Commutative Algebra II

Math 906 Spring 2026

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Warning!

Proceed with caution. These notes are under construction and are 100% guaranteed to contain typos. If you find any typos or errors, I will be most grateful to you for letting me know.

A note on external references

These notes make many direct references to my Commutative Algebra I and Homological Algebra notes, the content of which is assumed to be known to the reader.

Acknowledgements

These notes take much inspiration from other sources, including Tom Marley's notes from Spring 2024 Math 906, and especially from Bruns and Herzog's book *Cohen-Macaulay rings* [BH93].

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Contents

Chapter 0

Setup

Throughout, all rings are commutative with identity 1, all modules are unital, and all ring homomorphisms send 1 to 1. Moreover, all rings will be assumed to be noetherian.

When k is a field, the polynomial ring $R = k[x_1, \dots, x_n]$ can be given an $\mathbb{N}$-grading by setting $\deg(x_i) = d_i$ for some $d_i \in \mathbb{N}$. The most common $\mathbb{N}$-grading, also known as the **standard grading**, is the one where we declare $\deg(x_i) = 1$ for all i. Once we declare the degrees of the variables, we can extend that grading to all monomials as follows:

$$\deg\left(x_1^{a_1} \cdots x_n^{a_n}\right) = a_1 d_1 + \cdots + a_n d_n.$$

In this case, a **homogeneous element** in R is any k-linear combination of monomials of the same degree. We write R_i for the set of all homogeneous elements of degree i, which is an abelian group under addition, and note that

$$R = \bigoplus_i R_i.$$

Note also that $R_i R_j \subseteq R_{i+j}$ for all i and j. So when $R = k[x_1, \dots, x_n]$ is standard graded,

$$R_i = \bigoplus_{a_1 + \cdots + a_n = i} x_1^{a_1} \cdots x_n^{a_n}.$$

More generally, a **graded ring** is any ring that can be decomposed in pieces of this form, meaning that

$$R = \bigoplus_i R_i \qquad \text{and} \qquad R_i R_j \subseteq R_{i+j}.$$

The homogeneous elements of degree i are the elements in R_i. A graded R-module is an R-module M such that

$$M = \bigoplus_i M_i \qquad \text{and} \qquad R_i M_j \subseteq M_{i+j}.$$

A homomorphism of graded R-modules $\varphi \colon M \to N$ satisfying $\varphi(M_i) \subseteq N_{i+d}$ for all i is a **graded map** of degree d. A map of degree 0 is sometimes called **degree preserving**. Any graded map can be thought of as a map of degree 0 by shifting degrees. We write $M(-d)$ for the graded R-module that has M as the underlying R-module, but where the graded structure is given by taking $M(-d)_i = M_{i-d}$.

Note that 0 can be thought of as a homogeneous element of any degree; one sometimes declares $\deg(0) = -\infty$. An ideal I in R is a **homogeneous ideal** if it can be generated by homogeneous elements; one can show that this is equivalent to the condition

$$I = \bigoplus_i (I \cap R_i).$$

Finally, whenever I itself is homogeneous, the grading on R passes onto R/I, with

$$(R/I)_i = R_i/I_i.$$

We will be concerned with finitely generated $\mathbb{N}$-graded k-algebras R with $R_0 = k$, which are of the form $R = k[x_1, \dots, x_n]/I$ for some homogeneous ideal I. One nice feature of such rings is that while there might be many maximal ideals, there is only one *homogeneous* maximal ideal, which is given by

$$R_+ = \bigoplus_{i>0} R_i.$$

In many ways, the behavior of such a graded ring and its unique homogeneous maximal ideal R_+ is an analogue to the behavior of a local ring R and its unique maximal ideal ideal $\mathfrak{m}$, though one always needs to provide a separate proof for the graded and local versions.

We will summarize this as follows:

Setting 1 (Local/graded setting)**.** We will say that we are in the local/graded setting, or more precisely that $(R, \mathfrak{m})$ is local/graded to indicate one of the following holds:

- Local setting: R is a noetherian local ring with $\mathfrak{m}$ its unique maximal ideal.
- Graded setting: $R = k[x_1, \dots, x_d]/I$, where k is a field, $k[x_1, \dots, x_d]$ has the standard grading, and I is a homogeneous ideal, and $\mathfrak{m} = R_+$ is the unique homogeneous maximal ideal of R.

In the graded setting, unless otherwise stated, any statement about modules will refer to *graded* modules, and all module homomorphisms will be assumed to be graded of degree 0.

Chapter 1

Free resolutions and the Koszul complex

1.1 Minimal free resolutions

Given an R-module M, how do we describe it? We need to know a set of generators and the relations among those generators. Going further and asking for relations among the relations (treating the relations as generators for the module of relations), and relations among the relations among the relations, and so on, we construct a free resolution for M. Free resolutions play a key role in many important constructions, and encode a lot of interesting information about our module. For example, if the module came from some geometric setting, geometric information about the module gets reflected in the free resolution.

Definition 1.1. Let M be a module over a ring R. A **projective resolution** of M is a complex of projective R-modules F_i

$$F = \cdots \longrightarrow \underset{\mathrm{n}}{F_n} \longrightarrow \cdots \longrightarrow \underset{1}{F_1} \longrightarrow \underset{0}{F_0} \longrightarrow 0$$

such that $\mathrm{H}_i(F) = 0$ for all $i \neq 0$, together with an isomorphism $\mathrm{H}_0(F) \cong M$. When all the F_i are free, we say F is a **free resolution** for M. We will abuse notation and carelessly identify F with the corresponding augmented resolution, which is the exact sequence

$$F = \cdots \longrightarrow \underset{\mathrm{n}}{F_n} \longrightarrow \cdots \longrightarrow \underset{1}{F_1} \longrightarrow \underset{0}{F_0} \longrightarrow M \longrightarrow 0.$$

Remark 1.2. In the local setting, any finitely generated projective module must be free, by Theorem 4.55 from Homological Algebra. In fact, Kaplansky [Kap58] showed that *any* projective module over a noetherian local ring must be free. In the graded setting, one can also show (exercise!) that every bounded below graded projective must be free. Therefore, any projective resolution in the local/graded setting is in fact a free resolution.

Thus we will focus on free resolutions. We can think of a free resolution of M as an approximation of M by free modules. Since every module is a quotient of a free module, every module has a free resolution. Let us recall the construction we saw in Homological Algebra:

Construction 1.3 (Minimal free resolution)**.** Let M be a finitely generated module over R, where R is either local or graded as in our general setup. If M has β_0 many minimal generators, then we can write a surjective R-module homomorphism from R^{β_0} to M, say

$$R^{\beta_0} \xrightarrow{\pi_0} M$$

If π_0 is an isomorphism, then $M \cong R^{\beta_0}$ is a free module of rank β_0. Otherwise, π_0 has a nonzero kernel $\ker(\pi_0)$, which must also be a finitely generated module since R is noetherian. If $\ker(\pi_0)$ is minimally generated by β_1 elements, then we repeat this process and construct a surjective R-module map from R^{β_1} to $\ker(\pi_0)$, and compose it with the inclusion of $\ker(\pi_0)$ into R^{β_0}:

$$\begin{tikzcd} R^{\beta_1} \arrow[rr, dashed] \arrow[dr, two heads] & & R^{\beta_0} \arrow[r, two heads, "\pi_0"] & M \\ & \ker(\pi_0) \arrow[ur, hook] & & \end{tikzcd}$$

The elements in $\ker(\pi_0)$ are the **relations** on our chosen minimal generators for M: if M is generated by $m_1, \ldots, m_{\beta_0}$, we can take π_0 to be the map sending each canonical basis element e_i in R^{β_0} to m_i, and each $(r_1, \ldots, r_{\beta_0}) \in \ker(\pi_0)$ corresponds to a **relation** among the m_i, meaning

$$r_1 m_1 + \cdots + r_{\beta_0} m_{\beta_0} = 0.$$

Such relations are called **syzygies**[1] of M and the module $\ker(\pi_0)$ is the first syzygy module of M, which we will denote by $\Omega_1(M)$.

Continuing this process, we construct a free resolution for M:

$$\cdots \longrightarrow F_n \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \longrightarrow 0.$$

In the local/graded setting and when M is a finitely generated (graded) module, we can choose F_i at each step to have the minimal number of generators; in that case, we say that F is a **minimal free resolution** for M.

Back in Homological Algebra, we then showed the following remarkable facts:

- Every free resolution of M has a minimal free resolution of M as a direct summand.
- Any two minimal free resolutions of M are isomorphic complexes, thus we can talk about *the* minimal free resolution of M.
- As a consequence of the previous facts, the minimal free resolution of M must have the shortest length of any resolution for M, and M has a finite resolution if and only if the minimal free resolution of M is finite.
- A free resolution F of M with differential ∂ is minimal if and only if $\partial(F) \subseteq \mathfrak{m}F$. Thus if we fix bases for all the free modules F_i, the resolution is minimal if and only if all the entries in the matrices representing ∂ have all entries in $\mathfrak{m}$.

[1] Fun fact: in astronomy, a syzygy is an alignment of three or more celestial objects.

Definition 1.4. In general, any complex (F, ∂) satisfying $\partial(F) \subseteq \mathfrak{m}F$ is called a **minimal complex**.

Definition 1.5. Let M be an R-module. A finite projective resolution

$$F = \cdots \longrightarrow 0 \longrightarrow F_c \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

has length c if $F_c \neq 0$ and $F_i = 0$ for all $i \geqslant c$. A resolution F has infinite length if $F_i \neq 0$ for all $i \geqslant 0$. The **projective dimension** of M is

$$\operatorname{pdim}_R(M) := \inf\{c \mid M \text{ has a projective resolution of length } c\}.$$

Remark 1.6. As we noted in Remark 1.2, in the local/graded setting, any projective resolution is in fact a free resolution. Thus

$$\operatorname{pdim}_R(M) = \inf\{c \mid M \text{ has a free resolution of length } c\}.$$

Note that if a module M has a finite (minimal) free resolution, then the projective dimension of M is simply the length of the minimal free resolution for M. If the minimal free resolution of M is infinite, then $\operatorname{pdim} M = \infty$.

Remark 1.7. Note that $\operatorname{pdim}(M) = 0$ if and only if M is projective (and in the local/graded setting, free).

Definition 1.8. Consider a minimal free resolution F of M, and consider the notation in Construction 1.3. The ***i*th syzygy module of** M, denoted $\Omega_i(M)$, is defined to be the image of π_i or equivalently the kernel of π_{i-1}.

Note that $\Omega_i(M)$ is defined only up to isomorphism.

Remark 1.9. There are two conventions for what module $\Omega_i(M)$ corresponds to: one could instead take $\Omega_i(M) = \ker(\pi_i)$. One advantage of setting $\Omega_i(M) = \ker(\pi_i)$ is that this makes $\Omega_i(M)$ a submodule of F_i. But one advantage of the convention $\Omega_i(M) = \ker(\pi_{i-1})$ we have chosen is that the module of first syzygies of M, $\Omega_1(M)$, corresponds to the relations among the generators of M, which are the first set of relations we want to consider.

We recommend always checking the definition used in a particular reference.

Exercise 1. Show that for all R-modules M and all $i \geqslant 1$, if F if the minimal free resolution for M, then there is a natural short exact sequence

$$0 \longrightarrow \Omega_{i+1}(M) \longrightarrow F_i \longrightarrow \Omega_i(M) \longrightarrow 0.$$

Remark 1.10. Suppose that at some point when building a resolution following the procedure we described in Construction 1.3, we obtain an injective map of free modules. Then its kernel is trivial, so we obtain a finite free resolution.

The projective dimension of a finitely generated module can be infinite.

Example 1.11. Let k be a field and $R = k[x]/(x^2)$, which is a local ring with maximal ideal $\mathfrak{m} = (x)$. The residue field $k = R/\mathfrak{m}$ has infinite projective dimension: indeed, its minimal free resolution is

$$\cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow k \longrightarrow 0.$$

So even cyclic modules can have infinite projective dimension.

These invariants can give us some information about Ext and Tor, and vice-versa.

Remark 1.12. If $\operatorname{pdim}_R(M) = n$ is finite, then $\operatorname{Tor}_i^R(M, -) = 0$ and $\operatorname{Ext}_R^i(M, -) = 0$ for all $i > n$, since for all R-modules N, we can compute $\operatorname{Tor}_i^R(M, N)$ and $\operatorname{Ext}_R^i(M, N)$ using a free resolution for M of length n.

The ranks of the free modules in the minimal free resolution are particularly important invariants of M:

Definition 1.13 (Betti numbers)**.** Let F be the minimal free resolution of M. The i**th Betti number** of M is

$$\beta_i(M) = \operatorname{rank}(F_i).$$

Remark 1.14. Note that $\beta_0(M) = \mu(M) = \dim_k(M/\mathfrak{m}M)$.

Example 1.15. Let $R = k[\![x, y]\!]$ and $M = R/(x^2, xy)$. Let us write a minimal free resolution for M. First, we note that M is cyclic, so we start with

$$R \longrightarrow R/(x^2, xy) \longrightarrow 0.$$

In this case, the relations on the unique generator 1 in degree 0 are $x^2 \cdot 1 = 0$ and $xy \cdot 1 = 0$, so we proceed with

$$R^2 \xrightarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} R \longrightarrow R/(x^2, xy) \longrightarrow 0.$$

Now we need to find all relations among x^2 and xy, meaning all choices of a and b in R such that $ax^2 + bxy = 0$. We note that x is a regular element on R, so $ax^2 = -bxy \implies ax = -by$. But x and y are nonassociate irreducibles, so $a \in (y)$ and $b \in (x)$. Thus

$$\underline{y} \cdot x^2 + \underline{-x} \cdot xy = 0$$

is one of the relations we are looking for, and *all* other such relations are multiples of this one. This shows that we can continue our resolution by taking

$$R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} R \longrightarrow R/(x^2, xy) \longrightarrow 0.$$

Now note that R is a domain, so the leftmost map is in fact injective, and we are done. We conclude that

$$0 \longrightarrow \underset{2}{R} \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \underset{1}{R^2} \xrightarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} \underset{0}{R} \longrightarrow R/(x^2, xy) \longrightarrow 0$$

is a free resolution for $R/(x^2, xy)$. We also took as few generators at each step as possible, so this is a minimal free resolution. We can check this more precisely by noting that all the entries in our matrices are nonunits. In particular, we learn that $\operatorname{pdim}(R/(x^2, xy)) = 2$.

Example 1.16. Let $R = k[x, y, z]$ and $M = R/(xy, xz, yz)$. We claim that the minimal free resolution for M is

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} R \longrightarrow M \longrightarrow 0.$$

The Betti numbers of M are

$$\beta_0(M) = 1 \qquad \beta_1(M) = 3 \qquad \beta_2(M) = 2.$$

This is a special case of the Hilbert–Burch Theorem [Bur68], which tells us about the shape of the minimal free resolution of cyclic modules of projective dimension 2.

In the example above, we presented a minimal free resolution without justification. How could one check by hand that this is indeed the minimal free resolution? In theory, we would find the kernel of each differential map explicitly, which can be quite challenging, and then check that it does in fact match the image of the next differential. But in specific cases, we can take advantage of other facts about resolutions to determine some or all of the betti numbers, for example, which can help justifying a particular complex is in fact a resolution. As for minimality, we can simply check that all the entries in the matrices presented are in fact in the unique (homogeneous) maximal ideal, showing that the complex is minimal.

Over the course of this class, we will be discussing some helpful properties of projective dimension and betti numbers that will help us justify these things more carefully.

In the graded setting, we can take resolutions of graded free modules and graded maps. Including this graded information enhances the kind of information we can obtain from a resolution. In particular, we can consider graded betti numbers, which take into account not only the number of generators in each homological degree but also what their internal degree (R-degree) is. We will explore this in more detail in the next section.

1.2 Graded betti numbers

Let R be a standard graded k-algebra with $R_0 = k$ and homogeneous maximal ideal $\mathfrak{m} = R_+$. Let M be a graded R-module. To write a graded free resolution for M, we choose all maps to have degree 0, so that the graded free modules in each degree are sums of copies of shifts of R. We write $R(-d)$ for the R-module R but with a new grading, where

$$(R(-d))_i := R_{i-d}.$$

Definition 1.17. Let M be a graded R-module with minimal graded free resolution F. The (i,j)**th Betti number** of M, $\beta_{ij}(M)$, counts the number of generators of F_i in degree j. Thus

$$\beta_{i,j}(M) = \text{number of copies of } R(-j) \text{ in } F_i \quad \text{and} \quad F_i = \bigoplus_j R(-j)^{\beta_{ij}(M)}.$$

Remark 1.18. At each stage, the nonzero entries in the differential must be nonunits, and thus they are homogeneous elements of positive degree. Therefore, if

$$F_i = R(-d_1)^{\beta_{id_1}} \oplus \cdots \oplus R(-d_s)^{\beta_{id_s}}$$

with $d_1 \leqslant \cdots \leqslant d_s$, and $F_{i+1} \neq 0$, then the smallest possible shift in F_{i+1} is $d_1 + 1$. In particular, $\beta_{ij}(M) = 0$ for all $j < i$.

Definition 1.19. We often collect the Betti numbers of a module in its **Betti table**:

$\beta(M)$	0	1	2	$\cdots$
0	$\beta_{00}(M)$	$\beta_{11}(M)$	$\beta_{22}(M)$	
1	$\beta_{01}(M)$	$\beta_{12}(M)$	$\beta_{23}(M)$	
2	$\beta_{02}(M)$	$\beta_{13}(M)$		
$\vdots$			$\ddots$	

By convention, the entry corresponding to (i,j) in the Betti table of M contains $\beta_{i,i+j}(M)$, and *not* $\beta_{ij}(M)$. This is how Macaulay2 displays Betti tables.

Example 1.20. From the minimal resolution in Example 1.16, we can read the graded Betti numbers of M:

- $\beta_0(M) = 1$, since M is cyclic. The unique generator lives in degree 0, so $\beta_{0,0}(M) = 1$.
- $\beta_1(M) = 3$, and these three quadratic generators live in degree 2, so $\beta_{12} = 3$.
- $\beta_2(M) = 2$. These are linear syzygies on quadrics, living in degree 1+2=3, so $\beta_{23} = 2$.

Here is the graded free resolution of M:

$$0 \longrightarrow R(-3)^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}} R(-2)^3 \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} R \longrightarrow M \longrightarrow 0.$$

Notice that the graded shifts in lower homological degrees affect all the higher homological degrees as well. For example, when we write the map in degree 2, we only need to shift the degree of each generator by 1, but since our map now lands on $R(-2)^3$, we have to bump up degrees from 2 to 3, and write $R(-3)^2$. So again we have

$$\beta_{00} = 1, \beta_{12} = 3, \text{and } \beta_{23} = 2.$$

We can now collect the graded Betti numbers of M in its Betti table:

	0	1	2
0	1	–	–
1	–	3	2

Example 1.21. Let k be a field, $R = k[x, y]$, and consider the ideal

$$I = (x^2, xy, y^3)$$

which has two generators of degree 2 and one of degree 3, so there are graded Betti numbers β_{12} and β_{13}. The minimal free resolution for R/I is

$$0 \longrightarrow \begin{matrix} R(-3)^1 \\ \bigoplus \\ R(-4)^1 \end{matrix} \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y^2 \\ 0 & -x \end{pmatrix}} \begin{matrix} R(-2)^2 \\ \bigoplus \\ R(-3)^1 \end{matrix} \xrightarrow{\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}} R \longrightarrow R/I.$$

Thus

$$\begin{matrix} \beta_{23}(R/I) = 1 & \qquad & \beta_{12}(R/I) = 2 \\ \beta_{24}(R/I) = 1 & \qquad & \beta_{13}(R/I) = 1 \end{matrix}$$

and the Betti table of R/I is

$\beta(M)$	0	1	2
0	1	–	–
1	–	2	1
2	–	1	1

Even if all we know is the Betti numbers of M, there is lots of information we can extract about M. For more about the beautiful theory of free resolutions and syzygies, see [Eis05]. For a detailed treatment of graded free resolutions, see [Pee11].

Macaulay2. In Macaulay2, given an R-module M, we can ask for its minimal free resolution by running `res M`. If I is an ideal in R, note that `res I` returns a resolution for R/I. Currently, resolutions still default to type `ChainComplexes`, but the default will soon be replaced with `Complexes`. You can ask for a resolution of type `Complexes` by loading the `Complexes package` and asking for `freeResolution M`.

More on `ChainComplexes` vs `Complexes` in Macaulay2 can be found in Appendix A.

1.3 The residue field has the largest resolution

The residue field plays an especially important role in the story of free resolutions:

Exercise 2. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and M be a finitely generated R-module. Show that $\beta_i(M) = \dim_k \left(\operatorname{Tor}_i^R(M,k)\right) = \dim_k \left(\operatorname{Ext}_R^i(M,k)\right)$.

Remark 1.22. As a consequence of Exercise 2, we learn that

$$\operatorname{pdim}(M) = \sup\{i \mid \beta_i(M) \neq 0\} = \sup\{i \mid \operatorname{Ext}_R^i(M,k) \neq 0\} = \sup\{i \mid \operatorname{Tor}_R^i(M,k) \neq 0\}.$$

We can extend Exercise 2 to graded betti numbers once we realize that Tor and Ext of graded modules can also be given graded structures.

Remark 1.23. When R is a graded ring and M and N are graded R-modules, we can compute $\operatorname{Ext}_R^i(M,N)$ using a graded free resolution of M, and thus the Ext-modules inherit an R-graded structure.

Exercise 3. Let R be a standard graded finitely generated algebra over a field $k = R_0$ and let M be a graded R-module. Show that

$$\beta_{i,j}(M) = \text{number of copies of } R(-j) \text{ in } F_i = \dim_k \left(\operatorname{Tor}_i^R(M,k)_j\right) = \dim_k \left(\operatorname{Ext}_R^i(M,k)_{-j}\right).$$

The residue field has the largest free resolution possible.

Corollary 1.24. *Let $(R, \mathfrak{m}, k)$ be local/graded. Then for every finitely generated (graded) R-module M,*

$$\operatorname{pdim}_R(M) \leqslant \operatorname{pdim}_R(k).$$

Proof. It suffices to consider the case when $\operatorname{pdim}_R(k)$ is finite. When $i > \operatorname{pdim}_R(k)$, we must have $\operatorname{Tor}_i^R(M,k) = 0$, as noted in Remark 1.12. By Exercise 2, $\beta_i(M) = 0$ for all such i, so $\operatorname{pdim}(M) \leqslant \dim(k)$. □

Definition 1.25. Let R be a domain with fraction field Q. The **rank** of a finitely generated R-module M is defined as

$$\operatorname{rank} M := \dim_Q(M \otimes_R Q).$$

Exercise 4. Check that if M is a free module, then $\operatorname{rank}(M)$ is the free rank of M.

Exercise 5. Let M be a finitely generated module over a noetherian local domain. Show that

$$\beta_i(M) = \operatorname{rank}\left(\Omega_i(M)\right) + \operatorname{rank}\left(\Omega_{i+1}(M)\right).$$

Exercise 6. Show that if M has finite projective dimension over a domain, then

$$\sum_{i=0}^{\operatorname{pdim}(M)} (-1)^i \beta_i(M) = \operatorname{rank}(M).$$

If only we had an explicit minimal free resolution of k, maybe we could use it to say something about the minimal free resolutions of other finitely generated R-modules. With that goal in mind, we take a break from thinking about free resolutions and projective dimension to discuss a very important complex – perhaps *the* most important complex.

1.4 The Koszul complex

The Koszul complex is arguably the most important complex in commutative algebra (and beyond). It appears everywhere, and it is a very powerful yet elementary tool any homological algebraist needs in their toolbox. Every sequence of elements $x_1, \dots, x_n$ in any ring R gives rise to a Koszul complex.

