Commutative Algebra II

Math 906 Spring 2026

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Warning!

Proceed with caution. These notes are under construction and are 100% guaranteed to contain typos. If you find any typos or errors, I will be most grateful to you for letting me know.

A note on external references

These notes make many direct references to my Commutative Algebra I and Homological Algebra notes, the content of which is assumed to be known to the reader.

Acknowledgements

These notes take much inspiration from other sources, including Tom Marley's notes from Spring 2024 Math 906, and especially from Bruns and Herzog's book *Cohen-Macaulay rings* [BH93].

Contents

Chapter 0

Setup

Throughout, all rings are commutative with identity 1, all modules are unital, and all ring homomorphisms send 1 to 1. Moreover, all rings will be assumed to be noetherian.

When k is a field, the polynomial ring $R = k[x_1, \dots, x_n]$ can be given an $\mathbb{N}$-grading by setting $\deg(x_i) = d_i$ for some $d_i \in \mathbb{N}$. The most common $\mathbb{N}$-grading, also known as the **standard grading**, is the one where we declare $\deg(x_i) = 1$ for all i. Once we declare the degrees of the variables, we can extend that grading to all monomials as follows:

$$\deg\left(x_1^{a_1} \cdots x_n^{a_n}\right) = a_1 d_1 + \cdots + a_n d_n.$$

In this case, a **homogeneous element** in R is any k-linear combination of monomials of the same degree. We write R_i for the set of all homogeneous elements of degree i, which is an abelian group under addition, and note that

$$R = \bigoplus_i R_i.$$

Note also that $R_i R_j \subseteq R_{i+j}$ for all i and j. So when $R = k[x_1, \dots, x_n]$ is standard graded,

$$R_i = \bigoplus_{a_1+\cdots+a_n=i} x_1^{a_1} \cdots x_n^{a_n}.$$

More generally, a **graded ring** is any ring that can be decomposed in pieces of this form, meaning that

$$R = \bigoplus_i R_i \qquad \text{and} \qquad R_i R_j \subseteq R_{i+j}.$$

The homogeneous elements of degree i are the elements in R_i. A graded R-module is an R-module M such that

$$M = \bigoplus_i M_i \qquad \text{and} \qquad R_i M_j \subseteq M_{i+j}.$$

A homomorphism of graded R-modules $\varphi \colon M \to N$ satisfying $\varphi(M_i) \subseteq N_{i+d}$ for all i is a **graded map** of degree d. A map of degree 0 is sometimes called **degree preserving**. Any graded map can be thought of as a map of degree 0 by shifting degrees. We write $M(-d)$ for the graded R-module that has M as the underlying R-module, but where the graded structure is given by taking $M(-d)_i = M_{i-d}$.

Note that 0 can be thought of as a homogeneous element of any degree; one sometimes declares $\deg(0) = -\infty$. An ideal I in R is a **homogeneous ideal** if it can be generated by homogeneous elements; one can show that this is equivalent to the condition

$$I = \bigoplus_i (I \cap R_i).$$

Finally, whenever I itself is homogeneous, the grading on R passes onto R/I, with

$$(R/I)_i = R_i/I_i.$$

We will be concerned with finitely generated $\mathbb{N}$-graded k-algebras R with $R_0 = k$, which are of the form $R = k[x_1, \dots, x_n]/I$ for some homogeneous ideal I. One nice feature of such rings is that while there might be many maximal ideals, there is only one *homogeneous* maximal ideal, which is given by

$$R_+ = \bigoplus_{i>0} R_i.$$

In many ways, the behavior of such a graded ring and its unique homogeneous maximal ideal R_+ is an analogue to the behavior of a local ring R and its unique maximal ideal ideal $\mathfrak{m}$, though one always needs to provide a separate proof for the graded and local versions.

We will summarize this as follows:

Setting 1 (Local/graded setting)**.** We will say that we are in the local/graded setting, or more precisely that $(R, \mathfrak{m})$ is local/graded to indicate one of the following holds:

- Local setting: R is a noetherian local ring with $\mathfrak{m}$ its unique maximal ideal.
- Graded setting: $R = k[x_1, \dots, x_d]/I$, where k is a field, $k[x_1, \dots, x_d]$ has the standard grading, and I is a homogeneous ideal, and $\mathfrak{m} = R_+$ is the unique homogeneous maximal ideal of R.

In the graded setting, unless otherwise stated, any statement about modules will refer to *graded* modules, and all module homomorphisms will be assumed to be graded of degree 0.

Chapter 1

Free resolutions and the Koszul complex

1.1 Minimal free resolutions

Given an R-module M, how do we describe it? We need to know a set of generators and the relations among those generators. Going further and asking for relations among the relations (treating the relations as generators for the module of relations), and relations among the relations among the relations, and so on, we construct a free resolution for M. Free resolutions play a key role in many important constructions, and encode a lot of interesting information about our module. For example, if the module came from some geometric setting, geometric information about the module gets reflected in the free resolution.

Definition 1.1. Let M be a module over a ring R. A **projective resolution** of M is a complex of projective R-modules F_i

$$F = \cdots \longrightarrow \underset{\mathrm{n}}{F_n} \longrightarrow \cdots \longrightarrow \underset{1}{F_1} \longrightarrow \underset{0}{F_0} \longrightarrow 0$$

such that $\mathrm{H}_i(F) = 0$ for all $i \neq 0$, together with an isomorphism $\mathrm{H}_0(F) \cong M$. When all the F_i are free, we say F is a **free resolution** for M. We will abuse notation and carelessly identify F with the corresponding augmented resolution, which is the exact sequence

$$F = \cdots \longrightarrow \underset{\mathrm{n}}{F_n} \longrightarrow \cdots \longrightarrow \underset{1}{F_1} \longrightarrow \underset{0}{F_0} \longrightarrow M \longrightarrow 0.$$

Remark 1.2. In the local setting, any finitely generated projective module must be free, by Exercise 62 from Homological Algebra. In fact, Kaplansky [Kap58] showed that *any* projective module over a noetherian local ring must be free. In the graded setting, one can also show (exercise!) that every bounded below graded projective must be free. Therefore, any projective resolution in the local/graded setting is in fact a free resolution.

Thus we will focus on free resolutions. We can think of a free resolution of M as an approximation of M by free modules. Since every module is a quotient of a free module, every module has a free resolution. Let us recall the construction we saw in Homological Algebra:

Construction 1.3 (Minimal free resolution)**.** Let M be a finitely generated module over R, where R is either local or graded as in our general setup. If M has β_0 many minimal generators, then we can write a surjective R-module homomorphism from R^{β_0} to M, say

$$R^{\beta_0} \xrightarrow{\pi_0} M$$

If π_0 is an isomorphism, then $M \cong R^{\beta_0}$ is a free module of rank β_0. Otherwise, π_0 has a nonzero kernel $\ker(\pi_0)$, which must also be a finitely generated module since R is noetherian. If $\ker(\pi_0)$ is minimally generated by β_1 elements, then we repeat this process and construct a surjective R-module map from R^{β_1} to $\ker(\pi_0)$, and compose it with the inclusion of $\ker(\pi_0)$ into R^{β_0}:

$$\begin{tikzcd} R^{\beta_1} \arrow[rr, dashed] \arrow[dr, two heads] & & R^{\beta_0} \arrow[r, two heads, "\pi_0"] & M \\ & \ker(\pi_0) \arrow[ur, hook] & & \end{tikzcd}$$

The elements in $\ker(\pi_0)$ are the **relations** on our chosen minimal generators for M: if M is generated by $m_1, \ldots, m_{\beta_0}$, we can take π_0 to be the map sending each canonical basis element e_i in R^{β_0} to m_i, and each $(r_1, \ldots, r_{\beta_0}) \in \ker(\pi_0)$ corresponds to a **relation** among the m_i, meaning

$$r_1 m_1 + \cdots + r_{\beta_0} m_{\beta_0} = 0.$$

Such relations are called **syzygies**[1] of M and the module $\ker(\pi_0)$ is the first syzygy module of M, which we will denote by $\Omega_1(M)$.

Continuing this process, we construct a free resolution for M:

$$\cdots \longrightarrow F_n \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \longrightarrow 0.$$

In the local/graded setting and when M is a finitely generated (graded) module, we can choose F_i at each step to have the minimal number of generators; in that case, we say that F is a **minimal free resolution** for M.

Back in Homological Algebra, we then showed the following remarkable facts:

- Every free resolution of M has a minimal free resolution of M as a direct summand.
- Any two minimal free resolutions of M are isomorphic complexes, thus we can talk about *the* minimal free resolution of M.
- As a consequence of the previous facts, the minimal free resolution of M must have the shortest length of any resolution for M, and M has a finite resolution if and only if the minimal free resolution of M is finite.
- A free resolution F of M with differential ∂ is minimal if and only if $\partial(F) \subseteq \mathfrak{m}F$. Thus if we fix bases for all the free modules F_i, the resolution is minimal if and only if all the entries in the matrices representing ∂ have all entries in $\mathfrak{m}$.

[1]Fun fact: in astronomy, a syzygy is an alignment of three or more celestial objects.

Definition 1.4. In general, any complex (F, ∂) satisfying $\partial(F) \subseteq \mathfrak{m}F$ is called a **minimal complex**.

Definition 1.5. Let M be an R-module. A finite projective resolution

$$F = \cdots \longrightarrow 0 \longrightarrow F_c \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

has length c if $F_c \neq 0$ and $F_i = 0$ for all $i \geqslant c$. A resolution F has infinite length if $F_i \neq 0$ for all $i \geqslant 0$. The **projective dimension** of M is

$$\operatorname{pdim}_R(M) := \inf \{c \mid M \text{ has a projective resolution of length } c\}.$$

Remark 1.6. As we noted in Remark 1.2, in the local/graded setting, any projective resolution is in fact a free resolution. Thus

$$\operatorname{pdim}_R(M) = \inf \{c \mid M \text{ has a free resolution of length } c\}.$$

Note that if a module M has a finite (minimal) free resolution, then the projective dimension of M is simply the length of the minimal free resolution for M. If the minimal free resolution of M is infinite, then $\operatorname{pdim} M = \infty$.

Remark 1.7. Note that $\operatorname{pdim}(M) = 0$ if and only if M is projective (and in the local/graded setting, free).

Definition 1.8. Consider a minimal free resolution F of M, and consider the notation in Construction 1.3. The i**th syzygy module of** M, denoted $\Omega_i(M)$, is defined to be the image of π_i or equivalently the kernel of π_{i-1}.

Note that $\Omega_i(M)$ is defined only up to isomorphism.

Remark 1.9. There are two conventions for what module $\Omega_i(M)$ corresponds to: one could instead take $\Omega_i(M) = \ker(\pi_i)$. One advantage of setting $\Omega_i(M) = \ker(\pi_i)$ is that this makes $\Omega_i(M)$ a submodule of F_i. But one advantage of the convention $\Omega_i(M) = \ker(\pi_{i-1})$ we have chosen is that the module of first syzygies of M, $\Omega_1(M)$, corresponds to the relations among the generators of M, which are the first set of relations we want to consider.

We recommend always checking the definition used in a particular reference.

Exercise 1. Show that for all R-modules M and all $i \geqslant 1$, if F if the minimal free resolution for M, then there is a natural short exact sequence

$$0 \longrightarrow \Omega_{i+1}(M) \longrightarrow F_i \longrightarrow \Omega_i(M) \longrightarrow 0.$$

Remark 1.10. Suppose that at some point when building a resolution following the procedure we described in Construction 1.3, we obtain an injective map of free modules. Then its kernel is trivial, so we obtain a finite free resolution.

The projective dimension of a finitely generated module can be infinite.

Example 1.11. Let k be a field and $R = k[x]/(x^2)$, which is a local ring with maximal ideal $\mathfrak{m} = (x)$. The residue field $k = R/\mathfrak{m}$ has infinite projective dimension: indeed, its minimal free resolution is

$$\cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow k \longrightarrow 0.$$

So even cyclic modules can have infinite projective dimension.

