Linear Algebra

Math 314 Fall 2025

Lecture 9

Today's poll code:

DF6GU3

To do list:

- Webwork 2.6 due Friday
- Webwork 3.1 due next Tuesday

Office hours

Mondays 5–6 pm
Wednesdays 2–3 pm
in Avery 339 (Dr. Grifo)

Tuesdays 11–noon
Thursdays 1–2 pm
in Avery 337 (Kara)

Quiz on Friday

on Lectures 8—9

Midterm 1

in 2 weeks

Review:
linear transformations

Given a linear transformation $\mathrm{T}\colon \mathbb{R}^n \to \mathbb{R}^m$

its **standard matrix** is

$$A = \begin{bmatrix} \mathrm{T}(e_1) & \cdots & \mathrm{T}(e_n) \end{bmatrix}$$

so

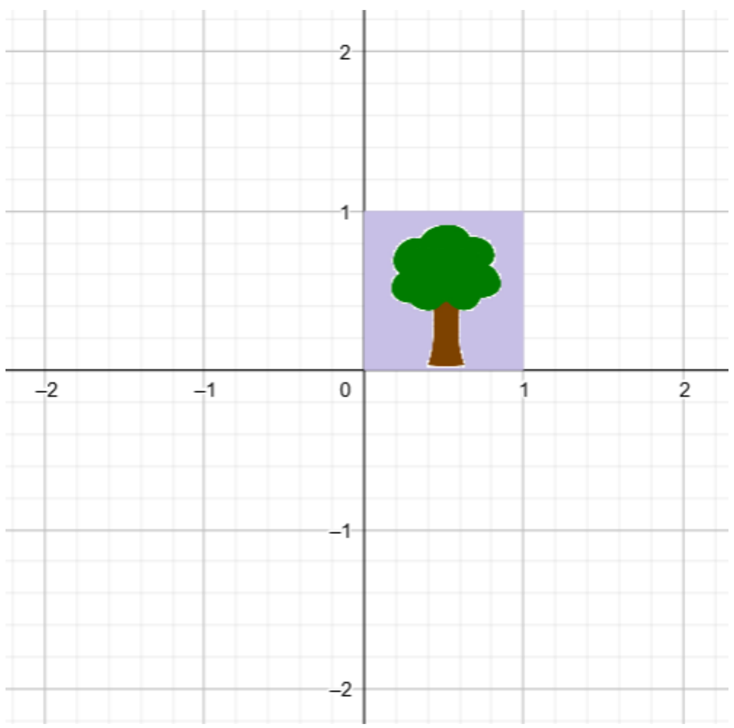
$T(x) = Ax$ for all vectors x

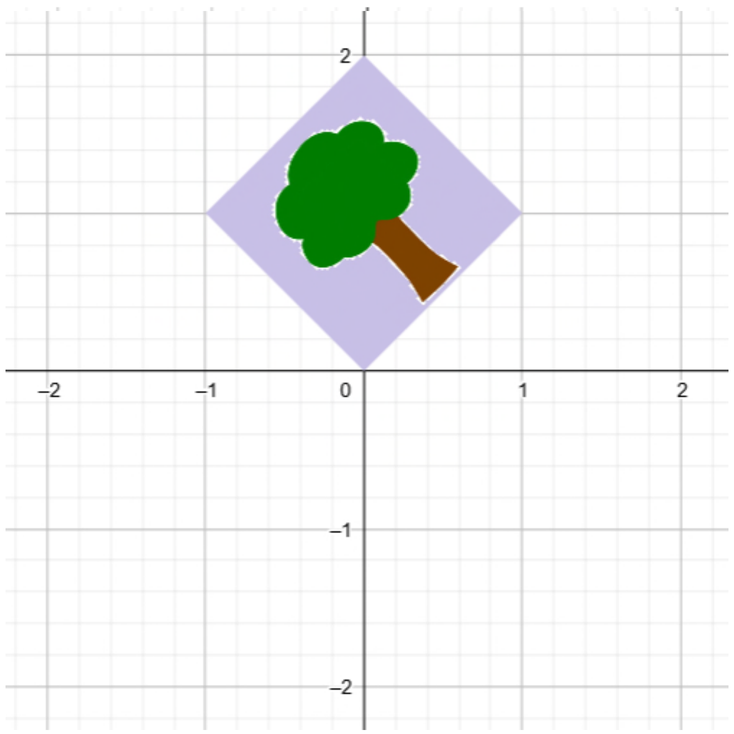
ith standard vector in $\mathbb{R}^n$

$$\mathbf{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \longrightarrow \text{position } i$$

Review: example from last Wednesday's class

Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that does the following:





