A stable version of Harbourne's Conjecture
Utah Commutative Algebra Seminar 30/08/2019

R regular ring ($R = k[x,y,z]$)
I radical
h = big height of I = $\max\{\operatorname{ht} Q : Q \in \operatorname{Ass}(I)\}$

n-th symbolic power of I

$$I^{(n)} = \bigcap_{Q \in \operatorname{Min}(I)} (I^n R_Q \cap R)$$

<u>theorem</u> (Zariski–Nagata)

$R = \mathbb{C}[x_1, \ldots, x_d]$

$$I^{(n)} = \{f \in R : f \text{ vanish up to order } n \text{ along } I\}$$

$$= \bigcap_{m \supseteq I} m^n$$

$$= \{f \in R : g \frac{\partial^{a_1 + \cdots + a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}}(f) \in I \quad \forall g \in R,\ a_1 + \cdots + a_d \leq n-1\}$$

<u>Notes</u>
1) $I^n \subseteq I^{(n)}$
2) $I^{(n+1)} \subseteq I^{(n)}$
3) $I^{(n)} = I^n$ for I = (regular sequence)
False in general.

Example $P = \ker\left(k[x,y,z] \longrightarrow k[\overset{x}{t^3}, \overset{y}{t^4}, \overset{z}{t^5}]\right)$ prime

$\deg x = 3$
$\deg y = 4$
$\deg z = 5$

$$= (\underbrace{x^3 - yz}_{\substack{f \\ \deg 9}}, \underbrace{y^2 - xz}_{\substack{g \\ \deg 8}}, \underbrace{z^2 - x^2y}_{\substack{h \\ \deg 10}})$$

$$P^2 \subsetneq P^{(2)} = P^2 R_P \cap R = \{ r \in R : sr \in P^2, s \notin P \}$$

$$\underbrace{f^2 - gh}_{\in P^2} = \underbrace{x}_{\notin P} q \Rightarrow q \in P^{(2)}, q \notin P^2$$

$\deg 18 = \deg 3 + \deg 15$ In P^2, elements have $\deg \geq 16$

But actually, $P^{(3)} \subseteq P^2$.

Containment Problem When is $I^{(a)} \subseteq I^b$?

theorem (Ein-Lazarsfeld-Smith 2001, Hochster-Huneke 2002, Ma-Schwede 2017)

$$I^{(hn)} \subseteq I^n \quad \text{for all } n \geq 1$$

$$\Rightarrow I^{((\dim R)n)} \subseteq I^n \quad \text{for all } n \geq 1$$

Example $P \sim (t^3, t^4, t^5)$ $h = 2$ $P^{(2n)} \subseteq P^n \ \forall n \Rightarrow P^{(4)} \subseteq P^2$.

<u>Question</u> (Huneke, 2000) P prime of ht 2 in RLR. Is $P^{(3)} \subseteq P^2$?

Conjecture (Harbourne, 2008) $I^{(hn-h+1)} \subseteq I^n$ for all $n \geq 1$

<u>Note</u> In char $p>0$, $I^{(hq-h+1)} \subseteq I^{[q]} \subseteq I^q$ $\qquad q=p^e$

<u>Example</u> (Dumnicki, Szemberg, Tutaj-Gasińska, 2013)

$\exists$ radical (not prime) ideal in $\mathbb{C}[x,y,z]$, $h=2$ st

$I^{(2n-1)} \not\subseteq I^n$ for $n=2$. $\quad (I^{(3)} \not\subseteq I^2)$

Harbourne's Conjecture does hold for:

- General points in $\mathbb{P}^2$ (Bocci–Harbourne) and $\mathbb{P}^3$ (Dumnicki)
- In char p, if R/I is F-pure. (G-Huneke)

<u>Stable Harbourne</u> $\quad I^{(hn-h+1)} \subseteq I^n$ for $n \gg 0$.

<u>Question</u> $\quad I^{(hn-h+1)} \subseteq I^n$ for some $n \overset{?}{\Longrightarrow}$ all $n \gg 0$?

<u>Remark</u> If yes, then we are done in char p.

Remark the answer is provided I verifies:

$$I^{(n+h)} \subseteq I\, I^{(n)} \quad \text{for all } n \geq 1$$

False in general, but

thm If R/I is F-pure, $I^{(n+h)} \subseteq I\, I^{(n)}$ for all $n \geq 1$

Theorem If $I^{(hk-h)} \subseteq I^k$ for some n then

$$I^{(hn-h)} \subseteq I^n \quad \text{for all } n \gg 0$$

+ Schwede in mixed char

No counterexamples to $I^{(hn-C)} \subseteq I^n$ for $n \gg 0$, C fixed.

Resurgence (Bocci - Harbourne) $\rho(I) = \sup\left\{ \frac{a}{b} : I^{(a)} \not\subseteq I^b \right\}$

$$1 \leq \rho(I) \leq h$$

Remark If $\rho(I) < h$, then Stable Harbourne holds.

$$\frac{hn-C}{n} > \rho(I) \implies I^{(hn-C)} \subseteq I^n$$

C fixed

$$\Updownarrow$$

$$n > \frac{C}{h-\rho(I)}$$

Question Can $\rho(I) = h$?

Theorem (G–Huneke–Mukundan)

(R, m) RLR

Assume R/I Cohen-Macaulay, $\dim(R/I) = 1$

(More generally, $I^{(n)} = I^n : m^\infty = \bigcup_{k \geq 1} I^n : m^k$)

If $I^{(hn-h+1)} \subseteq m I^n$ for some n, then $\rho(I) < h$.

(and stable Harbourne holds)

Applications

1) I homogeneous ideal, generated in degree $a < h$, char 0

2) Space monomial curves $P \sim (t^a, t^b, t^c)$

$P^{(3)} \subseteq m P^2$ (using Knödel–Schenzel–Zonsarow implicit generators for $P^{(3)}$)

$\Downarrow$

$\rho(P) < 2$

and

stable Harbourne