Linear Algebra

Math 314 Fall 2025

Lecture 20

Today's poll code:

C7DJE9

To do list:

- Webwork 5.2 now due today
- Webwork 6.1 due on Friday
- Webwork 6.2 due next Friday
- Study for the midterm!

Quiz on Friday

on eigenvalues

Office hours

Mondays 5–6 pm and Wednesdays 2–3 pm
Thursday 3:30 to 4:30 pm
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in Avery 339 (Dr. Grifo)

Tuesdays 11–noon
Thursdays 1–2 pm
in Avery 337 (Kara)

Quick Recap on diagonalization

A square matrix

A real number λ is an **eigenvalue** of A

if there exists a nontrivial solution x to the equation

$$Ax = \lambda x.$$

x is a **eigenvector** of A associated to λ if

$$Ax = \lambda x.$$

Recipe to find eigenvalues:

Find the solutions to the

characteristic equation of A

$$\det(A - \lambda I) = 0$$

A and B two $n \times n$ matrices

We say that A and B are **similar** if

there exists an invertible matrix P such that

$$A = PBP^{-1}$$

$$A \text{ is similar to } B \iff B \text{ is similar to } A$$

take $Q = P^{-1}$, get $B = QAQ^{-1}$

Properties of similarity:

Reflexivity: every square matrix A is similar to itself.

$$A = IAI^{-1}$$

Symmetry: if A is similar to B then B is similar to A.

$A = PBP^{-1}$ take $Q = P^{-1}$, get $B = QAQ^{-1}$

Transitivity:

A is similar to B and B similar to C $\Rightarrow A$ is similar to C

$A = PBP^{-1}$ $B = QCQ^{-1}$ Take $S = PQ$, get $A = (PQ)C(PQ)^{-1}$

Theorem. A $n \times n$ matrix

A is diagonalizable if and only if

the sum of the dimensions of its eigenspaces is n.

Important point:

The eigenspace associated to an eigenvalue λ always has dimension $\geqslant 1$.

so if all the eigenvalues of A have multiplicity 1 then A is diagonalizable.

The **algebraic multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

The **geometric multiplicity** of an eigenvalue λ is the dimension of the eigenspace corresponding to λ.

Theorem. A $n \times n$ matrix

A is diagonalizable if and only if

each eigenvalue of A has the same

algebraic and geometric multiplicity.

When A is diagonalizable:

$$A = PDP^{-1}$$

where

columns of P = linearly independent eigenvectors of A

Algorithm: find a basis for each eigenspace

D = diagonal with eigenvectors, in order

(the order in P and D needs to match)

Applications of diagonalization

not on the midterm

Theorem. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A.

If $A = PDP^{-1}$ with D a diagonal matrix,

and $\mathcal{B}$ is the basis for $\mathbb{R}^n$ whose elements are the columns of P,

then D is the matrix representing T in the basis $\mathcal{B}$.

Example: $\quad T\colon \mathbb{R}^2 \to \mathbb{R}^2 \quad T(x) = Ax$

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}.$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \longrightarrow A_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

Suppose A is diagonalizable, say

$$A = PDP^{-1}$$

Then

$$A^k = \underbrace{A \cdots A}_{k \text{ times}} = PD\underbrace{(P^{-1}P)}_{I}D\underbrace{(P^{-1}P)}_{I}\cdots\underbrace{(P^{-1}P)}_{I}DP^{-1} = P\underbrace{D\cdots D}_{k \text{ times}}P^{-1} = PD^kP^{-1}.$$

