Symbolic powers in mixed characteristic

joint with Alessandro De Stefani and Jack Jeffries

R Noetherian ring

$I = \sqrt{I}$ ideal in R

the nth symbolic powers of I is

$$I^{(n)} := \bigcap_{P \in \operatorname{Ass}(R/I)} \left(I^n R_P \cap R \right)$$

so when $I = P$ is prime, $P^{(n)} = \{ r \in R \mid sr \in P^n \text{ for some } s \notin P \}$

Symbolic powers appear as auxiliary tools in the proofs of Krull's Height Theorem, Hartshorne-Lichtenbaum Vanishing Theorem and other classical results

Facts
⓪ $P^{(n)}$ is the smallest P-primary ideal containing P^n
① $I^n \subseteq I^{(n)}$ $\forall n \geq 1$
② $I^{(n+1)} \subseteq I^{(n)}$ $\forall n \geq 1$
③ $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$ $\forall a, b \geq 1$
④ $I = \mathfrak{m}$ maximal $\Rightarrow \mathfrak{m}^{(n)} = \mathfrak{m}^n$ $\forall n \geq 1$
⑤ R Cohen-Macaulay, I ci $\Rightarrow I^n = I^{(n)}$
⑥ $I^n \neq I^{(n)}$ in general

Example $I = (xy, xz, yz) = (x,y) \cap (x,z) \cap (y,z)$

$$\underset{\substack{\cup\\ xyz}}{I^{(2)}} = (x,y)^2 \cap (x,z)^2 \cap (y,z)^2 \supsetneq \underset{\substack{\not\ni\\ xyz}}{I^2}$$

Ex X 3×3 generic matrix
k field
$I = I_2(X) \subseteq R = k[X]$
$\det(X) \in I^{(2)}$ but $\det(X) \notin I^2$
(I prime!)

theorem (Zariski's Main Lemma on Holomorphic Functions)
(Zariski, 1949, Nagata, 1962) (Eisenbud-Hochster, 1979)

P prime ideal in R fg k-algebra over a field k

$$P^{(n)} \supseteq \bigcap_{\substack{m \supseteq P \\ m \in \mathrm{mSpec}(R)}} m^n \qquad \text{for all } n.$$

When R is regular, we have equality:

$$P^{(n)} = \bigcap_{\substack{m \supseteq P \\ m \in \mathrm{mSpec}(R)}} m^n = \{ f \in R \mid f \text{ vanishes to order } n \text{ on } V(I) \}$$

theorem (Zariski, Nagata) (R, m) RLR, P prime

$$\Rightarrow P^{(n)} \subseteq m^n$$

We can write this using differential operators:

thm (Zariski–Nagata)

$R = \mathbb{C}[x_1, \dots, x_d]$

$I = \sqrt{I}$

$I^{(n)} = \{ f \in R \mid \partial/\partial x_1^{a_1} \cdots \partial/\partial x_d^{a_d} (f) \in I \text{ for all } a_1 + \cdots + a_d \leq n-1 \}$

Def R an A-algebra

the A-linear differential operators on R are given by:

$\mathcal{D}_{R|A} = \bigcup_{n \geq 0} \mathcal{D}^n_{R|A} \subseteq \mathrm{Hom}_A(R, R)$:

- $\mathcal{D}^0_{R|A} = R \cong \mathrm{Hom}_R(R, R)$
- $\delta \in \mathrm{Hom}_A(R, R)$ is in $\mathcal{D}^n_{R|A}$ if
  $[\delta, r] \in \mathcal{D}^{n-1}_{R|A}$ for all $r \in \mathcal{D}^0_{R|A}$

the A-linear nth differential power of I is

$I^{\langle n \rangle} := \{ f \in R \mid \partial(f) \in I \text{ for all } \partial \in \mathcal{D}^{n-1}_{R|A} \}$

theorem (Zariski–Nagata)

$R = k[x_1, \dots, x_d]$, k perfect field

$I = \sqrt{I}$

then $I^{\langle n \rangle} = I^{(n)}$ for all $n \geq 1$

Note When k is not perfect, we still have $I^{(n)} \subseteq I^{\langle n \rangle}$

Question Can such a theorem hold in mixed characteristic?

