Linear Algebra

Math 314 Fall 2025

Lecture 8

Today's poll code:

U6H5RJ

Office hours

Mondays 5–6 pm
Wednesdays 2–3 pm
in Avery 339 (Dr. Grifo)

Tuesdays 11–noon
Thursdays 1–2 pm
in Avery 337 (Kara)

To do list:

- Webwork 2.5 due tomorrow
- WebworK 2.6 due Friday

Quiz on Friday on Lectures 8—9

Midterm 1 in 2 weeks

Quick Recap

$$A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

$v_1, \ldots, v_n$ are linearly independent	$v_1, \ldots, v_n$ span $\mathbb{R}^m$
$\Updownarrow$	$\Updownarrow$
$Ax = 0$ has only one solution	$Ax = b$ has a solution for every b
$\Updownarrow$	$\Updownarrow$
A has pivots in every column	A has pivots in every row

Linear transformations

A function $T\colon \mathbb{R}^n \to \mathbb{R}^m$

is an assignment

each vector in $\mathbb{R}^n$ $\longrightarrow$ a vector in $\mathbb{R}^m$

Matrix transformations: Any $m \times n$ matrix A determines a function

$$T\colon \mathbb{R}^n \to \mathbb{R}^m$$

as follows: for each vector $x \in \mathbb{R}^n$,

$$T(x) = Ax.$$

Such a function is called a *matrix transformation*.

$$\mathbb{R}^{\#\text{columns}} \longrightarrow \mathbb{R}^{\#\text{rows}}$$

Two properties of matrix products:

- $A(v+w) = Av + Aw$
- $A(cv) = cAv$

So, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation then

- $T(v+w) = T(v) + T(w)$
- $T(cv) = cT(v)$

Any function with these properties is called a **linear transformation**.

linear transformations = matrix transformations

$$T(v + w) = T(v) + T(w)$$

$$T(cv) = cT(v)$$

for all u, v
and all constants c

$$T(x) = Ax$$

for all x

Given a linear transformation $\mathrm{T}\colon \mathbb{R}^n \to \mathbb{R}^m$

its **standard matrix** is

$$A = \begin{bmatrix} \mathrm{T}(e_1) & \cdots & \mathrm{T}(e_n) \end{bmatrix}$$

so

$T(x) = Ax$ for all vectors x

ith standard vector
in $\mathbb{R}^n$

$$\mathbf{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \longrightarrow \text{position } i$$

We've mostly shown examples of linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, but they can go from $\mathbb{R}^n$ to $\mathbb{R}^m$ for any choices of m and n.

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For example, let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation associated to the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$T \text{ sends } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ to } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ (first column of } A\text{)}$$

$$T \text{ sends } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ to } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ (second column of } A\text{)}$$

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$\Longrightarrow$ T moves xy-plane into the yz plane without stretching it in any way.

We can also define a linear transformation like $T : \mathbb{R}^7 \to \mathbb{R}^4$ by specifying some 4×7 matrix A, such as

$$A = \begin{bmatrix} 1 & -3 & 5 & 2 & 0 & 6 & 7 \\ 2 & \frac{3}{2} & 1 & 0 & 14 & -20 & \pi \\ 5 & e & 17 & 0 & 0 & 0 & 2 \\ 3 & 1 & 4 & 1 & 5 & 9 & 7 \end{bmatrix}.$$

Just don't ask me to visualize it!

Injective/surjective functions

A function $T\colon \mathbb{R}^n \to \mathbb{R}^m$

is an assignment

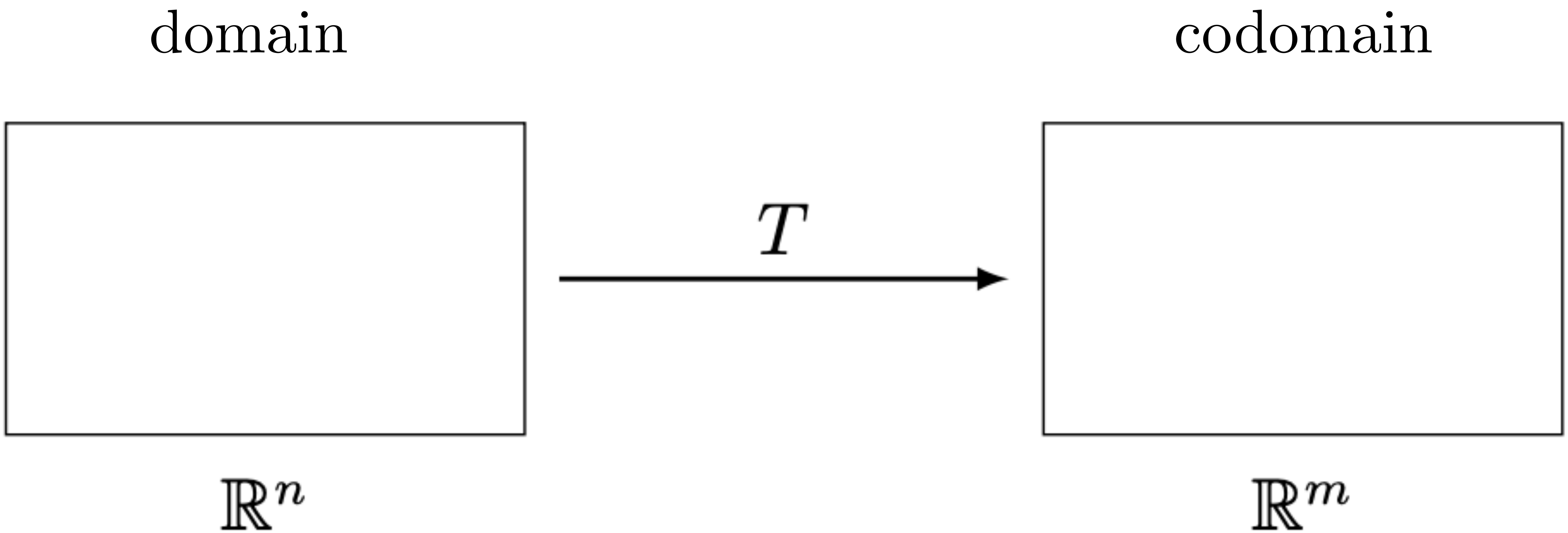
each vector in $\mathbb{R}^n$	$\longrightarrow$	a vector in $\mathbb{R}^m$

