Linear Algebra

Math 314 Fall 2025

Lecture 22

Today's poll code:

2H5GJN

To do list:

- Webwork 6.2 due Friday
- Webwork 7.1 due next Tuesday
- Webwork 7.2 due next Friday
- Find a lab team!

Quiz on Friday on diagonalizable matrices

Office hours

Mondays 5–6 pm
Wednesdays 2–3 pm
in Avery 339 (Dr. Grifo)

Tuesdays 11–noon
Thursdays 1–2 pm
in Avery 337 (Kara)

A few comments on the midterm

The dimension of a vector space is **a number**

A basis for a vector space is **a set of elements**

$\dim(V) =$ number of elements in a basis for V

$\dim(\mathbb{R}^n) = n$ Basis: $\{e_1, \ldots, e_n\}$

all $m \times n$ matrices $\dim(M_{m \times n}) = mn$

$\dim(\mathbb{P}_n) = n + 1$

polynomials $a_0 + a_1 t + \cdots + a_n t^n$

$\dim(\mathbb{P}) = \infty$

polynomials of any degree

The zero vector is never part of a basis for any vector space

A basis for V needs to be a set of elements of V

Example: The columns of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$ are elements of $\mathbb{R}^2$

The column space A is a subspace of $\mathbb{R}^2$

so any basis for col(A) will be a set of vectors in $\mathbb{R}^2$

Basis for col(A): $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

these are vectors in $\mathbb{R}^2$

$\dim(\mathrm{col}(A)) = 2$ and our basis has 2 vectors

A basis for V needs to be a set of elements of V

$$\text{Example: } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

one free variable $\implies$ $\dim(\mathrm{Nul}(B)) = 1$

$$\text{Basis: } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

this is one vector in $\mathbb{R}^2$

Vectors in $\mathrm{Nul}(B)$ are vectors x such that $Bx = 0$

$\implies$ $\mathrm{Nul}(B)$ is a subspace of $\mathbb{R}^2$

Find the eigenspace of $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ associated to the eigenvalue 3.

Eigenspace associated to 3 $=$ Null space of $A - 3I$

$$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{one free variable} \implies \dim(\mathrm{Nul}(A - 3I)) = 1$$

$$\text{one free variable} \implies \dim(E_3(A)) = 1$$

How to find a basis for the null space of $B = A - 3I$?

$\boxed{\text{general solution to } Bx = 0}$ $\begin{cases} x_1 \text{ free variable} \\ x_2 = 0 \end{cases} \implies x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Find the eigenspace of $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ associated to the eigenvalue 3.

Eigenspace associated to 3 $=$ Null space of $A - 3I$

$$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

one free variable $\implies \dim(E_3(A)) = 1$

How to find a basis for the null space of $B = A - 3I$?

$\boxed{\text{general solution to } Bx = 0}$

$$\begin{cases} x_1 \text{ free variable} \\ x_2 = 0 \end{cases} \implies x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

1 vector

The Withdraw deadline is on Friday

Midterm grades will posted by the end of the day tomorrow

If you're considering dropping this class:

- Talk to your advisor
- Let me know (urgently) if you'd like to talk to me

Lab 2

(To be released)

due after Thanksgiving

Groups of 2 or 3 students

Solo teams not allowed this time

Quick Recap
for Friday's quiz

A and B two $n \times n$ matrices

We say that A and B are **similar** if there exists an invertible matrix P such that

$$A = PBP^{-1}$$

A square matrix

A is **diagonalizable** if

it is similar to a diagonal matrix.

so

$$A = PDP^{-1}$$

D diagonal matrix

Theorem. A $n \times n$ matrix

A is diagonalizable if and only if

there exist n linearly independent eigenvectors for A.

Theorem. A $n \times n$ matrix

A is diagonalizable if and only if

the sum of the dimensions of its eigenspaces is n.

Important point:

The eigenspace associated to an eigenvalue λ always has dimension $\geqslant 1$.

so if all the eigenvalues of A have multiplicity 1 then A is diagonalizable.

The **algebraic multiplicity** of an eigenvalue λ is

its multiplicity as a root of the characteristic polynomial.

$$\rotatebox{90}{$\geqslant$}$$

The **geometric multiplicity** of an eigenvalue λ is

the dimension of the eigenspace corresponding to λ.

Theorem. A $n \times n$ matrix

A is diagonalizable if and only if

each eigenvalue of A has the same

algebraic and geometric multiplicity.

