Linear Algebra

Math 314 Fall 2025

Lecture 10

Today's poll code:

PD5XSA

To do list:

- Webwork 3.2 due Friday
- Study for the midterm!!

Office hours this week

Monday 5–6 pm and Wednesday 2–3 pm
Thursday 2—3 pm
Friday 2:30 to 3:30 pm
in Avery 339 (Dr. Grifo)

Tuesday 11–noon and Thursday 1–2 pm
in Avery 337 (Kara)

Midterm 1

on Monday

Quiz on Friday

on inverses

Vector spaces

TL;DR: can add vectors and multiply by scalars, with good properties.

A **vector space** is a nonempty set V, whose elements we call **vectors**, with rules for **addition** of vectors in V and **multiplication by scalars**, satisfying the following properties:

a) The addition $u + v$ of any vectors u and v in V is also a vector in V.
b) The multiplication cv of a vector v by a scalar c is a vector in V.
c) Commutativity: $u + v = v + u$ for all $u, v \in V$.
d) Associativity: $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$.
e) There is a **zero vector** in V, denoted 0, such that $0 + v = v + 0 = v$.
f) For every vector v there is a vector $-v$ such that $v + (-v) = 0$.
g) Distributivity: $c(u + v) = cu + cv$ and $(c + d)v = cv + dv$ for all $u, v \in V$ and all scalars c and d.
h) Associativity of multiplication by scalars: $c(dv) = (cd)v$.
i) $1v = v$.

Consequence: we also have

- $0v = \mathbf{0}$.
- $c\mathbf{0} = \mathbf{0}$.
- $-v = (-1)v$.

Vector space: can add vectors and multiply by scalars, with good properties.

Example: $\mathbb{R}^n$ with

addition:

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

multiplication by scalars

$$c \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} c\,u_1 \\ \vdots \\ c\,u_n \end{bmatrix}$$

Vector space: can add vectors and multiply by scalars, with good properties.

$$\mathrm{M}_{m\times n} = \{\text{all } m \times n \text{ matrices}\}$$

addition: $\qquad cA = [ca_{ij}]$

multiplication
by scalars $\qquad A + B := [a_{ij} + b_{ij}]$

Vector space: can add vectors and multiply by scalars, with good properties.

$\mathbb{S}$ = doubly infinite sequences of real numbers

$$\{y_n\} = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots)$$

addition:

$$\{y_n\} + \{z_n\} = \{y_n + z_n\} = (\ldots, y_{-2} + z_{-2}, y_{-1} + z_{-1}, y_0 + z_0, y_1 + z_1, y_2 + z_2, \ldots)$$

multiplication
by scalars

$$c\{y_n\} = \{cy_n\} = (\ldots, cy_{-2}, cy_{-1}, cy_0, cy_1, cy_2, \ldots)$$

appears in engineering applications in situations where a signal is measured in discrete time

Vector space: can add vectors and multiply by scalars, with good properties.

$\mathbb{P}_n$ = polynomials of degree at most n

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

t a variable $\qquad$ $a_0, \ldots, a_n$ real numbers

addition:

$$(a_0+a_1t+\cdots+a_nt^n)+(b_0+b_1t+\cdots+b_nt^n) = (a_0+b_0)+(a_1+b_1)t+\cdots+(a_n+b_n)t^n$$

multiplication by scalars

$$c(a_0 + a_1 t + \cdots + a_n t^n) = ca_0 + ca_1 t + \cdots + ca_n t^n$$

V vector space $\qquad$ W subset of V

W is a subspace of V if

1) The zero vector 0 is in W

2) W is <u>closed under addition</u>:

if u and v are in W, then $u + v$ is in W

3) W is <u>closed under multiplication by scalars</u>:

if v is in W, and c is any scalar, then cv is in W

A subspace is also a vector space!

