

Linear Algebra

Math 314 Fall 2025

Today's poll code:
U6H5RJ

Lecture 8

Office hours

Mondays 5–6 pm
Wednesdays 2–3 pm
in Avery 339 (Dr. Grifo)

Tuesdays 11–noon
Thursdays 1–2 pm
in Avery 337 (Kara)

To do list:

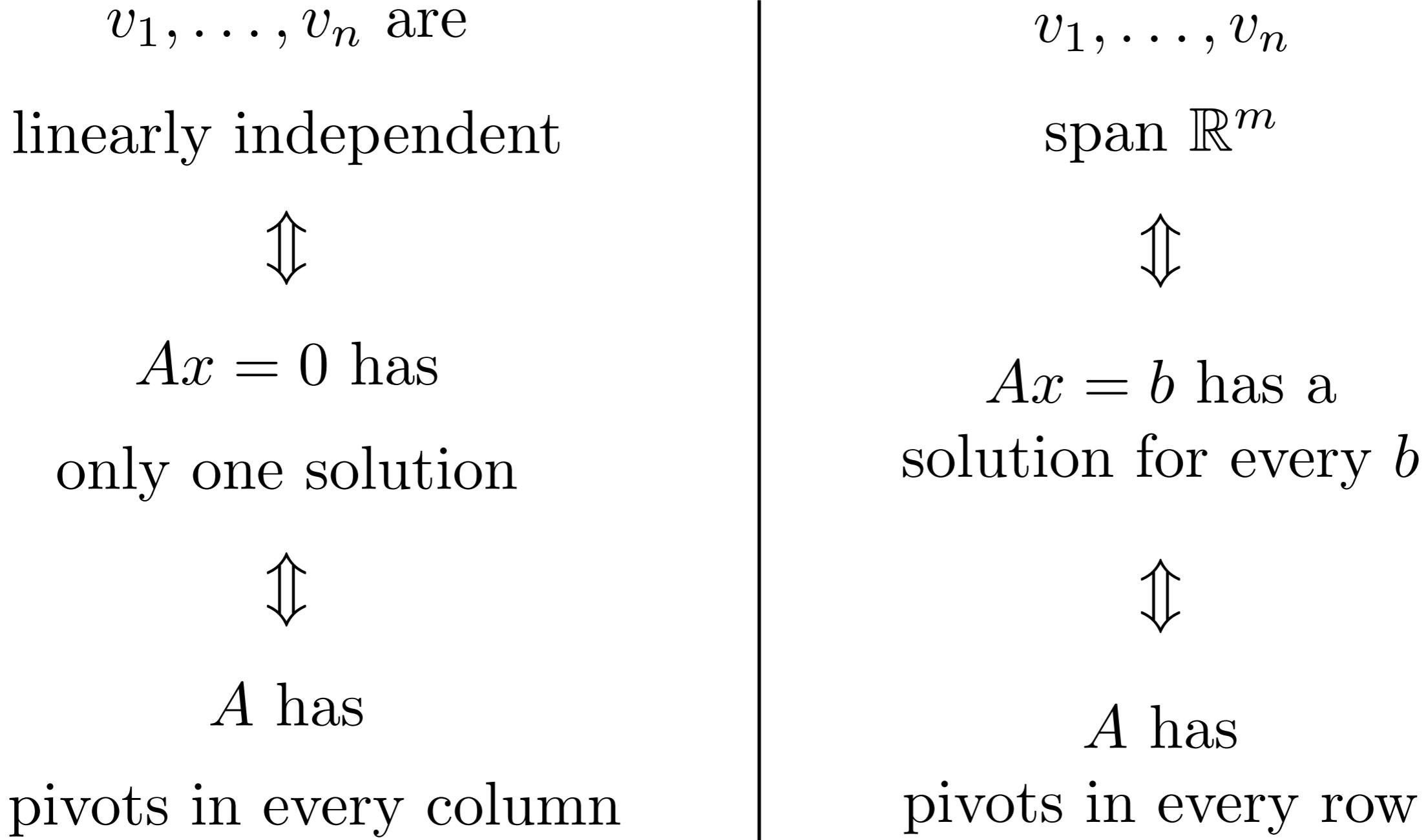
- Webwork 2.5 due tomorrow
- WebworK 2.6 due Friday

Quiz on Friday
on Lectures 8–9

Midterm 1
in 2 weeks

Quick Recap

$$A = [v_1 \quad \cdots \quad v_n]$$



Linear transformations

A function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

is an assignment

each vector
in \mathbb{R}^n



a vector
in \mathbb{R}^m

Matrix transformations: Any $m \times n$ matrix A determines a function

$$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

as follows: for each vector $x \in \mathbb{R}^n$,

$$T(x) = Ax.$$

Such a function is called a *matrix transformation*.

$$\mathbb{R}^{\#\text{columns}} \longrightarrow \mathbb{R}^{\#\text{rows}}$$

Two properties of matrix products:

- $A(v + w) = Av + Aw$
- $A(cv) = cAv$

So, if $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation then

- $T(v + w) = T(v) + T(w)$
- $T(cv) = cT(v)$

Any function with these properties is called a **linear transformation**.

linear transformations = matrix transformations



$$T(x) = Ax$$

for all x



$$T(v + w) = T(v) + T(w)$$

$$T(cv) = cT(v)$$

for all u, v
and all constants c

Given a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

its **standard matrix** is

$$A = [T(e_1) \quad \cdots \quad T(e_n)]$$

so

$$T(x) = Ax \text{ for all vectors } x$$

i th standard vector
in \mathbb{R}^n

$$\mathbf{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \xrightarrow{\text{position } i}$$

We've mostly shown examples of linear transformations from \mathbb{R}^2 to \mathbb{R}^2 , but they can go from \mathbb{R}^n to \mathbb{R}^m for any choices of m and n .

