Homework #1 volunteered solutions

Problem 1.
Let k be a field. Find a free resolution of $M = k[x,y]/(y)$ over $R = k[x,y]/(xy)$. Use it to compute $\operatorname{Ext}^i_R(M,k)$ for all $i \geq 0$, where we interpret k as a module via $k \cong R/(x,y)$.

Solution 1. (Takumi Murayama)
We claim that setting $P_i = R$ for all $i \geq 0$ and setting

$$d_i = \begin{cases} y & \text{for } i \text{ odd} \\ x & \text{for } i \text{ even} \end{cases}$$

the sequence

$$P_\bullet = \left\{ \cdots \longrightarrow P_3 \xrightarrow{y} P_2 \xrightarrow{x} P_1 \xrightarrow{y} P_0 \longrightarrow 0 \right\}$$

is a free resolution of M. Note that this is a complex since $xy = 0$ in R. Moreover, we have

$$\operatorname{coker}(P_1 \xrightarrow{y} P_0) \cong R/yR \cong k[x,y]/(y,xy) = M,$$

hence it suffices to show that the complex given above is exact away from degree 0.
Let i be odd, in which case we want to show

$$\ker(P_i \xrightarrow{y} P_{i-1}) = \operatorname{im}(P_{i+1} \xrightarrow{x} P_i).$$

The inclusion $\supseteq$ follows from the fact that $P_\bullet$ is a complex, so it suffices to show the reverse inclusion. Let $f \in \ker(P_i \xrightarrow{y} P_{i-1})$, and choose a representative $\tilde{f} \in k[x,y]$ for f. Then, $yf = 0 \in R$ implies $y\tilde{f} \in xy \cdot k[x,y]$. Since $k[x,y]$ is a unique factorization domain, this implies $\tilde{f} \in x \cdot k[x,y]$, hence $f \in \operatorname{im}(P_{i+1} \xrightarrow{x} P_i)$ as desired. On the other hand, if i is even, we want to show

$$\ker(P_i \xrightarrow{x} P_{i-1}) = \operatorname{im}(P_{i+1} \xrightarrow{y} P_i).$$

As before, it suffices to show the reverse inclusion. Let $f \in \ker(P_i \xrightarrow{x} P_{i-1})$, and choose a representative $\tilde{f} \in k[x,y]$ for f. Then, $xf = 0 \in R$ implies $x\tilde{f} \in xy \cdot k[x,y]$. Since $k[x,y]$ is a unique factorization domain, this implies $\tilde{f} \in y \cdot k[x,y]$, hence $f \in \operatorname{im}(P_{i+1} \xrightarrow{y} P_i)$ as desired.
We now compute $\operatorname{Ext}^i_R(M,k)$. Applying $\operatorname{Hom}_R(-,k)$ to $P_\bullet$, we obtain

$$\operatorname{Hom}_R(P_\bullet, k) = \left\{ 0 \longrightarrow \operatorname{Hom}_R(R,k) \xrightarrow{y} \operatorname{Hom}_R(R,k) \xrightarrow{x} \operatorname{Hom}_R(R,k) \xrightarrow{y} \cdots \right\}.$$

Since $\operatorname{Hom}_R(R,k) \simeq k$ and since x and y act as zero on k, we have

$$\operatorname{Hom}_R(P_\bullet, k) \simeq \left\{ 0 \longrightarrow k \xrightarrow{0} k \xrightarrow{0} k \xrightarrow{0} k \xrightarrow{0} \cdots \right\},$$

hence $\operatorname{Ext}^i_R(M,k) = k$ for all $i \geq 0$.

Problem 2. *Use the definition of Cohen-Macaulay to show that*

$$R = \frac{k\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}_{\mathfrak{m}}}{(uy - vx, uz - wx, vz - wy)}$$

is Cohen-Macaulay, where $\mathfrak{m}$ is the ideal generated by the (images of the) variables.

Solution 2. (David Schwein)
I claim that the sequence

$$u, y, z + w, x + w + v$$

is a regular system of parameters[1] for A. This shows both that $\dim A = 4$ and that A is Cohen-Macaulay.

[1]Beware that "regular SOP" is sometimes used to specify a SOP consisting of elements whose images are linearly independent mod $\mathfrak{m}^2$. The more awkward "SOP that is a regular sequence" is preferred.

To show that the sequence is regular, it is enough to prove the stronger statement that the sequence is regular in the ring B defined above.

