

# Linear Algebra

Math 314 Fall 2025

Today's poll code:

GDE2LU

Lecture 15

To do list:

## Office hours

Mondays 5–6 pm  
Wednesdays 2–3 pm  
in Avery 339 (Dr. Grifo)

Tuesdays 11–noon  
Thursdays 1–2 pm  
in Avery 337 (Kara)

- Lab 1 due on Friday
- Webwork 4.2 due next Wednesday
- Webwork 4.3 due Friday October 24

Quiz on Friday

on vector spaces

**Lab 1 due Friday!**

**You will turn in a pdf on gradescope  
Your pdf will include the code, your comments, and the output**

**For example, some of the outputs will be pictures!**

# **Recap:**

# **Bases and dimension**

A **basis**<sup>\*</sup> for a vector space  $V$  is  
a spanning set of linearly independent vectors.

The **dimension** of a vector space is  
the number of vectors in a basis.

Notation:  $\dim(V)$

\* The plural of basis is bases.

$$\dim(\mathbb{R}^n) = n \quad \text{Basis: } e_1, \dots, e_n$$

$$\dim(M_{m \times n}) = mn \quad \dim(\{0\}) = 0$$

all  $m \times n$  matrices

$$\dim(\mathbb{P}_n) = n + 1$$

polynomials  $a_0 + a_1t + \dots + a_nt^n$

Think of  $\mathbb{P}_n$  as  $\mathbb{R}^{n+1}$

$$a_0 + a_1t + \dots + a_nt^n$$



$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

**Column space**

**Null space**

$A m \times n$  matrix

The **column space** of  $A$  is

the span of the columns of  $A$

$$A = [a_1 \cdots a_n] \xrightarrow{\text{purple arrow}} \text{col}(A) = \text{span}(\{a_1, \dots, a_n\})$$

$\text{col}(A)$  is a subspace of  $\mathbb{R}^m$

Note:  $\text{col}(A) = \text{im}(T)$ , where  $T$  is the linear transformation  $T(x) = Ax$

The image of a linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$   
is a subspace of  $\mathbb{R}^m$

(and we also call it the column space of the standard matrix for  $T$ )

$A$   $m \times n$  matrix

$\text{col}(A) =$  the span of the columns of  $A$

Example:  $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\text{col}(A) = \text{span} \left( \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \right)$$

$A$   $m \times n$  matrix

The **null space** of  $A$  is

$$\text{Nul}(A) = \{x \text{ in } \mathbb{R}^n : Ax = 0\}$$

$\text{Nul}(A)$  is a subspace of  $\mathbb{R}^n$

The linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  given by

$$T(x) = Ax$$

has  $\text{Nul}(A) = \ker(T)$

The kernel of a linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

is a subspace of  $\mathbb{R}^n$

(and we also call it the null space of the standard matrix for  $T$ )

## **Theorem.**

The set of solutions to a homogeneous linear system

is a vector space.

Why? The solution set of  $Ax = 0$  is the same as the null space of  $A$ .

Example:

$$A = \begin{bmatrix} 1 & 4 & -5 & 2 \\ 0 & 2 & -4 & 0 \\ -1 & 1 & -5 & 2 \\ 3 & -1 & 11 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

general solution to  $Ax = 0$  is

$$x = t \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{where } t \text{ is any scalar}$$

so  $\text{Nul}(A) = \text{span} \left( \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\} \right).$

Example:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

general solution to  $Ax = 0$  is

$$x = a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \text{ } a, b, c \text{ any scalars}$$

$$\text{so } \text{Nul}(A) = \text{span} \left( \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$$

In general:

$$u \text{ in } \text{Nul}(A) \quad \text{means} \quad Aw = 0$$

Equivalently,  $w$  is a solution to the system  $Ax = 0$ .

$$u \text{ in } \text{col}(A) \quad \text{means} \quad u \text{ is a linear combination of the columns of } A$$

Equivalently, the system  $Ax = u$  has a solution.

Example:

$$A = \begin{bmatrix} 1 & 4 & -5 & 2 \\ 0 & 2 & -4 & 0 \\ -1 & 1 & -5 & 2 \\ 3 & -1 & 11 & 1 \end{bmatrix}$$

$$w = \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \text{ is in } \text{Nul}(A) \quad \text{means} \quad Aw = 0$$

Equivalently,  $w$  is a solution to the system  $Ax = 0$ .

$$u = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 2 \end{bmatrix} \text{ is in } \text{col}(A) \quad \text{means} \quad u \text{ is a linear combination of the columns of } A$$

Equivalently, the system  $Ax = u$  has a solution.