A $n \times n$ matrix

To diagonalize A (if possible):

Step 1: Find the eigenvalues of A.

Solve the equation $\det(A - \lambda I) = 0$.

The solutions are the eigenvalues.

Step 2: Find a basis for each eigenspace.

For each eigenvalue λ: find a basis for $\mathrm{Nul}(A - \lambda I)$.

$\mathrm{Nul}(A - \lambda I) =$ eigenspace associated to λ

If there are not enough vectors in these bases: A is not diagonalizable!

When A is diagonalizable:

$$A = PDP^{-1}$$

where

columns of P = linearly independent eigenvectors of A

Algorithm: find a basis for each eigenspace

D = diagonal with eigenvalues, in order

(the order in P and D needs to match)

Example:

$$\text{Diagonalize } A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \text{ if possible}$$

Step 1: Find the eigenvalues of A.

Example:

$$\text{Diagonalize } A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \text{ if possible}$$

Step 1: Find the eigenvalues of A.

Solve the equation $\det(A - \lambda I) = 0$.

Find the eigenvalues of $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = (1-\lambda)(-5-\lambda)(1-\lambda) - 27 - 27$$

$$- (9(-5-\lambda) - 9(1-\lambda) - 9(1-\lambda))$$

$$= -\lambda^3 - 3\lambda^2 + 4$$

$$= -(\lambda - 1)(\lambda + 2)^2$$

characteristic equation $\quad -(\lambda - 1)(\lambda + 2)^2 = 0$

Eigenvalues: 1 (with multiplicity 1) and -2 (with multiplicity 2)

Example:

$$\text{Diagonalize } A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \text{ if possible}$$

Step 1: Find the eigenvalues of A.

Solve the equation $\det(A - \lambda I) = 0$.

Eigenvalues: 1 (with multiplicity 1) and -2 (with multiplicity 2)

Example:

$$\text{Diagonalize } A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \text{ if possible}$$

Step 1: Find the eigenvalues of A.

Eigenvalues: 1 (with multiplicity 1) and -2 (with multiplicity 2)

Example:

$$\text{Diagonalize } A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \text{ if possible}$$

Step 1: Find the eigenvalues of A.

Eigenvalues: 1 (with multiplicity 1) and -2 (with multiplicity 2)

Step 2: Find three linearly independent eigenvalues for A.

Example:

$$\text{Diagonalize } A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \text{ if possible}$$

Step 1: Find the eigenvalues of A.

Eigenvalues: 1 (with multiplicity 1) and -2 (with multiplicity 2)

Step 2: Find three linearly independent eigenvalues for A.

Find a basis for $\text{Nul}(A - I)$ and a basis for $\text{Nul}(A + 2I)$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Eigenvalues: 1 with multiplicity 1 and -2 with multiplicity 2.

Find the eigenspaces for all the eigenvalues of A.

$$A - 1I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenspace associated to 1: span $\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}\right)$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Eigenvalues: 1 with multiplicity 1 and -2 with multiplicity 2.

Find the eigenspaces for all the eigenvalues of A.

$$A + 2I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenspace associated to -2: span $\left(\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$

Example:

$$\text{Diagonalize } A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \text{ if possible}$$

Step 1: Find the eigenvalues of A.

Eigenvalues: 1 (with multiplicity 1) and -2 (with multiplicity 2)

Step 2: Find three linearly independent eigenvalues for A.

Basis for the eigenspace associated to 1	$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$
Basis for the eigenspace associated to -2	$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Example:

$$\text{Diagonalize } A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \text{ if possible}$$

Eigenvalues: 1 (with multiplicity 1) and -2 (with multiplicity 2)

Basis for the eigenspace associated to 1

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Basis for the eigenspace associated to -2

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A is diagonalizable

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Example:

Diagonalize $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, if possible.

characteristic equation:

$$\begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$\iff (3-\lambda)^2 = 0$$

Eigenvalues: 3 with multiplicity 2

Eigenspace associated to 3:

$$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

one free variable $\implies \dim(E_3(A)) = 1 < 2$

A is not diagonalizable

Inner product

(also called the dot product)