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For example, let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation associated to the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

T sends $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (first column of A)

T sends $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (second column of A)

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T sends $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (second column of A)

$\Rightarrow T$ moves xy -plane into the yz plane without stretching it in any way.

We can also define a linear transformation like $T : \mathbb{R}^7 \rightarrow \mathbb{R}^4$ by specifying some 4×7 matrix A , such as

$$A = \begin{bmatrix} 1 & -3 & 5 & 2 & 0 & 6 & 7 \\ 2 & \frac{3}{2} & 1 & 0 & 14 & -20 & \pi \\ 5 & e & 17 & 0 & 0 & 0 & 2 \\ 3 & 1 & 4 & 1 & 5 & 9 & 7 \end{bmatrix}.$$

Just don't ask me to visualize it!

Injective/surjective functions

A function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

is an assignment

each vector
in \mathbb{R}^n



a vector
in \mathbb{R}^m

A function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ has

domain \mathbb{R}^n  inputs

and

codomain \mathbb{R}^m  where the
outputs live

domain



$$\mathbb{R}^n$$

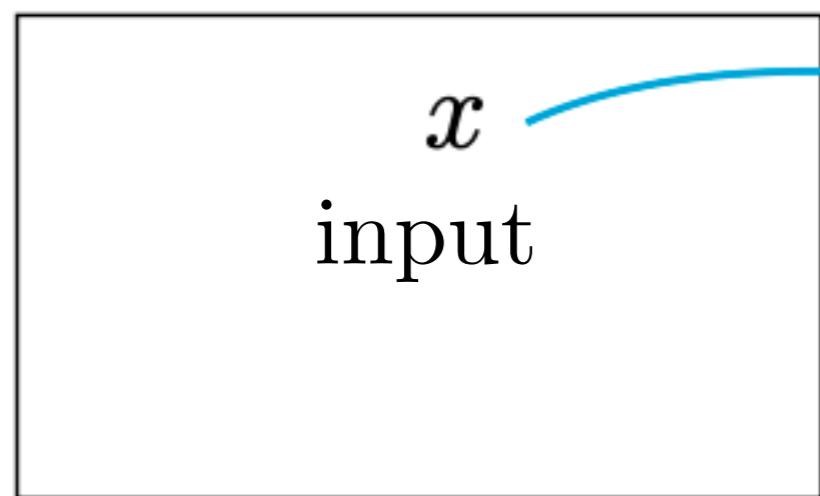
T

codomain



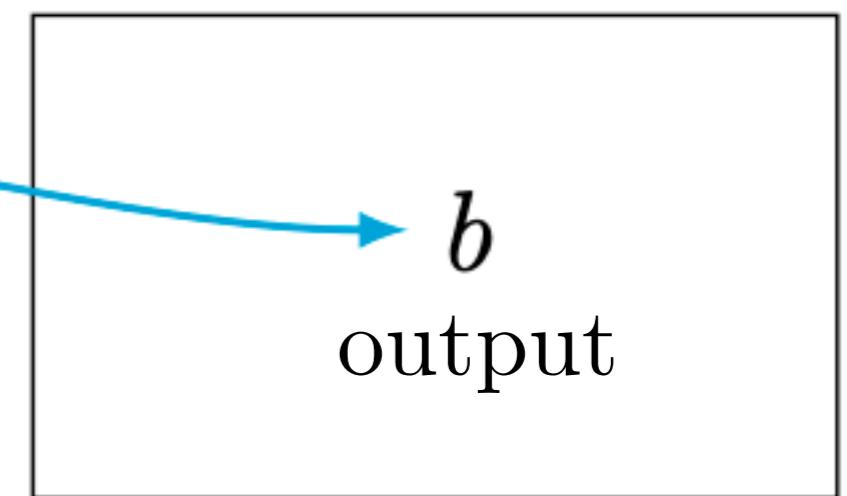
$$\mathbb{R}^m$$

domain



$$\mathbb{R}^n$$

codomain



$$\mathbb{R}^m$$

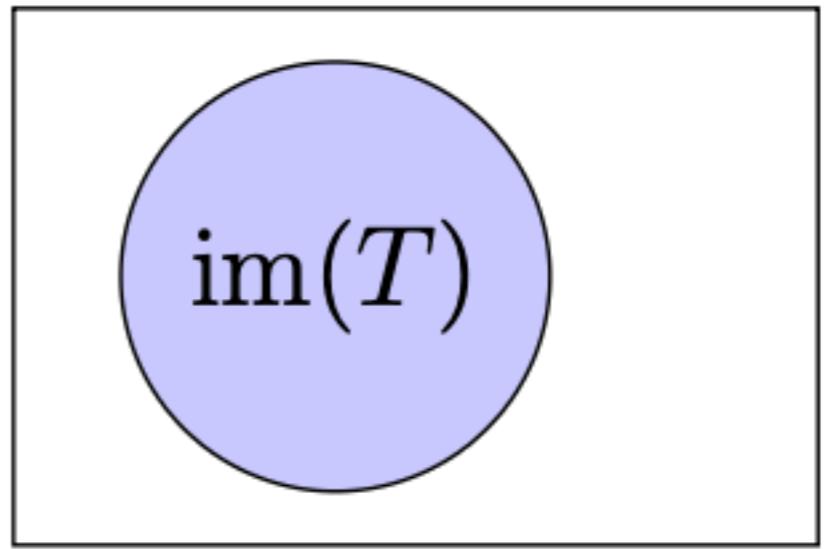
The **image** or **range** of a function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is

