Symbolic powers and the (stable) containment Problem
Wisconsin Algebra and Algebraic Geometry Seminar, 19/04/2019
(joint work with Craig Huneke and Vivek Mukundan)

<u>Example</u> geometry of the curve parametrized by $(t^3, t^4, t^5)$ in $\mathbb{C}^3$
$x \quad y \quad z$
$\updownarrow$
algebra of the ideal $(x^3 - yz,\ y^2 - xz,\ z^2 - x^2 y)$ in $\mathbb{C}[x,y,z]$

these are the polynomials that vanish on our curve
(which is the vanishing set of those polynomials)

But how do we measure vanishing?

A polynomial $f$ vanishes <u>up to order $n$</u> along $\underset{\mathbb{C}^d}{\overset{\text{variety}}{X}}$ $\longleftrightarrow$ on $\underset{R = \mathbb{C}[x_1, \ldots, x_d]}{\overset{\text{radical ideal}}{I}}$ if

- $f$ as a power series centered at all $x \in X$ has no terms of order $< n$.
- $g \dfrac{\partial^{a_1 + \cdots + a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}}(f) \in I$ for all $a_1 + \cdots + a_d \leqslant n-1$, $g \in R$
- $f \in \bigcap_{x \in X} m_x^n = \bigcap_{\substack{m \supseteq I \\ m \text{ max}}} m^n$ $\quad m_x$ maximal ideal corresponding to $x$

$$I^n = (f_1 \cdots f_n : f_i \in I)$$

the $n$-th symbolic power of the prime $\mathfrak{P}$ ($\Leftrightarrow$ irreducible variety)

$$\mathfrak{P}^{(n)} = \{ f \in R \; : \; \frac{sf}{s} = \frac{f}{1} \in R_{\mathfrak{P}} \}$$
$$s \notin \mathfrak{P}$$

$$= \{ f \in R : s f \in \mathfrak{P}^n , \; s \notin \mathfrak{P} \}$$

$$= \mathfrak{P}^n R_{\mathfrak{P}} \cap R \qquad \text{"polynomials locally in } \mathfrak{P}^n \text{"}$$

$I$ radical ideal $\qquad I = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_k$, $\mathfrak{P}_i$ prime

$$I^{(n)} = \bigcap_i I^n R_{\mathfrak{P}_i} \cap R$$
$$= \{ f \in R : \; sf \in I^n, \; s \notin \cup \mathfrak{P}_i \}$$

<u>theorem</u> (Zariski-Nagata)

$I \subseteq \mathbb{C}[x_1, \dots, x_d]$ radical ideal $\longleftrightarrow$ $X \subseteq \mathbb{C}^d$ variety

$$I^{(n)} = \{ f \in R : f \text{ vanishes up to order } n \text{ along } X \}$$

<u>Facts</u>

1) $I^n \subseteq I^{(n)}$

2) $I^{(n+1)} \subseteq I^{(n)}$

3) If $I = $ (reg seq), then $I^{(n)} = I^n \quad \forall n \geq 1$

In general $I^{(n)} \neq I^n$.

Example geometry of the curve parametrized by $(\underset{x}{t^3}, \underset{y}{t^4}, \underset{z}{t^5})$ in $\mathbb{C}^3$

$\updownarrow$

algebra of the ideal $P = (\underbrace{x^3 - yz}_{\substack{f \\ \deg 9}}, \underbrace{y^2 - xz}_{\substack{g \\ \deg 8}}, \underbrace{z^2 - x^2y}_{\substack{h \\ \deg 10}})$ in $\mathbb{C}[x,y,z]$

$\deg x = 3$
$\deg y = 4$
$\deg z = 5$

$$P^{(2)} \supseteq P^2 = (f^2, g^2, h^2, fg, gh, fh)$$

$\hookrightarrow$ degrees $\geq 16$

$$\underbrace{f^2 - gh}_{\substack{\in P^2 \\ \deg 18}} = \underbrace{x}_{\substack{\notin P \\ \downarrow \\ \deg 3}} q \implies q \in P^{(2)}$$

$\searrow \deg 15 \implies q \notin P^2$

But $P^{(3)} \subsetneq P^2 \subsetneq P^{(2)}$

Containment Problem when is $I^{(a)} \subseteq I^b$ ?

theorem (Ein-Lazarsfeld-Smith 2001, Hochster-Huneke 2002, Ma-Schwede 2017)

$R = k[x_1, \ldots, x_d]$, $k$ a field or $\mathbb{Z}_p$

$I = P_1 \cap \cdots \cap P_k$ radical ideal.

then $I^{(hn)} \subseteq I^n$ for all $n \geq 1$, where

$h =$ big height of $I = \max\{\operatorname{codim} P_i\}$

In our example, $h=2$, so $P^{(2n)} \subseteq P^n$ for all $n \geq 1$.
In particular, $P^{(4)} \subseteq P^2$. But we can do better: $P^{(3)} \subseteq P^2$.

