Symbolic powers in mixed characteristic

joint with Jack Jeffries and Alessandro De Stefani

R noetherian ring

$I = \sqrt{I}$ ideal in R

the nth symbolic power of I is

$$I^{(n)} := \bigcap_{P \in \mathrm{Ass}(R/I)} \left(I^n R_P \cap R \right)$$

$$P \text{ prime} \Rightarrow P^{(n)} = \{ f \in R \mid sf \in P^n, s \notin P \}$$

$$I = P_1 \cap \cdots \cap P_s \Rightarrow I^{(n)} = P_1^{(n)} \cap \cdots \cap P_s^{(n)}$$

Symbolic powers appear as auxiliary tools in the proofs of Krull's Height theorem, Hartshorne–Lichtenbaum Vanishing, etc and other classical results

Facts

0. $P^{(n)}$ is the smallest P-primary ideal containing P^n
1. $I^n \subseteq I^{(n)}$ for all $n \geq 1$
2. $I^{(n+1)} \subseteq I^{(n)}$ for all $n \geq 1$
3. $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$ for all $a, b \geq 1$

$\Rightarrow$ Can define a graded algebra $\bigoplus_{n \geq 0} I^{(n)} t^n \subseteq R[t]$
which is not always fg over R

4. m maximal $\Rightarrow m^{(n)} = m^n$ for all $n \geq 1$
5. R Cohen–Macaulay, I ci $\Rightarrow I^{(n)} = I^n$ for all $n \geq 1$
6. $I^n \neq I^{(n)}$ in general

Example $I = (xy, xz, yz) \subseteq k[x,y,z]$

$= (x,y) \cap (x,z) \cap (y,z)$

$I^{(2)} = (x,y)^2 \cap (y,z)^2 \cap (x,z)^2 \ni xyz$

But $\deg(xyz) = 3$ and every element in I^2 has degree ≥ 4

Example X 3×3 generic matrix

$I = I_2(X) \subseteq k[X]$, k field

$\det(X) \in I^{(2)}$ but $\det(X) \notin I^2$

(I is prime, and $x_i \det(X) \in I^2$)

there are many interesting open questions related to symbolic powers:

1. When is $I^n = I^{(n)}$?
2. When I is homogeneous:

   What is the smallest degree of an element in $I^{(n)}$?
3. (Eisenbud-Mazur, 1997) (R, m) RLR, $R \supseteq k$ field of char 0

   $I^{(2)} \subseteq m I$
4. Containment Problem: When is $I^{(a)} \subseteq I^b$?
5. When is $\oplus I^{(n)} t^n$ a noetherian ring?

theorem (Zariski-Nagata 1949, 1962, Eisenbud-Hochster 1979)

$R = k[x_1, \ldots, x_d]$, k field

$$I^{(n)} = \bigcap_{\substack{m \supseteq I \\ m \text{ max}}} m^n \qquad \text{for all } n \geq 1.$$

We can write this with differential operators:

R A-algebra

A-linear differential operators on R are

$$\mathcal{D}_{R|A} = \bigcup_{n\geq 0} \mathcal{D}^n_{R|A} \subseteq \operatorname{Hom}_A(R,R)$$

- $\mathcal{D}^0_{R|A} = \operatorname{Hom}_R(R,R)$
- $\partial \in \mathcal{D}^{n+1}_{R|A}$ if $[\partial, r] \in \mathcal{D}^n_{R|A}$ for all $r \in \mathcal{D}^0_{R|A}$

the A-linear <u>differential power</u> of I is

$$I^{\langle n \rangle_A} := \{ f \in R \mid \partial(f) \in I \text{ for all } \partial \in \mathcal{D}^{n-1}_{R|A} \}$$

<u>theorem</u> (Differential operators version of Zariski-Nagata)

k perfect field, $R = k[x_1, \dots, x_d]$, $I = \sqrt{I}$

$$I^{(n)} = I^{\langle n \rangle} \quad \text{for all } n \geq 1$$

But this doesn't work in mixed characteristic:

<u>Ex</u> $R = \mathbb{Z}[x]$ $\quad m = (2, x) \quad m^{(n)} = m^n$ for all $n \geq 1$

$$\partial(2) = 2\partial(1) \in (2) \subseteq m \quad \text{for all } \partial \in \mathcal{D}_{R|A}$$

so $2 \in m^{\langle n \rangle}$ for all n $\Rightarrow m^{(n)} \subsetneq m^{\langle n \rangle}$

