A Fedder type criterion over Gorenstein Rings (and symbolic powers)

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I radical ideal in a ring R

the n-th symbolic power of I is $I^{(n)} = \underbrace{\bigcap_{Q \in \mathrm{Min}(I)} (I^n R_Q \cap R)}_{\text{minimal part of } I^n}$

P prime ideal

$P^{(n)} = \{f \in R : sf \in P^n, s \notin P\}$

If $R = \mathbb{C}[x_1, \dots, x_d]$, the Zariski-Nagata theorem says

$I^{(n)} = \{f : f \text{ vanishes up to order } n \text{ on the variety defined by } I\}$

<u>Facts</u> 1) $I^{(n+1)} \subseteq I^{(n)}$

2) $I^n \subseteq I^{(n)}$

3) If $I = (\text{regular sequence})$, then $I^n = I^{(n)}$ $\forall n \geqslant 1$.

But in general, $I^n \neq I^{(n)}$.

<u>Example</u> $I = (xy, xz, yz) = (x,y) \cap (x,z) \cap (y,z) \subseteq k[x,y,z]$

$I^{(2)} = (x,y)^2 \cap (x,z)^2 \cap (y,z)^2 \ni \underbrace{xyz}_{\deg 3} \in I^{(2)} \setminus I^2$

Elements in I^2 have degree $\geqslant 4$ $\qquad \therefore I^2 \subsetneq I^{(2)}$

<u>Fun fact</u> $I^{(3)} \subseteq I^2$

Containment Problem When is $I^{(a)} \subseteq I^b$?

theorem (Ein-Lazarsfeld-Smith, 2001, Hochster-Huneke, 2002, Ma-Schwede, 2017)
R regular ring, excellent in mixed char
I radical ideal of big height h $(:= \max\{\text{ht } Q : Q \in \text{Ass}(I)\})$
then $I^{(hn)} \subseteq I^n$ for all $n \geqslant 1$

Example $I = (xy, xz, yz)$ has big height $h=2$
the theorem says $I^{(2n)} \subseteq I^n$ for all $n \geqslant 1$, and in particular $I^{(4)} \subseteq I^2$
But we can do better: $I^{(3)} \subseteq I^2$

Question (Huneke, 2000) P prime of height 2 in a RLR. Is $P^{(3)} \subseteq P^2$?

Conjecture (Harbourne, 2008) I radical ideal of big height h in a regular ring then $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$.

Note In char p, $I^{(hq-h+1)} \subseteq I^{[q]}$ for all $q = p^e$
this is the basis for Hochster and Huneke's work.

Example (Dumnicki-Szemberg-Tutaj-Gasińska, 2013, Harbourne-Seceleanu, 2015)
$I^{(3)} \nsubseteq I^2$ for $I = (x(y^k - z^k), y(z^k - x^k), z(x^k - y^k)) \subseteq k[x,y,z]$
$k \geqslant 3$ fixed constant
any field of char $\neq 2$

this corresponds to 12 points in $\mathbb{P}^2$ in very special position.

Harbourne's Conjecture holds for:

- Points in general position in $\mathbb{P}^2$ (Harbourne-Huneke) and $\mathbb{P}^3$ (Dumnicki)
- Squarefree monomial ideals

From now on: R is an F-finite domain of characteristic $p>0$

<u>theorem</u> (G– Huneke, 2017) R regular ring, I radical ideal of big height h

1) If R/I is F-pure, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$
2) If R/I is strongly F-regular ($h \geqslant 2$), $I^{((h-1)(n-1)+1)} \subseteq I^n$ for all $n \geqslant 1$

When $h=2$, $I^{(n)} = I^n$ for all $n \geqslant 2$.

Some char p basics: we can view R as a module via the Frobenius action
we write $F_* R$ for this module, or $F^e_* R$ for the action F^e

<u>Definition</u> An F-finite ring of char p is F-pure $\equiv$ F-split if the Frobenius map splits as a map of R-modules

$$\begin{tikzcd} R \arrow[r, "F^e"] & F^e_* R \arrow[l, dashed, bend left, "\text{splitting}"] \end{tikzcd} \text{ splits}$$

<u>Examples</u> of F-pure rings R/I, R regular:

- I squarefree monomial ideal
- $I = (t \times t$ minors of a generic $n \times m$ matrix$)$
- $R/I \cong$ Veronese ring
- $R/I \cong$ ring of invariants of a linearly reductive group.

Our goals Can we do this over a non-regular rings?

Main ingredients in our proof:

- Containments are local, so we might as well assume R is a local ring
- Fedder's Criterion (1983) (R, m) RLR of char p, I radical ideal

R/I is F-pure $\Leftrightarrow$ $(I^{[q]} : I) \not\subseteq m^{[q]}$ for some/all/large $q = p^e$.

