Symbolic powers
Tulane Colloquium 13/02/2020

<u>Fundamental Theorem of Arithmetic</u>

For every $n \in \mathbb{Z}$, $\exists$ primes $p_1, \ldots, p_k$, and $a_1, \ldots, a_k \geq 1$ st
$$n = \pm p_1^{a_1} \cdots p_k^{a_k},$$
and this is unique up to permutation.

<u>Example</u> In $\mathbb{Z}[\sqrt{-5}] = \mathbb{Z} + \mathbb{Z}\sqrt{-5}$,
$$6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$$ unique factorization fails!

to salvage this, there is primary decomposition
— a statement about ideals.

ideals in $\mathbb{Z}$ $\longrightarrow$ sets of multiples of a fixed integer
$$(n) = \{kn \mid k \in \mathbb{Z}\}$$

<u>Fundamental Theorem of Arithmetic</u>

For every ideal (n) in $\mathbb{Z}$ $\exists$ primes $p_1, \ldots, p_k$, and $a_1, \ldots, a_k \geq 1$ st
$$(n) = (p_1^{a_1}) \cap \cdots \cap (p_k^{a_k})$$
and this is unique up to permutation.

What's special about the ideals (p^a)? they are primary!

A proper ideal I is prime if $xy \in I \Rightarrow x \in I$ or $y \in I$

A proper ideal I is primary if $xy \in I \Rightarrow x \in I$ or $y^n \in I$ for some $n \geq 1$

Fact If I is primary, $\sqrt{I} = \{f : f^n \in I\}$ is prime.

$\sqrt{I}$ ← radical of I

so: Fundamental Theorem of Arithmetic

For every ideal (n) in $\mathbb{Z}$, $\exists$ primary ideals $(p_1^{a_1}), \dots, (p_k^{a_k})$ st

$$(n) = (p_1^{a_1}) \cap \cdots \cap (p_k^{a_k})$$

and this is unique up to permutation.

How do we write this as a theorem about other rings?

In $R = K[x_1, \dots, x_d]$ (K a field), ideals $\equiv$ systems of equations

ideal $I = (f_1, \dots, f_n) = \{g_1 f_1 + \cdots + g_n f_n \mid g_i \in R\}$

Theorem (Lasker 1905, Noether 1921)

For any ideal I in $R = k[x_1, \dots, x_d]$ there exist primary ideals st

$$I = \underbrace{Q_1 \cap \cdots \cap Q_k}_{\text{primary decomposition}}$$

(where we can choose $\sqrt{Q_i}$ all different, and we don't write unnecessary terms)

(Note that these $\sqrt{Q_i}$ are prime ideals that contain I)

The minimal components are uniquely determined:

Among the $\sqrt{Q_i}$ we will see every minimal prime containing I and the corresponding minimal components are unique.

Example

$$(x^2, xy) = \underbrace{\overset{\text{minimal}}{(x)}}_{\text{unique}} \cap \overset{\text{embedded}}{(x^2, xy, y^2)}$$

$$= (x) \cap \underbrace{(x^2, xy, y^n)}_{\text{can change}} \qquad n \geq 1$$

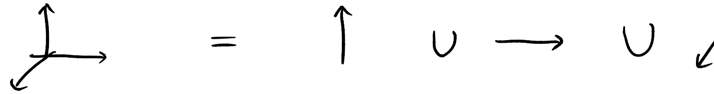
(x) $\qquad$ (x, y)

Example In $\mathbb{Z}[\sqrt{-5}]$,

$$(6) = \underset{\text{primary}}{(2)} \cap \underset{\text{not!}}{(3)} = \underbrace{(2) \cap (3, 1+\sqrt{-5}) \cap (3, 1-\sqrt{-5})}_{\text{primary decomposition unique!}}$$

Primary decomposition saves the day!

<u>Example</u> $(xy, xz, yz) = (x,y) \cap (x,z) \cap (y,z)$



So what are primary ideals, really? Powers of primes?

P prime:

$P^n = (f_1 \cdots f_n \mid f_i \in P)$ not always primary! (examples soon)

$P^n = $ minimal primary component with radical P $\cap$ embedded components

$\downarrow$

uniquely determined!

$\parallel$

$P^n \subseteq P^{(n)} = \{ f \mid sf \in P^n \text{ for some } s \notin P \}$

$=$ smallest primary ideal with radical P containing P^n

$I = P_1 \cap \cdots \cap P_k$ radical $\Rightarrow I^{(n)} = P_1^{(n)} \cap \cdots \cap P_k^{(n)}$

__theorem__ (Zariski–Nagata) $\quad R = \mathbb{C}[x_1, \ldots, x_d]$

$$P^{(n)} = \{ f \mid f \text{ vanishes to order } n \text{ along the variety defined by } P \}$$

$$= \left\{ f \in R \;\middle|\; \frac{\partial^{a_1 + \cdots + a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} f \in P \quad \forall a_1 + \cdots + a_d < n \right\}$$

- Can also do some version of this over any perfect field.

