Math 325-001 — Problem Set #7
Due: Monday, March 29 by midnight

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Suppose $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence such that $a_{100k} = 0$ for all $k \in \mathbb{N}$ (i.e., every 100-th term is 0). Prove, using only the definition of "Cauchy", that the sequence converges to 0.

(2) Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$a_n = 1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \cdots + (-1)^{n+1}\frac{1}{4n-3}$$

Equivalently, we may define this sequence recursively by $a_1 = 1$ and $a_n = a_{n-1} + (-1)^{n+1}\frac{1}{4n-3}$ for all $n \geq 2$.

- (a) Prove that the subsequence of even-indexed terms is strictly increasing and that the subsequence of odd-indexed terms is strictly decreasing; that is, prove $a_{2k} < a_{2k+2}$ for all $k \in \mathbb{N}$ and prove $a_{2k-1} < a_{2k+1}$ for all $k \in \mathbb{N}$.
- (b) Prove that every even-indexed term is less than every odd-indexed term; that is, show $a_{2k} < a_{2j-1}$ for all $k, j \in \mathbb{N}$.
- (c) Prove that the sequence is Cauchy and conclude that it therefore converges.
- (d) Bonus: What does it converge to?

(3) Using just the $\epsilon - \delta$ definition of the limit of a function, prove that

$$\lim_{x \to -3} \frac{x^2 + 5x + 6}{x + 3} = -1.$$

(4) Using just the $\epsilon - \delta$ definition of the limit of a function, prove that

$$\lim_{x \to 2} x^3 = 8.$$

Tip: Use that $x^3 - 8 = (x-2)(x^2 + 2x + 4)$ and that whenever $|x - 2| < 1$, we have $|x^2 + 2x + 4| < 19$.

(5) Let $a > 0$. Using just the $\epsilon - \delta$ definition of the limit of a function, prove that

$$\lim_{x \to a} \sqrt{x} = \sqrt{a}.$$

Tip: Use $|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \leq \frac{|x-a|}{\sqrt{a}}$.