Symbolic Powers vs Homological Algebra. (Kumunu FR 2018, on 29th April, 2018)

R regular ring $\qquad$ I radical ideal $\qquad$ P prime ideal

h = big height of I = $\max\{ ht\, Q : Q \in Ass(I) \}$

the n-th symbolic power of the prime P is

$$\begin{aligned} P^{(n)} &= P^n R_P \cap R \\ &= \{ f \in R : gf \in P^n \text{ for some } g \notin P \} \\ &= \{ f \in R : \tfrac{f}{g} \in P^n R_P \text{ for some } g \notin P \} \end{aligned}$$

= P-primary component in an irredundant primary decomposition of P^n

= functions that vanish up to order n in the variety defined by P.

$$I^{(n)} = \bigcap_{Q \in Min(I)} \left(I^n R_Q \cap R \right)$$

Properties
- $I^n \subseteq I^{(n)}$
- $I^{(n+1)} \subseteq I^{(n)}$
- $I^n \neq I^{(n)}$ in general

Symbolic powers are hard to study. Even determining a minimal set of generators for $I^{(n)}$ or what degrees they live in, can be very hard.

Example $\quad P = \ker\left(k[x,y,z] \longrightarrow k[t^3, t^4, t^5] \right)$ $\qquad$ $\deg x = 3$, $\deg y = 4$, $\deg z = 5$

$$= (\underbrace{x^3 - yz}_{\substack{f \\ \deg 9}}, \underbrace{y^2 - xz}_{\substack{g \\ \deg 8}}, \underbrace{z^2 - x^2 y}_{\substack{h \\ \deg 10}})$$

$$P^2 \ni \underbrace{f^2 - gh}_{\deg 18} = q\underbrace{x}_{\deg 3}, \quad x \notin P \Rightarrow q \in P^{(2)} \Rightarrow \deg q = 15$$

But all elements in P^2 have degree $\geq 16 \Longrightarrow q \notin P^2$

$\therefore P^2 \subsetneq P^{(2)}$

Fun facts: $\quad$ 1) $P^n \neq P^{(n)}$ for all n $\qquad$ 2) $P^{(3)} \subseteq P^2$

Containment Problem (Schenzel, 1980s) When is $I^{(a)} \subseteq I^{b}$?

Theorem (Ein-Lazarsfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)
Let I be a radical ideal of big height h in a regular local ring R. Then

$$I^{(hn)} \subseteq I^{n} \text{ for all } n \geq 1.$$

Question (Huneke, 2000) P prime of height 2 in a RLR.
Does that imply $P^{(3)} \subseteq P^{2}$?

Conjecture (Harbourne, ≤ 2008) I radical ideal of big height h in a regular ring

$$I^{(hn-h+1)} \subseteq I^{n}.$$

Example (Dumnicki, Szemberg, Tutaj-Gasińska, 2013) char $k \neq 2, 3$

$$I = (x(y^a - z^a), y(z^a - x^a), z(x^a - y^a)) \subseteq k[x,y,z]$$

$$= I_2 \begin{pmatrix} x^{a-1} & y^{a-1} & z^{a-1} \\ yz & xz & xy \end{pmatrix} \qquad I^{(3)} \not\subseteq I^2$$

But! this example is not prime. Huneke's question remains open.

In fact, Harbourne's Conjecture holds for:

- points in general position in $\mathbb{P}^2$ (Harbourne-Huneke) and $\mathbb{P}^3$ (Dumnicki)
- monomial ideals
- ideals defining F-pure rings (G-Huneke) meaning, R/I F-pure.

From now on: $R = k[x,y,z]$, k a field.

open Question (Huneke, 2000) P prime of height 2. Is $P^{(3)} \subseteq P^2$?

theorem (–) char $k \neq 3$, $P = \ker(k[x,y,z] \to k[t^a, t^b, t^c])$.

Fact (Herzog) $P = I_2\begin{pmatrix} x^{\alpha_3} & y^{\beta_1} & z^{\alpha_2} \\ z^{\alpha_1} & x^{\alpha_2} & y^{\beta_3} \end{pmatrix}$

So really we want to study ideals of the form $I = I_2\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\right)$ $a_i, b_i \in R$.

We will follow Alexandra Seceleanu's work – she found conditions implying $I^{(3)} \not\subseteq I^2$. But we will use the same ingredients to obtain $I^{(a)} \subseteq I^b$.

key ingredients $I^{(n)} = I^n : m^\infty$ for all $n \geq 1$

Consequence:

$$I^{(a)} \subseteq I^b$$

$$\iff H^0_m(R/I^a) \xrightarrow{0} H^0_m(R/I^b)$$

$$\iff \operatorname{Ext}^3(R/I^b, R) \xrightarrow{0} \operatorname{Ext}^3(R/I^b, R) \qquad \text{(local duality)}$$

$$\iff \underbrace{\operatorname{Ext}^2(I^b, R) \xrightarrow{0} \operatorname{Ext}^2(I^a, R)}_{\text{map induced by } I^a \subseteq I^b} \qquad \text{(Ext shifting)}$$

one option: Find resolutions to all I^n, then determine lifts for $I^{n+1} \subseteq I^n$:

$$\begin{array}{ccccccccc} 0 \to & F_2 & \to & F_1 & \to & F_0 & \to & I^n & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ 0 \to & G_2 & \to & G_1 & \to & G_0 & \to & I^{n+1} & \to 0 \end{array}$$

Apply $\operatorname{Hom}_R(-, R)$
Compute homology.

