Linear Algebra

Math 314 Fall 2025

Lecture 14

Today's poll code:

KJNMQ5

Office hours

Mondays 5–6 pm
Wednesdays 2–3 pm
in Avery 339 (Dr. Grifo)

Tuesdays 11–noon
Thursdays 1–2 pm
in Avery 337 (Kara)

To do list:

- Webwork 4.1 due tomorrow
- Lab 1 due on Friday
- Webwork 4.2 due next Wednesday

Quiz on Friday

on vector spaces

Bonus poll points

Miss at most 2 lectures between Lecture 14 and Lecture 30

$$\Downarrow$$

3 bonus poll points

Quick Recap

TL;DR: can add vectors and multiply by scalars, with good properties.

A **vector space** is a nonempty set V, whose elements we call **vectors**, with rules for **addition** of vectors in V and **multiplication by scalars**, satisfying the following properties:

a) The addition $u + v$ of any vectors u and v in V is also a vector in V.
b) The multiplication cv of a vector v by a scalar c is a vector in V.
c) Commutativity: $u + v = v + u$ for all $u, v \in V$.
d) Associativity: $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$.
e) There is a **zero vector** in V, denoted 0, such that $0 + v = v + 0 = v$.
f) For every vector v there is a vector $-v$ such that $v + (-v) = 0$.
g) Distributivity: $c(u + v) = cu + cv$ and $(c + d)v = cv + dv$ for all $u, v \in V$ and all scalars c and d.
h) Associativity of multiplication by scalars: $c(dv) = (cd)v$.
i) $1v = v$.

Consequence: we also have

- $0v = \mathbf{0}$.
- $c\mathbf{0} = \mathbf{0}$.
- $-v = (-1)v$.

Vector space: can add vectors and multiply by scalars, with good properties.

Example: $\mathbb{R}^n$ with

(component-wise) addition:

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

(component-wise) multiplication by scalars

$$c \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} c\, u_1 \\ \vdots \\ c\, u_n \end{bmatrix}$$

Vector space: can add vectors and multiply by scalars, with good properties.

$$\mathrm{M}_{n\times m} = \{\text{all } m \times n \text{ matrices}\}$$

(component-wise) addition: $$A + B := [a_{ij} + b_{ij}]$$

(component-wise) multiplication by scalars $$cA = [ca_{ij}]$$

V vector space $\qquad$ W subset of V

W is a subspace of V if

1) The zero vector 0 is in W

2) W is <u>closed under addition</u>:

if u and v are in W, then $u+v$ is in W

3) W is <u>closed under multiplication by scalars</u>:

if v is in W, and c is any scalar, then cv is in W

A subspace is also a vector space!

The quiz on Friday will include a question to show something is (not) a subspace

V vector space

$v_1, \ldots, v_p$ vectors in V

The **span** of $v_1, \ldots, v_p$ is

the set of all linear combinations of $v_1, \ldots, v_p$

$$\operatorname{span}(\{v_1, \ldots, v_p\}) = \{c_1 v_1 + \cdots + c_p v_p \mid c_i \in \mathbb{R}\}$$

Let $v_1, \ldots, v_p$ be vectors in $\mathbb{R}^n$.

span of $v_1, \ldots, v_p$ = set of all linear combinations of $v_1, \ldots, v_p$

$$\text{span}\left(\{v_1, \ldots, v_p\}\right) = \{c_1 v_1 + \cdots + c_p v_p \mid c_i \in \mathbb{R}\}$$

