Symbolic powers and the containment problem

UNL Commutative Algebra Seminar 18/09/2019

R regular ring

I radical ideal $= P_1 \cap \dots \cap P_k$

$h =$ big height of $I = \max\{ \operatorname{ht} P_i \}$ (= height if I prime)

the n-th symbolic power of I is given by

$$I^{(n)} = \bigcap_{P \in \operatorname{Min}(I)} \left(I^n R_P \cap R \right)$$

<u>theorem</u> (Zariski–Nagata) $R = \mathbb{C}[x_1, \dots, x_d]$

$$I^{(n)} = \{ f \in R : f \text{ vanish up to order } n \text{ along } I \}$$

$$= \bigcap_{m \supseteq I} m^n \qquad (m \text{ maximal})$$

$$= \left\{ f \in R : \frac{\partial^{a_1 + \dots + a_d}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} (f) \in I \text{ for all } a_1 + \dots + a_d \leq n-1 \right\}$$

<u>Notes</u>

1) $I^n \subseteq I^{(n)}$
2) $I^{(n+1)} \subseteq I^{(n)}$
3) $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$
4) $I^{(n)} = I^n$ for $I =$ (regular sequence)

False in general.

<u>Hard Problem</u> Characterize the ideals I for which $I^{(n)} = I^n$ for all $n \geq 1$.

Example $I = (xy, xz, yz) = (x,y) \cap (y,z) \cap (x,z) \subseteq k[x,y,z]$

$$I^{(2)} = (x,y)^2 \cap (y,z)^2 \cap (x,z)^2 \ni xyz$$

But $xyz \notin I^2$, since elements in $I^{(2)}$ have degree ≥ 4

$$I^{(3)} \subsetneq I^2 \subsetneq I^{(2)}$$

Example $P = \ker\left(k[a,b,c,d] \longrightarrow k[s^3, s^2t, st^2, t^3]\right)$

$$= \left(c^2 - bd, \; bc - ad, \; b^2 - ac\right)$$

height 2 prime, not a ci, yet $P^{(n)} = P^n$ for all n (why: tomorrow)

Example $P = \ker\left(k[x,y,z] \longrightarrow k[\overset{x}{t^3}, \overset{y}{t^4}, \overset{z}{t^5}]\right)$ prime

$\deg x = 3$
$\deg y = 4$
$\deg z = 5$

$$= \Big(\underbrace{x^3 - yz}_{\substack{f \\ \deg 9}}, \; \underbrace{y^2 - xz}_{\substack{g \\ \deg 8}}, \; \underbrace{z^2 - x^2y}_{\substack{h \\ \deg 10}}\Big)$$

$$P^2 \subsetneq P^{(2)} = P^2 R_P \cap R = \{r \in R : sr \in P^2, s \notin P\}$$

$$\underbrace{f^2 - gh}_{\in P^2} = \underbrace{x}_{\notin P} q \Rightarrow q \in P^{(2)}, \; q \notin P^2$$

$\deg 18 = \deg 3 + \deg 15$ In P^2, elements have $\deg \geq 16$

But actually, $P^{(3)} \subseteq P^2$.

Containment Problem When is $I^{(a)} \subseteq I^{b}$?

Theorem (Ein–Lazarsfeld–Smith 2001, Hochster–Huneke 2002, Ma–Schwede 2017)

$$I^{(hn)} \subseteq I^{n} \qquad \text{for all } n \geq 1$$

$$\Longrightarrow \quad I^{((\dim R)n)} \subseteq I^{n} \qquad \text{for all } n \geq 1$$

Examples

1) $h=2$, $P^{(2n)} \subseteq P^{n}$ $\forall n \geq 1 \Rightarrow I^{(4)} \subseteq I^{2}$. But $I^{(3)} \subseteq I^{2}$.

3) $P \sim (t^3, t^4, t^5)$ $h=2$ $P^{(2n)} \subseteq P^{n}$ $\forall n \Rightarrow P^{(4)} \subseteq P^{2}$.
But $P^{(3)} \subseteq P^{2}$.

Question (Huneke, 2000) P prime of ht 2 in RLR. Is $P^{(3)} \subseteq P^{2}$?

Conjecture (Harbourne, 2008) $I^{(hn-h+1)} \subseteq I^{n}$ for all $n \geq 1$

Why would he conjecture that?

1) Given h and n, we can build a monomial ideal I of big height h such that $I^{(hn-h)} \not\subseteq I^{n}$.