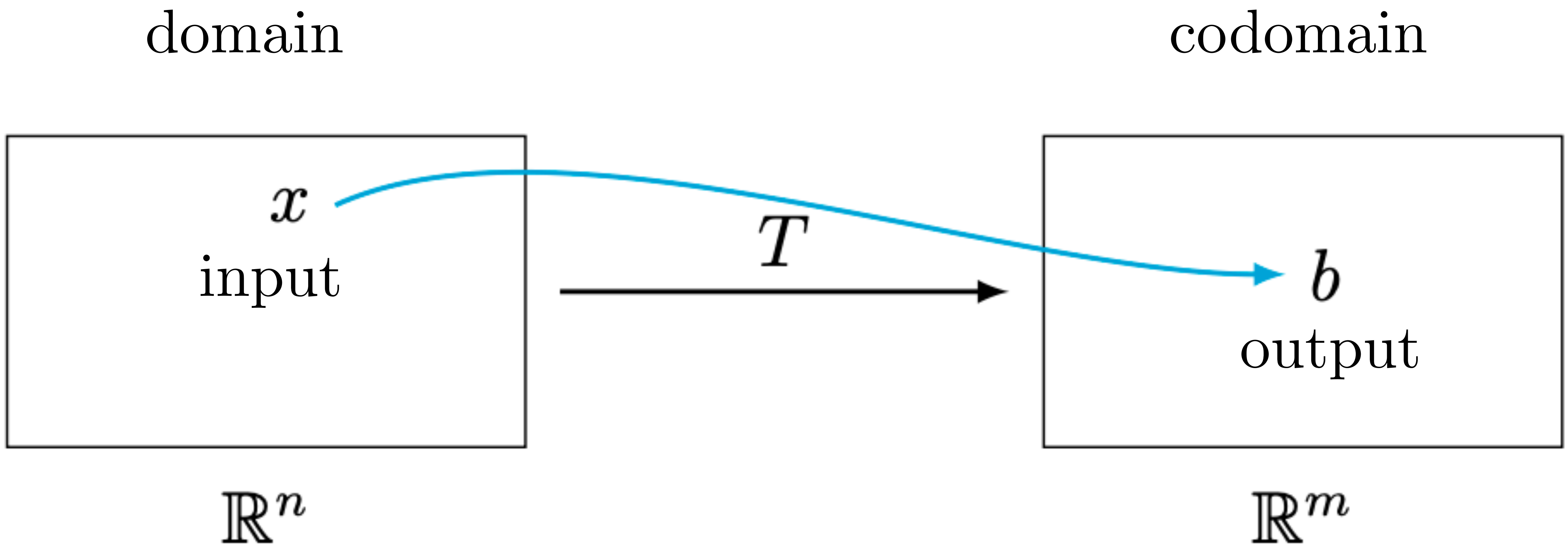
A function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ has

domain $\mathbb{R}^n$ $\longrightarrow$ inputs

and

codomain $\mathbb{R}^m$ $\longrightarrow$ where the outputs live

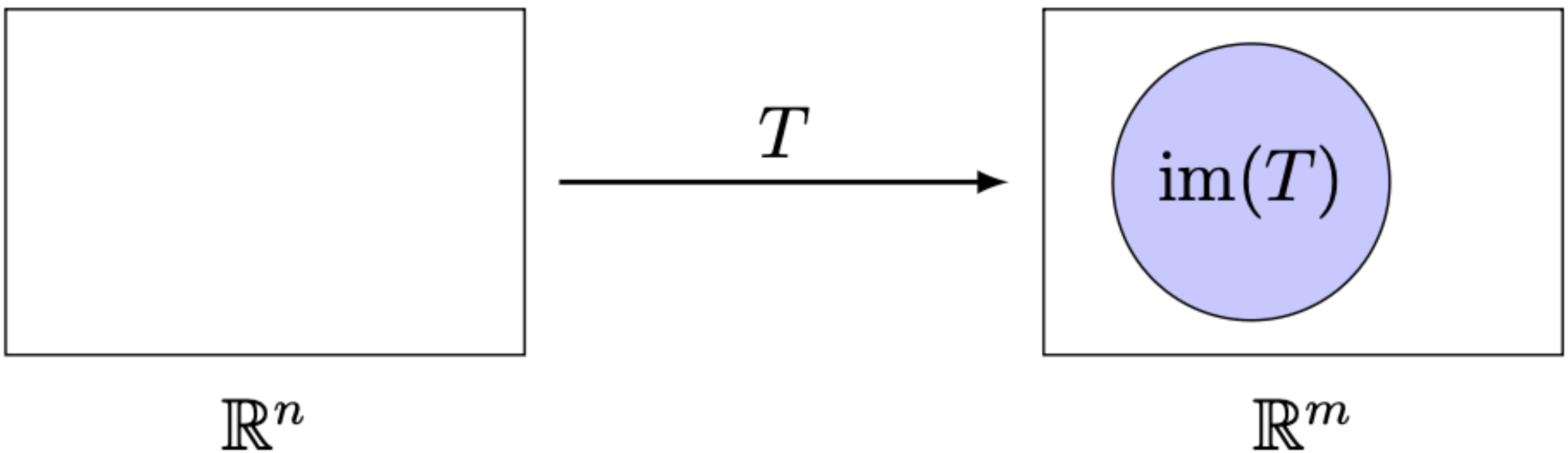

domain
codomain
T
$\mathbb{R}^n$
$\mathbb{R}^m$


domain
codomain
x
input
T
b
output
ℝn
ℝm

The **image** or **range** of a function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is

$$\operatorname{im}(T) := \{T(x) \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