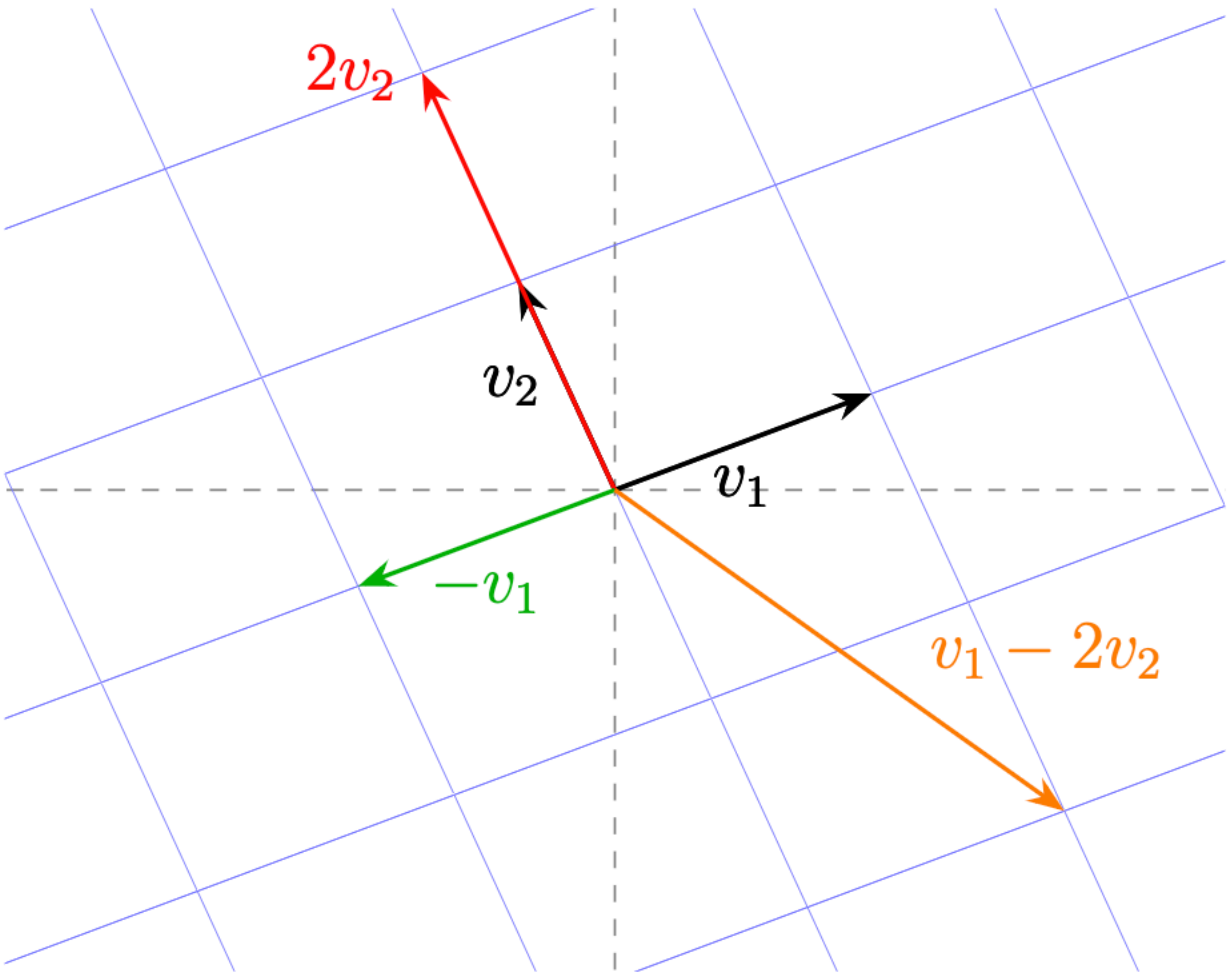
$$\text{span}(\{v_1\}) = \text{line determined by } v_1$$

if $v_1 \neq 0$

Let $v_1, \ldots, v_p$ be vectors in $\mathbb{R}^n$.

span of $v_1, \ldots, v_p$ = set of all linear combinations of $v_1, \ldots, v_p$

$$\text{span}\left(\{v_1, \ldots, v_p\}\right) = \{c_1 v_1 + \cdots + c_p v_p \mid c_i \in \mathbb{R}\}$$



$\text{span}(\{v_1, v_2\})$ = plane determined by v_1 and v_2

if $v_1 \neq 0$ and $v_2 \neq 0$ and v_1 and v_2 not multiples of each other

Theorem.

Given any vector space V, and vectors $v_1, \ldots, v_n$ in V,

the set $\text{span}(\{v_1, \ldots, v_n\})$ is a subspace of V.

Conversely, all subspaces of V are

spanned by some collection of vectors in V.

Example: $V = \mathbb{R}^4$

$$W = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \text{ any scalars} \right\}$$

is a subspace of $\mathbb{R}^4$

easier: all vectors in W can be written as

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{so} \quad W = \operatorname{span}\left(\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$$

is a subspace

linear combination of these vectors!

