Frobenius or: how I learned to stop worrying and love char p
BIKES Spring 2020 30/01/2020

Recommended intro to char p reading:

- Frobenius splittings in CA, by Karen Smith and Wenliang Zhang
- Karl Schwede's class notes
- Karen Smith's Winter 2019 Math 615 class notes

Why char p?

→ char p is fun, and there are many char p specific questions that are interesting

→ there are many questions about rings that contain a field that can be reduced to char p via a method call reduction to char p

Sometimes we can prove a statement about $\frac{k[x_1, \ldots, x_d]}{I}$, k any field, by:

1) Show that given a counterexample with char $k = 0$, we can build counterexamples in char p for infinitely many p

2) Prove our statement in char p.

this is often helpful since we can then use powerful tools we only have in char p.

R domain of char p

typical example: $R = \dfrac{k[x_1, \dots, x_d]}{I}$, $k = k^p$ a field of char p

or a localization of such a ring

Frobenius map $F: R \to R$ $\quad F(r) = r^p$

Freshman's dream $(a+b)^p = a^p + b^p \Rightarrow F$ is a ring homomorphism

What we may consider:

$R^p = F(R)$ = subring of R of all the elements that are p-powers

$R^{1/p} = \{ r^{1/p} : r \in R \}$ (subring of some algebraic closure of R

$F_* R$ = R with the R-module structure given via Frobenius

$$\underset{\substack{R \\ R\text{-mod action}}}{r} \cdot \underset{F_* R}{s} = \underbrace{F(r)s}_{\substack{\text{product} \\ \text{in } R \text{ now}}} = r^p s$$

$$\underbrace{\begin{array}{ccc} R & \xrightarrow{F} & R \\ \wr\| & \circlearrowleft & \wr\| \\ R & \hookrightarrow & R^{1/p} \end{array}}_{} \qquad \begin{array}{c} r \\ \downarrow \\ r^{1/p} \end{array}$$

we can identify the Frobenius map with this inclusion

Notation We will often consider powers of Frobenius, denoted F^e

Understanding R as an R^p-module

$\Updownarrow$

Understanding $R^{1/p}$ as an R-module

$\Updownarrow$

Understanding the R-module $F_* R$

and often, these are all the same as understanding the singularities of R

Stray observations

- Frobenius is a right exact functor (think $F_*(R) \otimes_R -$)
- Frobenius is exact $\iff$ R is regular (Kunz, 1969)
- the image of an ideal by Frobenius leads to Frobenius powers:

$$F(I) =: I^{[p]} = (f^p : f \in I) \qquad \text{more on these later}$$

Common assumption R is F-finite if $F_* R$ is a fg R-module

Example $k = k^p$ a perfect field is always F-finite

Fact F-finiteness is preserved by taking:

- quotients
- localizations
- fg algebras
- completions at a max ideal

$\therefore$ algebras of finite type over a perfect field are always F-finite

Frobenius measures singularities

0) R is reduced $\iff$ 0 is the only nilpotent element $\iff$ F is nilpotent

Note there is a condition called F-injective, and this is not it!

1) **theorem** (Kunz, 1969) (R, m) noetherian local ring
R is regular if and only if Frobenius is flat

Remark For a finite map between local rings, flat $\iff$ free

If (R, m) is F-finite,
R is regular $\iff$ $F_*(R)$ is a free R-module $\iff$ R is a free R^p-module

Example $\mathbb{F}_p[x_1, \dots, x_d]$ is a free module over $\mathbb{F}_p[x_1^p, \dots, x_d^p]$
with basis $\{ x_1^{a_1} \cdots x_d^{a_d} : 0 \leq a_i \leq p-1 \}$

Philosophy How singular R is $\iff$ how non-free F_*R is over R

2) R is F-split if F splits as a map of R-modules:

$$\begin{tikzcd} R \arrow[r, "F"] & R \arrow[l, dashed, bend left, "\phi"] \end{tikzcd} \qquad \phi \circ F = id_R$$

Equivalently, $\exists\, \varphi \in \mathrm{Hom}_R(F_*R, R)$ with $1 \in \mathrm{im}\, \varphi$
Equivalently, $\exists\, \chi \in \mathrm{Hom}_{R^p}(R, R^p)$ with $1 \in \mathrm{im}\, \chi$

Key idea : Nice properties pass onto direct summands.