Construction 1.26 (The Koszul complex)**.** The **Koszul complex** on one element $x \in R$ is the complex

$$\mathrm{kos}(x) := 0 \longrightarrow \underset{1}{R} \xrightarrow{x} \underset{0}{R} \longrightarrow 0.$$

More generally, given $x_1, \dots, x_n \in R$, the **Koszul complex** with respect to $x_1, \dots, x_n$ is the complex $\mathrm{kos}(x_1, \dots, x_n)$ defined inductively as

$$\mathrm{kos}(x_1, \dots, x_n) := \mathrm{kos}(x_1, \dots, x_{n-1}) \otimes_R \mathrm{kos}(x_n).$$

You will find different sign conventions for the Koszul complex in the literature, but at the end of the day they all lead to isomorphic complexes.

Example 1.27. The Koszul complex on $f, g \in R$ is given by

$$\mathrm{kos}(f,g) = \text{Totalization of} \quad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & R \arrow[r, "-g"] \arrow[d, "f"] & R \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & R \arrow[r, "g"] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & \end{tikzcd}$$

which is

$$0 \longrightarrow R \xrightarrow{\binom{-g}{f}} R^2 \xrightarrow{(f \quad g)} R \longrightarrow 0.$$

In general, computing the successive tensor products is a bit tedious. Instead, we will give an alternative way to think about the Koszul complex.

Definition 1.28. The **exterior algebra** $\bigwedge M$ on an R-module M is obtained by taking the the free R-algebra $R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots$, modulo the relations $x \otimes y = -y \otimes x$ and $x \otimes x = 0$ for all $x, y \in M$. We denote the product on $\bigwedge M$ by $a \wedge b$, and see $\bigwedge M$ as a graded algebra where the homogeneous elements in degree d consist of the image of $M^{\otimes d}$. This is a **skew commutative** algebra: for all homogeneous elements a and b

$$a \wedge b = (-1)^{\deg(a)\deg(b)} b \wedge a \quad \text{and} \quad a \wedge a = 0 \text{ whenever } a \text{ has odd degree.}$$

We denote the set of all homogeneous elements of degree n by $\bigwedge^n M$. Note also that this construction is functorial: a map $f\colon M \to N$ of R-modules induces a map

$$\begin{aligned} \textstyle\bigwedge M &\xrightarrow{\quad \wedge f \quad} \textstyle\bigwedge N \\ m_1 \wedge \cdots \wedge m_s &\longmapsto f(m_1) \wedge \cdots \wedge f(m_s). \end{aligned}$$

We will primarily use this construction in the case of free modules. When $M = R^n$ with basis $e_1, \ldots, e_n$, then for all $1 \leqslant s \leqslant n$

$$\wedge^s M \cong R^{\binom{n}{s}} \quad \text{with basis} \quad e_{i_1} \wedge \cdots \wedge e_{i_s} \text{ ranging over all } i_1 < i_2 < \cdots < i_s.$$

Definition 1.29 (The Koszul complex, again)**.** Let $x_1, \ldots, x_n$ be elements in R. The **Koszul complex** on $x_1, \ldots, x_n$ is the complex

$$\operatorname{kos}(x_1, \ldots, x_n) := \; 0 \longrightarrow \textstyle\bigwedge^n R^n \longrightarrow \textstyle\bigwedge^{n-1} R^n \longrightarrow \cdots \longrightarrow \textstyle\bigwedge^1 R^n \longrightarrow R \longrightarrow 0$$

with differential

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_s}) = \sum_{1 \leqslant p \leqslant s} (-1)^{p-1} x_{i_p} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_s}.$$

Exercise 7. Show that d as defined above is indeed a differential, meaning $d^2 = 0$.

Exercise 8. Check that our two definitions of the Koszul complex coincide.

Example 1.30. In the case of two elements, say x and y in R,

$$\operatorname{kos}(x, y) = \; 0 \longrightarrow \wedge^2 R^2 \longrightarrow \wedge^1 R^2 \longrightarrow R \longrightarrow 0$$

with $\partial(e_1) = x$, $\partial(e_2) = y$, $\partial(e_1 \wedge e_2) = xe_2 - ye_1$, so

$$\operatorname{kos}(x, y) = \; 0 \longrightarrow R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow 0.$$

Exercise 9. Write the Koszul complex on 3 elements f_1, f_2, f_3.

The previous exercise is must easier to do using the exterior algebra description of the Koszul complex.

Remark 1.31. You will find different sign conventions for the Koszul complex in the literature, but at the end of the day they all lead to isomorphic complexes.

Definition 1.32. Let M be an R-module and let $x_1, \dots, x_n \in R$. The **Koszul complex** on M with respect to $x_1, \dots, x_n$ is the complex

$$\operatorname{kos}(x_1, \dots, x_n; M) := \operatorname{kos}(x_1, \dots, x_n) \otimes_R M.$$

Remark 1.33. In general, the Koszul complex $\operatorname{kos}(x_1, \dots, x_n; M)$ looks like

$$0 \longrightarrow M \longrightarrow M^n \longrightarrow \cdots \longrightarrow M^{\binom{n}{i}} \longrightarrow \cdots \longrightarrow M^n \longrightarrow M \longrightarrow 0$$

and the nonzero entries in the differential matrices consist of our elements $x_1, \dots, x_n$ with carefully chosen some signs. The left most map, in degree n, is given by the matrix

$$\begin{pmatrix} x_1 \\ -x_2 \\ x_3 \\ \vdots \\ (-1)^{n+1} x_n \end{pmatrix}$$

and the rightmost map is

$$\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}.$$

Our exterior algebra description of the Koszul complex has the advantage that it indicates a bonus structure on our complex: it is also an algebra. In fact, this is the first big example of a DG algebra. While we will not have the chance to explore this further, this DG algebra structure on the Koszul complex plays a major role in commutative algebra.

We can also take advantage of this added structure to note that the Koszul complex is a functorial construction: given $\underline{x} = x_1, \dots, x_n \in R$ and $\underline{y} = y_1, \dots, y_m \in R$, any commutative diagram of R-module homomorphisms

$$\begin{tikzcd} R^n \arrow[r, "\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}"] \arrow[d, "\varphi_1"'] & R \arrow[d, "\varphi_0"] \\ R^m \arrow[r, "\begin{pmatrix} y_1 & \cdots & y_m \end{pmatrix}"'] & R \end{tikzcd}$$

extends to a map of complexes $\operatorname{kos}(\underline{x}) \longrightarrow \operatorname{kos}(\underline{y})$, by taking

$$\varphi_s \left(e_{i_1} \wedge \cdots \wedge e_{i_s} \right) = \varphi_1(e_{i_1}) \wedge \cdots \wedge \varphi_1(e_{i_1}).$$

The fact that this map commutes with taking the differential is an immediate consequence of the definition, which we leave as an exercise.

Notation 1. Given elements $x_1, \dots, x_n$ in a ring R, we often write $\underline{x} = x_1, \dots, x_n$.

The homology of the Koszul complex has some nice properties.

Definition 1.34. Let M be an R-module and $x_1, \dots, x_n \in R$. The i**th Koszul homology** module of M with respect to $x_1, \dots, x_d$, also called the **Koszul homology** on $x_1, \dots, x_d$ with coefficients in M, is the R-module

$$\mathrm{H}_i(x_1, \dots, x_d; M) := \mathrm{H}_i(\operatorname{kos}(x_1, \dots, x_d; M)).$$

Exercise 10. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and let $\underline{x}$ and $\underline{y}$ be two minimal generating sets for the same ideal. Show that $\operatorname{kos}(\underline{x})$ and $\operatorname{kos}(\underline{y})$ are isomorphic complexes.

Theorem 1.35. *Let R be a ring, $I = (x_1, \dots, x_n)$ be an ideal, and M an R-module.*

1) $\mathrm{H}_i(\underline{x}; M) = 0$ *whenever* $i < 0$ *or* $i > n$*.*

2) $\mathrm{H}_0(\underline{x}; M) = M/IM$*.*

3) $\mathrm{H}_n(\underline{x}; M) = (0 :_M I) = \operatorname{ann}_M(I)$*.*

4) Every Koszul homology module $\mathrm{H}_i(\underline{x}; M)$ *is killed by* $\operatorname{ann}_R(M)$*.*

5) Every Koszul homology module $\mathrm{H}_i(\underline{x}; M)$ *is killed by* I*.*

6) If M is a noetherian R-module, so is $\mathrm{H}_i(\underline{x}; M)$ *for every i.*

7) For every i, $\mathrm{H}_i(\underline{x}; -)$ *is a covariant additive functor* R***-Mod*** $\longrightarrow$ R***-Mod****.*

8) Every short exact sequence of R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

gives rise to a long exact sequence on Koszul homology,

$$\cdots \longrightarrow \mathrm{H}_1(\underline{x}; C) \longrightarrow \mathrm{H}_0(\underline{x}; A) \longrightarrow \mathrm{H}_0(\underline{x}; B) \longrightarrow \mathrm{H}_0(\underline{x}; C) \longrightarrow 0.$$

Proof.

1) Immediate from the definition, since the Koszul complex is only nonzero in homological degrees 0 through n.

2) Remark 1.33 tells us that

$$\mathrm{H}_0(\underline{x}; M) = \operatorname{coker}\left(M \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}} M \right) = M/IM.$$

3) Remark 1.33 above tells us that

$$\begin{aligned} \mathrm{H}_n(\underline{x}; M) &= \ker\left(M \xrightarrow{\begin{pmatrix} x_1 & -x_2 & x_3 & \cdots & (-1)^{n+1} x_n \end{pmatrix}^T} M^n \right) \\ &= \{ m \in M \mid rx_1 = rx_2 = \cdots = rx_n = 0 \} \\ &= (0 :_M I). \end{aligned}$$

4) In each homological degree, the Koszul complex is simply a direct sum of copies of M. So the modules in the complex $\operatorname{kos}(\underline{x}; M)$ are themselves already killed by $\operatorname{ann}_R(M)$, before we even take homology.

5) We are going to show something stronger: we will show that for all $a \in I$, multiplication by a on $\operatorname{kos}(\underline{x}; M)$ is nullhomotopic, which proves that multiplication by a is the zero map in $\mathrm{H}_i(\underline{x}; M)$. In fact, it is sufficient to show that multiplication by a is nullhomotopic on $\operatorname{kos}(\underline{x})$, since additive functors preserve the homotopy relation. To do this, we will explicitly use the multiplicative structure of the Koszul complex given by our description of the Koszul complex via exterior powers. Given $a \in I = (x_1, \dots, x_n)$, write $a = a_1 x_1 + \cdots + a_n x_n$. Consider the map

$$s_a \colon \operatorname{kos}(\underline{x}) \longrightarrow \Sigma^{-1} \operatorname{kos}(\underline{x})$$

given by multiplication by $a_1 e_1 \wedge \cdots \wedge a_n e_n$, meaning

$$s_a(e_{i_1} \wedge \cdots \wedge e_{i_t}) = \sum_{j=1}^{n} a_j e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_t}.$$

Now we claim this map s_a is a nullhomotopy for the map of complexes $\operatorname{kos}(\underline{x}) \longrightarrow \operatorname{kos}(\underline{x})$ given by multiplication by a in every component. To check that, it is sufficient to check this on basis elements, that is, we need only to check that

$$s_a \partial \left(e_{i_1} \wedge \cdots \wedge e_{i_t}\right) + \partial s_a \left(e_{i_1} \wedge \cdots \wedge e_{i_t}\right) = a e_{i_1} \wedge \cdots \wedge e_{i_t}.$$

Indeed, we have

$$\begin{aligned} s_a \partial \left(e_{i_1} \wedge \cdots \wedge e_{i_t}\right) &= s_a \left(\sum_{k=1}^{t} (-1)^{k+1} x_k e_{i_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{i_t} \right) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{t} (-1)^{k+1} a_j x_k e_j \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{i_t} \end{aligned}$$

and

$$\begin{aligned} \partial s_a \left(e_{i_1} \wedge \cdots \wedge e_{i_t}\right) &= \partial \left(\sum_{j=1}^{n} a_j e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_t} \right) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{t} (-1)^{k+2} a_j e_j \wedge e_{j_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_s} + \sum_{j=1}^{n} a_j x_j e_{i_1} \wedge \cdots \wedge e_{i_t} \\ &= -s_a d \left(e_{i_1} \wedge \cdots \wedge e_{i_t}\right) + \sum_{j=1}^{n} a_j x_j e_{i_1} \wedge \cdots \wedge e_{i_t} \end{aligned}$$

and since $a_1 x_1 + \cdots + a_n x_n = a$, we conclude that

$$(s_a \partial + \partial s_a) \left(e_{i_1} \wedge \cdots \wedge e_{i_t}\right) = a e_{i_1} \wedge \cdots \wedge e_{i_t}.$$

6) If M is noetherian, then so is M^k for any k, as well as any submodules of M^k and any of their quotients, by Theorem 1.53 from Commutative Algebra I. Each $\mathrm{H}_i(\underline{x}; M)$ is a subquotient of a direct sum of copies of M, so it must be noetherian.
7) Given an R-module homomorphism $f\colon M \to N$, we get an induced map

$$\operatorname{kos}(f)\colon \operatorname{kos}(\underline{x}; M) \longrightarrow \operatorname{kos}(\underline{x}; N)$$

by taking $\operatorname{kos}(\underline{x}) \otimes f$, so we set $\mathrm{H}_i(\underline{x}; f) = \mathrm{H}_i(\operatorname{kos}(\underline{x}) \otimes f)$. It is immediate to see that this assignment takes the identity on M to itself, and we leave it as an exercise to check that this definition preserves composition of homomorphisms.
8) Given any complex of free R-modules F, tensoring with F is an exact functor from R-modules to $\mathrm{Ch}(R)$. In particular, this applies to $\operatorname{kos}(\underline{x})$, so we get a short exact sequence of complexes

$$0 \longrightarrow \operatorname{kos}(\underline{x}) \otimes_R A \longrightarrow \operatorname{kos}(\underline{x}) \otimes_R B \longrightarrow \operatorname{kos}(\underline{x}) \otimes_R C \longrightarrow 0$$

The resulting long exact sequence in homology, as in Theorem 2.39 from Homological Algebra, is the long exact sequence we are looking for. □

In general, the koszul homology functor is neither left nor right exact; we will see examples later in Example 1.54.

Remark 1.36. Following our iterative definition of the Koszul complex, where

$$\operatorname{kos}(x_1, \dots, x_{n+1}; M) = \operatorname{kos}(x_1, \dots, x_n; M) \otimes \operatorname{kos}(x_{n+1}),$$

set $C := \operatorname{kos}(x_1, \dots, x_n; M)$, and note that by definition of tensor product of complexes we have

$$[\operatorname{kos}(x_1, \dots, x_{n+1}; M)]_i = C_{i-1} \otimes_R R \oplus C_i \otimes_R R \cong C_{i-1} \oplus C_i.$$

Let us explicitly write down the differential on $\operatorname{kos}(x_1, \dots, x_{n+1}; M)$ in terms of the differential on $\operatorname{kos}(x_1, \dots, x_n; M)$. Given $a \in C_{i-1}$, $b \in C_i$, $r \in R$ (in homological degree 1) and $s \in R$ (in homological degree 0), our differential is

$$\partial(a \otimes r + b \otimes s) = \partial(a) \otimes r + (-1)^{i-1} a \otimes (x_{n+1} r) + \partial(b) \otimes s + (-1)^i b \otimes 0,$$

so

$$\partial_i := \begin{pmatrix} \partial_C & 0 \\ (-1)^{i-1} x_{n+1} & \partial_C \end{pmatrix} : \begin{tikzcd} C_{i-1} \arrow[r, "\partial_C"] \arrow[dr, "(-1)^{i-1} x_{n+1}"] & C_{i-2} \\ \oplus & \oplus \\ C_i \arrow[r, "\partial_C"'] & C_{i-1}. \end{tikzcd}$$

Notice that this is exactly[2] the cone of the map $\operatorname{kos}(x_1, \dots, x_n; M) \xrightarrow{x_{n+1}} \operatorname{kos}(x_1, \dots, x_n; M)$ given by multiplication by x_{n+1} in every degree. This cone comes together with a short exact sequence

$$0 \longrightarrow \operatorname{kos}(x_1, \dots, x_n; M) \longrightarrow \operatorname{kos}(x_1, \dots, x_{n+1}; M) \longrightarrow \Sigma^{-1} \operatorname{kos}(x_1, \dots, x_n; M) \longrightarrow 0.$$

[2] Up to the sign convention differences we discussed in Theorem 6.20 from Homological Algebra.

This short exact sequence gives rise to a long exact sequence in homology, as described in Theorem 6.23 from Homological Algebra, where (up to sign) the connecting homomorphism is simply the map in homology induced by multiplication by x_{n+1}. Here is the degree n piece of that long exact sequence:

$$\cdots \longrightarrow \mathrm{H}_i(x_1, \ldots, x_{n+1}; M) \longrightarrow \mathrm{H}_{i-1}(x_1, \ldots, x_n; M) \xrightarrow{x_{n+1}} \mathrm{H}_{i-1}(x_1, \ldots, x_n; M) \longrightarrow \cdots.$$

Let us look at

$$\mathrm{H}_i(x_1, \ldots, x_n; M) \xrightarrow{x_{n+1}} \mathrm{H}_i(x_1, \ldots, x_n; M) \xrightarrow{\varphi} \mathrm{H}_i(x_1, \ldots, x_{n+1}; M) \xrightarrow{\psi} \mathrm{H}_{i-1}(x_1, \ldots, x_n; M).$$

By exactness,

$$\begin{aligned} &\ker\left(\mathrm{H}_i(x_1, \ldots, x_n; M) \xrightarrow{\varphi} \mathrm{H}_i(x_1, \ldots, x_{n+1}; M)\right) \\ &= \operatorname{im}(\mathrm{H}_i(x_1, \ldots, x_n; M) \xrightarrow{x_{n+1}} \mathrm{H}_i(x_1, \ldots, x_n; M)) \\ &= x_{n+1} \cdot \mathrm{H}_i(x_1, \ldots, x_n; M). \end{aligned}$$

By the First Isomorphism Theorem, φ induces an inclusion $\overline{\varphi}$

$$\frac{\mathrm{H}_i(x_1, \ldots, x_n; M)}{x_{n+1} \cdot \mathrm{H}_i(x_1, \ldots, x_n; M)} \overset{\overline{\varphi}}{\hookrightarrow} \mathrm{H}_i(x_1, \ldots, x_{n+1}; M)$$

with $\operatorname{im}(\overline{\varphi}) = \operatorname{im}(\varphi)$. Now we use the First Isomorphism Theorem and exactness repeatedly:

$$\begin{aligned} \operatorname{coker}(\overline{\varphi}) &= \frac{\mathrm{H}_i(x_1, \ldots, x_{n+1}; M)}{\operatorname{im}(\varphi)} && \text{by definition} \\ &= \frac{\mathrm{H}_i(x_1, \ldots, x_{n+1}; M)}{\ker(\psi)} && \text{by exactness} \\ &\cong \operatorname{im}(\psi) && \text{by the First Iso Theorem} \\ &= \operatorname{im}\left(\mathrm{H}_i(x_1, \ldots, x_{n+1}; M) \xrightarrow{\psi} \mathrm{H}_{i-1}(x_1, \ldots, x_n; M)\right) \\ &= \ker\left(\mathrm{H}_{i-1}(x_1, \ldots, x_n; M) \xrightarrow{x_{n+1}} \mathrm{H}_{n-1}(x_1, \ldots, x_n; M)\right) && \text{by exactness} \\ &= \operatorname{ann}_{\mathrm{H}_{i-1}(x_1, \ldots, x_n; M)}(x_{n+1}). \end{aligned}$$

And finally, we get the following short exact sequences induced by $\overline{\varphi}$:

$$0 \longrightarrow \tfrac{\mathrm{H}_i(x_1, \ldots, x_n; M)}{x_{n+1} \cdot \mathrm{H}_i(x_1, \ldots, x_n; M)} \longrightarrow \mathrm{H}_i(x_1, \ldots, x_{n+1}; M) \longrightarrow \operatorname{ann}_{\mathrm{H}_{i-1}(x_1, \ldots, x_n; M)}(x_{n+1}) \longrightarrow 0.$$

Definition 1.37. Let M be an R-module and consider a sequence of nonunit elements $\underline{f} \in R$. The **ith Koszul cohomology** on $\underline{f}$ with coefficients in M, also called the **Koszul cohomology** on M with respect to $\underline{f}$, is the R-module

$$\operatorname{kos}^i(\underline{f}; M) := \mathrm{H}^i(\operatorname{Hom}_R(\operatorname{kos}(\underline{f}), M)).$$

We write $\operatorname{kos}^i(\underline{f}) := \mathrm{H}^i(\operatorname{Hom}_R(\operatorname{kos}(\underline{f}), R))$.

However, this adds nothing new to the story: in fact, the Koszul complex is self-dual.

Theorem 1.38. *Let R be any ring and let $\underline{x} = x_1, \dots, x_n \in R$. The Koszul complex* $\operatorname{kos}(\underline{x})$ *is isomorphic to its dual* $\operatorname{Hom}_R(\operatorname{kos}(\underline{x}), R)$*. More generally,*

$$\operatorname{kos}(\underline{x}; M) \cong \operatorname{Hom}_R(\operatorname{kos}(\underline{x}), M).$$

In particular,

$$\mathrm{H}_i(\underline{x}; M) \cong \mathrm{H}^{n-i}(\underline{x}; M).$$

Proof sketch. In general, for any two R-modules M and N, the natural isomorphism

$$R \otimes_R M \cong M$$

leads to a homomorphism

$$\operatorname{Hom}_R(N, R) \otimes_R M \xrightarrow{\psi} \operatorname{Hom}_R(N, M)$$

that sends each simple tensor $f \otimes m$ to the R-module homomorphism $N \to M$ given by

$$n \mapsto f(n) \otimes m.$$

We leave the details to the reader. When N is a finitely generated free R-module, this map ψ is an isomorphism (exercise!), so for all $d \geqslant 1$

$$\operatorname{Hom}_R(R^d, R) \otimes_R M \cong \operatorname{Hom}_R(R^d, M).$$

This isomorphism is natural, and thus induces an isomorphism

$$\operatorname{Hom}_R(\operatorname{kos}(\underline{x}, R)) \otimes M \cong \operatorname{Hom}_R(\operatorname{kos}(\underline{x}, M).$$

Thus it suffices to construct an isomorphism

$$\operatorname{kos}(\underline{x}) \xrightarrow{\omega} \operatorname{Hom}_R(\operatorname{kos}(\underline{x}, R)).$$

Such a map corresponds to a commutative diagram

$$\begin{array}{ccccccccccccc}
\operatorname{kos}(\underline{x}): & & 0 & \longrightarrow & \bigwedge^n R^n & \longrightarrow & \bigwedge^{n-1} R^n & \longrightarrow \cdots \longrightarrow & \bigwedge^1 R^n & \longrightarrow & R & \longrightarrow & 0 \\
\big\downarrow w & & & & \big\downarrow w_n & & \big\downarrow w_{n-1} & & \big\downarrow w_1 & & \big\downarrow w_0 & & \\
\operatorname{Hom}_R(\operatorname{kos}(\underline{x}), R): & & 0 & \longrightarrow & R & \longrightarrow & \bigwedge^1 R^n & \longrightarrow \cdots \longrightarrow & \bigwedge^{n-1} R^n & \longrightarrow & R & \longrightarrow & 0.
\end{array}$$

For each subset $I = \{i_1, \dots, i_d\} \subseteq \{1, \dots, n\}$, write e_I for the homological degree d basis element of $\operatorname{kos}(\underline{x})$ given by

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_d} \in \bigwedge^i R^n.$$

We define $\omega_d(e_I) \in \operatorname{Hom}_R(\bigwedge^{n-i} R^n, R)$ to be the map given by

$$e_J \mapsto \begin{cases} e_I \wedge e_J & \text{if } I \cap J = \emptyset \\ 0 & \text{if } I \cap J \neq \emptyset. \end{cases}$$

We leave it as an exercise to check that this rule determines an isomorphism in each homological degree, and that the diagram commutes, proving the isomorphism between the Koszul complex and its dual. The isomorphism $\mathrm{H}_i(\underline{x}; M) \cong \mathrm{H}^{n-i}(\underline{x}; M)$ is a trivial consequence. $\square$

1.5 Regular sequences

The Koszul complex is closely tied to regular elements and regular sequences.

