

last time

Yoneda lemma

$\mathcal{G}$  locally small category

1 object in  $\mathcal{G}$

$F: \mathcal{G} \rightarrow \text{Set}$  covariant functor

then there is a bijection  $\text{Nat}(\text{Hom}_{\mathcal{G}}(A, -), F) \xrightarrow{\cong} F(A)$

Consequence Set  $F = \text{Hom}_{\mathcal{G}}(B, -)$ . then

$\text{Nat}(\text{Hom}_{\mathcal{G}}(A, -), \text{Hom}_{\mathcal{G}}(B, -)) \xrightarrow{\cong} \text{Hom}_{\mathcal{G}}(B, A)$

Theorem (Yoneda Embedding)  $\mathcal{G}$  locally small category

$$\mathcal{G} \longrightarrow \text{Set}^{\mathcal{G}^{\text{op}}}$$

$$A \longmapsto \text{Hom}_{\mathcal{G}}(-, A)$$

$$\mathcal{G}^{\text{op}} \longrightarrow \text{Set}^{\mathcal{G}}$$

$$A \longmapsto \text{Hom}_{\mathcal{G}}(-, A)$$

$$\begin{array}{ccc}
 A & \longmapsto & \text{Hom}_{\mathcal{G}}(-, A) \\
 f \downarrow & \longmapsto & \downarrow f^* \\
 B & \longmapsto & \text{Hom}_{\mathcal{G}}(-, B)
 \end{array}$$

$$\begin{array}{ccc}
 A & \longmapsto & \text{Hom}_{\mathcal{G}}(A, -) \\
 f \downarrow & \longmapsto & \downarrow f_* \\
 B & \longmapsto & \text{Hom}_{\mathcal{G}}(B, -)
 \end{array}$$

are embeddings.

Stegan Every locally small category embeds into a functor category to set

- A covariant functor  $F: \mathcal{G} \rightarrow \text{Set}$  is representable if it is naturally isomorphic to  $\text{Hom}_{\mathcal{G}}(A, -)$  for some  $A$
- A contravariant functor  $F: \mathcal{G} \rightarrow \text{Set}$  is representable if it is naturally isomorphic to  $\text{Hom}_{\mathcal{G}}(-, B)$  for some  $B$

Ex  $\text{Id}: \text{Set} \rightarrow \text{Set}$  is represented by  $1$  (singleton)

Why?  $X \in \text{Set}$

$$\begin{array}{lcl} X & \cong & \text{functions } 1 \rightarrow X \\ x & \mapsto & 1 \mapsto x \end{array} \equiv \text{Hom}_{\text{Set}}(1, X)$$

the following commutes:

$$\begin{array}{ccc} \text{Hom}_{\text{Set}}(1, X) & \xrightarrow{\cong} & X \\ f_* \downarrow & & \downarrow f \\ \text{Hom}_{\text{Set}}(1, Y) & \xrightarrow{\cong} & Y \end{array}$$

so our natural isomorphism is

$$x \mapsto (\text{Hom}_{\text{Set}}(1, X) \xrightarrow{\cong} X)$$

## Complex map / chain map

A map of complexes  $f: F \rightarrow G$  between complexes  $F, G$  is a sequence of  $R$ -module homomorphisms  $f_n: F_n \rightarrow G_n$  such that the diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & F_{n+1} & \xrightarrow{d_{n+1}} & F_n & \xrightarrow{d_n} & F_{n-1} \longrightarrow \dots \\ & & f_{n+1} \downarrow & & f_n \downarrow & & \\ \dots & \longrightarrow & G_{n+1} & \xrightarrow{d_{n+1}} & G_n & \xrightarrow{d_n} & G_{n-1} \longrightarrow \dots \end{array}$$

Commutes.

so

$$f_n d_{n+1} = d_{n+1} f_{n+1}$$

Ex: 0 map  $0: F_n \rightarrow G_n$  for all  $n$

identity:  $F_n \xrightarrow{=} F_n$  for all  $n$

Def the category of chain complexes of  $R$ -modules

$\mathbf{Ch}(R)$  or  $\mathbf{Ch}(R\text{-mod})$  has:

- objects all complexes of  $R$ -modules
- arrows all maps of complex

eg  $\mathbf{Ch}(\mathbf{Ab}) \cong \mathbf{Ch}(\mathbf{Z})$

Lemma  $h: F \rightarrow G$  map of complexes

then  $h$  induces  $R$ -module homomorphisms

$$\begin{aligned} \mathcal{B}_n(h) : \mathcal{B}_n(F) &\longrightarrow \mathcal{B}_n(G) \\ \mathcal{Z}_n(h) : \mathcal{Z}_n(F) &\longrightarrow \mathcal{Z}_n(G) \end{aligned} \quad \left( \text{recall: } H_n(F) = \frac{\mathcal{Z}_n(F)}{\mathcal{B}_n(F)} \right)$$

$$\implies H_n(h) : H_n(F) \longrightarrow H_n(G)$$

Proof

$$\begin{array}{ccccc} F_{n+1} & \xrightarrow{d_{n+1}} & F_n & \xrightarrow{d_n} & F_{n-1} \\ h_{n+1} \downarrow & & \downarrow h_n & & \downarrow h_{n-1} \\ G_{n+1} & \xrightarrow{d_{n+1}} & G_n & \xrightarrow{d_n} & G_{n-1} \end{array}$$

- If  $a \in \mathcal{B}_n(F) = \ker d_{n+1}^F$ , say  $a = d_{n+1}(b)$ ,  $b \in F_{n+1}$ ,  
 $h_n(a) = h_n d_{n+1}(b) = d_{n+1} h_{n+1}(b) \in \ker d_{n+1}^G = \mathcal{B}_n(G)$
- If  $a \in \mathcal{Z}_n(F) = \ker d_n^F$ , then

$$d_n h_n(a) = h_{n-1} \underbrace{d_n(a)}_{=0} = h_{n-1}(0) = 0$$

$$\Rightarrow h_n(a) \in \ker d_n^G = \mathcal{Z}_n(G)$$

$$\therefore H_n(F) = \frac{\mathcal{Z}_n(F)}{\mathcal{B}_n(F)} \longrightarrow \frac{\mathcal{Z}_n(G)}{\mathcal{B}_n(G)}$$

$H_n : \text{Ch}(R) \rightarrow R\text{-mod}$  is a functor!

$$F \longmapsto H_n(F)$$

$$\begin{array}{ccc} F & & H_n(F) \\ f \downarrow & \longmapsto & \downarrow H_n(f) \\ G & & H_n(G) \end{array}$$

Def A map of complexes is a quasi-iso if it induces an iso in homology, so  $H_n(f)$  is an iso for all  $n$ .

Ex:

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow & 0 \\ 0 \downarrow & & \downarrow \pi & & \downarrow 0 \\ 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \end{array}$$

is a quasi-iso  
but not an iso.

Def  $f, g : F \rightarrow G$  maps of complexes

A homotopy  $h$  between  $f$  and  $g$  is a sequence of maps

$$h_n : \tilde{F}_n \longrightarrow G_{n+1}$$

$$\begin{array}{ccccccc} \dots & \longrightarrow & F_{n+1} & \xrightarrow{d_{n+1}} & \tilde{F}_n & \longrightarrow & F_{n-1} \longrightarrow \dots \\ & & f_{n+1} \downarrow \lvert g_{n+1} & \swarrow h_n & f_n \downarrow \lvert g_n & \swarrow h_{n-1} & f_{n-1} \downarrow \lvert g_{n-1} \\ \dots & \longrightarrow & G_{n+1} & \xrightarrow{d_{n+1}} & G_n & \longrightarrow & G_{n-1} \longrightarrow \dots \end{array}$$

such that  $d_{n+1} h_n + h_{n-1} d_n = f_n - g_n$  for all  $n$ .

We say  $f$  and  $g$  are homotopic

Exercise: Homotopy is an equivalence relation.

Lemma: Homotopic maps induce the same map on homology

Proof: h homotopy between f and g: F → G

$$a \in Z_n(F) \Rightarrow f_n(a) - g_n(a) = d_{n+1} h_n(a) + \underbrace{p_{n-1} d_n(a)}_{=0} \in B_n(F)$$

∴  $f_n - g_n$  is 0 in homology!

$$H(f-g) = 0 \Rightarrow H(f) = H(g)$$

□

Null homotopic ≡ homotopic to 0 ⇒ induce 0-map on homology

$f: F \rightarrow G$  is a Homotopy Equivalence if there exists some  $g: G \rightarrow F$  such that  $fg$  and  $gf$  are homotopic to the identity  
(so a homotopy equivalence has an inverse up to homotopy)

homotopy equivalence ⇒ quasi-isomorphism

Why?  $H_n(f) H_n(g) = H_n(fg) = H_n(\text{id}) = \text{id}$

⇒  $H_n(f), H_n(g)$  are iso of R-modules

Warning: quasi-iso  $\not\Rightarrow$  homotopy equivalence.