

Def R ring
 M R -module

$p \in \text{Spec}(R)$ is a minimal prime of R if

$$I \subseteq Q \subseteq P \quad \Rightarrow \quad Q = I \\ Q \in \text{Spec}(R)$$

$$\text{Min}(I) = \{P \in \text{Spec}(R) \mid P \text{ minimal prime of } I\}$$

$$\text{Supp}(M) := \{P \in \text{Spec}(R) \mid M_P \neq 0\} \quad \text{support of } M$$

Facts $\text{Supp}(M) = \emptyset \iff M = 0$

$$\text{Supp}(M) = \sqrt{(\text{ann}(M))}$$

R Noetherian
 M fg R -mod $\Rightarrow |\text{Min}(I)| < \infty$

$p \in \text{Spec}(R)$ is an associated prime of M if

$$p = \text{ann}(m) \text{ for some } m \in M$$

$$\begin{array}{c} \Updownarrow \\ R/p \hookrightarrow M \end{array}$$

$$\text{Ass}(M) := \{p \in \text{Spec}(R) \mid p \text{ associated to } M\}$$

I ideal

associated primes of $I \equiv$ associated primes of R/I

Lemma $\text{Ass}(R/P) = \{P\}$ P prime

Proof $\text{ann}_R(\underbrace{x+P}_{\substack{\neq 0 \\ \Rightarrow x \notin P}}) = \{s \in R \mid sx \in P\} = P$ \downarrow
 P prime

Lemma R Noetherian ring
 M R -module

0) Every $\text{ann}(M)$ is contained in an associated prime of M

1) $\text{Ass}(M) = \emptyset \Leftrightarrow M = 0$

2) $\bigcup_{P \in \text{Ass}(M)} P = \{ \text{zero divisors on } M \}$

$$:= \{x \in R \mid xm = 0 \text{ for some } m \neq 0 \in M\}$$

If R and $M \neq 0$ are graded, M has an associated prime that is homogeneous

Proof $M = 0 \Rightarrow \text{Ass}(M) = \emptyset$ by definition. Let $M \neq 0$

If 0) holds, then $M \neq 0 \Rightarrow \exists m \neq 0 \Rightarrow \text{ann}(m) \subseteq \text{ass prime} (\text{exists!}) \Rightarrow 1)$

2) By definition, $P \in \text{Ass}(M) \Rightarrow P \subseteq \text{Z}(M)$.
 $x \in \text{Z}(M) \Rightarrow x \in \text{ann}(m) \subseteq \text{ass prime}$ $\Rightarrow 2)$

Proof of 0):

$M \neq 0 \Rightarrow S = \{\text{ann}(m) \mid m \in M, m \neq 0\} \neq \emptyset$

Any element in S is contained in a maximal element of S
(because R is Noetherian!)

Let $I = \text{ann}(m)$ be a maximal element in S

$$rs \in I, s \notin I \Rightarrow sm \neq 0 \Rightarrow \underbrace{\text{ann}(sm)}_{\in S} \supseteq \underbrace{\text{ann}(m)}_{\text{max in } S}$$

$$\Rightarrow \text{ann}(m) = \text{ann}(sm)$$

$$\text{so } r(sm) = \underbrace{(rs)}_{\in \text{ann}(m)} m = 0 \Rightarrow r \in \text{ann}(sm) = \text{ann}(m)$$

$$\Rightarrow r \in I$$

$$\therefore I \text{ is prime.} \Rightarrow I \in \text{Ass}(M)$$

Graded case: $\{ \text{ann}(m) \mid m \neq 0 \text{ is a homogeneous element} \}$

$$\begin{aligned} m \text{ homogeneous} & \Rightarrow f_{a_1} m + \dots + f_{a_n} m = 0 \Rightarrow \text{all } f_{a_i} m = 0 \\ f^m = 0 & \\ f = f_{a_1} + \dots + f_{a_n} & \quad \underbrace{\text{homogeneous}}_{\text{homogeneous}} \end{aligned}$$

\therefore the annihilators of homogeneous elements are homogeneous

Repeat the argument for the annihilators of homogeneous elements

Need: domma R 2-graded, I an ideal satisfying

for any homogeneous
 $r, s \in R$

$$rs \in I \Rightarrow r \in I \text{ or } s \in I$$

then I is prime.

