Slicing a square pizza

University of Michigan Math Club 28/03/2019

Mathematicians have definitions for everything...

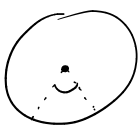
Definition (Pizza slicing)

Given a (square) pizza, an appropriate pizza slicing is one where

1) All slices are triangles
2) Two slices are either disjoint or share a vertex or a whole edge.
3) there are finitely many slices
4) Every slice has the same area.

Examples

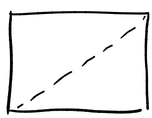
1) Traditional Pizza



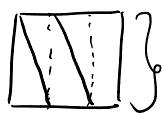
$\frac{2\pi}{n}$

Of course these aren't quite triangles...

2) Square pizza



2 slices



1

$1/n$

$2n$ slices

3) 3 slices on a square pizza? ?!

This question was posed in 1965 by Fred Richman, who was looking for a question to include in his master student's final exam.
He planned to include this question on the exam, but decided against it when he couldn't answer it. He came up with some partial results which he submitted to a journal, but the paper was rejected. The report said:
"The answer to this question is probably easy and well-known, even though I don't actually have an answer for it right now."
He eventually got his joint paper with John Thomas published in the American Mathematical Monthly, which caught the attention of Paul Monsky.

<u>Theorem</u> (Monsky, 1970)

There is no way to appropriately slice a square pizza into an odd number of slices.

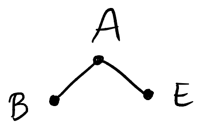
<u>Solution</u> Get odd friends.

<u>Our goal</u> Prove Monsky's theorem.

For that, we need to develop some unexpected tools...

Graph Set of vertices V and a set of edges E (both finite)
Each edge connects two distinct vertices
Between 2 vertices we have 0/1 edges

Example Adam is friends with Becca and Emma
Emma and Becca don't know each other



Handshaking Lemma

- the number of people at a party that have an odd number of friends is even.

OR more formally

- In a graph, the number of vertices of odd degree is even

degree = # edges out of a vertex

Example Emma and Becca (2) are friends with only 1 person.

Proof Write down the degrees of all the vertices and add them up

$N =$ sum of all degrees of all the vertices

Each edge was counted exactly twice!

$N = 2 \# \text{ edges}$

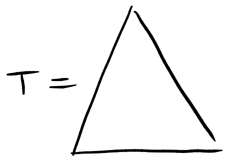
$\downarrow$

sum of however many even numbers and

evenly many odd numbers

Our example $N = 2+1+1 = 4 = 2 \cdot 2$

tool 2: Sperner's lemma



big triangle

Divide T into any finite number of triangles of ANY size st two triangles share a vertex, a whole edge, or are disjoint

Color every triangle vertex with Red, Blue or Green:

Coloring Rules:

- the vertices of T have all 3 colors
- Every small triangle vertex that lies on a side of T can only have 2 colors: the same as the vertices of T on that side
- No rules for inside vertices

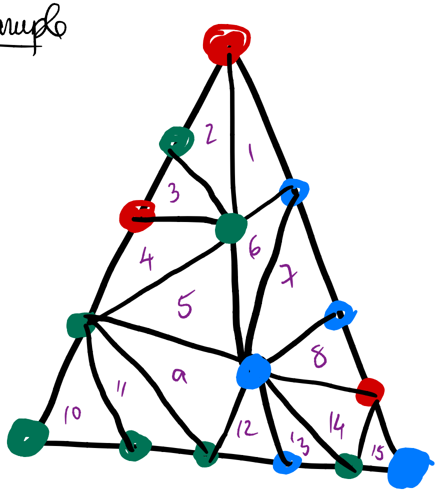
Theorem (Sperner's lemma) At least one of the small triangles has 3-colored vertices. In fact, the number of 3-colored small triangles is odd.

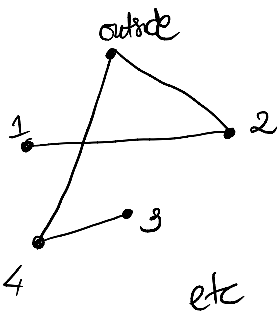
<u>Proof</u> Turn our picture into a graph.

Vertices = 1 vertex for each small triangle
1 vertex to represent "outside of T"

Edges = Connect two vertices if the corresponding triangles share a Red Green side

<u>Example</u>





Now focus on: Red-Green side of T

Along that side we only see red or green.

Eventually, as we move along from one end to the other the color changes. How many times? An odd number!

Every edge of our graph going through the outside vertex comes from a triangle on the Red Green side.

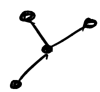
the outside vertex has odd degree!

By the Handshaking Lemma, there is an even number of vertices with odd degree $\Rightarrow$ so there is an odd number of vertices of odd degree <u>besides</u> the outside one!

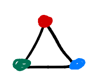
What is a vertex of odd degree in our graph?

<u>Possible</u> degrees: 0, 1, 2, 3

(since each small triangle touches at most 3 triangles)

Degree 3:  $\equiv$ 

this is impossible!!!

So odd degree $\equiv$ degree 1 $\equiv$  $\equiv$ 

<u>Conclusion</u> Our graph has an odd (≥ 1) number of "inside" vertices of degree 1

$\Longrightarrow$ there is an odd number of 3 colored triangles!

3^{rd} tool: 2-adic valuations

For each integer k, the 2-adic value of k is:

$$|k|_2 = \frac{1}{\text{largest power of 2 that divides } k}$$

Define $|0|_2 = 0$.