Informally: the set of actual outputs


T
$\mathrm{im}(T)$
$\mathbb{R}^n$
$\mathbb{R}^m$

T is **surjective** or **onto** if

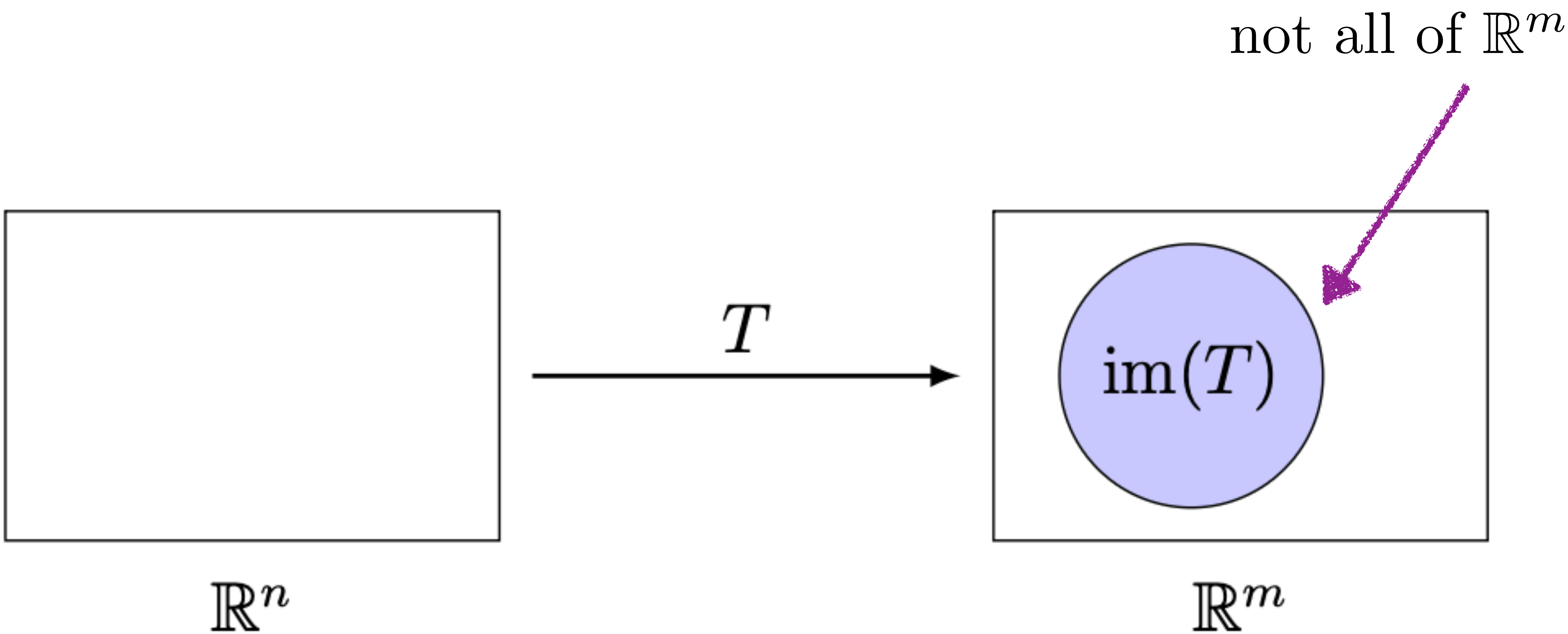
for every $b \in \mathbb{R}^m$ there exists *at least one* $x \in \mathbb{R}^n$

such that

$$T(x) = b$$

Equivalently: $\quad \operatorname{im}(T) = \mathbb{R}^m$

Informally: everything in the codomain is an actual output.