__theorem__ (De Stefani – G – Jeffries)

A version of Zariski–Nagata for $\mathbb{Z}[x_1, \ldots, x_d]$

__Example__ (space monomial curves) curve $\{(t^3, t^4, t^5) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}$

$$P = P(3,4,5) := (\underbrace{x^3 - yz}_{\substack{f \\ \deg 9}}, \underbrace{y^2 - xz}_{\substack{g \\ \deg 8}}, \underbrace{z^2 - x^2y}_{\substack{h \\ \deg 10}}) \text{ in } \mathbb{C}[x,y,z]$$

ideal

$$P^2 = (f^2, g^2, h^2, fg, fh, gh) \longrightarrow \deg \geq 16$$

but $\underbrace{f^2 - gh}_{\substack{\in P^2 \\ \deg 18}} = \underbrace{x}_{\substack{\notin P \\ = \deg 3 +}} q \Rightarrow \underbrace{q}_{\deg 15} \in P^{(2)} \quad \therefore q \notin P^2$

so $P^2 \subsetneq P^{(2)}$

__Fun fact__ the symbolic powers of space monomial curves $\{(t^a, t^b, t^c)\}$ can exhibit strange behavior: they can have unexpected elements in arbitrarily high degrees

Big Questions

1) Find generators for $I^{(n)}$.

Macaulay2 software package: Drabkin – G – Seceleanu – Stone with contributions from Andrew Conner and Diana Zhong

2) When is $I^n = I^{(n)}$?

Many sufficient conditions for $I^n = I^{(n)}$ $\forall n \geq 1$

No clean, efficient complete characterization for what classes of I have this property for all n.

Big Open Problem: a characterization over monomial ideals
Packing Problem

3) What degrees does $I^{(n)}$ live in?

Chudnovsky's Conjecture (open): lower bounds

4) Compare I^n and $I^{(n)}$

Containment Problem When is $I^{(a)} \subseteq I^b$?

<u>Theorem</u> (Ein-Lazarsfeld-Smith 2001, Hochster-Huneke 2002, Ma-Schwede 2017)

$R = k[x_1, \dots, x_d]$, k field or $k = \mathbb{Z}$

$$I^{(dn)} \subseteq I^n \quad \text{for all } n \geq 1$$

More precisely, if
- $I = P$ prime of codimension h
- $I = P_1 \cap \dots \cap P_k$, $h = \max\{\operatorname{codim} P_i\}$

then $I^{(hn)} \subseteq I^n$ for all $n \geq 1$.

<u>Example</u> $P = P(3,4,5)$ (defining the curve (t^3, t^4, t^5))

codim $3 - 1 = 2$

$\downarrow$

so $P^{(2n)} \subseteq P^n$ for all $n \geq 1$ and $P^{(4)} \subseteq P^2$. But actually $P^{(3)} \subseteq P^2$.

<u>Question</u> (Huneke, 2000) P prime of codim 2. Is $P^{(3)} \subseteq P^2$?

<u>Conjecture</u> (Harbourne, 2008) $I^{(hn-h+1)} \subseteq I^n$ for all $n \geq 1$.

<u>Fact</u> (Hochster-Huneke) In char p, $I^{(hq-h+1)} \subseteq I^q$ for all $q = p^e$

Conterexample (Dumnicki – Szemberg – Tutaj-Gasińska, 2013)
(Harbourne – Seceleanu, 2015)

$\exists$ family of radical ideals $I \subseteq k[x,y,z]$, char $k \neq 2$, $h=2$

$$I^{(3)} \not\subseteq I^2$$

But! these correspond to very special configurations

In fact, Harbourne's Conjecture holds for nice ideals:

- General points in $\mathbb{P}^2$ (Bocci – Harbourne) and $\mathbb{P}^3$ (Dumnicki)
- I squarefree monomial ideal
- R/I determinantal ring, Veronese ring, nice ring of invariants

} F-pure! (nice singularities)

strongly F-regular

Theorem (G-Huneke, 2017) $R = k[x_1, \ldots, x_d]$
(G-Ma-Schwede, 2019) R F-finite Gorenstein, pdim $I < \infty$.

- If R/I is F-pure, then I satisfies Harbourne's Conjecture
- If R/I is strongly F-regular, can substitute h by $h-1$.
So for $h=2$, get $I^n = I^{(n)}$ $\forall n \geq 1$.