These invariants can give us some information about Ext and Tor, and vice-versa.

Remark 1.12. If $\operatorname{pdim}_R(M) = n$ is finite, then $\operatorname{Tor}_i^R(M, -) = 0$ and $\operatorname{Ext}_R^i(M, -) = 0$ for all $i > n$, since for all R-modules N, we can compute $\operatorname{Tor}_i^R(M, N)$ and $\operatorname{Ext}_R^i(M, N)$ using a free resolution for M of length n.

The ranks of the free modules in the minimal free resolution are particularly important invariants of M:

Definition 1.13 (Betti numbers)**.** Let F be the minimal free resolution of M. The **ith Betti number** of M is

$$\beta_i(M) = \operatorname{rank}(F_i).$$

Remark 1.14. Note that $\beta_0(M) = \mu(M) = \dim_k(M/\mathfrak{m}M)$.

Example 1.15. Let $R = k[\![x, y]\!]$ and $M = R/(x^2, xy)$. Let us write a minimal free resolution for M. First, we note that M is cyclic, so we start with

$$R \longrightarrow R/(x^2, xy) \longrightarrow 0.$$

In this case, the relations on the unique generator 1 in degree 0 are $x^2 \cdot 1 = 0$ and $xy \cdot 1 = 0$, so we proceed with

$$R^2 \xrightarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} R \longrightarrow R/(x^2, xy) \longrightarrow 0.$$

Now we need to find all relations among x^2 and xy, meaning all choices of a and b in R such that $ax^2 + bxy = 0$. We note that x is a regular element on R, so $ax^2 = -bxy \implies ax = -by$. But x and y are nonassociate irreducibles, so $a \in (y)$ and $b \in (x)$. Thus

$$\underline{y} \cdot x^2 + \underline{-x} \cdot xy = 0$$

is one of the relations we are looking for, and *all* other such relations are multiples of this one. This shows that we can continue our resolution by taking

$$R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} R \longrightarrow R/(x^2, xy) \longrightarrow 0.$$

Now note that R is a domain, so the leftmost map is in fact injective, and we are done. We conclude that

$$0 \longrightarrow \underset{2}{R} \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \underset{1}{R^2} \xrightarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} \underset{0}{R} \longrightarrow R/(x^2, xy) \longrightarrow 0$$

is a free resolution for $R/(x^2, xy)$. We also took as few generators at each step as possible, so this is a minimal free resolution. We can check this more precisely by noting that all the entries in our matrices are nonunits. In particular, we learn that $\operatorname{pdim}(R/(x^2, xy)) = 2$.

Example 1.16. Let $R = k[x, y, z]$ and $M = R/(xy, xz, yz)$. We claim that the minimal free resolution for M is

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} R \longrightarrow M \longrightarrow 0.$$

The Betti numbers of M are

$$\beta_0(M) = 1 \qquad \beta_1(M) = 3 \qquad \beta_2(M) = 2.$$

This is a special case of the Hilbert–Burch Theorem [Bur68], which tells us about the shape of the minimal free resolution of cyclic modules of projective dimension 2.

In the example above, we presented a minimal free resolution without justification. How could one check by hand that this is indeed the minimal free resolution? In theory, we would find the kernel of each differential map explicitly, which can be quite challenging, and then check that it does in fact match the image of the next differential. But in specific cases, we can take advantage of other facts about resolutions to determine some or all of the betti numbers, for example, which can help justifying a particular complex is in fact a resolution. As for minimality, we can simply check that all the entries in the matrices presented are in fact in the unique (homogeneous) maximal ideal, showing that the complex is minimal.

Over the course of this class, we will be discussing some helpful properties of projective dimension and betti numbers that will help us justify these things more carefully.

In the graded setting, we can take resolutions of graded free modules and graded maps. Including this graded information enhances the kind of information we can obtain from a resolution. In particular, we can consider graded betti numbers, which take into account not only the number of generators in each homological degree but also what their internal degree (R-degree) is. We will explore this in more detail in the next section.

1.2 Graded betti numbers

Let R be a standard graded k-algebra with $R_0 = k$ and homogeneous maximal ideal $\mathfrak{m} = R_+$. Let M be a graded R-module. To write a graded free resolution for M, we choose all maps to have degree 0, so that the graded free modules in each degree are sums of copies of shifts of R. We write $R(-d)$ for the R-module R but with a new grading, where

$$(R(-d))_i := R_{i-d}.$$

Definition 1.17. Let M be a graded R-module with minimal graded free resolution F. The (i,j)**th Betti number** of M, $\beta_{ij}(M)$, counts the number of generators of F_i in degree j. Thus

$$\beta_{i,j}(M) = \text{number of copies of } R(-j) \text{ in } F_i \quad \text{and} \quad F_i = \bigoplus_j R(-j)^{\beta_{ij}(M)}.$$

Remark 1.18. At each stage, the nonzero entries in the differential must be nonunits, and thus they are homogeneous elements of positive degree. Therefore, if

$$F_i = R(-d_1)^{\beta_{id_1}} \oplus \cdots \oplus R(-d_s)^{\beta_{id_s}}$$

with $d_1 \leqslant \cdots \leqslant d_s$, and $F_{i+1} \neq 0$, then the smallest possible shift in F_{i+1} is $d_1 + 1$. In particular, $\beta_{ij}(M) = 0$ for all $j < i$.

Definition 1.19. We often collect the Betti numbers of a module in its **Betti table**:

$\beta(M)$	0	1	2	$\cdots$
0	$\beta_{00}(M)$	$\beta_{11}(M)$	$\beta_{22}(M)$	
1	$\beta_{01}(M)$	$\beta_{12}(M)$	$\beta_{23}(M)$	
2	$\beta_{02}(M)$	$\beta_{13}(M)$		
$\vdots$			$\ddots$	

By convention, the entry corresponding to (i,j) in the Betti table of M contains $\beta_{i,i+j}(M)$, and *not* $\beta_{ij}(M)$. This is how Macaulay2 displays Betti tables.

Example 1.20. From the minimal resolution in Example 1.16, we can read the graded Betti numbers of M:

- $\beta_0(M) = 1$, since M is cyclic. The unique generator lives in degree 0, so $\beta_{0,0}(M) = 1$.
- $\beta_1(M) = 3$, and these three quadratic generators live in degree 2, so $\beta_{12} = 3$.
- $\beta_2(M) = 2$. These are linear syzygies on quadrics, living in degree 1+2=3, so $\beta_{23} = 2$.

Here is the graded free resolution of M:

$$0 \longrightarrow R(-3)^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}} R(-2)^3 \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} R \longrightarrow M \longrightarrow 0.$$

Notice that the graded shifts in lower homological degrees affect all the higher homological degrees as well. For example, when we write the map in degree 2, we only need to shift the degree of each generator by 1, but since our map now lands on $R(-2)^3$, we have to bump up degrees from 2 to 3, and write $R(-3)^2$. So again we have

$$\beta_{00} = 1, \beta_{12} = 3, \text{and } \beta_{23} = 2.$$

We can now collect the graded Betti numbers of M in its Betti table:

	0	1	2
0	1	–	–
1	–	3	2

.

Example 1.21. Let k be a field, $R = k[x, y]$, and consider the ideal

$$I = (x^2, xy, y^3)$$

which has two generators of degree 2 and one of degree 3, so there are graded Betti numbers β_{12} and β_{13}. The minimal free resolution for R/I is

$$0 \longrightarrow \begin{matrix} R(-3)^1 \\ \bigoplus \\ R(-4)^1 \end{matrix} \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y^2 \\ 0 & -x \end{pmatrix}} \begin{matrix} R(-2)^2 \\ \bigoplus \\ R(-3)^1 \end{matrix} \xrightarrow{\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}} R \longrightarrow R/I.$$

Thus

$$\beta_{23}(R/I) = 1 \qquad \beta_{12}(R/I) = 2$$
$$\beta_{24}(R/I) = 1 \qquad \beta_{13}(R/I) = 1$$

and the Betti table of R/I is

$\beta(M)$	0	1	2
0	1	–	–
1	–	2	1
2	–	1	1

.

Even if all we know is the Betti numbers of M, there is lots of information we can extract about M. For more about the beautiful theory of free resolutions and syzygies, see [Eis05]. For a detailed treatment of graded free resolutions, see [Pee11].

Macaulay2. In Macaulay2, given an R-module M, we can ask for its minimal free resolution by running `res M`. If I is an ideal in R, note that `res I` returns a resolution for R/I. Currently, resolutions still default to type `ChainComplexes`, but the default will soon be replaced with `Complexes`. You can ask for a resolution of type `Complexes` by loading the `Complexes package` and asking for `freeResolution M`.

More on `ChainComplexes` vs `Complexes` in Macaulay2 can be found in Appendix A.

1.3 The residue field has the largest resolution

The residue field plays an especially important role in the story of free resolutions:

Exercise 2. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and M be a finitely generated R-module. Show that $\beta_i(M) = \dim_k \left(\operatorname{Tor}_i^R(M,k)\right) = \dim_k \left(\operatorname{Ext}_R^i(M,k)\right)$.

Remark 1.22. As a consequence of Exercise 2, we learn that

$$\operatorname{pdim}(M) = \sup\{i \mid \beta_i(M) \neq 0\} = \sup\{i \mid \operatorname{Ext}_R^i(M,k) \neq 0\} = \sup\{i \mid \operatorname{Tor}_R^i(M,k) \neq 0\}.$$

We can extend Exercise 2 to graded betti numbers once we realize that Tor and Ext of graded modules can also be given graded structures.

Remark 1.23. When R is a graded ring and M and N are graded R-modules, we can compute $\operatorname{Ext}_R^i(M,N)$ using a graded free resolution of M, and thus the Ext-modules inherit an R-graded structure.

Exercise 3. Let R be a standard graded finitely generated algebra over a field $k = R_0$ and let M be a graded R-module. Show that

$$\beta_{i,j}(M) = \text{number of copies of } R(-j) \text{ in } F_i = \dim_k \left(\operatorname{Tor}_i^R(M,k)_j\right) = \dim_k \left(\operatorname{Ext}_R^i(M,k)_{-j}\right).$$

The residue field has the largest free resolution possible.

Corollary 1.24. *Let $(R, \mathfrak{m}, k)$ be local/graded. Then for every finitely generated (graded) R-module M,*

$$\operatorname{pdim}_R(M) \leqslant \operatorname{pdim}_R(k).$$

Proof. When $i > \operatorname{pdim}_R(k)$, we must have $\operatorname{Tor}_i^R(M,k) = 0$, as noted in Remark 1.12. By Exercise 2, $\beta_i(M) = 0$. □

Definition 1.25. Let R be a domain with fraction field Q. The **rank** of a finitely generated R-module M is defined as

$$\operatorname{rank} M := \dim_Q(M \otimes_R Q).$$

Exercise 4. Check that if M is a free module, then $\operatorname{rank}(M)$ is the free rank of M.

Exercise 5. Let M be a finitely generated module over a noetherian local domain. Show that

$$\beta_i(M) = \operatorname{rank}\left(\Omega_i(M)\right) + \operatorname{rank}\left(\Omega_{i+1}(M)\right).$$

Exercise 6. Show that if M has finite projective dimension over a domain, then

$$\sum_{i=0}^{\operatorname{pdim}(M)} (-1)^i \beta_i(M) = \operatorname{rank}(M).$$

If only we had an explicit minimal free resolution of k, maybe we could use it to say something about the minimal free resolutions of other finitely generated R-modules. With that goal in mind, we take a break from thinking about free resolutions and projective dimension to discuss a very important complex – perhaps *the* most important complex.