The **inner product**

of $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is

$$u \bullet v = u^T v = u_1 v_1 + \cdots + u_n v_n.$$

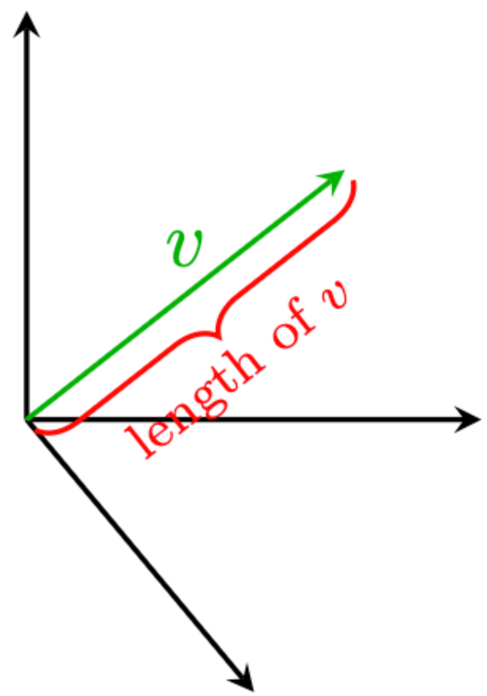
Example:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} = 1 \cdot 5 + 2 \cdot (-1) + 3 \cdot 0 = 5 - 2 = 3$$

Properties of the inner product:

1. $u \bullet v = v \bullet u$.
2. $(u + v) \bullet w = u \bullet w + v \bullet w$.
3. $(cu) \bullet v = v(u \bullet v) = u \bullet (cv)$.
4. $u \bullet u \geqslant 0$.
5. $u \bullet u = 0$ if and only if $u = 0$.

The **length** or **norm** of $v \in \mathbb{R}^n$ is the nonnegative real number



$$\|v\| = \sqrt{v \bullet v} = \sqrt{v^1 + \cdots + v_n^2}$$

If v is a vector in $\mathbb{R}^n$ and c is any scalar

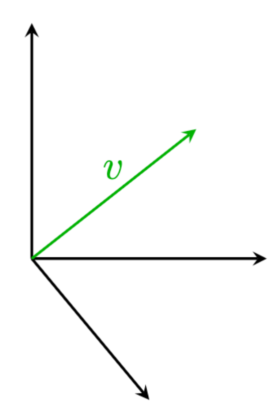
$$\|cv\| = |c| \cdot \|v\|$$

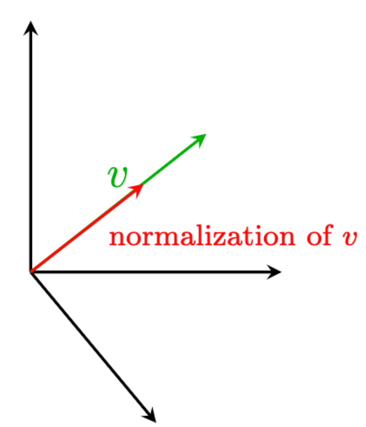
unit vector = vector with length 1

$v \neq 0$

normalization of $v = \dfrac{v}{\|v\|}$

= the unit vector with same direction as v





Orthogonal vectors

u and v vectors in $\mathbb{R}^n$

u and v are **orthogonal** if

$$u \bullet v = 0$$

Example: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are orthogonal

in $\mathbb{R}^2$

$$\boxed{\text{orthogonal vectors} = \text{perpendicular vectors}}$$

u and v are **orthogonal** if

$$u \bullet v = 0$$

in $\mathbb{R}^2$

$$\boxed{\text{orthogonal vectors} = \text{perpendicular vectors}}$$

Theorem (Pythagorean Theorem)**.** In $\mathbb{R}^2$,

u and v are orthogonal if and only if

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 .$$

A set of vectors in $\mathbb{R}^n$

$$\{v_1, \ldots, v_p\}$$

is an **orthogonal set** if

$$v_i \bullet v_j = 0 \text{ for all } i \neq j.$$

orthogonal basis = basis that is also an orthogonal set

Example: $e_1, \ldots, e_n$ is an orthogonal basis for $\mathbb{R}^n$

$$\boxed{\text{orthogonal} \implies \text{linearly independent}}$$

orthotogonal basis for $\mathbb{R}^n$ = n orthogonal vectors in $\mathbb{R}^n$

$= v_1, \ldots, v_n$ such that $\quad v_i \bullet v_j = 0$ for all $i \neq j$

Example: $e_1, \ldots, e_n$ is an orthogonal basis for $\mathbb{R}^n$

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix}$$

These form an orthogonal basis for $\mathbb{R}^3$, because

$$u_1 \bullet u_2 = 3 \cdot (-1) + 1 \cdot 2 + 1 \cdot 1 = 0$$

$$u_1 \bullet u_3 = 3 \cdot (-\frac{1}{2}) + 1 \cdot (-2) + 1 \cdot \frac{7}{2} = 0$$