Example $R = \mathbb{Z}[x]$

$\mathfrak{m} = (2, x)$

$\mathfrak{m}^{(n)} = \mathfrak{m}^n$ **but** $\mathfrak{m}^{\langle n \rangle} \supsetneq \mathfrak{m}^n$

because $\partial/\partial x\,(2) = 0$

even $\partial(2) = 2\,\partial(1) \in \mathfrak{m}$ for all $n \geq 1$, $\partial \in D^n_{R|\mathbb{Z}}$

$\Rightarrow\ 2 \in \mathfrak{m}^{\langle n \rangle}$ for all $n \geq 1$

theorem (De Stefani – G – Jeffries)

$A = \mathbb{Z}$ or a DVR with uniformizer $p \in \mathbb{Z}$

R localization of fg smooth A-algebra

Q prime ideal

If $Q \cap A = 0$, then $Q^{\langle n \rangle} = Q^{(n)}$ for all $n \geq 1$

when $Q \cap A \neq 0 \iff p \in Q$ prime integer

Need operators that decrease p-adic order:

$$\begin{cases} \delta(p^n) \in (p) \\ \delta(p^{n-1}) \notin (p) \end{cases} \quad \text{for } \delta \text{ of order } n-1$$

or

$$\begin{cases} \delta(p^n) \in (p^{n-1}) \\ \delta(p^n) \notin (p^n) \end{cases} \quad \text{for } \delta \text{ of order } 1$$

Definition (Buium, Joyal)
$p \in \mathbb{Z}$ prime
R ring where p is regular
$\delta : R \to R$ is a p-derivation if for all $x, y \in R$,

- $\delta(1) = 0$
- $\delta(xy) = \delta(x) y^p + x^p \delta(y) + p\delta(x)\delta(y)$
- $\delta(x+y) = \delta(x) + \delta(y) + \dfrac{x^p + y^p - (x+y)^p}{p}$

Note Having a p-derivation is equivalent to having a lift of the Frobenius map $R/p \xrightarrow{F} R/p$ to R

$\delta(x) = \dfrac{\phi(x) - x^p}{p}$ p-derivation $\iff$ $\Phi(x) = x^p + p\,\delta(x)$ lift of Frobenius

We do have p-derivations when:

- $\mathbb{Z}$: there is a unique p-derivation:

$$\delta(n) = \frac{n - n^p}{p}$$

In fact, any p-derivation δ on R must satisfy $\delta(n) = \dfrac{n - n^p}{p}$ for $n \in \mathbb{Z}$

- R a complete unramified DVR with perfect residue field
- $R = B[x_1, \dots, x_d]$, B admits a p-derivation
  set $\delta(x_i)$ freely, $\delta|_B$ = p-derivation on B

Note p-derivations decrease p-adic order (see Jack's talk)

$$\delta(p) = \frac{p-p^p}{p} = 1 - p^{p-1} \notin (p)$$

eg

$$\delta(p^n) = \underbrace{p^p \underbrace{\delta(p^{n-1})}_{\text{order } n-2}}_{\geqslant n} + \underbrace{p^{p(n-1)}}_{\text{order} \geqslant n} \delta(p) + \underbrace{\underbrace{p}_{1} \underbrace{\delta(p)}_{0} \underbrace{\delta(p^{n-1})}_{\text{order } n-2}}_{\text{order } n-1}$$

$$\delta(p^n v) \equiv p^{n-1} v \pmod{p^n}$$

$\therefore$ these are the maps we were looking for

Mixed differential powers (DS-G-J)

R ring with a p-derivation δ, $p \in I$ ideal

$$I^{\langle n \rangle_{mix}} = \{ f \in R \mid \delta^a \circ \partial(f) \in I,\ \partial \in D^b_{R|A},\ a+b \leq n-1 \}$$

theorem (De Stefani - G - Jeffries)

$A = \mathbb{Z}$ or DVR with uniformizer p

R localization of fg smooth A-algebra

R has a p-derivation δ

(eg $R = A[x_1, \dots, x_d]$, A unramified DVR)

Q prime ideal in R, $Q \ni p$

$A/p \hookrightarrow R_Q/QR_Q$ separable, eg if A/p is perfect

then $Q^{(n)} = Q^{\langle n \rangle_{mix}}$ for all $n \geqslant 1$.

Note the essentially smooth and separability hypotheses are necessary

Sketch ① show $Q^n \subseteq Q^{\langle n \rangle_{mix}}$
② $Q^{\langle n \rangle_{mix}}$ Q-primary
$\Longrightarrow Q^{(n)} \subseteq Q^{\langle n \rangle_{mix}}$
③ $Q^{\langle n \rangle_{mix}} R_Q = (QR_Q)^{\langle n \rangle_{mix}}$
④ (R, m) local $\Rightarrow m^{\langle n \rangle_{mix}} = m^n$

so $Q^{(n)} \subseteq Q^{\langle n \rangle_{mix}}$, need to show $Q^{\langle n \rangle_{mix}} \subseteq Q^{(n)}$

Enough to show after localizing at Q, since $\mathrm{Ass}(R/Q^{(n)}) = \{Q\}$

Corollary If R is also smooth over A,
$$Q^{(n)} = \bigcap_{\substack{m \supseteq Q \\ m \in \mathrm{mSpec}(R)}} m^n$$

Corollary $Q^{\langle n \rangle_{mix}}$ is independent of the choice of δ

Remark $R = A[x_1, \dots, x_d] \ni f$, δ p-derivation, $p \in Q$ prime
$(R/(f))_Q$ regular $\Leftrightarrow f \in Q^2 \Leftrightarrow f \in Q^{(2)} \Leftrightarrow \partial/\partial x_i f, \ \delta(f) \in Q$

