Linear Algebra

Math 314 Fall 2025

Today's poll code: JGHXEA

Lecture 7

Filling in for Dr. Grifo this week: Dr. Mark Walker (Avery 303)

Office hours

Tuesdays 11–noon
Thursdays 1–2 pm
in Avery 337 (Kara)

This Thursday (tomorrow) 4–5pm
in Avery 303 (Dr. Walker)

Come see how much nicer my office is than Dr. Grifo's

To do list:

- Webwork 2.4 due Friday
- Webwork 2.5 due Tuesday September 23
- Webwork 2.6 due Friday September 26

Quiz 4 on Friday
on lectures 6—7

Recap

- Any $m \times n$ matrix A determines a matrix transformation, $T : \mathbb{R}^n \to \mathbb{R}^m$, defined as $T(x) = Ax$.
- A *linear transformation* is any function $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfying two properties:
  - (Preserves addition) $T(u + v) = T(u) + T(v)$ for all vectors u and n.
  - (Preserves scaling) $T(cu) = cT(u)$ for all vectors u and scalars c.
- It turns out these are the same thing: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if it is given by a matrix.
- If we know T is a linear transformation, how to we find the matrix A that gives it?

For instance: The transformation T that rotates the plane by

90 degrees counter-clockwise is linear. What matrix gives it?

Theorem A function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation, meaning that there exists a matrix A such that

$$T(x) = Ax \quad \text{for all } x \in \mathbb{R}^n.$$

If T is a linear transformation, how do we go about finding A?

To find this matrix A, we do the following:

Definition [Standard matrix of a linear transformation]: Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$. Given a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$, consider the matrix

$$A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix}.$$

The matrix A is called the **standard matrix** of T and it satisfies

$$T(x) = Ax \qquad \text{for all } x \in \mathbb{R}^n.$$

Example: Recall that the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates the plane by 90 degrees is linear. So, it must be secretly given by a 2×2 matrix A. The previous theorem tells us how to find it: By recording $T(e_1)$ and $T(e_2)$.

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We have $T(e_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, so that the matrix for this transformation must be

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And thus

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

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Let's check this for a couple of vectors: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Poll question 1. Today's code: JGHXEA

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a matrix transformation and you know that

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\2\end{bmatrix} \text{ and } T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\4\end{bmatrix}.$$

Which of the following matrices gives T?

Watch out!

- (a) $A = \begin{bmatrix}1 & -3\\4 & 2\end{bmatrix}$
- (b) $A = \begin{bmatrix}1 & 4\\-3 & 2\end{bmatrix}$
- (c) $A = \begin{bmatrix}-3 & 1\\2 & 4\end{bmatrix}$
- (d) $A = \begin{bmatrix}2 & -3\\4 & 1\end{bmatrix}$

Recall:

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Why does this work?

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$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n.$$

Then

$$\begin{aligned} T(x) &= T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) \\ &= x_1 T(\mathbf{e}_1) + \cdots + x_n T(\mathbf{e}_n) && \text{since } T \text{ is a linear transformation} \\ &= Ax && \text{since the } T(\mathbf{e}_i) \text{ are the columns of } A. \end{aligned}$$

This shows that T is in fact a matrix transformation, with associated matrix A.

$$T(x) = Ax \text{ for all vectors } x$$

Recall: The standard matrix A of a linear transformation T is given by

$$A = [T(e_1)\, T(e_2)\, \cdots\, T(e_n)]$$

or, in words, the i-th column of A is the result of applying T to the i-th standard basis vector.

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Another example: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be dilation by a factor of 3: $T(x) = 3x$.
Then $T(e_1) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$, $T(e_3) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$.

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So $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, three times the identity matrix.

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We have

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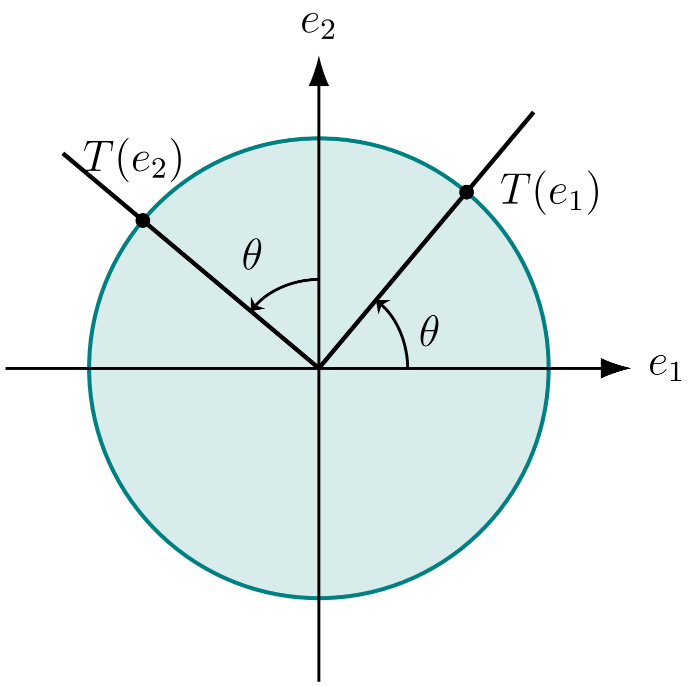
This allows us to find a formula for T:

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x\\-y\\-z\end{bmatrix}$$

Example (Rotation in the plane): Consider the function $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each point counterclockwise by an angle θ (in radians).