Example:

V any vector space

there are always two **trivial subspaces**:

The zero subspace: $\underbrace{\{\mathbf{0}\}}_{\text{zero vector}}$

The subspace V

Example:

$V = \mathbb{R}^2$

$$W = \text{all multiples of } \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

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$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ in } W \quad \checkmark$$

2) W is closed under addition:

$$v = \begin{bmatrix} c \\ 0 \end{bmatrix} \quad u = \begin{bmatrix} d \\ 0 \end{bmatrix} \quad v + u = \begin{bmatrix} c \\ 0 \end{bmatrix} + \begin{bmatrix} d \\ 0 \end{bmatrix} = \begin{bmatrix} c + d \\ 0 \end{bmatrix} \text{ in } W$$

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3) W is closed under multiplication by scalars:

$$v = \begin{bmatrix} c \\ 0 \end{bmatrix} \qquad d \text{ any scalar} \qquad dv = d \begin{bmatrix} c \\ 0 \end{bmatrix} = \begin{bmatrix} dc \\ 0 \end{bmatrix} \text{ in } W$$

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Example:

$$V = \mathbb{R}^2$$

v any vector

W = all multiples of v

is a subspace of $\mathbb{R}^2$

The vector space $\mathbb{R}^2$ is *not* a subspace of $\mathbb{R}^3$

just because $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$

but

$$W = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \text{ any scalars} \right\}$$

is indeed a subspace of $\mathbb{R}^3$

and it looks a lot like $\mathbb{R}^2$

$$W = \{(x, y) \in \mathbb{R}^2 \mid x, y \geqslant 0\}$$

Is W a subspace of $\mathbb{R}^2$?

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1) The zero vector 0 is in W yes, take $x = y = 0 \geqslant 0$

$$(0, 0) \in W$$

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2) W is closed under addition:

$v = (x, y)$ in W $\qquad v + w = (x + a, y + b)$ in W

$w = (a, b)$ in W

3) W is closed under multiplication by scalars:

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3) W is closed under multiplication by scalars:

$v = (x, y)$ in W $\quad x, y \geqslant 0$ $\quad$ Is $cv = (cx, cy)$ in W?

c any real number $\quad$ Is $cx \geqslant 0, cy \geqslant 0$?

$$W = \{(x, y) \in \mathbb{R}^2 \mid x, y \geqslant 0\}$$

Is W a subspace of $\mathbb{R}^2$?

1) The zero vector 0 is in W yes, take $x = y = 0 \geqslant 0$

$$(0, 0) \in W$$

✓

2) W is closed under addition:

$v = (x, y)$ in W $v + w = (x + a, y + b)$ in W

$w = (a, b)$ in W $x, y, a, b \geqslant 0 \implies x + a, y + b \geqslant 0$

✓

3) W is closed under multiplication by scalars:

$v = (x, y)$ in W $x, y \geqslant 0$ Is $cv = (cx, cy)$ in W?

c any real number Is $cx \geqslant 0, cy \geqslant 0$?

✗

$$\{(x, y) \in \mathbb{R}^2 \mid x, y \geqslant 0\}$$

is <u>not</u> a subspace of $\mathbb{R}^2$

because it is not closed for multiplication by scalars

eg, $(1, 0)$ is in W

but $(-1, 0) = -1 \cdot (1, 0)$ is not in W

Today’s poll code:

PD5XSA

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PD5XSA

The set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^3$

A. True

B. False

Today's poll code:

PD5XSA

$$\text{The set } \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ is a subspace of } \mathbb{R}^3$$

A. True

B. False

Today's poll code:

