DG-Structures on Minimal Free Resolutions of Fiber Products

k to be a field

$\underline{x} = x_1, \dots, x_m$

$\underline{y} = y_1, \dots, y_n$

$\underline{x}\,\underline{y} = \{ x_i y_j \mid i, j \}$

$I \subseteq \langle \underline{x} \rangle \subseteq k[\underline{x}]$

$J \subseteq \langle \underline{y} \rangle \subseteq k[\underline{y}]$

ideals

Fiber Product

$$\frac{k[\underline{x}]}{I} \times_k \frac{k[\underline{y}]}{J} \cong \frac{k[\underline{x}, \underline{y}]}{\langle I, \underline{x}\,\underline{y}, J \rangle}$$

Theorem (Nasseh, Sather-Wagstaff '17)

Let M, N be f.g. modules over the fiber product $R = S \times_k T$.

a.) If depth $S = 0$ or depth $T = 0$ and $0 = \operatorname{Tor}_i^R(M,N)$ for $i \geq 5$, then M or N is free over R.

b.) In general, if $\operatorname{Tor}_i^R(M,N) = 0 = \operatorname{Tor}_{i+1}^R(M,N)$ $i \geq 5$, then $\operatorname{pd}_R M \leq 1$ or $\operatorname{pd}_R N \leq 1$.

<u>Def</u>: Differential Graded Algebra is a complex A over R with a binary operation satisfying

- associative
- distributive
- unital
- graded commutative $ab = (-1)^{|a||b|} ba$
- Leibniz rule $\partial_{|a|+|b|}(ab) = \partial_{|a|}(a)b + (-1)^{|a|} a \partial_{|b|}(b)$.

$$\mathcal{X} = K^{k[x_1,x_2]}(x_1, x_2) = 0 \to k[\underline{x}] \xrightarrow{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}} k[\underline{x}]^2 \xrightarrow{(x_1 \ x_2)} k[\underline{x}] \to 0$$

$$\mathcal{Y} = K^{k[y_1,y_2,y_3]}(\underline{y}) =$$

$$0 \to k[\underline{y}] \xrightarrow{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \end{pmatrix}} k[\underline{y}]^3 \xrightarrow{\begin{pmatrix} -y_2 & -y_3 & 0 \\ y_1 & 0 & -y_3 \\ 0 & y_1 & y_2 \end{pmatrix}} k[\underline{y}]^3 \xrightarrow{(y_1 \ y_2 \ y_3)} k[\underline{y}] \to 0$$

$f_{12} \in Y_2 \qquad f_3 \in Y_1$

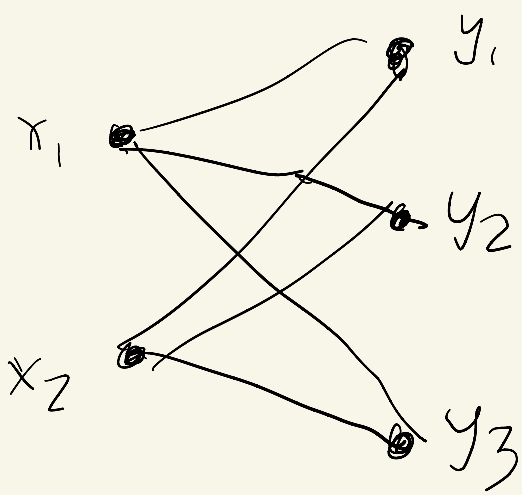
$f_{12} \cdot f_3 = f_{123}$

$f_{12} \sim f_1 \wedge f_2 \qquad f_{12} \cdot f_3 \sim (f_1 \wedge f_2) \wedge (f_3)$

f_3

Edge Ideals of Complete Bipartite graphs

$K_{2,3}$



$I(K_{2,3}) = \langle x_i y_j \mid i = 1,2,\ j = 1,2,3 \rangle$

$$\frac{k[\underline{x},\underline{y}]}{I(K_{2,3})} = \frac{k[\underline{x},\underline{y}]}{\langle \underline{x}\,\underline{y} \rangle} = k[\underline{x}] \times_k k[\underline{y}].$$

Daniel Visscher '06 gives an explicit min res of $\frac{k[\underline{x},\underline{y}]}{\langle \underline{x}\,\underline{y} \rangle}$ over $k[\underline{x},\underline{y}]$.