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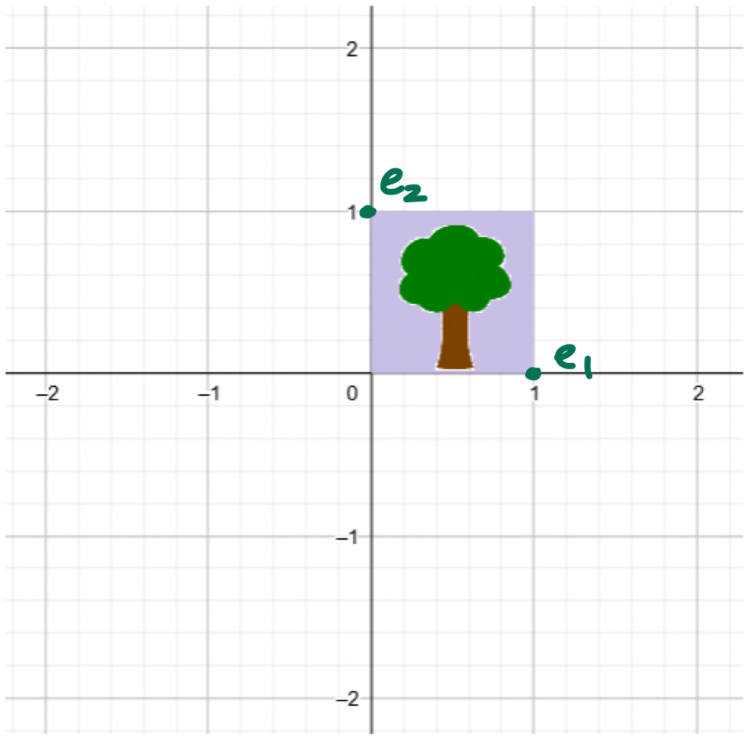

