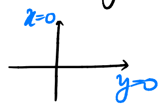
Symbolic powers and a story of algebra vs geometry

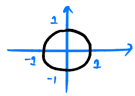
UIC Undergraduate Research Symposium 2/11/2019

Algebra $\longleftrightarrow$ Geometry

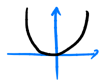
$xy = 0$



$x^2+y^2-1=0$



$x^2-y=0$



systems of polynomial equations in d variables $x_1, \dots, x_d$ $\longleftrightarrow$ nice subsets of d-dimensional space

eg: $\begin{cases} x_1 = 0 \\ \vdots \\ x_d = 0 \end{cases} \longleftrightarrow$ the point $(0, \dots, 0)$

<u>Defn</u>: A <u>variety</u> is a subset of points $V \subseteq \mathbb{C}^d$ that are precisely the zeroes of a system of polynomial equations

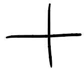
<u>Warning</u>! Different systems of equations can determine the same variety!

$$xy=0 \iff 42xy=0 \iff x^2y=0 \iff \begin{cases} x^2y^2=0 \\ 5(x+y)xy=0 \end{cases}$$

$f_1, \dots, f_n$ polynomials

$$\begin{cases} f_1 = 0 \\ \vdots \\ f_n = 0 \end{cases} \longmapsto \text{variety } V = \text{all common zeros of } f_1, \dots, f_n$$

What is the set I of all the polynomials f with $f(v) = 0$ for all $v \in V$?

Ex $x^2 y = 0 \quad \longmapsto \quad V =$  $\quad \longmapsto \quad I = \{(\text{any polynomial in } x \text{ and } y) \cdot xy\}$

$$\begin{cases} x-1=0 \\ y=0 \\ z=0 \end{cases} \longmapsto V = \{(1,0,0)\} \longmapsto I = \{g_1(x-1) + g_2 y + g_3 z \ : \ g_i \text{ any polynomial}\}$$

Hilbert's basis theorem

Every system of polynomial equations in d variables with coefficients in $\mathbb{R}/\mathbb{C}/$ your favorite field can be reduced to a finite number of equations.

In general: Algebra $\longrightarrow$ Geometry

$I = \{\text{polynomials } f : f \text{ vanishes along } V\} \longleftarrow$ variety V

(think of this as a giant system of equations)

$\downarrow$ Hilbert says we can find $f_1, \dots, f_n$

$I = (f_1, \dots, f_n)$

$\quad\quad ||$

$\{g_1 f_1 + \dots + g_n f_n : g_i \text{ polynomials}\} =:$ ideal generated by $f_1, \dots, f_n$

Hilbert's Nullstellensatz

Algebra $\longleftrightarrow$ Geometry

$$\underbrace{\mathbb{C}[x_1, \dots, x_d]}_{\substack{\text{polynomials with coefficients in } \mathbb{C} \\ \text{in the variables } x_1, \dots, x_d}} \longleftrightarrow \mathbb{C}^d$$

$$\underline{\text{radical}} \text{ ideals } \overset{1:1}{\longleftrightarrow} \text{ varieties}$$

$(f^n \in I \Rightarrow f \in I)$

$$(0) = \{0\} \longleftrightarrow \mathbb{C}^d$$

$$\mathbb{C}[x_1, \dots, x_d] \longleftrightarrow \emptyset$$

larger ideals $\longleftrightarrow$ smaller varieties

(maximal ideal: if we add any new equation we get everything) $\rightarrow$ $(x_1 - a_1, \dots, x_d - a_d) \longleftrightarrow$ point $\{(a_1, \dots, a_d)\}$

prime ideals $\longleftrightarrow$ irreducible varieties

$(fg \in I \Rightarrow f \in I \text{ or } g \in I)$ (<u>not</u> the union of 2 smaller varieties)

Moreover, if I is <u>the</u> radical ideal corresponding to the variety V,

$$I = \bigcap_{(a_1, \dots, a_d) \in V} (x_1 - a_1, \dots, x_d - a_d) = \bigcap_{\substack{m \supseteq I \\ \text{maximal ideal}}} m$$

So now we know varieties $\leftrightarrow$ polynomials that vanish at all points
But how do we measure that vanishing?

$$I \text{ radical ideal } \equiv V \text{ variety}$$

We say f vanishes to order n along V if the following equivalent conditions hold:

- the power series expansion of f around each $v \in V$ has no terms of order $< n$.