Lemma 1. $\operatorname{Spec} B$ *is irreducible, or in other words, every zerodivisor of* B *is nilpotent.*

Proof. It suffices to show that the scheme $X := \operatorname{Spec} B$ is irreducible. For this, note first that the localization of B at each of the variables is a localization of a polynomial ring. For example, $B_u = k[u^{\pm 1}, v, w, x]$. It follows that the complement $X - \mathfrak{m}$ of the origin is covered by six copies of the irreducible variety $(\mathbb{A} - \{0\}) \times \mathbb{A}^3$, each intersecting every other.[2] Therefore $X - \mathfrak{m}$ is irreducible. Since (in general) the closure of an irreducible open subset is open, the only way X could fail to be irreducible is if the closed point $\mathfrak{m}$ is also open. But the point $\mathfrak{m}$ is not open because we can connect it to other points of X via a line: the image of the map $\mathbb{A}^1 \to X$ defined on points by $t \mapsto (t, 0, 0; 0, 0, 0)$ (or on rings $B \to k[t]$ by sending x to t and all other variables to 0) contains $\mathfrak{m}$ and other points of X. Hence X is irreducible. $\square$

The function u is not nilpotent in B because there are points of $\operatorname{Spec} B$ at which it does not vanish. Therefore, by Lemma 1, u is a nonzerodivisor of B. Hence u is a regular sequence in B.

To show that u, y is a regular sequence in B it suffices to show that y is a nonzerodivisor in the ring

$$B_1 := B/(u) = \frac{k[v, w; x, y, z]}{(vx, wx, vz - wy)}.$$

Lemma 2. *The scheme* $\operatorname{Spec} B_1$ *has irreducible decomposition*

$$\operatorname{Spec} B_1 = V(x, vz - wy) \cup V(v, w) \subset \mathbb{A}^5.$$

Hence $\operatorname{Spec} B_1$ *is reduced.*

Proof. This amounts to showing that

$$(vx, wx, vz - wy) = (x, vz - wy) \cap (v, w).$$

The inclusion $\subseteq$ is clear. For the inclusion $\supseteq$, let

$$f = ax + b(vz - wy) = cv + dw \in (x, vz - wy) \cap (v, w).$$

Then $a \in (v, w)$, meaning that $f \in (vx, wx, vz - wy)$.

To finish the proof, note that both schemes in the decomposition are obviously integral. $\square$

Since B_1 is reduced by Lemma 2, the function y can be a nonzerodivisor only if it vanishes on some irreducible component of $\operatorname{Spec} B_1$. The decomposition above shows this to not be the case. Hence y is not a zerodivisor of B_1. Hence u, y is a regular sequence in B.

To show that $u, y, z + w$ is a regular sequence in B amounts to showing that $z + w$ is a nonzerodivisor in the ring

$$B_2 := B/(u, y) = \frac{k[v, w; x, z]}{(vx, wx, vz)}.$$

Lemma 3. *The scheme* $\operatorname{Spec} B_2$ *has irreducible decomposition*

$$\operatorname{Spec} B_2 = V(v, x) \cup V(x, z) \cup V(v, w) \subset \mathbb{A}^4.$$

Hence $\operatorname{Spec} B_2$ *is reduced.*

Proof. This amounts to showing that

$$(vx, wx, vz) = (v, x) \cap (x, z) \cap (v, w).$$

If $av + bx = cx + dz$ then $a \in (x, z)$, so that $(v, x) \cap (x, z) = (vz, x)$. Similarly, if $avz + bx = dv + dw$ then $b \in (v, w)$, so that $(vz, x) \cap (v, w) = (vx, wx, vz)$. $\square$

[2]In general, a topological space X is irreducible if it admits a finite cover $X = \bigcup_i X_i$ by open irreducible subsets X_i such that $X_i \cap X_j \neq \varnothing$.

Therefore, $z+w$ is a nonzerodivisor because it does not vanish on any component of $\operatorname{Spec} B_2$. Hence the sequence $u, y, z+w$ is regular on B_2.

