A test module for complete intersections
(joint work with Ben Briggs and Josh Pollitz)

throughout: $(R, \mathfrak{m}, k)$ local ring / fg positively graded k-algebra
$\mathfrak{m}$ unique maximal ideal / homogeneous maximal ideal

R is regular if $\mathfrak{m} = (x_1, \ldots, x_d)$, $d = \dim(R)$

We can always write

$\hat{R} \cong Q/I$, where
$\begin{cases} \bullet\ (Q, \mathfrak{n}, k) \text{ is a RLR} \\ \bullet\ I \subseteq \mathfrak{n}^2 \end{cases}$

$R \cong Q/I$, where
$\begin{cases} Q = k[x_1, \ldots, x_e] \\ \mathfrak{n} = (x_1, \ldots, x_e) \\ I \subseteq \mathfrak{n}^2 \end{cases}$

theorem (Auslander–Buchsbaum–Serre) TFAE:

- R is regular
- Every fg R-module M has finite projective dimension
- $\operatorname{pdim}_R(k) < \infty$ (pdim := projective dimension)

Example $R = k[x]/(x^2)$ $Q = k[x]$, $I = (x^2)$

$$\cdots \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \to k \to 0$$
$$\Longrightarrow \operatorname{pdim}_R(k) = \infty \Rightarrow R \text{ not regular}$$

But notice $\mu(I) = 1 = \operatorname{ht} I = \operatorname{codim} I \Rightarrow R$ is a ci

R is ci = complete intersection

$\iff \mu(I) = \mathrm{ht}\, I = \mathrm{codim}\, I$

$\iff$ I generated by a regular sequence

Example $\dfrac{k[x,y]}{(x^2, xy)}$ is not a ci $\quad \mathrm{ht}\, I = 1 < \mu(I) = 2$

Goal Give a homological characterization of cis, à la ABS

$\rightsquigarrow$ Need to move to the world of complexes

$\mathcal{D}(R) :=$ complexes of R-modules up to quasi-iso

$\cup\!\mathsf{I}$

$\mathcal{D}^f(R) :=$ complexes with fg homology

(analogous to $\mathrm{mod}(R) \subseteq \mathrm{Mod}(R)$)

M R-module $\iff$ complex $\quad 0 \to \overset{0}{M} \to 0$

$\cong$

projective resolution $\cdots \to P_2 \to P_1 \to P_0 \to 0$

$\mathrm{pdim}(M) < \infty \iff M \simeq$ bounded complex of fg projectives

=: small

so: theorem (Auslander–Buchsbaum–Serre) TFAE:

- R is regular
- Every $X \in \mathcal{D}^f(R)$ is small
- k is small

Def (Dwyer–Greenlees–Iyengar, 2005)

$X \in \mathcal{D}(R)$ is **proxy small** (ps) if:

① We can finitely build a small P from X, by

virtually small:
- shifting complexes
- taking direct summands
- if we can build two of $0 \to A \to B \to C \to 0$, we can build the third

② $\operatorname{Supp} P = \operatorname{Supp} X$

theorem (Pollitz, 2019) TFAE:
- R is ci
- Every $X \in \mathcal{D}^f(R)$ is proxy small

Notes ① $\Rightarrow$ is (Dwyer–Greenless–Iyengar, 2005)

② the original theorem is about local rings, but we can globalize it thanks to work of Letz

Corollary R is ci $\Longrightarrow$ every fg R-module is proxy small

Question $\Longleftarrow$?

Goal When R is not ci, construct fg R-module that is not ps.

Warning k is always proxy small
because it builds the Koszul complex

Another guess the conormal module I/I^2

theorem (Briggs, 2020) R is ci $\iff$ $\operatorname{pdim}_R(I/I^2) < \infty$

Unfortunately when R has an embedded deformation,
I/I^2 has a free summand $\Rightarrow$ proxy small
(R has an embedded deformation if $R \cong P/(f)$, f regular)

Possible motivation (Gheibi – Jorgensen – Takahashi, 2019)
R ci $\Rightarrow$ every fg R-module has finite quasiprojective dim
open $\Leftarrow$?
Fact (GJT) finite quasiprojective dimension $\Rightarrow$ proxy small

