

Frobenius or: how I learned to stop worrying and love char \neq

BIKES Spring 2020 30/01/2020

Recommended notes to char \neq reading:

- Frobenius splittings in CA, by Karen Smith and Wenzhang Zhang
- Karl Schwede's class notes
- Karen Smith's Winter 2019 Math 615 class notes

Why char \neq ?

→ char \neq is fun, and there are many char \neq specific questions that are interesting

→ there are many questions about rings that contain a field that can be reduced to char \neq via a method call reduction to char \neq

Sometimes we can prove a statement about $\frac{k[x_1, \dots, x_d]}{I}$, k any field, by:

- 1) Show that given a counterexample with $\text{char } k = 0$, we can build counterexamples in char \neq for infinitely many \neq
- 2) Prove our statement in char \neq .

this is often helpful since we can then use powerful tools we only have in char \neq .

\mathbb{R} domain of char \mathbb{R}

typical example : $R = \frac{k[x_1, \dots, x_d]}{I}$, $k = k^P$ a field of char P
 or a localization of such a ring

Frobenius map $F: \mathbb{R} \rightarrow \mathbb{R}$ $F(x) = x^p$

Freshman's dream $(a+b)^p = a^p + b^p \Rightarrow F$ is a ring homomorphism

What we may consider:

$\mathbb{R}^P = \mathbb{F}(R) =$ subring of R of all the elements that are p -powers

$\mathbb{R}^{1/p} = \{x^{1/p} : x \in \mathbb{R}\}$ (subring of some algebraic closure of \mathbb{R})

$\mathbb{F}_* R = R$ with the R -module structure given via Friedenius

$$\begin{array}{ccc} x \cdot s & = & F(x) s \\ \uparrow \cap & & \underbrace{}_{\text{product}} \\ R & \xrightarrow{*} & R \\ \text{R-mod action} & & \text{in } R \text{ now} \end{array}$$

$$\begin{array}{ccc} R & \xrightarrow{F} & R \\ \cong & \curvearrowleft C & \cong \\ R & \xrightarrow{\quad\quad\quad 1/p} & R \\ \{ & \text{purple wavy line} & \} \end{array}$$

we can identify the Frebenies map with this inclusion

Notation We will often consider powers of Frobenius, denoted F^e

Understanding \mathbb{R} as an \mathbb{R}^P -module

↓

Understanding $\mathbb{R}^{1/p}$ as an \mathbb{R} -module

↓

Understanding the \mathbb{R} -module $F_*\mathbb{R}$

and often, these are all the same as understanding the singularities of \mathbb{R}

Stay observations

- Frobenius is an exact functor
- the image of an ideal by Frobenius leads to Frobenius powers:
 $F(I) = I^{[p]} = (f^p : f \in I)$ more on these later

Common assumption \mathbb{R} is F-finite if $F_*\mathbb{R}$ is a fg \mathbb{R} -module

Example $k = k^p$ a perfect field is always F-finite

Fact F -finiteness is preserved by taking:

- quotients
 - localizations
 - fg algebras
 - completions at a max ideal
- ∴ algebras of finite type over a perfect field are always F-finite

Frobenius measures singularities

0) R is reduced $\Leftrightarrow 0$ is the only nilpotent element $\Leftrightarrow F$ is nilpotent

Note there is a condition called F -injective, and this is not it!

1) Theorem (Kunz, 1969) (R, \mathfrak{m}) noetherian local ring
 R is regular if and only if Frobenius is flat

Remark For a finite map between local rings, flat \Leftrightarrow free

If (R, \mathfrak{m}) is F -finite,

R is regular $\Leftrightarrow F_*(R)$ is a free R -module $\Leftrightarrow R$ is a free R^P -module

Example $\mathbb{F}_p[x_1, \dots, x_d]$ is a free module over $\mathbb{F}_p[x_1^p, \dots, x_d^p]$
with basis $\{x_1^{a_1} \dots x_d^{a_d} : 0 \leq a_i \leq p-1\}$

Philosophy How singular R is \Leftrightarrow how non-free F_*R is over R

2) R is F -split if F splits as a map of R -modules:

$$R \xrightarrow{\quad F \quad} R \quad \xleftarrow{\quad \phi \quad} \quad \phi \circ F = \text{id}_R$$

Equivalently, $\exists \psi \in \text{Hom}_R(F_*R, R)$ with $1 \in \text{im } \psi$

Equivalently, $\exists \chi \in \text{Hom}_{R^P}(R, R^P)$ with $1 \in \text{im } \chi$

Key idea: Nice properties pass onto direct summands.

