Symbolic powers and differential operators

joint with Alessandro De Stefani and Jack Jeffries

<u>Algebra</u>	$\longmapsto$	<u>Geometry</u>
$R = \mathbb{C}[x_1, \ldots, x_n]$	$\longmapsto$	$\mathbb{C}^d$
I an ideal in R	$\longmapsto$	points in $\mathbb{C}^d$ that are zeroes of all polynomials in I
P prime ideal	$\longmapsto$	irreducible variety X
m maximal ideal	$\longmapsto$	$X = \{1 \text{ point}\}$

the <u>n-th symbolic power</u> of P is given by

$$P^{(n)} = \{ f \in R : f \text{ vanishes up to order } n \text{ on } X \}$$

What does that mean?

1) For a maximal ideal $\equiv$ point, $m^{(n)} = m^n$.

In general, the n-th power of an ideal I is I^n, the ideal generated by all products $f_1 \cdots f_n$ of $f_i \in I$.

2) For a prime ideal $\underline{P}$,

$$\underline{P} = \{f : f(x) = 0 \ \forall x \in X\}$$

$$= \bigcap_{x \in X} m_x = \bigcap_{\substack{m \supseteq \underline{P} \\ \text{maximal}}} m$$

$\boxed{m_x := \text{maximal ideal corresponding to the point } x}$

and $$\underline{P}^{(n)} = \bigcap_{\substack{m \supseteq \underline{P} \\ \text{maximal}}} m^n$$

Problem: If X is infinite, computing $\underline{P}^{(n)}$ means taking an infinite intersection.

Can we describe $\underline{P}^{(n)}$ algebraically?

Vanishing at a point is a local property.

In commutative algebra, looking at something locally means localizing the ring.

$\leadsto$ Since $\underline{P}$ defines X, we only care about what happens in $R_{\underline{P}} = \left\{\frac{f}{g} : g \notin \underline{P}\right\}$

$$\underline{P}^{(n)} = \underline{P}^n R_{\underline{P}} \cap R = \{ f : gf \in \underline{P}^n, g \notin \underline{P} \}$$

= elements in R that "locally" live in $\underline{P}^n$.

Warning! this is not the same as $\underline{P}^n$.

Example $\underline{P} = \ker\left(\mathbb{C}[x,y,z] \longrightarrow \mathbb{C}[t^3, t^4, t^5]\right)$

$$\underline{P} = (\underbrace{x^3 - yz}_{\substack{f \\ \deg 9}}, \underbrace{y^2 - xz}_{\substack{g \\ \deg 8}}, \underbrace{z^2 - x^2 y}_{\substack{h \\ \deg 10}})$$

$\deg x = 3$
$\deg y = 4$
$\deg z = 5$

$$\underline{P}^2 = (f^2, g^2, h^2, fg, fh, gh) \qquad \deg \underline{P}^2 \geq 16$$

But $\underbrace{f^2 - gh}_{\deg 18} = xq \in \underline{P}^2,\ x \notin \underline{P} \Rightarrow \underbrace{q}_{\deg 18 - 3 = 15} \in \underline{P}^{(2)}$

So $q \in \underline{P}^{(2)}$, $q \notin \underline{P}^2$.

So $\underline{P}^2 \subsetneq \underline{P}^{(2)}$.

Recap: $P^{(n)} = P^n \cap R_P = \bigcap_{m \supseteq P} m^n$

Zariski - Nagata theorem

the fact that these two definitions coincide is the content of the Zariski-Nagata theorem.

this can also be described via differential operators:

$$\mathrm{Diff}_{R|\mathbb{C}} = \text{differential operators on } R$$
$$= \mathbb{C}\langle x_1, \dots, x_n, \partial/\partial x_1, \dots, \partial/\partial x_n \rangle$$

(not commutative)

More generally, if R is a k-algebra, we define

$$\mathrm{Diff}_{R|k} \subseteq \mathrm{Hom}_k(R,R)$$

$\mathrm{Diff}^0_{R|k} = \mathrm{Hom}_R(R,R)$ = multiplication by elements of R

$$\mathrm{Diff}^n_{R|k} = \{\delta \in \mathrm{Hom}_k(R,R) : [\delta, r] = \delta r - r\delta \in D^{n-1}_{R|k}\}$$

$$I^{\langle n \rangle} := \{f : \partial f \in I \text{ for all } \partial \in \mathrm{Diff}^{n-1}_{R|k}\}$$

theorem: (Dao, de Stefani, –, Huneke, Núñez-Betancourt)

$$I^{\langle n \rangle} = I^{(n)}$$ over any perfect field k, I prime

Note: this has been known for fields of char 0

Note this might fail if the ideal is not prime.

How about for polynomial rings over something other than a field – say, over $\mathbb{Z}$?

Example $R = \mathbb{Z}[x]$, $Q = (2)$.

$$Q^{(n)} = Q^n = (2^n)$$ for all n, but

$$Q^{\langle n \rangle} = (2)$$ for all n, since

$$\delta(2) = 2\delta(1) \in Q$$ for all differential operators δ of any order!

(if we take diff operators over $\mathbb{Z}$, so $\mathbb{Z}$-linear)

However:

<u>Theorem</u> (de Stefani, –, Jeffries)

$A = \mathbb{Z}$, $A = \mathbb{Z}_p$ or even $A =$ some DVR

If $R = A[x_1, \dots, x_n]$ and Q is a prime ideal with $Q \cap A = 0$, then $Q^{\langle n \rangle} = Q^{(n)}$ $\forall n$.