__Question__ (Huneke, 2000) $P$ prime of codim 2 in $k[x,y,z]$. Is $P^{(3)} \subseteq P^2$?

__Conjecture__ (Harbourne, 2008) $I^{(hn-h+1)} \subseteq I^n$ for all $n \geq 1$, $I$ radical

__Fact__ In char $p$, $I^{(hp^e-h+1)} \subseteq I^{[p^e]} \subseteq I^{p^e}$ $\forall e \geq 1$.

__Counterexample__ (Dumnicki – Szemberg – Tutaj-Gasińska, Harbourne – Seceleanu)
2013, 2015

$$I = (x(y^3-z^3),\ y(z^3-x^3),\ z(x^3-y^3)) \subseteq k[x,y,z]$$

char $\neq 3$

$h=2$, $I^{(3)} \not\subseteq I^2$.

But Harbourne's Conjecture does hold for:

- General points in $\mathbb{P}^2$ (Harbourne-Huneke) and $\mathbb{P}^3$ (Dumnicki)
- squarefree monomial ideals
- In char $p>0$, if $R/I$ is $F$-pure (G-Huneke)
  (eg, determinantal rings, Veronese)

__Stable Harbourne__ $I^{(hn-h+1)} \subseteq I^n$ for $n \gg 0$.

__Question__ If $I^{(hn-h+1)} \subseteq I^n$ for some $n \overset{?}{\Rightarrow}$ all $n \gg 0$?

In char $p$, a positive answer would finish the job.

Theorem (G) If $I^{(hk-h)} \subseteq I^k$ for some $k$, then $I^{(hn-h)} \subseteq I^n$ for $n \gg 0$

**Resurgence** (Bocci–Harbourne) $\rho(I) = \sup\{\frac{a}{b} : I^{(a)} \not\subseteq I^b\}$

**Facts** $\quad 1 \le \rho(I) \le h$

**Remark** If $\rho(I) < h$, then Harbourne stable holds.
In fact, for any $c > 0$, $I^{(hn-c)} \subseteq I^n$ for $n \gg 0$

**Theorem** (G–Huneke–Mukundan)
If $R/I$ is Cohen–Macaulay, $\dim(R/I) = 1$.
(More generally, if $I^{(n)} = (I^n)^{sat} = I^n : m^\infty = \bigcup_{k \ge 1} (I^n : m^k)$)
If $I^{(hn-h+1)} \subseteq m I^n$ for some $n$, then $\rho(I) < h$.
In particular, Harbourne stable holds.

**Applications**

① If $I$ is a homogeneous ideal generated in degree $a < h$, char 0.

**Sketch** $I^{(hn)} \underset{\substack{\uparrow \\ \text{use differential operators} \\ \text{Leibniz Rule}}}{\subseteq} m I^{(hn-1)} \subseteq m^2 I^{(hn-2)} \subseteq \dots \subseteq m^{hn-1} I$

$$m^{hn-1} I \subseteq m^{hn-1+a}$$

so $$I^{(hn)} \subseteq m^{hn-1+a} \cap I^n = m^{\overbrace{hn+a-1-an}^{\ge 1 \text{ if } a<h}} I^n$$

② Space monomial curves $P \sim (t^a, t^b, t^c)$ over a field $k$

<u>theorem</u> (G-Huneke-Mukundan)
Let $k$ be a field of char $k \neq 3$
$P = \ker (k[x,y,z] \longrightarrow k[t^a, t^b, t^c])$
$P^{(3)} \subseteq m P^2$, and $\rho(P) < h$.
In particular, $P$ verifies stable Harbourne.

<u>key ingredients</u>:

- $P^{(3)} \subseteq P^2$ (G)
- Implicit generators for $P^{(3)}$ (Knödel-Schenzel-Zonsarow)

<u>Conjecture</u> (Harbourne-Huneke)

$$I^{(hn)} \subseteq m^{hn-n} I^n \text{ for all } n \geq 1 \qquad (*)$$

<u>theorem</u> (G-Huneke-Mukundan)
If $(*)$ holds, then Harbourne stable holds.
(as long as $I^{(n)} = (I^n)^{sat}$)