Example:
$$A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\Downarrow$$

$$A^k = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{bmatrix}$$

Consider the sequence defined recursively by

$$f_0 = 1 \qquad f_1 = 1 \qquad f_{n+2} = f_n + f_{n+1}$$

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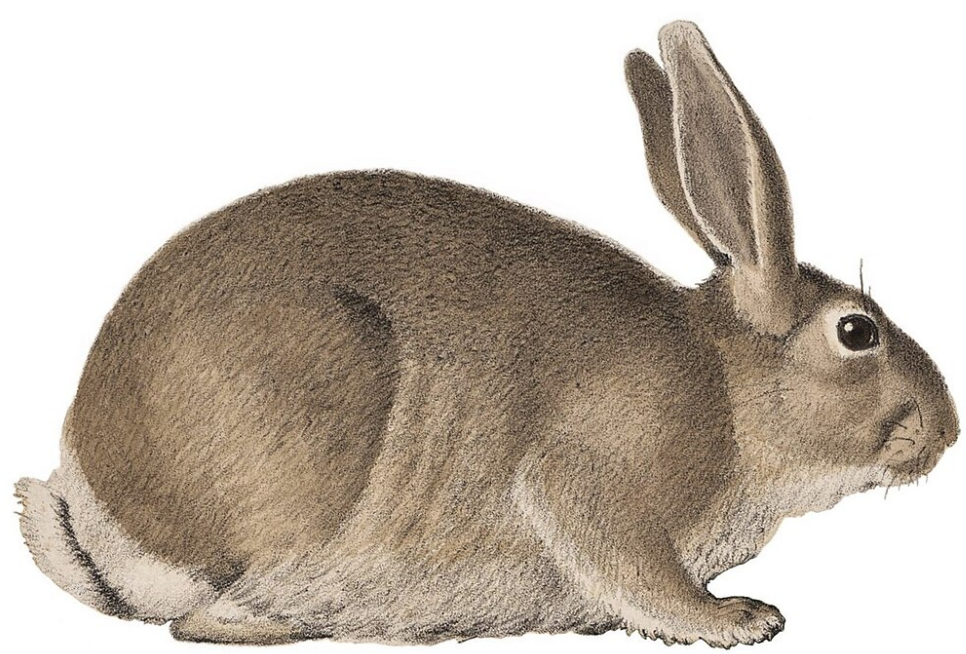
This is called the Fibonacci sequence, and it starts with

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$



Consider the sequence defined recursively by

$$f_0 = 1 \qquad f_1 = 1 \qquad f_{n+2} = f_n + f_{n+1}$$

Then

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} f_{n-1} \\ f_{n-2} \end{bmatrix}$$

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} f_1 \\ f_0 \end{bmatrix}$$

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Then

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

How do we compute

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \quad ?$$

Diagonalize!

Diagonalize $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\iff -\lambda(1-\lambda) - 1 = 0$$

$$\iff \lambda^2 - \lambda - 1 = 0$$

$$\iff \lambda = \tfrac{1 \pm \sqrt{1+4}}{2} = \tfrac{1 \pm \sqrt{5}}{2}$$

Eigenvalues: $\varphi_+ = \frac{1+\sqrt{5}}{2}$ and $\varphi_- = \frac{1-\sqrt{5}}{2}$

All eigenvalues have multiplicity 1, so our matrix is diagonalizable!

$$\text{Diagonalize } \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Eigenvectors:

$$\begin{bmatrix} \varphi_+ \\ 1 \end{bmatrix} \text{ is an eigenvector associated to } \varphi_+$$

$$\begin{bmatrix} \varphi_- \\ 1 \end{bmatrix} \text{ is an eigenvector associated to } \varphi_-$$

Diagonalization: take

$$P = \begin{bmatrix} \varphi_+ & \varphi_- \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} \varphi_+ & 0 \\ 0 & \varphi_- \end{bmatrix}$$

so

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} \varphi_+ & \varphi_- \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi_+^n & 0 \\ 0 & \varphi_-^n \end{bmatrix} \begin{bmatrix} \varphi_+ & \varphi_- \\ 1 & 1 \end{bmatrix}^{-1}$$

Can get a formula for f_n from here:

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Complex eigenvalues

not on the midterm

Some $n \times n$ matrices have less than n real eigenvalues or even none.

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Example: $$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

characteristic equation $\quad \lambda^2 + 1 = 0$

No real solutions $\implies$ no real eigenvalues

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But A does have two **complex** eigenvalues: i and $-i$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix} \longrightarrow$$

$\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is a complex eigenvector associated to i

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix} \longrightarrow$$

$\begin{bmatrix} 1 \\ i \end{bmatrix}$ is a complex eigenvector associated to $-i$.