(Remember from one variable: the coefficients of the power series expansion are determined by taking derivatives of each order and evaluating at the center)

- $$\frac{\partial^{a_1+\cdots+a_d}}{\partial x_1^{a_1}\cdots\partial x_d^{a_d}}(f)\Big|_{\text{evaluate at each } v\in V} = 0$$

$$\Updownarrow$$

$$\frac{\partial^{a_1+\cdots+a_d}}{\partial x_1^{a_1}\cdots\partial x_d^{a_d}}(f) \in I \quad \text{for all} \quad a_1+\cdots+a_d < n$$

- $$f \in \bigcap_{\substack{m \supseteq I \\ m \text{ max}}} m^n$$ where $I^n = (f_1 \cdots f_n : f_i \in I)$

n-th power of the ideal I

there's yet another description:

Symbolic powers

P prime ideal $\leftrightarrow$ V irreducible variety

the n-th symbolic power of P is

$$P^{(n)} = \{ f \in P : \ sf \in P^n \text{ for some } s \notin P \}$$

For a general variety V, $V = \underbrace{V_1 \cup \cdots \cup V_k}_{\text{irreducible}} \longleftrightarrow I = \underbrace{P_1 \cap \cdots \cap P_k}_{\text{primes}}$

$$I^{(n)} = P_1^{(n)} \cap \cdots \cap P_k^{(n)}$$

theorem (Zariski-Nagata) I radical ideal $\leftrightarrow$ variety V

$$I^{(n)} = \{ f \text{ vanishes to order } n \text{ along } V \}$$

Facts
0) $I^{(1)} = I$

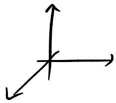
1) $I^n \subseteq I^{(n)}$ for any $n \geq 1$

2) $I^{(n+1)} \subseteq I^{(n)}$ for any $n \geq 1$

3) If $I = $ (some variables), then $I^n = I^{(n)}$ for all $n \geq 1$

Examples

1) $\begin{cases} xy = 0 \\ yz = 0 \\ xz = 0 \end{cases}$



x-axis $\cup$ y-axis $\cup$ z-axis

$$I = (xy, xz, yz) = \underbrace{(x,y)}_{z\text{ axis}} \cap \underbrace{(x,z)}_{y\text{-axis}} \cap \underbrace{(y,z)}_{x\text{-axis}}$$

$$I^2 = (x^2y^2, x^2z^2, y^2z^2, x^2yz, xy^2z, xyz^2)$$

$$I^{(2)} = \underset{\substack{\cup \\ xy}}{(x,y)^2} \cap \underset{\substack{\cup \\ xz}}{(x,z)^2} \cap \underset{\substack{\cup \\ yz}}{(y,z)^2} \ni xyz$$

But xyz has degree 3, and elements of $I^{(2)}$ have degree ≥ 4

2) $V = \{3\times 3 \text{ matrices of rank} \leq 1\} \subseteq \mathbb{C}^9$ is a variety!

$A = \begin{bmatrix} x_{11} & \cdots & x_{13} \\ & \ddots & \\ & & x_{33} \end{bmatrix}$ invertible $\Longleftrightarrow$ all 2×2 minors of A are 0

$I = (x_{11}x_{22} - x_{21}x_{12},$ other 2×2 minors$)$

$\det A \in I$ and $\partial/\partial x_{ij} (\det A) \in I \Rightarrow \det A \in I^{(2)}$

Again for degree reasons, $\det A \notin I^2$

(Fun fact: this one is an irreducible variety)

Open Problems and Difficult Questions

1) Given I, describe $I^{(n)}$ for all $n \geq 1$

Very hard, even with a powerful computer.

2) Characterize all the I such that $I^{(n)} = I^n$ for all $n \geq 1$.

Packing Problem Restrict this question to ideals generated by monomials

Open Problem $I^{(n)} = I^n$ for $n \leq d \Rightarrow I^{(n)} = I^n$ for all n

(known for monomials ideals by Montaño - Núñez - Betancourt

3) What is the smallest degree of an element in $I^{(n)}$?

Famous lower bound conjectured by Chudnovsky (still open)

4) Containment Problem When is $I^{(a)} \subseteq I^b$?

(this answers equality too, and gives lower bounds for degrees in $I^{(a)}$)

Theorem (Ein - Lazarsfeld - Smith 2001, Hochster - Huneke 2002, Ma - Schwede 2017)

$$I^{(dn)} \subseteq I^n \text{ for all } n \geq 1$$

where $d = \#$ variables