Want to Resolve $\frac{k[\underline{x},\underline{y}]}{I(K_{m,n})}$

$\langle \underline{x} \rangle \otimes_k \langle \underline{y} \rangle$

$$(X * Y)_i = \begin{cases} \left(X_{\geq 1} \otimes_k Y_{\geq 1}\right)_{i+1} & i \geq 1 \\ X_0 \otimes_k Y_0 & i = 0 \end{cases}$$

$$\partial_i^{X * Y} = \begin{cases} \partial_{i+1}^{X_{\geq 1} \otimes Y_{\geq 1}} & i \geq 2 \\ \partial_1^X \otimes \partial_1^Y & i = 1 \end{cases}$$

$$0 \to k[\underline{x},\underline{y}] \xrightarrow{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \\ -x_2 \\ x_1 \end{pmatrix}} k[\underline{x},\underline{y}]^5 \xrightarrow{\begin{pmatrix} -y_2 & -y_3 & 0 & 0 & 0 \\ y_1 & 0 & -y_3 & 0 & 0 \\ 0 & y_1 & y_2 & 0 & 0 \\ -x_2 & 0 & 0 & -y_3 & 0 \\ 0 & -x_2 & 0 & y_2 & 0 \\ 0 & 0 & -x_2 & -y_1 & 0 \\ x_1 & 0 & 0 & 0 & -y_3 \\ 0 & x_1 & 0 & 0 & y_2 \\ 0 & 0 & x_1 & 0 & -y_1 \end{pmatrix}} k[\underline{x},\underline{y}]^9 \longrightarrow$$

$$\begin{pmatrix} -x_2 & 0 & 0 & y_2 & y_3 & 0 & 0 & 0 & 0 \\ 0 & -x_2 & 0 & -y_1 & 0 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & -y_1 & -y_2 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 & y_2 & y_3 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & 0 & y_3 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & -y_2 \end{pmatrix} \longrightarrow \mathbb{R}[\underline{x},\underline{y}]^6$$

$$(x_1y_1 \quad x_1y_2 \quad x_1y_3 \quad x_2y_1 \quad x_2y_2 \quad x_2y_3) \longrightarrow k[\underline{x},\underline{y}] \longrightarrow 0$$

Thm: $X * Y$ resolves $\dfrac{k[\underline{x},\underline{y}]}{\langle \underline{x}\,\underline{y}\rangle}$ minimally over $k[\underline{x},\underline{y}]$.

Cor: $$P^{k[\underline{x},\underline{y}]}_{\frac{k[\underline{x},\underline{y}]}{\langle \underline{x}\,\underline{y}\rangle}}(t) = 1 + \frac{1}{t}\left(P^{k[\underline{x}]}_{k}(t) - 1\right)\left(P^{k[\underline{y}]}_{k}(t) - 1\right)$$

$$= 1 + \frac{1}{t}\left((1+t)^m - 1\right)\left((1+t)^n - 1\right)$$

$$(e_1 * f_2)(e_1 * f_3) = \cancel{e_1 e_1}^{0} * y_2 f_3 - e_1 \partial^x(e_1) * f_2 f_3$$

$$= -x_1 e_1 * f_2 f_3$$

Thm: If X is a DGA and Y is Koszul or Taylor, then $X * Y$ has a DGA structure given by

$$(e_I * f_\Omega)(e_J * f_\Gamma) = \mathbb{1}[\omega_1 \le \gamma_1 < \omega_2](-1)^{(|f_\Omega|-1)(|e_J|-1)}(e_I e_J * P_1(f_\Omega) f_\Gamma)$$
$$- \mathbb{1}[\omega_1 \le \gamma_1]\,\mathbb{1}[|e_J| = 1](e_I \partial^X(e_J) * f_\Omega f_\Gamma)$$
$$+ \mathbb{1}[\gamma_1 < \omega_1 < \gamma_2](-1)^{(|f_\Omega|-1)(|e_J|)}(e_I e_J * f_\Omega P_1(f_\Gamma))$$
$$- \mathbb{1}[\gamma_1 < \omega_1]\,\mathbb{1}[|e_I| = 1](-1)^{|f_\Omega|(|e_J|-1)}\partial^X(e_I) e_J * f_\Omega f_\Gamma$$

$$\Omega = \{\omega_1, \omega_2, \dots, \omega_\ell\}$$

$$\underbrace{\frac{k[x_1,x_2]}{\langle x_1^2,\, x_1x_2\rangle}}_{I''} \times_k k[y_1,y_2,y_3] \cong \frac{k[\underline{x},\underline{y}]}{\langle I,\, \underline{x}\,\underline{y}\rangle}$$

