Symbolic powers
University of Nebraska–Lincoln Colloquium 13/01/2020

Fundamental Theorem of Arithmetic

For every $n \in \mathbb{Z}$, $\exists$ primes $p_1, \ldots, p_k$, and $a_1, \ldots, a_k \geq 1$ st

$$n = \pm p_1^{a_1} \cdots p_k^{a_k},$$

and this is unique up to permutation.

Example In $\mathbb{Z}[\sqrt{-5}] = \mathbb{Z} + \mathbb{Z}\sqrt{-5}$,

$$6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$$ unique factorization fails!

to salvage this, there is primary decomposition
— a statement about ideals.

ideals in $\mathbb{Z}$ $\longrightarrow$ sets of multiples of a fixed integer

$$(n) = \{kn \mid k \in \mathbb{Z}\}$$

Fundamental Theorem of Arithmetic

For every ideal (n) in $\mathbb{Z}$ $\exists$ primes $p_1, \ldots, p_k$, and $a_1, \ldots, a_k \geq 1$ st

$$(n) = (p_1^{a_1}) \cap \cdots \cap (p_k^{a_k})$$

and this is unique up to permutation.

What's special about the ideals (p^a)? they are primary!

A proper ideal I is prime if $xy \in I \Rightarrow x \in I$ or $y \in I$

A proper ideal I is primary if $xy \in I \Rightarrow x \in I$ or $y^n \in I$ for some $n \geq 1$

Fact If I is primary, $\sqrt{I} = \{f : f^n \in I\}$ is prime.

radical of I

so: Fundamental Theorem of Arithmetic

For every ideal (n) in $\mathbb{Z}$, $\exists$ primary ideals $(p_1^{a_1}), \dots, (p_k^{a_k})$ st

$$(n) = (p_1^{a_1}) \cap \dots \cap (p_k^{a_k})$$

and this is unique up to permutation.

How do we write this as a theorem about other rings?

In $R = k[x_1, \dots, x_d]$ (k a field), ideals $\equiv$ systems of equations

ideal $I = (f_1, \dots, f_n) = \{g_1 f_1 + \dots + g_n f_n \mid g_i \in R)$

Theorem (Lasker 1905, Noether 1921)

For any ideal I in $R = k[x_1, \dots, x_d]$ there exist primary ideals st

$$I = \underbrace{Q_1 \cap \dots \cap Q_k}_{\text{primary decomposition}}$$

(where we can choose $\sqrt{Q_i}$ all different, and we don't write unnecessary terms)

(Note that these $\sqrt{Q_i}$ are prime ideals that contain I)

The minimal components are uniquely determined:

Among the $\sqrt{Q_i}$ we will see every minimal prime containing I and the corresponding minimal components are unique.

minimal   embedded

Example $(x^2, xy) = \boxed{(x)} \cap \boxed{(x^2, xy, y^2)}$

$= (x) \cap (x^2, xy, y^n) \quad n \geq 1$

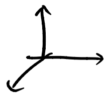
(x) unique   can change (x, y)

Example In $\mathbb{Z}[\sqrt{-5}]$,

$$(6) = \underset{\substack{\downarrow \\ \text{primary}}}{(2)} \cap \underset{\substack{\downarrow \\ \text{not!}}}{(3)} = \underbrace{(2) \cap (3, 1+\sqrt{-5}) \cap (3, 1-\sqrt{-5})}_{\substack{\text{primary decomposition} \\ \text{unique!}}}$$

Primary decomposition saves the day!

<u>Example</u> $(xy, xz, yz) = (x,y) \cap (x,z) \cap (y,z)$

 $= \uparrow \cup \longrightarrow \cup \swarrow$

So what are primary ideals, really? Powers of primes?

P prime:

$P^n = (f_1 \cdots f_n \mid f_i \in P)$ not always primary! (examples soon)

$P^n = \underbrace{\text{minimal primary component with radical } P}_{\substack{\downarrow \\ \text{uniquely determined!} \\ \|}} \cap$ embedded components

$P^n \subseteq P^{(n)} = \{ f \mid sf \in P^n \text{ for some } s \notin P \}$

$=$ smallest primary ideal with radical P containing P^n

$I = P_1 \cap \cdots \cap P_k$ radical $\Rightarrow I^{(n)} = P_1^{(n)} \cap \cdots \cap P_k^{(n)}$

theorem (Zariski–Nagata) $\quad R = \mathbb{C}[x_1, \dots, x_d]$

$$P^{(n)} = \{ f \mid f \text{ vanishes to order } n \text{ along the variety defined by } P \}$$

$$= \left\{ f \in R \;\middle|\; \frac{\partial^{a_1 + \cdots + a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} f \in P \quad \forall a_1 + \cdots + a_d < n \right\}$$

thm (De Stefani – G – Jeffries, 2018)