To show that $u, y, z+w, x+w+v$ is a regular sequence in B amounts to showing that $x+w+v$ is a nonzerodivisor in the ring

$$B_3 := B/(u, y, z+w) = \frac{k[v, w; x]}{(vw, vx, wx)}.$$

Lemma 4. *The scheme* $\operatorname{Spec} B_3$ *has irreducible decomposition*

$$\operatorname{Spec} B_3 = V(v, w) \cup V(v, x) \cup V(w, x) \subset \mathbb{A}^3.$$

Therefore $\operatorname{Spec} B_3$ *is reduced.*

Proof. This amounts to showing that

$$(vw, vx, wx) = (v, w) \cap (v, x) \cap (w, x).$$

As before, we know that $(v, w) \cap (v, x) = (v, wx)$. Then

$$(vw, vx, wx) = (v, wx) \cap (w, x)$$

because if $av + bwx = cw + dx$ then $a \in (w, x)$. □

Therefore, $x+w+v$ is a nonzerodivisor because it does not vanish on any component of $\operatorname{Spec} B_3$. Hence the sequence $u, y, z+w, x+w+v$ is regular on A. Moreover, since the quotient of A by the ideal generated by this sequence is a zero-dimensional ring, namely

$$\frac{k[v, w]}{(vw, v(v+w), w(v+w))} = \frac{k[v, w]}{(v, w)^2},$$

this regular sequence is a system of parameters for A. Hence A is Cohen-Macaulay.

Problem 3. *Find a free resolution of the cyclic module* $M = R/(uy - vx, uz - wx, vz - wy)$ *over*

$$R = k\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}_{\mathfrak{m}}.$$

Compute $\operatorname{Ext}^i_R(M, R)$ *for all* $i \geq 0$.

Solution 3. (Zhan Jiang)

The dimension of M is just the dimension of the ring in Problem 2, so $\operatorname{depth}_m(M) \leq \dim(M) = 4$. Meanwhile we can construct a regular sequence of length 4. So by Problem 2 we know that $\operatorname{depth}_m(M) = 4$. Since $\dim(R) = 6$, by Auslander-Buchsbaum formula, the projective dimension is of length 2. In particular, the minimal resolution of M is of length 2.

We have part of the minimal resolution:

$$R^3 \xrightarrow{e_i \mapsto \Delta_i} R \to M \to 0$$

where e_i is the element with 1 in the i^{th} place and zero everywhere else i.e. $e_1 = (1, 0, 0) \in R^3$. Then since any row of the matrix gives a relation on the minors, we have $(x, -y, z)$ and $(u, -v, w)$ in the kernel. We also know that the last term is free of rank $2 (= 3-1)$. So it has to be

$$0 \to R^2 \to R^3 \xrightarrow{e_i \mapsto \Delta_i} R \to M \to 0$$

where the left-side map sends the generator $(0, 1)$ to $(x, -y, z)$ and $(1, 0)$ to $(u, -v, w)$ respectively.

Now apply $\operatorname{Hom}_R(-, R)$ to the chain complex

$$0 \to R^2 \to R^3 \xrightarrow{e_i \mapsto \Delta_i} R \to 0$$

and use the identification $\operatorname{Hom}_R(R^{\oplus n}, R) \cong R^{\oplus n}$ we obtain a complex:

$$0 \leftarrow R^2 \xleftarrow{\psi} R^3 \xleftarrow{\phi} R \leftarrow 0$$

where $\phi(1) = (\Delta_1, \Delta_2, \Delta_3)$ and

$$\begin{aligned}\psi : e_1 &\mapsto (x, u)\\ e_2 &\mapsto (-y, -v)\\ e_3 &\mapsto (z, w)\end{aligned}$$

Now we compute

- $\mathrm{Ext}^0_R(M, R) = 0$: R is a domain so ϕ is injection. Hence $\mathrm{Ext}^0_R(M, R) = 0$
- $\mathrm{Ext}^1_R(M, R) = 0$: The kernel consists of elements (a, b, c) such that

  $$\begin{aligned}ax - by + cz &= 0\\ au - bv + cw &= 0\end{aligned}$$

  First of all, if one of the variable is zero, say $c = 0$, then $ax = by$ and $au = bv$. Hence $a = dy$ and $b = dx$. Then $dyu = dxv \Rightarrow d(yu - xv) = 0$. This implies that $d = 0$ as R is a domain. So we should have either $(a, b, c) = (0, 0, 0)$ or none of them is zero. Now instead of solving this in R^3, we'd like to consider this in $(k(u, v, w, x, y, z))^{\oplus 3}$ (the fraction field of R). Then any solution here would yield a solution $(\frac{a}{c}, \frac{b}{c})$ to the equations

  $$\begin{aligned}x\alpha - y\beta &= -z\\ u\alpha - v\beta &= -w\end{aligned}$$

  By Cramer's Rule we know that

  $$\alpha = \frac{\det\begin{vmatrix}-z & -y\\ -w & -v\end{vmatrix}}{\det\begin{vmatrix}x & -y\\ u & -v\end{vmatrix}} = \frac{vz - wy}{uy - vx}$$

  $$\beta = \frac{\det\begin{vmatrix}x & -z\\ u & -w\end{vmatrix}}{\det\begin{vmatrix}x & -y\\ u & -v\end{vmatrix}} = \frac{uz - xw}{uy - vx}$$