Example:

In $M_{2\times 2}$ (space of 2×2 matrices)

$$D = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \text{ any scalars} \right\}$$

diagonal matrices

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D = \operatorname{span}\left(\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \right)$$

so D is a subspace of $M_{2\times 2}$

V vector space

$v_1, \ldots, v_n$ vectors in V

$v_1, \ldots, v_n$ form a **spanning set** for V if

$$V = \operatorname{span}(\{v_1, \ldots, v_n\})$$

$$\mathbf{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \longrightarrow \text{position } i$$

ith standard vector

In $\mathbb{R}^n$

any vector v can be written as

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1 e_1 + \cdots + v_n e_n$$

so

$$\mathbb{R}^n = \operatorname{span}\left(\{e_1, \ldots, e_n\}\right)$$

$\{e_1, \ldots, e_n\}$ forms a spanning set for $\mathbb{R}^n$

Linearly independent vectors

V vector space

$v_1, \ldots, v_p$ vectors in V

$v_1, \ldots, v_p$ are **linearly independent** if the equation

$$x_1 v_1 + \cdots + x_p v_p = 0$$

has no nontrivial solutions.

$v_1, \ldots, v_p$ are **linearly dependent** if the equation

$$x_1 v_1 + \cdots + x_p v_p = 0$$

has nontrivial solutions

V vector space

$v_1, \ldots, v_p$ vectors in V

$v_1, \ldots, v_p$ are **linearly independent** if

$$c_1 v_1 + \cdots + c_p v_p = 0 \quad \Longrightarrow \quad c_1 = \cdots = c_p = 0$$

$v_1, \ldots, v_p$ are **linearly dependent** if

$$c_1 v_1 + \cdots + c_p v_p = 0$$

for some $c_1, \ldots, c_p$ not all zero

Theorem. Let V be a vector space.

Any set of vectors that contains the zero vector is linearly dependent.

$v_1, \ldots, v_p$ are **linearly dependent** if

$$c_1 v_1 + \cdots + c_p v_p = 0$$

for some $c_1, \ldots, c_p$ not all zero

Example:

in $\mathbb{R}^2$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ are linearly dependent

for example

$$2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0$$

$v_1, \ldots, v_p$ are **linearly dependent** if

$$c_1 v_1 + \cdots + c_p v_p = 0$$

for some $c_1, \ldots, c_p$ not all zero

Example:

in $\mathbb{R}^2$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ are linearly dependent

for example

$$2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0$$

Any two nonzero vectors that are scalar multiples of each other are linearly dependent.

In any vector space

Any two nonzero vectors that are scalar multiples of each other are linearly dependent.

v is any nonzero vector and $t \neq 1$

$$\Downarrow$$

v and tv are linearly dependent

$$t \cdot v + (-1) \cdot (tv) = 0$$

In any vector space

A set $\{v_1, \ldots, v_p\}$ of two or more vectors is linearly dependent

if and only if

one of the vectors is a linear combination of the others.

Note: this does <u>not</u> say that <u>every</u> v_i is a linear combination of the rest.

Example:

In $M_{2\times 2}$ (space of 2×2 matrices)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ are linearly independent}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ are linearly dependent}$$

$$-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example:

in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example:

in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

spanning set for $\mathbb{R}^2$

Example:

in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

spanning set for $\mathbb{R}^2$ linearly dependent

Example:

in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Example:

in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

not spanning set for $\mathbb{R}^2$

Example:

in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

not spanning set for $\mathbb{R}^2$

linearly independent

Example:

in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example:

in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

spanning set for $\mathbb{R}^2$

Example:

in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

spanning set for $\mathbb{R}^2$ linearly independent

Bases and dimension

A **basis*** for a vector space V is

a spanning set of linearly independent vectors.

The **dimension** of a vector space is

the number of vectors in a basis.

Notation: $\dim(V)$

* The plural of basis is bases.

The **dimension** of a vector space is

the number of vectors in a basis.

Notation: $\dim(V)$

Theorem. Let V be a vector space.

Every basis for V has the same number of elements.

$$\dim(\mathbb{R}^2) =$$

$$\dim(\mathbb{R}^3) =$$

$$\dim(\mathbb{R}^n) =$$

$$\dim(V) = \text{number of vectors in a basis for } V$$

$\dim(\mathbb{R}^2) = 2$

Basis: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\dim(\mathbb{R}^3) =$

$\dim(\mathbb{R}^n) =$

$\dim(V) = \text{number of vectors in a basis for } V$

$\dim(\mathbb{R}^2) = 2$

Basis: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\dim(\mathbb{R}^3) = 3$

Basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\dim(\mathbb{R}^n) =$

$$\dim(V) = \text{number of vectors in a basis for } V$$

$$\dim(\mathbb{R}^2) = 2 \qquad \text{Basis: } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\dim(\mathbb{R}^3) = 3 \qquad \text{Basis: } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\dim(\mathbb{R}^n) = n \qquad \text{Basis: } e_1, \ldots, e_n$$

$$\dim(V) = \text{number of vectors in a basis for } V$$

Warning:

The same vector space can have many different basis!