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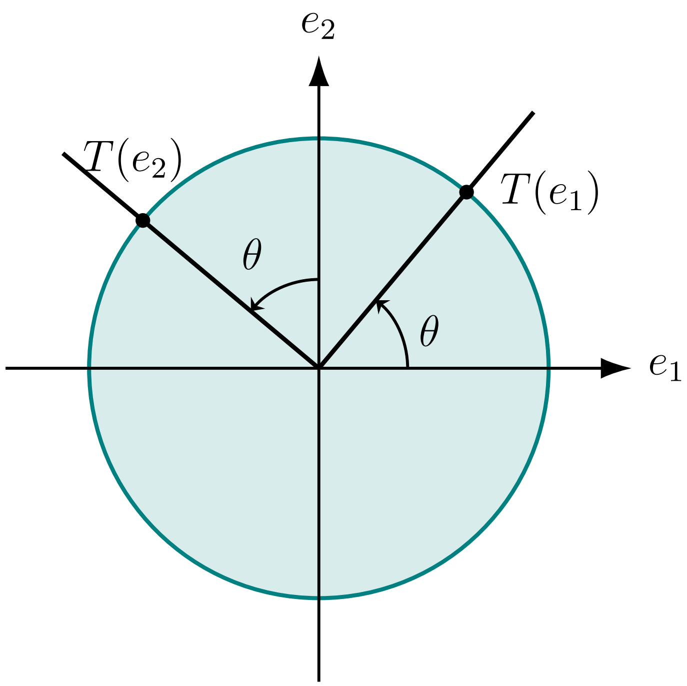

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Then using trigonometry, one can show that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \qquad \text{and} \qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}.$$

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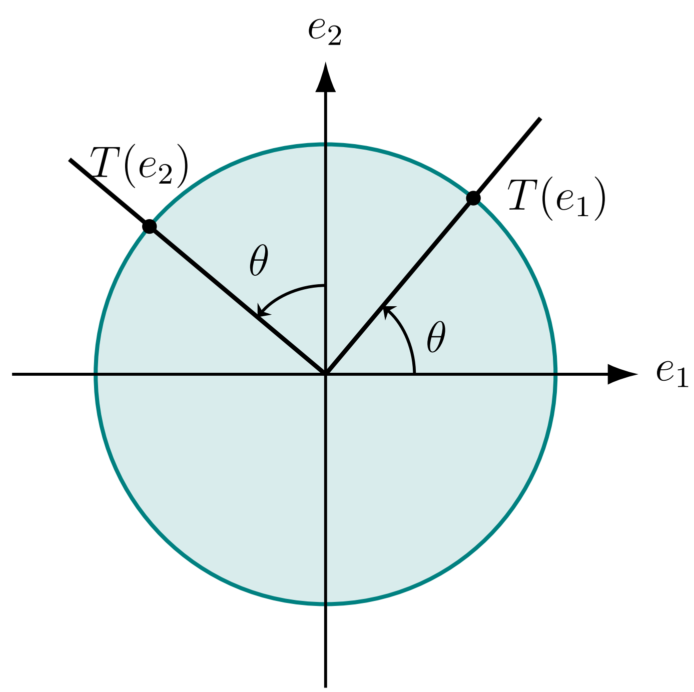

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Thus the standard matrix for this linear transformation is

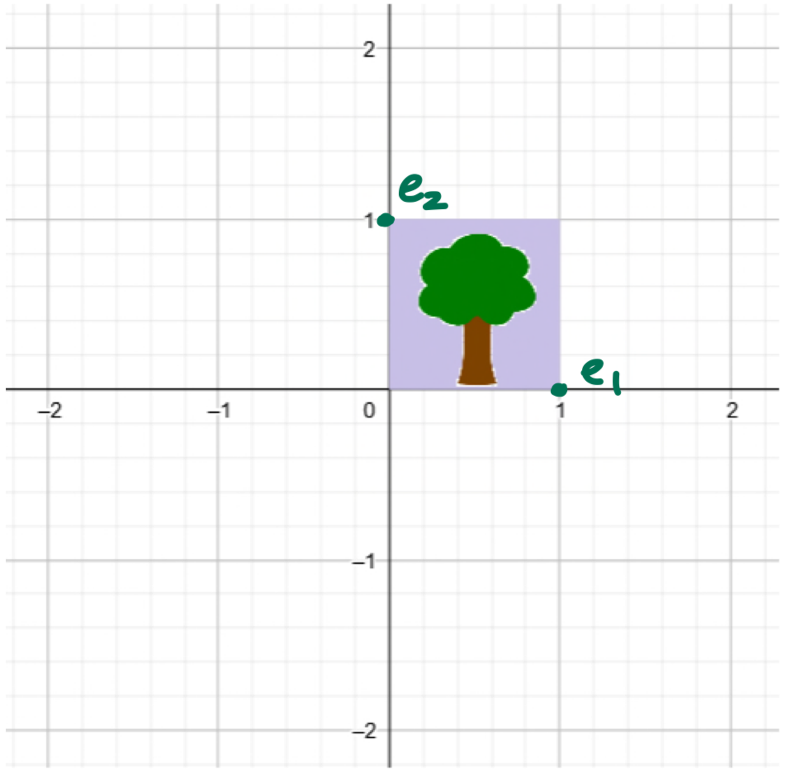
$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