1.4 The Koszul complex

The Koszul complex is arguably the most important complex in commutative algebra (and beyond). It appears everywhere, and it is a very powerful yet elementary tool any homological algebraist needs in their toolbox. Every sequence of elements $x_1, \dots, x_n$ in any ring R gives rise to a Koszul complex.

Construction 1.26 (The Koszul complex)**.** The **Koszul complex** on one element $x \in R$ is the complex

$$\operatorname{kos}(x) := 0 \longrightarrow \underset{1}{R} \xrightarrow{\;x\;} \underset{0}{R} \longrightarrow 0.$$

More generally, given $x_1, \dots, x_n \in R$, the **Koszul complex** with respect to $x_1, \dots, x_n$ is the complex $\operatorname{kos}(x_1, \dots, x_n)$ defined inductively as

$$\operatorname{kos}(x_1, \dots, x_n) := \operatorname{kos}(x_1, \dots, x_{n-1}) \otimes_R \operatorname{kos}(x_n).$$

You will find different sign conventions for the Koszul complex in the literature, but at the end of the day they all lead to isomorphic complexes.

Example 1.27. The Koszul complex on $f, g \in R$ is given by

$$\operatorname{kos}(f,g) = \text{Totalization of} \quad
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & R \arrow[r, "-g"] \arrow[d, "f"'] & R \arrow[r] \arrow[d, "f"] & 0 \\
 & 0 \arrow[r] & R \arrow[r, "g"'] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
 {\scriptstyle 2} & {\scriptstyle 1} & 0 & 0 & \\
 & & {\scriptstyle 0} & &
\end{tikzcd}$$

which is

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -g \\ f \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} f & g \end{pmatrix}} R \longrightarrow 0.$$

In general, computing the successive tensor products is a bit tedious. Instead, we will give an alternative way to think about the Koszul complex.

Definition 1.28. The **exterior algebra** $\bigwedge M$ on an R-module M is obtained by taking the the free R-algebra $R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots$, modulo the relations $x \otimes y = -y \otimes x$ and $x \otimes x = 0$ for all $x, y \in M$. We denote the product on $\bigwedge M$ by $a \wedge b$, and see $\bigwedge M$ as a graded algebra where the homogeneous elements in degree d consist of the image of $M^{\otimes d}$. This is a **skew commutative** algebra: for all homogeneous elements a and b

$$a \wedge b = (-1)^{\deg(a)\deg(b)} b \wedge a \quad \text{and} \quad a \wedge a = 0 \text{ whenever } a \text{ has odd degree.}$$

We denote the set of all homogeneous elements of degree n by $\bigwedge^n M$. Note also that this construction is functorial: a map $f\colon M \to N$ of R-modules induces a map

$$\begin{aligned} \bigwedge M &\xrightarrow{\quad \wedge f \quad} \bigwedge N \\ m_1 \wedge \cdots \wedge m_s &\longmapsto f(m_1) \wedge \cdots \wedge f(m_s). \end{aligned}$$

We will primarily use this construction in the case of free modules. When $M = R^n$ with basis $e_1, \dots, e_n$, then for all $1 \leqslant s \leqslant n$

$$\wedge^s M \cong R^{\binom{n}{s}} \quad \text{with basis} \quad e_{i_1} \wedge \cdots \wedge e_{i_s} \text{ ranging over all } i_1 < i_2 < \cdots < i_s.$$

Definition 1.29 (The Koszul complex, again)**.** Let $x_1, \dots, x_n$ be elements in R. The **Koszul complex** on $x_1, \dots, x_n$ is the complex

$$\operatorname{kos}(x_1, \dots, x_n) \coloneqq \; 0 \longrightarrow \bigwedge^n R^n \longrightarrow \bigwedge^{n-1} R^n \longrightarrow \cdots \longrightarrow \bigwedge^1 R^n \longrightarrow R \longrightarrow 0$$

with differential

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_s}) = \sum_{1 \leqslant p \leqslant s} (-1)^{p-1} x_{i_p} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_s}.$$

Exercise 7. Show that d as defined above is indeed a differential, meaning $d^2 = 0$.

Exercise 8. Check that our two definitions of the Koszul complex coincide.

Example 1.30. In the case of two elements, say x and y in R,

$$\operatorname{kos}(x, y) = \; 0 \longrightarrow \wedge^2 R^2 \longrightarrow \wedge^1 R^2 \longrightarrow R \longrightarrow 0$$

with $\partial(e_1) = x$, $\partial(e_2) = y$, $\partial(e_1 \wedge e_2) = xe_2 - ye_1$, so

$$\operatorname{kos}(x, y) = \; 0 \longrightarrow R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow 0.$$

Exercise 9. Write the Koszul complex on 3 elements f_1, f_2, f_3.

The previous exercise is must easier to do using the exterior algebra description of the Koszul complex.

Remark 1.31. You will find different sign conventions for the Koszul complex in the literature, but at the end of the day they all lead to isomorphic complexes.

Definition 1.32. Let M be an R-module and let $x_1, \dots, x_n \in R$. The **Koszul complex** on M with respect to $x_1, \dots, x_n$ is the complex

$$\operatorname{kos}(x_1, \dots, x_n; M) := \operatorname{kos}(x_1, \dots, x_n) \otimes_R M.$$

Remark 1.33. In general, the Koszul complex $\operatorname{kos}(x_1, \dots, x_n; M)$ looks like

$$0 \longrightarrow M \longrightarrow M^n \longrightarrow \cdots \longrightarrow M^{\binom{n}{i}} \longrightarrow \cdots \longrightarrow M^n \longrightarrow M \longrightarrow 0$$

and the nonzero entries in the differential matrices consist of our elements $x_1, \dots, x_n$ with carefully chosen some signs. The left most map, in degree n, is given by the matrix

$$\begin{pmatrix} x_1 \\ -x_2 \\ x_3 \\ \vdots \\ (-1)^{n+1} x_n \end{pmatrix}$$

and the rightmost map is

$$\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}.$$

Our exterior algebra description of the Koszul complex has the advantage that it indicates a bonus structure on our complex: it is also an algebra. In fact, this is the first big example of a DG algebra. While we will not have the chance to explore this further, this DG algebra structure on the Koszul complex plays a major role in commutative algebra.

We can also take advantage of this added structure to note that the Koszul complex is a functorial construction: given $\underline{x} = x_1, \dots, x_n \in R$ and $\underline{y} = y_1, \dots, y_m \in R$, any commutative diagram of R-module homomorphisms

$$\begin{tikzcd} R^n \arrow[r, "\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}"] \arrow[d, "\varphi_1"'] & R \arrow[d, "\varphi_0"] \\ R^m \arrow[r, "\begin{pmatrix} y_1 & \cdots & y_m \end{pmatrix}"'] & R \end{tikzcd}$$

extends to a map of complexes $\operatorname{kos}(\underline{x}) \longrightarrow \operatorname{kos}(\underline{y})$, by taking

$$\varphi_s\left(e_{i_1} \wedge \cdots \wedge e_{i_s}\right) = \varphi_1(e_{i_1}) \wedge \cdots \wedge \varphi_1(e_{i_1}).$$

The fact that this map commutes with taking the differential is an immediate consequence of the definition, which we leave as an exercise.

Notation 1. Given elements $x_1, \dots, x_n$ in a ring R, we often write $\underline{x} = x_1, \dots, x_n$.

The homology of the Koszul complex has some nice properties.

Definition 1.34. Let M be an R-module and $x_1, \dots, x_n \in R$. The **ith Koszul homology** module of M with respect to $x_1, \dots, x_d$, also called the **Koszul homology** on $x_1, \dots, x_d$ with coefficients in M, is the R-module

$$\mathrm{H}_i(x_1, \dots, x_d; M) := \mathrm{H}_i(\mathrm{kos}(x_1, \dots, x_d; M)).$$

Exercise 10. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and let $\underline{x}$ and $\underline{y}$ be two minimal generating sets for the same ideal. Show that $\mathrm{kos}(\underline{x})$ and $\mathrm{kos}(\underline{y})$ are isomorphic complexes.

Theorem 1.35. *Let R be a ring, $I = (x_1, \dots, x_n)$ be an ideal, and M an R-module.*

(a) $\mathrm{H}_i(\underline{x}; M) = 0$ *whenever $i < 0$ or $i > n$.*

(b) $\mathrm{H}_0(\underline{x}; M) = M/IM$.

(c) $\mathrm{H}_n(\underline{x}; M) = (0 :_M I) = \mathrm{ann}_M(I)$.

(d) Every Koszul homology module $\mathrm{H}_i(\underline{x}; M)$ *is killed by* $\mathrm{ann}_R(M)$.

(e) Every Koszul homology module $\mathrm{H}_i(\underline{x}; M)$ *is killed by I.*

(f) If M is a noetherian R-module, so is $\mathrm{H}_i(\underline{x}; M)$ *for every i.*

(g) For every i, $\mathrm{H}_i(\underline{x}; -)$ *is a covariant additive functor R-**Mod** $\longrightarrow$ R-**Mod**.*

(h) Every short exact sequence of R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

gives rise to a long exact sequence on Koszul homology,

$$\cdots \longrightarrow \mathrm{H}_1(\underline{x}; C) \longrightarrow \mathrm{H}_0(\underline{x}; A) \longrightarrow \mathrm{H}_0(\underline{x}; B) \longrightarrow \mathrm{H}_0(\underline{x}; C) \longrightarrow 0.$$

Proof.

(a) Immediate from the definition, since the Koszul complex is only nonzero in homological degrees 0 through n.

(b) Remark 1.33 tells us that

$$\mathrm{H}_0(\underline{x}; M) = \mathrm{coker}\left(M \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}} M \right) = M/IM.$$

(c) Remark 1.33 above tells us that

$$\begin{aligned} \mathrm{H}_n(\underline{x}; M) &= \ker\left(M \xrightarrow{\begin{pmatrix} x_1 & -x_2 & x_3 & \cdots & (-1)^{n+1} x_n \end{pmatrix}^T} M^n \right) \\ &= \{ m \in M \mid r x_1 = r x_2 = \cdots = r x_n = 0 \} \\ &= (0 :_M I). \end{aligned}$$

(d) In each homological degree, the Koszul complex is simply a direct sum of copies of M. So the modules in the complex $\operatorname{kos}(\underline{x}; M)$ are themselves already killed by $\operatorname{ann}_R(M)$, before we even take homology.

(e) We are going to show something stronger: we will show that for all $a \in I$, multiplication by a on $\operatorname{kos}(\underline{x}; M)$ is nullhomotopic, which proves that multiplication by a is the zero map in $\mathrm{H}_i(\underline{x}; M)$. In fact, it is sufficient to show that multiplication by a is nullhomotopic on $\operatorname{kos}(\underline{x})$, since additive functors preserve the homotopy relation. To do this, we will explicitly use the multiplicative structure of the Koszul complex given by our description of the Koszul complex via exterior powers. Given $a \in I = (x_1, \ldots, x_n)$, write $a = a_1 x_1 + \cdots + a_n x_n$. Consider the map

$$s_a \colon \operatorname{kos}(\underline{x}) \longrightarrow \Sigma^{-1} \operatorname{kos}(\underline{x})$$

given by multiplication by $a_1 e_1 \wedge \cdots \wedge a_n e_n$, meaning

$$s_a(e_{i_1} \wedge \cdots \wedge e_{i_t}) = \sum_{j=1}^{n} a_j e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_t}.$$

Now we claim this map s_a is a nullhomotopy for the map of complexes $\operatorname{kos}(\underline{x}) \longrightarrow \operatorname{kos}(\underline{x})$ given by multiplication by a in every component. To check that, it is sufficient to check this on basis elements, that is, we need only to check that

$$s_a \partial \left(e_{i_1} \wedge \cdots \wedge e_{i_t} \right) + \partial s_a \left(e_{i_1} \wedge \cdots \wedge e_{i_t} \right) = a e_{i_1} \wedge \cdots \wedge e_{i_t}.$$

Indeed, we have

$$\begin{aligned} s_a \partial \left(e_{i_1} \wedge \cdots \wedge e_{i_t} \right) &= s_a \left(\sum_{k=1}^{t} (-1)^{k+1} x_k e_{i_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{i_t} \right) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{t} (-1)^{k+1} a_j x_k e_j \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{i_t} \end{aligned}$$

and

$$\begin{aligned} \partial s_a \left(e_{i_1} \wedge \cdots \wedge e_{i_t} \right) &= \partial \left(\sum_{j=1}^{n} a_j e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_t} \right) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{t} (-1)^{k+2} a_j e_j \wedge e_{j_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_s} + \sum_{j=1}^{n} a_j x_j e_{i_1} \wedge \cdots \wedge e_{i_t} \\ &= -s_a d \left(e_{i_1} \wedge \cdots \wedge e_{i_t} \right) + \sum_{j=1}^{n} a_j x_j e_{i_1} \wedge \cdots \wedge e_{i_t} \end{aligned}$$

and since $a_1 x_1 + \cdots + a_n x_n = a$, we conclude that

$$(s_a \partial + \partial s_a) \left(e_{i_1} \wedge \cdots \wedge e_{i_t} \right) = a e_{i_1} \wedge \cdots \wedge e_{i_t}.$$

(f) If M is noetherian, then so is M^k for any k, as well as any submodules of M^k and any of their quotients, by Theorem 1.50 from Commutative Algebra I. Each $\mathrm{H}_i(\underline{x}; M)$ is a subquotient of a direct sum of copies of M, so it must be noetherian.