- For the strongly F-regular case, we use Glassbrenner's Criterion:

Glassbrenner's Criterion (1996) (R, m) F-finite RLR of char p, I radical ideal

R/I strongly F-regular $\Leftrightarrow$ $c(I^{[q]} : I) \not\subseteq m^{[q]}$ for all $q = p^e$ large

Roadmap: to show $I^{(a)} \subseteq I^b$:

1) $I^{(a)} \subseteq I^b \Leftrightarrow (I^b : I^{(a)}) = R \Leftrightarrow (I^b : I^{(a)}) \not\subseteq m$

2) show $(I^b : I^{(a)})^{[q]} \not\subseteq m^{[q]}$ fails by showing

$$\underbrace{J_q}_{\text{not contained in } m^{[q]} \text{ when } R/I \text{ is F-pure/strongly F-regular}} \subseteq (I^b : I^{(a)})^{[q]} \quad \text{for all large } q$$

When R is regular and R/I is F-pure, we take $J_q = (I^{[q]} : I)$

When R is regular and R/I is strongly F-regular, we take

$$J_q = \underbrace{(I^N : I^{(N)})}_{\text{always contains } c \notin \text{ minimal primes of } I} (I^{[q]} : I) \qquad N = (h-1)(n-1)+1$$

How about when R is not regular?

Why was Fedder's Criterion true?

$$R/I \text{ F-split} \iff \exists \text{ splitting } \phi \in \mathrm{Hom}_{R/I}(F^e_*(R/I), R/I)$$

$$1 \in \operatorname{im} \phi$$

R regular $\Downarrow$ free

$$\begin{tikzcd} F^e_* R \arrow[r, dashed, "\exists \tilde{\phi}"] \arrow[d] & R \arrow[d] \\ F^e_*(R/I) \arrow[r, "\phi"'] & R/I \end{tikzcd}$$

- When R is Gorenstein, $\mathrm{Hom}_R(F^e_* R, R)$ is cyclic, generated by the Grothendieck trace map Φ_e.

All maps in $\mathrm{Hom}_R(F^e_* R, R)$ look like $\Phi_e(F^e_* r \cdot -)$, $r \in R$

Fedder's Criterion says R/I F-pure

$$\iff 1 \in \text{image of some map } F^e_* R \to R$$

$$\iff \Phi_e(\underbrace{I^{[q]} : I}_{\text{elements in } R \text{ that define maps that descend to } R/I})$$

<u>Lemma</u> (G-Ma-Schwede) R F-finite Gorenstein ring, $pd(Q) < \infty$

1) All maps in $\mathrm{Hom}_{R/Q}(F^e_*(R/Q), R/Q)$ lift to $\mathrm{Hom}_R(F^e_* R, R)$

2) R/Q F-pure $\Rightarrow$ $\Phi_e(F^e_*(I_e(Q) : Q)) = R$

<u>theorem</u> (G–Ma–Schwede) R F-finite Gorenstein ring, $\mathrm{pd}(Q)<\infty$
Q radical ideal of big height h

1) R/Q F-pure $\Rightarrow$ $Q^{(hn-h+1)} \subseteq Q^n$ for all $n \geq 1$

2) R/Q strongly F-regular $\Rightarrow$ Q verifies Harbourne's Conj with $h-1$
$h \geq 2$

How about when $\mathrm{pd}(Q)=\infty$?

<u>Example</u> $R = k[x,y,z]/(xy-z^a)$, $Q=(x,z)$, $\mathrm{pd}(Q)=\infty$

R/Q is strongly F-regular, $h=1$ $\rightsquigarrow$ take $h=2$ in thm

the theorem would say $Q^{(n)}=Q^n$ for all n. But that's false!

In fact, $Q^{(an)}=(x^n) \neq Q^n$

But we can prove that if J = Jacobian ideal $=(x,y,z^{a-1})$

$$J^n Q^{(n)} \subseteq Q^n \quad \text{for all } n.$$

And in fact, in this case the smallest α such that

$$J^{\lfloor \alpha n \rfloor} Q^{(n)} \subseteq Q^n \quad \text{for all } n$$

is $\alpha = \dfrac{a-1}{a} \xrightarrow[a\to\infty]{} 1$.

__theorem__ (G-Ra-Schwede)

R Gorenstein F-finite ring of char $p>0$ containing k
Q radical ideal $\qquad J = Jac(R/k)$
$h = \max\{\mu(Q_P) : P \in Ass(R/Q)\}$

- If R/Q is F-pure, then $J^n Q^{(hn-h+1)} \subseteq Q^n$ for all $n \geq 1$.
- If R/Q is strongly F-regular, $J^n Q^{((h-1)(n-1)+1)} \subseteq Q^n$ for all $n \geq 1$.

__theorem__ (Hochster-Huneke, 2002, Takagi, 2006)

R reduced equidimensional affine algebra over a perfect field k of char p

$J = Jac(R/k)$ the Jacobian ideal of R over k

Q ideal in R, $Q \cap R^\circ \neq \emptyset$, $h = \max\{\mu(Q_P) : P \in Ass(Q)\}$

then $J^n Q^{(hn)} \subseteq Q^n$ for all $n \geq 1$.