$$\text{im}(T) := \{T(x) \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

Informally: the set of actual outputs



$$\xrightarrow{T}$$


$$\mathbb{R}^n$$
$$\mathbb{R}^m$$

T is **surjective** or **onto** if

for every $b \in \mathbb{R}^m$ there exists *at least one* $x \in \mathbb{R}^n$

such that

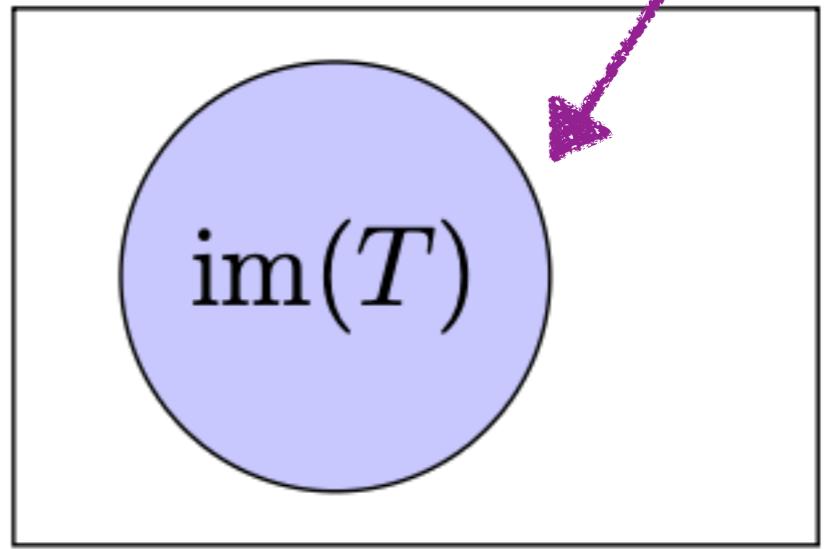
$$T(x) = b$$

Equivalently: $\text{im}(T) = \mathbb{R}^m$

Informally: everything in the codomain is an actual output.



T



\mathbb{R}^n

\mathbb{R}^m

not surjective

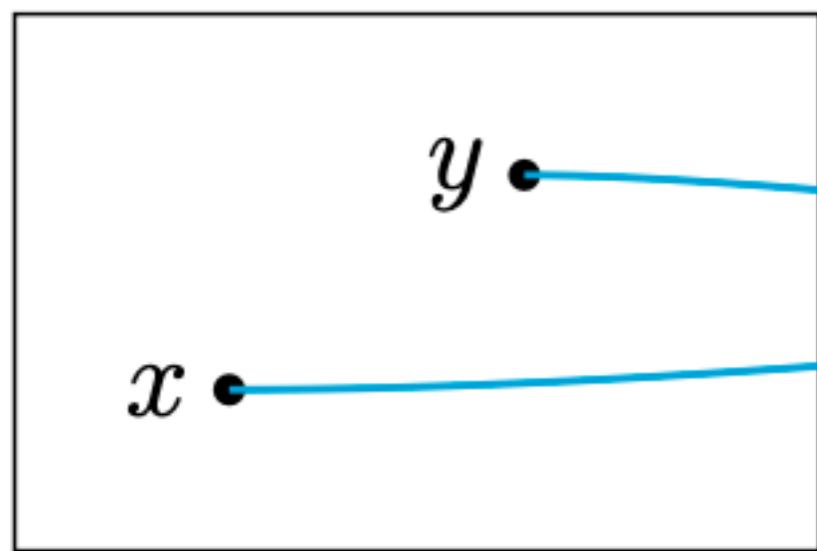
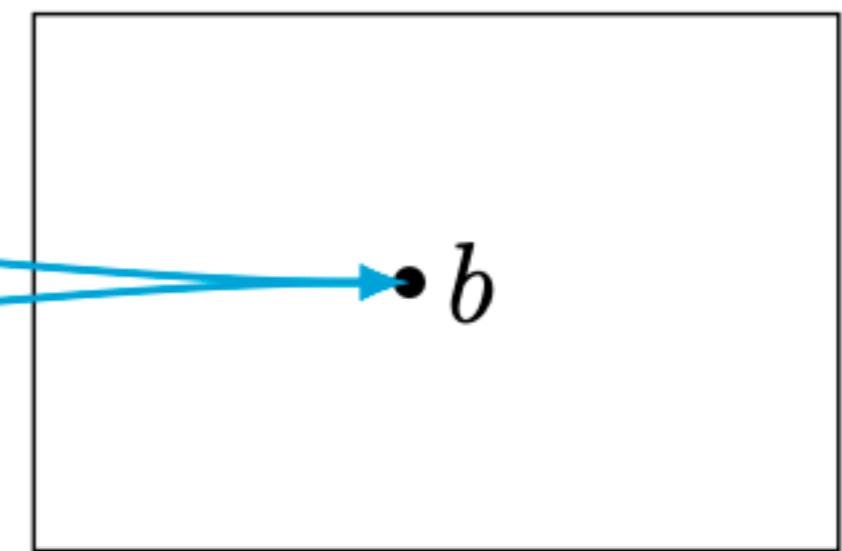
T is **injective** if