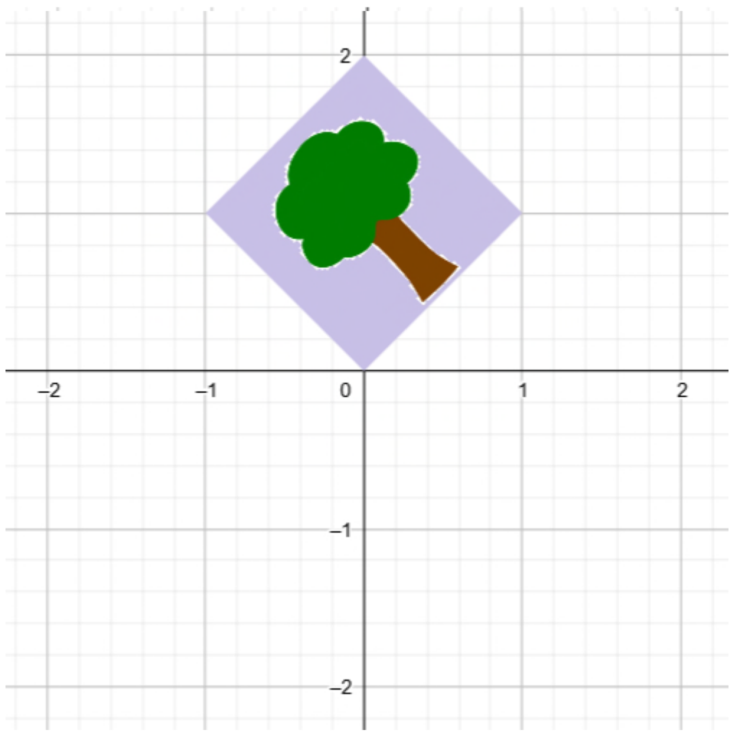

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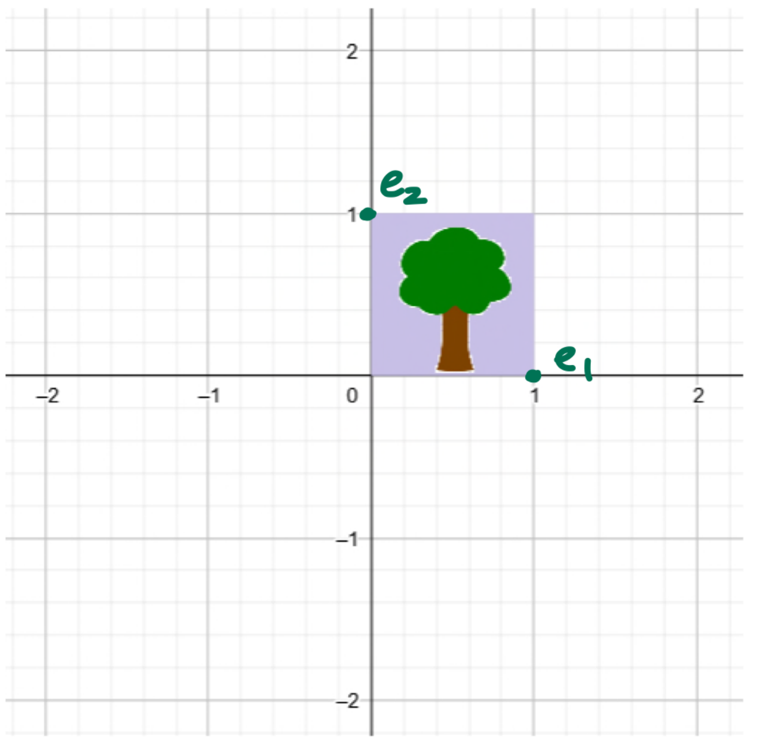
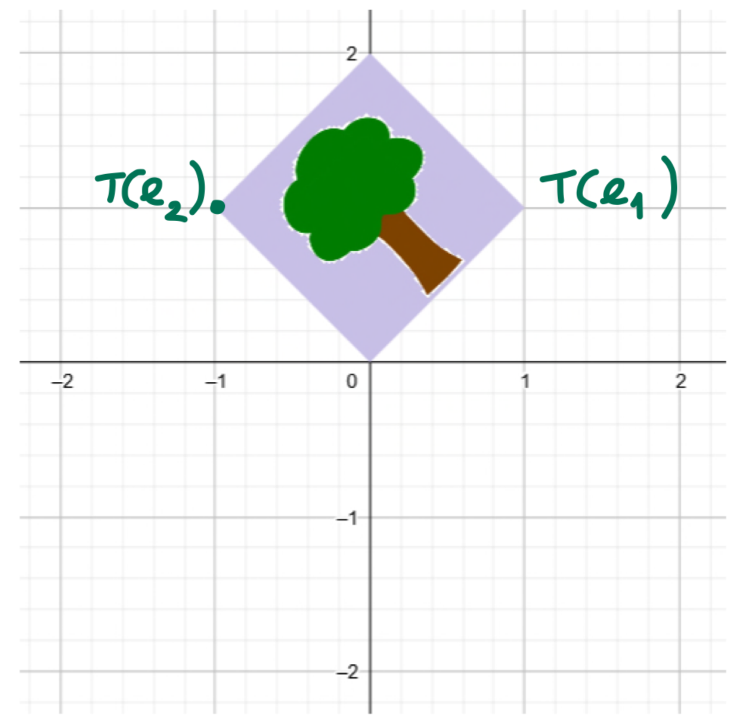
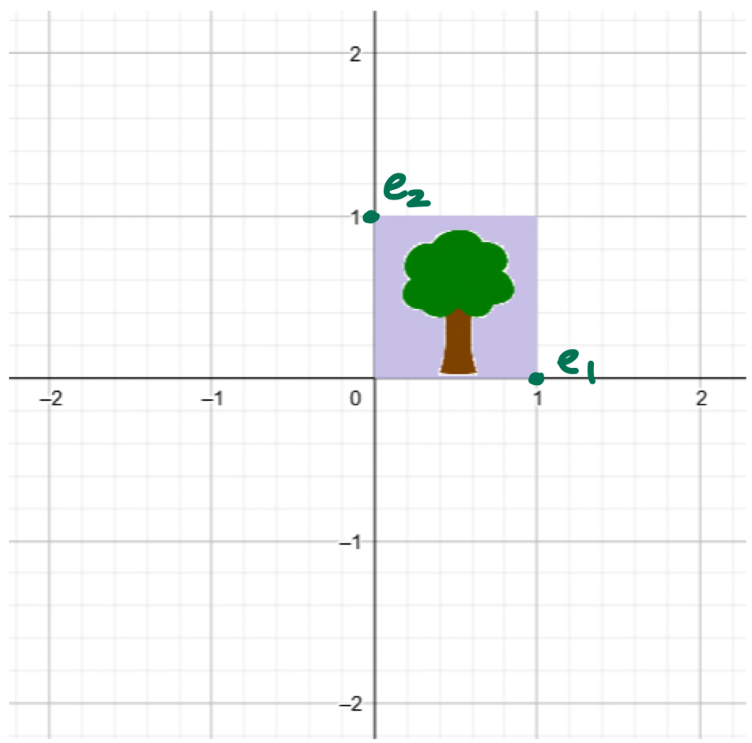
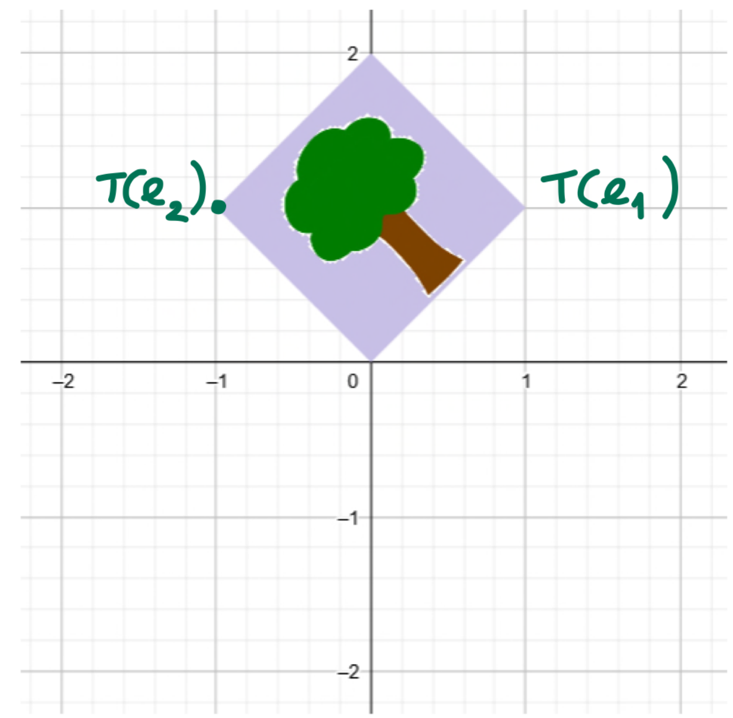




Review: example from last Wednesday's class

Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that does the following:

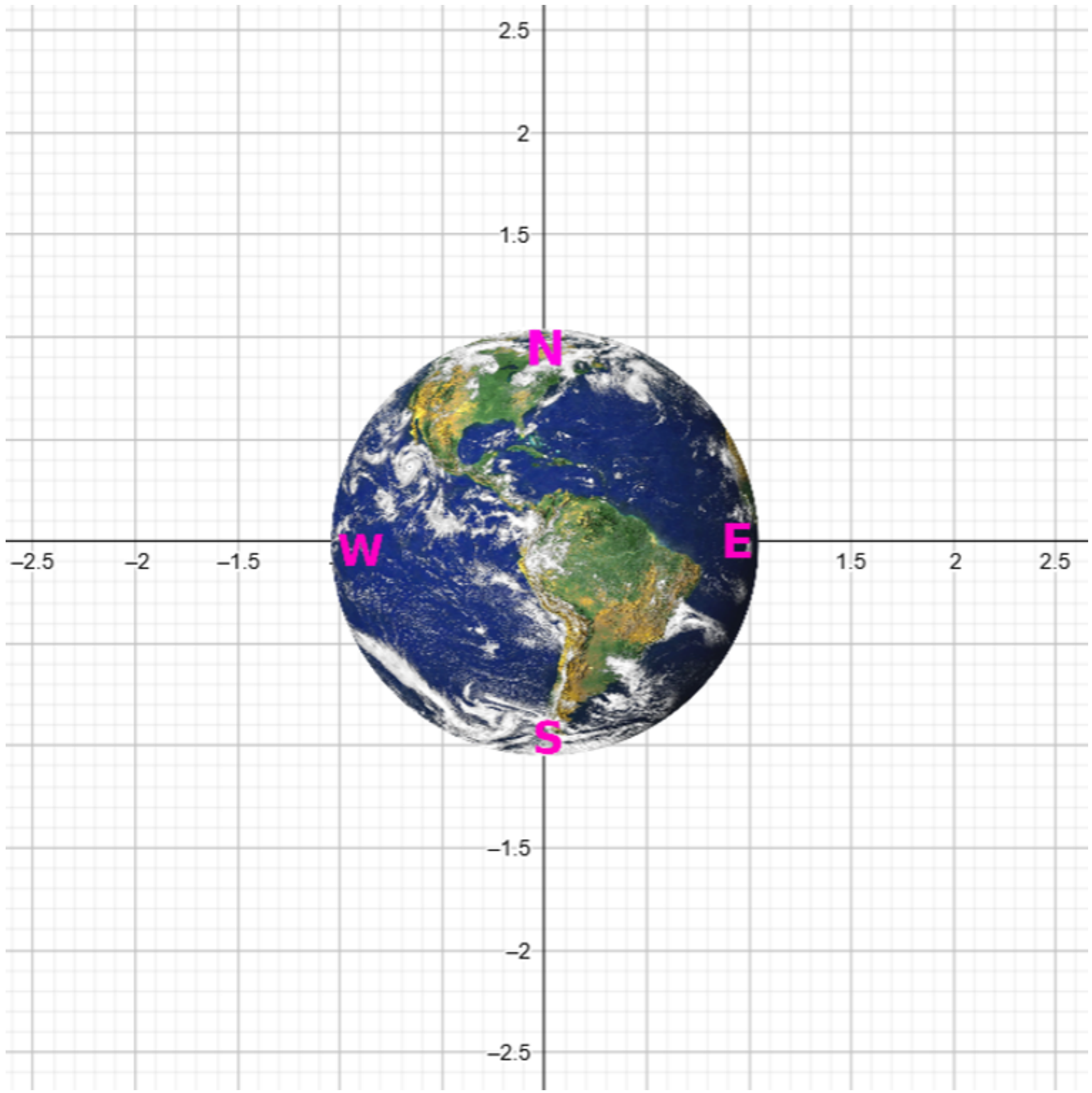




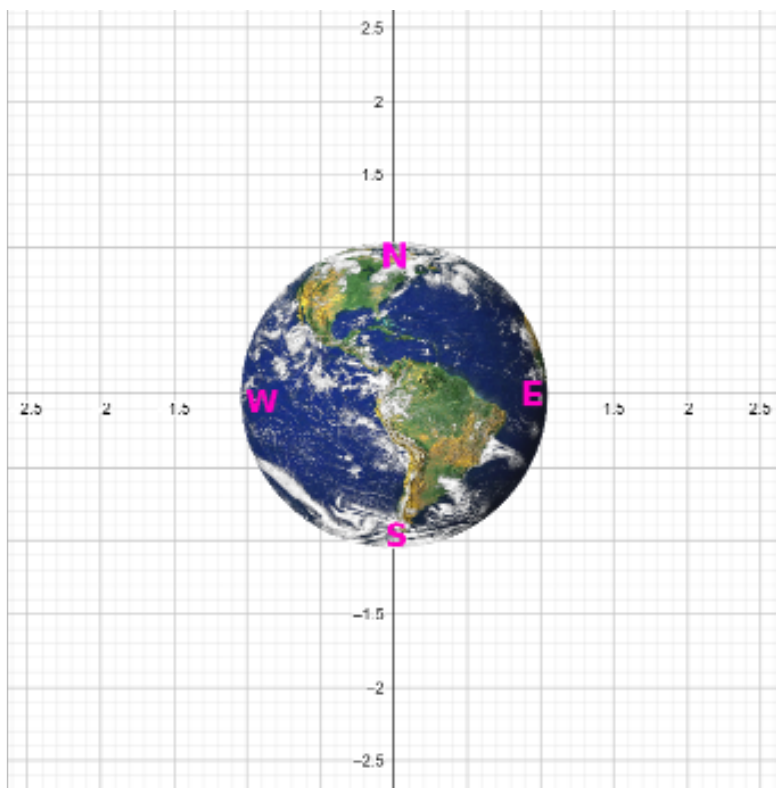
Note how we explicitly marked the images of $\mathbf{e}_1$ and $\mathbf{e}_2$. This is sufficient for us to find the standard matrix, and thus to completely describe the linear transformation: the matrix is

