Symbolic Powers and the Containment Problem

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Theorem (Lasker, 1905, Noether, 1921)
Every ideal in a Noetherian ring can be written as an intersection of primary ideals, called a primary decomposition.

Recall An ideal Q is primary if $ab \in Q \Rightarrow a \in Q$ or $b^n \in Q$ for some n.

$$\text{prime} \not\Leftarrow\!\!\Rightarrow \text{primary} \qquad \sqrt{\text{primary}} = \text{prime}$$

Example $I = (xy, yz, xz) = (x,y) \cap (y,z) \cap (x,z)$

Fact: We can always write $I = Q_1 \cap \cdots \cap Q_k$, Q_i primary ideals, with $\sqrt{Q_i} \neq \sqrt{Q_j}$ for all $i \neq j$, and such that no Q_i can be deleted.
In that case, $\{\sqrt{Q_1}, \ldots, \sqrt{Q_k}\} = \operatorname{Ass}(I)$

Reminder: An associated prime of I is a prime ideal Q such that for some $r \in R$, $r \notin I$,
$$Q = \{s \in R : sr \in I\} = \operatorname{ann}_R(r + I).$$

All the minimal primes of I are associated.
Associated primes that are not minimal are called embedded.

Associated primes appear all over in Commutative Algebra!

If we take a prime ideal P, P^n might not be primary. But we can take a primary decomposition of P^n, which will look like

$$P^n = \underbrace{Q_P}_{\text{component associated to } P} \cap \underbrace{Q_1 \cap \dots \cap Q_k}_{\text{components associated to embedded primes.}}$$

so $\mathrm{Ass}(P^n) = \{P, \underbrace{\sqrt{Q_1}, \dots, \sqrt{Q_k}}_{\text{primes containing } P}\}$

What is that component Q_P?

the <u>n-th symbolic power</u> of a prime ideal P is given by

$$\begin{aligned} P^{(n)} &= P^n R_P \cap R \\ &= P\text{-primary component of } P^n \\ &= \text{smallest primary ideal containing } P^n \text{ with radical } P \\ &= \{ f \in R : sf \in P^n \text{ for some } s \notin P \} \end{aligned}$$

these are <u>NOT</u> the same thing as the ordinary powers of P.

Example $\quad P = \ker\left(k[x,y,z] \longrightarrow k[t^3,t^4,t^5]\right) =: P(3,4,5)$

$$P = (\underbrace{x^3 - yz}_{\substack{f \\ \deg 9}},\ \underbrace{y^2 - xz}_{\substack{g \\ \deg 8}},\ \underbrace{z^2 - x^2y}_{\substack{h \\ \deg 10}})$$

$\deg x = 3$
$\deg y = 4$
$\deg z = 5$

$$\underbrace{f^2 - gh}_{\deg 18} = \underbrace{x}_{\deg 3} \underset{\deg 15}{q} \in P^2 \qquad x \notin P \implies q \in P^{(2)}$$

But elements in P^2 have degree $\geqslant 16$ (!) so $q \notin P^2$.
thus this proves that $P^2 \subsetneq P^{(2)}$.

Fun fact (this will come back to haunt us): $P^{(3)} \subseteq P^2$

theorem (Zariski–Nagata)
R regular ring containing a field k, P prime ideal in R

$$\text{then} \quad P^{(n)} = \bigcap_{\substack{m \supseteq P \\ m \text{ maximal}}} m^n .$$

this says that $P^{(n)}$ = functions that vanish up to order n along the variety defined by P

Sidenote this can be phrased in terms of differential operators.