(g) Given an R-module homomorphism $f\colon M \to N$, we get an induced map

$$\mathrm{kos}(f)\colon \mathrm{kos}(\underline{x}; M) \longrightarrow \mathrm{kos}(\underline{x}; N)$$

by taking $\mathrm{kos}(\underline{x}) \otimes f$, so we set $\mathrm{H}_i(\underline{x}; f) = \mathrm{H}_i(\mathrm{kos}(\underline{x}) \otimes f)$. It is immediate to see that this assignment takes the identity on M to itself, and we leave it as an exercise to check that this definition preserves composition of homomorphisms.

(h) Given any complex of free R-modules F, tensoring with F is an exact functor from R-modules to $\mathrm{Ch}(R)$. In particular, this applies to $\mathrm{kos}(\underline{x})$, so

$$0 \longrightarrow \mathrm{kos}(\underline{x}) \otimes_R A \longrightarrow \mathrm{kos}(\underline{x}) \otimes_R B \longrightarrow \mathrm{kos}(\underline{x}) \otimes_R C \longrightarrow 0$$

is a short exact sequence of complexes. The resulting long exact sequence in homology, as in Theorem 2.28 from Homological Algebra, is the long exact sequence we are looking for. □

Remark 1.36. Following our iterative definition of the Koszul complex, where

$$\mathrm{kos}(x_1, \dots, x_{i+1}; M) = \mathrm{kos}(x_1, \dots, x_i; M) \otimes \mathrm{kos}(x_{i+1}),$$

set $C := \mathrm{kos}(x_1, \dots, x_i; M)$, and note that by definition of tensor product of complexes we have

$$[\mathrm{kos}(x_1, \dots, x_{i+1}; M)]_n = C_{n-1} \otimes_R R \oplus C_n \otimes_R R \cong C_{n-1} \oplus C_n.$$

Let us explicitly write down the differential on $\mathrm{kos}(x_1, \dots, x_{i+1}; M)$ in terms of the differential on $\mathrm{kos}(x_1, \dots, x_i; M)$. Given $a \in C_{n-1}$, $b \in C_n$, $r \in R$ (in homological degree 1) and $s \in R$ (in homological degree 0), our differential is

$$\partial(a \otimes r + b \otimes s) = \partial(a) \otimes r + (-1)^{n-1} a \otimes (x_{i+1} r) + \partial(b) \otimes s + (-1)^n b \otimes 0,$$

so

$$\partial_n := \begin{pmatrix} \partial_C & 0 \\ (-1)^{n-1} x_{i+1} & \partial_C \end{pmatrix} : \begin{tikzcd} C_{n-1} \arrow[r, "\partial_C"] \arrow[dr, "(-1)^{n-1}x_{i+1}"] & C_{n-2} \\ \oplus & \oplus \\ C_n \arrow[r, "\partial_C"'] & C_{n-1}. \end{tikzcd}$$

Notice that this is exactly[2] the cone of the map $\mathrm{kos}(x_1, \dots, x_i; M) \xrightarrow{x_{i+1}} \mathrm{kos}(x_1, \dots, x_i; M)$ given by multiplication by x_{i+1} in every degree. The cone comes together with a short exact sequence

$$0 \longrightarrow \mathrm{kos}(x_1, \dots, x_i; M) \longrightarrow \mathrm{kos}(x_1, \dots, x_{i+1}; M) \longrightarrow \Sigma^{-1} \mathrm{kos}(x_1, \dots, x_i; M) \longrightarrow 0.$$

[2] Up to the sign convention differences we discussed in Theorem 6.19 from Homological Algebra.

This short exact sequence gives rise to a long exact sequence in homology, as described in Theorem 6.20 from Homological Algebra, where (up to sign) the connecting homomorphism is simply the map in homology induced by multiplication by x_{i+1}. Here is the degree n piece of that long exact sequence:

$$\cdots \longrightarrow \mathrm{H}_n(x_1, \dots, x_{i+1}; M) \longrightarrow \mathrm{H}_{n-1}(x_1, \dots, x_i; M) \xrightarrow{x_{i+1}} \mathrm{H}_{n-1}(x_1, \dots, x_i; M) \longrightarrow \cdots.$$

Let us look at

$$\mathrm{H}_n(x_1, \dots, x_i; M) \xrightarrow{x_{i+1}} \mathrm{H}_n(x_1, \dots, x_i; M) \xrightarrow{\varphi} \mathrm{H}_n(x_1, \dots, x_{i+1}; M) \xrightarrow{\psi} \mathrm{H}_{n-1}(x_1, \dots, x_i; M).$$

By exactness,

$$\begin{aligned} &\ker\left(\mathrm{H}_n(x_1, \dots, x_i; M) \xrightarrow{\varphi} \mathrm{H}_n(x_1, \dots, x_{i+1}; M)\right) \\ &= \mathrm{im}(\mathrm{H}_n(x_1, \dots, x_i; M) \xrightarrow{x_{i+1}} \mathrm{H}_n(x_1, \dots, x_i; M)) \\ &= x_{i+1} \cdot \mathrm{H}_n(x_1, \dots, x_i; M). \end{aligned}$$

By the First Isomorphism Theorem, φ induces an inclusion $\overline{\varphi}$

$$\frac{\mathrm{H}_n(x_1, \dots, x_i; M)}{x_{i+1} \cdot \mathrm{H}_n(x_1, \dots, x_i; M)} \xhookrightarrow{\overline{\varphi}} \mathrm{H}_n(x_1, \dots, x_{i+1}; M)$$

with $\mathrm{im}(\overline{\varphi}) = \mathrm{im}(\varphi)$. Now we use the First Isomorphism Theorem and exactness repeatedly:

$$\begin{aligned} \mathrm{coker}(\overline{\varphi}) &= \frac{\mathrm{H}_n(x_1, \dots, x_{i+1}; M)}{\mathrm{im}(\varphi)} && \text{by definition} \\ &= \frac{\mathrm{H}_n(x_1, \dots, x_{i+1}; M)}{\ker(\psi)} && \text{by exactness} \\ &\cong \mathrm{im}(\psi) && \text{by the First Iso Theorem} \\ &= \mathrm{im}\left(\mathrm{H}_n(x_1, \dots, x_{i+1}; M) \xrightarrow{\psi} \mathrm{H}_{n-1}(x_1, \dots, x_i; M)\right) \\ &= \ker\left(\mathrm{H}_{n-1}(x_1, \dots, x_i; M) \xrightarrow{x_{i+1}} \mathrm{H}_{n-1}(x_1, \dots, x_i; M)\right) && \text{by exactness} \\ &= \mathrm{ann}_{\mathrm{H}_{n-1}(x_1, \dots, x_i; M)}(x_{i+1}). \end{aligned}$$

And finally, we get the following short exact sequences induced by $\overline{\varphi}$:

$$0 \longrightarrow \tfrac{\mathrm{H}_n(x_1, \dots, x_i; M)}{x_{i+1} \cdot \mathrm{H}_n(x_1, \dots, x_i; M)} \longrightarrow \mathrm{H}_n(x_1, \dots, x_{i+1}; M) \longrightarrow \mathrm{ann}_{\mathrm{H}_{n-1}(x_1, \dots, x_i; M)}(x_{i+1}) \longrightarrow 0.$$

Definition 1.37. Let M be an R-module and consider a sequence of nonunit elements $\underline{f} \in R$. The **ith Koszul cohomology** on $\underline{f}$ with coefficients in M, also called the **Koszul cohomology** on M with respect to $\underline{f}$, is the R-module

$$\mathrm{kos}^i(\underline{f}; M) := \mathrm{H}^i(\mathrm{Hom}_R(\mathrm{kos}(\underline{f}), M)).$$

We write $\mathrm{kos}^i(\underline{f}) := \mathrm{H}^i(\mathrm{Hom}_R(\mathrm{kos}(\underline{f}), R))$.

However, this adds nothing new to the story: in fact, the Koszul complex is self-dual.

Theorem 1.38. *Let R be any ring and let $\underline{x} = x_1, \dots, x_n \in R$. The Koszul complex* $\operatorname{kos}(\underline{x})$ *is isomorphic to its dual* $\operatorname{Hom}_R(\operatorname{kos}(\underline{x}), R)$. *More generally,*

$$\operatorname{kos}(\underline{x}; M) \cong \operatorname{Hom}_R(\operatorname{kos}(\underline{x}), M).$$

In particular,

$$\mathrm{H}_i(\underline{x}; M) \cong \mathrm{H}^{n-i}(\underline{x}; M).$$

Proof sketch. In general, for any two R-modules M and N, the natural isomorphism

$$R \otimes_R M \cong M$$

leads to a homomorphism

$$\operatorname{Hom}_R(N, R) \otimes_R M \xrightarrow{\psi} \operatorname{Hom}_R(N, M)$$

that sends each simple tensor $f \otimes m$ to the R-module homomorphism $N \to M$ given by

$$n \mapsto f(n) \otimes m.$$

We leave the details to the reader. When N is a finitely generated free R-module, this map ψ is an isomorphism (exercise!), so for all $d \geqslant 1$

$$\operatorname{Hom}_R(R^d, R) \otimes_R M \cong \operatorname{Hom}_R(R^d, M).$$

This isomorphism is natural, and thus induces an isomorphism

$$\operatorname{Hom}_R(\operatorname{kos}(\underline{x}, R)) \otimes M \cong \operatorname{Hom}_R(\operatorname{kos}(\underline{x}, M).$$

Thus it suffices to construct an isomorphism

$$\operatorname{kos}(\underline{x}) \xrightarrow{\omega} \operatorname{Hom}_R(\operatorname{kos}(\underline{x}, R)).$$

Such a map corresponds to a commutative diagram

$$\begin{tikzcd}
\operatorname{kos}(\underline{x}): \arrow[d, "w"] & 0 \arrow[r] & \bigwedge^n R^n \arrow[r] \arrow[d, "w_n"] & \bigwedge^{n-1} R^n \arrow[r] \arrow[d, "w_{n-1}"] & \cdots \arrow[r] & \bigwedge^1 R^n \arrow[r] \arrow[d, "w_1"] & R \arrow[r] \arrow[d, "w_0"] & 0 \\
\operatorname{Hom}_R(\operatorname{kos}(\underline{x}), R): & 0 \arrow[r] & R \arrow[r] & \bigwedge^1 R^n \arrow[r] & \cdots \arrow[r] & \bigwedge^{n-1} R^n \arrow[r] & R \arrow[r] & 0.
\end{tikzcd}$$

For each subset $I = \{i_1, \dots, i_d\} \subseteq \{1, \dots, n\}$, write e_I for the homological degree d basis element of $\operatorname{kos}(\underline{x})$ given by

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_d} \in \bigwedge^i R^n.$$

We define $\omega_d(e_I) \in \operatorname{Hom}_R(\bigwedge^{n-i}, R)$ to be the map given by

$$e_J \mapsto \begin{cases} e_I \wedge e_J & \text{if } I \cap J = \emptyset \\ 0 & \text{if } I \cap J \neq \emptyset. \end{cases}$$

We leave it as an exercise to check that this rule determines an isomorphism in each homological degree, and that the diagram commutes, proving the isomorphism between the Koszul complex and its dual. The isomorphism $\mathrm{H}_i(\underline{x}; M) \cong \mathrm{H}^{n-i}(\underline{x}; M)$ is a trivial consequence. $\square$

1.5 Regular sequences

The Koszul complex is closely tied to regular elements and regular sequences.

