Ulrich Modules do not always Exist (Champ Sept 24)

(R, m, k) local (Noetherian), M module$/R$ always f.g.

Outline.

(1) Definitions + Basic Properties.

(2) Motivations / Applications.

(3) Survey of Major Existence Results.

(4) Counterexample (Existence)

(5) Open Questions.

I. Definitions.

(R, m, k) local (Noetherian) M f.g. module$/R$.

Def:) (MCM) M module$/R$. Then M is maximal CM (MCM) if $\operatorname{depth}_R(M) = \dim(R)$.

Useful Fact:) Every (part of) s.o.p of R is a (part of an) M-sequence, Def:) Hilbert-Samuel Multiplicity (HS-mult)

M module$/R$. The HS mult of M w.r.t. m is

$$e_m(M) = d! \lim_{n\to\infty} \frac{\ell(M/m^nM)}{n^d}.$$

$e_m(M) = e_R(M)$

where $d = \dim(M) = \dim(R/\mathrm{Ann}_R(M))$.

$e(R) = e_m(R)$

Recall:) $\ell(M/m^nM)$ agrees w/ a polyn

$$n \longmapsto \ell(M/m^nM)$$

in the variable n of degree $d = \dim(M)$ for $n \gg 0$

w/ leading term $\frac{\text{positive integer}}{d!} n^d$.

Useful Facts (R, m, k) local domain for rest of talk.
k infinite.

1) $e_m(M) = \underline{\text{rank}_R(M)} \cdot e_m(R)$.

v.s dim of $\text{frac}(R) \otimes_R M$.

2) Let $I \subseteq m$ minimal reduction, i.e. $\exists n$ s.t. $m^{n+1} = I m^n$. Then $e_m(M) = \ell(M/IM)$ if M is MCM.

Prop:) M MCM$/R$, then $e_R(M) \geq \nu_R(M)$

$\nu_R(M)$ = minimum # of generators.

Proof:) $e_R(M) = \ell(M/IM) \geq \ell(M/mM) = \nu_R(M)$. $\square$

$I \subseteq m$ minimal reduction.

Def:) (Ulrich module)

M is Ulrich if

1) M is MCM

2) $e_R(M) = \nu_R(M)$ "Ulrich condition".
"Ulrichness".

Observation:) M MCM.

Then M is Ulrich iff $\mathfrak{m}M = IM$ for any minimal reduction $I \subseteq \mathfrak{m}$.

Proof:) Observe:) $e_R(M) = \ell(M/IM)$

$$= \ell(M/\mathfrak{m}M) + \ell(\mathfrak{m}M/IM)$$

$$= \nu_R(M) + \ell(\mathfrak{m}M/IM.$$

M Ulrich iff $e_R(M) = \nu_R(M)$ iff $\underbrace{\ell(\mathfrak{m}M/IM)}_{=0}$

iff $\mathfrak{m}M = IM$.

II.) Motivations / Applications.

Introduced in 1984 by Bernd Ulrich.

Q:) When is a local CM ring Gorenstein.

B. Ulrich proved if R has an Ulrich module, then. R is Gorenstein iff $\mathrm{Ext}^i_R(M, k) = 0$

for $1 \leq i \leq \dim(R)$.

Problem:) It's really hard to find Ulrich modules.

Existence for complete local domains → Serre's positivity

→ don't need Ulrichness here. MCM enough.

→ Lech's conjecture.

→ "do use Ulrichness".

Lech's Conjecture (1960's)

Let $\varphi : (R, m, k) \to (S, n, l)$ flat local map. $(\varphi(m) \subseteq n)$. Then $e_m(R) \leq e_n(S)$.

Lech's Conjecture holds in following cases.

- Lech (1960) $\dim(R) \leq 2$.
- Ma (2019) in the equicharacteristic case, $\dim(R) = 3$.

  ↳ uses reduction to char p methods to get a much better bound:

  $$e_m(R) \leq \max\left\{1, \frac{d!}{2^d}\right\} \cdot e_n(S)$$

  where $d = \dim(R)$.

Plug in 3, and get

$$e_m(R) \leq \max\left\{1, \frac{6}{8}\right\} \cdot e_n(S) = e_n(S)$$

- Ma (2020) Standard graded rings / perfect field.

↳ strategy is to use weakly lim Ulrich

"Ulrichness" as a strategy towards Lech's conj.

First, known that we can reduce to the case where:

1) R complete local domains

2) $\dim(R) = \dim(S) = d$; (mS is primary to nS).

Claim:) If R has an Ulrich module, then Lech's conj holds.

Proof:) M be Ulrich $/R$ $\left(\begin{array}{l} M \text{ MCM.} \\ e_R(M) = \nu_R(M) \end{array}\right)$.

$$e_m(R) = \frac{1}{\operatorname{rank}_R(M)} \cdot e_m(M)$$

$$= \frac{1}{\operatorname{rank}_S(S \otimes_R M)} \cdot \nu_R(M)$$

$$= \frac{1}{\operatorname{rank}_S(S \otimes_R M)} \cdot \nu_S(S \otimes_R M).$$

$$\leq \frac{1}{\operatorname{rank}_S(S \otimes_R M)} \cdot e_S(S \otimes_R M)$$

$$= e_n(S). \quad \blacksquare$$

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don't need Ulrich but can use netsorns that approximate "Ulrichness".