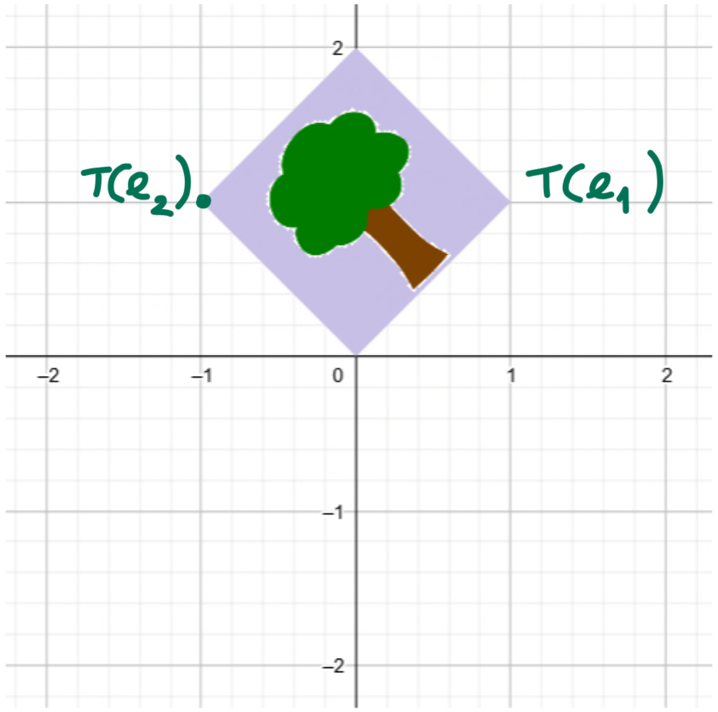
For instance, if we take $\theta = \pi/2$, then $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$ and thus the matrix that gives rotation by $\pi/2$ (90 degrees) counter-clockwise is

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

This is the same answer we got before.

Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that does the following:





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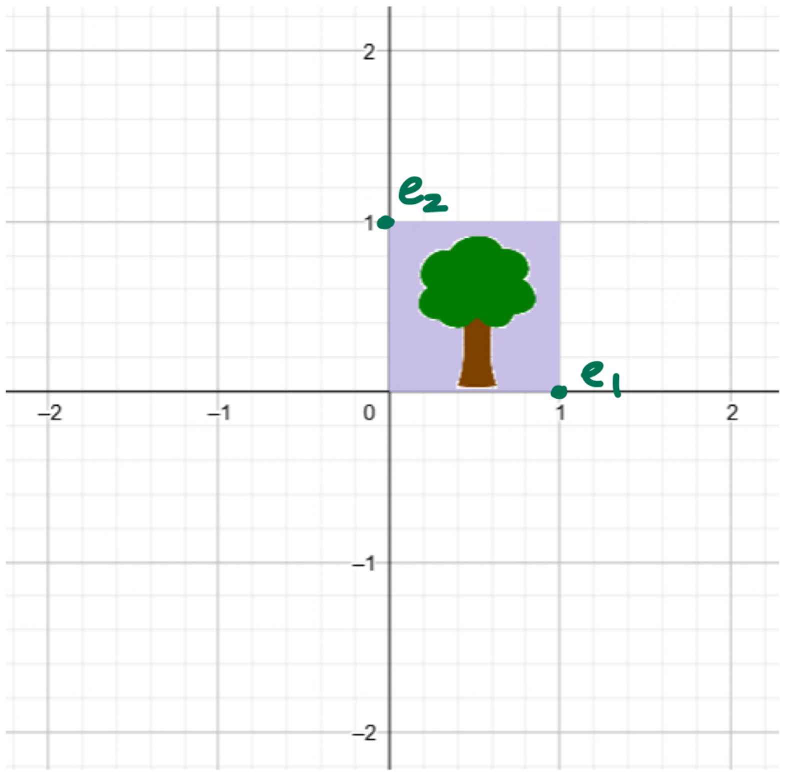
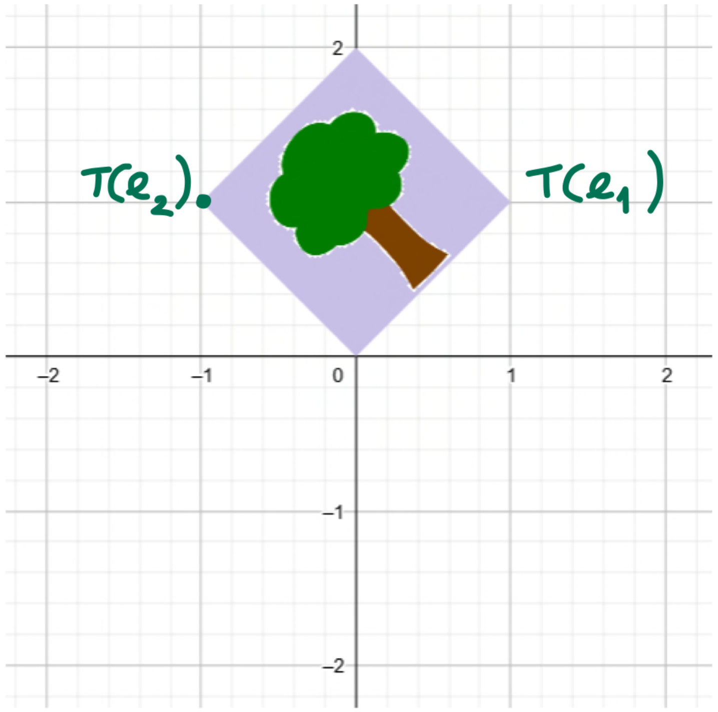
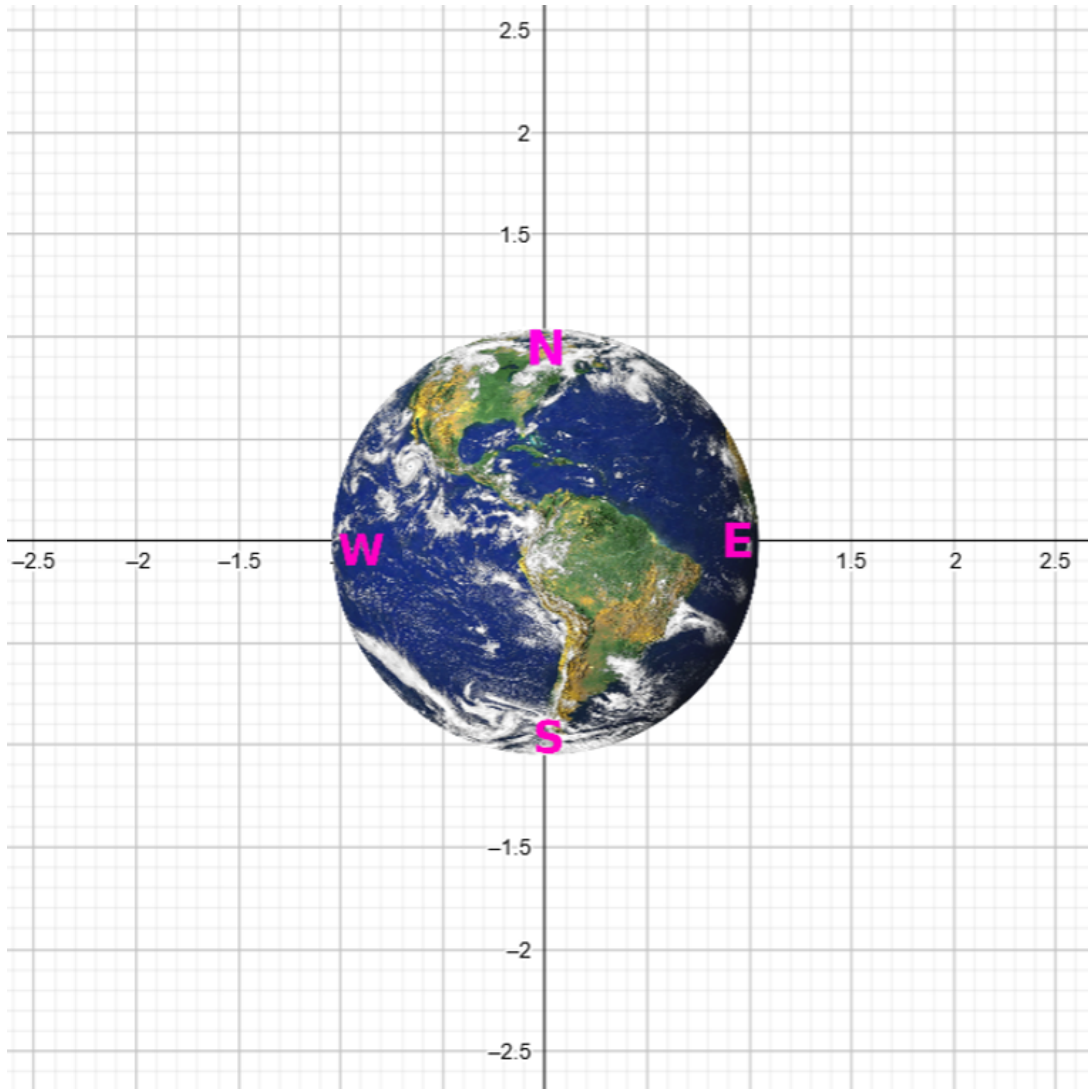