resolved by $K_{\underline{x}\,\underline{y}}$

want to resolve

$$0 \longrightarrow I\,\frac{k[\underline{x},\underline{y}]}{\langle \underline{x}\,\underline{y}\rangle} \longrightarrow \frac{k[\underline{x},\underline{y}]}{\langle \underline{x}\,\underline{y}\rangle} \longrightarrow \frac{k[\underline{x},\underline{y}]}{\langle I,\, \underline{x}\,\underline{y}\rangle} \longrightarrow 0$$

$$I \subseteq \langle \underline{x}\rangle \implies I\,\frac{k[\underline{x},\underline{y}]}{\langle \underline{x}\,\underline{y}\rangle} \cong I \otimes_k \frac{k[\underline{y}]}{\langle \underline{y}\rangle}$$

?

Assume $\mathcal{d}$ resolves $\dfrac{k[\underline{x}]}{I}$.

$\Phi : \Sigma^{-1}(\mathcal{d}_{\geq 1} \otimes_k Y)$ resolves $I \otimes_k \dfrac{k[\underline{y}]}{\langle \underline{y} \rangle}$

$$\Sigma^{-1}(\mathcal{d}_{\geq 1} \otimes_k Y) \longrightarrow X * Y$$

$$\begin{tikzcd}
\mathcal{d} = \arrow[d, "\phi"'] & 0 \arrow[r] & k[\underline{x}] \arrow[r, "\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}"] \arrow[d, "x_1"] & k[\underline{x}]^2 \arrow[r, "\begin{pmatrix} x_1^2 & x_1x_2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} x_1 & 0 \\ 0 & x_1 \end{pmatrix}"] & k[\underline{x}] \arrow[r] \arrow[d, "1"] & 0 \\
X = & 0 \arrow[r] & k[\underline{x}] \arrow[r, "\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}"'] & k[\underline{x}]^2 \arrow[r, "\begin{pmatrix} x_1 & x_2 \end{pmatrix}"'] & k[\underline{x}] \arrow[r] & 0
\end{tikzcd}$$

$$\Sigma^{-1}\left(d_{\geq 1} \otimes_R \gamma\right)$$

$$0 \longrightarrow k[\underline{x},\underline{y}] \xrightarrow{\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \\ -x_2 \\ x_1 \end{pmatrix}} k[\underline{x},\underline{y}]^5 \xrightarrow{\begin{pmatrix} -y_2 & -y_3 & 0 & 0 & 0 \\ y_1 & 0 & -y_3 & 0 & 0 \\ 0 & y_1 & y_2 & 0 & 0 \\ -x_2 & 0 & 0 & -y_3 & 0 \\ 0 & -x_2 & 0 & y_2 & 0 \\ 0 & 0 & -x_2 & -y_1 & 0 \\ x_1 & 0 & 0 & 0 & -y_3 \\ 0 & x_1 & 0 & 0 & y_2 \\ 0 & 0 & x_1 & 0 & -y_1 \end{pmatrix}} k[\underline{x},\underline{y}]^9 \longrightarrow$$

$$
\begin{pmatrix}
y_1 & y_2 & y_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_2 & 0 & 0 & y_2 & y_3 & 0 & 0 & 0 & 0 \\
0 & -x_2 & 0 & -y_1 & 0 & y_3 & 0 & 0 & 0 \\
0 & 0 & -x_2 & 0 & -y_1 & -y_2 & 0 & 0 & 0 \\
x_1 & 0 & 0 & 0 & 0 & 0 & y_2 & y_3 & 0 \\
0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & 0 & y_3 \\
0 & 0 & x_1 & 0 & 0 & 0 & 0 & -y_1 & -y_2
\end{pmatrix}
\longrightarrow k[x,y]^7 -
$$

$$\Phi(\alpha \otimes \beta) = \begin{cases} (-1)^{|\alpha|+|\beta|} \phi(\alpha) * \beta & |\beta| > 0 \\ \partial_1^{\alpha}(\alpha) * \beta & |\beta| = 0,\ |\alpha| = 1 \\ 0 & |\beta| = 0,\ |\alpha| > 1 \end{cases}$$

$$\underbrace{\begin{pmatrix} -x_2 & -y_1 & -y_2 & -y_3 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & -y_1 & -y_2 & -y_3 \end{pmatrix}}_{} \longrightarrow k[x,y]^2 \longrightarrow 0$$

$$\Phi_0 : k[\underline{x},\underline{y}]^2 \xrightarrow{\begin{pmatrix} x_1^2 & x_1x_2 \end{pmatrix}} k[\underline{x},\underline{y}]$$

$$\Phi_1 : k[\underline{x},\underline{y}]^7 \xrightarrow{\hspace{3cm}} k[\underline{x},\underline{y}]^6$$

