Differential powers in mixed characteristic
joint with Alessandro De Stefani and Jack Jeffries

R Noetherian A-algebra
I ideal in R

A-linear differential operators $\quad D_{R|A} := \bigcup_{n \geq 0} D^n_{R|A} \subseteq \operatorname{Hom}_A(R, R)$

- $D^0_{R|A} := \operatorname{Hom}_R(R, R)$
- $\partial \in D^n_{R|A}$ if $[\partial, r] \in D^{n-1}_{R|A}$ for all $r \in D^0_{R|A}$

the nth differential power of I is
$$I^{\langle n \rangle} := \{ f \in R \mid \partial f \in I \text{ for all } \partial \in D^n_{R|A} \}$$

why do we care? Here's one great reason:

<u>theorem</u> (Zariski – Nagata)
k perfect field
$R = k[x_1, \ldots, x_d]$ $\qquad I^{\langle n \rangle} = I^{(n)}$ for all $n \geq 1$
$I = \sqrt{I}$

(Here $I^{(n)} = \bigcap_{P \in \operatorname{Ass}(R/I)} (I^n R_P \cap R)$ is the nth symbolic power of I)

Another format: $\quad I^{(n)} = I^{\langle n \rangle} = \bigcap_{\substack{m \supseteq I \\ m \text{ max}}} m^n$

How about in mixed characteristic?

<u>Ex</u>: $R = \mathbb{Z}[x]$

$\mathfrak{m} = (x, 2)$

$$\mathfrak{m}^n = \mathfrak{m}^{(n)} \quad \text{for all } n \geq 1$$

but

$$\partial(2) = 2\partial(1) \in (2) \subseteq \mathfrak{m} \quad \text{for all } n \geq 1$$

$$\Longrightarrow \quad 2 \in \mathfrak{m}^{\langle n \rangle} \quad \text{for all } n \geq 1$$

From now on: $A = \mathbb{Z}$ or a DVR with uniformizer $p \in \mathbb{Z}$ prime

R an A-algebra

<u>Def</u>: (Joyal, Buium) A p-derivation on R is a map $\delta : R \longrightarrow R$ st:

1) $\delta(1) = 0$

2) $\delta(a+b) = \delta(a) + \delta(b) + \dfrac{a^p + b^p - (a+b)^p}{p}$ for all $a, b \in R$

3) $\delta(ab) = \delta(a) b^p + a^p \delta(b) + p\,\delta(a)\,\delta(b)$

<u>Note</u> $\delta(x) = \dfrac{\Phi(x) - x^p}{p}$ is a p-derivation on R $\Leftrightarrow$ $\Phi(x) = x^p + p\,\delta(x)$ is a lift of the Frobenius map on R/p

- $R = \mathbb{Z} \Rightarrow \delta(n) = \dfrac{n - n^p}{p}$ is the unique p-derivation on R
- If R a complete unramified DVR with perfect residue field, R has a p-derivation
- B has a p-derivation $\Longrightarrow$ $B[x_1, \ldots, x_d]$ has a p-derivation

Key point: p-derivations decrease p-adic order, in the sense that

$$\delta(p^k v) = p^{k-1} v^p \mod p^k \qquad \text{for } v \notin (p)$$

Def (De Stefani - G - Jeffries) suppose R has a p-derivation δ:
the nth mixed differential power of $I \subseteq R$ is

$$I^{\langle n \rangle_{mix}} := \{ f \in R \mid \delta^a \circ \partial (f) \in I \text{ for all } a+b<n,\ \partial \in D^b_{R|A} \}$$

thm (De Stefani - G - Jeffries, 2020)
$A = \mathbb{Z}$ or a DVR with uniformizer $p \in \mathbb{Z}$ prime
R localization of fg smooth A-algebra (with a p-derivation)

Q prime ideal

$$\begin{cases} \bullet \text{ If } Q \cap A = 0, \text{ then } Q^{(n)} = Q^{\langle n \rangle} \text{ for all } n \geq 1 \\ \bullet \text{ If } Q \ni p, \text{ then } Q^{(n)} = Q^{\langle n \rangle_{mix}} \text{ for all } n \geq 1 \text{ as long as:} \end{cases}$$

→ $A/p \hookrightarrow R_Q/QR_Q$ separable (eg A/p perfect)
→ R has a p-derivation (eg $R = A[x_1, \ldots, x_d]$, A unramified DVR)

Application:

theorem (Chevalley's lemma, 1943)
(R, m) complete local ring
$\{I_n\}$ decreasing family of ideals
If $\bigcap_{n \geq 1} I_n = 0$, then $\exists f: \mathbb{N} \to \mathbb{N}$ st $I_{f(n)} \subseteq m^n$
(so $\{I_n\}$ induces a finer topology than m^n)

special case: $I_n = I^{(n)}$

Uniform Chevalley Lemma (Huneke – Katz – Validashti, 2009)

(R, m) complete local domain

then there exists a constant C, not depending on I, such that

$$I^{(Cn)} \subseteq m^n \text{ for all } n \geq 1$$

Goal Find explicit bounds on C.

(this will give us uniform bounds on the m-adic order of elements in $I^{(n)}$)

theorem (Zariski – Nagata) When R is regular, $C = 1$ works.

theorem (Dao – De Stefani – G – Huneke – Núñez Betancourt)

k any field

$R = k[f_1, \dots, f_e] \xhookrightarrow{\oplus} S = k[x_1, \dots, x_d]$ graded direct summand

$m = (f_1, \dots, f_e)$, f_i homogeneous

$D = \max_i \{\deg f_i\}$

then $Q^{(Dn)} \subseteq m^n$ for all $n \geq 1$ and every homogeneous prime Q

Ingredients: Use the facts that: $R \underset{\curvearrowleft \, s}{\xhookrightarrow{\oplus}} S$

- $\delta \in D^n_{S|k} \Rightarrow s \circ \delta \in D^n_{R|k}$ (Àlvarez Montaner – Huneke – Núñez Betancourt)
- $Q^{(n)} \subseteq Q^{\langle n \rangle}$
- $\eta = (x_1, \dots, x_d)$ satisfies $\eta^n = \eta^{(n)} = \eta^{\langle n \rangle}$

<u>theorem</u> (De Stefani – Grifo – Jeffries)

A DVR with uniformizer p and A/p perfect

$R = A[f_1, \dots, f_e] \subseteq S = A[x_1, \dots, x_d]$, $\quad f_i$ homogeneous

$q = (f_1, \dots, f_e)$

$m = q + (p) \qquad\qquad D := \max\{\deg f_i\}$

① $\begin{cases} Q \subseteq q \text{ prime} \\ R \xhookrightarrow{\oplus} S \end{cases} \Longrightarrow Q^{(Dn)} \subseteq q^n$

for all $n \geq 1$

② $\begin{cases} Q \subseteq m \text{ prime}, \ Q \ni p \\ R/p \xhookrightarrow{\oplus} S/p \end{cases} \Longrightarrow Q^{(Dn)} \subseteq m^n$

<u>Idea</u> Even if R does not have a p-derivation, we can get by just using the p-derivation on S, by using

$$Q^{\{n\}} := \{ x \in R \mid \delta\left(\overline{S^{a}\partial(x)}S\right) \subseteq \overline{Q} \text{ for all } a+b<n, \ \partial \in D^{b}_{S|A} \}$$