Constructing non-proxy small modules
(joint with Benjamin Briggs and Josh Pollitz)

$(R, \mathfrak{n}, k)$ noetherian local ring

$Q/I \cong \hat{R}$ $\quad (Q, \mathfrak{m}, k)$ RLR, $\quad I \subseteq \mathfrak{n}^2$

theorem (Auslander–Buchsbaum, 1957, Serre, 1956) TFAE:
- R is regular
- Every fg R-module has $\operatorname{pdim}_R(M) < \infty$
- $\operatorname{pdim}_R k < \infty$

In the world of complexes:

$\mathcal{D}(R) :=$ complexes of R-modules up to quasi iso

$\mathcal{D}(R) \supseteq \mathcal{D}^f(R) :=$ complexes with fg homology

morally equivalent to

$\operatorname{Mod}(R) \supseteq \operatorname{mod}(R)$

M R-module $\iff$ complex $\quad 0 \to \overset{0}{M} \to 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wr\wr$ quasi iso

projective resolution of M $\quad \cdots \to P_1 \to P_0 \to 0$

$\operatorname{pd}_R(M) < \infty \iff M \simeq$ <u>bounded</u> complex of fg projectives (perfect)

An object of $\mathcal{D}(R)$ is <u>small</u> if it is quasiiso to a bounded complex of fg projective R-modules.

<u>Theorem</u> (Auslander–Buchsbaum–Serre) TFAE:

- R is regular
- Every fg R-module has $\mathrm{pdim}_R(M) < \infty$
- $\mathrm{pdim}_R\, k < \infty$
- Every object in $\mathcal{D}^f(R)$ is small.

<u>Def</u> (Dwyer–Greenlees–Iyengar, 2005)

$X \in \mathcal{D}(R)$ is <u>proxy small</u> if:

- We can finitely build a small P, by:

P is in $\mathrm{thick}(X)$
- → shifting complexes
- → taking direct summands
- → if we can build two of $0 \to A \to B \to C \to 0$ then we can build the third.

- $\mathrm{Supp}\, P = \mathrm{Supp}\, X$.

theorem (Pollitz, 2018)

- R is a complete intersection

$\Leftrightarrow$

- Every object in $D(R)$ is proxy small.

Remarks
- (Dwyer – Greenlees – Iyengar, 2005) proved $\Rightarrow$
- (Letz, 2019) proxy smallness is local

Consequence R ci $\Rightarrow$ Every fg R-module is proxy small

Question How about the converse?

Goal If R is not ci, construct a fg R-module that is not proxy small.

Remark k is always proxy small because k builds the koszul complex.

Sidenote Another motivation:

theorem (Gheibi – Jorgensen – Takahashi, 2019)
R ci $\Rightarrow$ every fg R-module has finite quasiprojective dim:
$P = (\cdots \to P_1 \to P_0 \to 0)$ projectives, $H_i(P) \cong M^{\oplus x_i}$

Q: Does the converse hold?

Fact (Ghosh – Jorgensen – Takahashi) finite gpd $\Rightarrow$ proxy small

So attaining our goal

Every fg R-mod has finite gpd

$\Downarrow$ GJT

Every fg R-mod is proxy small

$\Downarrow$ our goal

R is ci

Sidenote 2 theorem (Avramov - Gasharov - Peeva 1997)

R ci $\Leftrightarrow$ Every fg R-mod has finite CI-dim

$\Leftrightarrow$ k has finite CI-dim

Fact finite CI dimension $\Longrightarrow$ proxy small

$\not\Longleftarrow$

very much nope

Definition $R \xrightarrow{\text{flat}} R' \xleftarrow{\text{def}} Q$, $M' = M \otimes_R R'$

$$\text{CI-dim } M = \inf\{ \operatorname{pd}_Q M' - \operatorname{pd}_Q R' : R \to R' \leftarrow Q \text{ quasi-deformation} \}$$

theorem (Briggs – G – Pollitz)

If R is equipresented, then

Every $f \in I \setminus mI$ has the same m-adic order

R is a ci $\iff$ Every fg R-mod is proxy small

(if $|k| = \infty$) $\iff$ Every $R \twoheadrightarrow$ artinian ci is proxy small.