2) In char $p>0$, $I^{(hq-h+1)} \subseteq I^{[q]} \subseteq I^{q}$ $q=p^{e}$

this is a key point in Hochster and Huneke's work.

Counterexample (Dumnicki, Szemberg, Tutaj-Gasińska, 2013)

$\exists$ radical ideal (not prime) in $\mathbb{C}[x,y,z]$, $h=2$ st

$$I^{(2n-1)} \not\subseteq I^n \quad \text{for } n=2. \qquad (I^{(3)} \not\subseteq I^2)$$

$$I = (x(\cancel{y^3}^{\,c} - \cancel{z^3}^{\,c}),\ y(\cancel{z^3}^{\,c} - \cancel{x^3}^{\,c}),\ z(\cancel{x^3}^{\,c} - \cancel{y^3}^{\,c}))$$

(Harbourne-Seceleanu) $c \geq 3$, any field of char $\neq 2$

$$= I_2 \begin{pmatrix} x^{c-1} & y^{c-1} & z^{c-1} \\ yz & xz & xy \end{pmatrix}$$

other generalizations: Malara-Szpond, Ben Rabkin

What's special about these counterexamples?
they correspond to very special configurations of points in $\mathbb{P}^2$
But there might also be an algebraic reason for this...

thm (G) k field of char $\neq 3$

$$P = \ker(k[x,y,z] \longrightarrow k[t^a, t^b, t^c])$$

then $P^{(3)} \subseteq P^2$.

Notes 1) the symbolic Rees algebra of P, $\bigoplus_{n \geq 0} P^{(n)} t^n$, may not even be fg! this varies with (a,b,c)

2) Herzog (1970) showed these are of the form $I_2 \begin{pmatrix} x^{a_1} & y^{a_2} & z^{a_3} \\ z^{b_1} & x^{b_2} & y^{b_3} \end{pmatrix}$

thm (G-Huneke-Mukundan) k field of char $\neq 3$

$$I = I_2 \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \subseteq k[x,y,z]$$
of height 2.

If $(a_1, \dots, a_6)$ can be generated by at most 5 elements, then $I^{(3)} \subseteq I^2$.

Example $P \sim (t^a, t^b, t^c) \longrightarrow$ always get (x^p, y^q, z^r)

Example (Fermat configurations) See survey by Justyna Szpond

$$I = I_2 \begin{pmatrix} x^{c-1} & y^{c-1} & z^{c-1} \\ yz & xz & xy \end{pmatrix} \rightsquigarrow \text{get a } \underline{\underline{6}} \text{ generated ideal}$$

Example $$I = I_2 \begin{pmatrix} x^2 & xz & z^2 \\ yz & y^2 & xy \end{pmatrix}$$

In this case, $I^{(3)} \subseteq I^2$

(because xy divides $x^2y^2 - xyz^2$)

All of this heavily uses Seceleanu's work on 2-minors of 2×3 matrices

Harbourne's Conjecture does hold:

- For general points in $\mathbb{P}^2$ (Bocci – Harbourne) and $\mathbb{P}^3$ (Dumnicki)
- For other special configurations of points, eg star configurations
- For squarefree monomial ideals
- In char p, if R/I is F-pure
  Over a field of char 0, if R/I is of dense F-pure type

  this includes:

  - I is a squarefree monomial ideal
  - $I = I_t(X)$, t-minors of a generic matrix
  - $R/I \cong k[$all t-minors of a generic
  - R/I Veronese $= k[$all monomials of degree $d]$
  - R/I ring of invariants of linearly reductive groups

Two versions of Harbourne's Conjecture
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Harbourne's Conjecture $I^{(hn-h+1)} \subseteq I^n$ for all $n \geq 1$, h = big height of I

Today's Goal to discuss 2 different ways Harbourne's Conjecture is true

1) For nice ideals in char p, even if R not regular

Joint work with Linquan Ma and Karl Schwede

2) Stable Harbourne

Joint work with Craig Huneke and Vivek Mukundan

1) R domain of char $p > 0$

F is the Frobenius map $x \mapsto x^p$

$F^e_* R \equiv R$ with the R-module structure induced by F^e

R is F-finite if $F_*(R)$ is a fg R-module.
eg, if $k = k^p$, $R = k[x_1, \ldots, x_d]/J$ or localization of this

Will assume: R F-finite

A ring R is F-pure $\equiv$ F-split if F^e splits.

$$\begin{tikzcd} R \arrow[r, "F^e"] & F_*^e(R) \arrow[l, dotted, bend left, "\exists \emptyset \text{ splitting}"] \end{tikzcd}$$

<u>theorem</u> (Fedder, 1983) Let (R, m) be a regular local ring, I ideal in R

R/I F-pure $\Longleftrightarrow$ $(I^{[q]} : I) \not\subseteq m^{[q]}$ for all/some/large q

Interesting subclass: strongly F-regular rings ($\Rightarrow$ Cohen-Macaulay)

<u>Glassbrenner's Criterion</u> (1996) Let (R, m) be an F-finite RLR, I ideal in R.

