Problem Set 10

Due Wednesday, December 4

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly*. As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of ring or Lagrange's Theorem). Please do not refer to theorems by their number in the course notes, as that can change.

Problem 1. Let R be a ring.

(1.1) Prove that an ideal I of R is proper if and only if I contains no units.

(1.2) Assume R is commutative. Show that R is a field if and only if its only ideals are $\{0\}$ and R.

(1.3) Show that the only ideals of $R = \mathrm{Mat}_{2\times 2}(\mathbb{R})$ are $\{0\}$ and R, and yet R is not a division ring.

Problem 2. Define $N\colon \mathbb{C} \to \mathbb{R}$ to be the square of the complex norm; that is,

$$N(a+bi) = (a+bi)(a-bi) = a^2+b^2.$$

You can use without proof that N satisfies $N(\alpha\beta) = N(\alpha)N(\beta)$ for any $\alpha, \beta \in \mathbb{C}$.

(2.1) Show that the only units of $\mathbb{Z}[i]$ are ± 1 and $\pm i$.

(2.2) Prove that the only units of the ring $\mathbb{Z}[\sqrt{-5}]$ are ± 1.

(2.3) Are there units in $\mathbb{Z}[\sqrt{2}]$ other than ± 1?

Problem 3. Let a and b be nonzero integers. Prove that $(a,b) = (d)$ where $d = \gcd(a,b)$.

Problem 4. Let I and J be ideals of a commutative ring R with $1 \neq 0$. You can use without proof that $I+J$, $I\cap J$, and IJ are ideals of R.

(4.1) Show that $IJ \subseteq I \cap J$.

(4.2) Give an example where $IJ \neq I \cap J$.

(4.3) Suppose that $I+J = R$. Show that $IJ = I \cap J$.

(4.4) Suppose m and n are distinct maximal ideals of a commutative ring R. Prove that $mn = m \cap n$.

Hint: First consider $m+n$.

(4.5) Suppose that $I+J=R$. Show that there is a ring isomorphism $R/(I\cap J) \cong R/I \times R/J$.

Problem 5. Let $I = (2,x)$ in $R = \mathbb{Z}[x]$.

(5.1) Show that $\mathfrak{m} = (2,x)$ is a maximal ideal.

(5.2) Show that $(2,x)$ is not a principal ideal.