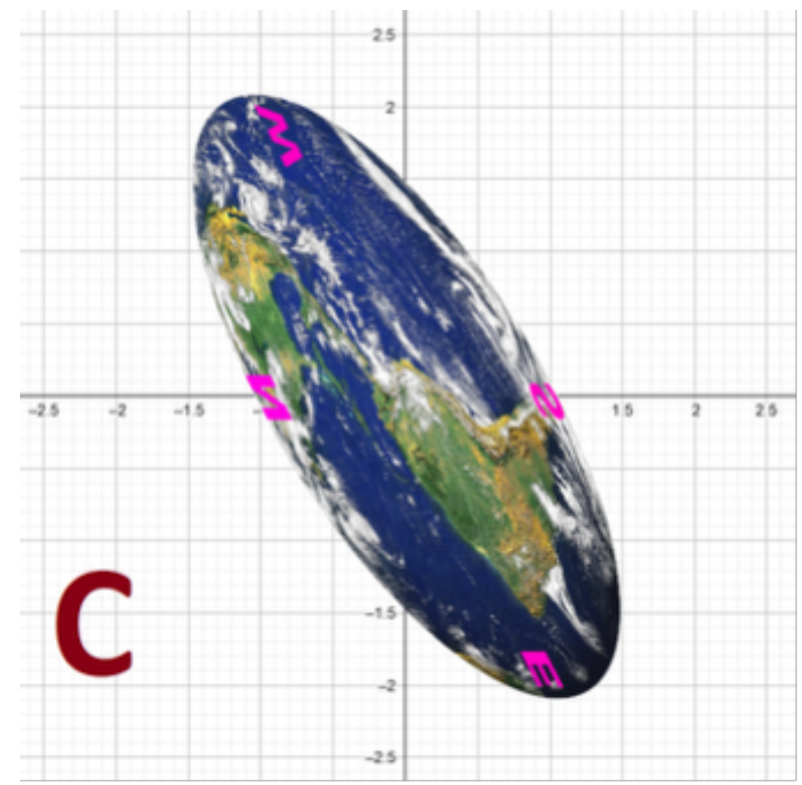
$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Let’s look at an example involving the earth:



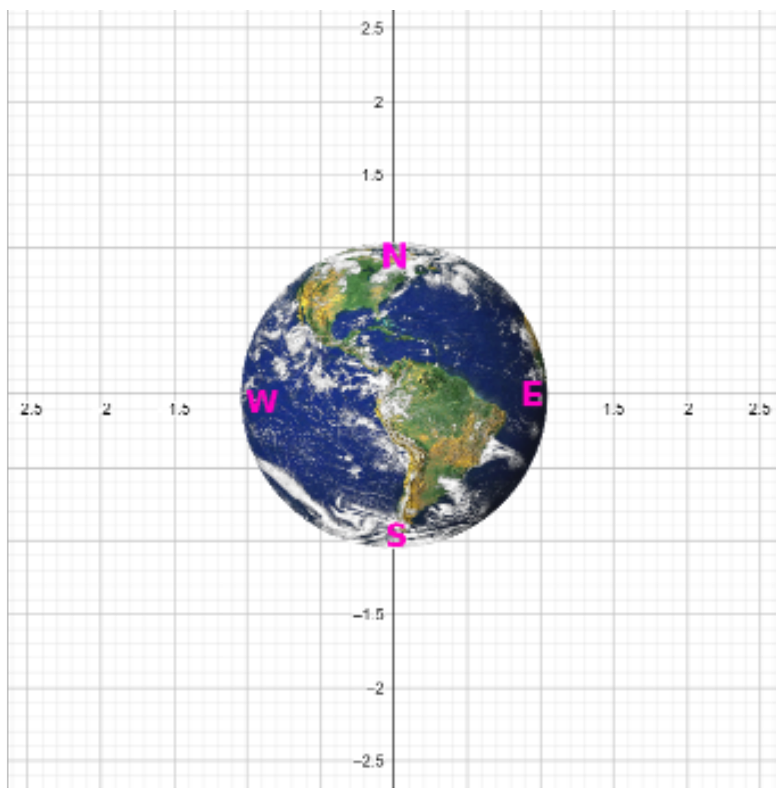
Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that does the following:

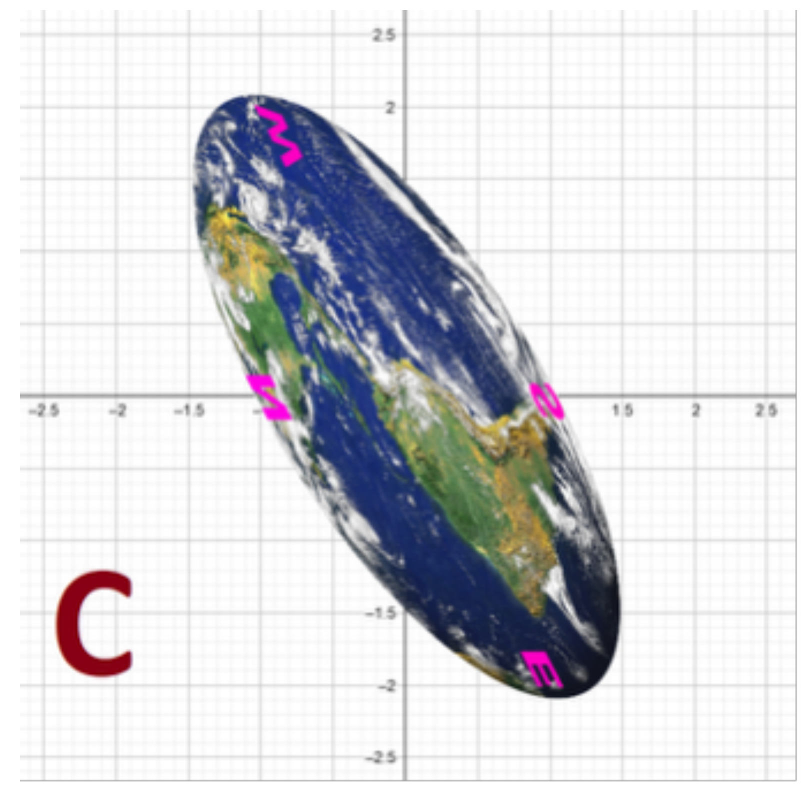




The standard matrix for T is

Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that does the following:





The standard matrix for T is $\begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$

Midterm 1

Midterm 1

October 6

Material for Midterm 1: Chapters 1, 2, 3

Study materials:

- Class notes (see canvas)
- Textbook
- Slides from lecture
- Poll questions
- Quizzes
- Webwork questions
- Study guide (quick summary)

Office hours

Mondays 5–6 pm and Wednesdays 2–3 pm
in Avery 339 (Dr. Grifo)

Tuesdays 11–noon and Thursdays 1–2 pm
in Avery 337 (Kara)

Special office hours next week

Monday 5–6 pm and Wednesday 2–3 pm
Thursday TBA
Friday 2:30 to 3:30 pm
in Avery 339 (Dr. Grifo)

Tuesday 11–noon and Thursday 1–2 pm
in Avery 337 (Kara)

Chapter 3

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad (i,j)\text{th entry of } A = a_{ij}$$

multiplying a matrix by a scalar: $\quad cA = [ca_{ij}]$

$$3 \cdot \begin{bmatrix} -1 & 1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -3 & 15 \end{bmatrix}$$

always defined

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad (i,j)\text{th entry of } A = a_{ij}$$

The sum $A + B$ is defined if

A and B have the same size $\left(\begin{array}{c} \#\text{ rows of } A = \#\text{ rows of } B \\ \#\text{ columns of } A = \#\text{ columns of } B \end{array} \right)$

sum of matrices: $\qquad A + B := [a_{ij} + b_{ij}]$

$$\begin{bmatrix} 3 & -1 \\ 2 & -11 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3+1 & -1+3 \\ 2+4 & -11+5 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 6 & -6 \end{bmatrix}$$

product of matrices: The product AB is defined if

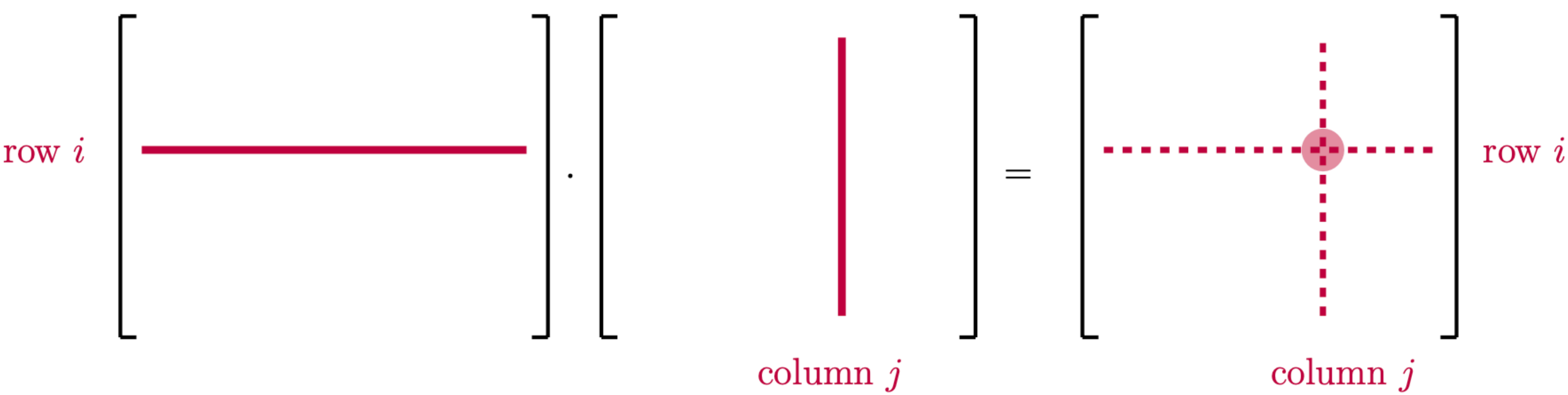
$$\#\text{ of columns of } A = \#\text{ of rows of } B$$

$$\#\text{ of rows of } AB = \#\text{ of rows of } A$$

$$\#\text{ of columns of } AB = \#\text{ of columns of } B$$

In AB:

$$(i,j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$



A $m \times n$ matrix

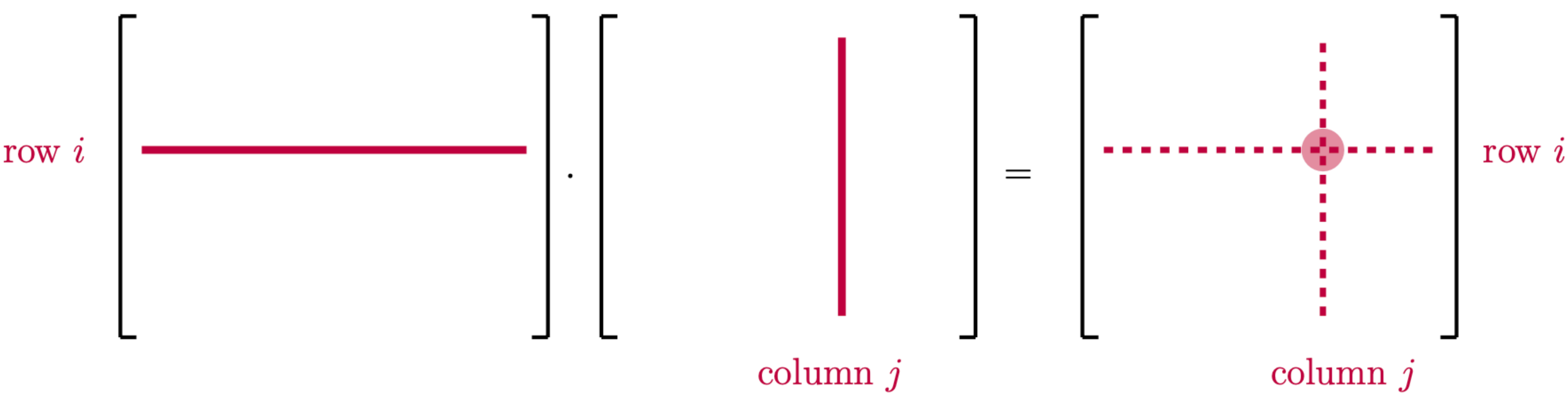
B $n \times p$ matrix

$\longrightarrow$ AB $m \times p$ matrix

Informally: $(m \times n)\ (n \times p) = m \times p$

In AB:

$$(i, j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$



$$\begin{array}{c} A\ \ m \times n \text{ matrix} \\ B\ \ n \times p \text{ matrix} \end{array} \longrightarrow AB\ \ m \times p \text{ matrix}$$

$$\text{Informally: } (m \times \cancel{n)\,(n} \times p) = m \times p$$

In AB:

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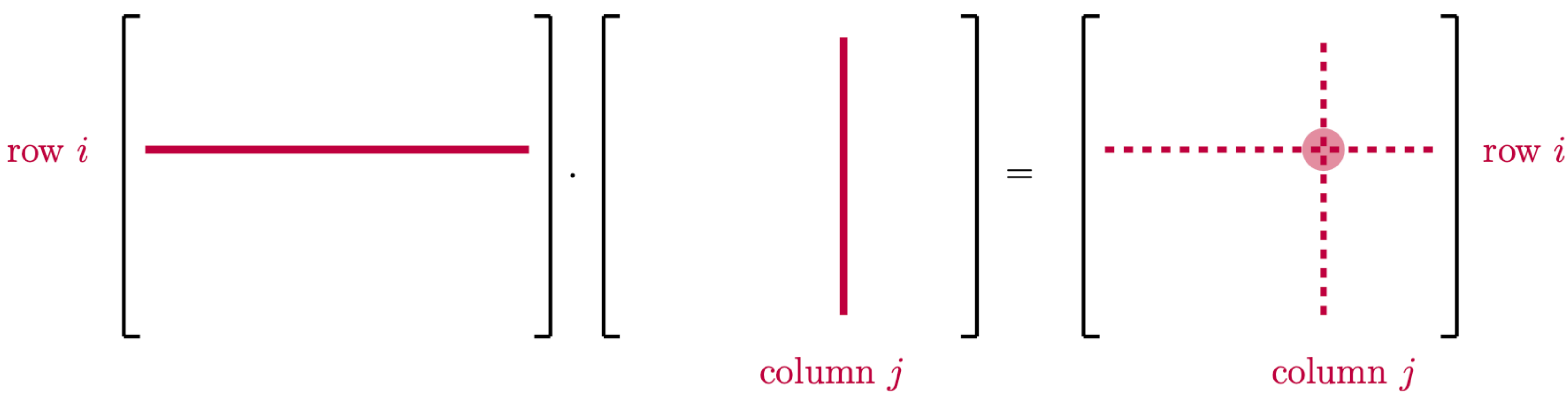

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad (i,j)\text{th entry of } A = a_{ij}$$

product of matrices: The product AB is defined if

$$\#\text{ of columns of } A = \#\text{ of rows of } B$$

$$AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ell} \end{bmatrix}$$

$$(i,j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

product of matrices: The product AB is defined if

$$\# \text{ of columns of } A = \# \text{ of rows of } B$$

$$(i,j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

ith row of A and jth column of B

eg: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{n\ell} \end{bmatrix}$

$$(1,1)\text{th entry} = a_{11}b_{11} + a_{i2}b_{21} + \cdots + a_{1n}b_{n1}$$

product of matrices: The product AB is defined if

$$\#\text{ of columns of } A = \#\text{ of rows of } B$$

$$(i,j)\text{th entry} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

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$$(m,2)\text{th entry} = a_{m1}b_{12} + a_{m2}b_{22} + \cdots + a_{mn}b_{n2}$$

Example:

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + 3 \times 1 & 2 \times 3 + 3 \times (-2) & 2 \times 6 + 3 \times 3 \\ 1 \times 4 - 5 \times 1 & 1 \times 3 + (-5) \times (-2) & 1 \times 6 + (-5) \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Example:

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Example:

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$

is undefined!

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$

3 columns $\neq$ 2 rows

Example:

$$\cancel{\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}}$$

is undefined!

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$$

3 columns $\neq$ 2 rows

Theorem. Let A, B, C be matrices,

Assume that the products AB and BC make sense.

1. <u>Associativity:</u> $(AB)C = A(BC)$.
2. <u>Left distributivity:</u> $A(B + C) = AB + AC$.
3. <u>Right distributivity:</u> $(B + C)A = BA + CA$.
4. For any scalar α, $\alpha(AB) = (\alpha A)B = A(\alpha B)$.
5. Let I_m denote the $m \times m$ identity matrix.