Definition (Buium, Joyal) $p \in \mathbb{Z}$ prime, regular on R

$\delta : R \to R$ is a p-derivation if for all $x, y \in R$:

- $\delta(1) = 0$
- $\delta(xy) = x^p \delta(y) + \delta(x) y^p + p\,\delta(x)\delta(y)$
- $\delta(x+y) = \delta(x) + \delta(y) + \dfrac{x^p + y^p - (x+y)^p}{p}$

Having a p-derivation $\Leftrightarrow$ Having a lift of Frobenius on R/p to R

$$\delta(x) = \frac{\phi(x) - x^p}{p} \quad \Longleftrightarrow \quad \Phi(x) = x^p + p\,\delta(x)$$

p-derivation $\qquad\qquad$ lift of Frobenius

We do have p-derivations for:

- $\mathbb{Z}$: there is a unique p-derivation $\delta(n) = \dfrac{n - n^p}{p}$
- R complete unramified DVR with perfect residue field
- $R = B[x_1, \dots, x_d]$, B admits a p-derivation
  set $\delta(x_i)$ freely, $\delta|_B$ p-derivation on B

<u>Note</u>: $a \notin (p)$ $\qquad \delta(ap^n) \in (p^{n-1}) \setminus (p^n)$

$\Rightarrow \delta$ decreases p-adic order

like differential operators decrease $(x_1, \dots, x_d)$-adic order

eg: $\delta(p) = \dfrac{p - p^p}{p} = 1 - p^{p-1} \notin (p)$

Mixed differential powers (De Stefani - G - Jeffries)

R ring with p-derivation δ, $p \in I$ ideal

$$I^{\langle n \rangle_{mix}} := \{ f \in R \mid \delta^a \circ \partial(f) \in I,\ \partial \in D^b_{R|A},\ a+b<n \}$$

theorem (De Stefani - G - Jeffries, 2020)

$A = \mathbb{Z}$ or DVR with uniformizer p

R localization of smooth A-algebra

R has a p-derivation (eg $R = A[x_1, \dots, x_d]$, A unramified DVR)

Q prime ideal, $Q \ni p$

$A/p \hookrightarrow R_Q / QR_Q$ separable (eg A/p perfect)

then $Q^{(n)} = Q^{\langle n \rangle_{mix}}$ for all $n \geq 1$

Note Essentially smooth and separability hypotheses are necessary

Proof sketch

$$\left.\begin{array}{l} ① \text{ show } Q^n \subseteq Q^{\langle n \rangle_{mix}} \\ ② \ Q^{\langle n \rangle_{mix}} \ \ Q\text{-primary} \end{array}\right\} \Rightarrow Q^{(n)} \subseteq Q^{\langle n \rangle_{mix}}$$

so need to show $Q^{\langle n \rangle_{mix}} \subseteq Q^{(n)}$

$\therefore$ sufficient to show after localizing at

$$\mathrm{Ass}(Q^{(n)}) = \mathrm{Ass}(Q) = \{Q\}$$

③ $Q^{\langle n \rangle_{mix}} R_Q = (QR_Q)^{\langle n \rangle_{mix}}$

④ (R, m) local $\Longrightarrow m^{\langle n \rangle_{mix}} = m^n$

Corollary R smooth over A

$$Q^{(n)} = \bigcap_{\substack{\mathfrak{m} \supseteq Q \\ \mathfrak{m} \text{ max}}} \mathfrak{m}^n$$

Corollary $Q^{\langle n \rangle_{\max}}$ is independent of the choice of δ

Example $R = \mathbb{Z}[x]$, $\mathfrak{m} = (2, x)$

$$\delta(2) = \frac{2 - 2^2}{2} = 1 - 2 = -1 \notin \mathfrak{m}$$

$$\Rightarrow 2 \notin \mathfrak{m}^{\langle 2 \rangle_{\max}} = \mathfrak{m}^{(2)}$$

Note We also got $I^{(n)} \subseteq I^{\langle n \rangle_{\max}}$ in general!
$\rightsquigarrow$ useful for applications in singular settings.

other fun applications : Jacobian Criterion

$R = A[x_1, \dots, x_d] \ni f$ δ p-derivation, $p \in Q$ prime

$\left(R/(f)\right)_Q$ regular $\Leftrightarrow \frac{f}{1} \in Q_Q^2 \Leftrightarrow f \in Q^{(2)} \Leftrightarrow \partial/\partial x_i (f), \delta(f) \in Q$

Tomorrow: Jack will tell us how to do this for any radical ideal I, not just $I = (f)$