  Hence $(a, b, c) \in R(\Delta_1, \Delta_2, \Delta_3)$. Therefore the sequence is exact at the middle and $\mathrm{Ext}^1_R(M, R) = 0$.
- $\mathrm{Ext}^2_R(M, R) = M \oplus M/((x, u)M + (y, v)M + (z, w)M)$: The image of ψ is the submodule of R^2 spanned by $(x, u), (y, v), (z, w)$. So $\mathrm{Ext}^2_R(M, R) = R^2/(R(x, u) + R(y, v) + R(z, w))$. But note that $v(x, u) - u(y, v)$ is zero hence $(\Delta_1, 0)$ is killed. Similarily we have $(\Delta_i, 0)$ and $(0, \Delta_i)$ are killed. So $\mathrm{Ext}^2_R(M, R) \cong M \oplus M/((x, u)M + (y, v)M + (z, w)M)$
- $\mathrm{Ext}^i_R(M, R) = 0 \forall i \geq 3$.

Problem 4. *Let R be a local ring. A f.g. module is called* maximal Cohen-Macaulay *or* MCM *if* depth $M = \dim R$. *Show that, if R has an MCM module, R is regular if and only if every f.g. MCM module over R is free.*

Solution 4. (Devlin Mallory)

First, say that R is regular, and let M be a finitely generated maximal Cohen–Macaulay module. Since R is regular, any finitely generated module has finite projective dimension, so Auslander–Buchsbaum implies

$$\operatorname{projdim} M = \operatorname{depth} R - \operatorname{depth} M = 0$$

(since regular rings are Cohen–Macaulay and thus $\operatorname{depth} R = \dim R$). Then M is projective over a local ring, hence free, and thus every finitely generated maximal Cohen–Macaulay module is free.

Conversely, say that R is Cohen–Macaulay and that every maximal Cohen–Macaulay module is free. We know regularity of R is equivalent to an arbitrary finitely generated R-module M having finite projective dimension, so let M be a finitely generated R-module. If $\operatorname{depth} M = \operatorname{depth} R$ then M is maximal Cohen-Macaulay, hence free, so we may assume $\operatorname{depth} M < \operatorname{depth} R$. Choose some surjection $f_1 : R^{\oplus n_1} \to M$, yielding a short exact sequence

$$0 \to \ker(f_1) \to R^{\oplus n_1} \longrightarrow f_1 M \to 0.$$

Thus, we have that

$$\operatorname{depth}(\ker(f_1)) \geq \min(\operatorname{depth} R^{\oplus n_1}, \operatorname{depth} M + 1) = \operatorname{depth} M + 1.$$

By downwards induction on depth, we thus have that $\ker f_1$ admits a finite free resolution, and thus so does M. Thus, R is regular.

Problem 5. *Compute the minimal injective resolution of $\mathbb{C}[x]$ as a $\mathbb{C}[x]$-module. Write each injective as a direct sum of indecomposable injectives. Use this to compute $\mathrm{H}^i_{(x)}(\mathbb{C}[x])$.*

Solution 5. (Eamon Quinlian)

Set $R = \mathbb{C}[x]$. Because R is a PID, an R-module is injective if and only if it is divisible. In particular, $K := \operatorname{Frac}(R)$ is an injective R-module, and so is K/R. One also observes that K is an essential extension of R. It follows that

$$0 \to R \to K \to K/R$$

is a minimal injective resolution for R.

We now obtain the direct sum decompositions. For K it is easy, since $K = E_R(R) = E_R(R/(0))$.

For K/R we claim that the map

$$\bigoplus_{\alpha \in \mathbb{C}} \frac{R[(x-\alpha)^{-1}]}{R} \to K/R$$

is an isomorphism.