If $R = \mathbb{C}[x_1, \dots, x_d]$, the ring of differential operators of R is given by

$$D_{R|k} = \mathbb{C}\langle \underbrace{x_1, \dots, x_d}_{\deg 0}, \underbrace{\partial/\partial x_1, \dots, \partial/\partial x_d}_{\deg 1} \rangle$$

$D^n_{R|\mathbb{C}}$ = differential operators of order $\leq n$

The <u>n-th differential power</u> of P is the ideal

$$P^{\langle n \rangle} = \{ r \in R : \partial(r) \in P \text{ for all } \partial \in D^{n-1}_{R|\mathbb{C}} \}$$

<u>theorem</u> (well-known / general proof in Dao, De Stefani, –, Huneke, Núñez Betancourt)

If P is a prime ideal in R, then $\underline{P^{\langle n \rangle}} = \underline{P^{(n)}}$ for all $n \geq 1$

<u>Sidenote</u> there is a general version of this for regular rings containing a perfect field.
In mixed characteristic, this theorem is false. However, there is a fix.
(De Stefani, –, Jeffries) Besides differential operators, need Buium's p-derivations

The n-th <u>symbolic</u> power of I is $I^{(n)} = \bigcap_{P \in \mathrm{Ass}(I)} (I^n R_P \cap R)$

<u>Properties</u>:

1) $I^n \subseteq I^{(n)}$
2) $I^{(a)} \subseteq I^{(b)}$ for $a \geq b$

Example $I = (xy, xz, yz) = (x,y) \cap (y,z) \cap (x,z)$

$$I^{(2)} = (x,y)^2 \cap (y,z)^2 \cap (x,z)^2 \ni xyz$$

But all elements in I^2 have degree $\geq 4 \Rightarrow xyz \notin I^2$

so $I^2 \subsetneq I^{(2)}$.

Fun fact: $I^{(3)} \subseteq I^2$.

Example: Given a finite set of points $X \subseteq \mathbb{P}^n$, symbolic powers are easy to compute:

$$I^{(n)} = \bigcap_{x \in X} m_x^n$$

Open Questions:

1) What ideals verify $I^{(n)} = I^n$?

Fact If I is generated by a regular sequence, then $I^{(n)} = I^n$ $\forall n \geq 1$

the converse is not true.

this is open even for squarefree monomial ideals!

There is a cash prize associated to solving this question.

2) How many generators does $I^{(n)}$ have?

How does that compare to the minimal number of generators of I^n?

Galetto, Geramita, Shin, Van Tuyl defined the symbolic defect to study this question.

3) When I is homogeneous, what degrees is $I^{(n)}$ generated in?
How do these relate to the degrees I^n is generated in?

Waldschmidt constant: $\hat{\alpha}(I) = \lim\limits_{n\to\infty} \frac{\alpha(I^{(n)})}{n} = \inf\limits_{n} \frac{\alpha(I^{(n)})}{n}$

$\alpha(I) := \min\{i : I_i \neq 0\}$

4) <u>Containment Problem</u> When is $I^{(a)} \subseteq I^b$?

$\Rightarrow$ this is the subject of the talk.

Might study this question asymptotically: $\rho(I) = \sup\{\frac{a}{b} : I^{(a)} \not\subseteq I^b\}$

what is $\rho(I)$? $\quad 1 \leq \rho(I) \leq h$

Characteristic p tools we will use in the talk:

R ring containing a field of characteristic $p>0$.
R comes equipped with the Frobenius homomorphism:

$$F: R \longrightarrow R \qquad\qquad (x+y)^p = x^p + y^p$$
$$x \longmapsto x^p$$

Given an ideal I, $I^{[p^e]} = (x^{p^e} : x \in I)$ is the e-th Frobenius power of I

We can view R as a module over itself via the Frobenius action, which we write $F^1_* R$, or $F^e_* R$ to for the e-th iteration of Frobenius

We say R is F-finite if $F^1_* R$ is a finitely generated R-module
eg, a quotient of a polynomial ring over a perfect field.

We say R is F-pure if F is a pure map: $M \otimes R \xrightarrow{id \otimes F} M \otimes R$ is injective for all R-modules M.

If R is F-finite, F-pure $\Leftrightarrow$ F-split: the Frobenius map splits. $R \xrightarrow{F} F^1_* R$ splitting

We say R F-finite is strongly F-regular if for all $f \in R$, f nonzerodivisor,

$$\exists e: \quad R \xrightarrow{F^e_*(f)} F^e_* R \quad \text{splits}$$

$$\{F\text{-pure rings}\} \supseteq \{\text{strongly } F\text{-regular}\} \supseteq \{\text{regular rings}\}$$
$$\Downarrow$$
Cohen-Macaulay

<u>talk</u>:

R regular ring, containing a field.