Definition 1.39. Let R be a ring and M be an R-module. An element $r \in R$ is **regular** on M if $rM \neq M$ and for any $m \in M$

$$rm = 0 \Rightarrow m = 0.$$

More generally, a sequence of elements $x_1, \dots, x_n$ is a **regular sequence on** M if

- $(x_1, \dots, x_n)M \neq M$, and
- for each i, the element x_i is regular on $M/(x_1, \dots, x_{i-1})M$.

When $M = R$, we drop the *on* M and say r is regular or $x_1, \dots, x_n$ is a regular sequence.

Remark 1.40. Suppose $(x_1, \dots, x_n)M \neq M$. Note that x_i is regular on $M/(x_1, \dots, x_{i-1})M$ if and only if

$$((x_1, \dots, x_{i-1})M :_M x_i) = (x_1, \dots, x_{i-1})M.$$

Remark 1.41. When $(R, \mathfrak{m}, k)$ is a noetherian local ring and $M \neq 0$ is finitely generated R-module, NAK[3] gives us $(x_1, \dots, x_n)M \neq M$ automatically for all $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$.

Example 1.42. Consider the polynomial ring $R = k[x_1, \dots, x_n]$ in n variables over a field k. The variables $x_1, \dots, x_n$ for a regular sequence on R.

Example 1.43. Let k be a field and $R = k[x, y, z]$. The sequence xy, xz is not regular on R, since xz is kills y on $R/(xy)$.

The order we write the elements in matters.

Example 1.44. Let k be a field and $R = k[x, y, z]$. We claim that $x, (x-1)y, (x-1)z$ is a regular sequence, while $(1-x)y, (1-x)z, x$ is not.

For the first claim, note that over $R/(x)$, we have $(x-1)y = -y$ and $(x-1)z = -z$. Since $x, -y, -z$ is a regular sequence on R, then so is $x, (x-1)y, (x-1)z$. In contrast, over $R/((x-1)y)$, the elements y and $(x-1)z$ are nonzero, but

$$(x-1)z \cdot y = 0.$$

In particular, $(x-1)y$ is not regular on $R/((1-x)y)$, and $(1-x)y, (1-x)z, x$ is not a regular sequence.

Remark 1.45. Suppose that $r \in R$ is such that $rM \neq M$. We claim that r is regular on M if and only if the first koszul homology vanishes, $\mathrm{H}_1(r; M) = 0$. Indeed,

$$\mathrm{H}_1(r; M) = \ker(M \xrightarrow{r} M) = (0 :_M r),$$

and by definition, r is regular on M if and only if $(0 :_M r) = 0$.

[3] Theorem 5.32 and Theorem 5.39 from Commutative Algebra I, the latter in the graded case.

In fact, the Koszul complex on a regular sequence is exact in all positive degrees.

Theorem 1.46. *If $\underline{x} = x_1, \ldots, x_n \in R$ is a regular sequence on the R-module M, then $\mathrm{H}_i(\underline{x}; M) = 0$ for all $i > 0$.*

Proof. We proceed by induction on the length of the sequence, noting that the case $n = 1$ is Remark 1.45. Now suppose that $\mathrm{H}_j(x_1, \ldots, x_i; M) = 0$ for all $j > 0$. The long exact sequence

$$\cdots \longrightarrow \mathrm{H}_n(x_1, \ldots, x_{i+1}; M) \longrightarrow \mathrm{H}_{n-1}(x_1, \ldots, x_i; M) \xrightarrow{x_{i+1}} \mathrm{H}_{n-1}(x_1, \ldots, x_i; M) \longrightarrow \cdots$$

from Remark 1.36 shows that we must have $\mathrm{H}_j(x_1, \ldots, x_{i+1}; M) = 0$ for all $j > 1$. Moreover, Remark 1.36 also gave us the short exact sequence

$$0 \longrightarrow \frac{\mathrm{H}_1(x_1, \ldots, x_i; M)}{x_{i+1} \cdot \mathrm{H}_1(x_1, \ldots, x_i; M)} \longrightarrow \mathrm{H}_1(x_1, \ldots, x_{i+1}; M) \longrightarrow \mathrm{ann}_{\mathrm{H}_0(x_1, \ldots, x_i; M)}(x_{i+1}) \longrightarrow 0.$$

Since x_{i+1} is regular on $M/(x_1, \ldots, x_i)M = \mathrm{H}_0(x_1, \ldots, x_i; M)$, $\mathrm{ann}_{\mathrm{H}_0(x_1, \ldots, x_i; M)}(x_{i+1}) = 0$. Moreover, $\mathrm{H}_1(x_1, \ldots, x_i; M) = 0$ by hypothesis. Therefore, in the last short exact sequence above only the middle term remains, which gives us $\mathrm{H}_1(x_1, \ldots, x_{i+1}; M) = 0$. □

Corollary 1.47. *If $x_1, \ldots x_n$ is a regular sequence on R, then the Koszul complex on $x_1, \ldots, x_n$ is a free resolution for $R/(x_1, \ldots, x_n)$. Moreover, if $(R, \mathfrak{m}, k)$ is either a local/graded, then $\mathrm{kos}(x_1, \ldots, x_n)$ is a minimal free resolution for $R/(x_1, \ldots, x_n)$.*

Proof. By Theorem 1.46, $P = \mathrm{kos}(x_1, \ldots, x_n)$ has $\mathrm{H}_i(P) = 0$ for all $i > 0$. This is a complex of free modules, and thus a free resolution of $\mathrm{H}_0(P)$, which by Theorem 1.35 is $R/(x_1, \ldots, x_n)$. Finally, in the local/graded case, the assumption that $\underline{x}$ is a regular sequence guarantees that $x_i \in \mathfrak{m}$ for all i. Since all the nonzero entries in the differentials (under the standard basis for the Koszul complex) are of the form $\pm x_i$, we conclude that the resolution must be minimal. □

There are three big theorems of Hilbert's every commutative algebraist must know: Hilbert's Basis Theorem, Hilbert's Nullstellensatz, and Hilbert's Syzygy Theorem. We are finally ready to prove the third.

Theorem 1.48 (Hilbert Syzygy Theorem)**.** *Every finitely generated graded module M over a polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k has finite projective dimension. In fact, $\mathrm{pdim}(M) \leqslant n$.*

Proof. By Corollary 1.47, the Koszul complex on the regular sequence $x_1, \ldots, x_d$ is a minimal free resolution for $k = R/(x_1, \ldots, x_n)$, so $\mathrm{pdim}(k) = n$. By Corollary 1.24, every finitely generated R-module M has

$$\mathrm{pdim}(M) \leqslant \mathrm{pdim}(k) = n.$$

□

It is natural to ask if Theorem 1.46 has a converse; over a nice enough ring, the answer is yes: the vanishing of Koszul homology does characterize regular sequences. In fact, more is true: the first Koszul homology completely determines whether we have a regular sequence.

Theorem 1.49. *Let $(R, \mathfrak{m}, k)$ be local/graded. Let $M \neq 0$ be a finitely generated R-module, and $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$. In the graded case, we assume that M is graded and $x_1, \dots, x_n$ are all homogeneous. The following are equivalent:*

(a) $\mathrm{H}_i(\underline{x}; M) = 0$ *for all* $i > 0$.

(b) $\mathrm{H}_1(\underline{x}; M) = 0$.

(c) $\underline{x}$ *is a regular sequence on* M.

Proof. The implication (a) ⇒ (b) is obvious, and (c) ⇒ (a) is Theorem 1.49. We will finish the proof by showing (b) ⇒ (c), which we will do by induction on the length n of $\underline{x}$.

The case $n = 1$ is Remark 1.45, so suppose that the statement holds for all sequences of length $n - 1 \geqslant 1$. Now Remark 1.36 gives us the short exact sequence

$$0 \longrightarrow \frac{\mathrm{H}_1(x_1,\dots,x_{n-1};M)}{x_n \cdot \mathrm{H}_1(x_1,\dots,x_n;M)} \longrightarrow \mathrm{H}_1(x_1, \dots, x_n; M) \longrightarrow \operatorname{ann}_{\mathrm{H}_0(x_1,\dots,x_{n-1};M)}(x_n) \longrightarrow 0.$$

By assumption, $\mathrm{H}_1(x_1, \dots, x_n; M) = 0$, so exactness tells us all the terms in the short exact sequence above must vanish.

The vanishing of the left term gives us $x_n \cdot \mathrm{H}_1(x_1, \dots, x_n; M) = \mathrm{H}_1(x_1, \dots, x_{n-1}; M)$. But $\mathrm{H}_1(x_1, \dots, x_n; M)$ is a finitely generated R-module by Theorem 1.35, and $x_n \in \mathfrak{m}$, so $\mathrm{H}_1(x_1, \dots, x_{n-1}; M) = 0$ by NAK[4]. By induction hypothesis, $x_1, \dots, x_{n-1}$ is a regular sequence on M.

From the same short exact sequence, we also get the vanishing of the rightmost term, which is

$$\operatorname{ann}_{\mathrm{H}_0(x_1,\dots,x_{n-1};M)}(x_n) = 0.$$

By definition, this means that x_n is regular on $\mathrm{H}_0(x_1, \dots, x_{n-1}; M) = M/(x_1, \dots, x_{n-1})M$. We conclude that $\underline{x}$ is a regular sequence on M. □

A corollary of Theorem 1.49 is that in a local/graded ring, the order of the elements in a regular sequence does not matter.

Corollary 1.50. *Let $(R, \mathfrak{m}, k)$ be local/graded. Let M be a finitely generated R-module, and consider $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$. In the graded case, we assume that M is graded and $x_1, \dots, x_n$ are all homogeneous. If the sequence $\underline{x}$ is regular on M, then so is any of its permutations.*

Proof. If $x_1, \dots, x_n$ is a regular sequence, then Theorem 1.46 gives us

$$\mathrm{H}_i(x_1, \dots, x_n; M) = 0 \quad \text{for all } i > 0.$$

By Exercise 10, the Koszul homology on $\underline{x}$ agrees with the Koszul homology on any permutation of $\underline{x}$, which must then also vanish. By Theorem 1.49, any permutation of $\underline{x}$ is a regular sequence. □

[4] Theorem 5.32 and Theorem 5.39 from Commutative Algebra I, the latter in the graded case.

In fact, we can extend this to any ring and any module under a reasonable assumption.