How do we do this? We use Rees Algebras!

The Rees Algebra of I is the graded algebra $\oplus I^n t^n \subseteq R[t]$

there is a graded map $R[T_1, T_2, T_3] \twoheadrightarrow \mathcal{R}(I)$

$T_i \longmapsto f_i t$

In general, determining the kernel of this map is hard, but our setting is nice:

$$\mathcal{R}(I) \cong R[T_1,T_2,T_3] / (\underbrace{a_1T_1+a_2T_2+a_3T_3}_{F}, \underbrace{b_1T_1+b_2T_2+b_3T_3}_{G})$$

$\rightsquigarrow$ Since $\mathcal{R}(I)$ is a complete intersection, the Koszul complex gives a free resolution for $\mathcal{R}(I)$ over $S=R[T_1,T_2,T_3]$:

$$0 \longrightarrow S(-2) \xrightarrow[\begin{bmatrix} F \\ G \end{bmatrix}]{} S(-1) \oplus S(-1) \xrightarrow[\begin{bmatrix} G & -F \end{bmatrix}]{} S \longrightarrow \mathcal{R}(I) \longrightarrow 0$$

$$S_n = R\{\text{monomials in } T_1, T_2, T_3 \text{ of degree } n\} \cong R^{\binom{n+2}{2}}$$

$$\begin{tikzcd}
0 \arrow[r] & R^{\binom{n}{2}} \arrow[r] & R^{\binom{n+1}{2}} \oplus R^{\binom{n+1}{2}} \arrow[r] & R^{\binom{n+2}{2}} \arrow[r] & I^n \arrow[r] & 0 \\
 & & & R^{\binom{n+3}{2}} \arrow[r] \arrow[u, dashed] & I^{n+1} \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

Now how do we lift $I^{n+1} \subseteq I^n$?

$$\mathcal{D} = f_1 \partial/\partial T_1 + f_2 \partial/\partial T_2 + f_3 \partial/\partial T_3 \qquad \text{Euler operator}$$

Induces a map on $\mathcal{R}(I)$ of degree -1 corresponding to $n\,(I^n \subseteq I^{n-1})$ in degree n:
$$\begin{aligned} I^n t^n &\longrightarrow I^{n-1} t^{n-1} \\ g t^n &\longmapsto (n-1)\, g\, t^{n-1} \end{aligned}$$

Example: $I^{(3)} \subseteq I^2$, char $k \neq 3$

$$\begin{tikzcd}
0 & R \arrow[l] \arrow[d, "C"'] & R^3 \oplus R^3 \arrow[l] \arrow[d] & R^6 \arrow[l] \arrow[d] & I^2 \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & R^3 \arrow[l] & R^6 \oplus R^6 \arrow[l, "E"] & R^{10} \arrow[l] & I^3 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

$$I^{(3)} \subseteq I^2 \iff \underset{C}{\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}} \in \operatorname{im} \underset{E}{\begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & a_1 & 0 & a_2 & a_3 & 0 & 0 & b_1 & 0 & b_2 & b_3 & 0 \\ 0 & 0 & a_1 & 0 & a_2 & a_3 & 0 & 0 & b_1 & 0 & b_2 & b_3 \end{pmatrix}}$$

Theorem (G–Huneke–Mukundan) If $\mu(a_1, a_2, a_3, b_1, b_2, b_3) \leq 5$, then $I^{(3)} \subseteq I^2$.

Example $I = I_2\left(\begin{bmatrix} x^{a-1} & y^{a-1} & z^{a-1} \\ yz & xz & xy \end{bmatrix}\right)$, $a \geq 3$ has $I^{(3)} \not\subseteq I^2$, but $\mu(xy, xz, yz, x^{a-1}, y^{a-1}, z^{a-1}) = 6$.

Theorem (–) If $P = \ker(k[x,y,z] \longrightarrow k[t^a, t^b, t^c])$, $P^{(5)} \subseteq P^3$ as long as char $k \neq 5, 2$.

Theorem (–) If $P = \ker(k[x,y,z] \longrightarrow k[t^a, t^b, t^c])$ for $a = 3$ or $4 \leq b \leq c$, then $P^{(4)} \subseteq P^3$ for char $k \neq 2$.

Example: when $P = \ker(k[x,y,z] \longrightarrow k[t^9, t^{11}, t^{14}])$, we have $P^{(4)} \not\subseteq P^3$. However, $P^{(2n-1)} \subseteq P^n$ $\forall n \geq 2$.
(In fact, for $n \geq 8$, $P^{(2n-2)} \subseteq P^n$)