not surjective

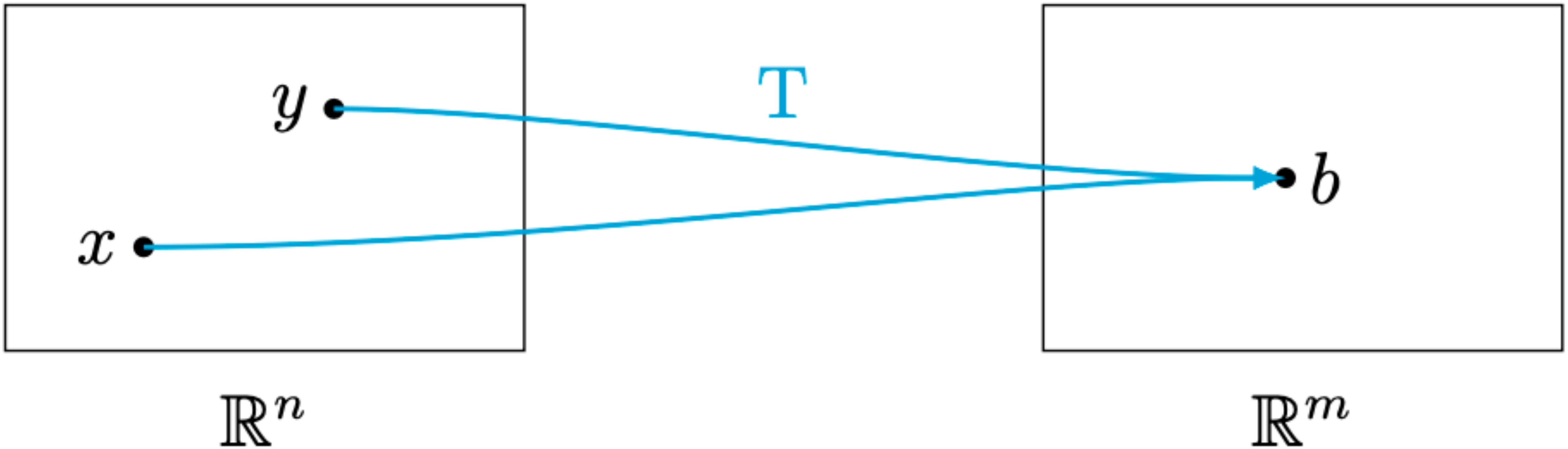
T is **injective** if

for each $b \in \mathbb{R}^m$ there exists *at most one* $x \in \mathbb{R}^n$

such that

$T(x) = b$

Equivalently: $T(x_1) = T(x_2) \implies x_1 = x_2.$



not injective

$$\text{bijective} = \text{injective} + \text{surjective}$$

Example: Consider the (nonlinear) function

$$T\colon \mathbb{R} \to \mathbb{R} \text{ given by } T(x) = x^2$$

Is it injective? Surjective? Bijective?

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Not surjective: $\quad$ the outputs are never negative!

Example: Consider the (nonlinear) function

$$T\colon \mathbb{R} \to \mathbb{R} \text{ given by } T(x) = x^2$$

Is it injective? Surjective? Bijective?

Not injective: $T(1) = 1 = T(-1)$

Not surjective: the outputs are never negative!

$$\operatorname{im}(T) = \{x \in \mathbb{R} \mid x \geqslant 0\}$$

For functions $f : \mathbb{R} \to \mathbb{R}$

f is **surjective**

if and only if

every horizontal line crosses the graph of f at least once.

f is **injective**

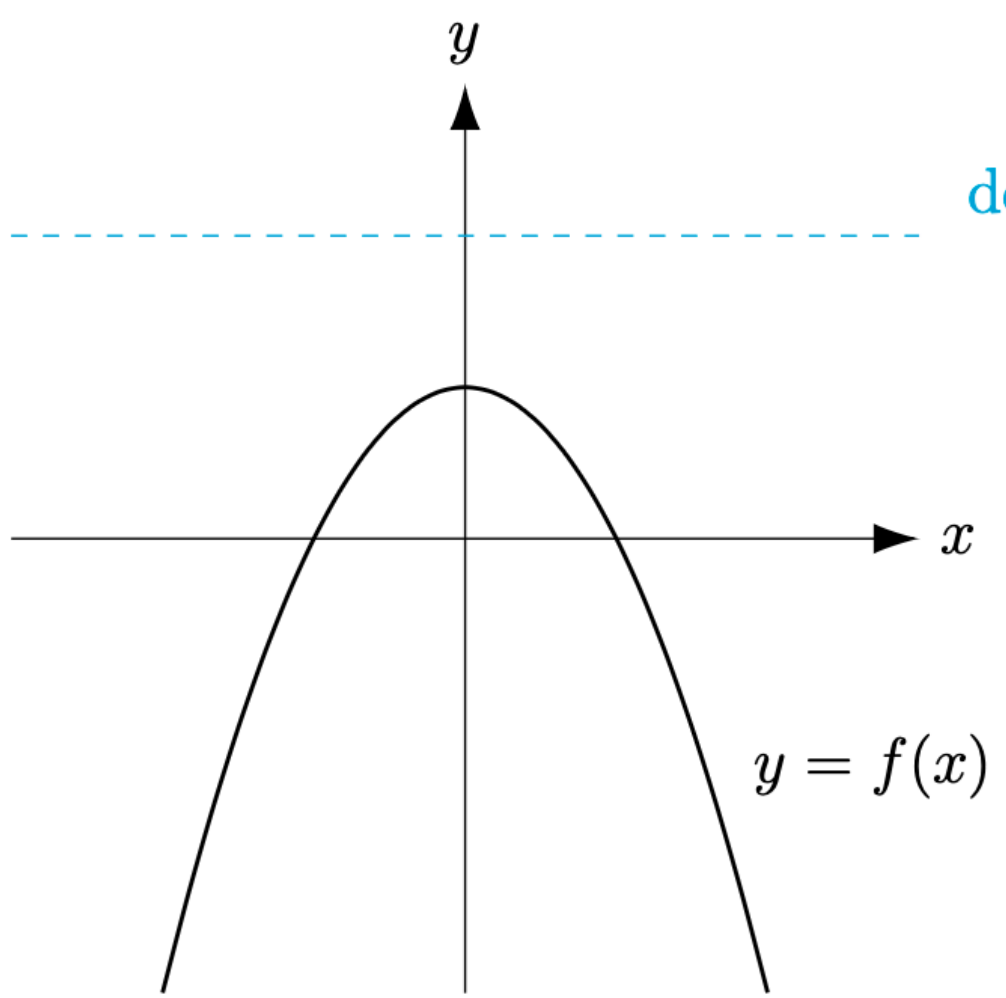
if and only if

every horizontal line crosses the graph of f at most once.