   Then $I_m A = A = AI_n$.

Warning: the order of the matrices in a product matters!

For example:

$$A = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 \cdot 2 + 1 \cdot 4 & 5 \cdot 0 + 1 \cdot 3 \\ (-1) \cdot 2 + 3 \cdot 4 & (-1) \cdot 0 + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ 10 & 9 \end{bmatrix}$$

$$\neq$$

$$BA = \begin{bmatrix} 2 \cdot 5 + 0 \cdot (-1) & 2 \cdot 1 + 0 \cdot 3 \\ 4 \cdot 5 + 3 \cdot (-1) & 4 \cdot 1 + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 17 & 13 \end{bmatrix}$$

The zero $m \times n$ matrix is

the $m \times n$ matrix whose entries are all 0.

We denote it simply by 0, if the size is clear from context.

Warning! Cancellation fails.

$AB = AC$ does <u>not</u> imply that $B = C$.

For example:

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix} \neq 0 \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

$$AB = \begin{bmatrix} 1 \times 0 + 0 \times 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Warning! Cancellation fails.

$AB = AC$ does <u>not</u> imply that $B = C$.

For example:

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \neq \quad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 \times 2 + 1 \times 0 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 1 \times 1 \end{bmatrix} = AC$$

The zero $m \times n$ matrix is

the $m \times n$ matrix whose entries are all 0.

We denote it simply by 0, if the size is clear from context.

Warning! Cancellation fails.

$AB = 0$ does <u>not</u> imply that $A = 0$ or $B = 0$

For example:

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix} \neq 0 \qquad\qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

$$AB = \begin{bmatrix} 1 \times 0 + 0 \times 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

A square matrix

Powers of A:

$$A^0 = I_n$$

$$A^1 = A \qquad A^2 = AA \qquad A^3 = AAA$$

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}}$$

A $m \times n$ matrix

The **transpose** of A is the $n \times m$ matrix A^{T}

whose rows are the columns of A

Example:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} \longrightarrow A^{\mathsf{T}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}$$

Theorem. Let A and B be matrices.

Assume A and B have appropriate sizes so that the following make sense.

1. $(A^\top)^\top = A$.
2. $(A+B)^\top = A^\top + B^\top$.
3. $(\alpha A)^\top = \alpha A^\top$ for any scalar α.
4. $(AB)^\top = B^\top A^\top$.

A $m \times n$ matrix

The **transpose** of A is the $n \times m$ matrix A^T

whose rows are the columns of A

Example:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} \qquad A^\mathsf{T} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix} \qquad (A^\mathsf{T})^\mathsf{T} = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

Today’s poll code:

DF6GU3

Today's poll code:

DF6GU3

A is a 2×3 matrix

B is a 3×5 matrix

AB is...

A. absurd

B. 2×5

C. 5×3

D. 3×3

E. 5×2

Today's poll code:

DF6GU3

A is a 2×3 matrix

B is a 3×5 matrix

BA is...

A. absurd

B. 2×5

C. 5×3

D. 3×3

E. 5×2

Today's poll code:

DF6GU3

A is a 2×3 matrix

B is a 3×5 matrix

$A + B$ is...

A. absurd

B. 2×5

C. 5×3

D. 3×3

E. 5×2

Today's poll code:

DF6GU3

If AB is a 3x4 matrix, how many rows rows A have?

A. 3

B. 4

C. unknown

A $n \times n$ (square) matrix

Powers of A:

$$A^2 = AA \qquad\qquad A^3 = AAA$$

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}}$$

$$A^1 = A \qquad\qquad A^0 = I_n$$

A $m \times n$ matrix

The **transpose** of A is

the $n \times m$ matrix whose rows are the columns of A

$$A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

column 1 column n

$$A^{\mathsf{T}} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \begin{array}{l} \longrightarrow \text{row } 1 \\ \\ \longrightarrow \text{row } n \end{array}$$

Example: $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$ $\quad A^{\mathsf{T}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}$

Theorem. Let A and B be matrices whose sizes make the following make sense.

1. $(A^\top)^\top = A$.
2. $(A+B)^\top = A^\top + B^\top$.
3. $(\alpha A)^\top = \alpha A^\top$ for any scalar α.
4. $(AB)^\top = B^\top A^\top$.

A $m \times n$ matrix

The **transpose** of A is

the $n \times m$ matrix whose rows are the columns of A

A square matrix A is **symmetric** if $A = A^{\mathsf{T}}$.

In practice: if $A = [a_{ij}]$

$$a_{ij} = a_{ji} \qquad \text{for all } i,\ j$$

This imposes no conditions on the diagonal entries a_{ii}

A square matrix A is **symmetric** if $A = A^{\mathsf{T}}$.

Example: if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $A^{\mathsf{T}} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

If A is symmetric,

A square matrix A is **symmetric** if $A = A^{\mathsf{T}}$.

Example: if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $A^{\mathsf{T}} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

If A is symmetric,

we must have $a_{12} = a_{21}$

a_{11} can be anything

a_{22} can be anything

$$\longrightarrow \quad A = \begin{bmatrix} x & z \\ z & y \end{bmatrix}$$

To do list:

- Webwork 2.6 due Friday
- WebworK 3.1 due next Tuesday

On Friday:

Quiz 4

at the beginning

of the recitation

on Lectures 8—9

Midterm 1

On October 6

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