

last time:

$\text{Hom}_R(-, M)$

and

are additive left exact functors

$\text{Hom}_R(M, -)$

meaning that for every short exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

of R -modules, the sequences

$$0 \rightarrow \text{Hom}_R(M, A) \rightarrow \text{Hom}_R(M, B) \rightarrow \text{Hom}_R(M, C)$$

and

$$0 \rightarrow \text{Hom}_R(C, M) \rightarrow \text{Hom}_R(B, M) \rightarrow \text{Hom}_R(A, M)$$

are exact.

However, $\text{Hom}_R(N, -)$ and $\text{Hom}_R(-, M)$ are not necessarily exact.

tensor products

M, N R -modules

A function $f: M \times N \rightarrow L$ is R -bilinear if:

- $f(m+m', n) = f(m, n) + f(m', n)$ for all $m, m' \in M, n \in N$
- $f(m, n+n') = f(m, n) + f(m, n')$ for all $m \in M, n, n' \in N$
- $f(rm, n) = f(m, rn) = rf(m, n)$ for all $r \in R, m \in M, n \in N$

Example the product on R is an R -bilinear function $R \times R \rightarrow R$

Def M, N R -modules

the tensor product $M \otimes_R N$ is the R -module $M \otimes_R N$ together with an R -bilinear map $\tau: M \times N \rightarrow M \otimes_R N$ satisfying the following universal property:

for every R -bilinear function $f: M \times N \rightarrow A$, there exists a unique R -module homomorphism $\hat{f}: M \otimes_R N \rightarrow A$ such that

$$\begin{array}{ccc} M \otimes_R N & \xrightarrow{\quad \hat{f} \quad} & A \\ \tau \uparrow & \searrow & \\ M \times N & \xrightarrow{\quad f \quad} & A \end{array}$$

Commutes

this universal property is encoded in the representable functor

$\mathcal{B}\text{lin}(M \times N; -) : R\text{-mod} \rightarrow \text{Set}$

$L \mapsto R\text{-bilinear maps } M \times N \rightarrow L$

$\begin{matrix} A \\ f \downarrow \\ B \end{matrix} \mapsto \text{postcomposition with } f$

via

$$\tau \in \mathcal{B}\text{lin}(M \times N, M \otimes_R N)$$

Thm Given any R -modules M, N , $M \otimes_R N$ exists

Sketch

$F :=$ free module on $M \times N$

$$S := \left(\begin{array}{c|c} (m, n+n') - (m, n) - (m, n') \\ (m+m', n) - (m, n) - (m', n) \\ (xm, n) - x(m, n) \\ (m, xn) - x(m, n) \end{array} \right) \quad \begin{array}{l} m, m' \in M \\ n, n' \in N \\ x \in R \end{array}$$

$$M \otimes_R N := F/S$$

$$\begin{aligned} M \times N &\xrightarrow{\tau} F/S \\ (m, n) &\mapsto m \otimes n \end{aligned}$$

$$m \otimes n := (m, n) + S \text{ in } F/S$$

Check: these satisfy our universal property □

Def A simple tensor in $M \otimes_R N$ is an element of the form $m \otimes n$.

The simple tensors generate $M \otimes_R N$, but they are not all the elements.

so every element in $M \otimes_R N$ is of the form $\sum_{i=1}^k m_i \otimes n_i$

such expressions are not unique.

Warning Sometimes $M \otimes_R N$ are unexpectedly 0

- to define maps out of $M \otimes_R N$, use the universal property.
- to show $M \otimes_R N \neq 0$, give a nonzero R-bilinear map $M \times N \rightarrow A$.
- two R-module maps $M \otimes_R N \rightarrow A$ are the same if they agree on simple tensors.

Lemma the tensor product is unique up to iso.

Lemma $A \xrightarrow{f} C, B \xrightarrow{g} D$ R-module maps

there exists a unique R-module homomorphism

$$f \otimes g : A \otimes_R B \longrightarrow C \otimes_R D$$
$$a \otimes b \longmapsto f(a) \otimes g(b)$$

Lemma $(f_2 \otimes g_2)(f_1 \otimes g_1) = (f_2 f_1) \otimes (g_2 g_1)$

Lemma $f, g \text{ iso} \Rightarrow f \otimes g \text{ iso}$

Theorem

$$\begin{array}{ccc} M \otimes_R - : R\text{-mod} & \longrightarrow & R\text{-mod} \\ A & & M \otimes_R A \\ f \downarrow & \longmapsto & \downarrow 1 \otimes f \\ B & & M \otimes_R B \end{array} \quad \text{and} \quad \begin{array}{ccc} - \otimes_R M : R\text{-mod} & \longrightarrow & R\text{-mod} \\ A & & A \otimes_R M \\ f \downarrow & \longmapsto & \downarrow f \otimes 1 \\ B & & B \otimes_R M \end{array}$$

are additive covariant functors.

