Cohomological support varieties

joint with Ben Briggs and Josh Pollitz

Idea: complex of
R-modules $M \longmapsto$ variety $V_R(M)$
encodes homological properties of M

Cohomological support varieties were first defined by Avramov in 1989 (inspired by work of Quillen) when R is ci
David Jorgensen extended the definition to any ring R
More recently, Josh Pollitz has been developing the general theory
Many others have contributed to establishing the theory of support varieties or have used it in their work, including:
Avramov–Buchweitz, Burke–Walker, Iyengar, Bergh–Pevtsova

throughout $k = \overline{k}$ field

- $R = Q/I$ where $Q = k[x_1, \dots, x_d]$, I homogeneous
$\mathfrak{m} = (x_1, \dots, x_d)$

OR

- $(R, \mathfrak{m}, k)$ noetherian local ring
$\hat{R} = Q/I$ where $(Q, \mathfrak{m}, k)$ is a RLR

$I = (f_1, \dots, f_n) \subseteq \mathfrak{m}^2$ $\qquad \mu(I) = n$

Definition 1 $I/\mathfrak{m}I = kf_1 \oplus \cdots \oplus kf_n \cong k^n$

Given $a = (a_1, \dots, a_n) \in I/\mathfrak{m}I$, write $Q_a := Q/(a_1 f_1 + \cdots + a_n f_n)$

$V_R(M) := \{0 \neq a \in I/\mathfrak{m}I \mid \operatorname{pdim}_{Q_a}(M) = \infty\} \cup \{0\} \subseteq \mathbb{A}^n$

<u>Notes</u> $V_R(M)$ is a union of lines through the origin
$\Rightarrow$ we can define $V_R(M)$ as a <u>projective</u> variety instead

Examples

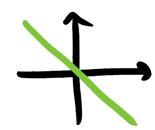
1) For all $a \in k^n$, $a \neq 0$, Q_a is singular $\Rightarrow pd_{Q_a}(k) = \infty$

$$\Rightarrow V_R(k) = k^n$$

2) $R = k[x,y]/(x^2, y^2)$ $\qquad\qquad$ $M = R/(x-y)$

$$pd_{Q_a}(M) = \begin{cases} \infty & a = (c,-c) \\ 1 & a \neq (c,-c) \end{cases}$$

$V_R(M) = \{(a_1, a_2) \mid a_1 + a_2 = 0\}$



<u>theorem</u> (Pollitz, 2020)

$V_R(R) = \{0\} \Leftrightarrow R$ is ci

$\Leftrightarrow I$ is generated by a regular sequence

$\Leftrightarrow n = \mu(I) = ht(I)$

Applications

- when R is ci, $V_R(M) \supseteq$ info on the betti/bass numbers of M, eg

(Avramov-Buchweitz, 2000) R ci, M, N fg R-modules

$$\mathrm{Ext}_R^{\gg 0}(M,N) = 0 \Leftrightarrow \mathrm{Ext}_R^{\gg 0}(N,M) = 0$$

(Briggs-McCormick-Pollitz, 2021) $\beta_{\gg 0}(M)$ quasipolynomial of period 2, same leading term

$\mathcal{D}(R) :=$ complexes of R-modules up to quasi-iso
$\cup\!|$
$\mathcal{D}^f(R) :=$ complexes of fg homology

module M $\Longleftrightarrow$ complex $0 \to \overset{\circ}{M} \to 0$

M <u>builds</u> N
$N \in \langle M \rangle$ $\Longleftrightarrow$ we can obtain N from M by finitely many ①, ②, ③
① shifts
② direct summands
③ if we can build 2 of $0 \to A \to B \to C \to 0$, then we can build the third

<u>Example</u> $M \in \langle R \rangle$ $\Longleftrightarrow$ $M \simeq 0 \to F_d \to \cdots \to F_0 \to 0$
$:=$ M is <u>small</u> finite rank free R-modules

<u>theorem</u> (Auslander – Buchsbaum, Serre) TFAE:
- R is regular
- Every fg R-module has $\operatorname{pdim}(M) < \infty$
- $\operatorname{pdim}(k) < \infty$
- Every $M \in \mathcal{D}^f(R)$ is small.

M is <u>proxy small</u> if $\exists P \in \langle M \rangle \cap \langle R \rangle$, $\operatorname{Supp} P = \operatorname{Supp} M$.

$\rightsquigarrow$ M builds a small complex (with the same support)

theorem (Pollitz, 2020) R ci $\Leftrightarrow$ Every $M \in D^f(R)$ is proxy small.

($\Rightarrow$) Dwyer – Greenlees – Iyengar

Note this gives a new solution to the localization problem

Question If R is not ci, is there a fg R-module that is not ps?

Bad guesses

① k is always proxy small
(it builds the Koszul complex on a minimal generating set for $\mathfrak{n}$)

② **theorem** (Briggs, 2021) R is ci $\Leftrightarrow$ $\mathrm{pdim}(I/I^2) < \infty$

I/I^2 might be proxy small when R is not ci
(If R has an embedded deformation, I/I^2 has a free summand)

theorem (Briggs – G – Pollitz) R not ci
Assume:
- R is equipresented (all generators of I have the same degree/order)

OR
- $\mathrm{span}_k(V_R(R)) = k^n$

then there exists fg R-module M that is <u>not</u> proxy small.
We can take M of the form

$$M \cong R/J \quad \text{artinian complete intersection}$$

Main tools (Facts by Pollitz)

- M proxy small $\Rightarrow V_R(R) \subseteq V_R(M)$ (easy, inspired by Hopkins-Neeman)
- R not ci $\Rightarrow V_R(R) \neq \{0\}$

Lemma (us) Can always find M with

$$V_R(M) \subseteq \text{hyperplane}$$ (determined by $f \in I/mI$ of minimal order/degree)

Question What may $V_R(R)$ look like?

theorem (Briggs-G-Pollitz)
If R is Golod but not ci, then $V_R(R) = k^n$
↑
Poincaré series of k maximal possible $\Leftrightarrow$ homotopy Lie algebra free

Given $Q \xrightarrow{\varphi} R$ local/graded map, Q any ring
Can extend the definition to $V_\varphi(R)$ in the obvious way

New definition of support $\hat{V}_\varphi(M)$
Given a free resolution $P \twoheadrightarrow M$ for M over Q
construct a 2-periodic complex

$$\cdots \xrightarrow{B} P_{\text{even}} \otimes k \xrightarrow{A} P_{\text{odd}} \otimes k \xrightarrow{B} P_{\text{even}} \otimes k \longrightarrow \cdots$$

where the maps depend on $a \in I/mI$

$\hat{V}_\varphi(M) = \{ a \in I/mI \mid \text{the complex is not exact} \}$

theorem (Briggs – G – Pollitz)

$$\{0\} \subseteq \hat{V}_\varphi(M) \subseteq V_\varphi(M) \subseteq k^n$$

ours ↑ classical

$=$ when $\operatorname{pdim} \varphi < \infty$

$\neq$ in general