R/I strongly F-regular $\Longleftrightarrow$ $c(I^{[q]} : I) \not\subseteq m^{[q]}$ for all large q, $c \notin$ min prime of I

<u>Examples</u> All the examples from yesterday, except quotients by monomial ideals

<u>theorem</u> (G-Huneke, 2017) Let R be a regular ring of char p.

1) If R/I is F-pure, then I verifies Harbourne's Conjecture.

2) If R/I is strongly F-regular, then I verifies Harbourne's Conjecture with h replaced by $h-1$, if $h \geq 2$.

When $h = 2$, $I^{(n)} = I^n$ for all $n \geq 1$

<u>Goal</u> Extend this to some non-regular rings

<u>Sketch</u>: R/I is F-pure/strongly F-regular $\Rightarrow I^{(a)} \subseteq I^b$

<u>Step 0</u> Reduce to the local case (R, m)

<u>Step 1</u> $I^{(a)} \subseteq I^b \Leftrightarrow (I^b : I^{(a)}) = R \Leftrightarrow (I^b : I^{(a)}) \not\subseteq m$

<u>Step 2</u> If $I \subseteq J$, then $I^{[q]} \subseteq J^{[q]}$.

Show that $(I^b : I^{(a)})^{[q]} \not\subseteq m^{[q]}$ for $q \gg 0$

by showing

$$\underbrace{J_q}_{\text{witness to } R/I \text{ f-pure/strongly F-regular.}} \subseteq (I^b : I^{(a)})^{[q]} \quad \text{for } q \gg 0$$

tools we need to remove the regularity assumption:

- Non-regular version of Feder/Glassbrenner Criteria
- Corresponding containment involving our colon ideal and our witness

<u>Gorenstein version of Fedder/Glassbrenner's Criteria</u>:

Back to the regular case:

- R/I F-pure $\Leftrightarrow \exists \phi \in Hom_{R/I}(F_*^e(R/I), R/I)$ $1 \in im\, \phi$
- Understand $Hom_R(F_*^e(R/I), R/I)$:

$$\begin{array}{ccc} \underline{\underline{\text{free}}}\ F^e_*(R) & \overset{\exists \tilde{\phi}}{\dashrightarrow} & R \\ \downarrow & & \downarrow \\ F^e_*(R/I) & \xrightarrow[\phi]{} & R/I \end{array}$$

Every $\phi \in \mathrm{Hom}_{R/I}(F^e_*(R/I), R/I)$ lifts to $\tilde{\phi}$ in $\mathrm{Hom}_R(F^e_* R, R)$.

- R Gorenstein $\Rightarrow$ $\mathrm{Hom}_R(F^e_*(R), R)$ is cyclic, generated by Φ_e

Every element looks like $\Phi_e(F^e_* r \cdot \text{—})$, $r \in R$

Fedder's Criterion: R/I F-pure $\Rightarrow$ $\Phi_e\Big(F^e_*\big(\underbrace{I^{[q]} : I}_{\substack{r \in R \text{ corresponding} \\ \text{to a map on } R \\ \text{that descends to } R/I}}\big)\Big) = R$

R Gorenstein $\Rightarrow$ $F^e_* R$ no longer free/projective

<u>Need</u> $\quad I_e(I) = \{ r \in R : \varphi(F^e_* r) \subseteq I,\ \varphi \in \mathrm{Hom}_R(F^e_* R, R) \}$

<u>Our criterion</u> (G-Ra-Schwede)

Let R be a Gorenstein F-finite ring of char $p > 0$, $\mathrm{pd}(I) < \infty$

1) Every $\phi \in \mathrm{Hom}_{R/I}(F^e_*(R/I), R/I)$ lifts to $\tilde{\phi} \in \mathrm{Hom}_R(F^e_* R, R)$.

2) If R/I is F-pure, then $\Phi_e(F^e_*(I_e(I) : I)) = R$ for all e.

3) If R/I is strongly F-regular, then $\Phi_e(F^e_*(c(I_e(I) : I))) = R$ for all $c \notin$ min prime of I, $e \gg 0$.