Definition 1.39. Let R be a ring and M be an R-module. An element $r \in R$ is **regular** on M if $rM \neq M$ and for any $m \in M$

$$rm = 0 \Rightarrow m = 0.$$

More generally, a sequence of elements $x_1, \dots, x_n$ is a **regular sequence on** M if

- $(x_1, \dots, x_n)M \neq M$, and
- for each i, the element x_i is regular on $M/(x_1, \dots, x_{i-1})M$.

When $M = R$, we drop the *on* M and say r is regular or $x_1, \dots, x_n$ is a regular sequence.

Remark 1.40. Suppose $(x_1, \dots, x_n)M \neq M$. Note that x_i is regular on $M/(x_1, \dots, x_{i-1})M$ if and only if

$$((x_1, \dots, x_{i-1})M :_M x_i) = (x_1, \dots, x_{i-1})M.$$

Remark 1.41. When $(R, \mathfrak{m}, k)$ is a noetherian local ring and $M \neq 0$ is finitely generated R-module, NAK[3] gives us $(x_1, \dots, x_n)M \neq M$ automatically for all $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$.

Example 1.42. Consider the polynomial ring $R = k[x_1, \dots, x_n]$ in n variables over a field k. The variables $x_1, \dots, x_n$ for a regular sequence on R.

Example 1.43. Let k be a field and $R = k[x, y, z]$. The sequence xy, xz is not regular on R, since xz kills y on $R/(xy)$.

The order we write the elements in matters.

Example 1.44. Let k be a field and $R = k[x, y, z]$. We claim that $x, (x-1)y, (x-1)z$ is a regular sequence, while $(x-1)y, (x-1)z, x$ is not.

For the first claim, note that over $R/(x)$, we have $(x-1)y = -y$ and $(x-1)z = -z$. Since $x, -y, -z$ is a regular sequence on R, then so is $x, (x-1)y, (x-1)z$. In contrast, over $R/((x-1)y)$, the elements y and $(x-1)z$ are nonzero, but

$$(x-1)z \cdot y = 0.$$

In particular, $(x-1)z$ is not regular on $R/((x-1)y)$, and $(x-1)y, (x-1)z, x$ is not a regular sequence.

Remark 1.45. Suppose that $r \in R$ is such that $rM \neq M$. We claim that r is regular on M if and only if the first koszul homology vanishes, $\mathrm{H}_1(r; M) = 0$. Indeed,

$$\mathrm{H}_1(r; M) = \ker(M \xrightarrow{r} M) = (0 :_M r),$$

and by definition, r is regular on M if and only if $(0 :_M r) = 0$.

[3] Theorem 5.32 and Theorem 5.39 from Commutative Algebra I, the latter in the graded case.

In fact, the Koszul complex on a regular sequence is exact in all positive degrees.

Theorem 1.46. *If $\underline{x} = x_1, \dots, x_n \in R$ is a regular sequence on the R-module M, then $\mathrm{H}_i(\underline{x}; M) = 0$ for all $i > 0$.*

Proof. We proceed by induction on the length of the sequence, noting that the case $n = 1$ is Remark 1.45.

Suppose that for some $n \geqslant 1$, all regular sequences $\underline{y}$ on M of length n have the property that $\mathrm{H}_i(\underline{y}; M) = 0$ for all $i > 0$. Let $\underline{x} = x_1, \dots, x_{n+1}$ be a regular sequence on M. Then $x_1, \dots, x_n$ is also a regular sequence on M, so by induction hypothesis

$$\mathrm{H}_i(x_1, \dots, x_n; M) = 0 \quad \text{for all } i > 0.$$

We saw in Remark 1.36 that viewing $\mathrm{kos}(\underline{x}; M)$ as the cone of multiplication by x_{n+1} on $\mathrm{kos}(x_1, \dots, x_n; M)$ gives us a long exact sequence in homology

$$\mathrm{H}_i(x_1, \dots, x_n; M) \xrightarrow{\varphi} \mathrm{H}_i(x_1, \dots, x_{n+1}; M) \xrightarrow{\psi} \mathrm{H}_{i-1}(x_1, \dots, x_n; M).$$

For all $i \geqslant 2$, the two homologies on the left and right vanish, by induction hypothesis, and thus the middle homology must vanish as well. We conclude that $\mathrm{H}_i(x_1, \dots, x_{n+1}; M) = 0$ for all $i > 1$.

Moreover, Remark 1.36 also gave us the short exact sequence

$$0 \longrightarrow \frac{\mathrm{H}_1(x_1, \dots, x_i; M)}{x_{i+1} \cdot \mathrm{H}_1(x_1, \dots, x_i; M)} \longrightarrow \mathrm{H}_1(x_1, \dots, x_{i+1}; M) \longrightarrow \mathrm{ann}_{\mathrm{H}_0(x_1, \dots, x_i; M)}(x_{i+1}) \longrightarrow 0.$$

By induction hypothesis, $\mathrm{H}_1(x_1, \dots, x_n; M) = 0$. Since x_{n+1} is regular on

$$M/(x_1, \dots, x_n)M = \mathrm{H}_0(x_1, \dots, x_n; M),$$

we must have

$$\mathrm{ann}_{\mathrm{H}_0(x_1, \dots, x_n; M)}(x_{n+1}) = 0.$$

Therefore, in the last short exact sequence above only the middle term remains, which gives us $\mathrm{H}_1(x_1, \dots, x_{n+1}; M) = 0$. $\square$

Corollary 1.47. *If $x_1, \dots x_n$ is a regular sequence on R, then the Koszul complex on $x_1, \dots, x_n$ is a free resolution for $R/(x_1, \dots, x_n)$. Moreover, if $(R, \mathfrak{m}, k)$ is local/graded, then $\mathrm{kos}(x_1, \dots, x_n)$ is a minimal free resolution for $R/(x_1, \dots, x_n)$.*

Proof. By Theorem 1.46, $P = \mathrm{kos}(x_1, \dots, x_n)$ has $\mathrm{H}_i(P) = 0$ for all $i > 0$. This is a complex of free modules, and thus a free resolution of $\mathrm{H}_0(P)$, which by Theorem 1.35 is $R/(x_1, \dots, x_n)$. Finally, in the local/graded case, the assumption that $\underline{x}$ is a regular sequence guarantees that $x_i \in \mathfrak{m}$ for all i. Since all the nonzero entries in the differentials (under the standard basis for the Koszul complex) are of the form $\pm x_i$, we conclude that the resolution must be minimal. $\square$

There are three big theorems of Hilbert's every commutative algebraist must know: Hilbert's Basis Theorem, Hilbert's Nullstellensatz, and Hilbert's Syzygy Theorem. We are finally ready to prove the third.

Theorem 1.48 (Hilbert's Syzygy Theorem)**.** *Every finitely generated graded module M over a polynomial ring $R = k[x_1, \dots, x_n]$ over a field k has finite projective dimension. In fact, $\operatorname{pdim}(M) \leqslant n$.*

Proof. By Corollary 1.47, the Koszul complex on the regular sequence $x_1, \dots, x_n$ is a minimal free resolution for $k = R/(x_1, \dots, x_n)$, so $\operatorname{pdim}(k) = n$. By Corollary 1.24, every finitely generated R-module M has

$$\operatorname{pdim}(M) \leqslant \operatorname{pdim}(k) = n.$$

□

It is natural to ask if Theorem 1.46 has a converse; over a nice enough ring, the answer is yes: the vanishing of Koszul homology does characterize regular sequences. In fact, more is true: the first Koszul homology completely determines whether we have a regular sequence.

Theorem 1.49. *Let $(R, \mathfrak{m}, k)$ be local/graded. Let $M \neq 0$ be a finitely generated R-module, and $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$. In the graded case, we assume that M is graded and $x_1, \dots, x_n$ are all homogeneous. The following are equivalent:*

1) $\mathrm{H}_i(\underline{x}; M) = 0$ *for all* $i > 0$.

2) $\mathrm{H}_1(\underline{x}; M) = 0$.

3) $\underline{x}$ *is a regular sequence on* M.

Proof. The implication (a) ⇒ (b) is obvious, and (c) ⇒ (a) is Theorem 1.49. We will finish the proof by showing (b) ⇒ (c), which we will do by induction on the length n of $\underline{x}$.

The case $n = 1$ is Remark 1.45, so suppose that the statement holds for all sequences of length n for some $n \geqslant 1$. Now Remark 1.36 gives us the short exact sequence

$$0 \longrightarrow \frac{\mathrm{H}_1(x_1,\dots,x_n;M)}{x_{n+1}\cdot \mathrm{H}_1(x_1,\dots,x_n;M)} \longrightarrow \mathrm{H}_1(x_1, \dots, x_{n+1}; M) \longrightarrow \operatorname{ann}_{\mathrm{H}_0(x_1,\dots,x_n;M)}(x_{n+1}) \longrightarrow 0.$$

By assumption, $\mathrm{H}_1(x_1, \dots, x_n; M) = 0$, so exactness tells us all the terms in the short exact sequence above must vanish.

The vanishing of the left term gives us $x_{n+1} \cdot \mathrm{H}_1(x_1, \dots, x_n; M) = \mathrm{H}_1(x_1, \dots, x_n; M)$. But $\mathrm{H}_1(x_1, \dots, x_n; M)$ is a finitely generated R-module by Theorem 1.35, and $x_n \in \mathfrak{m}$, so $\mathrm{H}_1(x_1, \dots, x_n; M) = 0$ by NAK.[4] By induction hypothesis, $x_1, \dots, x_n$ is a regular sequence on M. Note moreover that NAK also guarantees that $M/(\underline{x})M \neq 0$.

Finally, the vanishing of the rightmost term in our short exact sequence gives us

$$\operatorname{ann}_{\mathrm{H}_0(x_1,\dots,x_n;M)}(x_{n+1}) = 0.$$

By definition, this means that x_{n+1} is regular on $\mathrm{H}_0(x_1, \dots, x_n; M) = M/(x_1, \dots, x_n)M$. We conclude that $\underline{x}$ is a regular sequence on M. □

[4] Theorem 5.32 and Theorem 5.39 from Commutative Algebra I, the latter in the graded case.

A corollary of Theorem 1.49 is that in a local/graded ring, the order of the elements in a regular sequence does not matter.

Corollary 1.50. *Let $(R, \mathfrak{m}, k)$ be local/graded. Let M be a finitely generated R-module, and consider $\underline{x} = x_1, \ldots, x_n \in \mathfrak{m}$. In the graded case, we assume that M is graded and $x_1, \ldots, x_n$ are all homogeneous. If the sequence $\underline{x}$ is regular on M, then so is any of its permutations.*

Proof. If $x_1, \ldots, x_n$ is a regular sequence, then Theorem 1.46 gives us

$$\mathrm{H}_i(x_1, \ldots, x_n; M) = 0 \quad \text{for all } i > 0.$$

By Exercise 10, the Koszul homology on $\underline{x}$ agrees with the Koszul homology on any permutation of $\underline{x}$, which must then also vanish. By Theorem 1.49, any permutation of $\underline{x}$ is a regular sequence. □

In fact, we can extend this to any ring and any module under a reasonable assumption.

Lemma 1.51. *Let R be a ring and M an R-module. If x, y is a regular sequence on M and y is regular on M, then y, x is a regular sequence on M.*

Proof. Suppose that $xm = yn$ for some $m, n \in M$. Since x, y is a regular sequence on M and $yn \in (x)M$, we must have $n \in (x)M$, so there exists some $w \in M$ such that $n = xw$. But then $xm = yn = xyw$. Since x is regular on M, we conclude that $m = yw$, so $m \in (y)M$. In particular, this shows that x is regular on $M/(y)M$. □

Lemma 1.52. *Let $(R, \mathfrak{m})$ be a noetherian local ring. If $x_1, \ldots, x_n$ is a regular sequence on M, then so is $x_1^{a_1}, \ldots, x_n^{a_n}$ for any integers $a_i \geqslant 1$.*

Proof. Let $n = 1$. Suppose $x = x_1$ is a regular element on M. Given a nonzero $m \in M$ and $a \geqslant 1$,

$$x^a m = 0 \implies x x^{a-1} m = 0.$$

Since x is regular on M, we must have $x^{a-1}m = 0$. Repeating this $a-1$ times, we conclude that $xm = 0$, and $m = 0$.

Now consider any $n \geqslant 1$. Since x_n is a regular sequence on $M/(x_1, \ldots, x_{n-1})$ by the case of a sequence of length 1 we can now say $x_n^{a_n}$ is regular on $M/(x_1, \ldots, x_{n-1})$. Therefore, $x_1, \ldots, x_{n-1}, x_n^{a_n}$ is regular on M. By Corollary 1.50, we are allowed to permute the elements in our sequence. Now switch the order and repeat the argument with each x_i, until we conclude that $x_1^{a_1}, \ldots, x_n^{a_n}$ is also regular on M. □

Using all that we have learned about the koszul complex, we can now give examples showing that koszul homology is neither left exact nor right exact.

Example 1.53. Let $R = k[x]$ and consider $M = R/(x^2)$. Note that x is regular on R but not on M, so

$$\mathrm{H}_1(x; R) = 0 \quad \text{and} \quad \mathrm{H}_1(x; M) \neq 0.$$

On the other hand, the canonical projection $R \twoheadrightarrow M$ is surjective, while the induced map on the first koszul homology

$$0 = \mathrm{H}_1(x; R) \to \mathrm{H}_1(x; M) \neq 0$$

cannot be surjective. Thus $\mathrm{H}_1(x; -)$ is not right exact.

Example 1.54. Let $R = k[x]$ and note that x is regular on R, so we get a short exact sequence

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0.$$

On the one hand, $\mathrm{H}_0(x; R) = R/(x)$, but

$$\mathrm{H}_0\left(x; R \xrightarrow{x} R\right) = R/(x) \xrightarrow{x} R/(x)$$

is the zero map. In particular, $\mathrm{H}_0(x; -)$ is not left exact.

More generally, $\mathrm{H}_0(\underline{x}; -)$ is naturally isomorphic to $- \otimes_R R/(\underline{x})$, which is typically not left exact. You might enjoy finding examples of the failure of $\mathrm{H}_i(\underline{x}; -)$ to be left or right exact for other values of i. However, $\mathrm{H}_0(\underline{x}; -) \cong R/(\underline{x}) \otimes_R -$ is in fact right exact.

Theorem 1.55. *If $x_1, \ldots, x_n$ is a regular sequence on R, then* $\operatorname{height}(x_1, \ldots, x_n) = n$. *Moreover, every minimal prime of $(x_1, \ldots, x_n)$ has height n.*

Proof. By induction on n. When $n = 1$, x_1 is regular if and only if x_1 is not in the set of zero divisors of R. By Theorem 6.27 from Commutative Algebra I, this means x_1 is not in any associated prime of R, and in particular, x_1 is not in any of the minimal primes of R. Therefore, any prime containing x_1 must have height at least 1, so $\operatorname{height}(x_1) \geqslant 1$. By Krull's Height Theorem, Theorem 8.5 from Commutative Algebra I, we always have $\operatorname{height}(x_1) \leqslant 1$, so we conclude that $\operatorname{height}(x_1) = 1$.

Suppose that for some $n \geqslant 1$, all regular sequences of length n generate an ideal of height n. If $x_1, \ldots, x_{n+1}$ is a regular sequence on R, then in particular x_{n+1} is regular on the quotient $R/(x_1, \ldots, x_n)$. By the base case, the one-generated ideal $(x_1, \ldots, x_{n+1})/(x_1, \ldots, x_n)$ has height 1 in $R/(x_1, \ldots, x_n)$. By induction hypothesis, $\operatorname{height}(x_1, \ldots, x_n) = n$. We conclude that $\operatorname{height}(x_1, \ldots, x_{n+1}) = n + 1$.

We have now shown that $\operatorname{height}(x_1, \ldots, x_n) = n$, and thus every minimal prime of $x_1, \ldots, x_n$ has height at least n. But Krulls Height Theorem says more: it tells us that every minimal prime of $(x_1, \ldots, x_n)$ has height at most n. We conclude that every minimal prime of $x_1, \ldots, x_n$ has height exactly n. $\square$

The converse does not hold in general.

Example 1.56. Let $R = k[\![x, y]\!]/(x^2, xy)$. The element y is not in the unique minimal prime $(x)/(x^2, xy)$ of R, so $\operatorname{height}(y) = 1$. However, y is not regular.

We will later see that life is much more worth living over a Cohen-Macaulay ring, where indeed $\operatorname{height}(x_1, \ldots, x_n) = n$ if and only if $x_1, \ldots, x_n$ forms a regular sequence.

Remark 1.57. In the local/graded setting, another immediate consequence of Theorem 1.55 is that if $\underline{x} = x_1, \ldots, x_n$ form a regular sequence, then $\underline{x}$ form a minimal generating set for $(\underline{x})$: if not, then $(\underline{x})$ would have less than n generators, and by Krull's Height Theorem (Theorem 8.5 from Commutative Algebra I) $\operatorname{height}(\underline{x}) < n$.

But in fact, more is true: if $\underline{x} = x_1, \ldots, x_n$ is a regular sequence on any module M, then we claim that $\underline{x}$ form a minimal generating set for $(\underline{x})$. To see this, suppose that $\underline{x}$ is not a minimal generating set for $(\underline{x})$. Then some x_i is a combination of the others; since the order does not matter, we can rename the elements and assume $x_n = a_1 x_1 + \cdots + a_{n-1} x_{n-1}$. Then x_n cannot be regular on $M/(x_1, \ldots, x_{n-1})M$, as in fact it acts as zero on this quotient, and thus $\underline{x}$ is not a regular sequence on M.

1.6 Regular rings

Regular rings are the nicest possible kinds of rings, after fields.

Definition 1.58. The **embedding dimension** of a noetherian local ring $(R, \mathfrak{m}, k)$ is

$$\operatorname{embdim}(R) := \mu(\mathfrak{m}) = \dim_k \left(\mathfrak{m}/\mathfrak{m}^2\right).$$

To motivate the name embedding dimension, let us move to the setting of k algebras, and talk about the geometric reasons behind it. let $Q = k[x_1, \dots, x_d]$ be a polynomial ring over the field k, and let I be a homogeneous ideal in Q. The geometry of the variety $V(I)$ is reflected in the algebra of the k-algebra $R = Q/I$. In particular, its dimension is the dimension of the ring R/I. Moreover, $V(I)$ is a subvariety of the affine space $\mathbb{A}^d_k$ of dimension d. If $I \subseteq (x_1, \dots, x_d)^2$, then one can show that we cannot embedded $V(I)$ in $\mathbb{A}^n_k$ for any $n < d$. Thus d is the *embedding dimension* of $V(I)$, and thus of R. Now note that d is the minimal number of generators of the unique homogeneous maximal ideal $(x_1, \dots, x_d)/I$ of R.

Exercise 11. Let $(R, \mathfrak{m})$ be a noetherian local ring and let I be an ideal in R. Show that

$$\operatorname{embdim}(R) = \operatorname{embdim}(R/I) + \dim_k \left(\frac{I + \mathfrak{m}^2}{\mathfrak{m}^2}\right).$$

Remark 1.59. Let $(R, \mathfrak{m})$ be a noetherian local ring. By Krull's Height Theorem,

$$\operatorname{embdim}(R) = \mu(\mathfrak{m}) \geqslant \operatorname{height}(\mathfrak{m}) = \dim(R).$$

A noetherian local ring is regular whenever $\mathfrak{m}$ has the smallest possible number of minimal generators, or equivalently, R has the smallest possible embedding dimension.

Definition 1.60. A noetherian local ring $(R, \mathfrak{m}, k)$ is a **regular local ring** if $\mathfrak{m}$ is minimally generated by $\dim(R)$ many elements, that is

$$\operatorname{embdim}(R) = \dim(R).$$

If $\dim(R) = d$ and $\mathfrak{m} = (x_1, \dots, x_d)$, the elements $\underline{x} = x_1, \dots, x_d$ are called a **regular system of parameters**. We might write RLR as shorthand for regular local ring.

Example 1.61. Any field is a regular local ring: it has dimension 0 and its maximal ideal is generated by no elements.

Example 1.62. Given any field k, the power series ring $k[\![x_1, \dots, x_d]\!]$ is a regular local ring: its dimension is d and $x_1, \dots, x_d$ form a regular system of parameters.

The localization of a polynomial ring at the homogeneous maximal ideal is also regular: $R = k[x_1, \dots, x_d]_{(x_1,\dots,x_d)}$ has dimension d and $x_1, \dots, x_d$ form a regular system of parameters.

Example 1.63. The localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at the prime ideal (p) is a regular local ring: its dimension is $\operatorname{height}(p) = 1$ and the maximal ideal is generated by p. This is an example of a DVR (discrete valuation ring), which is a topic we will not have a chance to explore in detail. However, one of the many equivalent definitions of DVR is that it is a regular local ring of dimension 1.

The localization $\mathbb{Z}[x]_{(p,x)}$ is also a regular local ring: the dimension of R is 2 and p, x form a regular system of parameters.

Example 1.64. The local ring $R = k[\![x, y]\!]/(x^2, xy)$ is not regular: it has dimension 1 but embedding dimension 2.

Exercise 12. Let $(R, \mathfrak{m})$ be a noetherian local ring. Show that if $\underline{x} = x_1, \dots, x_n$ is a regular sequence in R, then $\dim(R/(\underline{x})) = \dim(R) - n$.

Exercise 13. Let $(R, \mathfrak{m})$ be a regular local ring. Show that if $x \in \mathfrak{m}$ and $x \notin \mathfrak{m}^2$, then $R/(x)$ is a regular local ring of $\dim(R/(x)) = \dim(R) - 1$.

Example 1.65. For a slightly more exotic example, let p be a prime integer and $Q = \mathbb{Z}_{(p)}[x]_{(p,x)}$, and let $R = Q/(p - x^2)$. This is a regular local ring: $\dim(R) = 1 = \operatorname{embdim}(R)$, as the maximal ideal is generated by x. Alternatively, we may note that $p - x^2 \in \mathfrak{m} \setminus \mathfrak{m}^2$, so Exercise 14 guarantees that R is a regular local ring.

Theorem 1.66. *Every regular local ring is a domain.*

Proof. We will do induction on $d = \dim(R)$. When $d = 0$, then $\mathfrak{m}$ must be generated by 0 elements, so $\mathfrak{m} = (0)$ and R is a field. In particular, R is a domain.

Now suppose that for some d, all regular local rings of dimension d are domains, and let R be a regular local rind of dimension $d + 1$. By the strong version of Prime Avoidance (see Lemma 3.29 from Commutative Algebra I), which says we can avoid one arbitrary ideal and any finite number of prime ideals, there exists

$$f \in \mathfrak{m} \setminus \mathfrak{m}^2 \bigcup_{P \in \operatorname{Min}(R)} P.$$

Now by Exercise 14, $R/(f)$ is a regular local ring of dimension $d - 1$, since $f \notin \mathfrak{m}^2$. By induction hypothesis, $R/(f)$ must be a domain, so (f) is prime. Note that we chose f to not be in any minimal prime, so (f) has height 1. In particular, there is a minimal prime P of R such that

$$(0) \subseteq P \subsetneq (f).$$

Now given any $y \in P$, $y = rf$ for some $r \in R$, and since $rf = y \in P$ and $f \notin P$, we must have $r \in P$. Thus $y \in (f)P$, and since this holds for all $y \in P$, we conclude that $P = (f)P$. But $f \in \mathfrak{m}$, so by NAK (see Theorem 5.34 from Commutative Algebra I) we must have $P = (0)$. In particular, (0) is a prime ideal. $\square$

Exercise 14. Let $(R, \mathfrak{m})$ be a regular local ring and let $x \in \mathfrak{m}$. Show that if $R/(x)$ is regular then $x \notin \mathfrak{m}^2$.

