

Theorem Let R be a ring. TFAE

- ① R is Noetherian and $\dim(R) = 0$
- ② R is a finite product of local Noetherian rings of dimension 0
- ③ $l_R(R) < \infty$
- ④ R is Artinian

Proof Have shown: $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$

$4 \Rightarrow 1$) Have shown: If R is Artinian, then:

- $\dim(R) = 0$
- $\text{Spec}(R) = \{m\}$ is finite

$\xrightarrow{\text{thm}}$ $R \cong R/q_1 \times \dots \times R/q_n$ ($R/q_i, m_i$) local of dimension 0

R Artinian $\Rightarrow R/q_i$ Artinian

Need to show: (R, m) Artinian, $\text{Spec}(R) = \{m\} \Rightarrow R$ Noetherian

$$m \supseteq m^2 \supseteq m^3 \supseteq \dots \text{ steps} \Rightarrow m^n = m^{n+1}$$

(Note: NAK doesn't apply \rightarrow don't know if m is fg)

If $m^n \neq 0 \Rightarrow S = \{I \subseteq m \mid I^{m^n} \neq 0\} \neq \emptyset$

R Artinian $\Rightarrow S$ has a minimal element \mathfrak{J}

$$\mathfrak{J} \in S \Rightarrow \exists x \in \mathfrak{J}, x m^n \neq 0 \Rightarrow (x) \in S \Rightarrow (x) = \mathfrak{J}$$

$$(x m)^n = x m^{n+1} = x m^n \neq 0 \Rightarrow x m \in S$$

$$\begin{array}{l} x\eta \subseteq (x) \\ x\eta \in S \end{array} \Rightarrow (x) = (x)\eta = \mathcal{J}$$

$$\left\{ \begin{array}{l} (x) = \eta(x) \\ (x) \text{ fg} \end{array} \right. \xrightarrow{\text{NAK}} (x) = 0 \Rightarrow \mathcal{J} = 0 \quad \downarrow$$

so $\eta^n = 0$, and if we take n smallest possible,

$$0 = \eta^n \subseteq \eta^{n-1} \subseteq \dots \subseteq \eta \subseteq R$$

has quotients $\cong R/\eta \Rightarrow R$ has finite length

$\Rightarrow R$ Artinian $R\text{-mod} \Rightarrow R$ Artinian ring

□

$$R \text{ ring} \quad R \text{ Artinian} \Leftrightarrow \left\{ \begin{array}{l} \dim R = 0 \\ \text{Noetherian} \end{array} \right. \Leftrightarrow l(R) < \infty$$

Warning!

M module

M Artinian

$$\begin{aligned} &\not\Rightarrow M \text{ Noetherian} \\ &\not\Rightarrow l(M) < \infty \end{aligned}$$

why? A module could be Artinian and infinitely generated

Krull's Height theorem

Theorem (Krull's Principal ideal theorem)

R Noetherian ring

Every minimal prime of (f) has height at most 1.

$$(\Rightarrow \text{ht}(f) \leq 1)$$

Proof Suppose $p \in \text{Spec}(R)$, $\text{ht}(p) > 1$, $p \in \text{Min}(f)$

$$\mathfrak{P}_0 \subsetneq \cdots \subsetneq \mathfrak{P}_n = p, \quad n \geq 2$$

- Localize at p ($\therefore p$ is the unique maximal ideal)
- mod out by \mathfrak{P}_0



(R, \mathfrak{m}) Noetherian local domain

$$\dim R \geq 2$$

$$\text{Min}(f) = \{\mathfrak{m}\}$$

Let $q \in \text{Spec}(R)$ with $0 \subsetneq q \subsetneq \mathfrak{m}$

$\bar{R} = R/(f)$ has $\dim(R/(f)) = 0 \Rightarrow \bar{R}$ Artinian

Consider $q^{(n)} = q$ -primary component in q^n

$$= q^n Rq \cap R$$

$$\bar{R} \text{ Artinian} \Rightarrow q\bar{R} \supseteq q^{(2)}\bar{R} \supseteq q^{(3)}\bar{R} \supseteq \dots \text{ stops}$$

$$\text{so for some } n \quad q^{(n)}\bar{R} = q^{(n+1)}\bar{R}$$

$$\text{In } R: \quad q^{(n)} + (f) = q^{(n+1)} + (f)$$

$\supseteq \text{ always}$

$$q^{(n)} \subseteq q^{(n+1)} + (f)$$

$$\text{Any } a \in q^{(n)} \quad \begin{matrix} a \\ \in q^{(n)} \end{matrix} = b + fx \quad \begin{matrix} x \in R, b \in q^{(n+1)} \\ q^{(n+1)} \\ \subseteq q^{(n)} \end{matrix} \Rightarrow fx \in q^{(n)}$$

$$\left\{ \begin{array}{l} f \in q^{(n)} \\ f \notin q \end{array} \right. \Rightarrow x \in q^{(n)} \Rightarrow a \in q^{(n+1)} + fq^{(n)}$$

$$q^{(n)} \subseteq q^{(n+1)} + fq^{(n)}$$

\Downarrow

$$q^{(n)} = q^{(n+1)} + fq^{(n)}$$

$$\Rightarrow q^{(n)} = q^{(n+1)} + \gamma q^{(n)} \xrightarrow{\text{NAk}} q^{(n)} = q^{(n+1)}$$

$$\text{so: } \bigcap_{k \geq 1}^{\infty} q^{(k)} = q^{(n)} \neq 0$$

\cup
 $q^n \neq 0 \text{ because } R \text{ domain!}$

$$\underline{\text{But}} \quad \bigcap_{m \geq 1} q^{(m)} = \bigcap_{m \geq 1} (q^m R_q \cap R) = \underbrace{\left(\bigcap_{m \geq 1} q^m R_q \right)}_{= 0} \cap R$$

