Symbolic powers

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Algebra $\longleftrightarrow$ Geometry

$R = \mathbb{C}[x_1, \dots, x_d]$ $\qquad$ $\mathbb{C}^d$

Hilbert's basis theorem $\quad$ ideal $I = (f_1, \dots, f_n) \longmapsto X = \{x \in \mathbb{C}^d : f(x) = 0 \ \forall f \in I\}$

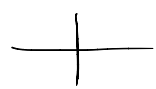
$= \{x \in \mathbb{C}^d : f_i(x) = 0 \ \forall i\}$

variety

$I = \{f \in R : f(x) = 0 \ \forall x \in I\}$ ideal $\longleftarrow\!\!| \quad$ variety $X \subseteq \mathbb{C}^d$

eg, $(x^2 - y) \longmapsto$

$(xy) \longmapsto$



$(x^2 y) \longmapsto$

Radical ideals $\longleftrightarrow$ Varieties

Prime ideals $\longleftrightarrow$ irreducible varieties

maximal ideals $\longleftrightarrow$ points

$m = (x_1 - a_1, \dots, x_d - a_d) \longleftrightarrow \{a = (a_1, \dots, a_d)\}$

Example curve parametrized by (t^3, t^4, t^5) in $\mathbb{C}^3$

$$\Updownarrow$$

prime ideal $P = (\underbrace{x^3 - yz}_{f}, \underbrace{y^2 - xz}_{g}, \underbrace{z^2 - x^2 y}_{h})$ in $\mathbb{C}[x,y,z]$

$$= \ker\left(\mathbb{C}[x,y,z] \longrightarrow \mathbb{C}[t^3, t^4, t^5] \subseteq \mathbb{C}[t]\right)$$

Algebra of P (or R/P) $\Leftrightarrow$ geometry of our curve

What polynomials vanish up to order n at P?

Given a radical ideal $I \leftrightarrow$ variety X
the polynomials that vanish up to order n along X are

$$\bigcap_{x \in X} m_x^n = \bigcap_{m \supseteq I} m^n$$

Problem: X is usually infinite - who wants to compute infinite intersections?

the n-th symbolic power of the prime ideal P is

$$P^{(n)} = \{ f \in R : f \text{ lives in } P^n \text{ locally} \}$$
$$= \{ f \in R : \frac{sf}{s} = \frac{f}{1} \in P^n \text{ in } R_P \}$$
$$= \{ f \in R : sf \in P^n \text{ for some } s \notin P \}$$

Theorem (Zariski–Nagata)

I radical ideal in $R = \mathbb{C}[x_1, \dots, x_d]$

$$\underbrace{I^{(n)}}_{\text{which we will define in full generality soon}} = \{ f \in R : f \text{ vanishes up to order } n \text{ on } I \}$$

$$= \bigcap_{m \supseteq I} m^n$$

differential operator version

$$= \{ f \in R : \partial(f) \in I \text{ for all } \partial \in D^n_{R|\mathbb{C}} \}$$

where $D^n_{R|\mathbb{C}}$ denotes differential operators of order $\leq n$

$$g \, \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{\partial^{a_d}}{\partial x_d^{a_d}} \, , \quad \underbrace{g \in R}_{\text{order } 0} \, , \quad a_1 + \cdots + a_d \leq n$$

Back to our example: $P = (\underbrace{x^3 - yz}_{\substack{f \\ \deg 9}}, \underbrace{y^2 - xz}_{\substack{g \\ \deg 8}}, \underbrace{z^2 - x^2 y}_{\substack{h \\ \deg 10}})$

$$\underbrace{f^2 - gh}_{\substack{\in P^2 \\ \deg 18}} = \boxed{\underbrace{x}_{\notin P}}_{\deg 3} \, \underbrace{q}_{\deg 15} \implies q \in P^{(2)}$$

$\deg x = 3$
$\deg y = 4$
$\deg z = 5$

So $P^2 \subsetneq P^{(2)}$

Actually, $P^{(2)} = (P^2, q)$

In general, given a radical ideal

$$I = P_1 \cap \dots \cap P_k \qquad P_i \text{ primes}$$

the n-th symbolic power of I is given by

$$I^{(n)} = \{f \in I : sf \in I^n \text{ for some } s \notin P_i \text{ for all } i\}$$

$$= \underbrace{\left(I^n R_{P_1} \cap R\right)}_{\substack{\text{primary component} \\ \text{of } I^n \text{ corresponding} \\ \text{to the minimal} \\ \text{prime } P_1}} \cap \dots \cap \left(I^n R_{P_k} \cap R\right)$$

$$= \bigcap_{P \in \mathrm{Min}(I)} \left(I^n R_P \cap R\right)$$

When P is prime, $P^{(n)} = P\text{-primary component of } P^n$

so $P^n = P^{(n)} \iff P^n$ is primary

In our example, $P \sim (t^3, t^4, t^5)$

can find it on M2

$$P^2 = \underbrace{P^{(2)}}_{\substack{P\text{-primary} \\ \text{minimal component}}} \cap \underbrace{Q_m}_{\substack{m\text{-primary} \\ \text{embedded component}}} \qquad m = (x,y,z)$$

Questions

1) When is $I^n = I^{(n)}$ for all / some n?

- If R is a Cohen-Macaulay ring and $I = $ (regular sequence), then $I^n = I^{(n)}$.

one way to think about this over $k[x_1, \dots, x_d]$:

height / codim of I $\leq$ minimal number of generators of I

$\uparrow$ if equality holds, then $I^n = I^{(n)}$

- Theorem (Huneke) over $k[x,y,z]$, if P is prime of height 2, $P^{(n)} = P^n$ for all n $\Leftrightarrow$ $P = (f,g)$

Example (Huckaba-Huneke; G-Huneke)

$P = \ker\left(k[x,y,z,w] \longrightarrow k[s^3, s^2t, st^2, t^3] \right)$

P has 3 minimal generators $>$ height 2.
$(4 - 2 = 2)$

Question Give explicit criteria on I that are equivalent to $I^n = I^{(n)}$ for all $n \geq 1$.

(Hochster, 64) gave a criterion, but it's very hard to apply (outside of special cases)

Open Question (Packing Problem)
I squarefree monomial ideal
$I^{(n)} = I^n$ for all $n \geq 1 \iff I$ is packed (combinatorial condition)

Open Question (Huneke) I ideal in a (regular) ring of dim d
If $I^{(n)} = I^n$ for all $n \leq \underbrace{\dim R}_{\text{or some variations on this}}$, is $I^n = I^{(n)}$ for all $n \geq 1$?

2) What degrees does $I^{(n)}$ live in?
or
What's is the smallest degree of a polynomial vanishing up to order n on a given variety?

Related Problem: Chudnovsky's Conjecture
Proposed lower bound for an asymptotic version of this.

$$\hat{\alpha}(I) = \lim_{n \to \infty} \frac{\alpha(I^{(n)})}{n} \overset{?}{\geq} \frac{\alpha(I) + N - 1}{N}$$

in $\mathbb{P}^N$

3) Give generators for $I^{(n)}$

Difficult computational questions

4) Compare I^n and $I^{(n)}$

One way to do this:

<u>Containment Problem</u> When is $I^{(a)} \subseteq I^b$?

In the regular case, nice bounds by ELS-HH-MS. In the Non-regular case, lots of work by Swanson, Huneke-Katz, Huneke-Katz-Validashti, Walker, Carvajal-Rojas-Smolkin, G-Ma-Schwede...

5) Characterize $I^{(n)}$

When are they Cohen-Macaulay? Gorenstein? Etc.