first column of the second matrix. Two purple arrows point downwards from the bottom of these circled columns towards the text below.

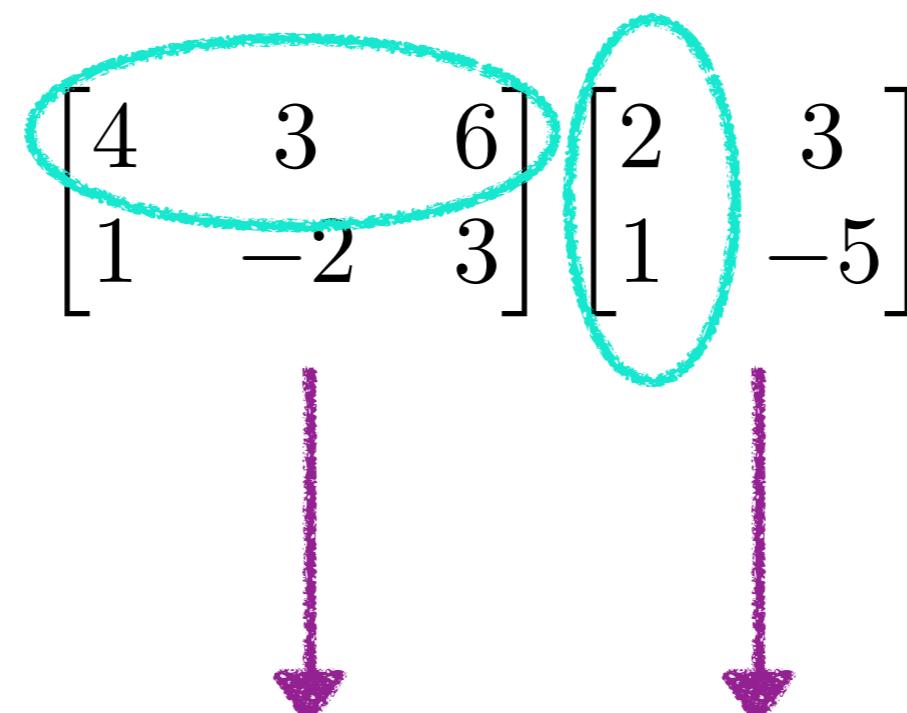
$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$

3 columns \neq 2 rows

Example:

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$

is undefined!

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$


3 columns \neq 2 rows

Theorem. Let A, B, C be matrices,

Assume that the products AB and BC make sense.

1. Associativity: $(AB)C = A(BC)$.
2. Left distributivity: $A(B + C) = AB + AC$.
3. Right distributivity: $(B + C)A = BA + CA$.
4. For any scalar α , $\alpha(AB) = (\alpha A)B = A(\alpha B)$.
5. Let I_m denote the $m \times m$ identity matrix.

Then $I_mA = A = AI_n$.

Warning: the order of the matrices in a product matters!

For example:

$$A = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 \cdot 2 + 1 \cdot 4 & 5 \cdot 0 + 1 \cdot 3 \\ (-1) \cdot 2 + 3 \cdot 4 & (-1) \cdot 0 + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ 10 & 9 \end{bmatrix}$$

≠

$$BA = \begin{bmatrix} 2 \cdot 5 + 0 \cdot (-1) & 2 \cdot 1 + 0 \cdot 3 \\ 4 \cdot 5 + 3 \cdot (-1) & 4 \cdot 1 + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 17 & 13 \end{bmatrix}$$

The zero $m \times n$ matrix is
the $m \times n$ matrix whose entries are all 0.

We denote it simply by 0, if the size is clear from context.

Warning! Cancellation fails.

$AB = AC$ does not imply that $B = C$.

For example:

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix} \neq 0 \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

$$AB = [1 \times 0 + 0 \times 1] = [0]$$

Warning! Cancellation fails.

$AB = AC$ does not imply that $B = C$.

