Linear Algebra

Today's poll code:

Math 314 Fall 2025

Lecture 15

To do list:

- Webwork 4.1 due tomorrow
- Lab 1 due on Friday
- Webwork 4.2 due next Wednesday

Quiz on Friday

on vector spaces

Office hours

Mondays 5–6 pm
Wednesdays 2–3 pm
in Avery 339 (Dr. Grifo)

Tuesdays 11–noon
Thursdays 1–2 pm
in Avery 337 (Kara)

Lab 1 due Friday!

You will turn in a pdf on gradescope
Your pdf will include the code, your comments, and the output

For example, some of the outputs will be pictures!

Recap:
Bases and dimension

A **basis**[*] for a vector space V is

a spanning set of linearly independent vectors.

The **dimension** of a vector space is

the number of vectors in a basis.

Notation: $\dim(V)$

[*] The plural of basis is bases.

$\dim(\mathbb{R}^n) = n$ Basis: $e_1, \dots, e_n$

$\dim(M_{m\times n}) = mn$

all $m \times n$ matrices

$\dim(\{0\}) = 0$

$\dim(\mathbb{P}_n) = n + 1$

polynomials $a_0 + a_1 t + \cdots + a_n t^n$

Think of $\mathbb{P}_n$ as $\mathbb{R}^{n+1}$

$$a_0 + a_1 t + \cdots + a_n t^n \longrightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Column space

Null space

A $m \times n$ matrix

The **column space** of A is

the span of the columns of A

$$A = [a_1 \cdots a_n] \longrightarrow \mathrm{col}(A) = \mathrm{span}\left(\{a_1, \ldots, a_n\}\right)$$

$\mathrm{col}(A)$ is a subspace of $\mathbb{R}^m$

Note: $\mathrm{col}(A) = \mathrm{im}(T)$, where T is the linear transformation $T(x) = Ax$

The image of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$

is a subspace of $\mathbb{R}^m$

(and we also call it the column space of the standard matrix for T)

A $m \times n$ matrix

$$\mathrm{col}(A) = \text{ the span of the columns of } A$$

Example: $$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{col}(A) = \mathrm{span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}\right)$$

A $m \times n$ matrix

The **null space** of A is

$$\mathrm{Nul}(A) = \{x \text{ in } \mathbb{R}^n : Ax = 0\}$$

$\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$

The linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ given by

$$T(x) = Ax$$

has $\mathrm{Nul}(A) = \ker(T)$

The kernel of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$

is a subspace of $\mathbb{R}^n$

(and we also call it the null space of the standard matrix for T)

Theorem.

The set of solutions to a homogeneous linear system is a vector space.

Why? The solution set of $Ax = 0$ is the same as the null space of A.

Example:

$$A = \begin{bmatrix} 1 & 4 & -5 & 2 \\ 0 & 2 & -4 & 0 \\ -1 & 1 & -5 & 2 \\ 3 & -1 & 11 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

general solution to $Ax = 0$ is

$$x = t \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{where } t \text{ is any scalar}$$

so $$\mathrm{Nul}(A) = \mathrm{span}\left(\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\} \right).$$

Example:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

general solution to $Ax = 0$ is

$$x = a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, a, b, c \text{ any scalars}$$

$$\text{so} \quad \mathrm{Nul}(A) = \mathrm{span}\left(\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$$

In general:

u in $\mathrm{Nul}(A)$ means $Aw = 0$

Equivalently, w is a solution to the system $Ax = 0$.

u in $\mathrm{col}(A)$ means u is a linear combination of the columns of A

Equivalently, the system $Ax = u$ has a solution.

Example: $$A = \begin{bmatrix} 1 & 4 & -5 & 2 \\ 0 & 2 & -4 & 0 \\ -1 & 1 & -5 & 2 \\ 3 & -1 & 11 & 1 \end{bmatrix}$$

$w = \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ is in $\mathrm{Nul}(A)$ means $Aw = 0$

Equivalently, w is a solution to the system $Ax = 0$.

$u = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 2 \end{bmatrix}$ is in $\mathrm{col}(A)$ means u is a linear combination of the columns of A

Equivalently, the system $Ax = u$ has a solution.

Finding a basis for the column space and the null space

To find a basis for the column space of A:

Step 1: Find the RREF of A.

Step 2: Collect the pivot columns of A.

Warning: Make sure to use the pivot columns of A, not of its reduced echelon form!

$$\dim(\mathrm{col}(A)) = \#\text{ of pivots in } A$$

$$A = \begin{bmatrix} 1 & 4 & -5 & 2 \\ 0 & 2 & -4 & 0 \\ -1 & 1 & -5 & 2 \\ 3 & -1 & 11 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 & -5 & 2 \\ 0 & 2 & -4 & 0 \\ -1 & 1 & -5 & 2 \\ 3 & -1 & 11 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dim(\mathrm{col}(A)) = 3$$

$$A = \begin{bmatrix} 1 & 4 & -5 & 2 \\ 0 & 2 & -4 & 0 \\ -1 & 1 & -5 & 2 \\ 3 & -1 & 11 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\dim(\mathrm{col}(A)) = 3$$

Basis for $\mathrm{col}(A)$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

To find a basis for the null space of A:

Step 1: Find the general solution for $Ax = 0$.

Step 2: Write the solution in parametric vector form.

Use one vector for each free variable.

Step 2: The vectors we used form a basis for $\mathrm{Nul}(A)$.

$$\dim(\mathrm{Nul}(A)) = \#\text{ free variables}$$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution to the homogeneous system $Ax = 0$ is

$$x = a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

a, b, c

any real numbers

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution to the homogeneous system $Ax = 0$ is

$$x = a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \qquad a, b, c \text{ any real numbers}$$

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } \mathrm{Nul}(A)$$

$$\dim(\mathrm{col}(A)) = \#\text{ of pivots in } A = \mathrm{rank}(A)$$

$$\dim(\mathrm{Nul}(A)) = \#\text{ free variables}$$

$$\dim(\mathrm{col}(A)) = \#\text{ of pivots in } A = \mathrm{rank}(A)$$

$$\dim(\mathrm{Nul}(A)) = \#\text{ free variables } = \text{nullity of } A$$

Theorem (Rank–Nullity Theorem)**.**

For any $m \times n$ matrix A,

$$\mathrm{rank}(A) + \dim(\mathrm{Nul}(A)) = n.$$

Today’s poll code:

GDE2LU

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The column space of

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

is a subspace of

A. $\mathbb{R}^2$ B. $\mathbb{R}^3$ C. $\mathbb{R}^4$ D. $M_{2\times 4}$

Today's poll code:

GDE2LU

The null space of

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is a subspace of

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Today's poll code:

GDE2LU

The column space of

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

has dimension

Today's poll code:

GDE2LU

The null space of

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

has dimension

Today's poll code:

GDE2LU

The column space of

$$M = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 11 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

has dimension

Today's poll code:

GDE2LU

The null space of

$$M = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 11 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

has dimension

Given polynomials in $\mathbb{P}_n$,

we can ask questions about them by

identifying them with vectors in $\mathbb{R}^{n+1}$

$$a_0 + a_1 t + \cdots + a_n t^n \longrightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

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