Knull Intersection

$$R \text{ domain} \Rightarrow 0 R_q \cap R = 0 \quad \rightarrow$$

$$\bigcap_{m \geq 1} q^{(m)} = 0 \quad \downarrow \quad \square$$

General version: need some kind of induction.

Lemma R Noetherian

$$P \subsetneq Q \subsetneq a \ni f$$

$$\text{then } \exists q' \ni f \quad P \subsetneq q' \subsetneq a$$

Proof $f \in P \Rightarrow f \in q = q' \checkmark \quad \text{Assume } f \notin P.$

- mod out by P
 - Localize at a
- $\left. \begin{array}{l} (\text{R}, m) \text{ local, } f \in \mathfrak{m}, f \neq 0 \\ \text{Need: } f \in q \subsetneq \mathfrak{m} \end{array} \right\}$

$$P \in \text{Min}(f) \xrightarrow[\text{Ideal thm}]{\text{Principal}} \text{ht } P \leq 1 \xrightarrow[\substack{\text{R domain} \\ f \neq 0}]{\text{ht } P = 1} \text{ht } P = 1$$

$\text{ht } a \geq 2 \Rightarrow \dim R \geq 2 \Rightarrow$ take any min prime over f .

Krull's Height theorem

R Noetherian

$$I = (f_1, \dots, f_n)$$

$$P \in \text{Min}(I) \Rightarrow \text{ht}(P) \leq n$$

$$(\Rightarrow \text{ht } I \leq n)$$

Proof Induction on n .

$n=0 \Rightarrow I=0 \Rightarrow$ minimal primes have height 0
 $n=1$ is Krull's Principal Ideal theorem

$$n > 2 \quad P \in \text{Min}(f_1, \dots, f_n)$$

$P_0 \subsetneq \dots \subsetneq P_h = P$ saturated chain

• Case 1 $f_1 \in P_1$

\Rightarrow apply induction hypothesis to $\bar{R} = R/f_1$

$$I\bar{R} = (\underbrace{f_2, \dots, f_n}_{n-1})\bar{R} \rightsquigarrow P_1\bar{R} \subsetneq \dots \subsetneq P_h\bar{R} \text{ has length } \leq n-1$$
$$\Rightarrow h-1 \leq n-1 \Rightarrow h \leq n \checkmark$$

• Case 2 $f_1 \notin P_1$ But $f \in P_h$, so use lemma repeatedly
get new chain

$$P \subsetneq Q_1 \subsetneq \dots \subsetneq Q_{h-1} \subsetneq P_h \quad f \in Q_1. \text{ Apply Case 1}$$

Notes

- this bound is sharp

$$\text{ht}(x_1, \dots, x_d) = d \text{ in } k[x_1, \dots, x_d]$$

If $\text{ht}(f_1, \dots, f_n) = n$, (f_1, \dots, f_n) is a complete intersection

- In $R = k[x, y, z]$, $\begin{matrix} (xy, nz) \\ \text{ht: } 1 \quad 1 \quad 2 \end{matrix} = (x) \cap (y, z)$
- An associated prime can have height $> \# \text{generators}$

$$R = \frac{k[x, y]}{(x^2, xy)} \quad I = (0) \quad 0 \text{ generators}$$

$$(x, y) \in \text{Ass}(R/I) \quad (x, y) = \text{ann } x$$

- Noetherianity is necessary

Theorem R Noetherian, $\dim R = d$

- 1) If p is a prime of height h , $\exists f_1, \dots, f_h \in p : p \in \text{Min}(f_1, \dots, f_h)$

- 2) I any ideal in R . $\exists f_1, \dots, f_{d+1} \in I$
 $\sqrt{I} = \sqrt{(f_1, \dots, f_{d+1})}$

- 3) (R, m) local / graded k -algebra, $k = R_0$,
 $\exists f_1, \dots, f_d \quad \sqrt{(f_1, \dots, f_d)} = m$

Corollary (R, m) Noetherian local ring

$$\dim(R) = \min \{ n \mid \sqrt{(f_1, \dots, f_n)} = m \} \leq \mu(m)$$

In particular, R has finite dimension.

Embedding dimension (R, m) local/graded k -alg with $R_0 = k$

$$\text{embdim}(R) := \mu(m)$$

Regular Local Ring if $\dim(R) = \text{embdim}(R)$

Corollary $k[x_1, \dots, x_d]$ is a regular local ring