Example $R = \mathbb{F}_p[x_1, \dots, x_d]$ is F-split:

$$R = \bigoplus_{0 \leq a_i \leq p-1} R^P \cdot x_1^{a_1} \cdots x_d^{a_d} \longrightarrow R^P \quad \text{sends } 1 \mapsto 1$$

projection onto the
 $x_1^{\circ} \cdots x_d^{\circ}$ coordinate

Fact All F-finite regular rings are F-split.

Indeed: $R = \bigoplus_n R^P \Rightarrow R^P$ is a direct summand of R (!)

Lemma Any direct summand of an F-split ring is F-split

$$\begin{array}{ccc}
 & \text{①} & \\
 & \curvearrowright \text{split} & \\
 R & \xrightarrow{\quad} & S \\
 \downarrow & & \downarrow \text{split} \\
 R^P & \xrightarrow{\quad} & S^P \\
 & \swarrow \text{split} & \\
 & \text{③} = \text{①}^P &
 \end{array}
 \quad \text{② splitting of } S^P \hookrightarrow S$$

the map $\xrightarrow{\quad \text{''} \quad}$ is a splitting of $R^P \rightarrow R$

Gallary Any Veronese is F-split

$k[\text{all monomials of degree } d \text{ in } v \text{ variables}] \subseteq k[v \text{ variables}]$

\downarrow
split!

Example $k[x, y, z]/(x^2 - y^2) \cong k[u^2, uv, v^2]$ is F-split

Def (Hochster-Roberts, 1974)

R is F-pure if Frobenius is a pure map:

$\text{For } 1: R \otimes M \longrightarrow R \otimes M$ is injective for all R -modules M

Fact If R is F-finite, F-pure \equiv F-split

Fedder's Criterion (Fedder, 1983) $(R, m) \text{ RLR, } I \text{ ideal in } R$

R F-pure $\Leftrightarrow (I^{[q]} : I) \not\subseteq m^{[q]}$ for all/some/large $q = p^e$

Remarks: • this is secretly a homological result!

• this can be tested by a computer, taking $q = p$

• F-finite is not necessary here!

Example Say I is monomial ideal

I not squarefree $\Rightarrow R/I$ not reduced $\Rightarrow R/I$ not F-pure

$$(I^{[p]} : I) = (g^{p-1} : g \text{ generator of } I) \subseteq m^{[p]}$$

$\therefore R/I$ F-pure
 $\Leftrightarrow I$ squarefree

if and only if
g all squarefree

Good philosophy Facts about squarefree monomial ideals extend to ideals defining F-pure rings.

Zoo of nice char & singularities

R regular

?

R F -pure

\mathbb{F}_*R free

\mathbb{F}_*R has a free summand

Def (Hochster - Huneke) R F -finite domain

R is strongly F -regular if

$\forall f \in R, f \neq 0 \exists e \quad R \rightarrow \mathbb{F}_*^e R \xrightarrow{F_*^e f} \mathbb{F}_*^e R$ splits

i.e. $\exists \varphi \in \text{Hom}_{\mathbb{F}_*^e R}(\mathbb{F}_*^e R, R)$ with $\varphi(F_*^e f) = 1$

Really: we can find free summands of $\mathbb{F}_*^e R$ in many ways

Fact Regular rings are strongly F -regular

claim Direct summands of strongly F -regular rings are strongly F -regular

Example I monomial ideal

R/I strongly F -regular $\Leftrightarrow I$ generated by variables

Example $k[x, y, z]/(x^2 + y^2 + z^2)$, $p \neq 2$ is strongly F -regular

Example $k[x, y, z]/(x^3 + y^3 + z^3)$ \rightarrow F -pure but
 $p \equiv 1 \pmod{3}$ not strongly F -regular

Strongly F-regular rings are Cohen-Macaulay and normal

Caution! F-pure rings are not necessarily Cohen-Macaulay.

Nice Consequence:

Theorem (Hochster-Roberts, 1974)

Rings of invariants of linearly reductive groups are Cohen-Macaulay.

This motivates the introduction of strong F-regularity

In char p , when G is finite and $p \nmid |G|$, $G \subset k[x_1, \dots, x_n]$

$$\begin{array}{ccc} R^G & \hookrightarrow & R \\ \text{algebra of elements} \\ \text{of } R \text{ fixed by } G & & \end{array} \xrightarrow{\quad r \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot r \quad} R^G \quad \text{is a splitting!}$$

If R^G is the ring of invariants of a linearly reductive group
there is a generalization of this map \Rightarrow a splitting

$\therefore R^G$ is strongly F-regular $\Rightarrow R^G$ is Cohen-Macaulay.

Hochster and Roberts then used reduction to char p to show
 R^G is Cohen-Macaulay even when $\text{char } k = 0$.