$\varphi_{12} \otimes 1$

$$\begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 \end{pmatrix}$$

$$\Phi_2 : k[\underline{x},\underline{y}]^9 \xrightarrow{-x_1 I_9} k[\underline{x},\underline{y}]^9$$

$$\Phi_3 : k[\underline{x},\underline{y}]^5 \xrightarrow{x_1 I_5} k[\underline{x},\underline{y}]^5$$

$$\Phi_4 : k[\underline{x},\underline{y}] \xrightarrow{-x_1} k[\underline{x},\underline{y}]$$

$$
\Phi \qquad
\begin{tikzcd}[column sep=huge, row sep=huge]
0 \arrow[r] & k[\underline{x},\underline{y}] \arrow[r, "-\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \\ -x_2 \\ x_1 \end{pmatrix}"] \arrow[d, "-x_1"'] & k[\underline{x},\underline{y}]^5 \arrow[r, "-\begin{pmatrix} -y_2 & -y_3 & 0 & 0 & 0 \\ y_1 & 0 & -y_3 & 0 & 0 \\ 0 & y_1 & y_2 & 0 & 0 \\ -x_2 & 0 & 0 & -y_3 & 0 \\ 0 & -x_2 & 0 & y_2 & 0 \\ 0 & 0 & -x_2 & -y_1 & 0 \\ x_1 & 0 & 0 & 0 & -y_3 \\ 0 & x_1 & 0 & 0 & y_2 \\ 0 & 0 & x_1 & 0 & -y_1 \end{pmatrix}"] \arrow[d, "x_1 I_5"'] & k[\underline{x},\underline{y}]^9 \arrow[r] \arrow[d, "-x_1 I_9"'] & {} \\
0 \arrow[r] & k[\underline{x},\underline{y}] \arrow[r, "\begin{pmatrix} y_3 \\ -y_2 \\ y_1 \\ -x_2 \\ x_1 \end{pmatrix}"'] & k[\underline{x},\underline{y}]^5 \arrow[r, "\begin{pmatrix} -y_2 & -y_3 & 0 & 0 & 0 \\ y_1 & 0 & -y_3 & 0 & 0 \\ 0 & y_1 & y_2 & 0 & 0 \\ -x_2 & 0 & 0 & -y_3 & 0 \\ 0 & -x_2 & 0 & y_2 & 0 \\ 0 & 0 & -x_2 & -y_1 & 0 \\ x_1 & 0 & 0 & 0 & -y_3 \\ 0 & x_1 & 0 & 0 & y_2 \\ 0 & 0 & x_1 & 0 & -y_1 \end{pmatrix}"] & k[\underline{x},\underline{y}]^9 \arrow[r] & {}
\end{tikzcd}
$$

$\underline{\text{Thm}}$: $\Phi : \Sigma^{-1}(\mathcal{C}\ell_{\geq 1} \otimes Y) \longrightarrow X * Y$

is a chain map and

$$\mathrm{Cone}(\Phi) \text{ resolves } \frac{k[x_-, y_-]}{\langle I, x_- y_- \rangle}$$

$\underline{\text{Thm}}$: $\mathrm{Cone}(\Phi)$ as above is a DG-module over $X * Y$ where

$$(e_I * f_\Omega)(\alpha \otimes f_\Gamma) = -\mathbb{1}[|e_I| = 1](-1)^{(|\alpha|-1)|f_\Omega|} \partial^X(e_I)\alpha \otimes f_\Omega f_\Gamma$$

$$+ \mathbb{1}[\omega_1 \leq \gamma_1](-1)^{|\alpha|(|f_\Omega|-1)+|f_\Gamma|} e_I \phi(\alpha) * f_\Omega f_\Gamma.$$

$$F \qquad \frac{k[x]}{I} \times_k \frac{k[y]}{J}$$

resolves

$$\partial_i^F = \begin{pmatrix} \partial_i^{X \otimes Y} & \Phi_{i-1} & \Psi_{i-1} \\ 0 & \partial_i^{d_{\geq 1} \otimes Y} & 0 \\ 0 & 0 & \partial_i^{X \otimes d_{\geq 1}} \end{pmatrix}$$