Examples

$$|3|_2 = \frac{1}{2^0} = 1$$

$$|4|_2 = \frac{1}{2^2} = \frac{1}{4}$$

$$|6|_2 = \frac{1}{2^1} = \frac{1}{2}$$

take $(a,b)=1$

the 2-adic value of a rational number $\frac{a}{b}$ is

$$\left|\frac{a}{b}\right|_k = \begin{cases} 1 & \text{if } a,b \text{ odd} \\ |a|_2 & \text{if } a \text{ is even} \\ \frac{1}{|b|_2} = \text{largest power of 2 dividing } b \end{cases} = \frac{|a|_2}{|b|_2}$$

Examples $\left|\frac{3}{20}\right|_2 = 4$ $\quad \left|\frac{20}{3}\right|_2 = \frac{1}{4}$

Idea smallest = most even

Properties 1) $|xy|_2 = |x|_2 \, |y|_2$

2) $|x+y|_2 \leq |x|_2 + |y|_2$

In fact, $|x+y|_2 \leq \max\{ |x|_2, |y|_2 \}$

with equality if and only if $|x|_2 \neq |y|_2$

Fun facts:

1) there is a p-adic absolute value for every prime p

2) the usual absolute value and the p-adic absolute are the only functions with those properties

theorem We can extend 2-adic absolute values to $\mathbb{R}$.

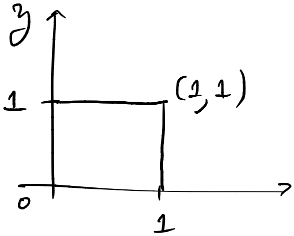
Remark this requires the Axiom of Choice!

Example $|\sqrt{2}|_2 \underset{\substack{\downarrow \\ \text{property 1}}}{=} \frac{1}{\sqrt{2}}$

Now we are ready to prove Monsky's theorem!

Proof of Monsky's theorem:

Step 1



Step 2

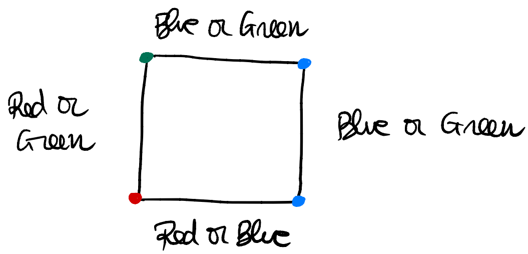
Divide into any number of triangles of any size

Step 3 Color every point in $\mathbb{R}^2$ as follows:

- Red: $|x|_2 < 1$ and $|y|_2 < 1$
- Blue: $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$
- Green: $|y|_2 \geq 1$ and $|y|_2 > |x|_2$

Notes

- (0, 0) Red
- (1, 0) Blue
- (0, 1) Green
- (1, 1) Blue



Step 4 Pretend our square is a triangle having a bad day.

(think of the Blue Green sides as one)

Sperner's lemma $\Rightarrow$ there is a 3-colored triangle!

Step 5 Look at the 3 color triangle.
Move it to have its red vertex at $(0,0)$

Step 6 Show this doesn't change the colors of the vertices

Red point used to be (a,b)
translation by $(x,y) \mapsto (x-a, y-b)$
(a,b) goes to $(0,0) \equiv$ Red

(a,b) red means $|a|_2 < 1$, $|b|_2 < 1$

So:
$$|x-a|_2 = |x+(-a)|_2 \leq \max\{|x|_2, |a|_2\}$$

For x with $|x|_2 \geq 1$, get $|x-a|_2 = |x|_2$
(Note: then $|x|_2 \neq |a|_2$)

- If both $|x|_2, |y|_2 \geq 1$, also $|y-b|_2 = |y|_2$
- If $|x|_2 < 1$ or $|y|_2 < 1$, $|x-a|_2$ or $|y-b|_2$ might change, but it <u>can only</u> decrease $\Rightarrow$ same color!

Step 7 We have a triangle of vertices $(0,0)$, (x_1, y_1), (x_2, y_2)
Red Blue Green

area $= \frac{1}{2} | x_1 y_2 - x_2 y_1 |$

$$|\text{area}|_2 = \left| \frac{1}{2} (x_1 y_2 - x_2 y_1) \right|_2 = \left|\frac{1}{2}\right|_2 |x_1 y_2 - x_2 y_1|_2$$
$$= 2 \, |x_1 y_2 - x_2 y_1|_2 \leq 2 \max\{ |x_1|_2 |y_2|_2 , |x_2|_2 |y_1|_2 \}$$

(x_1, y_1) blue $\equiv$ $|x_1|_2 \geq |y_1|_2$ and $|x_1|_2 \geq 1$

(x_2, y_2) green $\equiv$ $|y_2|_2 > |x_2|_2$ and $|y_2|_2 \geq 1$

$\Rightarrow |x_1|_2 \, |y_2|_2 > |y_1|_2 \, |x_2|_2$

so $|\text{area}|_2 = 2 |x_1|_2 \, |y_2|_2 \geq 2$

Conclusion Any way we divide the square into small triangles, one of them has area with $|\text{area}|_2 \geq 2$.

Final Step If there were n triangles of the same area, the area of each one would be $\frac{1}{n}$.

But $\left| \frac{1}{n} \right|_2 \underset{\substack{\downarrow \\ n \text{ odd}}}{=} 1 < 2$