Symbolic powers and the (stable) containment problem
Tulane Algebra and Combinatorics Seminar (12/02/2020)

<u>Setup</u> $R$ regular ring (eg, $R = k[x,y,z]$, $k$ field)

$$I = \underbrace{P_1 \cap \ldots \cap P_k}_{\text{primes}} = \sqrt{I}$$

$$h = \max\{\operatorname{ht} P_i : 1 \leq i \leq k\}$$

the $n$-th symbolic power of a prime $P$ is given by

$$P^{(n)} = \{f \in R \mid sf \in P^n,\ s \notin P\}$$

the $n$-th symbolic power of $I$ is given by

$$I^{(n)} = P_1^{(n)} \cap \ldots \cap P_k^{(n)} \qquad \text{Note: } I^n = (f_1 \cdots f_n \mid f_i \in I)$$

$$= \{f \in R \mid sf \in I^n,\ s \notin P_i \text{ for } 1 \leq i \leq k\}$$

$=$ minimal components in a primary decomposition of $I^n$

<u>theorem</u> (Zariski–Nagata) $R = \mathbb{C}[x_1, \ldots, x_d]$

$I^{(n)} = \{f \in R \mid f$ vanishes to order $n$ along the corresponding variety$\}$

$$= \left\{f \in R \;\middle|\; \frac{\partial^{a_1 + \cdots + a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}}(f) \in I \text{ for all } a_1 + \cdots + a_d < n\right\}$$

Properties: 0) $I^{(1)} = I$

1) $I^{(n+1)} \subseteq I^{(n)}$

2) $I^n \subseteq I^{(n)}$

3) If $I = (\text{reg seq})$, then $I^n = I^{(n)}$.

In general, $I^{(n)} \neq I^n$.

Example $I = (xy, xz, yz) = (x,y) \cap (y,z) \cap (x,z) \subseteq k[x,y,z]$

$$I^{(2)} = (x,y)^2 \cap (x,z)^2 \cap (y,z)^2 \ni xyz$$

But $xyz \notin I^2$ (because elements in $I^2$ have degree $\geq 4$)

$$\therefore \quad I^{(3)} \subseteq I^2 \subsetneq I^{(2)}$$

Tomorrow we can have $I^{(n)} \neq I^n$ even when $I$ is prime, eg

$$P = P(3,4,5) = \ker\left(k[x,y,z] \longrightarrow k[t^3, t^4, t^5]\right)$$

$P^n \neq P^{(n)}$ for all $n$

In fact, the symbolic powers of the primes ideals $P(a,b,c)$ are very interesting, and exhibit various kinds of strange behavior, eg

$\bigoplus_{n \geq 0} P^{(n)}$ (an algebra over $R$) is not always fg

**Containment Problem** When is $I^{(a)} \subseteq I^b$?

**Theorem** (Ein–Lazarsfeld–Smith 2001, Hochster–Huneke 2002, Ma–Schwede 2018)

$$I^{(hn)} \subseteq I^n \text{ for all } n$$

In particular, $I^{((\dim R)n)} \subseteq I^n$ for all $n$

**Example** $I = (xy, xz, yz) \to h=2 \to I^{(2n)} \subseteq I^n \ \forall n \geqslant 1$

In particular, $I^{(4)} \subseteq I^2$. But in fact, $I^{(3)} \subseteq I^2$.

**Example** $P = P(3,4,5) \to h=2 \to P^{(2n)} \subseteq P^n \ \forall n \geqslant 1$

But also $P^{(3)} \subseteq P^2$.

**Question** (Huneke, 2000) $P$ prime of codim 2 in a RLR. Is $P^{(3)} \subseteq P^2$?

**Conjecture** (Harbourne, 2008) $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$.

**Fact** In char $p$, $I^{(hq-h+1)} \subseteq \underset{(f^q \mid f \in I)}{I^{[q]}} \subseteq I^q$ for all $q = p^e$

**Counterexample** (Dumnicki–Szemberg–Tutaj-Gasińska, 2013)
(Harbourne–Seceleanu, 2015)

$\exists$ family of radical ideals $I \subseteq k[x,y,z]$, char $k \neq 2$, $h=2$

$$I^{(2n-1)} \nsubseteq I^n \quad \text{for } n=2$$

Harbourne's Conjecture holds for:

- General points in $\mathbb{P}^2$ (Bocci–Harbourne) and $\mathbb{P}^3$ (Dumnicki)
- $I$ squarefree monomial ideal
  $\cap$
- If $R/I$ is F-pure in char $p$ (G–Huneke)

eg, determinantal rings, Veronese rings

But there are <u>no</u> counterexamples to the following:

<u>Stable Harbourne Conjecture</u> $\quad I^{(hn-h+1)} \subseteq I^n$ for $n \gg 0$

<u>Question</u> If $I^{(hn-h+1)} \subseteq I^n$ for some $n \overset{?}{\Longrightarrow}$ all $n \gg 0$ ?

<u>Note</u> If yes, then we are done in prime characteristic!

the answer is yes if $I$ satisfies the stronger property

$$I^{(n+h)} \subseteq I\, I^{(n)} \qquad \forall n \geq 1 \text{ or for } n \gg 0$$

- this holds if $R/I$ is F-pure, for example
- False in general (Secelanu)

**Theorem** (G, G-Schwede in mixed char)

If $I^{(hk-h)} \subseteq I^k$ for some $k$, then $I^{(hn-h)} \subseteq I^n$ for all $n \gg 0$

**Example** $I = \bigcap_{i \neq j} (x_i, x_j) \subseteq k[x_1, \ldots, x_d]$

has $I^{(2n-2)} \not\subseteq I^n$ for all $n < v$

But also, $I^{(2v-2)} \subseteq I^v \implies I^{(2n-2)} \subseteq I^n \ \forall n \gg 0$

In fact, there are no counterexamples to $I^{(hn-c)} \subseteq I^n$ any fixed $c$, $n \gg 0$

**Def** (Bocci-Harbourne) the Resurgence of $I$ is given by

$$\rho(I) = \sup\left\{ \frac{a}{b} : I^{(a)} \not\subseteq I^b \right\}$$

$$1 \leq \rho(I) \leq h$$

$\uparrow$

ELS-HH-MS

If $\rho(I) < h$, then Harbourne stable holds!

$$\frac{hn-c}{n} > \rho(I) \implies I^{(hn-c)} \subseteq I^n$$

$$\Updownarrow$$

$$n > \frac{c}{h-\rho(I)}$$

**Open Question** If $h \geq 2$, Can $\rho(I) = h$?

If $\rho(I) < h$, we say $I$ has expected resurgence.

there are no known examples of ideals of unexpected resurgence

In fact, all known counterexamples to the original conjecture have expected resurgence.

Even if $I$ has unexpected resurgence, that does not imply $I$ fails stable Harbourne

**Goal** To give sufficient conditions for expected resurgence.

**thm** (G-Huneke-Mukundan)

$(R, m)$ RLR, containing a field or $R = k[x,y,z]$, $I$ homogeneous

Assume $I^{(n)} = I^n : m^\infty = \bigcup_{k \geq 1} I^n : m^k$ for all $n \geq 1$

(eg, $I$ prime of height $\dim R - 1$)

If $I^{(hn-h+1)} \subseteq m I^n$ for some $n$, then $\rho(I) < h$.

**Applications**

- $I$ homogeneous, generated in deg $a < h$, char $k = 0$
- $P = P(a,b,c)$ space monomial curve.

$P^{(3)} \subseteq m P^2 \implies \rho(P) < 2$ (For some subclass, $\rho(I) \leq \frac{4}{3}$)