Lemma $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ ses

$$\text{Ass}(L) \subseteq \text{Ass}(M) \subseteq \text{Ass}(L) \cup \text{Ass}(M)$$

Proof $p \in \text{Ass}(L) \Rightarrow R/p \subseteq L \subseteq M \xrightarrow{\text{assumption}} p \in \text{Ass}(M)$

$p \in \text{Ass}(M)$, say $p = \text{ann}(m)$

$\Rightarrow p \subseteq \text{ann}(rm)$ for all $r \in R$

Option 1 $rm \in L$ for some $r \notin p$

$$a(rm) = 0 \Leftrightarrow (ar)m = 0 \Rightarrow ar \in p \Rightarrow a \in p$$

$$\therefore \text{ann}(rm) = p$$

$$\therefore p \in \text{Ass}(N)$$

Option 2 $rm \notin L$ for all $r \notin p$

Let $n := \text{image of } m \text{ in } N$

$$\left\{ \begin{array}{l} pm = 0 \Rightarrow pn = 0 \Rightarrow p \subseteq \text{ann}(n) \\ rn = 0 \Rightarrow rm \in L \Rightarrow r \in p \end{array} \right. \Rightarrow \text{ann}(n) = p$$

$$\therefore p \in \text{Ass}(N)$$

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Note : Both these inclusions may be strict.

Theorem R Noetherian M fg R -module

there exists a (prime) filtration of M

$$M = M_t \supsetneq M_{t-1} \supsetneq M_{t-2} \supsetneq \dots \supsetneq M_1 \supsetneq M_0 = 0$$

where $M_i/M_{i-1} \cong R/P_i$ for some primes $P_i \in \text{Spec}(R)$

If R and M are \mathbb{Z} -graded, there exists a filtration with

$$M_i/M_{i-1} \cong R/P_i(t_i) \text{ for homogeneous primes } P_i, t_i \in \mathbb{Z}$$

Proof If $M \neq 0$, then $\text{Ass}(M) \neq 0$, so $\exists R/P_1 \cong M_1 \subseteq M$

If $M/M_1 \neq 0$, then $\text{Ass}(M) \neq 0$, so $\exists R/P_2 \cong \frac{M_2}{M_1} \subseteq R/M_1$

Continue this process: $M_0 = 0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots$

M Noetherian $\Rightarrow M$ Noetherian \Rightarrow chain stops.

Graded Case: if $P_i = \text{ann}(m_i)$, m_i homogeneous of degree t_i

$(R/P_i)(t_i) \xrightarrow{\cong} R m_i$ is degree preserving \blacksquare

Note to find m_i with $\text{ann}(m_i) = P_i$, look at $0 :_{M_i} P_i$

Our m_i is in there somewhere

Corollary R Noetherian M fg R -module

$M = M_t \supsetneq M_{t-1} \supsetneq \dots \supsetneq M_1 \supsetneq M_0 = 0$ prime filtration with $M_i/M_{i-1} \cong R/P_i$
then:

1) $\text{Ass}(M) \subseteq \{P_1, \dots, P_t\} \Rightarrow |\text{Ass}(M)| < \infty$

2) M graded $\Rightarrow \text{Ass}(M)$ is a finite set of homogeneous primes

Proof Just need to show $\text{Ass}(M) \subseteq \{P_1, \dots, P_t\}$

$$0 \rightarrow M_{i-1} \rightarrow M_i \rightarrow M_i/M_{i-1} \rightarrow 0 \text{ ses}$$

$$\begin{aligned} \Rightarrow \text{Ass}(M_i) &\subseteq \text{Ass}(M_{i-1}) \cup \text{Ass}(M_i/M_{i-1}) \\ &= \text{Ass}(M_{i-1}) \cup \{P_i\} \end{aligned}$$

(skip proof)

Theorem Associated primes localize: R Noetherian, M R -mod

$$\text{Ass}_{w^{-1}R}(w^{-1}M) = \{w^{-1}p \mid p \in \text{Ass}(M), p \cap w = \emptyset\}$$

Proof $p \in \text{Ass}(M), p \cap w = \emptyset \Rightarrow w^{-1}p$ prime in $w^{-1}R$

$$\begin{aligned} 0 \rightarrow R/P \rightarrow M &\implies 0 \rightarrow w^{-1}(R/P) \rightarrow w^{-1}M \\ &\qquad \qquad \qquad \text{IR} \\ \Rightarrow w^{-1}p &\in \text{Ass}(M) \qquad \qquad \qquad w^{-1}R/w^{-1}P \end{aligned}$$