Example $R = \mathbb{F}_p[x_1, \dots, x_d]$ is F-split:

$$R = \bigoplus_{0 \leq a_i \leq p-1} R^p \cdot x_1^{a_1} \cdots x_d^{a_d} \longrightarrow R^p \qquad \text{sends } 1 \mapsto 1$$

projection onto the $x_1^0 \cdots x_d^0$ coordinate

Fact All F-finite regular rings are F-split.

Indeed: $R = \bigoplus_n R^p \Longrightarrow R^p$ is a direct summand of R (!)

Lemma Any direct summand of an F-split ring is F-split

$$\begin{tikzcd}
R \arrow[r, hook, "\text{split}"] \arrow[r, bend left=50, "①"] \arrow[d, bend right=50, "\text{splitting}"'] & S \arrow[d, dashed, bend left=50, "② \text{ splitting of } S^p \hookrightarrow S"] \\
R^p \arrow[u, hook] \arrow[r] & S^p \arrow[u, hook, "\text{split}"'] \arrow[l, dashed, bend left=50, "③ = ①^p"]
\end{tikzcd}$$

the map $\dfrac{\longrightarrow}{\overset{\shortparallel}{\longrightarrow}}$ is a splitting of $R^p \longrightarrow R$

Corollary Any Veronese is F-split

k[all monomials of degree d in v variables] $\subseteq k$[v variables]

$\downarrow$ split!

Example $k[x, y, z] / (xz - y^2) \cong k[u^2, uv, v^2]$ is F-split

Def (Hochster–Roberts, 1974)
R is F-pure if Frobenius is a pure map:
$F \otimes 1 : R \otimes M \longrightarrow R \otimes M$ is injective for all R-modules M

Fact If R is F-finite, F-pure $\equiv$ F-split

Fedder's Criterion (Fedder, 1983) (R, m) RLR, I ideal in R

R F-pure $\iff (I^{[q]} : I) \not\subseteq m^{[q]}$ for all/some/large $q = p^e$

Remarks: • this is secretly a homological result!
• this can be tested by a computer, taking $q = p$
• F-finite is not necessary here!

Example Say I is monomial ideal
I not squarefree $\Rightarrow R/I$ not reduced $\Rightarrow R/I$ not F-pure

$(I^{[p]} : I) = (g^{p-1} : g \text{ generator of } I) \subseteq m^{[p]}$

if and only if g all squarefree

$\therefore R/I$ F-pure
$\iff I$ squarefree

Good philosophy Facts about squarefree monomial ideals extend to ideals defining F-pure rings.

Zoo of nice char p singularities

R regular $\Updownarrow$ $F_* R$ free

?

R F pure $\Updownarrow$ $F_* R$ has a free summand

Def (Hochster – Huneke) R F-finite domain

R is strongly F-regular if

$\forall f \in R, f \neq 0 \quad \exists e \quad R \longrightarrow F_*^e R \xrightarrow{F_*^e f} F_*^e R$ splits

ie $\exists \varphi \in \mathrm{Hom}_R(F_*^e R, R)$ with $\varphi(F_*^e f) = 1$

Really: we can find free summands of $F_*^e R$ in many ways

Fact Regular rings are strongly F-regular

Lemma Direct summands of strongly F-regular rings are strongly F-regular

Example I monomial ideal

R/I strongly F-regular $\Longleftrightarrow$ I generated by variables

Example $k[x,y,z]/(x^2+y^2+z^2)$, $p \neq 2$ is strongly F-regular

Example $k[x,y,z]/(x^3+y^3+z^3)$ $\longrightarrow$ F-pure but not strongly F-regular

$p \equiv 1 \pmod 3$

Strongly F-regular rings are Cohen-Macaulay and normal

Caution! F-pure rings are not necessarily Cohen-Macaulay.

<u>Nice consequence</u>:

<u>Theorem</u> (Hochster-Roberts, 1974)

Rings of invariants of linearly reductive groups are Cohen-Macaulay.

This predates the introduction of strong F-regularity

In char p, when G is finite and $p \nmid |G|$, $G \circlearrowright k[x_1, \dots, x_d]$

$$\begin{array}{ccccc} R^G & \hookrightarrow & R & \longrightarrow & R^G \\ \| & & r & \longmapsto & \frac{1}{|G|} \sum_{g \in G} g \cdot r \end{array}$$ is a splitting!

algebra of elements of R fixed by G

If R^G is the ring of invariants of a linearly reductive group there is a generalization of this map $\Rightarrow$ a splitting

$\therefore$ R^G is strongly F-regular $\Rightarrow$ R^G is Cohen-Macaulay.

Hochster and Roberts then used reduction to char p to show R^G is Cohen-Macaulay even when char $k = 0$.