Lemma 1.51. *Let R be a ring and M an R-module. If x, y is a regular sequence on M and y is regular on M, then y, x is a regular sequence on M.*

Proof. Suppose that $xm = yn$ for some $m, n \in M$. Since x, y is a regular sequence on M and $yn \in (x)M$, we must have $n \in (x)M$, so there exists some $w \in M$ such that $n = xw$. But then $xm = yn = xyw$. Since x is regular on M, we conclude that $m = yw$, so $m \in (y)M$. In particular, this shows that x is regular on $M/(y)M$. $\square$

Lemma 1.52. *Let $(R, \mathfrak{m})$ be a noetherian local ring. If $x_1, \dots, x_n$ is a regular sequence on M, then so is $x_1^{a_1}, \dots, x_n^{a_n}$ for any integers $a_i \geqslant 1$.*

Proof. Let $n = 1$. Suppose $x = x_1$ is a regular element on M. Given a nonzero $m \in M$ and $a \geqslant 1$,

$$x^a m = 0 \implies x x^{a-1} m = 0.$$

Since x is regular on M, we must have $x^{a-1}m = 0$. Repeating this $a - 1$ times, we conclude that $xm = 0$, and $m = 0$.

Now consider any $n \geqslant 1$. Since x_n is a regular sequence on $M/(x_1, \dots, x_{n-1})$ by the case of a sequence of length 1 we can now say $x_n^{a_n}$ is regular on $M/(x_1, \dots, x_{n-1})$. Therefore, $x_1, \dots, x_{n-1}, x_n^{a_n}$ is regular on M. By Corollary 1.50, we are allowed to permute the elements in our sequence. Now switch the order and repeat the argument with each x_i, until we conclude that $x_1^{a_1}, \dots, x_n^{a_n}$ is also regular on M. $\square$

Finally, we note a connection between regular sequences and height.

Theorem 1.53. *If $x_1, \dots, x_n$ is a regular sequence on R, then* $\operatorname{height}(x_1, \dots, x_n) = n$.

Proof. By induction on n. When $n = 1$, x_1 is regular if and only if x_1 is not in the set of zero divisors of R. By Theorem 6.27 from Commutative Algebra I, this means x_1 is not in any associated prime of R, and in particular, x_1 is not in any of the minimal primes of R. Therefore, any prime containing x_1 must have height at least 1, so $\operatorname{height}(x_1) \geqslant 1$. By Krull's Height Theorem, Theorem 8.5 from Commutative Algebra I, we always have $\operatorname{height}(x_1) \leqslant 1$, so we conclude that $\operatorname{height}(x_1) = 1$.

When $n \geqslant 2$, if $x_1, \dots, x_n$ is a regular sequence on R then in particular x_n is regular on the quotient $R/(x_1, \dots, x_{n-1})$. By case $n = 1$, the ideal $(x_1, \dots, x_n)/(x_1, \dots, x_{n-1})$ has height 1 in $R/(x_1, \dots, x_{n-1})$. By induction hypothesis, $\operatorname{height}(x_1, \dots, x_{n-1}) = n - 1$. We conclude that $\operatorname{height}(x_1, \dots, x_n) = n$. $\square$

The converse does not hold in general.

Example 1.54. Let $R = k[\![x, y]\!]/(x^2, xy)$. The element y is not in the unique minimal prime $(x)/(x^2, xy)$ of R, so $\operatorname{height}(y) = 1$. However, y is not regular.

We will later see that life is much more worth living over a Cohen-Macaulay ring.

1.6 Regular rings

Regular rings are the nicest possible kinds of rings, after fields.

Definition 1.55. A noetherian local ring $(R, \mathfrak{m}, k)$ is **regular** if $\mathfrak{m}$ is minimally generated by d elements, that is

$$\mu(\mathfrak{m}) = \dim(R).$$

Back in Commutative Algebra, when we discussed height and dimension, we saw that this is in fact the smallest possible value for the minimal number of generators of $\mathfrak{m}$; in general, $\mu(\mathfrak{m}) \geqslant d$. This is also called the embedding dimension of R.

Definition 1.56. The **embedding dimension** of a noetherian local ring $(R, \mathfrak{m}, k)$ is

$$\operatorname{embdim}(R) := \mu(\mathfrak{m}) = \dim_k \left(\mathfrak{m}/\mathfrak{m}^2\right).$$

Remark 1.57. By Krull's Height Theorem,

$$\operatorname{embdim}(R) = \mu(\mathfrak{m}) \geqslant \operatorname{height}(\mathfrak{m}) = \dim(R).$$

Thus a ring is regular if and only if $\mathfrak{m}$ has the smallest possible number of minimal generators, or equivalently, R has the smallest possible embedding dimension.

We are now ready to give a completely homological characterization of regular local rings. This characterization, first proved by Auslander and Buchsbaum and independently by Serre, solved a famous question called the Localization Problem.

Problem 1 (Localization Problem)**.** If R is a regular local ring, must R_P be regular for every prime P in R?

This is asking if being regular is a local property. A positive answer allows for a simple global definition of regularity:

Definition 1.58. A ring R is **regular** if R_P is a regular local ring for all prime ideals P in R.

Before we can get to this famous homological characterization of regular local rings, and the solution to the localization problem, we will need to sharpen our tools a bit.

Lemma 1.59. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and $F \xrightarrow{\varphi} G$ an R-module map between finitely generated free R-modules. If $\varphi \otimes_R k$ is injective, then φ splits as a map of R-modules.*

Proof. Let $F = R^n$, $G = R^m$, and let $\{e_1, \dots, e_n\}$ be the standard basis for F. Our assumption that $\varphi \otimes_R k$ is injective means that the images of $\varphi(e_1), \dots, \varphi(e_n)$ in $G/\mathfrak{m}G = k^m$ are linearly independent, so we can complete them to a basis $\varphi(e_1), \dots, \varphi(e_n), \overline{f_{n_1}}, \dots, \overline{f_m}$ for $G/\mathfrak{m}G$. By NAK (see Theorem 5.34 from Commutative Algebra I), we can lift those elements f_i so that $\varphi(e_1), \dots, \varphi(e_n), f_{n+1}, \dots, f_m$ is a basis for G. Now the projection onto $\varphi(e_1), \dots, \varphi(e_n)$ is a splitting for φ. $\square$

Theorem 1.60 (Serre). *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Moreover, if $\mu(\mathfrak{m}) = s$, then*

$$\dim_k(\operatorname{Tor}_i^R(k,k)) \geqslant \binom{s}{i}.$$

Proof. Let $x_1, \dots, x_s$ be minimal generators for $\mathfrak{m}$, and set $K := \operatorname{kos}(x_1, \dots, x_s)$. If F is a minimal free resolution for $\mathfrak{m}$, we claim that K_i is a direct summand of F_i.

By Theorem 5.18 from Homological Algebra, the identity map on k lifts to a map of complexes $\varphi\colon K \longrightarrow F$. We claim that the maps φ_i split, which will prove our claim that K_i is a direct summand of F_i. The map $\varphi_0\colon K_0 = R \longrightarrow F_0 = R$ must be the identity map, so clearly φ_0 splits.

We proceed by induction on i. Suppose we have shown that φ_{i-1} splits, say by a splitting ψ_{i-1}. By Lemma 1.59, it is enough to show that $\varphi_i \otimes_R k$ to be injective. To show our claim, we need to show that if $z \in K_i$ is such that $\varphi_i(z) \in \mathfrak{m}F_i$, then $z \in \mathfrak{m}K_i$. First, we note that $\partial_{i+1}\varphi_{i+1}(z) \in \mathfrak{m}^2 F_i$, since F is minimal and thus $\operatorname{im}\partial \subseteq \mathfrak{m}F$, by **??**. By commutativity, $\varphi_{i-1}d_i(z) \in \mathfrak{m}^2 F_i$. Since $\psi_{i-1}\varphi_{i-1} = 1_{K_{i-1}}$, we must have $d_i(z) = \psi_{i-1}\varphi_{i-1}d_i(z) \in \mathfrak{m}^2 K_{i-1}$. We claim that this implies $z \in \mathfrak{m}F_{i+1}$. To see that, we compute d_i explicitly: first we write z as a linear combination of our basis elements, say

$$z = \sum_{j_1 < \cdots < j_i} z_j e_{j_1} \wedge \cdots \wedge e_{j_i}$$

so that

$$d_i(z) = \sum_{j,k} \pm z_j x_{j_k} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{j_i}.$$

The one thing to keep track of here is the coefficients: rewriting this carefully in the basis for K_{i-1}, the coefficient for each basis element is a linear combination of terms of the form $z_j x_l$. We showed that $d_i(z) \in \mathfrak{m}^2 F_{i-1}$, so each appropriate combination of $z_j x_l$ is in $\mathfrak{m}^2$. We assumed that $x_1, \dots, x_s$ were minimal generators for $\mathfrak{m}$ to begin with, and thus a basis for $\mathfrak{m}/\mathfrak{m}^2$, so all our coefficients z_j must be in $\mathfrak{m}$. We conclude that indeed $z \in \mathfrak{m}F_i$, and thus that $\varphi_i \otimes_R k$ is injective. By Lemma 1.59, φ_i splits.

So we have shown that K_i is a direct summand of F_i for all i, which implies that the number of copies of R in F_i must be at least as large as the number of copies of R in K_i. More precisely,

$$\dim_k(F_i \otimes_R k) = \dim_k(\operatorname{Tor}_i^R(k,k)) \geqslant \dim_k(K_i \otimes_R k) = \binom{d}{i}. \quad \square$$

Replacing k by M in the previous result leads to a famous open question.

Conjecture 1.61 (Buchsbaum—Eisenbud, Horrocks). *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring of dimension d and M a finitely generated Artinian R-module of finite projective dimension. Then*

$$\beta_i(M) = \dim_k(\operatorname{Tor}_i^R(M,k)) \geqslant \binom{d}{i}.$$

While this remains an open question, there is much evidence to support it. For example, the conjecture predicts that

$$\sum_i \beta_i(M) \geqslant \sum_i \binom{d}{i} = 2^d.$$

This is known as the Total Rank Conjecture, and it was recently shown by Walker in almost all cases, and by Walker and Van

Theorem 1.62 (Walker, 2017)**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring of dimension d and characteristic not 2, $M \neq 0$ a finitely generated R-module of finite projective dimension, and $c = \operatorname{height}(\operatorname{ann}(M))$. Then*

$$\sum_i \beta_i(M) \geqslant \sum_i \binom{c}{i} = 2^c.$$

The famous homological characterization of regular rings that solved the localization problem is the following:

Theorem 1.63 (Auslander–Buchsbaum, Serre)**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring of dimension d. The following are equivalent:*

63) The residue field k has finite projective dimension.

63) Every finitely generated R-module has finite projective dimension.

63) The maximal ideal $\mathfrak{m}$ is generated by a regular sequence.

63) The maximal ideal $\mathfrak{m}$ is generated by d elements.

Proof. The implication 1.63 $\Longrightarrow$ 1.63 is obvious: just take $M = k$. The proof of 1.63 $\Longrightarrow$ 1.63 is essentially the same as Hilbert's Syzygy Theorem: $\beta_i(M) = \dim_k \operatorname{Tor}_i^k(M, k)$ for all i, and $\operatorname{Tor}_i^k(M, k) = 0$ for all $i > \operatorname{pdim}_R(k)$.

If $\mathfrak{m}$ is generated by a regular sequence, then the Koszul complex on that regular sequence is a minimal free resolution of k, by Corollary 1.47, so k has projective dimension d. This is 1.63 $\Longrightarrow$ 1.63.

Let's now show that 1.63 $\Longrightarrow$ 1.63. Set $\mathfrak{m} = (x_1, \dots, x_d)$. In fact, we will show something stronger: that $(0), (x_1), (x_1, x_2), \dots, (x_1, \dots, x_d)$ are distinct prime ideals in R. Notice in particular that this implies that $x_1, \dots, x_d$ form a regular sequence.

If $d = 0$, then $\mathfrak{m} = (\{\}) = (0)$, and there is nothing to prove. We proceed by induction on d, assuming that $d > 0$ and that we have shown that whenever $\mathfrak{m}$ is generated by $d - 1$ elements, $x_1, \dots, x_{d-1}$, the ideals $(0), (x_1), (x_1, x_2), \dots, (x_1, \dots, x_{d-1})$ are distinct prime ideals in R.