I radical ideal $\iff I = P_1 \cap \dots \cap P_k$, P_i prime

h = big height of I = largest height of an associated prime.
$= \max\{\operatorname{ht} P_i\} = \max\{\text{codim of an irreducible component}\}$

<u>Note</u>: If I is prime, h = height = codimension of the corresponding variety

the n-th symbolic power of I is $I^{(n)} = \bigcap_{P \in \operatorname{Ass}(I)} (I^n R_P \cap R) = \bigcap_{i=1}^{k} (I^n R_{P_i} \cap R)$

<u>Containment Problem</u> When is $I^{(a)} \subseteq I^b$?

<u>theorem</u> (Ein–Lazarsfeld–Smith, 2001, Hochster–Huneke, 2002, Ma–Schwede, 2017)
$\forall n \geq 1 \quad I^{(hn)} \subseteq I^n$

<u>Corollary</u> $\forall$ radical ideals I, $I^{(dn)} \subseteq I^n$, $d = \dim R$

<u>Open Question</u> Given a Noetherian ring R is there a constant $c = c(R)$ (not depending on R) such that $P^{(cn)} \subseteq P^n$ for all primes P?

<u>Example</u> $P = \ker(k[x,y,z] \to k[t^3,t^4,t^5])$, $h = 2$
$P^{(2n)} \subseteq P^n$. In particular, $P^{(4)} \subseteq P^2$. We said $P^{(3)} \subseteq P^2$.

<u>Example</u> $I = (x,y) \cap (x,z) \cap (y,z)$ $h = 2$
theorem says $I^{(4)} \subseteq I^2$, but actually $I^{(3)} \subseteq I^2$.

<u>Question</u> (Huneke, 2000) If P is a prime of height 2, is $P^{(3)} \subseteq P^2$?

<u>Note</u> this question is still open even over a polynomial ring in 3 variables.

Conjecture (Harbourne, ≤ 2006) $\quad I^{(hn-h+1)} \subseteq I^n \quad \forall n \geq 1$.

<u>Example</u>: When $h=2$, Harbourne's conjecture is $I^{(2n-1)} \subseteq I^n \quad \forall n \geq 1$

the conjecture does not hold for all radical ideals.

<u>Example</u> (Dumnicki, Szemberg, Tutaj-Gasińska, 2013)

$$I = (x(y^c - z^c), y(z^c - x^c), z(x^c - y^c)) \subseteq k[x,y,z]$$

For all $c \geq 3$, $I^{(3)} \nsubseteq I^2$.
When $k=3$, this corresponds to some specific configuration of 12 points

<u>Note</u> the original proof was for $c=3$ and $k=\mathbb{C}$, but Alexandra Seceleanu and Brian Harbourne extended the result.

this family of ideals corresponds to the Fermat Configurations of points.

there are other special configurations of points for which $I^{(3)} \nsubseteq I^2$.

these special configurations of points seem to have very particular properties. Lots of work has been devoted to studying them.

<u>Note</u>: there are no known counterexamples to $P^{(3)} \subseteq P^2$ for P prime.

Harbourne's Conjecture holds for:

- General sets of points in $\mathbb{P}^2$ (Harbourne-Huneke) and $\mathbb{P}^3$ (Dumnicki)
- Other special configurations of points (eg star configurations)
- Squarefree monomial ideals.

<u>Núñez Betancourt philosophy</u>: If a theorem holds for squarefree monomial ideals, it should hold for all ideals defining F-pure rings.

F-pure rings are a much larger class than just monomial ideals.

We will ask that R/I be an F-pure ring.

<u>Examples</u>

- I squarefree monomial ideal
- $I = I_t(X) = t \times t$ minors of the generic matrix X, $R = k[X]$
- $R/I \cong k[t\times t \text{ minors of } X]$
- $R/I \cong k[\text{all monomials in } n \text{ variables of degree } d]$ (Veronese ring)
- $R/I \cong$ nice ring of invariants.