For surjectivity, it suffices to show that we can hit every element of the form $1/f(x)$. Assume $f(x) \notin \mathbb{C}$ and that $f(x)$ is monic. Then because $\mathbb{C}$ is algebraically closed we have

$$f(x) = (x-\alpha_1)^{e_1} \cdots (x-\alpha_n)^{e_n}$$

For $i = 1, \ldots, n$ set $g_i(x) = f(x)/(x-\alpha_i)^{e_i}$ – in other words, the product of all the $(x-\alpha_j)^{e_j}$ where we omit the i-th one. Then $g_1, \ldots, g_n$ are coprime and thus we can find $h_1(x), \ldots, h_n(x)$ such that

$$1 = g_1(x)h_1(x) + \cdots g_n(x)h_n(x).$$

The following observation then proves surjectivity.

$$\frac{h_1(x)}{(x-\alpha_1)^{e_1}} + \cdots + \frac{h_n(x)}{(x-\alpha_n)^{e_n}} = \frac{g_1(x)h_1(x) + \cdots g_n(x)h_n(x)}{(x-\alpha_1)^{e_1} \cdots (x-\alpha_n)^{e_n}} = \frac{1}{f(x)}$$

For the injectivity, suppose

$$\frac{g_1(x)}{(x-\alpha_1)^{e_1}} + \cdots + \frac{g_n(x)}{(x-\alpha_n)^{e_n}} = f(x)$$

for some $g_i(x), f(x) \in \mathbb{C}[x]$. For each i, set

$$h_i(x) = \prod_{j \neq i} (x-\alpha_j)^{e_j}.$$

Then we may rewrite our equation as

$$g_1(x)h_1(x) + \cdots + g_n(x)h_n(x) = f(x)(x-\alpha_1)^{e_1} \cdots (x-\alpha_n)^{e_n}.$$

It follows that, for each i, $(x-\alpha_i)^{e_i}$ divides the left-hand side, thus divides $g_i(x)h_i(x)$, thus divides $g_i(x)$. Therefore all $g_i(x)/(x-\alpha_i)^{e_i}$ where in R to begin with, thus proving the injectivity of the map.

We now claim that

$$\frac{R[(x-\alpha)^{-1}]}{R} \cong E_R(R/(x-\alpha))$$

thus giving the required decomposition:

$$K/R \cong \bigoplus_{\alpha \in \mathbb{C}} E_R(R/(x-\alpha)).$$

By a change of variables we reduce to the case $\alpha = 0$. The module $R/(x)$ embeds into $R[x^{-1}]/R$ by $1 \mapsto x^{-1}$. The injectivity of $R[x^{-1}]/R$ follows because it is a direct summand of an injective module.

Alternatively, one could prove the injectivity by showing the module is divisible as follows: let $f(x)/x^n \in R[x^{-1}]/R$. We can clearly divide this by x. To divide it by some $(x - \beta$ where $\beta \neq 0$ divide x^n by $x-\beta$ to obtain an expression $1 = \gamma x^n + h(x)(x-\beta)$ where $\gamma \in \mathbb{C}$ (we regard this expression as an expression in $\mathbb{C}(x) = K$. Multiplying this expression by $f(x)/x^n$ and passing to the quotient by R, we obtain $f(x)/x^n = h(x)(x-\beta)/x^n$.

To show the extension is essential, observe that if $f(x)/x^n \in R[x^{-1}]/R$ for some $n \geq 1$ we have $x^{n-1}f(x)/x^n = f(x)/x \in \operatorname{im}(R/x)$.

Problem 6. *Let R be a Noetherian ring, I an ideal of R, and E an injective R-module. Show that $\Gamma_I(E)$ is injective, and compute its direct sum decomposition into indecomposables in terms I and the direct sum decomposition of E.*

Solution 6. (Devlin Mallory)

Let R be a noetherian ring, $I \subset R$ an ideal, and E an injective R-module. By our results on injective modules, E decomposes as the direct sum of indecomposables

$$E = \bigoplus_{P \in \operatorname{Spec} R} E_R(R/P)^{\mu(P)}$$

for some nonnegative integers $\mu(P)$. Clearly Γ_I commutes with direct sums, and since R is noetherian the direct sum of injectives is injective; thus it suffices to show that each $\Gamma_I(E_R(R/P))$ is injective. But this is immediate from our description of the structures of the $E_R(R/P)$: $E_R(R/P)$ is P-torsion and elements $R \setminus P$ act as automorphisms of $E_R(R/P)$, so either $I \subset P$ in which case $\Gamma_I(E_R(R/P)) = E_R(R/P)$ or $I \not\subset P$, in which case there's $r \in I \setminus P$ which acts as a unit on $E_R(R/P)$ and hence no power of I can annihilate any element of $E_R(R/P)$, so that $\Gamma_I(E_R(R/P)) = 0$. In either case, $\Gamma_I(E_R(R/P))$ is injective, and thus $\Gamma_I(E)$ is injective as well.