<u>Surjective:</u> f is surjective

if and only if

every horizontal line crosses the graph of f at least once.

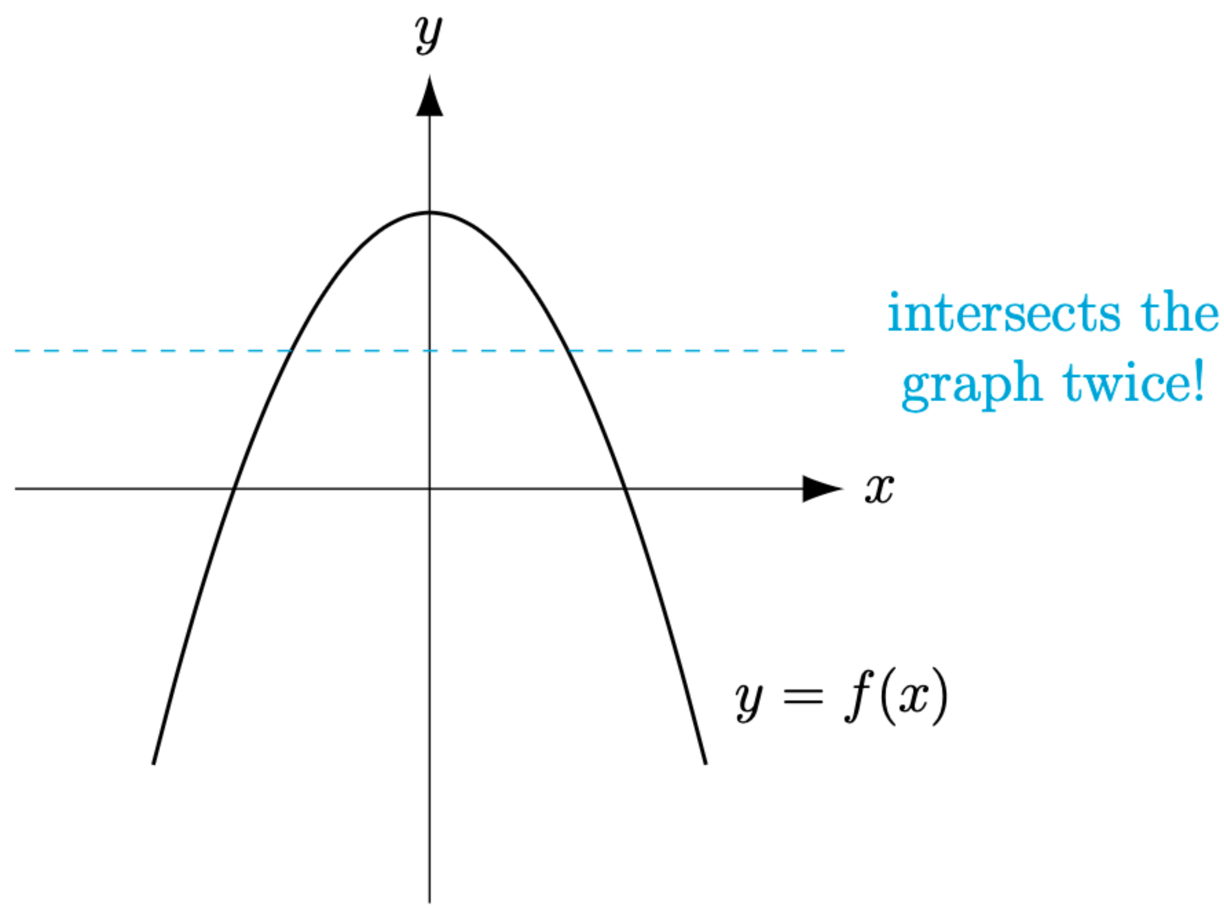


Example: f not surjective

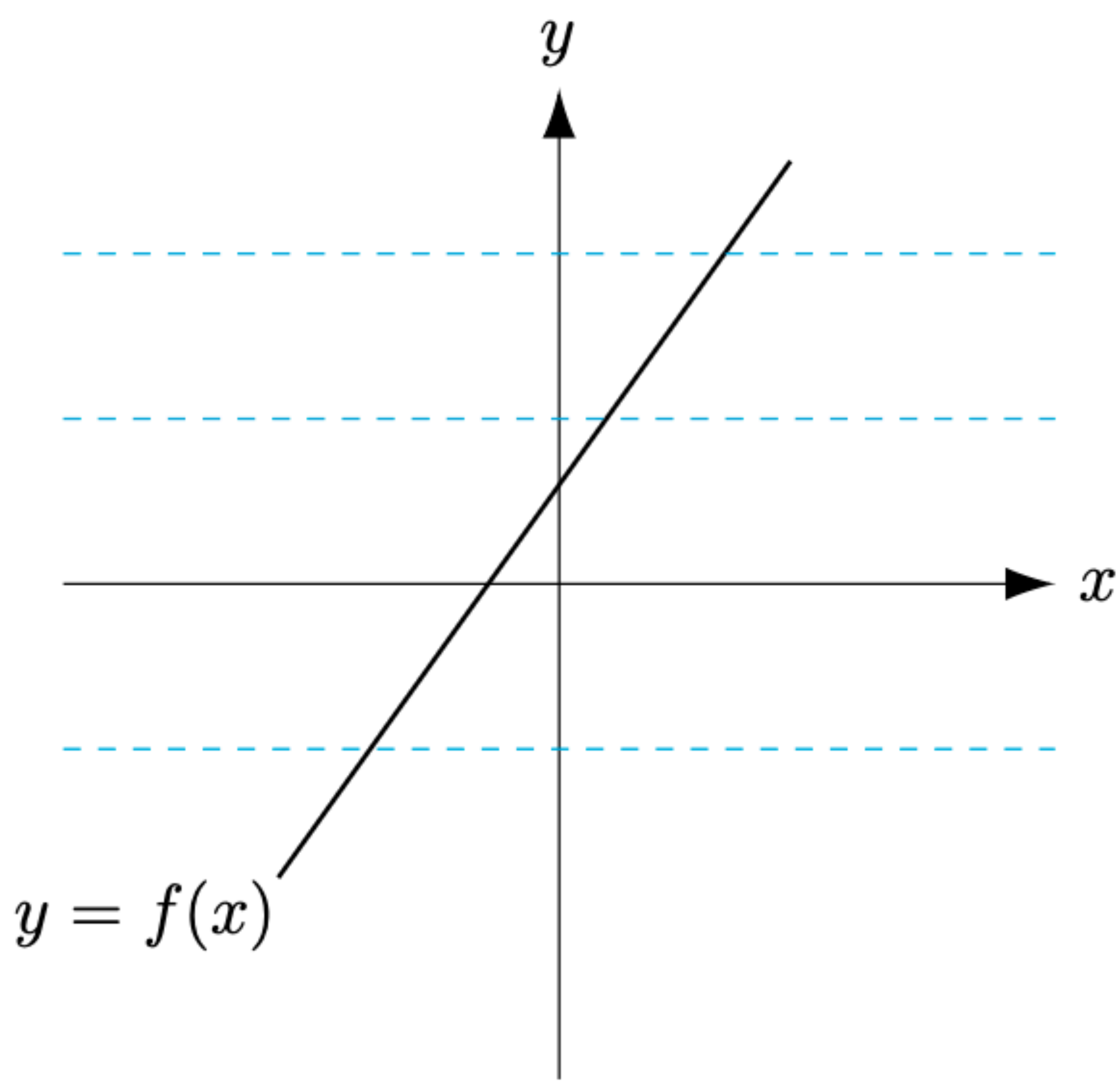
Injective: f is injective

if and only if

every horizontal line crosses the graph of f at most once.



Example: f not injective



Example: f bijective

The **kernel** of a linear transformation

$$T\colon \mathbb{R}^n \to \mathbb{R}^m$$

is the set

$$\ker(T) := \{x \in \mathbb{R}^n \mid T(x) = 0\}.$$

Note: we always have $0 \in \ker(T)$

Theorem. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

T is injective if and only if the equation

$$T(x) = 0$$

has only the trivial solution $x = 0$.

Equivalently, T is injective if and only if $\ker(T) = \{0\}$.

Theorem. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

T is surjective $\iff$ the columns of A span $\mathbb{R}^m$

T is injective $\iff$ the columns of A are linearly independent

Theorem. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

$$T \text{ is surjective} \iff A \text{ has a pivot in every row}$$

$$T \text{ is injective} \iff A \text{ has a pivot in every column}$$

Example: The identity map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$$

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$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$$

T is surjective: for any $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ we have $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$

T is injective: $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) \implies \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$

T is bijective

Example: The identity map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$$

has standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

T is surjective: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has a pivot in every row

T is injective: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has a pivot in every column

T is bijective

Example: The map $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix}$$

Example: The map $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix}$$

T is not surjective: for example, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \notin \operatorname{im}(T)$.

T is injective:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) \implies \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = \begin{bmatrix} u \\ v \\ u+v \end{bmatrix} \implies \begin{cases} x = u \\ y = v \end{cases}$$

Example: The map $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} \quad \text{has standard matrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

T is not surjective:

T is injective:

Example: The map $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

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T is not surjective: there is no pivot on the last row

T is injective:

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T is not surjective: there is no pivot on the last row

T is injective: pivot in every column