Midterm 2
On Monday November 10

What is on the midterm?

- Vector spaces (chapter 4)
- Determinants (chapter 5)
- Eigenvalues and eigenvectors (6.1)

Midterm 2

on Monday in lecture

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Review
on vector spaces

TL;DR: can add vectors and multiply by scalars, with good properties.

A **vector space** is a nonempty set V, whose elements we call **vectors**, with rules for **addition** of vectors in V and **multiplication by scalars**, satisfying the following properties:

a) The addition $u + v$ of any vectors u and v in V is also a vector in V.
b) The multiplication cv of a vector v by a scalar c is a vector in V.
c) Commutativity: $u + v = v + u$ for all $u, v \in V$.
d) Associativity: $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$.
e) There is a **zero vector** in V, denoted 0, such that $0 + v = v + 0 = v$.
f) For every vector v there is a vector $-v$ such that $v + (-v) = 0$.
g) Distributivity: $c(u + v) = cu + cv$ and $(c + d)v = cv + dv$ for all $u, v \in V$ and all scalars c and d.
h) Associativity of multiplication by scalars: $c(dv) = (cd)v$.
i) $1v = v$.

Consequence: we also have

- $0v = \mathbf{0}$.
- $c\mathbf{0} = \mathbf{0}$.
- $-v = (-1)v$.

V vector space $\qquad$ W subset of V

W is a subspace of V if

1) The zero vector 0 is in W

2) W is <u>closed under addition</u>:

if u and v are in W, then $u + v$ is in W

3) W is <u>closed under multiplication by scalars</u>:

if v is in W, and c is any scalar, then cv is in W

A subspace is also a vector space!

Example: $\quad V = \mathbb{R}^4$

$$W = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \text{ any scalars} \right\}$$

is a subspace of $\mathbb{R}^4$

easier: all vectors in W can be written as

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{so} \quad W = \operatorname{span}\left(\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$$

is a subspace

linear combination of these vectors!

Example:

In $M_{2\times 2}$ (space of 2×2 matrices)

$$D = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \text{ any scalars} \right\}$$ diagonal matrices

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D = \operatorname{span}\left(\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \right)$$

so D is a subspace of $M_{2\times 2}$

A **basis**[*] for a vector space V is

a spanning set of linearly independent vectors.

The **dimension** of a vector space is

the number of vectors in a basis.

Notation: $\dim(V)$

* The plural of basis is bases.

$\dim(\mathbb{R}^n) = \ n$ Basis: $e_1, \ldots, e_n$

$\dim(M_{m\times n}) = \ mn$

all $m \times n$ matrices

$\dim(\{0\}) = \ 0$

$\dim(\mathbb{P}_n) = n+1$

polynomials $a_0 + a_1 t + \cdots + a_n t^n$

Think of $\mathbb{P}_n$ as $\mathbb{R}^{n+1}$

$$a_0 + a_1 t + \cdots + a_n t^n \longrightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$W = \operatorname{span}\left(\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\\0\end{bmatrix}, \begin{bmatrix}4\\5\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\7\end{bmatrix}\right\}\right)$$

What is the dimension of W?

W is the same as the column space of

$$\begin{bmatrix}1 & 2 & 4 & 0\\0 & 3 & 5 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 7\end{bmatrix}$$

$$W = \operatorname{span}(\{v_1, \ldots, v_n\}) \text{ subspace of } \mathbb{R}^m$$

How to find a basis for W?

Find a basis for the column space of

$$A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

<u>Step 1</u>: Find the RREF of A.

<u>Step 2</u>: Collect the pivot columns of A.

Warning: Make sure to use the pivot columns of A, not of its reduced echelon form!

$$\boxed{\dim(\operatorname{col}(A)) = \#\text{ of pivots in } A}$$

To find a basis for the null space of A:

Step 1: Find the general solution for $Ax = 0$.

