A stable version of Harbourne's Conjecture

R regular ring
I radical ideal
$h =$ big height of $I = \max\{ \operatorname{ht} P : P \in \operatorname{Min}(I)\}$

n-th symbolic power of I $\qquad I^{(n)} = \bigcap_{P \in \operatorname{Min}(I)} \left(I^n R_P \cap R \right)$

<u>Containment Problem</u> When is $I^{(a)} \subseteq I^b$

<u>theorem</u> (Ein-Lazarsfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)

$$I^{(hn)} \subseteq I^n \qquad \forall n \geqslant 1.$$

<u>Question</u> (Huneke, 2000) P prime of height 2 in a RLR. Is $P^{(3)} \subseteq P^2$?

Conjecture (Harbourne, 2008?) $\quad I^{(hn-h+1)} \subseteq I^n \qquad \forall n \geqslant 1.$

Good reasons to think so: in char $p > 0$,

$$I^{(hq-h+1)} \subseteq I^{[q]} \qquad \forall q = p^e$$

But unfortunately, counter examples exist.

<u>Example</u> (Dumnicki, Szemberg, Tutaj-Gasińska, 2013)

$$I = \left(x\left(y^{\overset{c}{\cancel{3}}} - z^{\overset{c}{\cancel{3}}}\right), \; y\left(z^{\overset{c}{\cancel{3}}} - x^{\overset{c}{\cancel{3}}}\right), \; z\left(x^{\overset{c}{\cancel{3}}} - y^{\overset{c}{\cancel{3}}}\right)\right) \subseteq \overset{k}{\cancel{\mathbb{Q}}}[x, y, z]$$

$$I^{(3)} \nsubseteq I^2$$

Harbourne - Seceleanu

so really, $\quad I^{(2n-1)} \subseteq I^n$ fails for $n = 2$.

But $I^{(hn-h+1)} \subseteq I^n$ for all $n \geq 1$ holds for nice classes of ideals:

- General points in $\mathbb{P}^2$ (Harbourne-Huneke) and $\mathbb{P}^3$ (Dumnicki)
- Nice configurations of points (eg, star configurations)
- In char p, if R/I is an F-pure ring (G-Huneke)

this is an easy to check condition:

When $R = k[x_1, \ldots, x_d]$, $m = (x_1, \ldots, x_d)$, I ideal in R.

$$R/I \text{ F-pure} \iff (I : I^{[p]}) \not\subseteq m^{[p]}$$

<u>Notation</u>: $I^{[p]} = (f^p : f \in I)$

<u>Examples</u>:
- I squarefree monomial ideal

also strongly F-regular
$$\begin{cases} \circ \ R/I \text{ Veronese } k[\text{all monomials of deg } d \text{ in } v \text{ variables}] \\ \circ \ R = k[X],\ I = I_t(X),\quad X\ n\times m \text{ matrix, any } t \\ \circ \ R/I = k[\text{all } t\times t \text{ minors of } n\times m \text{ matrix}] \\ \circ \ R/I = \text{normal semigroup ring} \\ \circ \ R/I \text{ nice ring of invariants} \end{cases}$$

<u>theorem</u> (G-Huneke) If R/I is strongly F-regular, $I^{((h-1)(n-1)+1)} \subseteq I^n$ $\forall n \geq 1$

This is Harbourne's Conjecture if we replace h by $h-1$.

<u>Corollary</u> $h = 2 \Rightarrow I^{(n)} = I^n$ $\forall n \geq 1$

eg $P = \ker\left(k[a,b,c,d] \longrightarrow k[s^3, s^2t, st^2, t^3]\right)$
3 generated height 2 prime.

On the other hand, there are no counterexamples to the following:

Harbourne's Conjecture (stable version) $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$.

Example $I =$ ideal defining the Fermat, Klein or Wiman configurations

$I^{(2n-1)} \subseteq I^n$ for all $n \geqslant 3$, since $\rho(I) = \frac{3}{2}$

(Dumnicki, Harbourne, Szemberg, Tutaj-Gasińska ; Bauer, Di Rocco, Harbourne, Huizenga, Seceleanu, Szemberg)
Fermat / Klein, Wiman

Resurgence of I (Bocci-Harbourne) $\rho(I) = \sup\left\{\frac{a}{b} : I^{(a)} \not\subseteq I^b\right\}$

If $\rho(I) < h$, then in $I^{(hn-h+1)} \subseteq I^n$ for all

$$\frac{hn-h+1}{n} > \rho(I) \iff n > \frac{h-1}{h-\rho(I)}$$

Question Are there ideals I with $\rho(I) = h$?

Even if such ideals exist, they might still verify $I^{(hn-h+1)} \subseteq I^n$ $\forall n \gg 0$

Question If $I^{(hk-h+1)} \subseteq I^k$ for some k, does that imply $I^{(hn-h+1)} \subseteq I^n$ $\forall n \gg 0$?

Theorem R regular ring containing a field, I radical of big height h.
If $I^{(hk-h)} \subseteq I^k$ for some k, then $I^{(hn-h)} \subseteq I^n$ $\forall n \gg 0$

key ingredient $I^{(hn+a_1+\cdots+a_n)} \subseteq I^{(a_1+1)} \cdots I^{(a_n+1)}$ $\forall a_i \geqslant 0, n \geqslant 1$

Eg Can apply this theorem if $I^{(4)} \subseteq I^3$.

- there exist primes of height 2 in dim 3 with $I^{(4)} \subseteq I^3$ whose symbolic Rees algebra is not even noetherian
- true for $\ker(k[x,y,z] \to k[t^a, t^b, t])$ for $a = 3$ or $4 < b < c$
- true for many $\ker(k[x,y,z] \to k[t^9, t^b, t^c])$, but not all (weird problem)

Question Given $C > 0$, is there N such that $I^{(hn-C)} \subseteq I^n$ $\forall n \geq N$?

Question If $I^{(hk-C)} \subseteq I^k$ for some k, does that imply $I^{(hn-C)} \subseteq I^n$ $\forall n \gg 0$?

Partial Answer: Yes if

- $\rho(I) < h$, or
- $I^{(n+h)} \subseteq I I^{(n)}$ $\forall n \geq 1$, or at least $n \gg 0$.

But:

- this cannot hold for all I, since for $n = 1$, $h = 2$ get $I^{(3)} \subseteq I^2$.
- Does not even hold eventually for all I: some Fermat configurations fail this (Seceleanu) for n arbitrarily large.

Theorem If R/I is F-pure, then $I^{(n+h)} \subseteq I I^{(n)}$ $\forall n \geq 1$.

so if $I^{(hk-C)} \subseteq I^k$ for some k, then $I^{(hn-C)} \subseteq I^n$ for all $n \gg 0$.