A version of Zariski–Nagata in mixed characteristic (eg, $R = \mathbb{Z}[x_1, \dots, x_d]$)

Example (space monomial curves) curve $\{(t^3, t^4, t^5) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}$

ideal $P = (\underbrace{x^3 - yz}_{f}, \underbrace{y^2 - xz}_{g}, \underbrace{z^2 - x^2 y}_{h})$ in $\mathbb{C}[x, y, z]$

deg 9 $\qquad$ deg 8 $\qquad$ deg 10

$$P^2 = (f^2, g^2, h^2, fg, fh, gh) \longrightarrow \deg \geq 16$$

but $\quad \underbrace{f^2 - gh}_{\in P^2} = \underset{P}{x}\, q \implies q \in P^{(2)}$

deg 18 $=$ deg 3 $+$ $\qquad$ deg 15 $\qquad \therefore q \notin P^2$

so $P^2 \subsetneq P^{(2)}$

Fun fact: the symbolic powers of space monomial curves $\{(t^a, t^b, t^c)\}$ can exhibit strange behavior: they can have unexpected elements in arbitrarily high degrees

Big Questions

1) Find generators for $I^{(n)}$.

Macaulay2 software package: Drabkin – G – Seceleanu – Stone
with contributions from Andrew Conner and Diana Zhong

2) When is $I^n = I^{(n)}$?

Many sufficient conditions for $I^n = I^{(n)}$ $\forall n \geq 1$

No clean, efficient complete characterization for what classes of I have this property for all n.

Big open Problem: a characterization over monomial ideals
Packing Problem

3) What degrees does $I^{(n)}$ live in?

Chudnovsky's Conjecture (open): lower bounds

4) Compare I^n and $I^{(n)}$

Containment Problem When is $I^{(a)} \subseteq I^b$?

Theorem (Ein–Lazarsfeld–Smith 2001, Hochster–Huneke 2002, Ma–Schwede 2017)

$R = k[x_1, \dots, x_d]$, k field or $\mathbb{Z}[x_1, \dots, x_d]$

$$I^{(dn)} \subseteq I^n \text{ for all } n \geq 1, \text{ ideal } I$$

More precisely:

P prime of codim h $\Rightarrow$ $P^{(hn)} \subseteq P^n$ $\forall n \geq 1$

$I = P_1 \cap \dots \cap P_k$ radical, $h =$ largest codim P_i $\Rightarrow$ $I^{(hn)} \subseteq I^n$ $\forall n \geq 1$.

Example $P \sim (t^3, t^4, t^5)$ codim 2 so $P^{(2n)} \subseteq P^n$ $\forall n$, $P^{(4)} \subseteq P^2$

Question (Huneke, 2000) P prime of codim 2. Is $P^{(3)} \subseteq P^2$?

Theorem (G) In char $\neq 3$, $P \sim (t^a, t^b, t^c)$ $\forall a,b,c$ $P^{(3)} \subseteq P^2$

Notes:
- the skeleton of the proof follows work of Seceleanu
- these curves are very interesting:
  their symbolic powers exhibit all sorts of bad behavior

$\bigoplus_{n \geq 1} I^{(n)}$ symbolic Rees algebra — not always fg

Conjecture (Harbourne, 2008) $I^{(hn-h+1)} \subseteq I^n \ \forall n \geqslant 1$

Fact (Hochster-Huneke) this holds in char p for $n = p^e$.

Counterexample (Dumnicki - Szemberg - Tutaj - Gasińska)
$\exists$ radical I, $h=2$ such that $I^{(3)} \not\subseteq I^2$ in $\mathbb{C}[x,y,z]$
(Harbourne - Seceleanu) Extended this to a family, any field (char $\neq 2$)

But Harbourne's Conjecture holds for nice radical ideals:

- General points in $\mathbb{P}^2$ (Bocci - Harbourne) and $\mathbb{P}^3$ (Dumnicki)
- I squarefree monomial ideal
- R/I determinantal ring, Veronese ring, nice ring of invariants

$\Big\}$ F-pure! (nice singularities)

$\rightarrow$ strongly F-regular

Theorem (G-Huneke, 2017) $R = k[x_1, \dots, x_d]$
(G-Ma-Schwede, 2019) R F-finite Gorenstein, pdim $I < \infty$.

- If R/I is F-pure, then I satisfies Harbourne's Conjecture
- If R/I is strongly F-regular, can substitute h by $h-1$.
So for $h=2$, get $I^n = I^{(n)} \ \forall n \geqslant 1$.