<u>theorem</u> (G-Ma-Schwede) Let R be an F finite Gorenstein ring, $\mathrm{pd}(I) < \infty$

1) If R/I is F-pure, then I verifies Harbourne's Conjecture.

2) If R/I is strongly F-regular, then I verifies Harbourne's Conjecture with h replaced by $h-1$.

<u>Warning</u> $\mathrm{pd}(I) < \infty \not\Rightarrow \mathrm{pd}(I^{(n)}) < \infty$
this is an obstacle we luckily can jump through

the issue is we have no control over $\mathrm{Ass}\left(R/(I^{(n)})^{[q]}\right)$

<u>Great news:</u> $I_e(-)$ preserves associated primes

<u>Moral</u> $I_e(-)$ is a better replacement for $(\ )^{[q]}$

How about infinite projective dimension? Not all Cartier maps lift, but they lift after pre-multiplication by something in the Jacobian

<u>theorem</u> (G-Ma-Schwede) R Gorenstein F-finite ring $/ k = k^p$
J = Jacobian ideal (R/k)

If R/I is F-pure, then $J^n I^{(hn-h+1)} \subseteq I^n$ for all $n \geq 1$

If R/I is strongly F-regular, then $J^{2n} I^{((h-1)(n-1)+1)} \subseteq I^n$ for all $n \geq 1$.

<u>Hopes and dreams:</u> Replace J^{2n} by J^n.

2) Stable Harbourne $I^{(hn-h+1)} \subseteq I^n$ for $n \gg 0$.

Question $I^{(hn-h+1)} \subseteq I^n$ for some $n \overset{?}{\Longrightarrow}$ all $n \gg 0$?

Remark If yes, then we are done in char p.

Remark the answer is yes provided I verifies:

$$I^{(n+h)} \subseteq I\, I^{(n)} \qquad \text{for all } n \geq 1$$

False in general, but

thm If R/I is F-pure, $I^{(n+h)} \subseteq I\, I^{(n)}$ for all $n \geq 1$

Also true for R F-finite, Gorenstein, and $\operatorname{pd}(I) < \infty$ (G-Ma-Schwede)

theorem If $I^{(hk-h)} \subseteq I^k$ for some n then

$$I^{(hn-h)} \subseteq I^n \quad \text{for all } n \gg 0$$

+ Schwede in mixed char

No counterexamples to $I^{(hn-C)} \subseteq I^n$ for $n \gg 0$, C fixed.

Resurgence (Bocci-Harbourne) $\rho(I) = \sup\left\{ \frac{a}{b} : I^{(a)} \not\subseteq I^b \right\}$

$$1 \leq \rho(I) \leq h \qquad \text{always}$$

Remark If $\rho(I) < h$, then Stable Harbourne holds.

$$\frac{hn-c}{n} > \rho(I) \implies I^{(hn-c)} \subseteq I^n$$

c fixed

$$\Updownarrow$$

$$n > \frac{c}{h-\rho(I)}$$

Question Can $\rho(I) = h$?

Theorem (G–Huneke–Mukundan)

(R,m) RLR containing a field

Assume $I^{(n)} = I^n : m^\infty = \bigcup_{k\geq 1} I^n : m^k$)

(eg R/I Cohen-Macaulay, $\dim(R/I) = 1$)

If $I^{(hn-h+1)} \subseteq m I^n$ for some n, then $\rho(I) < h$.

(and stable Harbourne holds)

Applications

1) I homogeneous ideal, generated in degree $a < h$, char 0

Sketch of proof: char 0, using differential operators

$$I^{(hn-h+1)} \subseteq \mathfrak{m} I^{(hn-h+1)} \subseteq \cdots \subseteq \mathfrak{m}^{hn-h} I \subseteq \mathfrak{m}^{hn-h+a}$$

so

$$I^{(hn-h+1)} \subseteq \mathfrak{m}^{hn-h+a} \cap I^{n-1} \quad \text{by ELS-HH-MS}$$

$$\subseteq \mathfrak{m}^{hn-h+a-an-a} I$$

$$hn-h+a-an-a = n(h-a)-h \geq 1 \quad \text{if} \quad a<h,\ n>h$$

2) Space monomial curves $P \sim (t^a, t^b, t^c)$, char $\neq 3$

$$P^{(3)} \subseteq \mathfrak{m} P^2$$ (using Knödel–Schenzel–Zonsarow implicit generators for $P^{(3)}$)

$\Downarrow$

$\rho(P) < 2$
and
~~stable~~ Harbourne