Symbolic powers and the containment problem
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Theorem (Lasker, 1905, Noether, 1921)
Every ideal I in a noetherian ring R has a primary decomposition

$$I = Q_1 \cap \dots \cap Q_k \qquad Q_i \text{ primary}$$

Q primary if $ab \in Q,\ a \notin Q \Rightarrow b \in \sqrt{Q}$

Facts
- $\sqrt{\text{primary}} = \text{prime}$
- Can take primary components st $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$ and no Q_i can be deleted
- $\{\sqrt{Q_1}, \dots, \sqrt{Q_j}\} = \text{Ass}(I)$

Warning P prime $\not\Rightarrow P^n$ primary
but

$$P^n = \underbrace{Q_P}_{P\text{-primary}} \cap \underbrace{Q_1 \cap \dots \cap Q_k}_{\text{embedded components}}$$

the n-th symbolic power of P is

$$\begin{aligned} P^{(n)} &= P\text{-primary component of } P^n \\ &= P^n R_P \cap R \\ &= \{ f \in R : sf \in P^n, \text{ for some } s \notin P \} \end{aligned}$$

Theorem (Zariski - Nagata)

P prime ideal in $\mathbb{C}[x_1, \dots, x_d]$

$P^{(n)} = \{f \in R : f$ vanishes up to order n on the variety defined by $P\}$

$$= \bigcap_{x \in X} m_x^n = \bigcap_{m \supseteq P} m^n$$

$$= \{ f \in R : \underbrace{g \frac{\partial^{a_1 + \cdots + a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}}}_{\text{differential operator of order } a_1 + \cdots + a_d} (f) \in P \text{ for all } a_1 + \cdots + a_d \leq n-1, g \in R \}$$

Example C = curve parametrized by (t^3_x, t^4_y, t^5_z) in $\mathbb{C}^3$

$\Updownarrow$

$$P = (\underbrace{x^3 - yz}_{\substack{f \\ \deg 9}}, \underbrace{y^2 - xz}_{\substack{g \\ \deg 8}}, \underbrace{z^2 - x^2 y}_{\substack{h \\ \deg 10}})$$

$\deg x = 3$
$\deg y = 4$
$\deg z = 5$

$$P^2 = (f^2, g^2, h^2, fg, gh, fg)$$

these all vanish up to order 2 on C ... but there's more!

$$\underbrace{f^2 - gh}_{\substack{\in P^2 \\ \deg 18}} = \underbrace{x}_{\substack{\notin P \\ \deg 3}} \underset{\deg 15}{q} \Rightarrow q \in P^{(2)} \setminus P^2$$

Every element in P^2 has degree ≥ 16

$P^2 \subsetneq P^{(2)}$

Fun fact: $P^{(3)} \subseteq P^2$

I radical ideal $\iff I = \underbrace{P_1 \cap \cdots \cap P_k}_{\text{primes}} \iff I \sim$ variety in $\mathbb{C}^d$

the n-th symbolic power of I is

$I^{(n)}$ = collect all primary components of I^n corresponding to minimal primes

$$= \bigcap_{P \in \mathrm{Min}(I)} (I^n R_P \cap R)$$

$$= \{ f \in R : sf \in I^n \text{ for some } s \notin \text{any minimal prime of } I \}$$

<u>More generally</u>

1) $I^n \subseteq I^{(n)}$

2) $I^{(n+1)} \subseteq I^{(n)}$

3) In general, $I^n \neq I^{(n)}$
For special I, eg if $I = (\text{regular sequence})$, $I^n = I^{(n)}$ $\forall n \geq 1$

4) Zariski-Nagata holds more generally for any radical ideal, over any field.

<u>Difficult Questions</u>

1) When is $I^n = I^{(n)}$?

2) Give generators for $I^{(n)}$.

3) What degrees does $I^{(n)}$ live in?

Containment Problem When is $I^{(a)} \subseteq I^b$?

Theorem (Ein-Lazarsfeld-Smith 2001, Hochster-Huneke 2002, Ma-Schwede 2017)

Let I be a radical ideal in $k[x_1, \dots, x_d]$, k field or $\mathbb{Z}$
then $I^{(dn)} \subseteq I^n$ for all $n \geq 1$.

More generally, this holds for any regular ring of dim d.

Can replace d by the following:

h = big height of I = largest codimension of a component of I

so: $I^{(hn)} \subseteq I^n$ for all $n \geq 1$.

Question (Huneke, 2000) P prime of codim 2 in $k[x_1, \dots, x_d]$.
Is $P^{(3)} \subseteq P^2$?

Example $P \sim (t^3, t^4, t^5)$ has codim 2

theorem says $P^{(4)} \subseteq P^2$ (and $P^{(2n)} \subseteq P^n$)

But actually, $P^{(3)} \subseteq P^2$, which is better.

Theorem (-) $P \sim (t^a, t^b, t^c)$ over a field of char $\neq 3$
then $P^{(3)} \subseteq P^2$.

Conjecture (Harbourne, 2008) I radical ideal of big height h in $k[x_1, \dots, x_d]$

$$I^{(hn-h+1)} \subseteq I^n \text{ for all } n \geq 1.$$

Fact In char $p>0$, $I^{(hq-h+1)} \subseteq I^{[q]} \subseteq I^q$ for all $q = p^e$.

Example (Dumnicki, Szemberg, Tutaj-Gasińska 2013, Harbourne-Seceleanu, 2015)

$$I = (x(y^a - z^a), y(z^a - x^a), z(x^a - y^a)) \subseteq k[x, y, z]$$

char $k \neq 2$

$h = 2$ but $I^{(3)} \not\subseteq I^2$

not a prime ideal! But corresponding to points in $\mathbb{P}^2$ in very special position

Harbourne's Conjecture does hold for:

- General points in $\mathbb{P}^2$ (Harbourne-Huneke) and $\mathbb{P}^3$ (Dumnicki)
- squarefree monomial ideals
- determinantal ideals: $I = I_t(X_{n \times m})$ generic $n \times m$ matrix
- $R/I \cong k[t \times t$ minors of generic $n \times m$ matrix$]$
- $R/I \cong k[$all monomials of degree d in v variables$]$

(the last three: all strongly F-regular; all five: all F-pure)

Theorem (G-Huneke, 2017)

Let I be a radical ideal in a regular ring of char $p > 0$.

- If R/I is F-pure, then I verifies Harbourne's Conjecture
- If R/I is strongly F-regular, then I verifies Harbourne's Conjecture with $h-1$ replacing h.