We are now ready to give a completely homological characterization of regular local rings. This characterization, first proved by Auslander and Buchsbaum and independently by Serre, solved a famous question called the Localization Problem.

Problem 1 (Localization Problem)**.** If R is a RLR, must R_P be a RLR for every prime P?

This question asks if being regular is a local property. A positive answer allows for the following global definition of regularity, as we will later see. But before we can get to this famous homological characterization of regular local rings, and the solution to the localization problem, we will need to sharpen our tools a bit.

Lemma 1.67. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and $F \xrightarrow{\varphi} G$ an R-module map between finitely generated free R-modules. If $\varphi \otimes_R k$ is injective, then $\varphi(F)$ if a free direct summand of G, and φ splits as a map of R-modules.*

Proof. Let $F = R^n$ with standard basis $\{e_1, \dots, e_n\}$, and let $G = R^m$. Our assumption that $\varphi \otimes_R k$ is injective implies that the images of $\varphi(e_1), \dots, \varphi(e_n)$ in $G/\mathfrak{m}G = k^m$ are linearly independent, so we can complete them to a basis $\overline{\varphi(e_1)}, \dots, \overline{\varphi(e_n)}, \overline{f_{n+1}}, \dots, \overline{f_m}$ for $G/\mathfrak{m}G$. By NAK (see Theorem 5.34 from Commutative Algebra I), we can lift those elements f_i to G so that $\varphi(e_1), \dots, \varphi(e_n), f_{n+1}, \dots, f_m$ is a basis for G. In particular, taking

$$H = \bigoplus_{i=n+1}^{m} Rf_i$$

the free module $\varphi(F) \cong F$ is a direct summand of G, via

$$G = \varphi(F) \oplus H.$$

Note that $\varphi(F)$ is a free module of rank n, and thus isomorphic to F. In particular, the map $\psi \colon G \to F$ given by projection onto $\varphi(F)$ is a splitting for φ. $\square$

Theorem 1.68 (Serre)**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. If $\mu(\mathfrak{m}) = s$, then*

$$\beta_i(k) \geqslant \binom{s}{i}.$$

Proof. Let $x_1, \dots, x_s$ be minimal generators for $\mathfrak{m}$, and set $K := \operatorname{kos}(x_1, \dots, x_s)$. If F is a minimal free resolution for $\mathfrak{m}$, we claim that K_i is a direct summand of F_i.

By Theorem 5.19 from Homological Algebra, the identity map on k lifts to a map of complexes $\varphi \colon K \longrightarrow F$. We claim that the maps φ_i split, which will prove our claim that K_i is a direct summand of F_i. The map $\varphi_0 \colon K_0 = R \longrightarrow F_0 = R$ must be the identity map, so φ_0 splits.

We proceed by induction on i. Suppose we have shown that φ_{i-1} splits, say by a splitting ψ_{i-1}. To be precise, this means that $\psi_{i-1}\varphi_{i-1} = \mathrm{id}_{F_{i-1}}$. By Lemma 1.67, it is enough to show that $\varphi_i \otimes_R k$ to be injective. To show $\varphi_i \otimes_R k$ is injective, we need to show that if

$z \in K_i$ is such that $\varphi_i(z) \in \mathfrak{m}F_i$, then $z \in \mathfrak{m}K_i$. First, we note that F is minimal and thus $\operatorname{im}\partial \subseteq \mathfrak{m}F$, by Theorem 5.9 from Homological Algebra, so

$$\partial_i \varphi_i(z) \in \mathfrak{m}^2 F_i.$$

By commutativity, $\varphi_{i-1}\partial_i(z) \in \mathfrak{m}^2 F_i$. Since $\psi_{i-1}\varphi_{i-1} = \mathrm{id}_{K_{i-1}}$, we must have

$$\partial_i(z) = \psi_{i-1}\varphi_{i-1}\partial_i(z) \in \mathfrak{m}^2 K_{i-1}.$$

We claim that this implies $z \in \mathfrak{m}K_i$. To see that, we compute ∂_i explicitly: first we write z as a linear combination of our basis elements, say

$$z = \sum_{j_1 < \cdots < j_i} z_j e_{j_1} \wedge \cdots \wedge e_{j_i}$$

so that

$$\partial_i(z) = \sum_{j,k} \pm z_j x_{j_k} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{j_i}.$$

Now let us keep track of is the coefficients: rewriting this carefully in the basis for K_{i-1}, the coefficient for each basis element is a linear combination of terms of the form $z_j x_l$. We showed that $\partial_i(z) \in \mathfrak{m}^2 F_{i-1}$, so each appropriate combination of $z_j x_l$ is in $\mathfrak{m}^2$. Thus each such combination of $z_j x_l$ is zero in $\mathfrak{m}/\mathfrak{m}^2$. We assumed that $x_1, \ldots, x_s$ were minimal generators for $\mathfrak{m}$ to begin with, and thus a basis for $\mathfrak{m}/\mathfrak{m}^2$, so that the images of our coefficients z_j must all be zero in $R/\mathfrak{m}$. Therefore, all our coefficients z_j must be in $\mathfrak{m}$. We conclude that indeed $z \in \mathfrak{m}F_i$, and thus that $\varphi_i \otimes_R k$ is injective. By Lemma 1.67, φ_i splits.

We have shown that K_i is a direct summand of F_i for all i, and thus

$$\beta_i(k) = \dim_k(\operatorname{Tor}_i^R(k,k)) = \dim_k(F_i \otimes_R k) \geqslant \dim_k(K_i \otimes_R k) = \binom{s}{i}. \quad \square$$

Theorem 1.68 does not hold if we replace k by another cyclic module R/I and $\mu(\mathfrak{m})$ by $\mu(I)$. However, there is a closely related conjectured inequality that remains open:

Conjecture 1.69 (Buchsbaum—Eisenbud, Horrocks)**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring of dimension d and M a finitely generated Artinian R-module with* $\operatorname{pdim}(M) < \infty$*. Then*

$$\beta_i(M) \geqslant \binom{d}{i}.$$

While this remains an open question, there is much evidence to support it. For example, the conjecture predicts the following inequality, previously know as the Total Rank Conjecture, which was recently shown by Walker in almost all cases, and by Walker and VandeBogert in characteristic 2.

Theorem 1.70 (Walker, 2017 [Wal17], VandeBogert–Walker, 2025 [VW25]). *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring of dimension d and characteristic not 2, $M \neq 0$ a finitely generated R-module of finite projective dimension, and $c = \operatorname{height}(\operatorname{ann}(M))$. Then*

$$\sum_i \beta_i(M) \geqslant \sum_i \binom{c}{i} = 2^c.$$

We are finally ready for the famous homological characterization of regular rings that solved the Localization Problem:

Theorem 1.71 (Auslander–Buchsbaum, Serre [AB57, Ser56]). *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring of dimension d. The following are equivalent:*

1) The residue field k has finite projective dimension.

2) Every finitely generated R-module has finite projective dimension.

3) The maximal ideal $\mathfrak{m}$ is generated by a regular sequence.

4) The maximal ideal $\mathfrak{m}$ is generated by d elements.

Proof. The implication $2 \Rightarrow 1$ is obvious, as $M = k$ is a finitely generated R-module. We proved $1 \Rightarrow 2$ in Corollary 1.24.

($3 \Rightarrow 1$) If $\mathfrak{m}$ is generated by a regular sequence $\underline{x}$, then the Koszul complex on $\underline{x}$ is a minimal free resolution of k, by Corollary 1.47. Therefore, k has projective dimension $\mu(\mathfrak{m})$. By Theorem 1.55, the length $\mu(\mathfrak{m})$ of that regular sequence is the height of $\mathfrak{m}$, which is d.

($1 \Rightarrow 4$) Suppose $\operatorname{pdim}(k) < \infty$. We claim that $\operatorname{pdim}(k) \leqslant \dim(R) = d$. By contradiction, suppose $\operatorname{pdim}(k) > d$ but $\operatorname{pdim}(k) < \infty$. Choose a maximal regular sequence $y_1, \ldots, y_t \in \mathfrak{m}$. By Theorem 1.55, $t \leqslant d$. Moreover, every element in $\mathfrak{m}$ is a zerodivisor on $R/(y_1, \ldots, y_t)$, or else we could add to our regular sequence. So $\mathfrak{m}$ is contained in the union of the zerodivisors on $R/(y_1, \ldots, y_t)$, which by Theorem 6.27 from Commutative Algebra I is the same as the union of the associated primes of $R/(y_1, \ldots, y_t)$. By Prime Avoidance (see Lemma 3.29 from Commutative Algebra I), $\mathfrak{m}$ must be contained in some associated prime of $R/(y_1, \ldots, y_t)$. But $\mathfrak{m}$ is maximal, so $\mathfrak{m}$ *is* an associated prime of $R/(y_1, \ldots, y_t)$. Equivalently, $k = R/\mathfrak{m}$ embeds into $R/(y_1, \ldots, y_t)$. This gives us a short exact sequence

$$0 \longrightarrow k \longrightarrow R/(y_1, \ldots, y_t) \longrightarrow M \longrightarrow 0.$$

We get a long exact sequence for Tor by Theorem 6.31 from Homological Algebra:

$$\cdots \longrightarrow \operatorname{Tor}_{i+1}^R(M, k) \longrightarrow \operatorname{Tor}_i^R(k, k) \longrightarrow \operatorname{Tor}_i^R(R/(y_1, \ldots, y_t), k) \longrightarrow \cdots.$$

We know $t = \operatorname{pdim}(R/(y_1, \ldots, y_t))$, by Corollary 1.47, so $\operatorname{Tor}_i^R(R/(y_1, \ldots, y_t), k) = 0$ for all $i > t$. But $t \leqslant d < \operatorname{pdim}(k)$, so in particular $\operatorname{Tor}_i^R(R/(y_1, \ldots, y_t), k) = 0$ for $i = \operatorname{pdim}(k)$.

Moreover, Corollary 1.24 says that $\operatorname{pdim}(M) \leqslant \operatorname{pdim}(k)$ for all finitely generated R-modules M, so in particular $\operatorname{Tor}_{i+1}^R(M,k) = 0$ for $i = \operatorname{pdim}(k)$. But this is impossible: our long exact sequence would then have $\operatorname{Tor}_{\operatorname{pdim}(k)}^R(k,k) \neq 0$ sandwiched between two zero modules.

Now suppose that $\operatorname{pdim}(k) < \infty$. We have seen that this implies that $\operatorname{pdim}(k) \leqslant \dim(R) = d$. By Theorem 1.68 and Exercise 2, for all i

$$\beta_i(k) = \dim_k(\operatorname{Tor}_i^R(k,k)) \geqslant \binom{\mu(\mathfrak{m})}{i}.$$

Since $\beta_i(k) = 0$ for all $i > \operatorname{pdim}(k)$, we must have

$$\mu(\mathfrak{m}) \leqslant \operatorname{pdim}(k) \leqslant \dim(R) = d.$$

But $\operatorname{height}(\mathfrak{m}) = \dim(R) = d$, so by Krull's Height theorem, Theorem 8.5 from Commutative Algebra I, $\mu(\mathfrak{m}) \geqslant d$. We conclude that $\mathfrak{m}$ is generated by exactly d elements, as desired.

($4 \Rightarrow 3$) Let $\mathfrak{m} = (x_1, \dots, x_d)$. If $d = 0$, then $\mathfrak{m} = (\{\}) = (0)$, and there is nothing to prove.

We proceed by induction on d, assuming that $d \geqslant 1$ and that we have shown that whenever $\mathfrak{m}$ is generated by $d-1$ elements $x_1, \dots, x_{d-1}$, then $x_1, \dots, x_{d-1}$ form a regular sequence.

Since there is at least one minimal generator x_1 and $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$, by Exercise 14 we know that $R/(x_1)$ is a regular local ring of dimension $d-1$. Therefore, by induction hypothesis $x_2, \dots, x_d$ form a regular sequence on $R/(x_1)$. Moreover, by Theorem 1.66 R is a domain then x_1 is a regular element in R, so we conclude that $x_1, \dots, x_d$ form a regular sequence. □

Our proof also showed the following:

Corollary 1.72. *Every regular local ring* $(R, \mathfrak{m}, k)$ *has* $\operatorname{pdim}(k) = \dim R$. *Therefore, for all finitely generated R-modules M,*

$$\operatorname{pdim}(M) \leqslant \dim(R).$$

Now we can solve the Localization Problem.

Exercise 15. Show that if R is a regular local ring, then R_P is regular for every prime P.

This allows for the following global definition of regular ring:

Definition 1.73. A noetherian ring R is **regular** if R_P is a regular local ring for all prime ideals P.

Remark 1.74. If we want to show that a particular ring (not necessarily local) is regular, it is sufficient to show that $R_\mathfrak{m}$ is a regular local ring for every maximal ideal $\mathfrak{m}$: given any prime P, there is a maximal ideal $\mathfrak{m}$ containing P, and $R_P \cong (R_\mathfrak{m})_P$ is now a localization of the regular local ring $R_\mathfrak{m}$, which by Exercise 15 must be a regular local rings.

Remark 1.75. In a principal ideal domain R that is not a field, every maximal ideal is generated by exactly one element, so R is regular.

Hilbert's Syzygy Theorem (see Theorem 1.48) tells us that finitely generated *graded* modules over a polynomial ring $R = k[x_1, \dots, x_d]$ have finite projective dimension, but this is not quite enough to conclude that polynomial rings are regular. Nevertheless, they are indeed regular rings. This is elementary when k is algebraically closed:

Example 1.76. Let $R = k[x_1, \dots, x_d]$ with k an algebraically closed field. The maximal ideals in R are precisely those of the form $\mathfrak{m} = (x_1 - a_1, \dots, x_d - a_d)$, which are all generated by d elements, and they all have height d. Therefore, $R_\mathfrak{m}$ is a regular local ring for all maximal ideals $\mathfrak{m}$, and thus R is a regular ring.

The general case, however, requires a bit more work:

Theorem 1.77. *Every polynomial ring $R = k[x_1, \dots, x_d]$ over a field k is a regular ring.*

Proof. It is sufficient to show that $R_\mathfrak{m}$ is a regular local ring for every maximal ideal $\mathfrak{m}$. We are going to show that every maximal ideal is generated by d elements, which implies that $\mathfrak{m}_\mathfrak{m}$ is also generated by d elements. Since the height of every maximal ideal is d, by Theorem 7.46 from Commutative Algebra I, this will imply that $R_\mathfrak{m}$ is a regular local ring.

We will use induction on d to show that every maximal ideal is generated by exactly d elements. When $d = 1$, $k[x]$ is a principal ideal domain, and as noted in Remark 1.75, we are done.

When $d > 1$, we can do a change of variables such as in Lemma 7.40 from Commutative Algebra I so that $\mathfrak{m}$ has a minimal generator that is monic in x_d. Set $\mathfrak{n} := \mathfrak{m} \cap k[x_1, \dots, x_{d-1}]$. Then $k[x_1, \dots, x_{d-1}]/\mathfrak{n} \longrightarrow R/\mathfrak{m}$ is an integral extension, as x_d satisfies a monic polynomial with coefficients in $k[x_1, \dots, x_{d-1}]$. Now Theorem 1.44 from Commutative Algebra I says that in an integral extension of domains $A \subseteq B$, A is a field if and only if B is a field. Since $R/\mathfrak{m}$ is a field, then $L = k[x_1, \dots, x_{d-1}]/\mathfrak{n}$ is also a field. Therefore, $\mathfrak{n}$ is a maximal ideal in $k[x_1, \dots, x_{d-1}]$, so by induction hypothesis it must be generated by $d-1$ elements. Consider the map

$$\begin{aligned} R &\longrightarrow L[x_d] = \tfrac{k[x_1, \dots, x_{d-1}]}{\mathfrak{n}}[x_n] \cong R/\mathfrak{n}R \\ x_i &\longmapsto x_i. \end{aligned}$$

The image of $\mathfrak{m}$ in $L[x_d]$ is a maximal ideal, and $L[x_d]$ is a polynomial ring over a field in one variable, so by the case $d = 1$ the $\mathfrak{m}L[x_d] = \mathfrak{m}/\mathfrak{n}R$ must be generated by 1 element. Therefore, $\mathfrak{n}$ is generated by $d - 1$ elements and $\mathfrak{m}/\mathfrak{n}R$ is generated by 1 element, so $\mathfrak{m}$ is generated by $d - 1 + 1 = d$ elements. □

In fact, this is a corollary of the following more general fact, which we will not prove:

Theorem 1.78. *If R is a regular ring, then the polynomial ring $R[x_1, \dots, x_n]$ is also regular.*

Our proof of Theorem 1.77 also showed that over any field k, every maximal ideal in $k[x_1, \dots, x_d]$ is generated by exactly d elements. This does not extend to all regular rings: while all maximal ideals in a regular ring of dimension d must be *locally* generated by at most d elements, *globally* they might need more than d generators.

Exercise 16. Let $R = \mathbb{Z}[\sqrt{-5}] \cong \mathbb{Z}[x]/(x^2+5)$. Show that R is a one-dimensional regular ring, but not all maximal ideals are principal.

Definition 1.79. Let R be a noetherian ring. The **global dimension** of R is

$$\operatorname{gldim}(R) := \sup\{\operatorname{pdim}_R(M) \mid M \text{ is a finitely generated } R\text{-module}\}.$$

In fact, one can show that if for some integer $n \geqslant 0$, all finitely generated R-modules have projective dimension at most n, then *every* R-module has projective dimension at most n. Therefore, the supremum in the definition of global dimension can be taken over all R-modules. Time permitting, we might prove this later.

We can now rewrite the Auslander–Buchsbaum–Serre theorem as follows:

Theorem 1.80. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Then $\operatorname{pdim}(k) < \infty$ if and only if $\operatorname{gldim}(R) < \infty$. Moreover, if $\operatorname{gldim}(R) < \infty$, then we must have $\operatorname{gldim}(R) = \dim(R)$.*

The global dimension matches the dimension more generally. For that, we need a lemma:

Lemma 1.81. *If R is a regular ring, then $\operatorname{gldim}(R) \leqslant \dim(R)$.*

Proof. There is nothing to prove when $\dim(R)$ is infinite, so assume $\dim(R) = d$ is finite. Let F be a free resolution for the finitely generated R-module M. Let P be any prime ideal of R, and

$$\Omega := \ker\left(F_{d-1} \xrightarrow{\partial_{d-1}} F_{d-2}\right).$$

Localization is exact, so $F \otimes_R R_P$ is a free resolution for M_P and Ω_P is the kernel of $\partial_{d-1} \otimes_R R_P$. By assumption, R_P is a regular local ring of dimension at most d, so by Theorem 1.71 we know that the projective dimension of M_P over R_P is at most d. Any resolution for M_P is a direct sum of a minimal resolution and a trivial complex, and a minimal resolution of M_P has length at most d. Thus Ω_P must be a free module over R_P.

This holds for any prime ideal P, so Ω is locally free. By Theorem 4.61 from Homological Algebra, Ω must be projective, and thus the complex

$$0 \longrightarrow \Omega \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_0$$

is a projective resolution for M, of length d. We conclude that $\operatorname{pdim}_R(M) \leqslant d$. $\square$

Theorem 1.82. *A noetherian ring R is regular of dimension d if and only if $\operatorname{gldim}(R) = d$.*

Proof. ($\Rightarrow$) If that R is a regular ring of dimension d, then by Lemma 1.81 we know that $\operatorname{gldim}(R) \leqslant d$. Let $\mathfrak{m}$ be a maximal ideal of height d. By assumption, $R_\mathfrak{m}$ is a regular local ring of dimension d. Since localization is exact, localizing a finite resolution for $R/\mathfrak{m}$ gives us a finite resolution for $(R/\mathfrak{m})_\mathfrak{m} \cong R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$, which may however not be minimal. Therefore,

$$d \geqslant \operatorname{gldim}(R) \geqslant \operatorname{pdim}_R(R/\mathfrak{m}) \geqslant \operatorname{pdim}_{R_\mathfrak{m}}\left(\frac{R_\mathfrak{m}}{\mathfrak{m}_\mathfrak{m}}\right) = \dim\left(R_\mathfrak{m}\right) = d.$$

We conclude that $\operatorname{gldim}(R) = \dim(R)$.

($\Leftarrow$) Suppose R has finite global dimension d. Then for every maximal ideal $\mathfrak{m}$, $R/\mathfrak{m}$ has projective dimension at most d. Since localization is exact and finitely generated projectives over a noetherian ring are locally free, localizing a finite projective resolution for $R/\mathfrak{m}$ gives us a finite projective resolution for $(R/\mathfrak{m})_\mathfrak{m} \cong R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$, and therefore $R_\mathfrak{m}$ is a regular local ring by Theorem 1.71. We have shown that $R_\mathfrak{m}$ is a RLR for all maximal ideals $\mathfrak{m}$, and thus R is a regular ring.

Our argument also shows that

$$d = \operatorname{gldim}(R) \geqslant \operatorname{pdim}_R(R/\mathfrak{m}) \geqslant \operatorname{pdim}_{R_\mathfrak{m}}\left(\frac{R_\mathfrak{m}}{\mathfrak{m}_\mathfrak{m}}\right) = \dim(R_\mathfrak{m}) = \operatorname{height}(\mathfrak{m}).$$

In particular, $\dim(R)$ is finite, and bounded above by d. By Lemma 1.81,

$$d = \operatorname{gldim}(R) \leqslant \dim(R).$$

We conclude that $\dim(R) = d$. □

Remark 1.83. Note that there are examples of noetherian regular rings with infinite Krull dimension, and their global dimension is thus also infinite. Nagata's Example 7.7 from Commutative Algebra I is one such ring.

However, if R is regular (even of infinite Krull dimension), then every finitely generated R-module has finite projective dimension; the issue is that there might be finitely generated R-modules of arbitrarily high projective dimension.

Macaulay2. Theorem 1.82 gives us a hint at how to decide if a module has infinite projective dimension: if its projective dimension is at least $d + 1$, then it must be infinite. This is particularly helpful for computations using Macaulay2: while the computer will only compute finitely many steps of the resolution, when looking at a graded R-module, if the computer returns a minimal resolution with at least $\dim(R) + 1$ steps, then we know that M must have infinite projective dimension.

We end this chapter with one more property of regular local rings: they are all UFDs.

Theorem 1.84 (Auslander–Buchsbaum, 1959)**.** *Any regular local ring is a UFD.*

However, not all regular rings are UFDs, only RLRs.

Example 1.85. The regular ring $R = \mathbb{Z}[\sqrt{-5}]$ from Exercise 16 is not a UFD:

$$6 = 2 \times 3 \quad \text{and} \quad 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

are two factorizations as products of irreducibles.

Definition 1.86. A homomorphism of local rings $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a **local map** if $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$.

Exercise 17. Let $R \to S$ be a flat local homomorphism of noetherian local rings. If S is regular then R is regular.

Regularity is preserved by taking the completion at the maximal ideal.

Theorem 1.87. *For any noetherian local ring $(R, \mathfrak{m})$, its completion $\widehat{R}$ at $\mathfrak{m}$ is regular if and only if R is regular.*

Proof. Suppose that R is regular. Then

$$\mu(\mathfrak{m}) = \dim(R) = \dim(\widehat{R}).$$

Moreover, the maximal ideal of $\widehat{R}$ is $\mathfrak{m}\widehat{R}$, so it must be generated by at most $\mu(\mathfrak{m}) = \dim(\widehat{R})$ many elements. We conclude that $\widehat{R}$ is also a regular local ring.

The converse follows from the fact that $\widehat{R}$ is a flat R-module and Exercise 17. □

We have seen that regular rings are very nice. It turns out that up to completion, *every* noetherian local ring is a quotient of a regular ring. More precisely, every *complete* local ring is a quotient of a regular local ring.