Example: The map $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$$ has standard matrix

Example: The map $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{has standard matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Example: The map $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{has standard matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

T is surjective:

T is not injective:

Example: The map $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{has standard matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

T is surjective: pivot in every row

T is not injective:

Example: The map $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{has standard matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

T is surjective: pivot in every row

T is not injective: there is no pivot on the third column

Example: The map $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{has standard matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

T is surjective: for any $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ we have $T\left(\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$

T is not injective: $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

Example:

The linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ with standard matrix

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

Example:

The linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ with standard matrix

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

no pivot on this row

$\Downarrow$

T is not surjective

no pivot on this column $\Longrightarrow$ T is not injective

Today’s poll code:

U6H5RJ

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U6H5RJ

The linear transformation with standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

A. Injective and surjective

B. Injective but not surjective

C. Not injective but surjective

D. Not injective and not surjective

Today's poll code:

U6H5RJ

The linear transformation with standard matrix $\begin{bmatrix} 1 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

A. Injective and surjective

B. Injective but not surjective

C. Not injective but surjective

D. Not injective and not surjective

Today's poll code:

U6H5RJ

The linear transformation with standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ is

A. Injective and surjective

B. Injective but not surjective

C. Not injective but surjective

D. Not injective and not surjective

Today's poll code:

U6H5RJ

The linear transformation with standard matrix $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ is

A. Injective and surjective

B. Injective but not surjective

C. Not injective but surjective

D. Not injective and not surjective

Chapter 3

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad (i,j)\text{th entry of } A = a_{ij}$$

multiplying a matrix by a scalar: $\qquad cA = [ca_{ij}]$

$$3 \cdot \begin{bmatrix} -1 & 1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -3 & 15 \end{bmatrix}$$

always defined

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad (i,j)\text{th entry of } A = a_{ij}$$

