Constructing non-proxy small modules
(joint with Ben Briggs and Josh Pollitz)

$(R, \mathfrak{n}, k)$ noetherian local ring

$Q/I \cong \hat{R}$ $\quad (Q, \mathfrak{m}, k)$ RLR, $I \subseteq \mathfrak{m}^2$

<u>theorem</u> (Auslander – Buchsbaum, 1957, Serre, 1956) TFAE:

- R is regular
- Every fg R-module has finite projdim.
- k has finite projdim

In the world of complexes, a fg R-module M can be viewed as

$$\begin{array}{lcl} & \overset{0}{} & \\ \qquad 0 \to & M & \to 0 \\ & \wr\wr & \text{quasi-iso} \\ \underbrace{\cdots \to P_1 \to P_0}_{\text{complex of fg projectives}} & & \to 0 \end{array}$$

projective resolution of M

$M \cong$ <u>bounded</u> complex of fg projectives $\iff$ $\mathrm{pd}_R(M) < \infty$

An object in $\mathcal{D}(R)$ is <u>small</u> if it is quasiiso to a bounded complex of fg projective R-modules.

$\mathcal{D}(R) \supseteq \mathcal{D}^f(R) :=$ complexes with fg homology

morally equivalent to

$\mathrm{Mod}(R) \supseteq \mathrm{mod}(R)$

<u>theorem</u> (Auslander–Buchsbaum, Serre)

- R is regular
- Every fg R-module has finite projdim.
- k has finite projdim
- Every object in $\mathcal{D}^f(R)$ is small

<u>Def</u> (Dwyer – Greenlees – Iyengar, 2005)

$X \in \mathcal{D}(R)$ is <u>proxy small</u> if

- We can finitely build a small P from M, by:
  - → shifting complexes
  - → taking direct summands
  - → if we can build two of $0 \to A \to B \to C \to 0$, we can build the third
- $\mathrm{Supp}\, X = \mathrm{Supp}\, P$

thm (Pollitz, 2018)

- R is a complete intersection

$\Updownarrow$

- Every object in $\mathcal{D}(R)$ is proxy small

(Dwyer – Greenlees – Iyengar proved $\Rightarrow$
(Letz, 2019): proxy smallness is a local property)

Consequence R ci $\Rightarrow$ Every fg R-module is proxy small.

Question How about the converse?

Remark k is always proxy small
(it builds the Koszul complex)

theorem (Briggs – G – Pollitz)
If R is equipresented, then — Every element in I/mI has the same m-adic order

R is a ci $\Leftrightarrow$ Every fg R-module is proxy small
(if $|k| = \infty$) $\Leftrightarrow$ Every $R \twoheadrightarrow$ artinian ci is proxy small

In fact, the proof is constructive

Strategy: ① Given ideals $J_1, \dots, J_n$ in Q st:

- $J_i \supseteq I$ • Q/J_i is ci • Q/J_i is proxy small over R

- $\bigcap_i \ker\left(I/\mathfrak{m}I \longrightarrow J/\mathfrak{m}J_i \right) = 0$

then R is ci

② If $f \in I \setminus \mathfrak{m}I$ has minimal $\mathfrak{m}$-adic order we can build a ci $J \supseteq I$ having $f \in J \setminus \mathfrak{m}J$

$$\Longleftrightarrow f \in \ker\left(I/\mathfrak{m}I \longrightarrow J/\mathfrak{m}J \right)$$

③ Need to be able to find enough of these f

Example 1) $R = \dfrac{k[\![x,y]\!]}{(x^2, xy)}$

$M = Q/(x^2, y)$ $\qquad\qquad M_2 = Q/(xy, x+y)$

one of these is not proxy small!

(Fun fact: none of these is proxy small)

What's behind all this? Support varieties!

M R-module $\longrightarrow$ cohomological support variety $V_R(M) \subseteq k^n$

Facts • R ci $\iff$ $V_R(R) = \emptyset$ (Pollitz, 2018)

• M fg proxy small R-module $\Rightarrow$ $V_R(R) \subseteq V_R(M)$

Lemma (BGP) J ci with $J \supseteq I$. Then

$$V_R(R/J) = \ker\left(I/\mathfrak{m}I \longrightarrow J/\mathfrak{m}J \right)$$

so if $\bigcap_i V_R(Q/J_i) \not\supseteq V_R(R)$, then some Q/J_i not proxy small

the real condition in our main theorem is

if $d = \min\{\text{order } f : f \in I, f \neq 0\}$, $n = \mu(I)$

$$\underbrace{\dim_k\left(\frac{\mathfrak{m}^{d+1} \cap I}{\mathfrak{m}I}\right)}_{\substack{\shortparallel \\ \#\text{ min gens of order} > \text{minimal} \\ \leq n-1}} < \underbrace{\dim_k\left(\operatorname{span} V_R(R)\right)}_{\leq n}$$

Can also do: Stanley–Reisner rings