$Q \in \text{Ass}(w^{-1}M) \rightsquigarrow Q = w^{-1}p \text{ for some } p \in \text{Spec } R, p \cap w = \emptyset$

$$R \text{ Noetherian} \Rightarrow p = (f_1, \dots, f_n) \Rightarrow Q = \left(\frac{f_1}{1}, \dots, \frac{f_n}{1}\right)$$

$$Q = \text{ann}\left(\frac{x}{w}\right) = \text{ann}\left(\frac{r}{1}\right)$$

$$\Rightarrow \frac{f_i}{1} \cdot \frac{x}{1} = \frac{0}{1} \Rightarrow w_i f_i x = 0 \text{ for some } w_i \in w$$

$$\Rightarrow w f_i x = 0 \text{ for all } i, w = w_1 \cdots w_n$$

$$\Rightarrow p(wx) = 0$$

Claim : $p = \text{ann}(wx)$

Why? If $v \in \text{ann}(wx)$, then

$$vwx = 0 \iff w(vx) = 0 \iff \frac{vx}{1} = \frac{0}{1} \text{ in } w^{-1}R$$
$$\Rightarrow \frac{v}{1} \in \text{ann}\left(\frac{x}{1}\right) = w^{-1}p \Rightarrow t v \in p \text{ for some } t \in w$$

$\begin{array}{c} t \notin p \\ p \text{ prime} \end{array} \Rightarrow v \in p$!

$$\therefore p \in \text{Ass}(M)$$

□

Corollary R Noetherian, M R -module

$$1) \text{Supp}(M) = \bigcup_{p \in \text{Ass}(M)} V(p)$$

$$2) M \neq 0 \Rightarrow \text{Min}(\text{ann}(M)) \subseteq \text{Ass}(M)$$

In particular, $\text{Min}(I) \subseteq \text{Ass}(R/I)$

Proof 1) $p \in \text{Ass}(M)$, say $p = \text{ann}(m)$

$q \in V(p) \Rightarrow p_q \subseteq q_q$, and $Rq/pq \neq 0 \Rightarrow q \in \text{Supp}(R/p)$

$$0 \rightarrow R/p \rightarrow M \xrightarrow{\text{exact}} 0 \rightarrow \underbrace{(R/p)_q}_{\neq 0} \rightarrow M_q \text{ exact}$$
$$\Rightarrow q \in \text{Supp}(M)$$

$q \notin \bigcup_{p \in \text{Ass}(M)} V(p) \Rightarrow q \neq p \text{ for all } p \in \text{Ass}(M)$

$\Rightarrow p \cap (R \setminus q) \neq \emptyset \text{ for all } p \in \text{Ass}(M)$

$\Rightarrow \text{Ass}_{R_q}(M_q) = \emptyset$

$\Rightarrow M_q = 0$

2) $V(\text{ann}(M)) = \text{Supp}(M) = \bigcup_{p \in \text{Ass}(M)} V(p)$

\Rightarrow minimal elements agree

\Rightarrow the minimal primes of $\text{ann}(M)$ are all in $\bigcup_{p \in \text{Ass}(M)} V(p)$

\Rightarrow minimal primes must be associated

$\therefore P$ minimal prime of $M \Rightarrow P \in \text{Ass}(M)$
(over $\text{ann}(M)$)

Associated primes that are not minimal = embedded

Prime avoidance R any ring \mathcal{J} ideal in R
 I_1, \dots, I_n ideal in R , I_i prime for $i > 2$

$$\mathcal{J} \not\subseteq I_i \text{ for all } i \Rightarrow \mathcal{J} \not\subseteq \bigcup_{i=1}^n I_i$$

$$\Leftrightarrow \mathcal{J} \subseteq \bigcup_{i=1}^n I_i \Rightarrow \mathcal{J} \subseteq I_i \text{ for some } i$$

Proof Induction on n

$n=1 \rightarrow$ nothing to show

$n \geq 2$: by induction, can find

$$a_i \notin \bigcup_{j \neq i} I_j \quad a_i \in \mathcal{J} \quad \text{for each } i$$

If $a_i \notin I_i$, we are done. So assume $a_i \in I_i$ for all i

$$a := \underbrace{a_n + a_1 \dots a_{n-1}}_{\notin I_i \in I_i, i < n} \in \mathcal{J} \Rightarrow a \notin I_i \text{ for all } i < n$$

$$a = \underbrace{a_n}_{\in I_n} + \underbrace{a_1 \dots a_{n-1}}_{\in I_n} \Leftrightarrow \underbrace{\in I_n}_{\in I_n} \quad \text{Suppose } a \in I_n \Rightarrow a_1 \dots a_{n-1} \in I_n$$

$$n=2 \Leftrightarrow a_1 \in I_2 \not\subseteq I_1 \Rightarrow a \notin I_n$$

$$n > 2 \Rightarrow I_n \text{ prime} \Rightarrow a_i \in I_n \text{ for some } i \not\subseteq I_1 \dots I_{n-1} \quad \therefore a \notin I_i \text{ for all } i$$

Graded prime avoidance R N -graded

I_i, \mathcal{J} all homogeneous

$\mathcal{J} \not\subseteq I_i$ for all $i \Rightarrow$ there exists a homogeneous element in \mathcal{J} not in $\bigcup_{i=1}^n I_i$