<u>Problem</u> when $Q \cap A \neq 0$, say $Q \cap A$ contains a prime integer p, we need maps that decrease p-adic order, so that

$$\begin{cases} \delta(p^n) \in (p) & \text{for } \delta \text{ of order } n-1 \\ \delta(p^{n+1}) \notin (p) & \text{for } \delta \text{ of order } n-1 \end{cases}$$

or

$$\begin{cases} \delta(p^n) \in (p^{n-1}) \\ \delta(p^n) \notin (p^n) \end{cases} \quad \text{for } \delta \text{ of order } 1$$

Let R be a ring on which $p \in \mathbb{Z}$ (prime) is a nonzerodivisor, meaning $ps \neq 0$ for all $s \neq 0$ in R.
(so this includes the cases above with $R = A[x_1, \dots, x_n]$)

<u>Definition</u> (Buium) A <u>p-derivation</u> on R is a map $\delta : R \to R$ such that:

1) $\delta(xy) = \delta(x)y^p + x^p\delta(y) + p\delta(x)\delta(y)$

2) $\delta(x+y) = \delta(x) + \delta(y) + C_p(x,y)$

where $C_p(x,y) = \dfrac{x^p + y^p - (x+y)^p}{p}$.

<u>Note</u>: Having a p-derivation is equivalent to having a lift of the Frobenius map $S/pS \to S/pS$, $x \mapsto x^p$.
(Note: S/pS is a ring of char p)

If ϕ is a lift of Frobenius,

$$\delta(x) = \frac{\phi(x) - x^p}{p}$$ is a p-derivation.

p-derivations don't always exist, but they do exist over nice rings (like polynomial rings)

As maps, these are a little unusual from the Commutative Algebra perspective: They are not even additive. But they do decrease p-adic order.

Over $\mathbb{Z}$, there is only one p-derivation:

(Fermat difference operator) $\delta(n) = \frac{n - n^p}{p}$

Ex: $\delta(p) = 1 - p^{p-1} \notin (p)$ (p-adic order 0)

$\delta(p^2) = \underbrace{2p^p\delta(p)}_{\text{p-adic order } p} + \underbrace{p\,\delta(p)^2}_{\text{p-adic order } 1}$ p-adic order 1

$\delta(p^t)$ has p-adic order $t-1$

Ex: $R = \mathbb{Z}[x]$, $Q = (2)$, $\phi(x) = x^p$, $\phi(n) = n$

$\delta(2) = 1 - 2^{2-1} = -1 \notin (2) = Q$, $2 \notin Q^{(2)}$

the n-th mixed differential power of Q is

$$Q^{\langle n \rangle_{mix}} = \{ f : \delta^s \circ \partial (f) \in Q, \ \partial \in D^t_{R|A}, \ s+t \leq n-1 \}$$

<u>Note</u> A priori, this definition depends on the choice of δ, but in fact it is independent of δ.

<u>Theorem</u> (de Stefani, –, Jeffries)

$A = \mathbb{Z}$, $A = \mathbb{Z}_p$ or even $A =$ some DVR

If $R = A[x_1, \dots, x_n]$ and Q is a prime ideal with $Q \cap A \neq 0$, then

$$Q^{(n)} = Q^{\langle n \rangle_{mix}} = \bigcap_{\substack{m \supseteq Q \\ \max}} m^n .$$

More generally, this holds for any smooth R-algebra such that:

- R has a p-derivation δ
- $A/pA \to R/pR$ is a separable extension

Sketch of proof:

- $Q^{\langle n\rangle_{mix}}$ is a Q-primary ideal
- $Q^n \subseteq Q^{\langle n\rangle_{mix}}$
- Corollary: $Q^{(n)} \subseteq Q^{\langle n\rangle_{mix}}$
- $Q^{\langle n\rangle_{mix}} R_Q = (QR_Q)^{\langle n\rangle_{mix}}$
- $m^{\langle n\rangle_{mix}} = m^n$ when (R,m) local
- Since $Q^{\langle n\rangle_{mix}}$ and $Q^{(n)}$ are both Q-primary ideals, it is enough to show equality after localizing at Q.

to show some of these properties, we write

$$I^{\langle n\rangle_{mix}} = \bigcap_{a+b\leq n+1} \left(I^{\langle a\rangle_\delta}\right)^{\langle b\rangle}$$

and study the properties of taking $(-)^{\langle a\rangle_\delta}$ and $(-)^{\langle b\rangle}$

Example of a ring with no p-derivations

$S = \mathbb{Z}[x_1, \dots, x_n]$, $R = S/(f-p)$, for some $f \in (x_1, \dots, x_n)^2$

there is no p-derivation on R.

Suppose δ is a p-derivation on R. Then $p \equiv f$ as elements of R,

$\delta(p) = \delta(f) \in (p, x_1, \dots, x_n)$, since $f \in (x_1, \dots, x_n)^2$

But $\delta(p) = 1 - p^{p-1} \notin (x_1, \dots, x_n, p)$

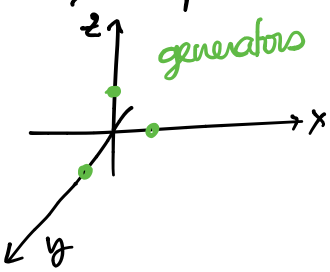
In particular, $R = \mathbb{Z}[x]/(x^2 - p)$ has no p-derivations.

More generally, a (not necessarily reducible) variety X corresponds to some radical ideal

$$I = \underbrace{P_1 \cap \dots \cap P_k}_{\text{prime ideals corresponding to each irreducible component}}$$

$$I^{(n)} = \left(I^n R_{P_1} \cap R\right) \cap \dots \cap \left(I^n R_{P_k} \cap R\right)$$

Example: $I = (x,y) \cap (y,z) \cap (x,z)$



$$= (xy, xz, yz)$$

$$I^{(2)} = (x,y)^2 \cap (y,z)^2 \cap (x,z)^2$$