Theorem 1.88 (Cohen's Structure Theorem)**.** *Every complete noetherian local ring $(R, \mathfrak{m})$ can be written as a quotient of a regular local ring Q, where $Q = V[\![x_1, \dots, x_d]\!]$ with V a field or a complete characteristic zero discrete valuation ring whose maximal ideal is generated by a prime number p.*

This amazing theorem was I. S. Cohen's PhD thesis. If a local ring R is not complete, we can always take its *completion*, which is now a quotient of a regular local ring. When our local ring R contains a field k, Cohen's Structure Theorem actually says that R is of the form $R = k[\![x_1, \dots, x_d]\!]/I$.

Chapter 2

Cohen-Macaulay rings

2.1 Depth

We now get back to regular sequences to talk about how long they can be.

Definition 2.1. Let I be an ideal in a noetherian ring R and M be an R-module. The **I-depth** of M is the maximal length of a regular sequence on M consisting of elements in I, denoted $\operatorname{depth}_I(M)$. The **grade** of an ideal I, denoted $\operatorname{grade}(I)$, is the maximal length of a regular sequence inside I, meaning that

$$\operatorname{grade}(I) = \operatorname{depth}_I(R).$$

When $(R, \mathfrak{m})$ is a local ring, the **depth** of M is $\operatorname{depth}(M) = \operatorname{depth}_{\mathfrak{m}}(M)$.

Remark 2.2. In a noetherian local ring $(R, \mathfrak{m})$, $\operatorname{depth}(R) = \operatorname{grade}(\mathfrak{m})$.

A priori, two maximal regular sequences on M inside I may have different lengths; we will soon see that is not the case. Before we prove that, we already have an upper bound for depth.

Remark 2.3. If $x_1, \dots, x_n$ is a regular sequence on R inside I, we saw in Theorem 1.55 that $\operatorname{height}(x_1, \dots, x_n) = n$, so $\operatorname{grade}(I) \leqslant \operatorname{height}(I)$. In particular, $\operatorname{depth}(R) \leqslant \dim(R)$.

Example 2.4. Let k be any field and $R = k[\![x_1, \dots, x_d]\!]$. The variables $x_1, \dots, x_d$ form a regular sequence, so $\operatorname{depth}(R) \geqslant d$. On the other hand, $\operatorname{depth}(R) \leqslant \dim(R) = d$, so $\operatorname{depth}(R) = d$.

Lemma 2.5. *Let R be a noetherian ring, I an ideal in R, and $M \neq 0$ a finitely generated R-module such that $IM \neq M$. There exists $r \in I$ which is regular on M if and only if $I \nsubseteq P$ for all $P \in \operatorname{Ass}(M)$. If $(R, \mathfrak{m})$ is local/graded, and M is a graded module in the graded setting, $\mathfrak{m} \in \operatorname{Ass}(M)$ if and only if there are no regular elements on M.*

Proof. If $r \in I$ is regular on M, then r is not in the union of the associated primes of M, by Theorem 6.27 from Commutative Algebra I. As a consequence, I cannot be contained in any associated prime of M.

Conversely, recall that M has finitely many associated primes, by Corollary 6.35 from Commutative Algebra I. If I is not contained in any associated prime of M, then by Prime Avoidance, Lemma 3.29 from Commutative Algebra I, it also cannot be contained in the union of the associated primes of M. Recall Theorem 6.27 from Commutative Algebra I, which says that the union of the associated primes is the set of zero divisors. We conclude that I contains some regular element on M.

In the graded case, recall that all associated primes of a graded module must be homogeneous, and thus they are all contained in the homogeneous maximal ideal $\mathfrak{m}$. Thus in the local/graded case, $\mathfrak{m} \in \operatorname{Ass}(M)$ if and only if $\mathfrak{m}$ is contained in some associated prime of M. We conclude that $\mathfrak{m} \in \operatorname{Ass}(M)$ if and only if there are no regular elements on M. □

We can construct maximal regular sequences explicitly.

Construction 2.6. Let R be a noetherian ring, I an ideal in R, and $M \neq 0$ a finitely generated R-module with $IM \neq M$. To construct a regular sequence on M inside I, we start by finding a regular element on M inside I. Either I is contained in some associated prime of M, in which case every element in I is a zerodivisor on M, or there exists an element $x_1 \in I$ not in any associated prime of M, by Prime Avoidance, Lemma 3.29 from Commutative Algebra I. In the graded case, homogeneous Prime Avoidance guarantees we can find such x_1 that is homogeneous. Such an element x_1 is regular on M, since the union of the associated primes of M is precisely the set of zerodivisors, by Theorem 6.27 from Commutative Algebra I.

Now we repeat the process: either I is contained in some associated prime of $M/(x_1)M$, in which case there are no regular elements on $M/(x_1)M$ inside I, or we can find $x_2 \in I$ not in any associated prime of $M/(x_1)M$, so x_2 is necessarily regular on $M/(x_1)M$. At each step, $(x_1, \dots, x_i)M \subsetneq (x_1, \dots, x_{i+1})M$, and since M is noetherian, the process must stop. In the graded setting, we have in particular proved we can construct a maximal regular sequence that consists entirely of homogeneous elements.

In fact, any regular sequence on M inside of I leads to such an increasing sequence of submodules of M, so all such sequences are finite.

Lemma 2.7. *Let R be a noetherian ring and M and N be finitely generated R-modules. If $a \in \operatorname{ann}_R(M)$ and $b \in \operatorname{ann}_R(N)$, then $a \operatorname{Ext}^i_R(M,N) = 0$ and $b \operatorname{Ext}^i_R(M,N) = 0$ for all i.*

Proof. When $i = 0$, we want to show that $a \in \operatorname{ann}(\operatorname{Hom}_R(M,N))$ and $b \in \operatorname{ann}(\operatorname{Hom}_R(M,N))$. Given any $f \in \operatorname{Hom}_R(M,N)$ and any $m \in M$, we know $am = 0$, so

$$af(m) = f(am) = f(0) = 0.$$

We conclude that $af = 0$. Moreover, b kills every element in N, so $bf = 0$. Now let $P \longrightarrow M$ be a projective resolution on M, and $N \longrightarrow E$ be an injective resolution of N. Then

$$\begin{aligned} \operatorname{Ext}^i_R(M,N) &= \mathrm{H}^i\left(0 \longrightarrow \operatorname{Hom}_R(P_0,N) \longrightarrow \operatorname{Hom}_R(P_1,N) \longrightarrow \operatorname{Hom}_R(P_2,N) \longrightarrow \cdots \right) \\ &= \mathrm{H}^i\left(0 \longrightarrow \operatorname{Hom}_R(M,E^0) \longrightarrow \operatorname{Hom}_R(M,E^1) \longrightarrow \operatorname{Hom}_R(M,E^2) \longrightarrow \cdots \right), \end{aligned}$$

so $\operatorname{Ext}^i_R(M,N)$ is a subquotient of both $\operatorname{Hom}_R(P_i,N)$ and $\operatorname{Hom}_R(M,E^i)$. We conclude that a and b both kill $\operatorname{Ext}^i_R(M,N)$. □

Theorem 2.8. *Let R be a noetherian ring, I be an ideal in R, and M be a finitely generated R-module with $M \neq IM$. All maximal regular sequences on M inside I have the same length.*

Proof. Let $x_1, \dots, x_n \in I$ and $y_1, \dots, y_\ell \in I$ both be maximal regular sequences on M, and assume $n \leqslant \ell$.

When $n = 0$, every element in I is a zerodivisor on M, so $\ell = 0$.

When $n = 1$, we cannot extend the regular sequence x_1 further, so every element in I is a zerodivisor on $M/(x_1)M$. By Lemma 2.5, $I \subseteq P$ for some $P \in \operatorname{Ass}(M/(x_1)M)$. Thus there exists some $m \in M$, $m \notin (x_1)M$, such that $I \subseteq ((x_1)M :_R m)$, so $Im \subseteq (x_1)M$.

In particular, since $y_1 \in I$, we have $y_1 m = x_1 a$ for some $a \in M$. If $a \in (y_1)M$, then we would have

$$y_1 m = x_1 a \in (x_1 y_1)M.$$

Thus for some $n \in M$, we have

$$y_1 m = y_1 x_1 n \implies y_1(m - x_1 n) = 0.$$

Since y_1 is regular on M, that would imply $m \in (x_1)M$, which is a contradiction. We conclude that $a \notin (y_1)M$. Moreover,

$$(x_1)I \cdot a = I(x_1 a) = I(y_1 m) = y_1(Im) \subseteq (y_1)(x_1)M = (x_1)(y_1)M.$$

Since x_1 is a regular element on M, we must have $Ia \subseteq (y_1)M$. Therefore, $a \in M$ is an element that both satisfies $a \notin (y_1)M$ and $Ia \subseteq (y_1)M$, so every element in I kills a in $M/(y_1)M$. Therefore, every element in I is a zerodivisor on $M/(y_1)M$. This proves we cannot extend the regular sequence y_1, and thus $\ell = 1$.

We proceed by induction on n. Now assume that $n > 1$ and $\ell > n$. In particular, I contains a regular element on $M/(x_1, \dots, x_i)$ for all $i < n$ and a regular element on $M/(y_1, \dots, y_j)$ for all $j < \ell$, so by Prime Avoidance[1] we can pick $c \in I$ that avoids both all the (finitely many) associated primes of $M/(x_1, \dots, x_i)M$ for all $i < n$ and $M/(y_1, \dots, y_j)M$ for all $j < \ell$. In particular, $x_1, \dots, x_{n-1}, c$ and $y_1, \dots, y_n, c$ are both regular sequences on M. Now x_n and c are both regular sequences on $M/(x_1, \dots, x_{n-1})$, so the case $n = 1$ says $x_1, \dots, x_{n-1}, c$ is also a maximal regular sequence on M. Now by Lemma 1.51, $x_1, \dots, c, x_{n-1}$ is also a regular sequence on M, since c is also regular on $M/(x_1, \dots, x_{n-2})$, and so on, until we conclude that $c, x_1, \dots, x_{n-1}$ is a regular sequence on M. Similarly, $c, y_1, \dots, y_n$ is a regular sequence on M. Notice in fact that $c, x_1, \dots, x_{n-1}$ is maximal inside I, or else we could increase its size, move c back to after x_{n-1}, and obtain a contradiction. Therefore, $x_1, \dots, x_{n-1}$ and $c, y_1, \dots, y_n$ are both regular sequences on $M/(c)M$, and $x_1, \dots, x_{n-1}$ is maximal. But by induction hypothesis, all maximal regular sequences on $M/(c)M$ inside I have the same length, which would say that the length of $y_1, \dots, y_n$, n, is at most $n - 1$. This is a contradiction, so we conclude that $\ell = n$. $\square$

Remark 2.9. Theorem 2.8 tells us that $\operatorname{depth}_I(M)$ is not just the largest length of a regular sequence on M inside I: it is in fact the length of *any* maximal regular sequence on M inside I, where maximal simply means we cannot extend it to be any longer.

[1]See Lemma 3.29 from Commutative Algebra I.

We will now show that depth can be described in a purely homological way.

Theorem 2.10. *Let R be a noetherian ring and M a finitely generated R-module. Then*

$$\operatorname{depth}_I(M) = \min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

Proof. When $\operatorname{depth}_I(M) = 0$, there is no regular sequence on M inside I. By Lemma 2.5, $I \subseteq P$ for some $P \in \operatorname{Ass}(M)$. We have an inclusion $R/P \hookrightarrow M$, so consider the composition

$$R/I \twoheadrightarrow R/P \hookrightarrow M.$$

This is a nonzero homomorphism of R-modules, so $\operatorname{Ext}^0(R/I, M) = \operatorname{Hom}(R/I, M) \neq 0$. On the other hand, if $\operatorname{Hom}_R(R/I, M) = \operatorname{Ext}^0(R/I, M) \neq 0$, then there exists a nonzero R-module homomorphism $R/I \longrightarrow M$. But to choose an R-module homomorphism $R/I \longrightarrow M$ is the same as choosing an element in M that is killed by I, so I contains no nonzero divisors on M and $\operatorname{depth}_I(M) = 0$.

We proceed by induction on $\operatorname{depth}_I(M) = n$, assuming the statement holds whenever $\operatorname{depth}_I(M) < n$. Let $x_1, \ldots, x_n$ be a maximal regular sequence on M inside I. Then $x_2, \ldots, x_n$ is a maximal regular sequence on $M/(x_1)M$, so by induction we know

$$n - 1 = \min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

Applying $\operatorname{Hom}_R(R/I, -)$ to the short exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot x_1} M \longrightarrow M/(x_1)M \longrightarrow 0$$

we get a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^{i-1}_R(R/I, M/(x_1)M) \longrightarrow \operatorname{Ext}^i_R(R/I, M) \xrightarrow{x_1} \operatorname{Ext}^i_R(R/I, M) \longrightarrow \cdots.$$

We know $\operatorname{Ext}^{n-1}_R(R/I, M/(x_1)M) \neq 0$ and $\operatorname{Ext}^i_R(R/I, M/(x_1)M) = 0$ for $i < n-1$. Therefore, whenever $i < n - 1$,

$$\operatorname{Ext}^i_R(R/I, M) \xrightarrow{x_1} \operatorname{Ext}^i_R(R/I, M)$$

is an isomorphism. However, $x_1 \in I = \operatorname{ann}(R/I)$, so $\operatorname{ann}(R/I) \subseteq \operatorname{ann}(\operatorname{Ext}^i_R(R/I, M))$ by Lemma 2.7. Therefore, $\operatorname{Ext}^i_R(R/I, M) = 0$ for all $i < n-1$. Moreover, multiplication by x_1 is the zero map on $\operatorname{Ext}^i_R(R/I, M)$ for any i, also by Lemma 2.7. Finally, we have an exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^{n-1}_R(R/I, M) \xrightarrow{x_1} \operatorname{Ext}^{n-1}_R(R/I, M) \longrightarrow \operatorname{Ext}^{n-1}_R(R/I, M/(x_1)M) \longrightarrow \cdots.$$

where the multiplication by x_1 maps are 0, so our exact sequence is

$$0 \to \operatorname{Ext}^{n-1}_R(R/I, M) \xrightarrow{0} \operatorname{Ext}^{n-1}_R(R/I, M) \to \underbrace{\operatorname{Ext}^{n-1}_R(R/I, M/x_1 M)}_{\neq 0} \to \operatorname{Ext}^n_R(R/I, M) \xrightarrow{0} \cdots.$$

In particular, $\operatorname{Ext}^{n-1}(R/I, M) = 0$ and $\operatorname{Ext}^n(R/I, M) \neq 0$. We conclude that

$$\operatorname{depth}_I(M) = n = \min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}. \quad \square$$

For yet another homological characterization of depth, we turn to Koszul homology.

Theorem 2.11 (Depth sensitivity of the Koszul complex)**.** *Let R be a noetherian ring and M be finitely generated R-module. Given any $\underline{x} = x_1, \dots, x_n$ such that $(\underline{x})M \neq M$,*

$$\operatorname{depth}_{(\underline{x})}(M) = \max\{r \mid \mathrm{H}_i(\underline{x}; M) = 0 \textit{ for all } i > n - r\}.$$

So we can measure $\operatorname{depth}_I(M)$ by looking at the first nonzero Koszul homology we see when we start counting *from the top*.

$$\begin{array}{lcccccccccc} \operatorname{kos}(\underline{x}; M) & 0 \to & M & \to \cdots \to & M^{\binom{n}{n-r+1}} & \to & M^{\binom{n}{n-r}} & \to \cdots \to & M & \to 0 \\ \mathrm{H}(\underline{x}; M) & & 0 & \cdots & 0 & & \neq 0. & & & \end{array}$$

Proof. We are going to show that if $I = (\underline{x})$ contains elements $\underline{y} = y_1, \dots, y_m$ such that $\underline{y}$ is a regular sequence on M, then

$$\mathrm{H}_{n-i+1}(\underline{x}; M) = 0 \text{ for all } i = 1, \dots, m \text{ and } \mathrm{H}_{n-m}(\underline{x}; M) \cong \operatorname{Ext}^m_R(R/I, M).$$

This will prove the theorem: depth is the largest possible m we could take, and Theorem 2.10 says $\operatorname{Ext}^{\operatorname{depth}}_R(R/I, M) \neq 0$.

We proceed by induction on m. When $m = 0$, Theorem 1.35 says that

$$\mathrm{H}_n(\underline{x}; M) \cong (0 :_M I) \cong \operatorname{Hom}_R(R/I, M) \cong \operatorname{Ext}^m_R(R/I, M),$$

and we are done. When $m > 0$, the short exact sequence

$$0 \longrightarrow M \xrightarrow{y_1} M \longrightarrow M/(y_1)M \longrightarrow 0$$

induces a long exact sequence in koszul homology (see Theorem 1.35)

$$\cdots \longrightarrow \mathrm{H}_{i+1}(\underline{x}; M/(y_1)M) \longrightarrow \mathrm{H}_i(\underline{x}; M) \xrightarrow{y_1} \mathrm{H}_i(\underline{x}; M) \longrightarrow \mathrm{H}_i(\underline{x}; M/(y_1)M) \longrightarrow \cdots.$$

Now by Theorem 1.35, I kills $\mathrm{H}_i(\underline{x}; M)$, so the multiplication by y_1 map is zero in the long exact sequence above, which must then break into short exact sequences

$$0 \longrightarrow \mathrm{H}_i(\underline{x}; M) \longrightarrow \mathrm{H}_i(\underline{x}; M/(y_1)M) \longrightarrow \mathrm{H}_{i-1}(\underline{x}; M) \longrightarrow 0.$$

Since $y_2, \dots, y_m$ form a regular sequence of $m - 1$ elements on $M/(y_1)M$ inside $(\underline{x})$, so by induction hypothesis

$$\mathrm{H}_{n-i+1}(\underline{x}; M/(y_1)M) = 0 \text{ for all } i = 1, \dots, m-1$$

and

$$\mathrm{H}_{n-m+1}(\underline{x}; M/(y_1)M) \cong \operatorname{Ext}^{m-1}_R(R/I, M/(y_1)M).$$

Therefore, for all $i \leqslant m-1$, we have an exact sequence

$$0 \longrightarrow \mathrm{H}_{n-i+1}(\underline{x};M) \longrightarrow \underbrace{\mathrm{H}_{n-i+1}(\underline{x};M/(y_1)M)}_{0} \longrightarrow \mathrm{H}_{n-i}(\underline{x};M) \longrightarrow 0.$$

This implies the other two terms are zero also. The left hand side gives us $\mathrm{H}_{n-i+1}(\underline{x};M)=0$ for all $i=1,\dots,m-1$. The right hand side gives us $\mathrm{H}_{n-m+1}(\underline{x};M)=\mathrm{H}_{n-(m-1)}(\underline{x};M)=0$. Therefore, $\mathrm{H}_{n-i+1}(\underline{x};M)=0$ for all $i=1,\dots,m$.

Moreover, we have a short exact sequence

$$0 \longrightarrow \underbrace{\mathrm{H}_{n-m+1}(\underline{x};M)}_{0} \longrightarrow \mathrm{H}_{n-m+1}(\underline{x};M/(y_1)M) \longrightarrow \mathrm{H}_{n-m}(\underline{x};M) \longrightarrow 0.$$

so by induction hypothesis

$$\mathrm{H}_{n-m}(\underline{x};M) \cong \mathrm{H}_{n-m+1}(\underline{x};M/(y_1)M) \cong \mathrm{Ext}^{m-1}_R(R/I,M/(y_1)M).$$

Finally, we claim that

$$\mathrm{Ext}^m_R(R/I,M) \cong \mathrm{Ext}^{m-1}_R(R/I,M/(y_1)M).$$

To show that, consider the short exact sequence

$$0 \longrightarrow M \xrightarrow{y_1} M \longrightarrow M/(y_1)M \longrightarrow 0,$$

and the long exact sequence induced by applying $\mathrm{Ext}^i_R(R/I,-)$:

$$\cdots \longrightarrow \mathrm{Ext}^i_R(R/I,M) \xrightarrow{y_1} \mathrm{Ext}^i_R(R/I,M) \longrightarrow \mathrm{Ext}^i_R(R/I,M/(y_1)M) \longrightarrow \cdots.$$

We assumed we have a regular sequence on M of length m inside I, so $\operatorname{depth}_I(M) \geqslant m$. Therefore, $\mathrm{Ext}^{m-1}_R(R/I,M)=0$ by Theorem 2.10. Since $y_1 \in I$, by Lemma 2.7 multiplication by y_1 on $\mathrm{Ext}^i_R(R/I,M)$ is the zero map, so we get an exact sequence

$$0 \longrightarrow \mathrm{Ext}^{m-1}_R(R/I,M/(y_1)M) \longrightarrow \mathrm{Ext}^m_R(R/I,M) \longrightarrow 0.$$

Therefore,

$$\mathrm{Ext}^m_R(R/I,M) \cong \mathrm{Ext}^{m-1}_R(R/I,M/(y_1)M),$$

which finishes our proof. □

Remark 2.12. If $I=(x_1,\dots,x_n)$ and $\operatorname{depth}_I(M)=n$, then by Theorem 2.11 the Koszul complex $\operatorname{kos}(\underline{x};M)$ must be exact. This does not necessarily say that $\underline{x}$ is a regular sequence, only that there exists some regular sequence on M of length n inside I. However, that implication does hold in the local or graded setting, by Theorem 1.49.

Another useful property of depth is how it behaves across regular sequences; we leave the proof for the next problem set.

Theorem 2.13 (The Depth Lemma)**.** *Let R be a noetherian local ring and consider a short exact sequence of nonzero finitely generated R-modules*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Then the following hold:

$$\begin{aligned} \operatorname{depth}(B) &\geqslant \min\{\operatorname{depth}(A), \operatorname{depth}(C)\} \\ \operatorname{depth}(A) &\geqslant \min\{\operatorname{depth}(B), \operatorname{depth}(C)+1\} \\ \operatorname{depth}(C) &\geqslant \min\{\operatorname{depth}(B), \operatorname{depth}(A)-1\}. \end{aligned}$$

Lemma 2.14. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and $M \neq 0$ a finitely generated R-module of finite projective dimension. If $x \in R$ is regular on both R and M, then $M/(x)M$ has finite projective dimension over $R/(x)$, given by*

$$\operatorname{pdim}_{R/(x)}(M/(x)) = \operatorname{pdim}_R(M).$$

Proof. Since x is regular on R, the quotient $R/(x)$ is resolved by the Koszul complex, by Corollary 1.47, so $\operatorname{pdim}(R/(x)) = 1$. Therefore, $\operatorname{Tor}_i^R(R/(x), M) = 0$ for all $i \geqslant 2$. Let

$$0 \longrightarrow R^{\beta_n} \xrightarrow{\varphi_n} R^{\beta_{n-1}} \xrightarrow{\varphi_{n-1}} \cdots \longrightarrow R^{\beta_1} \xrightarrow{\varphi_1} R^{\beta_0} \xrightarrow{\varphi_0} M \longrightarrow 0$$

be a minimal free resolution for M. In particular, if we choose basis for each R^{β_i}, the entries in the matrices representing φ_i are all in $\mathfrak{m}$. Applying $-\otimes_R R/(x)$, we get a complex

$$0 \longrightarrow (R/(x))^{\beta_n} \longrightarrow (R/(x))^{\beta_{n-1}} \longrightarrow \cdots \longrightarrow (R/(x))^{\beta_0} \longrightarrow M/xM \longrightarrow 0$$

which is exact at $M \otimes_R R/(x) \cong M/xM$, since tensor is right exact, and whose homology is otherwise given by $\operatorname{Tor}_i^R(R/(x), M)$. In particular, our complex is exact for all $i \geqslant 2$, since we have seen that $\operatorname{Tor}_i^R(R/(x), M) = 0$ for all $i \geqslant 2$. The only remaining homology with a chance of being interesting is given by

$$\operatorname{Tor}_1^R(R/(x), M) = \mathrm{H}_1(M \otimes (0 \to R \xrightarrow{x} R \to 0)) = \mathrm{H}_1(0 \to M \xrightarrow{x} M \to 0) = (0 :_M x).$$

By assumption, x is regular on M, so $\operatorname{Tor}_1^R(R/(x), M) = (0 :_M x) = 0$. So the complex above is exact, and thus a free resolution for M/xM over $R/(x)$. In fact, the maps in this free resolution for $M/(x)M$ were obtained by tensoring φ with $R/(x)$, so we can obtain matrices representing each map by taking the matrix representing φ_i and setting all the entries in (x) equal to 0. In particular, all the entries are still in $\mathfrak{m}/(x)$, and our resolution for M/xM over $R/(x)$ is minimal. We conclude that

$$\operatorname{pdim}_{R/(x)}(M/xM) = \operatorname{pdim}_R(M). \qquad \square$$

We now have all the tools we need to prove a very useful formula relating depth and projective dimension.