$$u_2 \bullet u_3 = -1 \cdot (-\frac{1}{2}) + 2 \cdot (-2) + 1 \cdot \frac{7}{2} = 0$$

and

$$\boxed{\text{orthogonal} \implies \text{linearly independent}}$$

Why do we like orthogonal bases for $\mathbb{R}^n$?

Theorem. W subspace of $\mathbb{R}^n$

$S = \{u_1, \ldots, u_p\}$ orthogonal basis for W

For each y in W, the unique weights $c_1, \ldots, c_p$ such that

$$y = c_1 u_1 + \cdots + c_p u_p$$

are given by

$$\boxed{c_j = \frac{y \bullet u_j}{u_j \bullet u_j} \text{ for each } j = 1, \ldots, p.}$$

$\mathcal{B} = \{u_1, \ldots, u_n\}$ orthogonal basis for $\mathbb{R}^n$

v any vector in $\mathbb{R}^n$

$$[v]_{\mathcal{B}} = \begin{bmatrix} \frac{v \bullet u_1}{u_1 \bullet u_1} \\ \vdots \\ \frac{v \bullet u_n}{u_n \bullet u_n} \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix}$$

orthogonal basis for $\mathbb{R}^3$

$$y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$$

$$\frac{y \bullet u_1}{u_1 \cdot u_1} = \frac{18 + 1 - 8}{11} = 1$$

$$\frac{y \bullet u_2}{u_2 \cdot u_2} = \frac{-6 + 2 - 8}{6} = -2$$

$$\frac{y \bullet u_3}{u_3 \cdot u_3} = \frac{-3 - 2 - 28}{\frac{33}{2}} = -2$$

$$[y]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$$y = u_1 - 2u_2 - 2u_3$$

Orthonormal vectors

A set of vectors in $\mathbb{R}^n$

$$\{v_1, \ldots, v_p\}$$

is an **orthogonal set** if

$v_i \bullet v_j = 0$ for all $i \neq j$.

A set of vectors in $\mathbb{R}^n$

$$\{v_1, \ldots, v_p\}$$

is a **orthonormal set** if

$v_i \bullet v_j = 0$ for all $i \neq j$

and $\|v_i\| = 1$ for all i.

$$\boxed{\text{orthogonal} \implies \text{linearly independent}}$$

orthonormal basis for $\mathbb{R}^n$ = n orthonormal vectors

$$= v_1, \ldots, v_n \text{ such that } \begin{array}{c} v_i \bullet v_j = 0 \text{ for all } i \neq j \\ \|v_i\| = 1 \text{ for all } i \end{array}$$

Example: $e_1, \ldots, e_n$ is an orthonormal basis for $\mathbb{R}^n$

Orthonormal bases are the best bases:

Theorem. W subspace of $\mathbb{R}^n$

$S = \{u_1, \ldots, u_p\}$ orthonormal basis for W

For each y in W, the unique weights $c_1, \ldots, c_p$ such that

$$y = c_1 u_1 + \cdots + c_p u_p$$

are given by

$$\boxed{c_j = y \bullet u_j \text{ for each } j = 1, \ldots, p.}$$

Today’s poll code:

2H5GJN

The set

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

is

A. an orthogonal set

B. not an orthogonal set

The set

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

is

A. an orthonormal set

B. not an orthonormal set

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is

A. a unit vector

B. not a unit vector

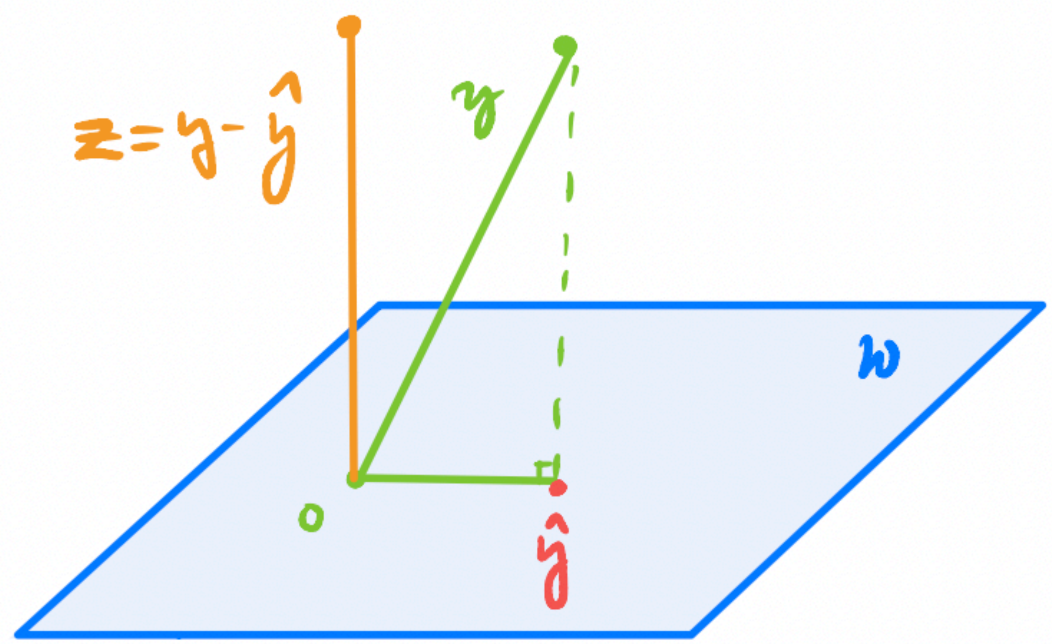
Orthogonal projections

Orthogonal projection

W subspace of $\mathbb{R}^n$

y any vector in $\mathbb{R}^n$

$\hat{y} =$ orthogonal projection of y in W

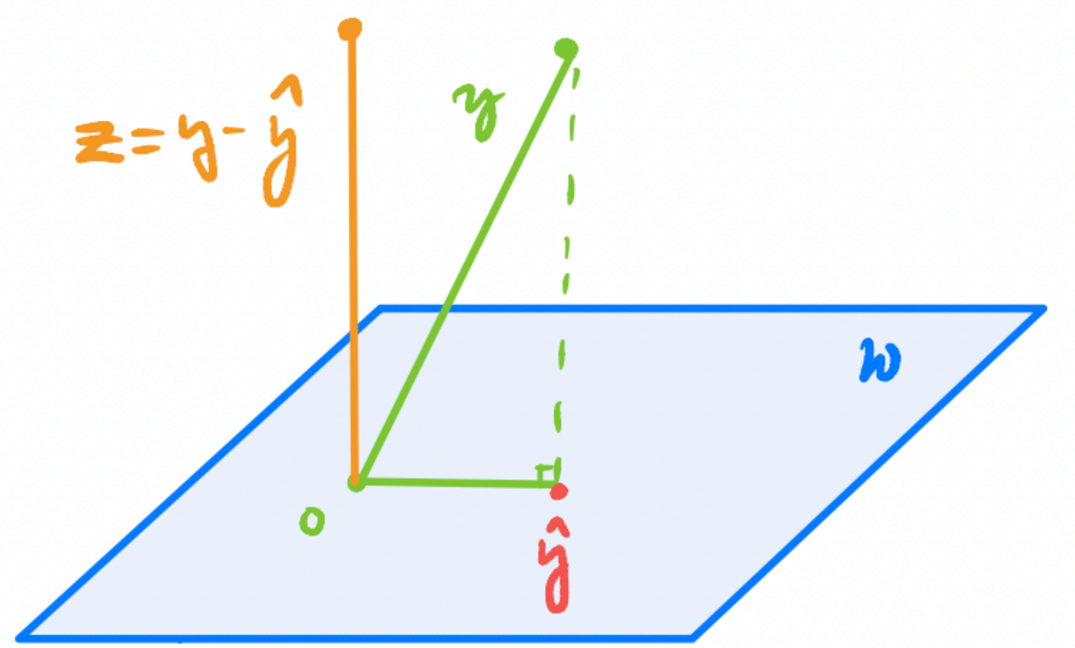


$\hat{y}$ is the unique vector with the following properties:

$y - \hat{y}$ is orthogonal to W

$\hat{y}$ is the unique vector in W that is closest to y.

$$\text{proj}_W(y) = \hat{y} = \text{orthogonal projection of } y \text{ at } W$$