Sidenote (Avramov – Gasharov – Peeva, 1997)
R ci $\iff$ every fg R-module has finite CI-dimension
$\iff$ k has finite CI-dimension

finite CI-dimension $\overset{\Longrightarrow}{\not\Longleftarrow}$ proxy small

Theorem (Briggs – G – Pollitz) Suppose R is equipresented meaning that every $f \in I \setminus mI$ has the same m-adic order then

R is ci $\iff$ Every fg R-module is proxy small

$\iff$ Every $R \twoheadrightarrow$ artinian ci is proxy small
$(|k| = \infty)$

Strategy

Main tool Cohomological support varieties

$\hat{R} \cong Q/I \qquad (Q, m, I)$ RLR $\qquad I = (f_1, \dots, f_n) \qquad \mu(I) = n$

M R-module $\rightsquigarrow V_R(M) \subseteq k^n$ variety

$$V_R := I/mI \cong k^n$$

Def: $V_R(M) := \{ [f] \in V_R \mid \operatorname{pd}_{Q/f} \hat{M} = \infty \text{ or } [f] = 0 \}$

Note $V_R(M)$ is intrinsic to M, and does not depend on choices

Appears in various formats in work of
Avramov, Avramov – Buchweitz, Burke – Walker, Jorgensen, Pollitz

Facts (Pollitz)

- R is ci $\iff V_R(R) = 0$
- M proxy small R-module $\Rightarrow V_R(R) \subseteq V_R(M)$

Goal Construct fg R-module with $V_R(M) \not\supseteq V_R(R)$

How? Construct fg R-modules $M_1, \dots, M_t$ such that $t \le n$

① can compute $V_R(M_i)$

② $V_R(M_1) \cap \dots \cap V_R(M_t) = 0$

Lemma (Briggs – G – Pollitz)

If $J \supseteq I$, J ci, then

$$V_R(Q/J) = \ker\left(I/\mathfrak{m}I \longrightarrow J/\mathfrak{m}J\right)$$

~ minimal generators of I that are not minimal generators of J

Corollary If $f \in I \setminus \mathfrak{m}I$, and $J \supseteq I$ is a ci, $f \in J \setminus \mathfrak{m}J$, then

$V_R(Q/J) \subseteq$ hyperplane not containing the f-direction

Hopes and dreams Can we find enough different directions?

Lemma If $f \in I \setminus \mathfrak{m}I$ has minimal $\mathfrak{m}$-adic order, we can construct ci $J \supseteq I$ with $f \in J \setminus \mathfrak{m}J$.

$\Longrightarrow$ can do this when I is equipresented

Examples

1) $R = \dfrac{k[x,y]}{(x^2, xy)}$ $\qquad Q = k[x,y], \quad I = (x^2, xy)$

$M = Q/(x^2, y)$ $\qquad N = Q/(xy, x+y)$

$$V_R(M) = \ker \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} x^2 \\ y \end{array} \\ \begin{array}{cc} x^2 & xy \end{array} & \end{array} = \operatorname{span}\{(0,1)\}$$

$$V_R(N) = \ker \begin{array}{cc} \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} x+y \\ xy \end{array} \\ \begin{array}{cc} x^2 & xy \end{array} & \end{array} = \operatorname{span}\{(1,1)\}$$

$$x^2 = x(x+y) - xy$$

$V_R(M) \cap V_R(N) = (0) \implies M$ or N are not proxy small

2) $\operatorname{char} k \neq 2$

$Q = k[x,y,z], \quad I = (x^2+y^2+z^2, xyz, x^3)$

$\mathfrak{a}_1 = (x^2+y^2+z^2, y, x^3)$ $\qquad \mathfrak{a}_2 = (x^2-2z, xyz, y+z)$

$V_R(Q/\mathfrak{a}_1) = \operatorname{span}\{(0,1,0)\}$ $\qquad (0,\ell,0) \notin V_R(Q/\mathfrak{a}_2)$

$\implies$ one of these is not proxy small

<u>Updated theorem</u> we can do all rings such that

$$\sigma := \min\{ m\text{-adic order of } f \mid f \in I,\ f \neq 0 \}$$

$$\underbrace{\dim_k \left(\frac{m^{\sigma+1} \cap I}{m I} \right)}_{\#\text{ min gens of order } > \sigma} < \underbrace{\dim_k \left(\operatorname{span} V_R(R) \right)}_{\leq n = \mu(I)}$$

Example 2) does not satisfy this, and yet we can still fulfill our goal. (both numbers are 2)