"Soft" example:) (1999, Hanes).

If $\{M_n\}$ MCM s.t. $\dfrac{\nu_R(M_n)}{e_R(M_n)} \longrightarrow 1$, then Lech's conjecture holds. ∎.

III.) Survey of Major Existences

(1) 1987. (Brennan, Herzog, Ulrich).

R standard graded, 2-dim CM domain (k infinite).

(2) (1991) Herzog, Ulrich, Backelin.

R strict ~~complete~~ complete intersection, i.e. gr(R) is a complete intersection.

(3) (1999) (2003).

↪ Eisenbud, Schreyer, Weyman

char 0 all veronese subrings of the polynomial ring.

(4) (2004) Bruns, Römer, Wiebe. generic determinantal rings.

Counterexample :)

Thm:) (Yhee, 2019) Ulrich modules do not always exist for (complete) local domain.

Counterexample:) $R = k[x^n, x^{n+1}, x^n y, y^n, y^{n+1}, xy^n, xy]_m$

$n \geq 2$.

For simplicity, stick w/ $n=3$.

Q:) How can we classify MCM modules $/R$.

Prop:) (R, m, k) excellent local domain
↳ Complete local domain).

Let S be the S_2-ification of R.

If S is local, the any MCM module M/R.

is an MCM module $/S$.

Recall: S_2-ification S of R is a module finite extension $R \subseteq S$ s.t for and $f \in S - R$, the ideal

$$R :_R f = \{ a \in R \mid af \in R \}$$

is ht ≥ 2.

Proof: $\Rightarrow$) M MCM $/R$.

Enough to show that for any $f \in S - R$ and $m \in M$, there is a well-defined action.

$f \cdot m \in M$.

$\operatorname{ht}(R :_R f) \geq 2 \Rightarrow \exists\, u, v$ in $R :_R f$ s.t.

u, v is part of a s.o.p. for R.

M MCM $\Rightarrow u, v$ is regular sequence on M.

$$v \cdot ((uf) \cdot m) = u \cdot (vf) \cdot m \quad \in vM.$$

$\Rightarrow (rf)\cdot m \in rM$.

Let $n \in M$ s.t. $(rf)\cdot m = r\cdot n$.

Define $f\cdot m = n$. ∎

Counterexample :)

$R = k[x^3, x^4, x^3y, y^3, y^4, xy^3, xy]_m$.

$\cap$

$S = k[x,y]_{(x,y)}$.

$\underline{x^3, y^3}$ multiply x into R.

" " y into R.

$\Rightarrow x, y \in S_2$-ification.

$\Rightarrow S$ is the S_2-ification.

MCM module/S look like $S^{\oplus n}$.

Enough to show that S is not Ulrich$/R$.

$\rightsquigarrow (xy, x^3-y^3) \subseteq m$ minimal reduction in R.

<u>Remark</u>:) M MCM, then M Ulrich iff $mM = IM$ where I minimal reduction.

$$(xy, x^3-y^3)S \neq (x^3, x^4, x^3y, y^3, y^4, xy^3, xy)S.$$

$x^3 \notin$?

$\Rightarrow$ S not Ulrich $/R$.

$\Rightarrow$ R has no Ulrich module.

<u>Counterexample</u>:) (localization).

$$R = k[s^4, sx^3, x^4, x^3y, sy^3, y^4, xy^3, s^2xy]_m.$$

$\subseteq k[s,x,y]_{(s,x,y)}$

$p = \varphi^{-1}(x, y)$

$$R_p = k(s^4)\left[\left(\frac{x}{s}\right)^3, \left(\frac{x}{s}\right)^4, \left(\frac{x}{s}\right)^3\left(\frac{y}{s}\right), \left(\frac{y}{s}\right)^4, \left(\frac{y}{s}\right)^3, \left(\frac{x}{s}\right)\left(\frac{y}{s}\right)^3, \left(\frac{x}{s}\right)\left(\frac{y}{s}\right)\right]$$

$R \subseteq k[s, x, y]^{(4)}_{n}$. $\leftarrow$ has an Ulrich module M.

$e(R) = 16 = e(S)$

M is MCM $/R$. ✓.

$$16 \cdot \operatorname{rank}_R(M) = e_R(M) \geq \nu_R(M) \geq \nu_S(M) = e_S(M) = 16 \cdot \operatorname{rank}_S(M).$$

$\Rightarrow$ $e_R(M) = \nu_R(M)$ and M Ulrich$/R$.

Some drawbacks :) Counterexample is not CM.

Open Questions :) Do all CM ~~domains~~ local rings have an Ulrich module?