$\{u_1, \ldots, u_p\}$ orthogonal basis for W

Then

$$\boxed{\text{proj}_W\,(y) = \frac{y \bullet u_1}{u_1 \bullet u_1} u_1 + \cdots + \frac{y \bullet u_p}{u_p \bullet u_p} u_p}$$

u and v vectors in $\mathbb{R}^n$

$$\text{projection of } v \text{ onto } u = \begin{array}{c}\text{projection of } v \text{ onto} \\ \text{the line spanned by } u\end{array}$$

Example: $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\{u_1, u_2\}$ orthogonal basis for $W = \mathrm{span}\,(\{u_1, u_2\})$

Why? Because

$$u_1 \bullet u_2 = 2 \cdot (-2) + 5 \cdot 1 + (-1) \cdot 1 = -4 + 5 - 1 = 0$$

and

$$\boxed{\text{orthogonal} \implies \text{linearly independent}}$$

Example: $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\{u_1, u_2\}$ orthogonal basis for $W = \text{span}\left(\{u_1, u_2\}\right)$

$$\begin{aligned}
\text{proj}_W(y) = \hat{y} &= \frac{y \bullet u_1}{u_1 \bullet u_1} u_1 + \frac{y \bullet u_2}{u_2 \bullet u_2} u_2 \\
&= \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix}
\end{aligned}$$

Example: $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

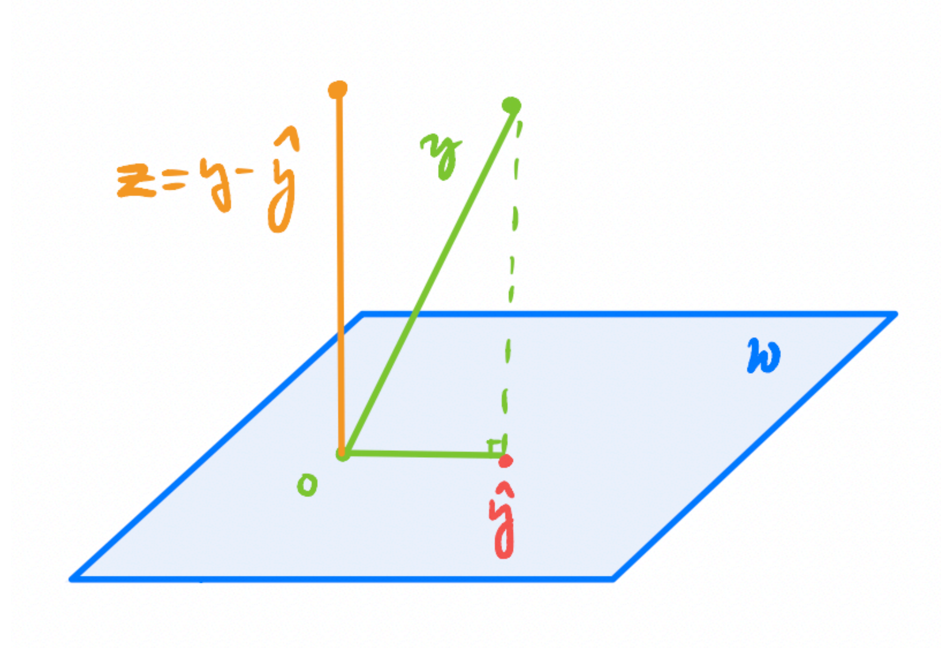
$\{u_1, u_2\}$ orthogonal basis for $W = \operatorname{span}(\{u_1, u_2\})$

$$\operatorname{proj}_W(y) = \hat{y} = \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix}$$ is in W

$$z = y - \hat{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{7}{5} \\ 0 \\ \frac{14}{5} \end{bmatrix}$$ is orthogonal to W

Theorem.