For example:

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \neq \quad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$AB = [2 \times 2 + 1 \times 0] = [2] = [1 \times 1 + 1 \times 1] = AC$$

The zero $m \times n$ matrix is

the $m \times n$ matrix whose entries are all 0.

We denote it simply by 0, if the size is clear from context.

Warning! Cancellation fails.

$AB = 0$ does not imply that $A = 0$ or $B = 0$

For example:

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix} \neq 0 \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

$$AB = [1 \times 0 + 0 \times 1] = [0]$$

A square matrix

Powers of A :

$$A^0 = I_n$$

$$A^1 = A \qquad A^2 = AA \qquad A^3 = AAA$$

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}}$$

A $m \times n$ matrix

The **transpose** of A is the $n \times m$ matrix A^T
whose rows are the columns of A

Example:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} \quad \xrightarrow{\text{purple arrow}} \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}$$

Theorem. Let A and B be matrices.

Assume A and B have appropriate sizes so that the following make sense.

1. $(A^T)^T = A$.
2. $(A + B)^T = A^T + B^T$.
3. $(\alpha A)^T = \alpha A^T$ for any scalar α .
4. $(AB)^T = B^T A^T$.

A $m \times n$ matrix

The **transpose** of A is the $n \times m$ matrix A^\top
whose rows are the columns of A

Example:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} \quad A^\top = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix} \quad (A^\top)^\top = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

Today's poll code:

DF6GU3

Today's poll code:

DF6GU3

A is a 2×3 matrix

B is a 3×5 matrix

AB is...

A. absurd

B. 2×5

C. 5×3

D. 3×3

E. 5×2

Today's poll code:

DF6GU3

A is a 2×3 matrix

B is a 3×5 matrix

BA is...

A. absurd

B. 2×5

C. 5×3

D. 3×3

E. 5×2

Today's poll code:

DF6GU3

A is a 2×3 matrix

B is a 3×5 matrix

$A + B$ is...

A. absurd

B. 2×5

C. 5×3

D. 3×3

E. 5×2

Today's poll code:

DF6GU3

If AB is a 3×4 matrix, how many rows does A have?

- A. 3
- B. 4
- C. unknown

A $n \times n$ (square) matrix

Powers of A :

$$A^2 = AA$$

$$A^3 = AAA$$

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}}$$

$$A^1 = A$$

$$A^0 = I_n$$

A m × n matrix

The transpose of A is

the $n \times m$ matrix whose rows are the columns of A

$$A = [v_1 \quad \cdots \quad v_n]$$

↓ ↓

column 1 column n

$$A^T = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \xrightarrow{\hspace{1cm}} \text{row } 1$$
$$\xrightarrow{\hspace{1cm}} \text{row } n$$

$$\text{Example: } A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} \quad A^\top = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}$$

Theorem. Let A and B be matrices whose sizes make the following make sense.

1. $(A^T)^T = A$.
2. $(A + B)^T = A^T + B^T$.
3. $(\alpha A)^T = \alpha A^T$ for any scalar α .
4. $(AB)^T = B^T A^T$.

A $m \times n$ matrix

The **transpose** of A is

the $n \times m$ matrix whose rows are the columns of A

A square matrix A is **symmetric** if $A = A^\top$.

In practice: if $A = [a_{ij}]$

$$a_{ij} = a_{ji} \quad \text{for all } i, j$$

This imposes no conditions on the diagonal entries a_{ii}

A square matrix A is **symmetric** if $A = A^\top$.

Example: if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $A^\top = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

If A is symmetric,

A square matrix A is **symmetric** if $A = A^\top$.

Example: if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $A^\top = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

If A is symmetric,

we must have $a_{12} = a_{21}$

a_{11} can be anything

a_{22} can be anything



$$A = \begin{bmatrix} x & z \\ z & y \end{bmatrix}$$

To do list:

- Webwork 2.5 due Friday
- WebworK 3.1 due next Tuesday

On Friday:

Quiz 4

**at the beginning
of the recitation**

on Lectures 8—9

Midterm 1

On October 6

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