Step 2: Write the solution in parametric vector form.

Use one vector for each free variable.

Step 2: The vectors we used form a basis for $\mathrm{Nul}(A)$.

$$\boxed{\dim(\mathrm{Nul}(A)) = \#\text{ free variables}}$$

$$\dim(\mathrm{col}(A)) = \#\text{ of pivots in } A = \mathrm{rank}(A)$$

$$\dim(\mathrm{Nul}(A)) = \#\text{ free variables } = \text{nullity of } A$$

Theorem (Rank–Nullity Theorem)**.**

For any $m \times n$ matrix A,

$$\mathrm{rank}(A) + \dim(\mathrm{Nul}(A)) = n.$$

Review on determinants

Theorem.

A square matrix

$$\det(A) \neq 0$$

$$\Updownarrow$$

A is invertible

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\boxed{A^{-1} = \frac{1}{\det A}\ \mathrm{adj}(A).}$$

Theorem.

If A is a triangular matrix, then

$$\det(A) = \text{product of the main diagonal entries}$$

$$A = \begin{bmatrix} -1 & -1 & 7 & 5 \\ 0 & -5 & 42 & 2 \\ 0 & 0 & 2 & 13 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

A upper triangular

$$\Downarrow$$

$$\det(A) = (-1) \cdot (-5) \cdot 2 \cdot 5 = 50$$

Effect of Elementary Row Operations on the determinant:

$$A \xrightarrow[\text{add a multiple of a row of } A \text{ to another row}]{\text{Replace}} B \qquad \det(B) = \det(A)$$

$$A \xrightarrow[\text{switch two rows of } A]{\text{Swap}} B \qquad \det(B) = -\det(A)$$

$$A \xrightarrow[\text{multiply a row of } A \text{ by } c]{\text{Rescale}} B \qquad \det(B) = c\det(A)$$

Theorem.

A square matrix

1. $\det(A^\mathsf{T}) = \det(A)$.
2. $\det(AB) = \det(A)\det(B)$.
3. $\det(cA) = c^n \det(A)$.

Warning! No formula for $\det(A + B)$

Practice problems

Today's poll code:

C7DJE9

Regular polls for today:

1 point total for answering at least 1 question today

Practice problems

up to 6 bonus points

up to 5 points for 5 correct answers

1 bonus point if all correct

Today's poll code:

C7DJE9

7×5 matrix M with 3 pivots

What is the dimension of the space of solutions to

$$Mx = 0$$

?

Today's poll code:

C7DJE9

Is it possible for a homogeneous system of 5 equations in 6 variables to have only the trivial solution?

A. Yes

B. No

Today's poll code:

C7DJE9

Is it possible for a homogeneous system

of 6 equations in 5 variables to have

only the trivial solution?

A. Yes

B. No

Today's poll code:

C7DJE9

Is it possible for the solutions to a homogeneous systems with 7 equations in 9 variables to be all contained in a plane?

A. Yes

B. No

Today's poll code:

C7DJE9

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The null space of A is a subspace of

A. $\mathbb{R}^5$

B. $\mathbb{R}^6$

C. $\mathbb{R}^3$

D. $M_{5\times 6}$

E. $M_{6\times 5}$

Today's poll code:

C7DJE9

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The dimension of the null space of A is

Today's poll code:

C7DJE9

Which of these is a basis for

the space of diagonal 2×2 matrices?

A. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

C. $M_{2\times 2}$

D. $\left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \text{ any real number} \right\}$

Today's poll code:

C7DJE9

A and B $n \times n$ matrices

$$\det(AB) = 17$$

A. A is invertible

B. A is not invertible

C. Not enough information

Today's poll code:

C7DJE9

A and B $n \times n$ matrices

$$\det(AB) = 0$$

A. A is invertible

B. A is not invertible

C. Not enough information

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Lab 2

(To be released)

due after Thanksgiving

Groups of 2 or 3 students

Solo teams not allowed this time

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not on the midterm

due after the midterm

Quiz on Friday

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