Theorem 2.15 (Auslander–Buchsbaum Formula)**.** *Let $(R,\mathfrak{m},k)$ be a noetherian local ring and $M \neq 0$ a finitely generated R-module of finite projective dimension. Then*

$$\operatorname{depth}(M) + \operatorname{pdim}_R(M) = \operatorname{depth}(R).$$

Proof. Suppose $\operatorname{depth}(R) = 0$. In that case, the claim is that $\operatorname{pdim}(M) = 0 = \operatorname{depth}(M)$. First, note that the fact that $\operatorname{depth}(R) = 0$ implies immediately that $\mathfrak{m} \in \operatorname{Ass}(R)$, by Lemma 2.5, so $\mathfrak{m}$ kills some nonzero $r \in R$. Consider a minimal free resolution for M, say

$$0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0.$$

Suppose $n > 0$, so that $\varphi_n \neq 0$. By minimality, $\varphi_n(F_n) \subseteq \mathfrak{m}F_{n-1}$, so

$$\varphi_n(r,0,\cdots,0) = r\varphi_n(1,0,\cdots,0) \in r\mathfrak{m} = 0$$

and φ_n is not injective. This is a contradiction, so we must have $n = 0$, and M is free, say $M \cong R^n$. Therefore, $\operatorname{pdim}_R(M) = 0$ and $\operatorname{depth}(M) = \operatorname{depth}(R^n) = \operatorname{depth}(R) = 0$. This proves the formula holds whenever $\operatorname{depth}(R) = 0$.

Now assume that $\operatorname{depth}(M) = 0$. Set $n = \operatorname{pdim}(M)$ and $t = \operatorname{depth}(R)$, and fix a maximal regular sequence $x_1, \ldots, x_t \in \mathfrak{m}$. By Corollary 1.47, $\operatorname{pdim}(R/(x_1,\ldots,x_t)) = t$. Our goal is to show that $n = t$. Notice that $\operatorname{Tor}_i^R(R/(x_1,\ldots,x_t), M)$ can be computed via minimal free resolutions for either M or $R/(x_1,\ldots,x_t)$, so $\operatorname{Tor}_i^R(R/(x_1,\ldots,x_t), M)$ vanishes for $i > \min\{t,n\}$. We are going to show that both $\operatorname{Tor}_t^R(R/(x_1,\ldots,x_t), M) \neq 0$ and $\operatorname{Tor}_n^R(R/(x_1,\ldots,x_t), M) \neq 0$, which proves that $n = t$.

By Corollary 1.47, the Koszul complex is a minimal free resolution for $R/(x_1,\ldots,x_t)$, so the last map in the minimal free resolution looks like

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} x_1 & -x_2 & x_3 & \cdots & (-1)^{t+1}x_t \end{pmatrix}^T} R^t.$$

Applying $-\otimes_R M$ gives

$$0 \longrightarrow M \xrightarrow{\begin{pmatrix} x_1 & -x_2 & x_3 & \cdots & (-1)^{t+1}x_t \end{pmatrix}^T} M^t.$$

Therefore,

$$\operatorname{Tor}_t^R(R/(x_1,\ldots,x_t), M) = \bigcap_{i=1}^{t} \ker(M \xrightarrow{\pm x_i} M).$$

Our assumption that $\operatorname{depth}(M) = 0$ says that there are no regular elements on M, so by Lemma 2.5, we must have $\mathfrak{m} \in \operatorname{Ass}(M)$. Therefore, there exists a nonzero element $m \in M$ such that $\operatorname{ann}(m) = \mathfrak{m} \supseteq (x_1,\ldots,x_t)$, so

$$m \in \bigcap_{i=1}^{t} \ker(M \xrightarrow{\pm x_i} M) = \operatorname{Tor}_t^R(R/(x_1,\ldots,x_t), M)$$

and $\operatorname{Tor}_t^R(R/(x_1,\ldots,x_t), M) \neq 0$.

On the other hand, to compute $\operatorname{Tor}_n^R(R/(x_1, \dots, x_t), M)$ we can take a minimal free resolution of M, say

$$0 \longrightarrow R^{\beta_n} \xrightarrow{\varphi_n} R^{\beta_{n-1}} \xrightarrow{\varphi_{n-1}} \cdots \longrightarrow R^{\beta_1} \xrightarrow{\varphi_1} R^{\beta_0} \xrightarrow{\varphi_0} M \longrightarrow 0,$$

and apply $- \otimes_R R/(\underline{x})$, so that $\operatorname{Tor}_n^R(R/(x_1, \dots, x_t), M)$ is the kernel of

$$(R/(x_1, \dots, x_t))^{\beta_n} \longrightarrow (R/(x_1, \dots, x_t))^{\beta_{n-1}}.$$

Our assumption that $x_1, \dots, x_t$ is a maximal regular sequence on R implies that any other element in R is a zerodivisor on $R/(x_1, \dots, x_t)$, and $\operatorname{depth}(R/(x_1, \dots, x_t)) = 0$. In particular, $\mathfrak{m} \in \operatorname{Ass}(R/(x_1, \dots, x_t))$, so there exists some $r \notin (x_1, \dots, x_t)$ such that $\mathfrak{m}r \subseteq (x_1, \dots, x_t)$. The map

$$(R/(x_1, \dots, x_t))^{\beta_n} \longrightarrow (R/(x_1, \dots, x_t))^{\beta_{n-1}}$$

is given by multiplication by a matrix whose entries are all in $\mathfrak{m}$, so is nonzero, meaning $\operatorname{Tor}_n^R(R/(x_1, \dots, x_t), M) \neq 0$.

So we have shown the theorem holds in two situations: when $\operatorname{depth}(R) = 0$ and when $\operatorname{depth}(M) = 0$. So now we assume that both $t = \operatorname{depth}(R) > 0$ and $n = \operatorname{pdim}(M) > 0$, and assume we have shown the theorem holds when $\operatorname{depth}(R) \leqslant t-1$ and $\operatorname{pdim}(M) \leqslant n-1$.

By Prime Avoidance, Lemma 3.29 from Commutative Algebra I, we can find $x \in \mathfrak{m}$ that avoids both the associated primes of M and R, so x is both regular on M and on R. By Lemma 2.14,

$$\operatorname{pdim}_{R/(x)}(M/(x)) = \operatorname{pdim}_R(M).$$

Now notice that we picked x to be regular on M, so that $\operatorname{depth}(M/xM) = \operatorname{depth}(M) - 1$. Similarly, x is regular on R, so $\operatorname{depth}(R/(x)) = \operatorname{depth}(R) - 1$. Using the assumption that the formula holds for $M/(x)$ over $R/(x)$, we conclude that

$$\begin{aligned} &\operatorname{depth}(M/(x)M) + \operatorname{pdim}_{R/(x)}(M/(x)M) = \operatorname{depth}(R/(x)) \\ \iff &\operatorname{depth}(M) - 1 + \operatorname{pdim}_R(M) = \operatorname{depth}(R) - 1 \\ \iff &\operatorname{depth}(M) + \operatorname{pdim}_R(M) = \operatorname{depth}(R). \quad \square \end{aligned}$$

This formula is very useful. For example, when doing explicit computations, it is often easier to compute a minimal free resolution for M than to compute its depth. If we happen to know $\operatorname{depth}(R)$, one can deduce $\operatorname{depth}(M)$ by computing $\operatorname{pdim}_R(M)$.

Remark 2.16. One of the consequences of Theorem 2.15 is that if a finitely generated R-module M has finite projective dimension, then

$$\operatorname{pdim}(M) \leqslant \operatorname{depth}(R) \leqslant \dim R.$$

Remark 2.17. Suppose M is a free R-module. In particular, M has finite projective dimension 0, so by the Auslander–Buchsbaum formula we have

$$\operatorname{depth}(M) = \operatorname{depth}(R).$$

Remark 2.18. Another consequence of the Auslander–Buchsbaum formula is that over a noetherian local ring of depth zero, every finitely generated module is either free or has infinite projective dimension.

Remark 2.19. Let $(R, \mathfrak{m})$ be a noetherian local ring and let $x \in \mathfrak{m}$ be a regular element. The short exact sequence

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0$$

is an augmented minimal free resolution for $R/(x)$, so we must have $\operatorname{pdim}(R/(x)) = 1$. By the Auslander–Buchsbaum formula,

$$\operatorname{depth}(R/(x)) = \operatorname{depth}(R) - 1.$$

Of course this also follows immediately from the fact that all maximal regular sequences have the same length, as any maximal regular sequence of length n on R that contains x gives a maximal regular sequence of length $n-1$ on $R/(x)$, and lifting any maximal regular sequence of length $n-1$ on $R/(x)$ back to R will give us a regular sequence on R of length n.

There is also a graded version of the Auslander–Buchsbaum formula.

Definition 2.20. Let $(R, \mathfrak{m})$ be a graded ring. For any finitely generated graded R-module M, we set $\operatorname{depth}(M) = \operatorname{depth}_{\mathfrak{m}}(M)$, and call it simply the **depth** of M.

Remark 2.21. By Construction 2.6, we can always pick a maximal regular sequence on M that consists entirely of homogeneous elements. But all maximal regular sequences have the same length, by Theorem 2.8, so we can also describe $\operatorname{depth}(M)$ as the maximal length of a homogeneous regular sequence on M.

Remark 2.22. Let $(R, \mathfrak{m})$ be a graded ring, as in Setting 1, and let M be a nonzero finitely generated graded R-module. First, we claim that $\mathfrak{m}$ must be in the support of M. Indeed, by Theorem 6.10 from Commutative Algebra I, $\operatorname{Supp}(M)$ is the set of primes containing $\operatorname{ann}(M)$, and one can show that the annihilator of M must be a homogeneous ideal, and thus contained in $\mathfrak{m}$.

Now consider a minimal graded resolution (F, ∂) for M over R. Localizing at $\mathfrak{m}$ gives us a free resolution for $M_{\mathfrak{m}}$, which remains minimal as the image of $\partial_{\mathfrak{m}}$ is contained in $\mathfrak{m}_{\mathfrak{m}} F_{\mathfrak{m}}$. This shows that

$$\operatorname{pdim}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \operatorname{pdim}_R(M).$$

The Ext modules $\operatorname{Ext}^i_R(R/\mathfrak{m}, M)$ inherit a graded structure, and thus

$$\operatorname{Ext}^i_R(R/\mathfrak{m}, M) = 0 \iff \left(\operatorname{Ext}^i_R(R/\mathfrak{m}, M)\right)_{\mathfrak{m}} = 0.$$

By Theorem 6.35 from Homological Algebra, these Ext modules localize well, and

$$\left(\operatorname{Ext}^i_R(R/\mathfrak{m}, M)\right)_{\mathfrak{m}} \cong \operatorname{Ext}^i_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}, M_\mathfrak{m}).$$

Therefore,

$$\operatorname{Ext}^i_R(R/\mathfrak{m}, M) = 0 \iff \operatorname{Ext}^i_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}, M_\mathfrak{m}) = 0.$$

By Theorem 2.10, the vanishing of these Ext modules measures the depth of M over R and of $M_\mathfrak{m}$ over $R_\mathfrak{m}$. We conclude that $\operatorname{depth}(M)$ over R and $\operatorname{depth}(M_\mathfrak{m})$ over $R_\mathfrak{m}$ agree.

We can now deduce graded versions of the local results in this chapter. For example, the Auslander–Buchsbaum also holds in the graded setting:

$$\operatorname{depth}(M) + \operatorname{pdim}_R(M) = \operatorname{depth}(R).$$

Moreover, $\operatorname{depth}(R) \leqslant \operatorname{height}(\mathfrak{m}) \leqslant \dim(R)$.

We can also compare the depth of any module with its dimension. To prove that, we will need the following:

Theorem 2.23 (Ischebeck's Theorem)**.** *Let $(R, \mathfrak{m})$ be a noetherian local ring and M and N finitely generated R-modules. Then $\operatorname{Ext}^i_R(M, N) = 0$ for all $i < \operatorname{depth}(N) - \dim(M)$.*

Proof. We will give a proof by induction on $\dim(M)$. We will combine this induction with the following reduction: we claim that we can reduce the problem to all modules of the form R/P with P prime.

Let M be any finitely generated R-module and assume we have already proved the statement for all modules of the form R/P with P a prime and $\dim(R/P) \leqslant \dim(M)$. Fix a prime filtration of M, meaning we get an ascending chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$$

such that $M_i/M_{i-1} \cong R/P_i$ for some primes P_i. Such a prime filtration exists by Theorem 6.33 from Commutative Algebra I. Notice also that by the construction in Theorem 6.33 from Commutative Algebra I, all the P_i contain $\operatorname{ann}(M)$, so $\dim(R/P_i) \leqslant \dim(M)$.

Our assumption is that $\operatorname{Ext}^j_R(R/P_i, N) = 0$ for all $j < \operatorname{depth}(N) - \dim(R/P_i)$ for each P_i. But $\dim(R/P_i) \leqslant \dim(M)$, so in particular for all i we know that

$$\operatorname{Ext}^j_R(M_i/M_{i-1}, N) = 0 \quad \text{for } j < \operatorname{depth}(N) - \dim(M).$$

In particular, since $R/P_1 \cong M_1/M_0 \cong M_1$, we know that

$$\operatorname{Ext}^j_R(M_1, N) = 0 \quad \text{for } j < \operatorname{depth}(N) - \dim(M).$$

Now for each $i \geqslant 1$, break the filtration into short exact sequences

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow R/P_i \longrightarrow 0$$

and look at the long exact sequence we get when we apply $\operatorname{Hom}_R(-, N)$:

$$\cdots \to \operatorname{Ext}_R^{j-1}(R/P_i, N) \to \operatorname{Ext}_R^j(M_i, N) \to \operatorname{Ext}_R^j(M_{i-1}, N) \to \operatorname{Ext}_R^j(R/P_i, N) \to \cdots.$$

If $j = 0 < \operatorname{depth}(N) - \dim(M)$, then $\operatorname{Ext}_R^0(R/P_i, N) = 0$ and the long exact sequence gives us

$$0 \longrightarrow \operatorname{Ext}_R^0(M_i, N) \longrightarrow \operatorname{Ext}_R^0(M_{i-1}, N) \longrightarrow 0,$$

and thus an isomorphism $\operatorname{Ext}_R^j(M_i, N) \cong \operatorname{Ext}_R^j(M_{i-1}, N)$. For $1 \leqslant j < \operatorname{depth}(N) - \dim(M)$, we know that $\operatorname{Ext}_R^j(R/P_i, N) = 0 = \operatorname{Ext}_R^{j-1}(R/P_i, N)$, so we also get an isomorphism

$$\operatorname{Ext}_R^j(M_i, N) \cong \operatorname{Ext}_R^j(M_{i-1}, N).$$

But $\operatorname{Ext}_R^j(M_1, N) = 0$, so inductively we conclude that $\operatorname{Ext}_R^j(M_i, N) = 0$ for all i. In particular, since $M = M_n$, we conclude that

$$\operatorname{Ext}_R^j(M_i, N) = 0 \text{ for all } j < \operatorname{depth}(N) - \dim(M).$$

Now that we have this reduction at our disposal, we proceed to prove the statement for a general M by induction.

When $\dim(M) = 0$, the reduction argument above tells us we need only to consider the case when $M = R/P$ and $\dim(R/P) = 0$. In that case, we must have $P = \mathfrak{m}$, so by Theorem 2.10 we have $\operatorname{Ext}_R^i(R/\mathfrak{m}, N) = 0$ for all $i < \operatorname{depth}(N) = \operatorname{depth}(N) - \dim(R/P)$.

Suppose that for some $d \geqslant 1$, we have proved the statement holds for all R-modules of dimension $d-1$, and let $\dim(M) = d$. Again, we need only to prove the statement holds for $M = R/P$. Since $\dim(R/P) > 0$, then $P \neq \mathfrak{m}$, so we can pick $x \in \mathfrak{m}$ with $x \notin P$. Since the image of x in the domain R/P is nonzero, we know by Exercise 12 that

$$\dim(R/(P + (x)) = \dim(R/P) - 1.$$

By induction hypothesis,[2] we have $\operatorname{Ext}_R^i(R/(P + (x)), N) = 0$ for all

$$i < \operatorname{depth}(N) - \dim(R/P + (x)) = \operatorname{depth}(N) - \dim(R/P) + 1.$$

Moreover, the short exact sequence

$$0 \longrightarrow R/P \xrightarrow{x} R/P \longrightarrow R/(P + (x)) \longrightarrow 0$$

[2]Note that this is the step that required us to set up our induction so carefully, as $P + (x)$ is not necessarily a prime ideal anymore.

gives rise to the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_R\left(\tfrac{R}{P+(x)}, N\right) \longrightarrow \operatorname{Ext}^i_R(R/P, N) \xrightarrow{x} \operatorname{Ext}^i_R(R/P, N) \longrightarrow \operatorname{Ext}^{i+1}_R\left(\tfrac{R}{P+(x)}, N\right) \longrightarrow \cdots.$$

Therefore, for all $i < \operatorname{depth}(N) - \dim(R/P)$ we have both

$$\operatorname{Ext}^i_R(R/(P+(x)), N) = 0 \operatorname{Ext}^{i+1}_R(R/P+(x), N).$$

Thus the exact sequence above gives us an isomorphism

$$\operatorname{Ext}^i_R(R/P, N) \xrightarrow{x} \operatorname{Ext}^i_R(R/P, N).$$

In particular, $\operatorname{Ext}^i_R(R/P, N) = x \cdot \operatorname{Ext}^i_R(R/P, N)$. Moreover, by Theorem 6.36 from Homological Algebra, the module $\operatorname{Ext}^i_R(R/P, N)$ is finitely generated, so $\operatorname{Ext}^i_R(R/P, N) = 0$ by NAK (see Theorem 5.32 from Commutative Algebra I). This completes the proof of the case $M = R/P$ for a prime ideal P. □

We already know that $\operatorname{depth}(R) \leqslant \dim(R)$. We can now obtain a generalization of this inequality for all R-modules.

Corollary 2.24. *Let $(R, \mathfrak{m})$ be a noetherian local ring and M a finitely generated R-module. For every associated prime P of M, $\operatorname{depth}(M) \leqslant \dim(R/P)$. In particular,*

$$\operatorname{depth}(M) \leqslant \dim(M).$$

Proof. Let P be an associated prime of M. By Theorem 2.23, $\operatorname{Ext}^i_R(R/P, M) = 0$ for all $i < \operatorname{depth}(M) - \dim(R/P)$. But every element in P is a zerodivisor on M, so $\operatorname{depth}_P(M) = 0$. Moreover, by Theorem 2.10, $\operatorname{Ext}^0_R(R/P, M) \neq 0$. We conclude that $\operatorname{depth}(M) \leqslant \dim(R/P)$. Finally, note that

$$\begin{aligned} \dim(M) &= \max\{\dim(R/P) \mid P \in \operatorname{Min}(M)\} \\ &= \max\{\dim(R/P) \mid P \in \operatorname{Ass}(M)\} \end{aligned}$$

so $\operatorname{depth}(M) \leqslant \dim(M)$. □

Next, we prove a useful lemma we will use later:

Lemma 2.25. *Let $(R, \mathfrak{m})$ be a noetherian local ring and M a finitely generated R-module. Given any ideal I in R, $\operatorname{depth}_I(M) = \operatorname{depth}_{\sqrt{I}}(M)$.*

Proof. On the one hand, $I \subseteq \sqrt{I}$, so

$$\operatorname{depth}_I(M) \leqslant \operatorname{depth}_{\sqrt{I}}(M).$$

On the other hand, if $x_1, \ldots, x_n$ is a maximal regular sequence on M inside $\sqrt{I}$, then there exists $a_1, \ldots, a_n > 0$ such that $x_1^{a_1}, \ldots, x_n^{a_n} \in I$, but by Lemma 1.52 $x_1^{a_1}, \ldots, x_n^{a_n}$ is a regular sequence on M. □

Exercise 18. Let R be a noetherian ring and W be a multiplicatively closed subset. Show that if $x_1, \dots, x_n \notin W$ form a regular sequence in R, then $\frac{x_1}{1}, \dots, \frac{x_n}{1}$ is also a regular sequence in $W^{-1}R$.

Exercise 19. Let R be a noetherian ring, and let P be a prime ideal containing an ideal I.

1) Show that $\operatorname{grade}(I_P) \geqslant \operatorname{grade}(I)$.

2) Give an example where $\operatorname{grade}(I_P) > \operatorname{grade}(I)$.

2.2 Cohen-Macaulay rings

> Life is really worth living in a noetherian ring R when all the local rings have the property that every system of parameters is an R-sequence. Such a ring is called Cohen-Macaulay (C-M for short).
>
> Mel Hochster, page 887 of [Hoc78]

Irvin Cohen and Francis Macaulay were two big influences in the early days of commutative algebra. Cohen-Macaulay rings, named after the two of them, are by some measure the largest class of nice rings commutative algebraists study. They are on the border of being just nice enough to make life is easier, and just broad enough to contain many interesting examples. One of the main reference books in any commutative algebraist's shelf is dedicated to Cohen-Macaulay rings specifically [BH93]. In this section, we will see some of the reasons why life really is worth living in a Cohen-Macaulay ring.

Given a local ring R,

$$\operatorname{depth}(R) \leqslant \dim(R) \leqslant \operatorname{embdim}(R).$$

When the second inequality is an equality, we have a regular ring. When the first inequality is an equality, our ring is Cohen-Macaulay.

More generally, for an R-module M, we proved in Corollary 2.24 that

$$\operatorname{depth}(M) \leqslant \dim(M).$$

When equality happens, we say M is a Cohen-Macaulay module.

Definition 2.26. A noetherian local ring R is **Cohen-Macaulay** if $\operatorname{depth}(R) = \dim(R)$. More generally, an R-module M is **Cohen-Macaulay** if $\operatorname{depth}(M) = \dim(M)$. We say that M is a **maximal Cohen-Macaulay module**, or **MCM** for short, if $\operatorname{depth}(M) = \dim(R)$.

Remark 2.27. A local ring is Cohen-Macaulay if it is a Cohen-Macaulay module over itself.

Example 2.28. All artinian rings are Cohen-Macaulay, as the inequalities

$$0 = \dim(R) \geqslant \operatorname{depth}(R)$$

automatically imply $\operatorname{depth}(R) = 0$.

Theorem 2.29. *Regular local rings are Cohen-Macaulay.*

Proof. Let $(R, \mathfrak{m})$ be a regular local ring. By Auslander–Buchsbaum–Serre, Theorem 1.71, $\mathfrak{m}$ is generated by a regular sequence of length $\dim(R)$, so $\operatorname{depth}(R) = \dim(R)$. □

But not all Cohen-Macaulay rings are regular.

Example 2.30. The artinian local ring $k[x]/(x^2)$ is Cohen-Macaulay but not regular.

Example 2.31. Every 1-dimensional domain is Cohen-Macaulay, since any nonzero nonunit is a regular element.

Exercise 20. Show that $R = k[\![x, y, z]\!]/(xy, xz)$ is not Cohen-Macaulay.

Exercise 21. Let $R = k[\![x, y]\!]/(x^2, xy)$ and $M = R/(x)$. Show M is an MCM module but R is not Cohen-Macaulay.