Note how we explicitly marked the images of $\mathbf{e}_1$ and $\mathbf{e}_2$. This is sufficient for us to find the standard matrix, and thus to completely describe the linear transformation: the matrix is

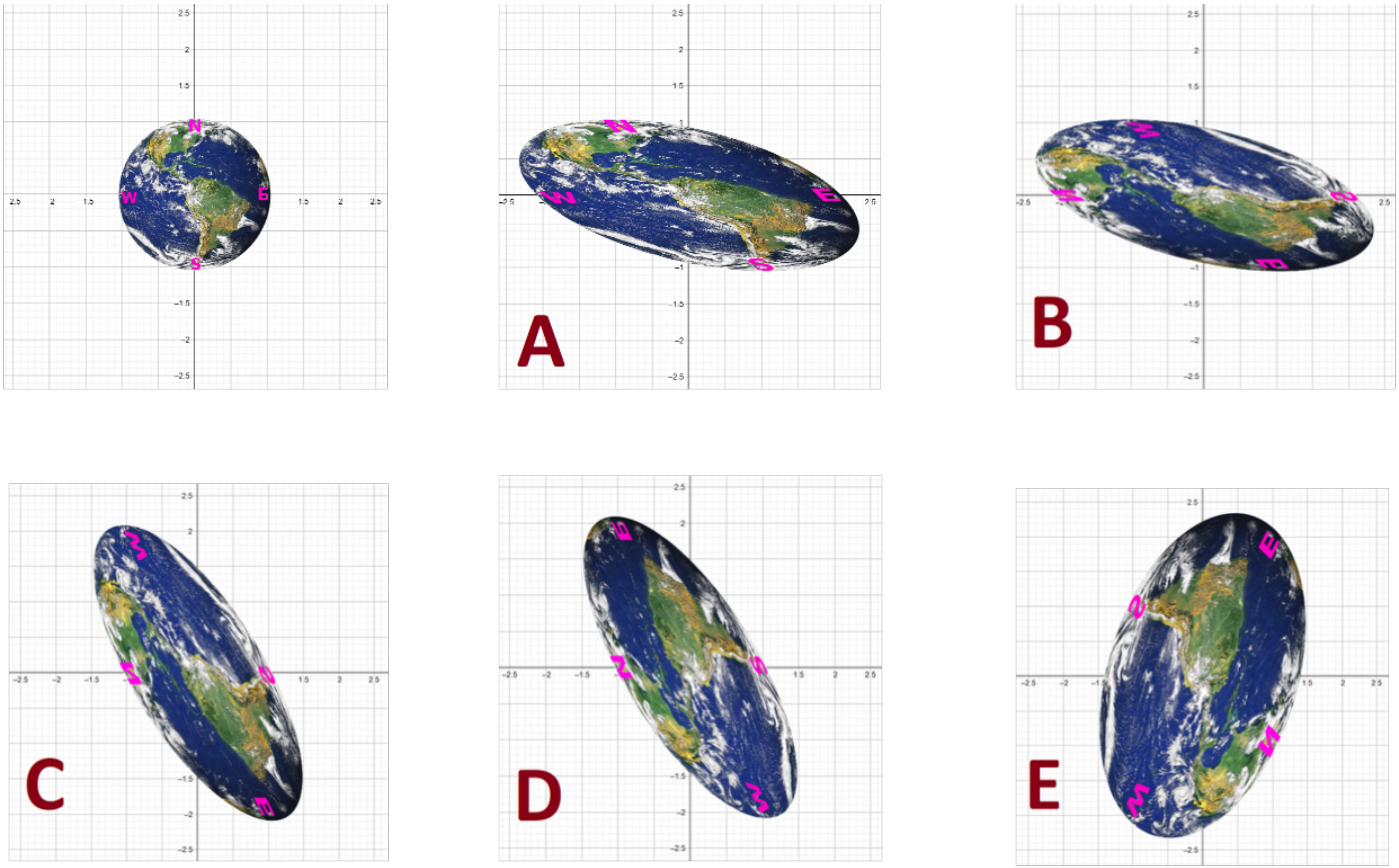
$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Let's look at example involving the earth:



Poll question 2. Today's code: JGHXEA

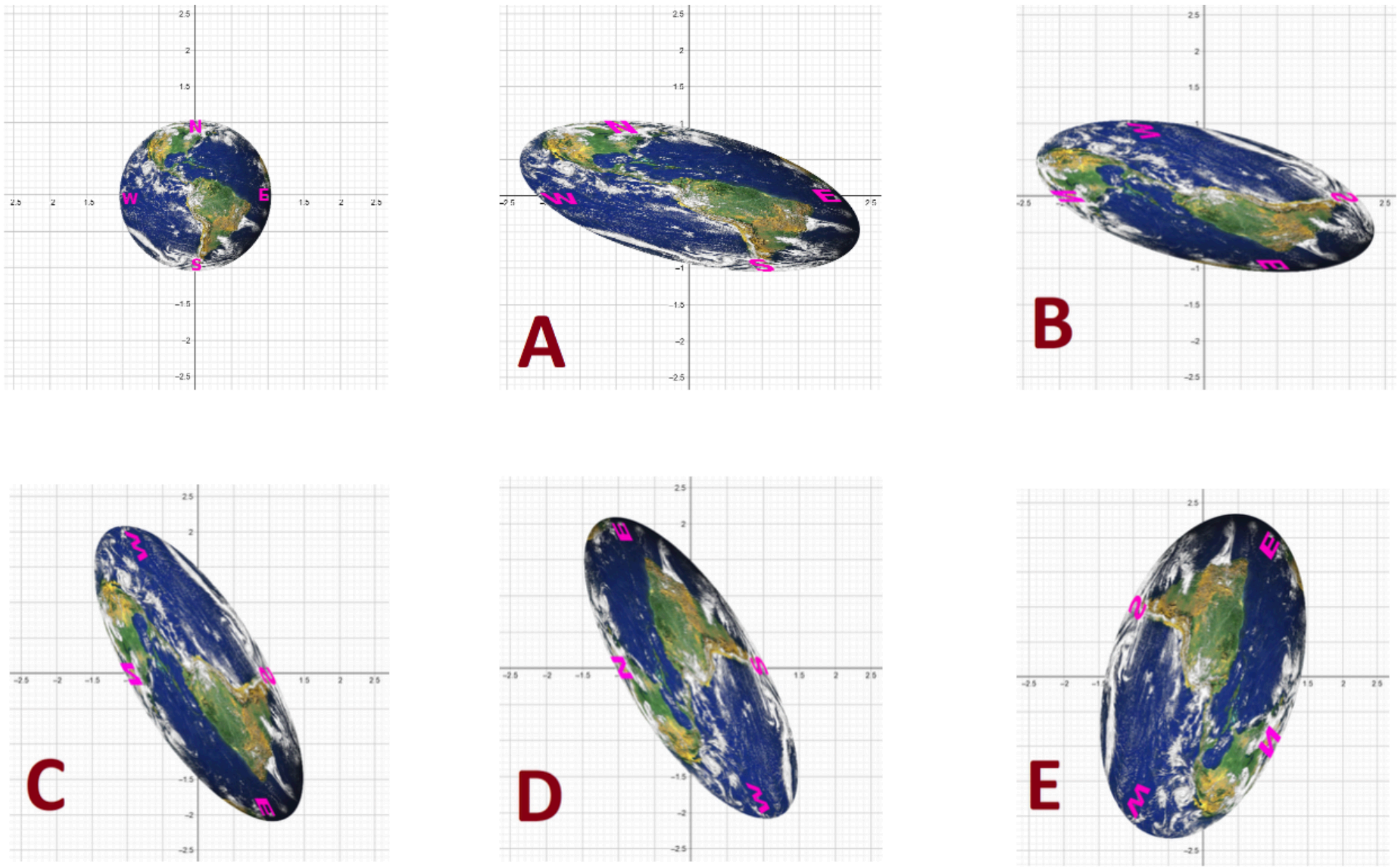
If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with standard matrix $A = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$ which shows how the earth is transformed?



Poll question 3. Today's code: JGHXEA

Different matrix!

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with standard matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ which shows how the earth is transformed?



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Then T sends $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (first column of A) and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (second column of A), and from this we can determine what T does in general:

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T moves xy-plane into the yz plane without stretching it in any way.

We can also define a linear transformation like $T : \mathbb{R}^7 \to \mathbb{R}^4$ by specifying some 4×7 matrix A, such as

$$A = \begin{bmatrix} 1 & -3 & 5 & 2 & 0 & 6 & 7 \\ 2 & \frac{3}{2} & 1 & 0 & 14 & -20 & \pi \\ 5 & e & 17 & 0 & 0 & 0 & 2 \\ 3 & 1 & 4 & 1 & 5 & 9 & 7 \end{bmatrix}.$$

Just don't ask me to visualize it!