**Finding a basis for  
the column space  
and the null space**

To find a basis for the column space of  $A$ :

Step 1: Find the RREF of  $A$ .

Step 2: Collect the pivot columns of  $A$ .

**Warning:** Make sure to use the pivot columns of  $A$ ,  
not of its reduced echelon form!

$$\dim(\text{col}(A)) = \# \text{ of pivots in } A$$

$$A = \begin{bmatrix} 1 & 4 & -5 & 2 \\ 0 & 2 & -4 & 0 \\ -1 & 1 & -5 & 2 \\ 3 & -1 & 11 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 & -5 & 2 \\ 0 & 2 & -4 & 0 \\ -1 & 1 & -5 & 2 \\ 3 & -1 & 11 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dim(\text{col}(A)) = 3$$

$$A = \begin{bmatrix} 1 & 4 & -5 & 2 \\ 0 & 2 & -4 & 0 \\ -1 & 1 & -5 & 2 \\ 3 & -1 & 11 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dim(\text{col}(A)) = 3$$

$$A = \left[ \begin{array}{cccc} 1 & 4 & -5 & 2 \\ 0 & 2 & -4 & 0 \\ -1 & 1 & -5 & 2 \\ 3 & -1 & 11 & 1 \end{array} \right] \sim \left[ \begin{array}{cccc} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\dim(\text{col}(A)) = 3$$

Basis for  $\text{col}(A)$ :

$$\left\{ \left[ \begin{array}{c} 1 \\ 0 \\ -1 \\ 3 \end{array} \right], \left[ \begin{array}{c} 4 \\ 2 \\ 1 \\ -1 \end{array} \right], \left[ \begin{array}{c} 2 \\ 0 \\ 2 \\ 1 \end{array} \right] \right\}$$

To find a basis for the null space of  $A$ :

Step 1: Find the general solution for  $Ax = 0$ .

Step 2: Write the solution in parametric vector form.

Use one vector for each free variable.

Step 2: The vectors we used form a basis for  $\text{Nul}(A)$ .

$$\dim(\text{Nul}(A)) = \# \text{ free variables}$$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution to the homogeneous system  $Ax = 0$  is

$$x = a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$a, b, c$   
any real numbers

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution to the homogeneous system  $Ax = 0$  is

$$x = a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad \begin{array}{l} a, b, c \\ \text{any real numbers} \end{array}$$

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{is a basis for } \text{Nul}(A)$$

$$\dim(\text{col}(A)) = \# \text{ of pivots in } A = \text{rank}(A)$$

$$\dim(\text{Nul}(A)) = \# \text{ free variables}$$

$$\dim(\text{col}(A)) = \# \text{ of pivots in } A = \text{rank}(A)$$

$$\dim(\text{Nul}(A)) = \# \text{ free variables} = \text{nullity of } A$$

**Theorem** (Rank–Nullity Theorem).

For any  $m \times n$  matrix  $A$ ,

$$\text{rank}(A) + \dim(\text{Nul}(A)) = n.$$

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The column space of

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

is a subspace of

- A.  $\mathbb{R}^2$
- B.  $\mathbb{R}^3$
- C.  $\mathbb{R}^4$
- D.  $M_{2 \times 4}$

Today's poll code:

GDE2LU

The null space of

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

is a subspace of

- A.  $\mathbb{R}^2$
- B.  $\mathbb{R}^3$
- C.  $\mathbb{R}^4$
- D.  $M_{2 \times 4}$

Today's poll code:

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The column space of

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

has dimension

Today's poll code:

GDE2LU

The null space of

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

has dimension

Today's poll code:

GDE2LU

The column space of

$$M = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 11 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

has dimension

Today's poll code:

GDE2LU

The null space of

$$M = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 11 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

has dimension

Given polynomials in  $\mathbb{P}_n$ ,

we can ask questions about them by

identifying them with vectors in  $\mathbb{R}^{n+1}$

$$a_0 + a_1 t + \cdots + a_n t^n$$



$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

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