Remark 2.32. If M is a finitely generated R-module of finite projective dimension, then by the Auslander–Buchsbaum formula, Theorem 2.15, we have $\operatorname{depth}(M) \leqslant \operatorname{depth}(R)$. If M is a finitely generated MCM module of finite projective dimension, then the inequalities

$$\operatorname{depth}(M) \leqslant \operatorname{depth}(R) \leqslant \dim(R)$$

must be equalities throughout, and thus R itself must be Cohen-Macaulay. Moreover, by the Auslander–Buchsbaum formula, Theorem 2.15,

$$\operatorname{pdim}(M) = \operatorname{depth}(R) - \operatorname{depth}(M) = 0.$$

Thus M must be free.

Theorem 2.33. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let $\underline{x} = x_1, \ldots, x_n$ be a regular sequence. The ring R is Cohen-Macaulay if and only if $R/(\underline{x})$ is Cohen-Macaulay.*

Proof. It is sufficient to prove the case when $n = 1$, since we can then apply the statement repeatedly. Let x be a regular element on R. By Exercise 12, $\dim(R/(x)) = \dim(R) - 1$. By Remark 2.19, $\operatorname{depth}(R/(x)) = \operatorname{depth}(R) - 1$. Thus R is Cohen-Macaulay if and only if $R/(x)$ is Cohen-Macaulay. $\square$

Many rings with *nice* singularities are Cohen-Macaulay. For example, Hochster and Roberts famously showed [HR74] that rings of invariants of any finite group G over a field k of characteristic not dividing $|G|$ are Cohen-Macaulay. Their proof used prime characteristic techniques, introducing what is now a very important class of characteristic p singularities, and which are essentially homological in nature.

Hochster's quote above pointed us to another characterization of Cohen-Macaulayness, for which we will need to recall the notion of a system of parameters.

Definition 2.34. Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension d. A sequence of d elements $x_1, \ldots, x_d$ is a **system of parameters** or **SOP** if $\sqrt{(x_1, \ldots, x_d)} = \mathfrak{m}$. We say that elements $x_1, \ldots, x_t$ are **parameters** if they are part of a system of parameters.

In the graded case, a **homogeneous system of parameters** is a system of parameters consisting only of homogeneous elements.

By Theorem 8.13 from Commutative Algebra I, every local (or graded) ring admits a system of parameters, and these can be useful in characterizing the dimension of a noetherian local ring, or the height of a prime in a noetherian ring. Most importantly to our current endeavours, we can characterize Cohen-Macaulayness in terms of SOPs.

Theorem 2.35. *Let $(R, \mathfrak{m})$ be a noetherian local ring. The following are equivalent:*

1) *R is Cohen-Macaulay.*

2) *Some system of parameters in R is a regular sequence on R.*

3) *Every system of parameters in R is a regular sequence on R.*

Proof. The implications $3 \Rightarrow 2 \Rightarrow 1$ are obvious.

To show $1 \Rightarrow 3$, suppose R is Cohen-Macaulay and let $\underline{x} = x_1, \ldots, x_d \in R$ be a system of parameters, meaning $\sqrt{(\underline{x})} = \mathfrak{m}$. By Lemma 2.25 and the fact that R is Cohen-Macaulay,

$$\operatorname{depth}_{(\underline{x})}(R) = \operatorname{depth}_{\mathfrak{m}}(R) = d.$$

By the depth sensitivity of the Koszul complex, Theorem 2.11, $\mathrm{H}_i(\underline{x}) = 0$ for all $i \geqslant 1$. By Theorem 1.49, this implies that $\underline{x}$ is a regular sequence. □

Theorem 2.36. *Let $(R, \mathfrak{m})$ be a noetherian local ring. For all Cohen-Macaulay R-modules M and all $P \in \operatorname{Ass}(M)$,*

$$\operatorname{depth}(M) = \dim(R/P).$$

In particular, if R is a Cohen-Macaulay local ring, then R has no embedded primes, and all maximal chains of primes in R have the same length: for all $P \in \operatorname{Min}(R) = \operatorname{Ass}(R)$,

$$\dim(R/P) = \dim(R).$$

Proof. By the inequality on depth and dimension of associated primes from Corollary 2.24,

$$\dim(M) = \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}(M)\} \geqslant \min\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}(M)\} \geqslant \operatorname{depth}(M).$$

Since M is Cohen-Macaulay, equality holds throughout, so $\dim(M) = \dim(R/P)$ for all $P \in \operatorname{Ass}(M)$. □

Theorem 2.36 implies that Cohen-Macaulay rings are equidimensional.

We saw in Remark 2.3 that we always have $\operatorname{grade}(I) \leqslant \operatorname{height}(I)$. We will be proving that equality holds for any ideal in a Cohen-Macaulay ring. We start by proving this for prime ideals, which will allow us to show that the Cohen-Macaulay property localizes.

Theorem 2.37. *Every prime ideal P a Cohen-Macaulay local ring has $\operatorname{grade}(P) = \operatorname{height}(P)$.*

Proof. By Remark 2.3, we know that $\operatorname{grade}(P) \leqslant \operatorname{height}(P)$. We will show by induction on the height of P that there is a regular sequence contained in P of length equal to the height of P. If P is a minimal prime, there is nothing to show.

Suppose that for some $h \geqslant 1$, the statement holds for all primes P of height $h - 1$ in any Cohen-Macaulay local ring. Let P be a prime of height h in a Cohen-Macaulay local ring R. Since P is not minimal, it is not contained in the union of the minimal primes, hence not in the union of the associated primes by Theorem 2.36. Thus, there is a regular element $x \in P$.

By Theorem 2.33, $R/(x)$ is a Cohen-Macaulay ring, and $\operatorname{height}(P/(x)) = \operatorname{height}(P) - 1 = h - 1$. By induction hypothesis, $P/(x)$ contains a regular sequence $\underline{y}$ of length $h - 1$, and thus P contains the regular sequence $x, \underline{y}$ of length h. □

We are now ready to show that the Cohen-Macaulay property localizes.

Theorem 2.38. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and P be a prime ideal in R. Then R_P is a Cohen-Macaulay local ring.*

Proof. By Theorem 2.37, $\operatorname{grade}(P) = \operatorname{height}(P)$. By Exercise 19, $\operatorname{grade}(P_P) \geqslant \operatorname{grade}(P)$. On the other hand, grade is always bounded above by height, by Remark 2.3, so

$$\operatorname{height}(P) = \operatorname{height}(P_P) \geqslant \operatorname{grade}(P_P) \geqslant \operatorname{grade}(P) = \operatorname{height}(P).$$

We conclude that

$$\operatorname{depth}(R_P) = \operatorname{grade}(P_P) = \operatorname{height}(P) = \dim(R_P)$$

and thus R_P is Cohen-Macaulay. □

We can now give a global definition of Cohen-Macaulayness.

Definition 2.39. A noetherian ring R is Cohen-Macaulay if R_P is Cohen-Macaulay for all primes P. An R-module M is Cohen-Macaulay if M_P is Cohen-Macaulay for all primes P.

Remark 2.40. Since localizations of local Cohen-Macaulay rings are Cohen-Macaulay by Theorem 2.38, to check that a ring R is Cohen-Macaulay it suffices to check that $R_\mathfrak{m}$ is Cohen-Macaulay for all maximal ideals $\mathfrak{m}$.

Corollary 2.41. *All regular rings are Cohen-Macaulay. In particular, any polynomial ring over a field is Cohen-Macaulay.*

Theorem 2.42 (Dimension formula)**.** *Let R be a Cohen-Macaulay ring, and let $P \subseteq Q$ be primes. Then*

$$\operatorname{height}(Q) - \operatorname{height}(P) = \dim(R_Q/PR_Q).$$

Proof. The localization R_Q is a Cohen-Macaulay local ring of dimension $\operatorname{height}(Q)$. Set $h = \operatorname{height}(P) = \operatorname{height}(PR_Q)$. By Theorem 2.37, we can pick h many elements $x_1, \dots, x_h \in P$ such that their image in R_Q is a regular sequence inside PR_Q.

Therefore, $R_Q/(x_1, \dots, x_h)R_Q$ is Cohen-Macaulay by Theorem 2.33, and by Exercise 12

$$\dim(R_Q/(x_1, \dots, x_h)R_Q) = \dim(R_Q) - h = \operatorname{height}(Q) - \operatorname{height}(P).$$

By Theorem 1.55, $\operatorname{height}(x_1, \dots, x_h) = h$, and since P has height h and contains $(x_1, \dots, x_h)$, P must be a minimal prime over $(x_1, \dots, x_h)$. Since $R_Q/(x_1, \dots, x_h)R_Q$ is Cohen-Macaulay, we know by Theorem 2.36 that

$$\dim(R_Q/(x_1, \dots, x_h)R_Q) = \dim(R_Q/PR_Q).$$

We conclude that

$$\operatorname{height}(Q) - \operatorname{height}(P) = \dim(R_Q/PR_Q) = \dim(R_Q/(x_1, \dots, x_h)R_Q).$$ □

Theorem 2.43. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, and let I be any proper ideal in R. Then*
$$\dim(R/I) = \dim(R) - \operatorname{height}(I).$$

Proof. Applying Theorem 2.42 to $Q = \mathfrak{m}$ and noticing that $R_{\mathfrak{m}} = R$ gives us

$$\dim(R) - \operatorname{height}(P) = \operatorname{height}(\mathfrak{m}) - \operatorname{height}(P) = \dim(R/P).$$

Hence, the formula holds when $I = P$ is prime.

For an arbitrary ideal I, fix a minimal prime P of I such that $\operatorname{height}(P) = \operatorname{height}(I)$. Note that P/I is a minimal prime of R/I, so

$$\dim(R/I) \geqslant \dim(R/P).$$

Then

$$\dim(R/I) + \operatorname{height}(I) \geqslant \dim(R/P) + \operatorname{height}(I) = \dim(R/P) + \operatorname{height}(P) = \dim(R).$$

On the other hand, if Q is a minimal prime of I with $\dim(R/I) = \dim(R/Q)$, then $\operatorname{height}(I) \leqslant \operatorname{height}(Q)$, and

$$\dim(R/I) + \operatorname{height}(I) \leqslant \dim(R/I) + \operatorname{height}(Q) = \dim(R/Q) + \operatorname{height}(Q) = \dim(R). \quad \square$$

Definition 2.44. An ideal I is **unmixed** if it has no embedded primes, that is, if all its associated primes are minimal. An ideal I is **height unmixed** if every associated prime of I has the same fixed height.

Note that height unmixed implies unmixed.

Lemma 2.45. *In a Cohen-Macaulay local ring $(R, \mathfrak{m})$, all ideals generated by a regular sequence are height unmixed: given a regular sequence $x_1, \ldots, x_n$, every associated prime of $(x_1, \ldots, x_n)$ must have height n.*

Proof. Let P be an associated prime of $(x_1, \ldots, x_n)$. By Theorem 6.38 from Commutative Algebra I, P_P is an associated prime of $(x_1, \ldots, x_n)R_P$.

By Theorem 2.38, R_P is Cohen-Macaulay. Moreover, the images of $x_1, \ldots, x_n$ in R_P also form a regular sequence in R_P, by Exercise 19, so by Theorem 2.33, $R_P/(\underline{x})R_P$ is also a Cohen-Macaulay ring. By Theorem 2.36, Cohen-Macaulay rings have no associated primes, so P_P must be a minimal prime $R_P/(\underline{x})_P$, and thus p is a minimal prime of $(\underline{x})$.

By Krull's Height Theorem, Theorem 8.5 from Commutative Algebra I, $\operatorname{height}(P) \leqslant n$. By Remark 2.3, $\operatorname{height}(P) \geqslant \operatorname{grade}(P) \geqslant n$, so $\operatorname{height}(P) = n$. $\square$

Theorem 2.46. *Let R be a noetherian ring. The following are equivalent:*

1) R is Cohen-Macaulay.

2) Every ideal I in R contains a regular sequence of length $\operatorname{height}(I)$*.*

3) For all ideals I, $\operatorname{grade}(I) = \operatorname{height}(I)$*.*

Proof. Note htat 2 says that $\operatorname{grade}(I) \geqslant \operatorname{height}(I)$ for all ideals I. Since we proved in Remark 2.3 that $\operatorname{grade}(I) \leqslant \operatorname{height}(I)$ for all ideals I, we conclude that 2 and 3 are equivalent.

To show 3 $\Rightarrow$ 1, consider a maximal ideal $\mathfrak{m}$ in R. By assumption, $\operatorname{grade}(\mathfrak{m}) = \operatorname{height}(\mathfrak{m})$, so

$$\operatorname{depth}(R_{\mathfrak{m}}) \geqslant \operatorname{grade}(\mathfrak{m}_{\mathfrak{m}}) \geqslant \operatorname{grade}(\mathfrak{m}) = \dim(R_{\mathfrak{m}}).$$

Thus $R_{\mathfrak{m}}$ is Cohen-Macaulay for all maximal ideals $\mathfrak{m}$, and so R must be Cohen-Macaulay.

To show 1 $\Rightarrow$ 3, let $x_1, \dots, x_n$ be a maximal regular sequence inside I. The elements of I must all be zerodivisors on $R/(x_1, \dots, x_n)$, by maximality, so by Lemma 2.5 the ideal I must be contained in some associated prime P of $R/(x_1, \dots, x_n)$. By Lemma 2.45, P has height n, so $\operatorname{height}(I) \leqslant \operatorname{height}(P) = n$. But $I \supseteq (x_1, \dots, x_n)$ and $\operatorname{height}(x_1, \dots, x_n) = n$ by Theorem 1.55, so $\operatorname{height}(I) = n = \operatorname{grade}(I)$. $\square$

Corollary 2.47. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring, and let I be a proper ideal in R. The following are equivalent:*

1) The ideal I is generated by a regular sequence.

2) Any minimal generating set for I is a regular sequence.

3) $\mu(I) = \operatorname{height}(I)$.

Proof. By Theorem 2.46, $\operatorname{grade}(I) = \operatorname{height}(I)$. If I is generated by a regular sequence, then that regular sequence must have length $\operatorname{height}(I)$, so I must be generated by exactly $\operatorname{height}(I)$ elements. This proves 1 $\Rightarrow$ 3.

Suppose I is generated by $h = \operatorname{height}(I)$ many elements, say $x_1, \dots, x_h$. By Theorem 2.11, since $\operatorname{grade}(I) = \operatorname{height}(I)$ we must have $\mathrm{H}_i(\underline{x}) = 0$ for all $i \geqslant 1$. By Theorem 1.49, the vanishing of koszul homology tells us that $\underline{x}$ itself must be a regular sequence. Since this holds for any minimal generating set for I, this proves that 3 implies both 1 and 2. $\square$

The characterization of Cohen-Macaulayness in Theorem 2.46 says $\operatorname{grade}(I) = \operatorname{height}(I)$ is always false if the ring is not Cohen-Macaulay. But this is elementary: note that any noetherian local ring $(R, \mathfrak{m})$ that is not Cohen-Macaulay necessarily has

$$\operatorname{height}(\mathfrak{m}) = \dim(R) > \operatorname{depth}(R) = \operatorname{grade}(\mathfrak{m}).$$

Example 2.48. Let $R = k[\![x, y]\!]/(x^2, xy)$. Since $\mathfrak{m} = \operatorname{ann}(x)$, we see that $\operatorname{depth}(R) = 0$, and in particular $\operatorname{grade}(\mathfrak{m}) = 0$. However, $\dim(R) = \operatorname{height}(\mathfrak{m}) = 1$, so $\operatorname{height}(\mathfrak{m}) \neq \operatorname{grade}(\mathfrak{m})$.

By Lemma 2.45 and Corollary 2.47, in a Cohen-Macaulay ring every ideal I generated by $\operatorname{height}(I)$ many elements is unmixed. In fact, this characterizes Cohen-Macaulay rings.

Theorem 2.49 (Unmixedness Theorem)**.** *A noetherian ring R is Cohen-Macaulay if and only if every ideal $I = (a_1, \dots, a_n)$ with* $\operatorname{height}(I) = n$ *is unmixed.*

Proof. ($\Rightarrow$) Let R be Cohen-Macaulay and $I = (x_1, \dots, x_n)$ has $\operatorname{height}(I) = n$. Fix $P \in \operatorname{Ass}(I)$. By Theorem 2.38, R_P is Cohen-Macaulay. Therefore, $\operatorname{grade}(I_P) = \operatorname{height}(I_P) = n$. Since I_P is generated by the n elements $\frac{a_1}{1}, \dots, \frac{a_n}{1}$, by Corollary 2.47 we conclude that $\frac{a_1}{1}, \dots, \frac{a_n}{1}$ is a regular sequence on R_P. Therefore, IR_P is generated by a regular sequence, and thus by Lemma 2.45 it has no embedded primes. By Theorem 6.38 from Commutative Algebra I, P_P is an associated prime of $(x_1, \dots, x_n)R_P$, and thus P_P must be a minimal prime of $(x_1, \dots, x_n)R_P$. We conclude that P itself was a minimal prime of I.

($\Leftarrow$) Now suppose that R has the property that every ideal $I = (a_1, \dots, a_n)$ with height n is unmixed. Note that this assumption passes on to any localization of R at a prime ideal, and that we need only to show that R_P is Cohen-Macaulay for all primes P. This reduces the problem to the local case.

Let R be a noetherian local ring such that every ideal $I = (a_1, \dots, a_n)$ with $\operatorname{height}(I) = n$ is unmixed. Let $d = \dim(R)$, and consider a system of parameters $a_1, \dots, a_d$ on R. By assumption, for each i all the associated primes of $(a_1, \dots, a_i)$ have height i. In particular, a_{i+1} is not in any associated prime of $R/(a_1, \dots, a_i)$ for each i. We conclude that $a_1, \dots, a_d$ is a regular sequence, and R must be Cohen-Macaulay. $\square$

Theorem 2.50. *Let $A \subseteq R$ be a module-finite extension of noetherian local rings. If A is a regular local ring, then R is Cohen-Macaulay if and only if R is free as an A-module.*

Proof. Module-finite extensions are integral, and thus $d = \dim(R) = \dim(A) = \operatorname{depth}(A)$. By Theorem 1.71, Since A is regular then R has finite projective dimension as an A-module, so the Auslander–Buchsbaum formula, Theorem 2.15, applies to R as an A-module. Thus

$$R \text{ is Cohen-Macaulay} \iff \operatorname{depth}(R) = d \iff \operatorname{pdim}_A(R) = 0 \iff R \text{ if free over } A. \quad \square$$

Next, we consider Cohen-Macaulayness in the graded setting.

Lemma 2.51. *Let $(R, \mathfrak{m})$ be graded, as in Setting 1. If $R_{\mathfrak{m}}$ is Cohen-Macaulay, then every homogeneous system of parameters in R is a regular sequence.*

Proof. Let $\underline{x} = x_1, \dots, x_n$ be a homogeneous system of parameters in R, so that $\sqrt{(\underline{x})} = \mathfrak{m}$. Note that its localization $\underline{y} = y_1, \dots, y_n$ with $y_i = \frac{x_i}{1}$ is a system of parameters for $R_{\mathfrak{m}}$, as

$$\sqrt{(\underline{y})} = \left(\sqrt{(\underline{x})}\right)_{\mathfrak{m}} = \mathfrak{m}_{\mathfrak{m}}.$$

Since $R_{\mathfrak{m}}$ is Cohen-Macaulay, Theorem 2.35 says that $\underline{y}$ is a regular sequence on $R_{\mathfrak{m}}$. If $\underline{x}$ is not a regular sequence on R, then for some i we would have $x_i \in P$ for some associated prime of $R/(x_1, \dots, x_{i-1})$. The associated primes of a graded module must be all homogeneous, so $P \subseteq \mathfrak{m}$. In particular, $P_{\mathfrak{m}}$ is associated to the localization

$$(R/(x_1, \dots, x_{i-1}))_{\mathfrak{m}} \cong R_{\mathfrak{m}}/(y_1, \dots, y_{i-1})$$

and $y_i \in P_{\mathfrak{m}}$. But this contradicts the fact that $\underline{y}$ is a regular sequence, so we conclude that $\underline{x}$ is a regular sequence on R. $\square$

Definition 2.52. Let R be a standard graded finitely generated k-algebra over a field k. A **graded Noether normalization** of R is a subalgebra $A = k[x_1, \dots, x_t] \subseteq R$ with $x_1, \dots, x_t \in R$ homogeneous and algebraically independent over k and $A \subseteq R$ module-finite.

Theorem 2.53. *Let k be a field and R a standard graded finitely generated k-algebra. Let S be a graded Noether normalization of R. Then R is a Cohen-Macaulay ring if and only if R is a free S-module.*

Proof. Fix algebraically independent homogeneous $x_1, \dots, x_d \in R$ with $S = k[x_1, \dots, x_d]$.

($\Rightarrow$) Suppose that R is Cohen-Macaulay with homogeneous maximal ideal $\mathfrak{m}$. Since $x_1, \dots, x_d$ are homogeneous in R, we have $(x_1, \dots, x_d)R \subseteq \mathfrak{m}$ and thus $(x_1, \dots, x_d) \subseteq \mathfrak{m} \cap R$, so $\mathfrak{m} \cap R = (x_1, \dots, x_d)$. By Incomparability, Theorem 7.32 from Commutative Algebra I, $\mathfrak{m}$ must be a minimal prime over $(x_1, \dots, x_d)$ in R. Since the extension is integral, we know that $\dim(R) = \dim(S) = d$. Thus $x_1, \dots, x_d$ is a homogeneous system of parameters in S. Moreover, since R is Cohen-Macaulay then so is $R_{\mathfrak{m}}$, and so $\underline{x}$ must be a regular sequence on R by Lemma 2.51. We conclude that the depth of R as a graded S-module is d. By the graded Auslander–Buchsbaum formula in Remark 2.22,

$$\operatorname{pdim}_S(R) = \operatorname{depth}(S) - \operatorname{depth}(R) = 0.$$

Therefore, R is a projective graded S-module. As we noted in Remark 1.2, bounded below graded projectives are free, and thus R must be a free S-module.

($\Leftarrow$) Suppose that R is a free S-module. Let P be any maximal ideal of R, and let $Q = P \cap S$. We need to show that R_P is Cohen-Macaulay. First, note that since $S \subseteq R$ is an integral extension, Q must also be maximal, by Lemma 7.30 from Commutative Algebra I.

Since S is regular, S_Q is Cohen-Macaulay. Since R is a free S-module, then the localization R_Q of R at Q as an S-module is a free S_Q-module. We can also view R_Q as the localization of the ring R at the multiplicatively closed subset of R given by $W = R \setminus Q$. Since $W \subseteq R \setminus P$, then R_P is a further localization of R_Q. Thus we get ring homomorphisms $S_Q \to R_Q \to R_P$ where R_Q is free as an S_Q-module, and R_P is a flat module over R_Q. We conclude that R_P is a flat S_Q-module, as

$$- \otimes_{S_Q} R_P \cong (- \otimes_{R_Q} R_P) \circ (- \otimes_{S_Q} R_Q)$$

is a composition of exact functors. This shows that R_P is a flat module over S_Q.

Fix a system of parameters $\underline{w}$ in S_Q. Since S_Q is Cohen-Macaulay, $\underline{w}$ is a regular sequence on S_Q. Thus $\operatorname{kos}(\underline{w}; S_Q)$ is exact in degrees 1 and above. By flatness,

$$\operatorname{kos}(\underline{w}; S_Q) \otimes_{S_Q} R_P \cong \operatorname{kos}(\underline{w}; R_P)$$

is also exact in degrees 1 and above, so $\underline{w}$ is a regular sequence on R_P by Theorem 1.49.

Finally, note that $\underline{w}$ has length $\dim(S_Q) = \operatorname{height}(Q)$. Since $S \subseteq R$ is integral and $Q = P \cap S$, we know that $\dim(R_P) = \operatorname{height}(P) \leqslant \operatorname{height}(Q)$, by Corollary 7.34 from Commutative Algebra I. On the other hand, since $\underline{w}$ was a regular sequence on R_P of length $\operatorname{height}(Q)$, we must also have $\dim(R_P) \geqslant \operatorname{height}(Q)$. Therefore, $\underline{w}$ is a regular sequence in R_P of length $\dim(R_P)$, and thus R_P must be a Cohen-Macaulay ring. $\square$

Remark 2.54. When k is an infinite field, any a standard finitely generated k-algebra R has a graded Noether Normalization, by Theorem 7.44 from Commutative Algebra I.