for each $b \in \mathbb{R}^m$ there exists *at most one* $x \in \mathbb{R}^n$

such that

$$T(x) = b$$

Equivalently: $T(x_1) = T(x_2) \implies x_1 = x_2.$


$$\mathbb{R}^n$$

$$\mathbb{R}^m$$

not injective

bijective = injective + surjective

Example: Consider the (nonlinear) function

$$T: \mathbb{R} \rightarrow \mathbb{R} \text{ given by } T(x) = x^2$$

Is it injective? Surjective? Bijective?

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Not injective:

$$T(1) = 1 = T(-1)$$

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Not surjective: the outputs are never negative!

$$\text{im}(T) = \{x \in \mathbb{R} \mid x \geq 0\}$$

For functions $f: \mathbb{R} \rightarrow \mathbb{R}$

f is **surjective**

if and only if

every horizontal line crosses the graph of f at least once.

f is **injective**

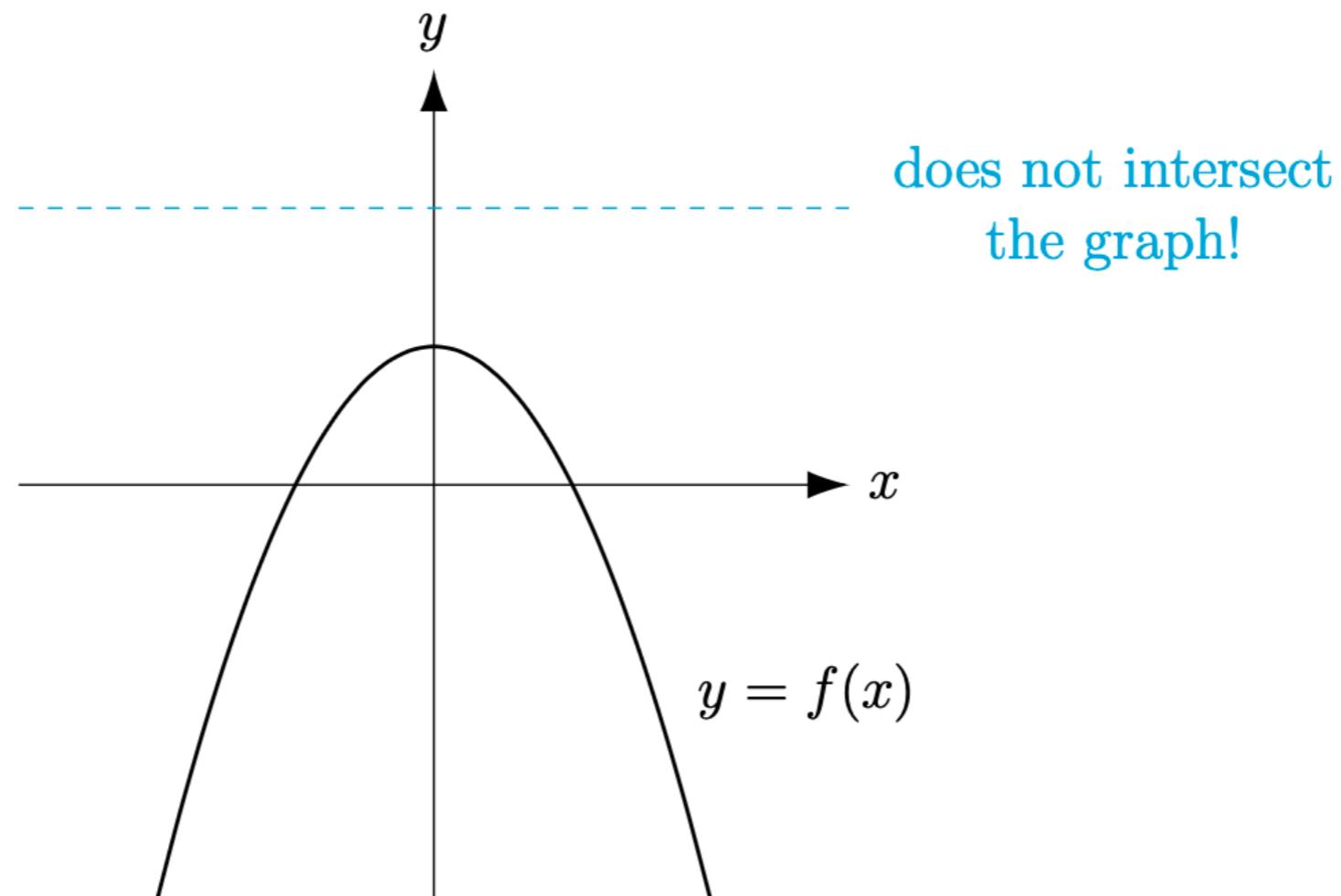
if and only if

every horizontal line crosses the graph of f at most once.

Surjective: f is surjective

if and only if

every horizontal line crosses the graph of f at least once.