Fedder's Criterion (1983)

(R, m) local ring of characteristic $p > 0$, I ideal in R.

$$R/I \text{ is F-pure} \iff (I^{[q]} : I) \not\subseteq m^{[q]} \quad \forall q = p^e$$

Theorem (—, Huneke, 2017)

If R/I is F-pure, then $I^{(hn-h+1)} \subseteq I \quad \forall n \geq 1$.

Proof

- $I \subseteq J$ is a local statement, meaning it is enough to show it holds after localizing at all prime ideals (associated primes of J)
- As a consequence, it is enough to show the case (R, m) local.
- $I \subseteq J \iff J : I = \{ r \in R : rI \subseteq J \} = R$.
- $(I^{[q]} : I) \subseteq (I^{(hn-h+1)} : I^n)^{[q]}$ for $q = p^e \gg 0$ $\quad (*)$

the theorem follows: if $I^{(hn-h+1)} \not\subseteq I^n$, then $(I^{(hn-h+1)} : I^n) \subseteq m$, so $I^{[q]} : I \subseteq m^{[q]} \Rightarrow R/I$ is not F-pure.

$(*)$ is the real content of the theorem.

Is this result best possible?

Example $\quad I = \bigcap_{i \neq j} (x_i, x_j) = (x_1 \cdots \hat{x}_i \cdots x_v : 1 \leq i \leq v) \subseteq k[x_1, \dots, x_v]$

Take $n < v$.

$$\underbrace{(x_1 \cdots x_v)^{n-1}} \in I^{(2n-2)} = \bigcap_{i \neq j} (x_i, x_j)^{2n-2} \notin I^n$$

$$\text{degree} = v(n-1) = vn - v \underset{\text{if } n<v}{<} nv - n = n(v-1) = \text{smallest degree in } I^n$$

this is a squarefree monomial ideal $\Rightarrow$ it defines an F-pure ring.

However, if we ask for more than R/I F-pure, we can get better results.

Theorem (–, Huneke, 2017) $h \geq 2$, R F-finite.
If R/I is strongly F-regular, then $I^{((h-1)(n-1)+1)} \subseteq I^n$ $\forall n \geq 1$.

<u>Idea</u> Use Glassbrenner's Criterion

<u>Glassbrenner's Criterion</u> (1996) (R, m) F-finite
R/I strongly F-regular $\iff$ for all $c \notin$ a minimal prime of I, $c(I^{[q]} : I) \not\subseteq m^{[q]}$ for all $q = p^e$

The key step in proving this theorem is the following lemma:

$$\underbrace{(I^d : I^{(d)})}_{\text{always contains an element } c \notin \text{ minimal prime of } I} (I^{[q]} : I) \subseteq (I I^{(d-h+1)} : I^{(d)})^{[q]} \quad \text{for all } q = p^e, \; d \geq h-1$$

If $I^d = I^{(d)}$, then $(I^d : I^{(d)}) = R$.

If $I^d \neq I^{(d)}$, then $I^d = I^{(d)} \cap$ (something containing such c)

All the examples we gave above are strongly F-regular, except for squarefree monomial ideals.

Corollary (–, Huneke, 2017)
If P is a prime ideal of height 2 defining a strongly F-regular ring, then

$$P^{(n)} = P^n \quad \text{for all } n \geq 1.$$

Example: $P = \ker\left(k[a,b,c,d] \longrightarrow k[s^3, s^2t, st^2, t^3] \right)$, char > 3

is a prime of height 2, not generated by a regular sequence.

$P = (b^2 - ac,\ c^2 - bd,\ bc - ad)$ theorem $\Rightarrow$ $P^{(n)} = P^n$

Example (Singh) $R = k[a,b,c,d]$, e a fixed integer

$P = I_2\begin{pmatrix} a^2 & b & d \\ c & a^2 & b^n - d \end{pmatrix}$ prime of height 2, R/P strongly F-regular.

$$P^{(n)} = P^n \quad \forall n \geq 1.$$

However, we do not know if our result is best possible when $h \geq 3$.

Moral: Harbourne's Conjecture holds for nice classes of ideals.

Moreover, maybe there are other versions of the conjecture that hold for all ideals.