The sum $A + B$ is defined if

A and B have the same size $\left(\begin{array}{c} \#\text{ rows of } A = \#\text{ rows of } B \\ \#\text{ columns of } A = \#\text{ columns of } B \end{array} \right)$

sum of matrices: $A + B := [a_{ij} + b_{ij}]$

$$\begin{bmatrix} 3 & -1 \\ 2 & -11 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3+1 & -1+3 \\ 2+4 & -11+5 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 6 & -6 \end{bmatrix}$$

product of matrices: The product AB is defined if

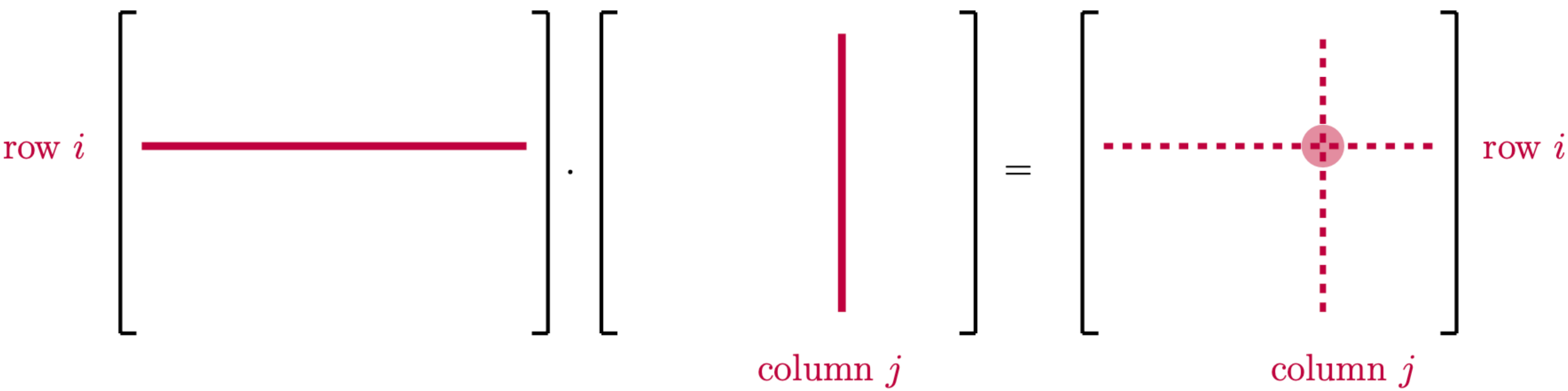
$$\#\text{ of columns of } A = \#\text{ of rows of } B$$

$$\#\text{ of rows of } AB = \#\text{ of rows of } A$$

$$\#\text{ of columns of } AB = \#\text{ of columns of } B$$

In AB:

$$(i,j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$



$A \ m \times n$ matrix

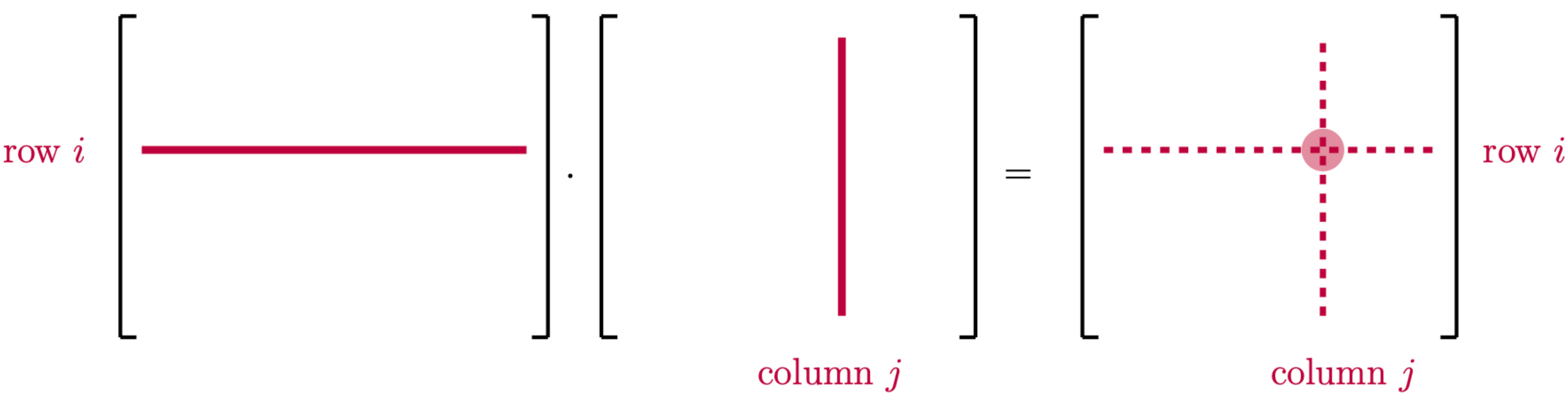
$B \ n \times p$ matrix

$\longrightarrow$ $AB \ m \times p$ matrix

Informally: $(m \times n)\ (n \times p) = m \times p$

In AB:

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$



A $m \times n$ matrix

B $n \times p$ matrix

$\longrightarrow$ AB $m \times p$ matrix

Informally: $(m \times \cancel{n)\,(n} \times p) = m \times p$

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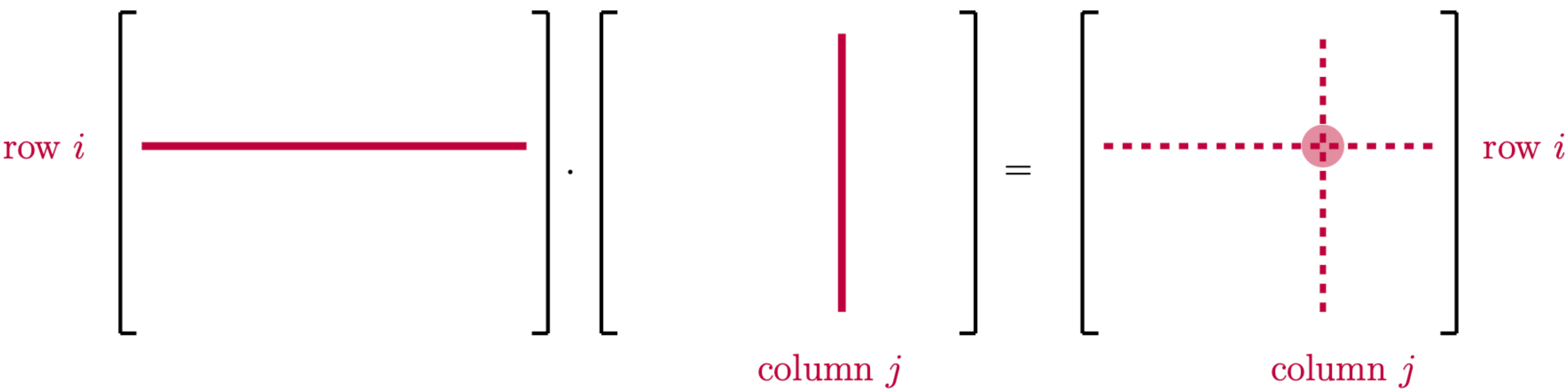

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

(i, j)th entry of $A = a_{ij}$

product of matrices: The product AB is defined if

$$\# \text{ of columns of } A = \# \text{ of rows of } B$$

$$AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ell} \end{bmatrix}$$

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

product of matrices: The product AB is defined if

$\#$ of columns of $A = \#$ of rows of B

$$(i,j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

ith row of A and jth column of B

eg: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ell} \end{bmatrix}$

$$(1,1)\text{th entry} = a_{11}b_{11} + a_{i2}b_{21} + \cdots + a_{1n}b_{n1}$$

product of matrices: The product AB is defined if

$\#$ of columns of $A = \#$ of rows of B

(i,j)th entry $= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

ith row of A and jth column of B

eg: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ell} \end{bmatrix}$

$(m,2)$th entry $= a_{m1}b_{12} + a_{m2}b_{22} + \cdots + a_{mn}b_{n2}$

Example:

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + 3 \times 1 & 2 \times 3 + 3 \times (-2) & 2 \times 6 + 3 \times 3 \\ 1 \times 4 - 5 \times 1 & 1 \times 3 + (-5) \times (-2) & 1 \times 6 + (-5) \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

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$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$

is undefined!

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$

3 columns $\neq$ 2 rows

Example:

$$\cancel{\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}}$$

is undefined!

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$

3 columns $\neq$ 2 rows

Theorem. Let A, B, C be matrices,

Assume that the products AB and BC make sense.

1. Associativity: $(AB)C = A(BC)$.
2. Left distributivity: $A(B + C) = AB + AC$.
3. Right distributivity: $(B + C)A = BA + CA$.
4. For any scalar α, $\alpha(AB) = (\alpha A)B = A(\alpha B)$.
5. Let I_m denote the $m \times m$ identity matrix.

   Then $I_m A = A = AI_n$.

Warning: the order of the matrices in a product matters!

For example:

$$A = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 \cdot 2 + 1 \cdot 4 & 5 \cdot 0 + 1 \cdot 3 \\ (-1) \cdot 2 + 3 \cdot 4 & (-1) \cdot 0 + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ 10 & 9 \end{bmatrix}$$

$$\neq$$

$$BA = \begin{bmatrix} 2 \cdot 5 + 0 \cdot (-1) & 2 \cdot 1 + 0 \cdot 3 \\ 4 \cdot 5 + 3 \cdot (-1) & 4 \cdot 1 + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 17 & 13 \end{bmatrix}$$

The zero $m \times n$ matrix is

the $m \times n$ matrix whose entries are all 0.

We denote it simply by 0, if the size is clear from context.

Warning! Cancellation fails.

$AB = AC$ does <u>not</u> imply that $B = C$.

For example:

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix} \neq 0 \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

$$AB = \begin{bmatrix} 1 \times 0 + 0 \times 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Warning! Cancellation fails.

$AB = AC$ does <u>not</u> imply that $B = C$.

For example:

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \neq \quad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 \times 2 + 1 \times 0 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 1 \times 1 \end{bmatrix} = AC$$

The zero $m \times n$ matrix is

the $m \times n$ matrix whose entries are all 0.

We denote it simply by 0, if the size is clear from context.

Warning! Cancellation fails.

$AB = 0$ does <u>not</u> imply that $A = 0$ or $B = 0$

For example:

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix} \neq 0 \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

$$AB = \begin{bmatrix} 1 \times 0 + 0 \times 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

To do list:

- Webwork 2.5 due tomorrow
- WebworK 2.6 due Friday

On Friday:

Quiz 4

at the beginning

of the recitation

on Lectures 8—9

Midterm 1

in 2 weeks

Office hours

Mondays 5–6 pm and Wednesdays 2–3 pm
in Avery 339 (Dr Grifo)

Tuesdays 11–noon and Thursdays 1–2 pm
in Avery 337 (Kara)