We can use these types of ideas to say something about symbolic powers even when R does not necessarily have a p-derivation

theorem (Chevalley's lemma, 1943)
(R, m) complete local ring
$\{I_n\}$ decreasing family of ideals

If $\bigcap_{n \geq 0} I_n = 0$, then $\exists f : \mathbb{N} \to \mathbb{N}$ st $I_{f(n)} \subseteq m^n$
(so $\{I_n\}$ induces a finer topology than $\{m^n\}$)

Important Special Case $I_n = I^{(n)}$

Uniform Chevalley Lemma (Huneke - Katz - Validashti, 2009)

(R, m) complete local domain

then $I^{(Cn)} \subseteq m^n$ for all $n \geq 1$

where C is independent of I (!)

Note Finding an explit C would give a uniform lower bound on the m-adic order of elements of $I^{(n)}$.

Note Zariski - Nagata: When R is regular, we can take $C=1$ $(I^{(n)} \subseteq m^n)$

Explicit bounds for some singular rings

theorem (Dao - De Stefani - G - Huneke - Núñez Betancourt)

k any field

$R = k[f_1, \cdots, f_e] \xhookrightarrow{\oplus} S = k[x_1, \ldots, x_d]$

$m = (\underbrace{f_1, \ldots, f_e}_{\text{homogeneous in } R})$ $\quad D = \max\{\deg f_i\}$ $\quad \eta := (x_1, \ldots, x_d)$

then $Q^{(Dn)} \subseteq m^n$

for every $n \geq 1$ and every homogeneous prime Q in R

<u>key idea</u> $R \underset{s}{\hookrightarrow} S$ $\quad s$ splitting

- Show $\eta^{\langle n \rangle} \cap R \subseteq \mathfrak{m}^n$
- $\partial \in D^n_{S|k} \implies s \circ \partial \in D^n_{R|k}$ (Àlvarez Montaner – Huneke – Núñez Betancourt)
- Show $Q^{(n)} \subseteq \eta^n \cap R$ using the fact that

$$\eta^n = \eta^{(n)} = \eta^{\langle n \rangle}$$

$\eta^{(n)} = \eta^{\langle d \rangle}$ for $\eta = (\underline{x})$

$$f \notin \eta^n \cap R \implies \partial f \notin \eta \text{ for some } \partial \in D^n_{S|k}$$

$$\implies s \circ \partial (f) \notin \mathfrak{m} \supseteq Q \implies f \notin Q^{\langle n \rangle} \supseteq Q^{(n)}$$

Main obstruction to copying this idea to mixed characteristic:

to define mixed differential powers in R, we need a p-derivation!

<u>Q</u>: If $R \overset{\oplus}{\hookrightarrow} S$ and S has a p-derivation, must R have a p-derivation?

<u>Setting</u> A DVR with uniformizer p with a p-derivation (eg, complete)

$R = A[f_1, \ldots, f_e] \subseteq S = A[x_1, \ldots, x_d]$, f_i homogeneous

δ p-derivation on S

$R/p \underset{s}{\overset{\oplus}{\hookrightarrow}} S/p$,

$\overline{(\ )} := (\ )/p$

Q prime ideal containing p

$$Q^{\{n\}} := \{ x \in R \mid s\left(\overline{\delta^a \partial(x) S}\right) \subseteq \overline{Q} \text{ for all } a+b<n,\ \partial \in D^b_{S|A} \}$$

Lemma $Q^{\{n\}}$ is a Q-primary ideal $\supseteq Q^{(n)}$

Theorem (De Stefani - G - Jeffries)

$R = A[f_1, \dots, f_e] \subseteq S = A[x_1, \dots, x_d]$, f_i homogeneous

$\mathfrak{q} = (f_1, \dots, f_e)R$

$\mathfrak{m} = \mathfrak{q} + (p)$ $\qquad D := \max\{\deg f_i\}$

① $\begin{cases} Q \subseteq \mathfrak{q} \text{ prime ideal} \\ R \overset{\oplus}{\hookrightarrow} S \end{cases} \Longrightarrow Q^{(Dn)} \subseteq \mathfrak{q}^n$

② $\begin{cases} Q \subseteq \mathfrak{m} \text{ prime containing } p \\ \overline{R} \overset{\oplus}{\hookrightarrow} \overline{S} \end{cases} \Longrightarrow Q^{(Dn)} \subseteq \mathfrak{m}^n$

We do have examples showing these can be sharp.

Lemma If R has a p-derivation

$\overline{R} \overset{\oplus}{\hookrightarrow} \overline{S}$

Every $\partial \in D^n_{R|A}$ extends to $D^n_{S|A}$ $\Longrightarrow Q^{\{n\}} \subseteq Q^{\langle n \rangle_{mix}}$

Question Are $Q^{\{n\}} = Q^{\langle n \rangle_{mix}}$ for all n?