Injective and Surjection Functions

A function $T\colon \mathbb{R}^n \to \mathbb{R}^m$

is an assignment

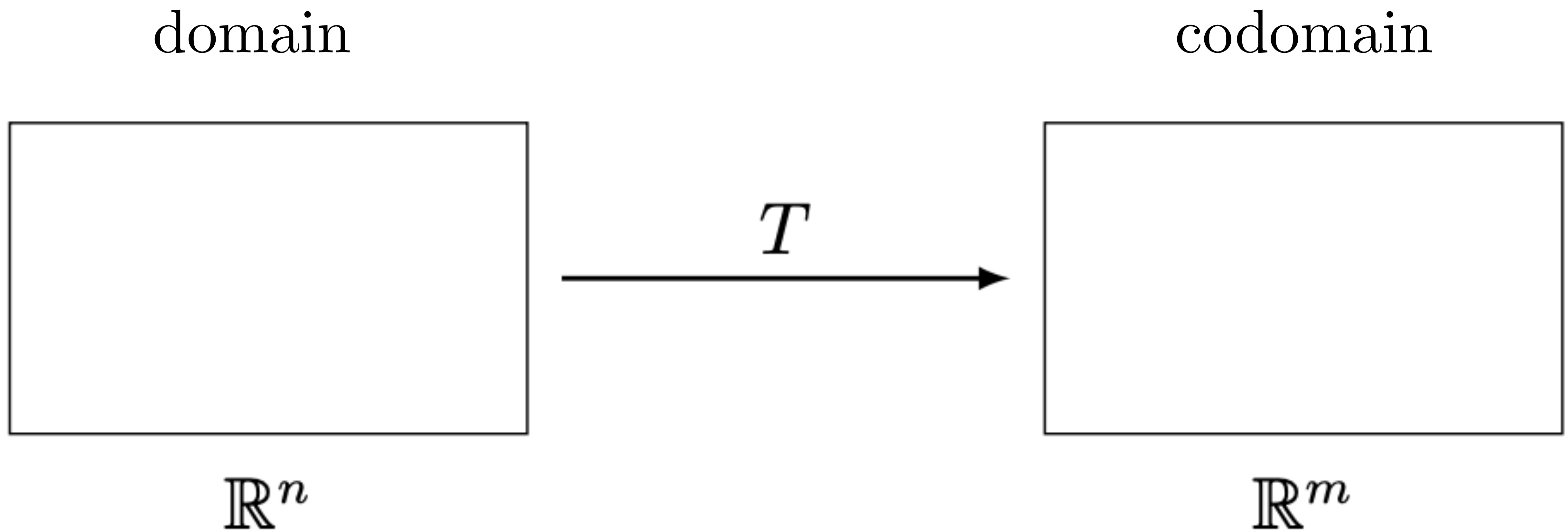
each vector in $\mathbb{R}^n$ $\longrightarrow$ a vector in $\mathbb{R}^m$

A function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ has

domain $\mathbb{R}^n$ $\longrightarrow$ inputs

and

codomain $\mathbb{R}^m$. $\longrightarrow$ where the outputs live


domain
codomain
T
$\mathbb{R}^n$
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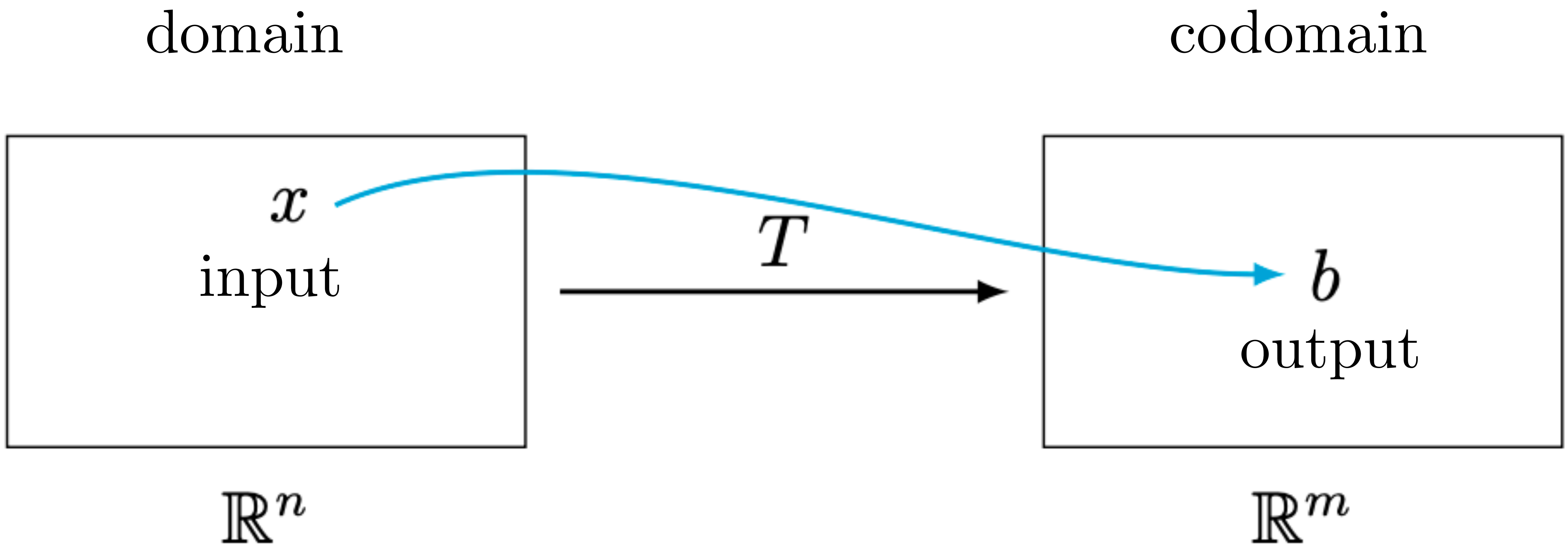
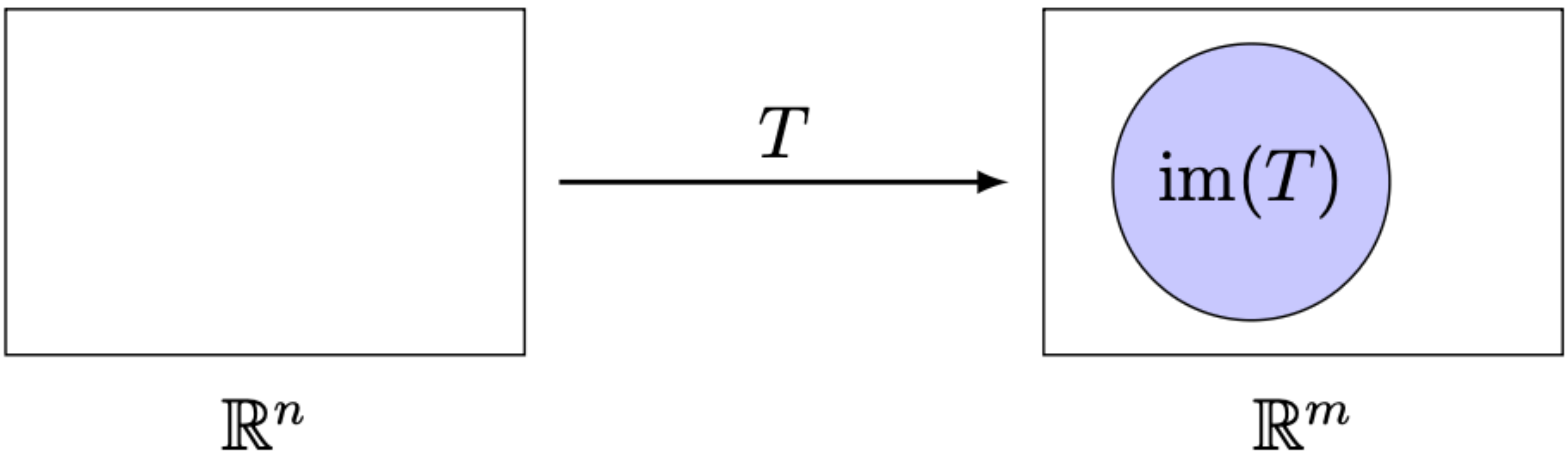