When $d > 0$, $\mathfrak{m}$ is not a minimal prime. By Prime Avoidance Theorem 3.29 from Commutative Algebra I,

$$\mathfrak{m} \not\subseteq \bigcup_{P \in \operatorname{Min}(R)} P,$$

and using Theorem 3.30 from Commutative Algebra I we can find an element

$$y_1 = x_1 + r_2 x_2 + \cdots + r_d x_d \notin \bigcup_{P \in \operatorname{Min}(R)} P.$$

Now we can replace x_1 by y_1, since $\mathfrak{m} = (y_1, x_2, \dots, x_d)$, so we can assume that x_1 is not in any minimal prime. By Krull's Height Theorem **??**, $\operatorname{height}(x_1) \leqslant 1$, so $\dim(R/(x_1)) = \operatorname{height}(\mathfrak{m}/(x_1)) \geqslant d-1$. By construction, $\mathfrak{m}/(x_1)$ is generated by $d-1$ elements, so again by Krull's Height Theorem, $\dim(R/(x_1)) = \operatorname{height}(\mathfrak{m}/(x_1)) \leqslant d-1$. We conclude that $\dim(R/(x_1)) = d-1$. By induction hypothesis, $(x_1)/(x_1), \dots, (x_1, \dots, x_d)/(x_1)$ are distinct prime ideals in $R/(x_1)$. Therefore, $(x_1), (x_1, x_2), \dots, (x_1, \dots, x_d)$ are distinct prime ideals in R.

Now we claim that R is a domain, which will show that $(0) \subsetneq (x_1)$ is also a prime ideal. First, note that x_1 is not contained in any minimal prime, but (x_1) is a prime ideal, so there exists some minimal prime $P \subsetneq (x_1)$. Given any $y \in P \subseteq (x_1)$, we can write $y = rx_1$ for some r. By construction, $x_1 \notin P$, so we must have $r \in P$. But we just showed that every element in P is of the form rx_1, so $P = x_1 P$. By NAK (see Theorem 5.32 from Commutative Algebra I), $P = (0)$. We conclude that R is a domain, and this finishes the proof of 1.63 $\Longrightarrow$ 1.63.

Finally, all that's left to show is 1.63 $\Longrightarrow$ 1.63. We claim that $\operatorname{pdim}_R(k) < \infty$ implies $\operatorname{pdim}_R(k) \leqslant \dim(R) = d$. If the claim holds, then Theorem 1.60 and **??** say that

$$\beta_i(k) = \dim_k(\operatorname{Tor}_i^R(k,k)) \geqslant \binom{\mu(\mathfrak{m})}{i}$$

for all i. Since $\beta_i(k) = 0$ for all $i > \operatorname{pdim}_R(k)$, we must have

$$\mu(\mathfrak{m}) \leqslant \operatorname{pdim}_R(k) \leqslant \dim(R) = d.$$

But $\operatorname{height}(\mathfrak{m}) = \dim(R) = d$, so by Krull's Height theorem, $\mu(\mathfrak{m}) \leqslant d$. We conclude that $\mathfrak{m}$ is generated by exactly d elements, which is precisely 1.63. So all we have left to do is to prove the claim that $\operatorname{pdim}_R(k) < \infty$ implies $\operatorname{pdim}_R(k) \leqslant \dim(R) = d$.

By contradiction, suppose $\operatorname{pdim}_R(k) > d$ but $\operatorname{pdim}_R(k) < \infty$. Choose a maximal regular sequence $y_1, \dots, y_t \in \mathfrak{m}$. By Theorem 1.53, $t \leqslant d$.

Since our regular sequence $y_1, \dots, y_t$ was chosen to be maximal inside $\mathfrak{m}$, every element in $\mathfrak{m}$ is a zerodivisor on $R/(y_1, \dots, y_t)$, or else we could increase our regular sequence. So $\mathfrak{m}$ is contained in the union of the zerodivisors on $R/(y_1, \dots, y_t)$, which by **??** is the same as the union of the associated primes of $R/(y_1, \dots, y_t)$. By Prime Avoidance **??**, $\mathfrak{m}$ must be contained in some associated prime of $R/(y_1, \dots, y_t)$. But $\mathfrak{m}$ is maximal, so $\mathfrak{m}$ *is* an associated prime of $R/(y_1, \dots, y_t)$. Equivalently, $k = R/\mathfrak{m}$ embeds into $R/(y_1, \dots, y_t)$. This gives us some short exact sequence

$$0 \longrightarrow k \longrightarrow R/(y_1, \dots, y_t) \longrightarrow M \longrightarrow 0.$$

Let us look at the corresponding long exact sequence for Tor, which is Theorem 6.27 from Homological Algebra:

$$\cdots \longrightarrow \operatorname{Tor}_{i+1}^R(M,k) \longrightarrow \operatorname{Tor}_i^R(k,k) \longrightarrow \operatorname{Tor}_i^R(R/(y_1, \dots, y_t), k) \longrightarrow \cdots.$$

We know $t = \operatorname{pdim}_R(R/(y_1, \dots, y_t))$, by Corollary 1.47, so $\operatorname{Tor}_i^R(R/(y_1, \dots, y_t), k) = 0$ for $i > t$. But $t \leqslant d < \operatorname{pdim}_R(k)$, so in particular $\operatorname{Tor}_i^R(R/(y_1, \dots, y_t), k) = 0$ for $i = \operatorname{pdim}_R(k)$. Moreover, Corollary 1.24 says that $\operatorname{pdim}_R(M) \leqslant \operatorname{pdim}_R(k)$ for any finitely generated R-module M, so in particular $\operatorname{Tor}_{i+1}^R(M,k) = 0$ for $i = \operatorname{pdim}_R(k)$. But this is impossible: our long exact sequence would then have $\operatorname{Tor}_{\operatorname{pdim}_R(k)}^R(k,k) \neq 0$ sandwiched between two zero modules. $\square$

Our proof also showed the following:

Corollary 1.64. *Every regular local ring is a domain.*

Corollary 1.65. *Every regular local ring $(R, \mathfrak{m}, k)$ has $\operatorname{pdim}_R(k) = \dim R$.*

Now we can solve the localization problem very easily.

Exercise 11. If R is a regular local ring, then R_P is a regular local ring for every prime P.

Remark 1.66. If we want to show that a particular ring (not necessarily local) is regular, it is sufficient to show that $R_\mathfrak{m}$ is a regular local ring for every maximal ideal $\mathfrak{m}$ — this will imply that R_P is a localization of a regular local ring for every prime P.

Exercise 12. Show that every principal ideal domain is a regular ring.

We have shown that finitely generated *graded* modules over a polynomial ring $k[x_1, \dots, x_d]$ have finite projective dimension, but this is not quite enough to conclude that polynomial rings are regular.

Theorem 1.67. *Every polynomial ring $R = k[x_1, \dots, x_d]$ over a field k is a regular ring.*

Proof. It is sufficient to show that $R_\mathfrak{m}$ is a regular local ring for every maximal ideal $\mathfrak{m}$. We are going to show that every maximal ideal is generated by d elements, which implies that $\mathfrak{m}_\mathfrak{m}$ is also generated by n elements. Since the height of every maximal ideal is d, by Theorem 7.46 from Commutative Algebra I, this will imply that $R_\mathfrak{m}$ is a regular local ring.

When k is algebraically closed, the maximal ideals in R are precisely those of the form $(x_1 - a_1, \dots, x_d - a_d)$, which are all generated by d elements. For the general case, we use induction on d, noting that $d = 1$ is trivial, since $k[x]$ is a principal ideal domain. When

$d > 1$, we can do a change of variables such as in Theorem 7.40 from Commutative Algebra I and assume that $\mathfrak{m}$ has a minimal generator that is monic in x_d. Let $\mathfrak{n} := \mathfrak{m} \cap k[x_1, \dots, x_{d-1}]$. Then $k[x_1, \dots, x_{d-1}]/\mathfrak{n} \longrightarrow R/\mathfrak{m}$ is an integral extension, but since $R/\mathfrak{m}$ is a field, so is $k[x_1, \dots, x_{d-1}]/\mathfrak{n}$, by **??** from Commutative Algebra I. Therefore, $\mathfrak{n}$ is a maximal ideal in $k[x_1, \dots, x_{d-1}]$, so by induction hypothesis it must be generated by $d-1$ elements. Now consider the image of $\mathfrak{m}$ via the map

$$\begin{aligned} R &\longrightarrow \tfrac{k[x_1,\dots,x_{d-1}]}{\mathfrak{n}}[x_n] \cong R/\mathfrak{n}R \\ x_i &\longmapsto x_i, \end{aligned}$$

which is now a maximal ideal in a polynomial ring over a field in one variable, which by the case $d = 1$ must be generated by 1 element. Now $\mathfrak{n}$ is generated by $n-1$ elements and $\mathfrak{m}/\mathfrak{n}R$ is generated by 1 element, so $\mathfrak{m}$ is generated by $n - 1 + 1 = n$ elements. $\square$

We have seen that regular rings are very nice. Modulo some technical conditions, it turns out that *every* noetherian local ring is a quotient of a regular ring. More precisely, every *complete* local ring is a quotient of a regular local ring, although we have unfortunately not discussed completeness. If a local ring R is not complete, we can always take its *completion*, which is now a quotient of a regular local ring. This very important fact is the Cohen Structure Theorem.[5] When our local ring R contains a field k, the Cohen Structure Theorem actually says that R is a quotient of $k[\![x_1, \dots, x_d]\!]$ for some d.

The nice things we proved about regular local rings have analogues in any regular ring, not necessarily local. For example, when R is a regular ring of dimension d, then it is still true that every finitely generated R-module has projective dimension at most d, even if R is not local; if R is not regular, then it has finitely generated modules with infinite projective dimension.

[5]In fact, this amazing theorem was I. S. Cohen's PhD thesis!

Appendix A

Macaulay2

There are several computer algebra systems dedicated to algebraic geometry and commutative algebra computations, such as Singular (more popular among algebraic geometers), CoCoA (which is more popular with european commutative algebraists, having originated in Genova, Italy), and Macaulay2. There are many computations you could run on any of these systems (and others), but we will focus on Macaulay2 since it's the most popular computer algebra system among US based commutative algebraists.

Macaulay2, as the name suggests, is a successor of a previous computer algebra system named Macaulay. Macaulay was first developed in 1983 by Dave Bayer and Mike Stillman, and while some still use it today, the system has not been updated since its final release in 2000. In 1993, Daniel Grayson and Mike Stillman released the first version of Macaulay2, and the current stable version if Macaulay2 1.16.

Macaulay2, or M2 for short, is an open-source project, with many contributors writing packages that are then released with the newest Macaulay2 version. Journals like the *Journal of Software for Algebra and Geometry* publish peer-refereed short articles that describe and explain the functionality of new packages, with the package source code being peer reviewed as well.

The National Science Foundation has funded Macaulay2 since 1992. Besides funding the project through direct grants, the NSF has also funded several Macaulay2 workshops — conferences where Macaulay2 package developers gather to work on new packages, and to share updates to the Macaulay2 core code and recent packages.

A.1 Getting started

A Macaulay2 session often starts with defining some ambient ring we will be doing computations over. Common rings such as the rationals and the integers can be defined using the commands `QQ` and `ZZ`; one can easily take quotients or build polynomial rings (in finitely many variables) over these. For example,

```
i1 : R = ZZ/101[x,y]

o1 = R
```

```
o1 : PolynomialRing
```

and

```
i1 : k = ZZ/101;

i2 : R = k[x,y];
```

both store the ring $\mathbb{Z}/101$ as R, with the small difference that in the second example Macaulay2 has named the coefficient field k. One quirk that might make a difference later is that if we use the first option and later set k to be the field $\mathbb{Z}/101$, our ring R is *not* a polynomial ring over k. Also, in the second example we ended each line with a `;`, which tells Macaulay2 to run the command but not display the result of the computation — which is in this case was simply an assignment, so the result is not relevant. Lines indicated with as in, where n is some integer, are input lines, whereas lines with an on indicate output lines.