PD5XSA

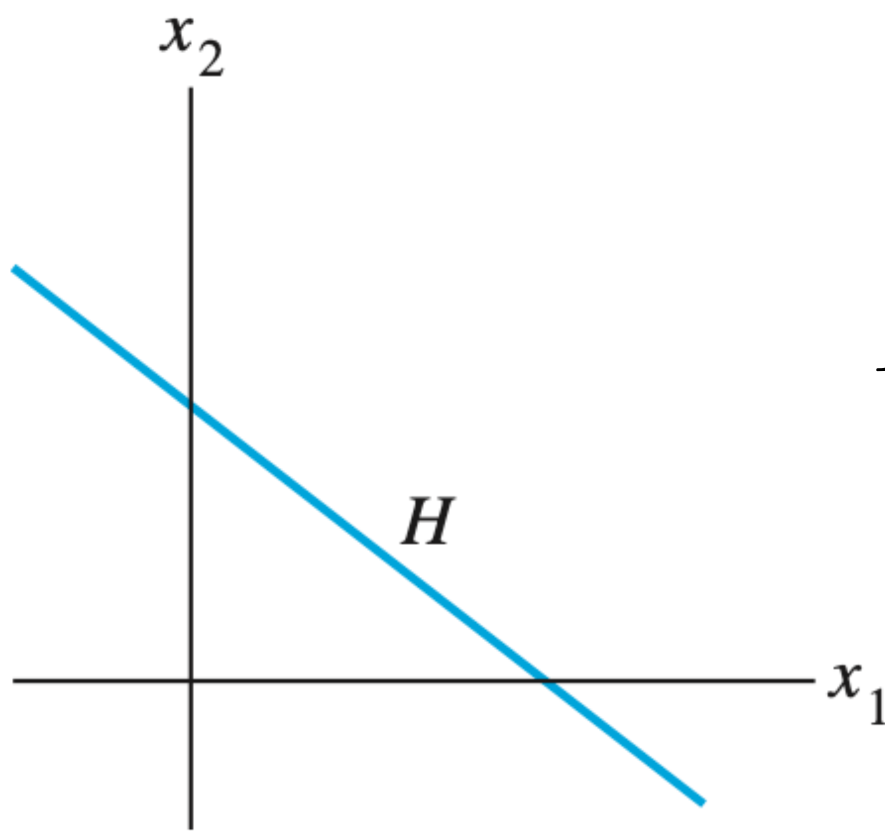
$$\text{The set } \left\{ \begin{bmatrix} a \\ a \\ 0 \end{bmatrix} : a \text{ any scalar} \right\} \text{ is a subspace of } \mathbb{R}^3$$

A. True

B. False

Today's poll code:

PD5XSA



H is a subspace of $\mathbb{R}^2$

A. True

B. False

Today's poll code:

PD5XSA

$$H = \left\{ t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : t \text{ any scalar} \right\}$$

is a subspace of $\mathbb{R}^3$

A. True

B. False

Today's poll code:

PD5XSA

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \text{ any scalars} \right\}$$

is a subspace of $\mathrm{M}_{2\times 2}$

A. True

B. False

Midterm 1 a week from today!

Midterm 1 on Monday

Material for Midterm 1: Chapters 1, 2, 3

Study materials:

- Class notes (see canvas)
- Textbook
- Slides from lecture
- Poll questions
- Quizzes
- Webwork questions
- Study guide (quick summary)
- Practice problems (to be discussed on Friday!)

Midterm 1

on Monday in lecture

On the day of the midterm:

- Arrive a few minutes early if you can
- Know your NUID!
- Write your name and NUID on the cover page
- Leave one empty seat between you and the student next to you
- No calculators or notes allowed
- Write only on the front side of each page
- There are extra pages at the end of the midterm you can use
- Scratch paper will be provided for you (you can't use your own)
- Only material you need to bring: writing utensils

Name:

NUID:

Math 314 Fall 2025

Midterm 1

Duration: 50 minutes.

1. Answer each question in the space provided. If you require more space, please use one of the blank pages at the end of the exam, but indicate that you have done so in the original answer space. Do not write in the backs of the pages. Please note that the exams will be scanned one-sided!
2. Remember to show all your work.
3. No calculators, notes, phones, computers, smartwatches, or any other outside assistance allowed.
4. Please remove your smartwatch for the duration of the exam.

Best of luck!

Lab 1

Due Friday, October 17

To be discussed in Recitation on October 10

Groups of up to 3 students

Start early!

To do list:

- Webwork 3.2 due Friday
- Find a group for Lab 1
- Study for the midterm!

Best of luck!

Midterm 1

On October 6

Quiz on Friday

on inverses

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Monday 5–6 pm and Wednesday 2–3 pm
Thursday 2–3 pm
Friday 2:30–3:30 pm
in Avery 339 (Dr. Grifo)

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