Example: $\mathbb{R}^2$

Basis 1: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Basis 2: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Basis 3: $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Basis 4: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$v_1, \ldots, v_n \text{ form a basis for } \mathbb{R}^n$$

$$\Updownarrow$$

$$A \sim I$$

$$\Updownarrow$$

$$A \text{ has a pivot in every row and column}$$

$$A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

$$\dim(M_{2\times 2}) =$$

all 2×2 matrices

$$\dim(M_{m\times n}) =$$

all $m \times n$ matrices

$$\underbrace{\dim(M_{2\times 2})}_{\text{all } 2\times 2 \text{ matrices}} = 4$$

Basis: $$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\underbrace{\dim(M_{m\times n})}_{\text{all } m\times n \text{ matrices}} =$$

$$\underbrace{\dim(M_{2\times 2})}_{\text{all } 2\times 2 \text{ matrices}} = 4$$

Basis: $$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\underbrace{\dim(M_{m\times n})}_{\text{all } m\times n \text{ matrices}} = mn$$

Basis: all the matrices with one 1
and zeroes everywhere else

$$\dim(\{0\}) = 0$$

$$\dim(\mathbb{P}^1) =$$

polynomials $a + bt$

$$\dim(\mathbb{P}^2) =$$

polynomials $a + bt + ct^2$

$$\dim(\mathbb{P}^n) =$$

polynomials $a_0 + a_1 t + \cdots + a_n t^n$

$$\dim(\mathbb{P}) =$$

polynomials of any degree

$$\dim(\mathbb{P}^1) = \ 2$$

polynomials $a + bt$

Basis: $1, t$

$$\dim(\mathbb{P}^2) =$$

polynomials $a + bt + ct^2$

$$\dim(\mathbb{P}^n) =$$

polynomials $a_0 + a_1 t + \cdots + a_n t^n$

$$\dim(\mathbb{P}) =$$

polynomials of any degree

$$\dim(\mathbb{P}^1) = \ 2$$

polynomials $a + bt$

Basis: $1, t$

$$\dim(\mathbb{P}^2) = \ 3$$

polynomials $a + bt + ct^2$

Basis: $1, t, t^2$

$$\dim(\mathbb{P}^n) =$$

polynomials $a_0 + a_1 t + \cdots + a_n t^n$

$$\dim(\mathbb{P}) =$$

polynomials of any degree

$$\dim(\mathbb{P}^1) = \ 2$$

polynomials $a + bt$

Basis: $1, t$

$$\dim(\mathbb{P}^2) = \ 3$$

polynomials $a + bt + ct^2$

Basis: $1, t, t^2$

$$\dim(\mathbb{P}^n) = \ n$$

polynomials $a_0 + a_1 t + \cdots + a_n t^n$

Basis: $1, t, \cdots, t^n$

$$\dim(\mathbb{P}) =$$

polynomials of any degree

$$\dim(\mathbb{P}^1) = \ 2$$

polynomials $a + bt$

Basis: $1, t$

$$\dim(\mathbb{P}^2) = \ 3$$

polynomials $a + bt + ct^2$

Basis: $1, t, t^2$

$$\dim(\mathbb{P}^n) = \ n$$

polynomials $a_0 + a_1 t + \cdots + a_n t^n$

Basis: $1, t, \cdots, t^n$

$$\dim(\mathbb{P}) = \infty$$

polynomials of any degree

Basis: $1, t, t^2, \ldots$

A vector space is **finite-dimensional** if its dimension is finite.

$\mathbb{R}^n$ is finite-dimensional

$\mathbb{P}$ is infinite-dimensional

Today’s poll code:

KJNMQ5

Today's poll code:

KJNMQ5

What is the dimension of $\mathbb{R}^{2025}$?

Today's poll code:

KJNMQ5

What is the dimension of the space $M_{2\times 3}$ of 2×3 matrices?

Theorem.

Linearly independent vectors can always be extended to a basis.

A spanning set always contains a basis.

Theorem (Basis theorem)**.**

Let V be a vector space of dimension n.

Any set of n linearly independent vectors is a basis for V.

Any set of n vectors that spans V is a basis for V.

