Symbolic powers Bridges 2021

You know nothin' (about) Symbolic Powers (Day 3)

Theorem (Hochster, 2016)

We don't know (almost) anything about symbolic powers.

Here are some big open problems about symbolic powers:

I. Equality Problem When is $I^n = I^{(n)}$?

① Given I, for which n do we have $I^n = I^{(n)}$?

→ not a reasonable question in general

② Fix R (eg $k[x_1, \ldots, x_d]$).

Which ideals I satisfy $I^n = I^{(n)}$ for all n ?

There is a theorem of Hochster from 1964 giving necessary and sufficient conditions on I, but it is not pratical.

Monomial ideals I is a squarefree monomial ideal if it is generated by monomials of the form $x_{i_1} \cdots x_{i_n}$, where $i_k \neq i_j$ for $j \neq k$.

A monomial ideal I is <u>packed</u> if whenever we

- → set some variables $= 0$,
- → set some variables $= 1$,
- → do nothing to some variables

the resulting ideal J has codimension c, and contains c many monomials with no common variables ($\equiv$ a regular sequence of c monomials)

<u>Ex</u>: $I = (xy, xz, yz)$ has codimension 2

but any 2 monomials have a common variable $\Rightarrow$ not packed

<u>Conjecture</u> (<u>Packing Problem</u>)

Let I be a monomial ideal in $k[x_1, \ldots, x_d]$.

I satisfies $I^n = I^{(n)}$ for all $n \geq 1$ if and only if I is <u>packed</u>

③ Is it sufficient to check $I^n = I^{(n)}$ for finitely many values of n?

<u>Theorem</u> (Montaño — Núñez Betancourt, 2018)

I squarefree monomial ideal generated by μ elements

If $I^{(n)} = I^n$ for $n \leq \lceil \frac{\mu}{2} \rceil$, then $I^{(n)} = I^n$ for all $n \geq 1$.

II. Finite Generation of Symbolic Rees Algebras

$I^{(a)} I^{(b)} \subseteq I^{(a+b)}$ for all a, b

$\Longrightarrow$ can form a <u>graded algebra</u>

$\mathcal{R}_S(I) := \bigoplus_{n \geq 0} I^{(n)} t^n \subseteq R[t]$ the <u>symbolic Rees algebra</u> of I

<u>Problem</u> Is $\mathcal{R}_S(I)$ always finitely generated?

Equivalently, is there d such that for all n,

$$I^{(n)} = \sum_{a_1 + 2a_2 + \cdots + d a_d = n} I^{a_1} (I^{(2)})^{a_2} \cdots (I^{(d)})^{a_d}$$

<u>Answer</u> No! $\mathcal{R}_S(I)$ can be finitely generated or not

Deciding what's the case is very hard!

<u>Example</u> (Goto-Nishida-Watanabe, 1994)

For some choices of a, b, c, the symbolic Rees algebra of (t^a, t^b, t^c) can be infinitely generated

(Huneke, Srinivasan, Cutkosky, many others) for some choices of a, b, c, the symbolic Rees algebra is fg!

II. Degrees

When I is homogeneous, $I^{(n)}$ is also homogeneous for all $n \geq 1$

Def $\alpha(I) :=$ minimal degree of a nonzero homogeneous element in I

Question: What is $\alpha(I^{(n)})$ and how does it grow with n?

$$k[x_0, \dots, x_d] \longleftrightarrow \mathbb{P}^d$$

homogeneous $I = \sqrt{I}$ $\qquad$ projective varieties

$I \neq (x_0, \dots, x_d)$

$$(a_j x_i - a_i x_j \mid i \neq j) \longleftrightarrow \{(a_0 : \cdots : a_d)\}$$

So when I corresponds to $\{P_1, \dots, P_s\} \subseteq \mathbb{P}^d$

$\alpha(I^{(n)}) :=$ minimal degree of a homogeneous polynomial vanishing to order n at $P_1, \dots, P_s$

$=$ smallest degree of a hypersurface passing through $P_1, \dots, P_s$ with multiplicity n

Conjecture (Chudnovsky, 1981)

If I defines s points in $\mathbb{P}^N$, then

$$\frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I) + N - 1}{N}$$

Theorem (Bisui – G – Hà – Nguyễn)

Chudnovsky's Conjecture holds for a $\underbrace{\text{general}}_{}$ set of $s \geq 4^N$ points

for "most" sets of points

(the sets of s points in $\mathbb{P}^N$ are parametrized by a topological space called the Hilbert scheme of s points. the theorem holds in an open dense set of the Hilbert scheme of s points)

How does one prove theorems like this?
Studying variations of the Containment problem.

Containment Problem When is $I^{(a)} \subseteq I^b$?

Theorem (Ein-Lazarsfeld-Smith 2001, Hochster-Huneke 2002, Ma-Schwede 2018)

$R = k[x_1, \ldots, x_d]$, k field or $\mathbb{Z}$ or $\mathbb{Z}_p$

$I = \sqrt{I} = P_1 \cap \cdots \cap P_s$

$h := \max_i \{\operatorname{codim} P_i\}$

then $I^{(hn)} \subseteq I^n$ for all $n \geq 1$

$\left(\Longrightarrow \quad I^{(dn)} \subseteq I^n \quad \text{for all } n \geq 1 \right)$

Example $I = (xy, xz, yz) \Rightarrow h=2 \Rightarrow I^{(2n)} \subseteq I^n \Rightarrow I^{(4)} \subseteq I^2$

Question (Huneke, 2000) What if P is a prime with $h=2$.
Is $P^{(3)} \subseteq P^2$?

theorem (G, 2020) True for P defining (t^a, t^b, t^c) in char $\neq 3$

Conjecture (Harbourne, 2008) $I^{(hn-h+1)} \subseteq I^n$ for all $n \geq 1$

theorem (Dumnicki – Szemberg – Tutaj-Gasińska, 2013) False
$\rightsquigarrow$ Constructed 12 points in $\mathbb{P}^2$ that don't satisfy $I^{(3)} \subseteq I^2$.
Extended by Harbourne-Seceleanu, and many others

But Harbourne's Conjecture is satisfied by
- $\longrightarrow$ squarefree monomial ideals
- $\longrightarrow$ general points in $\mathbb{P}^2$ (Bocci – Harbourne) and $\mathbb{P}^3$ (Dumnicki)
- $\longrightarrow$ ideals defining F-pure rings (G-Huneke, 2019)

$\left\{ \begin{array}{l} \bullet\ R/I \simeq k[\text{all monomials of degree } d \text{ in } v \text{ vars}] \quad \text{Veronese} \\ \bullet\ I = t\text{-minors of a generic } n \times m \text{ matrix} \\ \bullet\ \text{nice rings of invariants} \end{array} \right.$