key technical tool: Cohomological Support

M R-module $\rightsquigarrow$ $V_R(M) \subseteq k^n$ $\qquad \mu(I) = n$

actual definition: $V_R = I/mI \cong k^n$

$$V_R(M) = \{ [f] \in V_R : \operatorname{pd}_{Q/f} \hat{M} = \infty \text{ or } [f] = 0 \}$$

(Avramov, Avramov–Buchweitz, Burke–Walker, Jorgensen, Pollitz)

Notes V_R intrinsic to R, does not depend on our choice of (Q, m, k).

Facts (Pollitz)

- $V_R(R) = 0 \iff R$ ci
- M proxy small $\implies V_R(R) \subseteq V_R(M)$

Goal Build fg R-modules with $V_R(M) \not\supseteq V_R(R)$ when R is not ci.

More precisely: We will construct fg R-mods for which:

1) Can compute $V_R(M_1), \dots, V_R(M_t)$

2) $V_R(M_1) \cap \dots \cap V_R(M_t) = \emptyset$ (or small)

$\Longrightarrow$ one of $M_1, \dots, M_t$ is not proxy small.

Recall: $\hat{R} \cong Q/I$, $I \subseteq m^2$, Q RLR, $\mu(I) = n$

to solve 1:

Lemma If $I \subseteq J$, J ci, then for $M = Q/J$ (R-mod)

$$V_R(M) = \ker\Big(\underset{\substack{\cong \\ k^n}}{I/mI} \longrightarrow \underset{\substack{\cong \\ k^{\operatorname{embdim} R}}}{J/mJ} \Big) \subseteq k^n$$

Note: If I and J have no common minimal generators,

$$V_R(M) = \ker(0\text{-map}) = k^n$$

$\Rightarrow$ useless.

to solve 2:

Lemma If $f \in I \setminus mI$ has minimal m-adic order, we can build ci $J \supseteq I$ with $f \in J \setminus mJ$.

$$\Leftrightarrow f \notin \ker(I/mI \longrightarrow J/mJ)$$

to solve the problem, need to find <u>enough</u> of these f.

When R is equipresented, every $f \in I \setminus \mathfrak{m}I$ has minimal order.

<u>Example</u> $\quad R = \dfrac{k[[x,y]]}{(x^2, xy)} \qquad Q = k[[x,y]]$

$M_1 = Q/(x^2, y)$

$$\ker \begin{pmatrix} k\cdot x^2 + k\cdot xy \longrightarrow kx^2 + ky \\ x^2 \longmapsto x^2 \\ xy \longmapsto 0 \end{pmatrix} = k\cdot xy$$

1-dim

$M_2 = Q/(xy, x+y)$

$$\ker \begin{pmatrix} kx^2 + kxy \longrightarrow kxy + k(x+y) \\ x^2 \longmapsto 0 \\ xy \longmapsto xy \end{pmatrix} = k\cdot x^2$$

1-dim

$\therefore \quad \ker A_1 \cap \ker A_2 = 0$

$\Longrightarrow$ one of M_1 or M_2 is not proxy small.

actually both are not proxy small.

So our constructions builds $\leq n$ modules, one of which is not proxy small. Can test it by computing $V_R(R)$ with Macaulay2.

Macaulay2 package ThickSubcategories (Dotz – G – Pollitz)

<u>theorem</u> (Briggs – G – Pollitz)

R ci $\Leftrightarrow$ Every fg R-module is proxy small

as long as:

$$d := \min\{\operatorname{order} f : f \in I,\ f \neq 0\}, \quad n = \mu(I)$$

$$\underbrace{\underbrace{\dim_k\left(\frac{m^{d+1} \cap I}{mI}\right)}_{\substack{\shortparallel \\ \#\text{ minimal generators of } I \\ \text{of order} > \text{minimal}}}}_{\leq n-1} < \underbrace{\dim_k\left(\operatorname{span} V_R(R)\right)}_{\leq n}$$

<u>theorem</u> Can do Stanley-Reisner rings.