Singularities in char p: Frobenius is your friend

this talk is based on:

Frobenius splittings in Commutative Algebra, by Karen Smith and Wenliang Zhang

R a ring of char p

Frobenius map: $F: R \to R \quad r \mapsto r^p$ is a homomorphism, since

$$(a+b)^p = a^p + \sum_{k=1}^{p-1} \underbrace{\binom{p}{k}}_{=0} a^k b^{p-k} + b^p = a^p + b^p$$

$$R^p = F(R) = \text{subring of } R \text{ of all elements that are } p\text{-th powers}$$

we might also want to talk about (R domain)

$R \subseteq R^{1/p} = \{x^{1/p} : x \in R\}$ (subring of some algebraic closure of R)

we can identify the Frobenius map with this inclusion instead

$$\begin{tikzcd} R \arrow[r, "F"] \arrow[d, phantom, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & R \arrow[d, phantom, "\cong"] & r \arrow[d] \\ R \arrow[r, hook] & R^{1/p} & r^{1/p} \end{tikzcd}$$

Understanding R as an R^p-module $\Leftrightarrow$ understanding $R^{1/p}$ as an R-module

$\Leftrightarrow$ understanding R as a module over itself via the Frobenius action

Notation $F_* R = R$ with the R-module structure given by Frobenius

Notation I ideal: $I^{[p]} = (f^p : f \in I) = $ image of I by Frobenius

R is nicer when it is F-finite, meaning R is finitely generated over R^p.

A perfect field k (meaning $k = k^p$) is F-finite

F-finiteness is preserved by:

- taking quotients
- localization
- completion at a maximal ideal
- finitely generated algebras

So if R is fg over an F-finite ring, R is F-finite.

How does F measure singularities?

1) F measures whether R is reduced.

R is reduced $\iff$ 0 is the only nilpotent element $\iff$ F is injective

Note there is a condition called F-injective, and this is not it.

2) F measures regularity

Theorem (Kunz, 1969) Let (R, m) be a noetherian local ring. R is regular if and only if F is a flat map.

When (R, m) is F-finite, this says R is regular if and if R is free over R^p

Example $\mathbb{F}_p[x_1, \dots, x_d]$ is a free module over $\mathbb{F}_p[x_1^p, \dots, x_d^p]$, with basis $\{x_1^{a_1} \cdots x_d^{a_d},\ 0 \leq a_i < p\}$

so we can classify how far R is from being regular by how far R is from being free over R^p.

3) When R^p is a direct summand of R, R is nice

R is F-split if F splits, meaning that

$$\begin{tikzcd} R \arrow[r, "F"] & R \arrow[l, bend left, dashed, "\phi"] \end{tikzcd} \qquad \phi \circ F = id_R \qquad \text{as maps of } R\text{-modules.}$$

Equivalently, $\exists\, \phi \in \mathrm{Hom}_{R^p}(R, R^p)$ st $\phi(1) = 1$.
Equivalently, $\exists\, \phi \in \mathrm{Hom}_R(R^{1/p}, R)$ st $\phi(1) = 1$

Many nice properties pass onto direct summands

<u>Example</u> $R = \mathbb{F}_p[x_1, \dots, x_d]$ is F-split:

$$R = \bigoplus_{0 \leq a_1 + \dots + a_d < p} R^p \cdot x_1^{a_1} \cdots x_d^{a_d} \underset{\substack{\text{projection onto} \\ \text{the } x_1^0 \cdots x_d^{a_d} \text{ coordinate}}}{\twoheadrightarrow} R_p\, x_1^0 \cdots x_d^{a_d} \qquad \text{sends } 1 \mapsto 1.$$

<u>Fact</u> All F-finite regular rings are F-split:
An F-finite regular local ring R is free, so $R = \bigoplus_{\substack{\text{some} \\ \text{copies}}} R^p \Rightarrow R^p$ is a direct summand of R

So R regular in this context means R completely decomposes into copies of R.

<u>Fact</u> Any direct summand of an F-split ring is F-split.

<u>Proof</u>

splitting
||
$R \hookrightarrow S$ + blue + green

$$\begin{tikzcd}
R \arrow[r, hook, "\text{split}"] \arrow[d, dashed, bend right] & S \arrow[l, dashed, bend right, "\phi \text{ induces } \phi^p"'] \arrow[d, dashed, bend left, "\phi_S"] \\
R^p \arrow[u, hook] \arrow[r] & S^p \arrow[u, hook, "\text{split}"'] \arrow[l, dashed, bend left, "\phi^p"]
\end{tikzcd}$$

<u>Example</u> Any Veronese $k[\text{all monomials of degree } d \text{ in } v \text{ variables}] \overset{\text{split}}{\subseteq} k[v \text{ variables}]$ is F-split

eg $\quad k[x,y,z]/(xz - y^2) \cong k[u^2, uv, v^2]$

Definition (Hochster–Roberts, 1974)
R is F-pure if F is a pure map, meaning that for all R-modules M,

$$F \otimes 1 : R \otimes M \longrightarrow R \otimes M \quad \text{is injective.}$$

When R is F-finite, F-pure $\Leftrightarrow$ F-split.