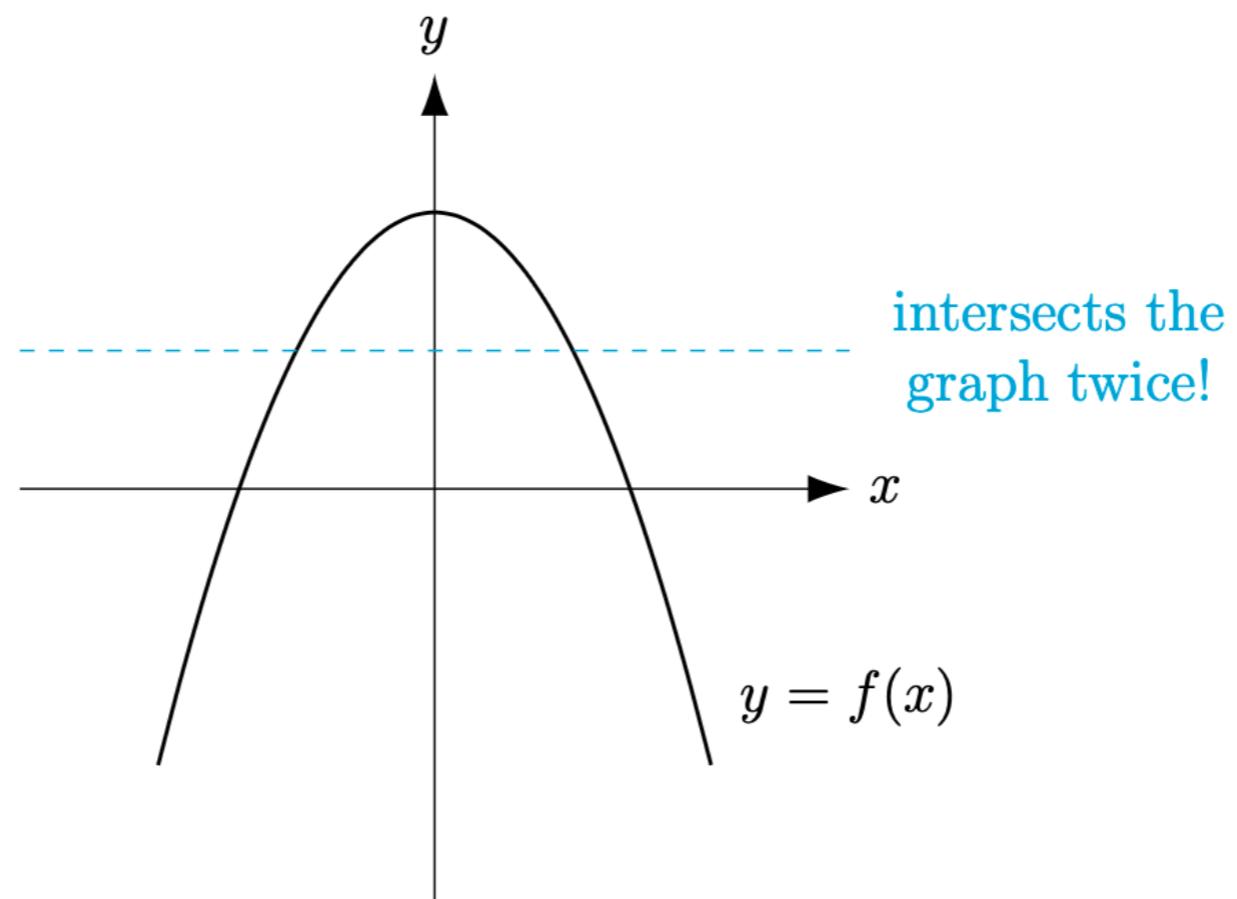


Example: f not surjective

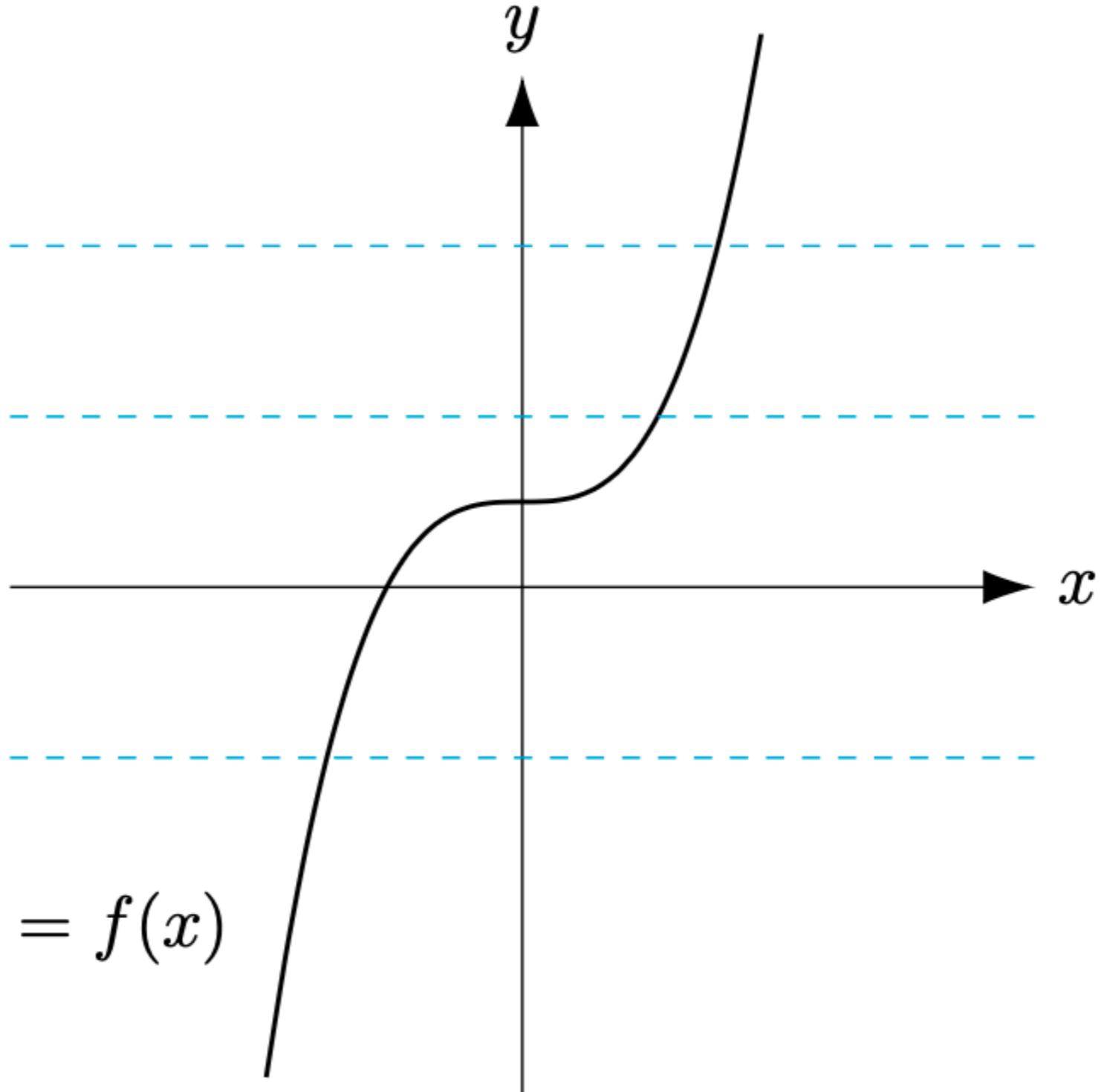
Injective: f is injective

if and only if

every horizontal line crosses the graph of f at most once.



Example: f not injective



Example: f bijective

The **kernel** of a linear transformation

$$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

is the set

$$\ker(T) := \{x \in \mathbb{R}^n \mid T(x) = 0\}.$$

Note: we always have $0 \in \ker(T)$

Theorem. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation.

T is injective if and only if the equation

$$T(x) = 0$$

has only the trivial solution $x = 0$.

Equivalently, T is injective if and only if $\ker(T) = \{0\}$.

Theorem. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation with standard matrix A .

T is surjective \iff the columns of A span \mathbb{R}^m

T is injective \iff the columns of A are linearly independent

Theorem. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation with standard matrix A .

$$T \text{ is surjective} \iff A \text{ has a pivot in every row}$$

$$T \text{ is injective} \iff A \text{ has a pivot in every column}$$

Example: The identity map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T \begin{pmatrix} [a] \\ [b] \end{pmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Example: The identity map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} a \\ b \end{bmatrix}$$

T is surjective: for any $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ we have $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$

T is injective: $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) \implies \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$

T is bijective

Example: The identity map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} a \\ b \end{bmatrix}$$

has standard matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

T is surjective: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has a pivot in every row

T is injective: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has a pivot in every column

T is bijective

Example:

The map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x \\ y \\ x + y \end{bmatrix}$$

Example:

The map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x + y \end{bmatrix}$$

T is not surjective: for example, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \notin \text{im}(T)$.

T is injective:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) \implies \begin{bmatrix} x \\ y \\ x + y \end{bmatrix} = \begin{bmatrix} u \\ v \\ u + v \end{bmatrix} \implies \begin{cases} x = u \\ y = v \end{cases}$$

Example:

The map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix}$$

has standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

T is not surjective:

T is injective:

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T is not surjective: there is no pivot on the last row

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T is not surjective: there is no pivot on the last row

T is injective: pivot in every column

Example:

The map $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

has standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Example:

The map $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by

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has standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

T is surjective: for any $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ we have $T \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

T is not injective: $T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = T \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

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T is surjective: pivot in the third column

T is not injective:

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The map $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

has standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

T is surjective: pivot in the third column

T is not injective: there is no pivot on the third column

Example:

The linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with standard matrix

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$



Example:

The linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with standard matrix

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \xrightarrow{\hspace{10em}} \quad \text{no pivot on this row}$$



↓
 T is not surjective

no pivot on this column $\implies T$ is not injective

Today's poll code:

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U6H5RJ

The linear transformation with standard matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
 is

- A. Injective and surjective
- B. Injective but not surjective
- C. Not injective but surjective
- D. Not injective and not surjective

Today's poll code:

U6H5RJ

The linear transformation with standard matrix

$$\begin{bmatrix} 1 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
 is

- A. Injective and surjective
- B. Injective but not surjective
- C. Not injective but surjective
- D. Not injective and not surjective

Today's poll code:

U6H5RJ

The linear
transformation with
standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{is}$$

A. Injective and
surjective

B. Injective but
not surjective

C. Not injective
but surjective

D. Not injective
and not surjective

Today's poll code:

U6H5RJ

The linear
transformation with
standard matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$
 is

- A. Injective and surjective
- B. Injective but not surjective
- C. Not injective but surjective
- D. Not injective and not surjective

Chapter 3

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (i, j)\text{th entry of } A = a_{ij}$$

multiplying a matrix by a scalar: $cA = [ca_{ij}]$

$$3 \cdot \begin{bmatrix} -1 & 1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -3 & 15 \end{bmatrix}$$

always defined

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (i, j)\text{th entry of } A = a_{ij}$$

The sum $A + B$ is defined if

A and B have the same size $\begin{pmatrix} \# \text{ rows of } A = \# \text{ rows of } B \\ \# \text{ columns of } A = \# \text{ columns of } B \end{pmatrix}$

sum of matrices:

$$A + B := [a_{ij} + b_{ij}]$$

$$\begin{bmatrix} 3 & -1 \\ 2 & -11 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3+1 & -1+3 \\ 2+4 & -11+5 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 6 & -6 \end{bmatrix}$$

product of matrices: The product AB is defined if

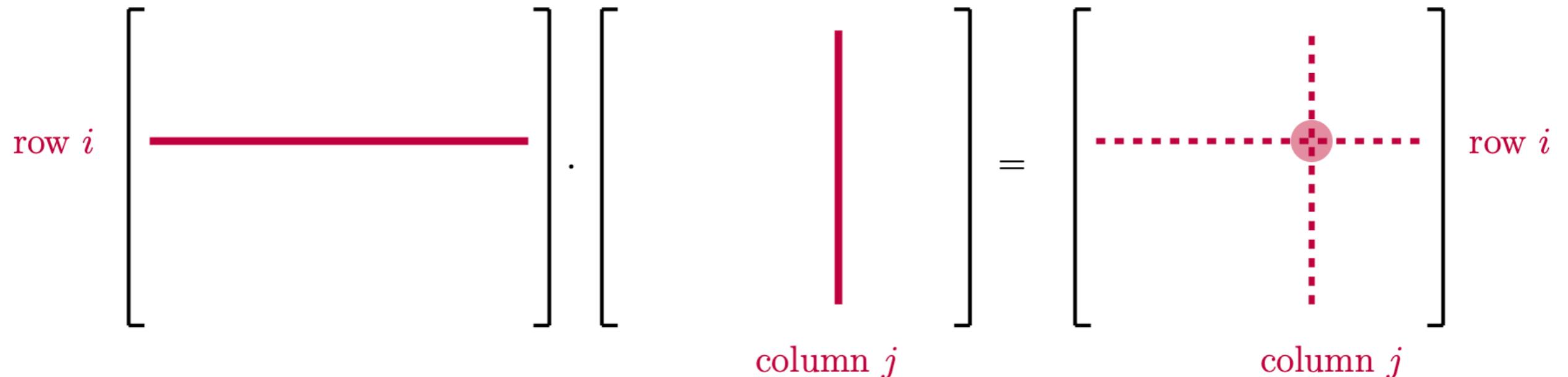
$$\# \text{ of columns of } A = \# \text{ of rows of } B$$

$$\# \text{ of rows of } AB = \# \text{ of rows of } A$$

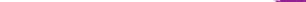
$$\# \text{ of columns of } AB = \# \text{ of columns of } B$$

In AB :

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$



A m × n matrix

 AB $m \times p$ matrix

B $n \times p$ matrix

Informally: $(m \times n) (n \times p) = m \times p$

In AB :

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$\text{row } i \begin{bmatrix} & & \\ & \text{---} & \\ & & \end{bmatrix} \cdot \begin{bmatrix} & & \\ & \text{---} & \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ & \text{---} & \\ & & \end{bmatrix} \text{row } i$$

column j

A $m \times n$ matrix

B $n \times p$ matrix

\longrightarrow AB $m \times p$ matrix

Informally: $(m \times \cancel{n}) (\cancel{n} \times p) = m \times p$

In AB :

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$\begin{matrix} \text{row } i \\ \left[\begin{array}{c} \text{---} \\ \cdot \\ \end{array} \right] \end{matrix} \cdot \begin{matrix} \text{column } j \\ \left[\begin{array}{c} | \\ \cdot \\ | \end{array} \right] \end{matrix} = \begin{matrix} \text{row } i \\ \left[\begin{array}{c} \dots \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \end{array} \right] \end{matrix}$$

column j

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (i, j)\text{th entry of } A = a_{ij}$$

product of matrices: The product AB is defined if

$$\# \text{ of columns of } A = \# \text{ of rows of } B$$

$$AB = [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p]$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ell} \end{bmatrix}$$

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

product of matrices: The product AB is defined if

$$\# \text{ of columns of } A = \# \text{ of rows of } B$$

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

i th row of A and j th column of B

eg:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ell} \end{bmatrix}$$

$$(1, 1)\text{th entry} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}$$

product of matrices: The product AB is defined if

$$\# \text{ of columns of } A = \# \text{ of rows of } B$$

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

i th row of A and j th column of B

eg:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$
$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ell} \end{bmatrix}$$

$$(m, 2)\text{th entry} = a_{m1}b_{12} + a_{m2}b_{22} + \cdots + a_{mn}b_{n2}$$

Example:

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + 3 \times 1 & 2 \times 3 + 3 \times (-2) & 2 \times 6 + 3 \times 3 \\ 1 \times 4 - 5 \times 1 & 1 \times 3 + (-5) \times (-2) & 1 \times 6 + (-5) \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + 3 \times 1 & 2 \times 3 + 3 \times (-2) & 2 \times 6 + 3 \times 3 \\ 1 \times 4 - 5 \times 1 & 1 \times 3 + (-5) \times (-2) & 1 \times 6 + (-5) \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

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Example:

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$

is undefined!