<u>Example</u> (Fermat configurations of points)

$$I = (x(y^n - z^n),\ y(z^n - x^n),\ z(x^n - y^n)) \subseteq k[x,y,z]$$

Harbourne's Conjecture would say $I^{(2n-1)} \subseteq I^n$

- this fails when $n = 2$.
- Actually this holds for $n \geq 3$.

<u>Conjecture</u> (A stable version of Harbourne)
Given a radical ideal I in a regular ring, there exists N such that $I^{(hn-h+1)} \not\subseteq I^n$ for all $n \geq N$.

How would we prove this? One could be to show the following:

<u>Question</u>: If $I^{(hk-h+1)} \subseteq I^k$ holds for some value of k, does that imply that $I^{(hn-h+1)} \subseteq I^n$ holds for all $n \gg 0$?

<u>Comment</u>: Methods by Hochster and Huneke show that in characteristic p,
$$I^{(hq-h+1)} \subseteq I^{[q]} \subseteq I^q \qquad q = p^e.$$
So if the answer to this question is yes, then we are done in char p.

<u>theorem</u> (–) If $I^{(hk-h)} \subseteq I^k$ for some value of k, then $I^{(hn-h)} \subseteq I^n$ for all $n \gg 0$.

<u>Comment</u>: the statement is not for all $n \geq k$, but rather for $n \geq hk$.

Example When $h=2$, If $I^{(4)} \subseteq I^3$ then $I^{(2n-2)} \subseteq I^n$ for all $n \geq 6$.

Example $P = P(t^3, t^4, t^5)$ $\quad P^{(4)} \subseteq P^3 \Rightarrow P^{(2n-2)} \subseteq P^n \ \forall n \geq 6$.

Soon we will see other classes of ideals verifying this condition.

Another approach:

Definition (Bocci–Harbourne, 2010) the resurgence of I is given by

$$\rho(I) = \sup\left\{\frac{a}{b} : I^{(a)} \not\subseteq I^b\right\}$$

Remark $1 \leq \rho(I) \leq h$

Observation If $\rho(I) < h$, then for any $C > 0$ $\exists N$ such that $I^{(hn-C)} \subseteq I^n$ for all $n \geq N$.

since: $\dfrac{hn - C}{n} > \rho(I) \iff n > \dfrac{C}{h - \rho(I)}$ has a solution.

Example (Fermat configuration) $I^{(2n-1)} \subseteq I^n$ for all $n \geq 3$.

$\rho(I) = \frac{3}{2}$ (Dumnicki, Harbourne, Nagel, Seceleanu, Szemberg, Tutaj-Gasińska)

then $I^{(2n-1)} \subseteq I^n$ holds for all $n > \dfrac{1}{2 - \frac{3}{2}} = 2$.

Open Question Is there a radical ideal I of bigheight h such that $\rho(I) = h$?

Note Such an ideal might still verify the stable version of Harbourne's Conjecture
eg, if $I^{(hn-h)} \not\subseteq I^n$ for all n, then $\rho(I) = h$

How about primes? Does Harbourne's Conjecture hold for primes?
this is still open, even in dimension 3.

Where to start? $P = P(a,b,c)$
Note: the symbolic powers of these ideals are a lot more interesting than one might expect. For instance, we might study the symbolic Rees Algebra of I, $\bigoplus_{n \geq 0} I^{(n)} t^n \subseteq R[t]$ A famous question of Cowsik asked if this is always finitely generated (for a prime ideal) Paul Roberts answered negatively.
Whether or not the symbolic Rees Algebra of $P(a,b,c)$ is Noetherian depends on a,b,c.

theorem (–) $P = P(a,b,c)$ in char > 5 then $P^{(3)} \subseteq P^2$ and $P^{(5)} \subseteq P^3$.

theorem (–) If $a \leq b \leq c$, $P = P(a,b,c)$, and $a = 3,4$, then $P^{(4)} \subseteq P^3$.

$\Rightarrow P^{(2n-2)} \subseteq P^n$.

Note: However, $P^{(4)} \subseteq P^3$ does not hold for all choices of a, b, c.
the smallest counterexample is $9, 11, 14$.