Example: in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

spanning set for $\mathbb{R}^2$ linearly dependent

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \cancel{v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

basis for $\mathbb{R}^2$

Example: in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

spanning set for $\mathbb{R}^2$ linearly dependent

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \cancel{v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

basis for $\mathbb{R}^2$

Example: in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

spanning set for $\mathbb{R}^2$ linearly dependent

$$\cancel{v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

basis for $\mathbb{R}^2$

Example: in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

spanning set for $\mathbb{R}^2$ linearly dependent

$$\cancel{v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \qquad v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

basis for $\mathbb{R}^2$

Example: in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

spanning set for $\mathbb{R}^2$ linearly dependent

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \cancel{v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

basis for $\mathbb{R}^2$

Example: in $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

spanning set for $\mathbb{R}^2$ linearly dependent

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \cancel{v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

NOT a basis for $\mathbb{R}^2$

Example: Consider the subspace of $\mathbb{R}^4$

$$W = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \text{ any real numbers} \right\}$$

Example: Consider the subspace of $\mathbb{R}^4$

$$W = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \text{ any real numbers} \right\}$$

$$W = \operatorname{span}\left(\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix} \right\} \right)$$

Example: Consider the subspace of $\mathbb{R}^4$

$$W = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \text{ any real numbers} \right\}$$

$$W = \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix} \right\} \right)$$

$\dim(W) \leqslant 4$ (since W is spanned by 4 vectors)

But what is the dimension of W?

Example: Consider the subspace of $\mathbb{R}^4$

$$W = \operatorname{span}\left(\left\{\begin{bmatrix}1\\5\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}6\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}0\\4\\-1\\5\end{bmatrix}\right\}\right)$$

$$-2\begin{bmatrix}-3\\0\\1\\0\end{bmatrix} = \begin{bmatrix}6\\0\\-2\\0\end{bmatrix} \quad\Rightarrow\quad W = \operatorname{span}\left(\left\{\begin{bmatrix}1\\5\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\4\\-1\\5\end{bmatrix}\right\}\right)$$

Example: Consider the subspace of $\mathbb{R}^4$

$$W = \text{span}\left(\left\{\begin{bmatrix}1\\5\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}6\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}0\\4\\-1\\5\end{bmatrix}\right\}\right)$$

$$-2\begin{bmatrix}-3\\0\\1\\0\end{bmatrix} = \begin{bmatrix}6\\0\\-2\\0\end{bmatrix} \quad \Rightarrow \quad W = \text{span}\left(\left\{\begin{bmatrix}1\\5\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\4\\-1\\5\end{bmatrix}\right\}\right)$$

$$\text{Is } \left\{\begin{bmatrix}1\\5\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\4\\-1\\5\end{bmatrix}\right\} \text{ a basis for } W?$$

Example: Consider the subspace of $\mathbb{R}^4$

$$W = \operatorname{span}\left(\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix} \right\}\right)$$

Is $\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix} \right\}$ a basis for W?

Example: Consider the subspace of $\mathbb{R}^4$

$$W = \operatorname{span}\left(\left\{\begin{bmatrix}1\\5\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}6\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}0\\4\\-1\\5\end{bmatrix}\right\}\right)$$

Is $\left\{\begin{bmatrix}1\\5\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\4\\-1\\5\end{bmatrix}\right\}$ linearly independent?

Example: Consider the subspace of $\mathbb{R}^4$

$$W = \operatorname{span}\left(\left\{\begin{bmatrix}1\\5\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}6\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}0\\4\\-1\\5\end{bmatrix}\right\}\right)$$

Is $\left\{\begin{bmatrix}1\\5\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\4\\-1\\5\end{bmatrix}\right\}$ linearly independent?

$$\begin{bmatrix}1 & -3 & 0 & \\ 5 & 0 & 0 & 4 \\ 0 & 1 & -1 & \\ 0 & 0 & 5 & \end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

pivot in every column

$\Downarrow$

yes

Example: Consider the subspace of $\mathbb{R}^4$

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$$\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix} \right\} \text{ is a basis for } W$$

$$\Downarrow$$

$$\dim(W) = 3$$

Given polynomials in $\mathbb{P}_n$,

we can ask questions about them by

identifying them with vectors in $\mathbb{R}^{n+1}$

$$a_0 + a_1 t + \cdots + a_n t^n \longrightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

To do list:

- Webwork 4.1 due Tuesday
- Lab 1 due Friday
- Webwork 4.2 due next Wednesday

Quiz on Friday

on subspaces

Office hours this week

Mondays 5–6 pm and Wednesdays 2–3 pm
in Avery 339 (Dr Grifo)

Tuesdays 11–noon and Thursdays 1–2 pm
in Avery 337 (Kara)