Lemma there is a natural isomorphism $M \otimes_R N \cong N \otimes_R M$

Lemma $(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C)$

Thm there is a natural isomorphism

$$M \otimes_R (\bigoplus_i N_i) \xrightarrow{\cong} \bigoplus_{i \in I} (M \otimes_R N_i)$$

on both M and the family $\{N_i\}_{i \in I}$.

Lemma there is a natural isomorphism between

$R \otimes_R -$ and $\text{Id}: R\text{-mod} \rightarrow R\text{-mod}$.

Sketch $R \times M \longrightarrow M$ is R -bilinear
 $(r, m) \longmapsto rm$ and surjective

\Rightarrow induces homomorphism $R \otimes_R M \xrightarrow{\varphi_M} M$

$M \xrightarrow{f_M} R \otimes_R M$ is an inverse to φ_M
 $m \longmapsto 1 \otimes m$

$$R \otimes_R M \xrightarrow{\varphi_M} M$$

the diagram $1 \otimes g \downarrow$ $\downarrow g$ commutes

$$R \otimes_R N \longleftrightarrow N$$

$$\varphi_N$$

□

Examples

$$\textcircled{1} \quad \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q} = 0$$

$$a \otimes p = a \otimes \frac{ap}{2} = \underbrace{(2a)}_{=0} \otimes \frac{p}{2} = 0$$

all simple tensor 0 \Rightarrow tensor product is 0

$$\textcircled{2} \quad \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0 \quad \text{because all simple tensors are 0:}$$

$$\begin{aligned} & \left(\frac{p}{q} + \mathbb{Z} \right) \otimes \left(\frac{a}{b} + \mathbb{Z} \right) \\ &= \left(\frac{bp}{bq} + \mathbb{Z} \right) \otimes \left(\frac{a}{b} + \mathbb{Z} \right) \\ &= \left(\frac{p}{bq} + \mathbb{Z} \right) \otimes b \left(\frac{a}{b} + \mathbb{Z} \right) \\ &= \left(\frac{p}{bq} + \mathbb{Z} \right) \otimes \underbrace{\left(a + \mathbb{Z} \right)}_{=0} = 0 \end{aligned}$$

$$\textcircled{3} \quad 2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \neq 0 \quad \text{because}$$

$$\begin{aligned} 2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} &\longrightarrow \mathbb{Z}/2\mathbb{Z} \\ (a, b) &\longmapsto \frac{ab}{2} \end{aligned}$$

is \mathbb{Z} -bilinear, so it induces an \mathbb{R} -module homomorphism

$$2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \mathbb{Z}/2\mathbb{Z}$$

$$\text{and } f(2 \otimes 1) = \frac{2 \cdot 1}{2} = 1 \neq 0$$

Theorem $M \otimes_R -$ is right exact, meaning that for every seq

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$$

we get an exact sequence

$$M \otimes_R A \xrightarrow{1 \otimes i} M \otimes_R B \xrightarrow{1 \otimes p} M \otimes_R C \rightarrow 0$$

Proof ① $(1 \otimes p)(1 \otimes i) = 0$

$$(1 \otimes p)(1 \otimes i) = 1 \otimes \underbrace{pi}_{=0} = 0$$

② $1 \otimes p$ is surjective

Given any $m_1 \otimes c_1 + \dots + m_n \otimes c_n \in M \otimes_R C$,

pick $b_1, \dots, b_n \in B$ such that $p(b_i) = c_i$ (p surjective)

then $(1 \otimes p)(m_1 \otimes b_1 + \dots + m_n \otimes b_n) = m_1 \otimes c_1 + \dots + m_n \otimes c_n$.

③ $\ker(1 \otimes p) = \text{im}(1 \otimes p)$

let $I := \text{im}(1 \otimes p) \subseteq \ker(1 \otimes p)$.