W subspace of $\mathbb{R}^n$



Any vector y in $\mathbb{R}^n$ can be written uniquely as

$$y = \hat{y} + z$$

in W

$\hat{y} = \mathrm{proj}_W(y)$

orthogonal to W

Example: $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\{u_1, u_2\}$ orthogonal basis for $W = \operatorname{span}(\{u_1, u_2\})$

$$\operatorname{proj}_W(y) = \hat{y} = \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix} \quad \text{is in } W$$

$$z = y - \hat{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{7}{5} \\ 0 \\ \frac{14}{5} \end{bmatrix} \quad \text{is orthogonal to } W$$

$$y = \hat{y} + z$$

in W

orthogonal to W

Best approximation:

$\mathrm{proj}_W(y) = \hat{y} =$ orthogonal projection of y at W

is the closest point to y in W

meaning

$$\|y - \hat{y}\| < \|y - w\|$$

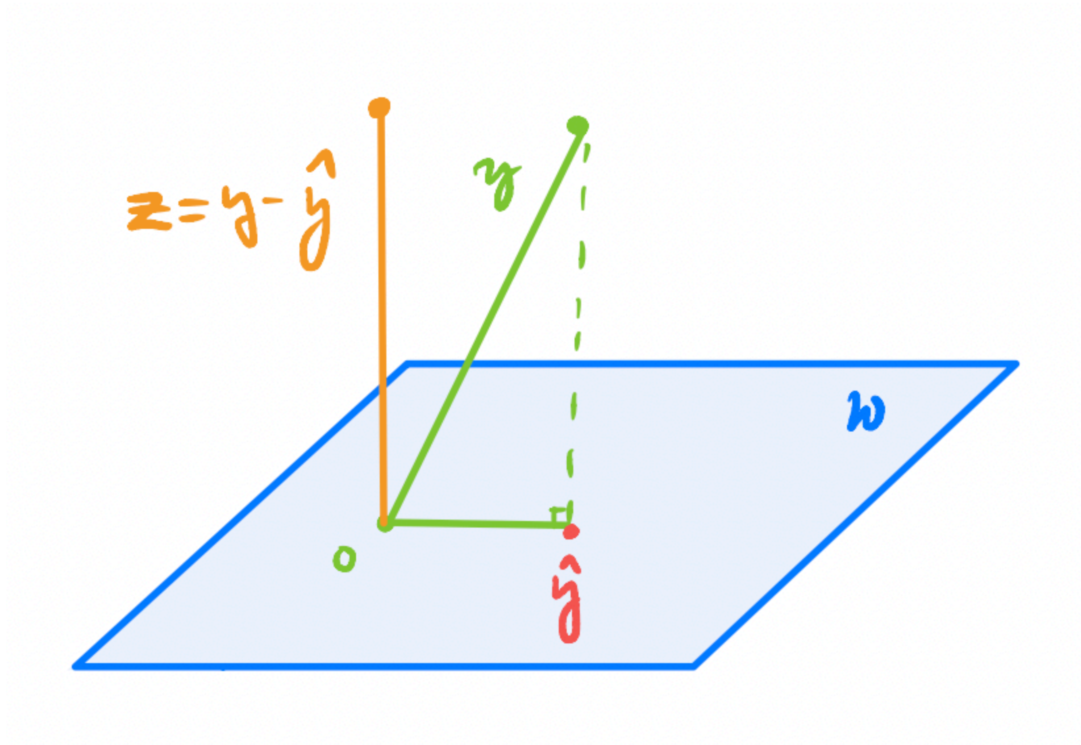
for all vectors $w \neq \hat{y}$ in W

Example: $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\{u_1, u_2\}$ orthogonal basis for $W = \operatorname{span}(\{u_1, u_2\})$

What is the closest point to y in W?

$$\operatorname{proj}_W(y) = \hat{y} = \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix}$$



The **distance** of y to W is

$$\|y - \text{proj}_W(y)\|$$

The **distance** of y to W is

$$\|y - \text{proj}_W(y)\|$$

Example: $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\{u_1, u_2\}$ orthogonal basis for $W = \text{span}\left(\{u_1, u_2\}\right)$

The distance of y to W is

$$\|y - \text{proj}_W(y)\| = \sqrt{\left(\tfrac{7}{5}\right)^2 + \left(\tfrac{14}{5}\right)^2} = \frac{\sqrt{245}}{5}$$

Note: if y is in W, then

$$\text{proj}_W(y) = y$$

and y is the closest point to y in W

The distance of y to W is

$$\|y - y\| = 0$$

To do list:

- Webwork 6.2 due Friday
- Webwork 7.1 due next Tuesday
- Webwork 7.2 due next Friday
- Find a lab team!

Quiz on Friday on diagonalizable matrices

Office hours

Mondays 5–6 pm and Wednesdays 2–3 pm
in Avery 339 (Dr Grifo)

Tuesdays 11–noon and Thursdays 1–2 pm
in Avery 337 (Kara)