domain
codomain
x
input
T
b
output
$\mathbb{R}^n$
$\mathbb{R}^m$

The **image** or **range** of a function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is

$$\operatorname{im}(T) := \{T(x) \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

Informally: the set of actual outputs


T
$\mathrm{im}(T)$
$\mathbb{R}^n$
$\mathbb{R}^m$

T is **surjective** or **onto** if

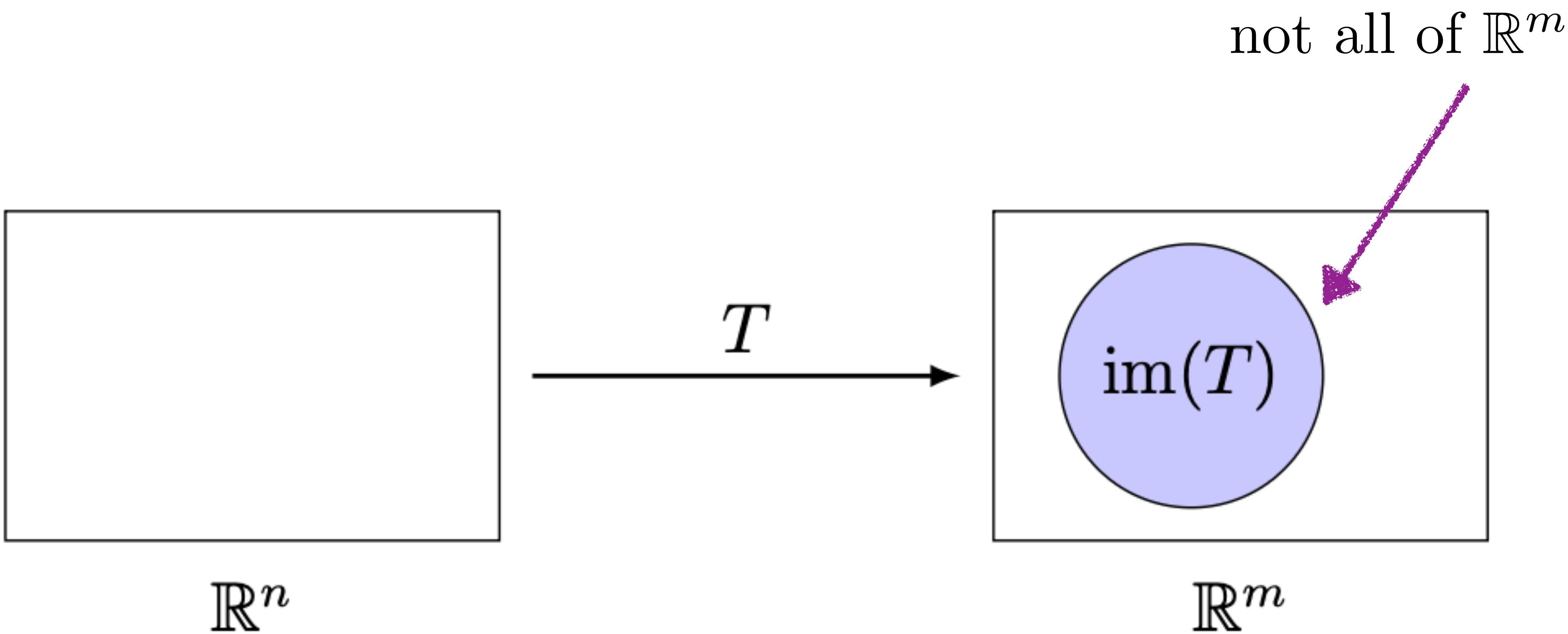
for every $b \in \mathbb{R}^m$ there exists *at least one* $x \in \mathbb{R}^n$

such that

$$T(x) = b$$

Equivalently: $\quad \operatorname{im}(T) = \mathbb{R}^m$

Informally: everything in the codomain is an actual output.