$1 \otimes p$ induces a map $M \otimes_R B / I \xrightarrow{q} M \otimes_R C$

Consider the canonical projection $\pi: M \otimes_R B \rightarrow M \otimes_R B / I$

By construction: $q \circ \pi = 1 \otimes p$

$$\begin{array}{ccc} M \times C & \xrightarrow{f} & M \otimes_R B / I \\ (m, c) & \longmapsto & m \otimes b \end{array}$$

where $p(b) = c$

- f is R -bilinear (exercise)

- f is well-defined:

If $p(b) = p(b') = c$, then

$$b - b' \in \ker p = \text{im } i \Rightarrow m \otimes (b - b') \in \text{im } (1 \otimes i) = I$$

$$\text{so } m \otimes b - m \otimes b' \in I$$

$$\Rightarrow f \text{ induces } R\text{-module map } M \otimes_R C \xrightarrow{\hat{f}} M \otimes_R B / I$$

Claim $\hat{f}q = \text{id}_{M \otimes_R B / I}$

$$\hat{f}q \left(\sum_{i=1}^n m_i \otimes b_i \right) = \hat{f} \left(\sum_{i=1}^n m_i \otimes p(b_i) \right) = \sum_{i=1}^n m_i \otimes b_i$$

Conclusion: q is injective

$$q \pi = 1 \otimes p$$

q injective

$$\Rightarrow \ker(1 \otimes p) = \ker(q \pi) = \ker \pi = \text{im}(1 \otimes i)$$

But tensor is not necessarily exact

Example $0 \rightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \rightarrow 0$

Apply $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} -$:

$$\underbrace{\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}}_{\cong \mathbb{Z}/2} \xrightarrow{1 \otimes i} \underbrace{\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q}}_{=0} \rightarrow \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

Hom-tensor adjunction there is a natural isomorphism

$$\begin{aligned} \text{Hom}_R(M \otimes_R N, P) &\cong \text{Hom}_R(M, \text{Hom}_R(N, P)) \\ f &\longmapsto (m \mapsto (n \mapsto f(m \otimes n))) \\ m \otimes n &\mapsto g(m)(n) \quad \longleftrightarrow \quad g \end{aligned}$$

What does this mean?

It says $(- \otimes_R M, \text{Hom}_R(M, -))$ is an adjoint pair.

two covariant functors $\mathcal{C} \rightleftarrows \mathcal{D}$ form an adjoint pair if

$$\text{Hom}_{\mathcal{D}}(F(C), D) \cong \text{Hom}_{\mathcal{C}}(C, G(D)) \quad \text{naturally on } C, D.$$

we say F is a left adjoint of G , G is a right adjoint of F

"the slogan is "adjoint functors arise everywhere"" Mac Lane

Idea An adjoint functor is a way of giving the most efficient solution to a problem in a functional way

Example

Problem Given a set I , what is the most efficient functional way of getting an R -module out of I ?

Solution take the free R -module on I , $\bigoplus_I R$.

Fact $R\text{-mod} \begin{array}{c} \xrightarrow{\text{Forget}} \\ \xleftarrow{\text{Free}} \end{array} \text{Set}$

(Free, Forget) form an adjoint pair.

thm $R \xrightarrow{f} S$ ring homomorphism

$\Leftrightarrow S$ is an R -algebra

$\Leftrightarrow S$ is a ring and an R -module via $r \cdot s = f(r) \cdot s$

$S \otimes_R - : R\text{-mod} \rightarrow S\text{-mod}$ is an additive functor.

Def the functor above is called Extension of scalars

$$f_! : R\text{-mod} \rightarrow S\text{-mod}$$

Def the reduction of scalars functor

$f^*: S\text{-mod} \rightarrow R\text{-mod}$

$$\begin{array}{ccc} M & \xrightarrow{\quad} & \text{view } M \text{ as an } R\text{-mod} \\ & & x \cdot M := f(x) \cdot M \end{array}$$

$$\begin{array}{ccc} M & & \\ g \downarrow & \xrightarrow{\quad} & \text{view } g \text{ as a homomorphism} \\ N & & \text{of } R\text{-modules} \end{array}$$

Theorem (Extension of scalars, Reduction of scalars)
form an adjoint pair

Idea Reduction of scalars solves the problem

Make an R -module out of a given S -module

Extension of scalars solves the problem

Make an S -module out of a given R -module

this is

thm If M is an R -module, and N is an S -module,

$$\text{Hom}_R(M, f^*N) \cong \text{Hom}_S(M \otimes_R S, N)$$

211

$$\text{Hom}_R(M, \text{Hom}_S(S, N))$$