We can now do all sorts of computations over our ring R. We can define ideals in R, and use them to either define a quotient ring S of R or an R-module M, as follows:

```
 i3 : I = ideal(x^2,y^2,x*y)

               2   2
o3 = ideal (x , y , x*y)

o3 : Ideal of R

i4 : M = R^1/I

o4 = cokernel | x2 y2 xy |

                                 1
o4 : R-module, quotient of R

i5 : S = R/I

o5 = S

o5 : QuotientRing
```

It is important to note that while R is a ring, R^1 is the R-module R — this is a very important difference for Macaulay2, since these two objects have different types. So S defined above is a ring, while M is a module. Notice that Macaulay2 stored the module M as the cokernel of the map

$$R^3 \xrightarrow{\begin{bmatrix} x^2 & y^2 & xy \end{bmatrix}} R$$

Note also that there is an alternative syntax to write our ideal I from above, as follows:

```
i15 : I = ideal"x2,xy,y2"

               2        2
o15 = ideal (x , x*y, y )

o15 : Ideal of R
```

When you make a new definition in Macaulay2, you might want to pay attention to what ring your new object is defined over. For example, now that we defined this ring S, Macaulay2 has automatically taken S to be our current ambient ring, and any calculation or definition we run next will be considered over S and not R. If you want to return to the original ring R, you must first run the command `use S`.

If you want to work over a finitely generated algebra over one of the basic rings you can define in Macaulay2, and your ring is not a quotient of a polynomial ring, you want to rewrite this algebra as a quotient of a polynomial ring. For example, suppose you want to work over the 2nd Veronese in 2 variables over our field k from before, meaning the algebra $k[x^2, xy, y^2]$. We need 3 algebra generators, which we will call a, b, c, corresponding to x^2, xy, and y^2:

```
 i11 : U = k[a,b,c]

o11 = U

o11 : PolynomialRing

i12 : f = map(R,U,{x^2,x*y,y^2})
                    2        2
o12 = map(R,U,{x , x*y, y })

o12 : RingMap R <--- U

i13 : J = ker f

             2
o13 = ideal(b  - a*c)

o13 : Ideal of U

i14 : T = U/J

o14 = T

o14 : QuotientRing
```

Our ring T at the end is isomorphic to the 2nd Veronese of R, which is the ring we wanted.

A.2 Basic commands

Many Macaulay2 commands are easy to guess, and named exactly what you would expect them to be named. If you are not sure how to use a certain command, you can run `viewHelp` followed by the command you want to ask about; this will open an html file with the documentation for the method you asked about. Often, googling "Macaulay2" followed by descriptive words will easily land you on the documentation for whatever you are trying to do.

Here are some basic commands you will likely use:

- `ideal`$(f_1, \dots, f_n)$ will return the ideal generated by $f_1, \dots, f_n$. Here products should be indicated by $*$, and powers with ^. If you'd rather not use ^ (this might be nice if you have lots of powers), you can write `ideal`$(f_1, \dots, f_n)$ instead.

- `map`$(S, R, f_1, \dots, f_n)$ gives a ring map $R \to S$ if R and S are rings, and R is a quotient of $k[x_1, \dots, x_n]$. The resulting ring map will send $x_i \mapsto f_i$. There are many variations of `map` — for example, you can use it to define R-module homomorphisms — but you should carefully input the information in the required format. Try `viewHelp map` in Macaulay2 for more details

- `ker`(f) returns the kernel of the map f.

- `I + J` and `I*J` return the sum and product of the ideals I and J, respectively.

- `A = matrix`$\{\{a_{1,1}, \dots, a_{1,n}\}, \dots, \{a_{m,1}, \dots, a_{m,n}\}\}$ returns the matrix

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}$$

A.3 Complexes in Macaulay2

There are two different ways to do computations involving complexes in Macaulay2: using `ChainComplexes`, or the new (and still incomplete) `Complexes` package. To use `Complexes`, you must first load the `Complexes` package, while the `ChainComplexes` methods are automatically loaded with Macaulay2.

A.3.1 Chain Complexes

To create a new chain complex by hand, we start by setting up R-module maps.

```
i1 : R = QQ[a,b];

i2 : d1 = map(R^1, R^2, {{a,b}})
```

```
o2 = | a b |
              1       2
o2 : Matrix R  <--- R

i3 : d2 = map(R^2, R^1, {{-b},{a}})

o3 = | -b |
     | a  |
              2       1
o3 : Matrix R  <--- R
```

Keep in mind that the syntax of map is a bit funny: we write `map(target,source,matrix)`. To make sure we set up the next map in a way that is composable with d_1, we can use the methods `source` and `target`:

```
i3 : d1 = map(source d0, R^1, {{-b},{a}})

o3 = | -b |
     | a  |
              2       1
o3 : Matrix R  <--- R
```

We can also double check our maps do indeed map a complex, by checking the composition $d_1 \circ d_2$:

```
i4 : d1 * d2 == 0

o4 = true
```

So now we are ready to set up our new chain complex.

```
i5 : C = new ChainComplex

o5 = 0

o5 : ChainComplex

i6 : C#0 = target d1

      1
o6 = R

o6 : R-module, free

i7 : C#1 = target d2

      2
```

```
o7 = R

o7 : R-module, free

i8 : C#2 = source d2

      1
o8 = R

o8 : R-module, free
```

Given a chain complex `C`, we can ask Macaulay2 what our complex is by simply running the name of the complex:

```
i9 : C

      1      2      1
o9 = R  <-- R  <-- R

     0      1      2

o9 : ChainComplex
```

Or we can ask for a better visual description of the maps, using `C.dd`:

```
i10 : C.dd

           1          2
o10 = 0 : R  <----- R  : 1
                0

           2          1
      1 : R  <----- R  : 2
                0

o10 : ChainComplexMap
```

We can also set up the same complex in a more compact way, by simply feeding the maps we want in order. Macaulay2 will automatically place the first map with the target in homological degree 0 and the source in degree 1.

```
11 : D = chainComplex(d1,d2)

       1      2      1
o11 = R  <-- R  <-- R

      0      1      2
```

```
o11 : ChainComplex
```

Notice this is indeed the same complex.

```
i12 : D.dd

           1               2
o12 = 0 : R  <----------- R  : 1
                | a b |

           2              1
      1 : R  <---------- R  : 2
                | -b |
                | a  |

o12 : ChainComplexMap
```

We can also ask Macaulay2 to compute the homology of our complex:

```
i13 : HH D

o13 = 0 : cokernel | a b |

      1 : subquotient (| b  |, | -b |)
                       | -a |  | a  |
      2 : image 0

o13 : GradedModule
```

Or we could simply ask for the homology in a specific degree:

```
i14 : HH_0 D

o14 = cokernel | a b |
                              1
o14 : R-module, quotient of R
```

A.3.2 The Complexes package

To use this functionality, you must first load the `Complexes` package.

```
i15 : needsPackage "Complexes";

o15 = Complexes

o15 : Package
```

We can use our maps from above to set up a complex with the same maps. We feed a list of the maps we want to use to the method `complex`.

```
i16 : F = complex({d1,d2})

       1      2      1
o16 = R  <-- R  <-- R

      0      1      2

o16 : Complex
```

We cam read off the maps and the homology in our complex using the same commands as we use with `chainComplexes`, although the information returned gets presented in a slightly different fashion.

```
i17 : HH F

o17 = cokernel | a b | <-- subquotient (| b  |, | -b |) <-- image 0
                                        | -a |  | a  |
      0                                                      2
                          1

o17 : Complex

i18 : F.dd

          1              2
o18 = 0 : R  <----------- R  : 1
               | a b |

          2             1
      1 : R  <---------- R  : 2
               | -b |
               | a  |

o18 : ComplexMap
```

If we want to set up our complex starting in a different homological degree, we can do the following:

```
i19 : G = complex({d1,d2}, Base => 7)

       1      2      1
o19 = R  <-- R  <-- R
```

```
      7      8      9

o19 : Complex

i20 : H = complex({d1,d2}, Base => -13)

       1       2       1
o20 = R   <-- R   <-- R

      -13     -12     -11

o20 : Complex
```

A.3.3 Maps of complexes

Suppose we are given two complexes `C` and `D` and a map of complexes `f`: `C` $\longrightarrow$ `D`. The routine `map` can be used to define f using `chainComplexes`: it receives the target `D`, the source `D`, and a function `f` that returns f_i when we compute `f(i)`.

```
i1 : R = QQ[a,b];

i2 : c1 = map(R^0,R^1,0);

                      1
o2 : Matrix 0 <--- R

i3 : c2 = map(R^1, R^2, {{a,b}});

             1       2
o3 : Matrix R  <--- R

i4 : c3 = map(R^2, R^1, {{-b},{a}});

             2       1
o4 : Matrix R  <--- R

i5 : c4 = map(R^1, R^0, 0);

             1
o5 : Matrix R  <--- 0

i6 : C = chainComplex(c1,c2,c3,c4);

i7 :
     d1 = map(R^0,R^1,0);
```

```
                     1
o7 : Matrix 0 <--- R

i8 : d2 = id_(R^1);

              1       1
o8 : Matrix R  <--- R

i9 : d3 = map(R^1, R^0, 0);

              1
o9 : Matrix R  <--- 0

i10 : d4 = map(R^0, R^0, 0);

o10 : Matrix 0 <--- 0

i11 : D = chainComplex(d1,d2,d3,d4)

              1      1
o11 = 0 <-- R  <-- R  <-- 0 <-- 0

      0     1      2      3     4

o11 : ChainComplex

i12 :
      f0 = map(R^0, R^0, 0);

o12 : Matrix 0 <--- 0

i13 : f1 = map(R^1, R^1, matrix{{0_R}});

               1       1
o13 : Matrix R  <--- R

i14 : f2 = map(R^2, R^1, {{b},{-a}});

               2       1
o14 : Matrix R  <--- R

i15 : f3 = map(R^1, R^0, 0);

               1
```

```
o15 : Matrix R  <--- 0

i16 : f4 = map(R^0, R^0, 0);

o16 : Matrix 0 <--- 0

i17 : f = map(C,D,i -> if i==0 then f0 else(
      if i==1 then f1 else (
      if i==2 then f2 else (
      if i == 3 then f3 else (
      if i==4 then f4)))))

o17 = 0 : 0 <----- 0 : 0
               0

           1         1
      1 : R  <----- R  : 1
                0

           2              1
      2 : R  <---------- R  : 2
                | b  |
                | -a |
           1
      3 : R  <----- 0 : 3
                0

      4 : 0 <----- 0 : 4
               0

o17 : ChainComplexMap
```

Here's what we can do if we prefer to write a list with the maps in `f`:

```
i18 : f = map(C,D,i -> {f0,f1,f2,f3,f4}_i)

o18 = 0 : 0 <----- 0 : 0
               0

           1         1
      1 : R  <----- R  : 1
                0

           2              1
      2 : R  <---------- R  : 2
```

```
                | b  |
                | -a |

           1
      3 : R  <----- 0 : 3
                0

      4 : 0 <----- 0 : 4
               0

o18 : ChainComplexMap
```

If we prefer to do the same with the `Complexes` package, one advantage is that `map` *does* receive `(target, source, list of maps)`.

```
i42 : C = complex({c1,c2,c3,c4});

i43 : D = complex({d1,d2,d3,d4});

i44 : f = map(C,D,{f0,f1,f2,f3,f4})

           2              1
o44 = 2 : R  <---------- R  : 2
                | b  |
                | -a |

o44 : ComplexMap
```

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