not surjective

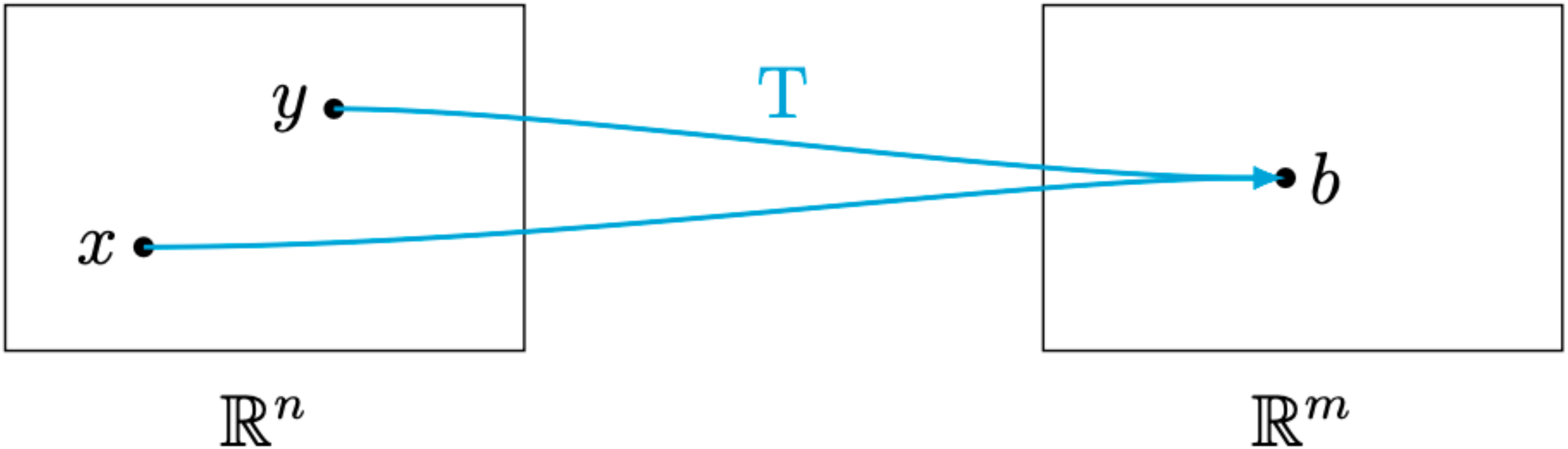
T is **injective** if

for each $b \in \mathbb{R}^m$ there exists *at most one* $x \in \mathbb{R}^n$

such that

$T(x) = b$

Equivalently: $T(x_1) = T(x_2) \implies x_1 = x_2.$



not injective

$$\text{bijective} = \text{injective} + \text{surjective}$$

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- Let $T : \mathbb{R} \to \mathbb{R}$ be the (non linear) transformation $T(x) = e^x$. Is it surjective? Is it injective? Is it bijective?

The **kernel** of a linear transformation

$$T\colon \mathbb{R}^n \to \mathbb{R}^m$$

is the set

$$\ker(T) := \{x \in \mathbb{R}^n \mid T(x) = 0\}.$$

Note: we always have $0 \in \ker(T)$

Theorem. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

T is injective if and only if the equation

$$T(x) = 0$$

has only the trivial solution $x = 0$.

Equivalently, T is injective if and only if $\ker(T) = \{0\}$.

Theorem. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

$$T \text{ is surjective} \iff \text{the columns of } A \text{ span } \mathbb{R}^m$$

$$T \text{ is injective} \iff \begin{array}{c}\text{the columns of } A\\ \text{are linearly independent}\end{array}$$

Theorem. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A.

$$T \text{ is surjective} \iff A \text{ has a pivot in every row}$$

$$T \text{ is injective} \iff A \text{ has a pivot in every column}$$

Example: The identity map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$$

T is surjective: for any $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ we have $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$

T is injective: $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) \implies \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$

T is bijective

Example: The identity map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$$

has standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

T is surjective: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has a pivot in every row

T is injective: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has a pivot in every column

T is bijective

Example: The map $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix}$$

T is not surjective: for example, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \notin \operatorname{im}(T)$.

T is injective:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) \implies \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = \begin{bmatrix} u \\ v \\ u+v \end{bmatrix} \implies \begin{cases} x = u \\ y = v \end{cases}$$

Example: The map $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix}$$ has standard matrix $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

T is not surjective: there is no pivot on the last row

T is injective: pivot in every column

Example: The map $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{has standard matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

T is surjective: for any $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ we have $T\left(\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$

T is not injective: $$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

Example: The map $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{has standard matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

T is surjective: pivot in the third column

T is not injective: there is no pivot on the third column

Example:

The linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ with standard matrix

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

no pivot on this row

$\Downarrow$

T is not surjective

no pivot on this column $\Longrightarrow$ T is not injective

To do list:

- Webwork 2.4 due Friday
- Webwork 2.5 due Tuesday September 23
- Webwork 2.6 due Friday September 26

On Friday:

Quiz 4

at the beginning

of the recitation

on Lectures 6—7

Office hours

This Thursday (tomorrow) 4–5pm
in Avery 303 (Dr. Walker)

Come see how much
nicer my office is than
Dr. Grifo's

Tuesdays 11–noon and
Thursdays 1–2 pm
in Avery 337 (Kara)