The diagram shows two matrices. The first matrix has three columns, and the second matrix has two columns. Cyan circles highlight the first two columns of the first matrix and the first column of the second matrix. Two purple arrows point downwards from the bottom of these circled columns towards the text below.

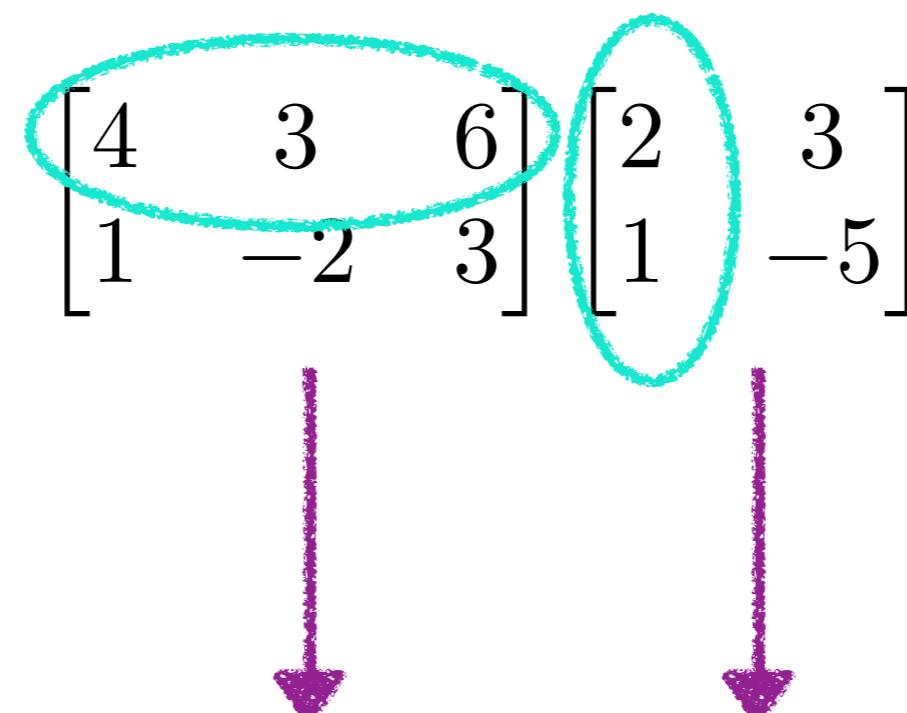
$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$

3 columns \neq 2 rows

Example:

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$

is undefined!

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$


3 columns \neq 2 rows

Theorem. Let A, B, C be matrices,

Assume that the products AB and BC make sense.

1. Associativity: $(AB)C = A(BC)$.
2. Left distributivity: $A(B + C) = AB + AC$.
3. Right distributivity: $(B + C)A = BA + CA$.
4. For any scalar α , $\alpha(AB) = (\alpha A)B = A(\alpha B)$.
5. Let I_m denote the $m \times m$ identity matrix.

Then $I_mA = A = AI_n$.

Warning: the order of the matrices in a product matters!

For example:

$$A = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 \cdot 2 + 1 \cdot 4 & 5 \cdot 0 + 1 \cdot 3 \\ (-1) \cdot 2 + 3 \cdot 4 & (-1) \cdot 0 + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ 10 & 9 \end{bmatrix}$$

≠

$$BA = \begin{bmatrix} 2 \cdot 5 + 0 \cdot (-1) & 2 \cdot 1 + 0 \cdot 3 \\ 4 \cdot 5 + 3 \cdot (-1) & 4 \cdot 1 + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 17 & 13 \end{bmatrix}$$

The zero $m \times n$ matrix is
the $m \times n$ matrix whose entries are all 0.

We denote it simply by 0, if the size is clear from context.

Warning! Cancellation fails.

$AB = AC$ does not imply that $B = C$.

For example:

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix} \neq 0 \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

$$AB = [1 \times 0 + 0 \times 1] = [0]$$

Warning! Cancellation fails.

$AB = AC$ does not imply that $B = C$.

For example:

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \neq \quad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$AB = [2 \times 2 + 1 \times 0] = [2] = [1 \times 1 + 1 \times 1] = AC$$

The zero $m \times n$ matrix is

the $m \times n$ matrix whose entries are all 0.

We denote it simply by 0, if the size is clear from context.

Warning! Cancellation fails.

$AB = 0$ does not imply that $A = 0$ or $B = 0$

For example:

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix} \neq 0 \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

$$AB = [1 \times 0 + 0 \times 1] = [0]$$

To do list:

- Webwork 2.5 due tomorrow
- WebworK 2.6 due Friday

**Midterm 1
in 2 weeks**

On Friday:

Quiz 4

**at the beginning
of the recitation
on Lectures 8—9**

Office hours

**Mondays 5–6 pm and Wednesdays 2–3 pm
in Avery 339 (Dr Grifo)**

**Tuesdays 11–noon and Thursdays 1–2 pm
in Avery 337 (Kara)**