Test modules for the complete intersection property
(joint with Ben Briggs and Josh Pollitz)

$(R, \mathfrak{n}, k)$ Noetherian local ring

$Q/I \cong \hat{R}$ $\quad$ $(Q, \mathfrak{m}, k)$ RLR, $I \subseteq \mathfrak{n}^2$

<u>theorem</u> (Auslander–Buchsbaum, 1957, Serre, 1956) TFAE:

- R is regular
- Every fg R-module has $\operatorname{pdim}_R(M) < \infty$
- $\operatorname{pdim}_R(k) < \infty$

In the world of complexes:

$\mathcal{D}(R) :=$ complexes of R-modules up to quasi-isomorphism

$\cup\!\!\mid$

$\mathcal{D}^f(R) :=$ complexes with fg homology

(Morally, think of this as a parallel to $\operatorname{Mod}(R) \supseteq \operatorname{mod}(R)$)

M R-module $\iff$ complex $0 \to \overset{0}{M} \to 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cong$

free resolution of M $\quad \cdots \to P_1 \to P_0 \to 0$

$\operatorname{pd}_R(M) < \infty \iff M \cong$ bounded complex of fg projectives

$X \in \mathcal{D}(R)$ is <u>small</u> if $X \cong$ bounded complex of fg projectives

Theorem (Auslander–Buchsbaum, 1957, Serre, 1956) TFAE:

- R is regular
- Every fg R-module has $\operatorname{pdim}_R(M) < \infty$
- $\operatorname{pdim}_R(k) < \infty$
- Every $X \in \mathcal{D}^f(R)$ is small

Def (Dwyer–Greenlees–Iyengar, 2005)

$X \in \mathcal{D}^f(R)$ is <u>proxy small</u> if

- We can finitely build a small P from X, by

  P is in $\operatorname{thick}(X)$:
  - shifting complexes
  - taking direct summands
  - if we can build two of $0 \to A \to B \to C \to 0$ then we can build the third

- $\operatorname{Supp} P = \operatorname{Supp} X$

Theorem (Pollitz, 2018)

R is a complete intersection $\iff$ Every $X \in \mathcal{D}^f(R)$ is proxy small.

Remark Dwyer–Greenlees–Iyengar showed $\Rightarrow$

<u>Consequence</u> R ci $\Rightarrow$ every fg R-module is proxy small.

<u>Question</u> How about the converse?

<u>Goal</u> If R is not ci, construct non-proxy small fg R-mod M

<u>Remark</u> k is <u>always</u> proxy small
(k builds the koszul complex, which is perfect)

<u>Sidenote</u> If we can do this, we will answer a question of
Ghosh – Jorgensen – Takahashi
about whether

R is ci $\Leftrightarrow$ all fg R-modules have finite quasiprojective dimension

<u>theorem</u> (Briggs – G – Pollitz)
If R is

- equipresented (every $f \in I \setminus mI$ has the same m-adic order)
- Stanley–Reisner (so $R = k[x_1, \ldots, x_d]/$ squarefree mon ideal)

R is ci $\Leftrightarrow$ every fg R-module is proxy small

$\Leftrightarrow$ every $R \twoheadrightarrow$ artinian ci is proxy small
($|k| = \infty$)

key technical tool $\qquad$ cohomological support

$\hat{R} \cong Q/I$, $\mu(I) = n$

M R-module $\rightsquigarrow$ $V_R(M) \subseteq k^n$

$$V_R := I/\mathfrak{m}I \cong k^n$$

Def: $V_R(M) := \{ [f] \in V_R \mid \operatorname{pd}_{Q/f} \hat{M} = \infty \text{ or } [f] = 0 \}$

Note this is intrinsic to M, and does not depend on our choices

(Avramov, Avramov-Buchweitz, Burke-Walker, Jorgensen, Pollitz)

Facts (Pollitz):

- $V_R(R) = 0 \iff R$ ci
- M proxy small fg R-mod $\implies V_R(R) \subseteq V_R(M)$

Goal Construct fg R-mod M with $V_R(M) \not\supseteq V_R(R)$

How? Construct fg R-modules $M_1, \ldots, M_t$ st:

1. Can compute $V_R(M_1), \ldots, V_R(M_t)$
2. $V_R(M_1) \cap \cdots \cap V_R(M_t) = \emptyset$

Basic idea: take $f \in I \setminus \mathfrak{m}I$ and construct a ci $J \supseteq I$ with $f \in J \setminus \mathfrak{m}J$. Then

$V_R(Q/J) \subseteq$ hyperplane determined by f

Need: sufficiently many different directions

But we can only construct these cis when f has minimal order

When R is equipresented, can choose whatever directions we want!

the actual condition in our theorem is:

if $\sigma := \min\{\, \mathfrak{m}\text{-adic order } f \mid f \in I, f \neq 0 \,\}$

$$\underbrace{\dim_k\left(\frac{\mathfrak{m}^{\sigma+1} \cap I}{\mathfrak{m} I}\right)}_{\#\text{ min gens of order} > \sigma} < \underbrace{\dim_k\left(\operatorname{span} V_R(R)\right)}_{\leq n = \mu(I)}$$

<u>Example</u> $\quad R = \dfrac{k[\![x,y]\!]}{(x^2, xy)} \qquad Q = k[\![x,y]\!]$

$$M = Q/(x^2, y) \qquad N = Q/(xy, x+y)$$

one of these is not proxy small