

last time

Long Exact Sequence in Homology

A short exact sequence in $\text{Ch}(R)$

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{P} C \rightarrow 0$$

induces a long exact sequence

$$\dots \rightarrow H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{P_*} H_n(A) \xrightarrow{\partial} H_{n-1}(C) \rightarrow \dots$$

Given $c \in Z_{n+1}(C) = \ker(d_{n+1}^C : C_{n+1} \rightarrow C_n)$

$$\begin{array}{ccc} b \in B_{n+1} & \longrightarrow & c \in C_{n+1} \\ \downarrow d_{n+1}^B & & \\ a \in A_n & \longmapsto & d_{n+1}^B(b) \end{array}$$

$$\partial(c) = a + \text{im } d_{n+1}^A \in H_{n+1}(A)$$

Theorem Any commutative diagram in $\text{Ch}(R)$

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{P} C \rightarrow 0$$

$$0 \rightarrow A' \xrightarrow{i'} B' \xrightarrow{P'} C' \rightarrow 0$$

with exact rows induces a commutative diagram with exact rows

$$\begin{array}{ccccccc} \dots & \rightarrow & H_{n+1}(C) & \xrightarrow{\partial} & H_n(A) & \xrightarrow{i_*} & H_n(B) \xrightarrow{P_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \dots \\ & & h_* \downarrow & & f_* \downarrow & & g_* \downarrow \\ \dots & \rightarrow & H_{n+1}(C') & \xrightarrow{\partial'} & H_n(A') & \xrightarrow{i'_*} & H_n(B') \xrightarrow{P'_*} H_n(C') \xrightarrow{\partial'} H_{n-1}(A') \rightarrow \dots \end{array}$$

Proof LES in homology \Rightarrow exact rows

$$H_n \text{ functorial} \Rightarrow \begin{array}{ccccc} H_n(A) & \xrightarrow{i_n} & H_n(B) & \xrightarrow{p_n} & H_n(C) \\ f_* \downarrow & & g_* \downarrow & & h_* \downarrow \\ H_n(A') & \xrightarrow{i'_n} & H_n(B) & \xrightarrow{p'_n} & H_n(C') \end{array} \quad \text{commute}$$

$$H_n(C) \xrightarrow{\partial} H_{n-1}(A)$$

Need to show: $\begin{array}{ccc} h_* \downarrow & & \downarrow f_* \\ H_n(C') & \xrightarrow{\partial'} & H_{n-1}(A') \end{array} \quad \text{commutes}$

Let $c \in \ker(d_n : C_n \rightarrow C_{n-1})$. Need:

$$\textcircled{L} \quad f_{n-1} \partial(c) = \partial' h_n(c) \quad \textcircled{R}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{p_n} & C_n & \longrightarrow 0 \\ & & \downarrow d_n & & \downarrow d_n & & \downarrow d_n & \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow{i_{n-1}} & B_{n-1} & \xrightarrow{p_{n-1}} & C_{n-1} & \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow i'_n & & \downarrow p'_n & \\ 0 & \longrightarrow & A'_n & \xrightarrow{i'_n} & B'_n & \xrightarrow{p'_n} & C'_n & \longrightarrow 0 \\ & & \downarrow d_n & & \downarrow f_* & & \downarrow h_* & \\ 0 & \longrightarrow & A'_{n-1} & \xrightarrow{i''_n} & B'_{n-1} & \xrightarrow{p''_{n-1}} & C'_{n-1} & \longrightarrow 0 \end{array}$$

\textcircled{L} :

$$\begin{aligned} \partial(c) &= a + \text{im } d_n \in H_{n-1}(A), \text{ where } i_{n-1}(a) = d_n(b), p_n(b) = c \\ \Rightarrow f_{n-1} \partial(c) &= f_{n-1}(a + \text{im } d_n) = f_{n-1}(a) + \text{im } d'_n \in H_{n-1}(A') \end{aligned}$$

(R) to compute $\partial' h_n(c)$, we:

Step 1 Find

$$b' \in B'_n \xrightarrow{p'_n} h_n(c) \in C'_n$$

By commutativity of

$$\begin{array}{ccc} B_n & \xrightarrow{p_n} & C_n \\ g_n \downarrow & & \downarrow h_n \\ B'_n & \xrightarrow{p'_n} & C'_n \end{array}$$

can choose $b' = g_n(b)$ where $p_n(b) = c$ (from (L))

because $p'_n(b') = p'_n g_n(b) = h_n p_n(b) = h_n(c)$

Step 2 take $a' \in A'_{n-1}$ such that

$$\begin{array}{ccc} b' \in B'_n & \xrightarrow{p'_n} & h_n(c) \\ d_n \downarrow & & \\ a' \in A'_{n-1} & \mapsto & d_n(b') \end{array}$$

set $\partial' h_n(c) := a' + \text{im } d_n \in H_{n-1}(A')$

By commutativity of

$$\begin{array}{ccc} B_n & \xrightarrow{d_n} & B_{n-1} \\ g_n \downarrow & & \downarrow g_{n-1} \\ B'_n & \xrightarrow{d'_n} & B_{n-1} \end{array}$$

$$d_n(b') = d_n g_n(b) = g_{n-1} d_n(b)$$

Our choice of a in ⑥ satisfies

$$d_n(b') = g_{n-1} d_n(b) = g_{n-1} i_{n-1}(a)$$

and by commutativity of

$$\begin{array}{ccc} A_{n-1} & \xrightarrow{i_{n-1}} & B_{n-1} \\ f_{n-1} \downarrow & & \downarrow g_{n-1} \\ A'_n & \xrightarrow{i'_{n-1}} & B'_n \end{array}$$

we have

$$i'_{n-1} f_{n-1}(a) = g_{n-1} i_{n-1}(a) = d_n(b')$$

above
↓

so $f_{n-1}(a)$ satisfies

$$i'_{n-1} f_{n-1}(a) = d_n(b')$$

so can choose $a' := f_{n-1}(a)$, and

$$\partial' h_n(c) = a' + im d_n = f_{n-1}(a) + im d_n \in H_{n-1}(A')$$

so: $\partial' h_n(c) = f_{n-1} \partial(c)$ as we wanted !

Comments:

- ① the LES in homology can often be used to find the homology of a particular complex by fitting it into a SES of complexes with some well-known or easier to understand complexes.
- ② later we will see how LES naturally pop up everywhere. In particular, we will study derived functors such as Tor and Ext, which are constructed via the homology of a particular complex, and which must then induce a LES from any SES.
- ③ Any LES breaks into SES:

$$\cdots \rightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \rightarrow \cdots \quad \text{LES}$$

breaks into

$$\begin{array}{ccccc}
 & \circlearrowleft & & & \\
 & \downarrow \ker f_n & & & \\
 C_{n+1} & \xrightarrow{f_{n+1}} & C_n & \xrightarrow{f_n} & C_{n-1} \\
 & \searrow & \downarrow & & \\
 & & \text{coker } f_{n+1} & & \\
 & & \searrow & & 0 \\
 & & & &
 \end{array}$$

$\ker f_n = \text{im } f_{n+1} = \ker(C_n \rightarrow \text{coker } f_n)$
 so
 $0 \rightarrow \ker f_n \rightarrow C_n \rightarrow \text{coker } f_{n+1} \rightarrow 0$
 is exact.