Coming up:
Hochster and Roberts introduced this notion in proving rings of invariants are Cohen-Macaulay

Fedder's Criterion (1983) (R, m) regular local ring, I ideal in R

$$R/I \text{ is F-pure} \Leftrightarrow (I^{[q]} : I) \nsubseteq m^{[q]} \quad \text{for one/all/all large } q = p^e$$

Note this can be tested by a computer! Note Does not require F-finite

Example If I is a monomial ideal, R/I is F-pure if and only if I is squarefree

I not squarefree $\Rightarrow$ not reduced and all F-pure rings are reduced.

But also,

$$(I^{[p]} : I) = (\text{ideal generated by the } (p-1)\text{th powers of the generators of } I)$$
$$\subseteq (x_1^p, \dots, x_d^p) \Leftrightarrow \text{the generators of } I \text{ are squarefree}$$

Example $R = k[x,y]/(xy(x-y))$ $\quad I = (xy(x-y)) \subseteq k[x,y]$

$$(I^{[p]} : I) = (x^{p-1} y^{p-1} \underbrace{(x-y)^{p-1}}) \subseteq (x^p, y^p)$$

R is NOT F-pure $\quad$ all terms have x or y, as $p-1 \geq 1$

Regular $\Updownarrow$ R completely splits over R^p | what's in between? | F-split $\Updownarrow$ R^p is a summand of R

An F-finite domain R is <u>strongly F-regular</u> if for all $f \in R$, $f \neq 0$, $\exists e$ such that

$R \longrightarrow R^{1/p^e} \xrightarrow{f^{1/p^e}} R^{1/p^e}$ splits ie, $\exists \phi \in \mathrm{Hom}_R(R^{1/p^e}, R)$: $\phi(f^{1/p^e}) = 1$

Morally: R has many summands that are isomorphic to R^{1/p^e}

<u>Fact</u> Regular rings are strongly F-regular

<u>Proof</u> $f \neq 0$
Since R^{1/p^e} is a fg R-module, f^{1/p^e} is part of a minimal generating set for R^{1/p^e} if and only if $f^{1/p^e} \notin m R^{1/p^e}$, or equivalently $f \notin m^{[p^e]}$.

$\bigcap_e m^{[p^e]} \subseteq \bigcap_e m^e = 0 \Rightarrow f \notin m^{[p^e]}$ for some $e \Rightarrow f$ minimal generator for R^{1/p^e}
R^{1/p^e} free R-module, so $\exists \phi \in R^{1/p^e} \longrightarrow R$ sending f^{1/p^e} to 1.

<u>Glassbrenner's Criterion</u> (1996) (R, m) F-finite regular local ring, $I \subseteq R$ ideal

R/I strongly F-regular $\Leftrightarrow$ $\forall c \in R$ not in a minimal prime of I
$c(I^{[q]} : I) \nsubseteq m^{[q]}$ for some/all/all large $q = p^e$

<u>Fact</u> I squarefree monomial ideal in a polynomial ring R.
R/I strongly F-regular $\Leftrightarrow$ I generated by variables

"Proof" Strongly F-regular rings are products of domains, so R/I is a domain.
If $x_i x_j$ divides a generator of I, R/I is not a domain

Example $R = k[x,y,z]/(x^2+y^2+z^2)$, $p \neq 2$ is strongly F-regular

Example $R = k[x,y,z]/(x^3+y^3+z^3)$ R is not strongly F-regular for $p \equiv 1 \pmod 3$ although it is F-pure.

Fact Direct summands of strongly F-regular rings are strongly F-regular

theorem (HH89) Strongly F-regular rings are Cohen-Macaulay and normal.

Corollary Direct summands of regular rings are Cohen-Macaulay.

Caution! Not all F-pure rings are Cohen-Macaulay, although many are.

Nice Consequence Rings of invariants of linearly reductive groups are Cohen-Macaulay

this a theorem of Hochster and Roberts from 1974, when they introduced F-purity.

linearly reductive groups are algebraic group with the property that every finite dimensional representation is completely reducible.

In char p, this includes $G =$ tori, finite groups of order not divisible by p, extensions of these

If G acts on $R = k[x_1, \ldots, x_d]$, $R^G =$ subring of elements of R fixed by G

$$\begin{aligned} R^G \hookrightarrow R &\longrightarrow R^G \\ r &\longmapsto \frac{1}{|G|} \sum_{g \in G} g \cdot r \end{aligned}$$

is a splitting if G is a finite group

rings of invariants of linearly reductive groups have a generalization of this map.

$\therefore$ R^G is strongly F-regular $\Rightarrow$ R^G is Cohen-Macaulay.

Hochster and Roberts used reduction to char p to show R^G is CM even in char 0.