Theorem 2.55. *Let $(R, \mathfrak{m})$ be graded, as in Setting 1. If $R_{\mathfrak{m}}$ is Cohen-Macaulay, then R is Cohen-Macaulay.*

Proof. Let $\underline{x} = x_1, \dots, x_n$ be a homogeneous system of parameters in R. By Lemma 2.51, $\underline{x}$ is a regular sequence in R.

We claim that $k[\underline{x}] \subseteq R$ is an integral extension. Consider the R-module $R/(\underline{x})$. Since $\mathfrak{m} = \sqrt{(\underline{x})}$, there is some N such that $\mathfrak{m}^N \subseteq (\underline{x})$. In particular, the graded ring $R/(\underline{x})$ has dimension 0, as it is killed by $\mathfrak{m}^N$. Therefore, $R/(\underline{x})$ is a finite-dimensional k-vector space. Since R is a bounded below graded S-module, by graded NAK, Theorem 5.41 from Commutative Algebra I, we conclude that R is finitely generated as a module over $k[\underline{x}]$, generated by lifts of a basis for $R/(\underline{x})R$.

We claim that the homogeneous system of parameters $\underline{x}$ is algebraically independent. If not, then $k[\underline{x}] \subseteq R$ would be an integral extension with $\dim(k[\underline{x}]) < n \leqslant \dim(R)$, which is impossible. Thus $S = k[x_1, \dots, x_n]$ is a graded Noether normalization of R.

Now we claim that R is a free S-module. First, note that by construction the graded depths match: $\operatorname{depth}(R) = n = \operatorname{depth}(S)$. By the Auslander–Buchsbaum Formula, Theorem 2.15, which we saw in Remark 2.22 applies to this graded version of depth, we must have

$$\operatorname{pdim}_S(R) = \operatorname{depth}(S) - \operatorname{depth}(R) = 0.$$

Therefore, R is a projective S-module. As noted in Remark 1.2, bounded below graded projectives are free, so R is a free S-module. By Theorem 2.53, R is Cohen-Macaulay. □

Remark 2.56. Let $(R, \mathfrak{m})$ be a graded ring as in Setting 1. We saw in Remark 2.22 that there is a graded version of the Auslander–Buchsbaum formula, where we consider the depth of a graded module M to be the largest length of a regular sequence on M consisting of homogeneous elements. We also saw that this graded depth matches the depth of the module localized at the homogeneous maximal ideal $\mathfrak{m}$, that is $\operatorname{depth}(M) = \operatorname{depth}_{R_{\mathfrak{m}}}(M_m)$.

Theorem 2.55 tells us that R is Cohen-Macaulay if and only if $R_{\mathfrak{m}}$ is Cohen-Macaulay. By definition, $R_{\mathfrak{m}}$ is Cohen-Macaulay if and only if $\operatorname{depth}(R_{\mathfrak{m}}) = \dim(R_{\mathfrak{m}}) = \operatorname{height}(\mathfrak{m})$. We conclude that R is Cohen-Macaulay if and only if $\operatorname{depth}(R) = \operatorname{height}(\mathfrak{m})$.

Corollary 2.57. *Let $(R, \mathfrak{m})$ be a graded as in Setting 1. The following are equivalent:*

1) R is Cohen-Macaulay.

2) $R_{\mathfrak{m}}$ is Cohen-Macaulay.

3) Any homogeneous system of parameters for R forms a regular sequence.

4) Some homogeneous system of parameters for R forms a regular sequence.

Proof. Note that $1 \Rightarrow 2$ by definition, $2 \Rightarrow 3$ is Lemma 2.51, and $3 \Rightarrow 4$ is obvious. If some homogeneous system of parameters forms a regular sequence, then the graded depth of R is maximal: $\operatorname{depth}(R) = \operatorname{height}(\mathfrak{m})$. This implies that R is Cohen-Macaulay by Remark 2.56. □

Next, we will show that Cohen-Macaulay rings are catenary. Let us recall the definition.

Definition 2.58. A ring R is **catenary** if for all prime ideals P and Q, there exists a saturated chains of prime ideals

$$P = P_0 \subsetneq \cdots \subsetneq P_n = Q$$

and all such chains have the same length.

Remark 2.59. Given a noetherian ring R, and two prime ideals $P \subseteq Q$, by Krull's Height Theorem $\operatorname{height}(Q)$ is finite, so all chains of primes from P to Q must in particular be finite, so that we can find a saturated chain of primes from P to Q. Moreover, all chains of prime ideals

$$P = P_0 \subsetneq \cdots \subsetneq P_n = Q$$

correspond to chains of prime ideals in R_Q. Therefore, to show that R is catenary it suffices to show that all localizations of R at a prime ideal are catenary.

Now suppose that R is a catenary noetherian local ring, and fix prime ideals $P \subseteq Q$. For any other prime ideal $P \subseteq \mathfrak{a} \subseteq Q$, note that all saturated chains of prime ideals from P to $\mathfrak{a}$ have length $\operatorname{height}(\mathfrak{a}/P)$, and all saturated chains of prime ideals from $\mathfrak{a}$ to Q have length $\operatorname{height}(Q/\mathfrak{a})$. Putting together two such chains gives us a saturated chain of prime ideals from P to Q that goes through $\mathfrak{a}$, of length $\operatorname{height}(\mathfrak{a}/P) + \operatorname{height}(Q/\mathfrak{a})$. On the other hand, all saturated chains of prime from P to Q have the same length, which must then be $\operatorname{height}(Q/P)$. We conclude that if R is catenary, then for all prime ideals $P \subseteq \mathfrak{a} \subseteq Q$

$$\operatorname{height}(Q/P) = \operatorname{height}(Q/\mathfrak{a}) + \operatorname{height}(\mathfrak{a}/P).$$

Conversely, suppose that R is a noetherian ring and for all prime ideals $P \subseteq \mathfrak{a} \subseteq Q$,

$$\operatorname{height}(Q/P) = \operatorname{height}(Q/\mathfrak{a}) + \operatorname{height}(\mathfrak{a}/P).$$

We claim that R is catenary. If not, consider prime ideals $P \subseteq Q$ of smallest possible $\operatorname{height}(Q/P)$ such that there are saturated chains of prime ideals

$$P = P_0 \subsetneq \cdots \subsetneq P_n = Q$$

of different length. First, note that such a saturated chain of prime ideals of maximal length must have length $\operatorname{height}(Q/P)$. Now consider any other chain

$$P = \mathfrak{a}_0 \subsetneq \mathfrak{a}_1 = \mathfrak{a} \cdots \subsetneq \mathfrak{a}_m = Q$$

any prime ideal $P \subsetneq \mathfrak{a} \subsetneq Q$. By our choice of P and Q, all saturated chains from P to $\mathfrak{a}$ have the same length, which must be $\operatorname{height}(\mathfrak{a}/P)$, and all saturated chains of primes from $\mathfrak{a}$ to Q have the same length, which must be $\operatorname{height}(Q/\mathfrak{a})$. But then

$$n = \operatorname{height}(Q/P) = \operatorname{height}(Q/\mathfrak{a}) + \operatorname{height}(\mathfrak{a}/P) = 1 + (m-1) = m.$$

We conclude that R must have been catenary to begin with.

In summary: to show that a noetherian local ring is catenary, we need only to show that

$$\operatorname{height}(Q/P) = \operatorname{height}(Q/\mathfrak{a}) + \operatorname{height}(\mathfrak{a}/P)$$

for all prime ideals $P \subseteq \mathfrak{a} \subseteq Q$.

Theorem 2.60. *Cohen-Macaulay rings are catenary.*

Proof. Let $P \subseteq Q$ be prime ideals in a Cohen-Macaulay ring R. Then R_Q is Cohen-Macaulay, and we need only to prove that R_Q is catenary. Note that $\operatorname{height}(Q_Q) = \operatorname{height}(Q)$ and $\operatorname{height}(P_Q) = \operatorname{height}(P)$. By Theorem 2.42 we know

$$\dim(R_Q/PR_Q) = \operatorname{height}(Q) - \operatorname{height}(P).$$

Notice that

$$\dim(R_Q/PR_Q) = \dim\left((R/P)_Q\right) = \operatorname{height}(Q/P).$$

Therefore,

$$\operatorname{height}(Q/P) = \operatorname{height}(Q) - \operatorname{height}(P).$$

Given any other prime $\mathfrak{a}$ with $P \subseteq \mathfrak{a} \subseteq Q$, applying the same idea to $P \subseteq \mathfrak{a}$ and $\mathfrak{a} \subseteq Q$ gives us

$$\operatorname{height}(\mathfrak{a}/P) = \operatorname{height}(\mathfrak{a}) - \operatorname{height}(P) \quad \text{and} \quad \operatorname{height}(Q/\mathfrak{a}) = \operatorname{height}(Q) - \operatorname{height}(\mathfrak{a}).$$

Therefore,

$$\operatorname{height}(Q/P) = \operatorname{height}(Q/\mathfrak{a}) + \operatorname{height}(\mathfrak{a}/P).$$

Now note that R is a noetherian local rind and thus all the heights involved are finite. $\square$

Theorem 2.61. *For all noetherian local rings $(R, \mathfrak{m})$, if R is Cohen-Macaulay then its completion $\widehat{R}$ at the maximal ideal is Cohen-Macaulay.*

Proof. Recall that $\dim(R) = \dim(\widehat{R})$. Fix a system of parameters $\underline{x}$ in R, and note that since $\mathfrak{m}\widehat{R}$ is the maximal ideal of $\widehat{R}$, then the image of $\underline{x}$ in $\widehat{R}$ is also a system of parameters. Since R is Cohen-Macaulay, $\underline{x}$ is a regular sequence on R. Thus by Theorem 1.46 $\operatorname{kos}(\underline{x}; R)$ is exact in degrees 1 and above. Since $\widehat{R}$ is a flat R-module,

$$\operatorname{kos}(\underline{w}; R) \otimes_R \widehat{R} \cong \operatorname{kos}(\underline{x}; \widehat{R})$$

is also exact in degrees 1 and above. By Theorem 1.49, $\underline{x}$ is a regular sequence on $\widehat{R}$. By Theorem 2.35, $\widehat{R}$ must be Cohen-Macaulay. $\square$

Appendix A

Macaulay2

There are several computer algebra systems dedicated to algebraic geometry and commutative algebra computations, such as Singular (more popular among algebraic geometers), CoCoA (which is more popular with european commutative algebraists, having originated in Genova, Italy), and Macaulay2. There are many computations you could run on any of these systems (and others), but we will focus on Macaulay2 since it's the most popular computer algebra system among US based commutative algebraists.

Macaulay2, as the name suggests, is a successor of a previous computer algebra system named Macaulay. Macaulay was first developed in 1983 by Dave Bayer and Mike Stillman, and while some still use it today, the system has not been updated since its final release in 2000. In 1993, Daniel Grayson and Mike Stillman released the first version of Macaulay2, and the current stable version if Macaulay2 1.16.

Macaulay2, or M2 for short, is an open-source project, with many contributors writing packages that are then released with the newest Macaulay2 version. Journals like the *Journal of Software for Algebra and Geometry* publish peer-refereed short articles that describe and explain the functionality of new packages, with the package source code being peer reviewed as well.

The National Science Foundation has funded Macaulay2 since 1992. Besides funding the project through direct grants, the NSF has also funded several Macaulay2 workshops — conferences where Macaulay2 package developers gather to work on new packages, and to share updates to the Macaulay2 core code and recent packages.

A.1 Getting started

A Macaulay2 session often starts with defining some ambient ring we will be doing computations over. Common rings such as the rationals and the integers can be defined using the commands `QQ` and `ZZ`; one can easily take quotients or build polynomial rings (in finitely many variables) over these. For example,

```
i1 : R = ZZ/101[x,y]

o1 = R
```

```
o1 : PolynomialRing
```

and

```
i1 : k = ZZ/101;

i2 : R = k[x,y];
```

both store the ring $\mathbb{Z}/101$ as R, with the small difference that in the second example Macaulay2 has named the coefficient field k. One quirk that might make a difference later is that if we use the first option and later set k to be the field $\mathbb{Z}/101$, our ring R is *not* a polynomial ring over k. Also, in the second example we ended each line with a `;`, which tells Macaulay2 to run the command but not display the result of the computation — which is in this case was simply an assignment, so the result is not relevant. Lines indicated with as in, where n is some integer, are input lines, whereas lines with an on indicate output lines.

We can now do all sorts of computations over our ring R. We can define ideals in R, and use them to either define a quotient ring S of R or an R-module M, as follows:

```
 i3 : I = ideal(x^2,y^2,x*y)

                2    2
o3 = ideal (x , y , x*y)

o3 : Ideal of R

i4 : M = R^1/I

o4 = cokernel | x2 y2 xy |

                                                1
o4 : R-module, quotient of R

i5 : S = R/I

o5 = S

o5 : QuotientRing
```

It is important to note that while R is a ring, R^1 is the R-module R — this is a very important difference for Macaulay2, since these two objects have different types. So S defined above is a ring, while M is a module. Notice that Macaulay2 stored the module M as the cokernel of the map

$$R^3 \xrightarrow{\begin{bmatrix} x^2 & y^2 & xy \end{bmatrix}} R$$

Note also that there is an alternative syntax to write our ideal I from above, as follows:

```
i15 : I = ideal"x2,xy,y2"

               2        2
o15 = ideal (x , x*y, y )

o15 : Ideal of R
```

When you make a new definition in Macaulay2, you might want to pay attention to what ring your new object is defined over. For example, now that we defined this ring S, Macaulay2 has automatically taken S to be our current ambient ring, and any calculation or definition we run next will be considered over S and not R. If you want to return to the original ring R, you must first run the command `use S`.

If you want to work over a finitely generated algebra over one of the basic rings you can define in Macaulay2, and your ring is not a quotient of a polynomial ring, you want to rewrite this algebra as a quotient of a polynomial ring. For example, suppose you want to work over the 2nd Veronese in 2 variables over our field k from before, meaning the algebra $k[x^2, xy, y^2]$. We need 3 algebra generators, which we will call a, b, c, corresponding to x^2, xy, and y^2:

```
 i11 : U = k[a,b,c]

o11 = U

o11 : PolynomialRing

i12 : f = map(R,U,{x^2,x*y,y^2})
                 2        2
o12 = map(R,U,{x , x*y, y })

o12 : RingMap R <--- U

i13 : J = ker f

              2
o13 = ideal(b  - a*c)

o13 : Ideal of U

i14 : T = U/J

o14 = T

o14 : QuotientRing
```

Our ring T at the end is isomorphic to the 2nd Veronese of R, which is the ring we wanted.

A.2 Basic commands

Many Macaulay2 commands are easy to guess, and named exactly what you would expect them to be named. If you are not sure how to use a certain command, you can run `viewHelp` followed by the command you want to ask about; this will open an html file with the documentation for the method you asked about. Often, googling "Macaulay2" followed by descriptive words will easily land you on the documentation for whatever you are trying to do.

Here are some basic commands you will likely use:

- `ideal`$(f_1, \dots, f_n)$ will return the ideal generated by $f_1, \dots, f_n$. Here products should be indicated by $*$, and powers with ^. If you'd rather not use ^ (this might be nice if you have lots of powers), you can write `ideal`$(f_1, \dots, f_n)$ instead.

- `map`$(S, R, f_1, \dots, f_n)$ gives a ring map $R \to S$ if R and S are rings, and R is a quotient of $k[x_1, \dots, x_n]$. The resulting ring map will send $x_i \mapsto f_i$. There are many variations of `map` — for example, you can use it to define R-module homomorphisms — but you should carefully input the information in the required format. Try `viewHelp map` in Macaulay2 for more details

- `ker`(f) returns the kernel of the map f.

- `I + J` and `I*J` return the sum and product of the ideals I and J, respectively.

- `A = matrix`$\{\{a_{1,1}, \dots, a_{1,n}\}, \dots, \{a_{m,1}, \dots, a_{m,n}\}\}$ returns the matrix

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}$$

A.3 Complexes in Macaulay2

There are two different ways to do computations involving complexes in Macaulay2: using `ChainComplexes`, or the new (and still incomplete) `Complexes` package. To use `Complexes`, you must first load the `Complexes` package, while the `ChainComplexes` methods are automatically loaded with Macaulay2.

A.3.1 Chain Complexes

To create a new chain complex by hand, we start by setting up R-module maps.

```
i1 : R = QQ[a,b];

i2 : d1 = map(R^1, R^2, {{a,b}})
```

```
o2 = | a b |
              1        2
o2 : Matrix R  <--- R

i3 : d2 = map(R^2, R^1, {{-b},{a}})

o3 = | -b |
     | a  |
              2        1
o3 : Matrix R  <--- R
```

Keep in mind that the syntax of map is a bit funny: we write `map(target,source,matrix)`. To make sure we set up the next map in a way that is composable with d_1, we can use the methods `source` and `target`:

```
i3 : d1 = map(source d0, R^1, {{-b},{a}})

o3 = | -b |
     | a  |
              2        1
o3 : Matrix R  <--- R
```

We can also double check our maps do indeed map a complex, by checking the composition $d_1 \circ d_2$:

```
i4 : d1 * d2 == 0

o4 = true
```

So now we are ready to set up our new chain complex.

```
i5 : C = new ChainComplex

o5 = 0

o5 : ChainComplex

i6 : C#0 = target d1

      1
o6 = R

o6 : R-module, free

i7 : C#1 = target d2

      2
```

```
o7 = R

o7 : R-module, free

i8 : C#2 = source d2

      1
o8 = R

o8 : R-module, free
```

Given a chain complex `C`, we can ask Macaulay2 what our complex is by simply running the name of the complex:

```
i9 : C

      1      2      1
o9 = R  <-- R  <-- R

     0      1      2

o9 : ChainComplex
```

Or we can ask for a better visual description of the maps, using `C.dd`:

```
i10 : C.dd

           1          2
o10 = 0 : R  <----- R  : 1
                0

           2          1
      1 : R  <----- R  : 2
                0

o10 : ChainComplexMap
```

We can also set up the same complex in a more compact way, by simply feeding the maps we want in order. Macaulay2 will automatically place the first map with the target in homological degree 0 and the source in degree 1.

```
11 : D = chainComplex(d1,d2)

       1      2      1
o11 = R  <-- R  <-- R

      0      1      2
```

```
o11 : ChainComplex
```

Notice this is indeed the same complex.

```
i12 : D.dd

           1               2
o12 = 0 : R  <----------- R  : 1
                | a b |

           2              1
      1 : R  <---------- R  : 2
                | -b |
                | a  |

o12 : ChainComplexMap
```

We can also ask Macaulay2 to compute the homology of our complex:

```
i13 : HH D

o13 = 0 : cokernel | a b |

      1 : subquotient (| b  |, | -b |)
                       | -a |  | a  |
      2 : image 0

o13 : GradedModule
```

Or we could simply ask for the homology in a specific degree:

```
i14 : HH_0 D

o14 = cokernel | a b |
                               1
o14 : R-module, quotient of R
```

A.3.2 The Complexes package

To use this functionality, you must first load the `Complexes` package.

```
i15 : needsPackage "Complexes";

o15 = Complexes

o15 : Package
```

We can use our maps from above to set up a complex with the same maps. We feed a list of the maps we want to use to the method `complex`.

```
i16 : F = complex({d1,d2})

       1      2      1
o16 = R  <-- R  <-- R

      0      1      2

o16 : Complex
```

We cam read off the maps and the homology in our complex using the same commands as we use with `chainComplexes`, although the information returned gets presented in a slightly different fashion.

```
i17 : HH F

o17 = cokernel | a b | <-- subquotient (| b  |, | -b |) <-- image 0
                                        | -a |  | a  |
      0                                                    2
                         1

o17 : Complex

i18 : F.dd

          1               2
o18 = 0 : R  <----------- R  : 1
                | a b |

          2              1
      1 : R  <---------- R  : 2
                | -b |
                | a  |

o18 : ComplexMap
```

If we want to set up our complex starting in a different homological degree, we can do the following:

```
i19 : G = complex({d1,d2}, Base => 7)

       1      2      1
o19 = R  <-- R  <-- R
```

```
      7      8      9

o19 : Complex

i20 : H = complex({d1,d2}, Base => -13)

       1       2       1
o20 = R   <-- R   <-- R

      -13     -12     -11

o20 : Complex
```

A.3.3 Maps of complexes

Suppose we are given two complexes `C` and `D` and a map of complexes `f: C` $\longrightarrow$ `D`. The routine `map` can be used to define f using `chainComplexes`: it receives the target `D`, the source `D`, and a function `f` that returns f_i when we compute `f(i)`.

```
i1 : R = QQ[a,b];

i2 : c1 = map(R^0,R^1,0);

                     1
o2 : Matrix 0 <--- R

i3 : c2 = map(R^1, R^2, {{a,b}});

             1       2
o3 : Matrix R  <--- R

i4 : c3 = map(R^2, R^1, {{-b},{a}});

             2       1
o4 : Matrix R  <--- R

i5 : c4 = map(R^1, R^0, 0);

             1
o5 : Matrix R  <--- 0

i6 : C = chainComplex(c1,c2,c3,c4);

i7 :
     d1 = map(R^0,R^1,0);
```

```
                    1
o7 : Matrix 0 <--- R

i8 : d2 = id_(R^1);

             1       1
o8 : Matrix R  <--- R

i9 : d3 = map(R^1, R^0, 0);

             1
o9 : Matrix R  <--- 0

i10 : d4 = map(R^0, R^0, 0);

o10 : Matrix 0 <--- 0

i11 : D = chainComplex(d1,d2,d3,d4)

             1      1
o11 = 0 <-- R  <-- R  <-- 0 <-- 0

      0     1      2      3     4

o11 : ChainComplex

i12 :
      f0 = map(R^0, R^0, 0);

o12 : Matrix 0 <--- 0

i13 : f1 = map(R^1, R^1, matrix{{0_R}});

              1       1
o13 : Matrix R  <--- R

i14 : f2 = map(R^2, R^1, {{b},{-a}});

              2       1
o14 : Matrix R  <--- R

i15 : f3 = map(R^1, R^0, 0);

              1
```

```
o15 : Matrix R  <--- 0

i16 : f4 = map(R^0, R^0, 0);

o16 : Matrix 0 <--- 0

i17 : f = map(C,D,i -> if i==0 then f0 else(
      if i==1 then f1 else (
      if i==2 then f2 else (
      if i == 3 then f3 else (
      if i==4 then f4)))))

o17 = 0 : 0 <----- 0 : 0
               0

           1          1
      1 : R  <----- R  : 1
                0

           2               1
      2 : R  <---------- R  : 2
                | b  |
                | -a |
           1
      3 : R  <----- 0 : 3
                0

      4 : 0 <----- 0 : 4
               0

o17 : ChainComplexMap
```

Here's what we can do if we prefer to write a list with the maps in `f`:

```
i18 : f = map(C,D,i -> {f0,f1,f2,f3,f4}_i)

o18 = 0 : 0 <----- 0 : 0
               0

           1          1
      1 : R  <----- R  : 1
                0

           2               1
      2 : R  <---------- R  : 2
```

```
                | b  |
                | -a |

          1
      3 : R  <----- 0 : 3
                0

      4 : 0 <----- 0 : 4
               0

o18 : ChainComplexMap
```

If we prefer to do the same with the `Complexes` package, one advantage is that `map` *does* receive `(target, source, list of maps)`.

```
i42 : C = complex({c1,c2,c3,c4});

i43 : D = complex({d1,d2,d3,d4});

i44 : f = map(C,D,{f0,f1,f2,f3,f4})

            2              1
o44 = 2 : R  <---------- R  : 2
                | b  |
                | -a |

o44 : ComplexMap
```

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