This also immediately describes the decomposition of $\Gamma_I(E)$ into indecomposables:

$$\Gamma_I(E) = \bigoplus_{P \in V(I)} E_R(R/P)^{\mu(P)},$$

i.e., the sum only over the primes containing I.

Problem 7. *Let $(A,\nu) \to (R,\mathfrak{m})$ be a local homomorphism*[3] *of local rings. Assume that $A/\nu \cong R/\mathfrak{m}$. Suppose that there exists an ideal J of R such that the composed map $A \to R \to R/J$ is an isomorphism.*[4] *Show that $E_R(R/\mathfrak{m}) \cong \operatorname{Hom}_A^{J\text{-cts}}(R, E_A(A/\nu))$. Use this to give an explicit description of $E_R(R/\mathfrak{m})$ when $R = \mathbb{Z}_p[\![\underline{x}]\!]$, where $\mathbb{Z}_p$ denotes the p-adic integers.*

Solution 7 (Zhan Jiang). Let $k = R/m = A/\nu$. First we notice that the direct limit is actually a union: since $E_A(k)$ is an injective module over A, the exact functor $\operatorname{Hom}_A(-, E_A(k))$ takes the surjections $R/J^n \twoheadrightarrow R/J^{n-1}$ to injections $\operatorname{Hom}_A(R/J^{n-1}, E_A(k)) \hookrightarrow \operatorname{Hom}_A(R/J^n, E_A(k))$.

Next we show that $\operatorname{Hom}_A^{\text{J-cts}}(R, E_A(k))$ is an essential extension of $\operatorname{Hom}_A(R/J, E_A(k))$. For any element ϕ in $\operatorname{Hom}_A^{\text{J-cts}}(R, E_A(k))$, it lies in some $\operatorname{Hom}_A(R/J^n, E_A(k))$. Choose minimal n so then $\phi : R/J^n \to E_A(k)$ doesn't factor through $R/J^{n-1} \to E_A(k)$. In particular we have $J^n \subsetneq J^{n-1}$. Let $r \in J^{n-1} \setminus J^n$ such that $\phi(r) \neq 0$, then $r \cdot \phi$ is nonzero and it factors through $R/J \to E_A(k)$: for any element $s \in J$, we have $r \cdot \phi(s) = \phi(rs)$ and since $rs \in J^n$, $\phi(rs) = 0$. So $r \cdot \phi \in \operatorname{Hom}_A(R/J, E_A(k))$. So $\operatorname{Hom}_A^{\text{J-cts}}(R, E_A(k))$ is an essential exteion of $\operatorname{Hom}_A(R/J, E_A(k))$.

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[3]This means that the image of ν is contained in $\mathfrak{m}$.

[4]We call A a *ring retract* of R in this setting.

The A-linear map $k \to E_A(k)$ is also an R-linear map and this is an essential extension. Since $E_A(k) = \mathrm{Hom}_A(A, E_A(k))$, we conclude that $k \hookrightarrow \mathrm{Hom}_A^{\text{J-cts}}(R, E_A(k))$ is an essential extension.

Now we want to show that $\mathrm{Hom}_A^{\text{J-cts}}(R, E_A(k))$ is an injective R-module. Let $I \subseteq R$ be an ideal and there is a map $I \to \mathrm{Hom}_A^{\text{J-cts}}(R, E_A(k))$. Since I is finitely generated, this map factors through $I \to \mathrm{Hom}_A(R/J^n, E_A(k)) \hookrightarrow \mathrm{Hom}_A^{\text{J-cts}}(R, E_A(k))$. Since $\mathrm{Hom}_A(R/J^n, E_A(k))$ is killed by J^n. The map continues to factor through $I \to I/J^nI \to \mathrm{Hom}_A(R/J^n, E_A(k))$. Now since R is Noetherian local, we have $\cap_{n=1}^{\infty} J^n \subseteq \cap_{n=1}^{\infty} m^n = 0$, hence there is some $N > n$ such that $I \cap J^N \subseteq J^n I \Rightarrow I/(I \cap J^N) \twoheadrightarrow I/J^nI$, so we have

$$\begin{tikzcd}
I/IJ^n \arrow[r] & \mathrm{Hom}_A(R/J^n, E_A(k)) \\
I/(I \cap J^N) \arrow[u, two heads] \arrow[r, hook] & R/J^N \arrow[u, dotted]
\end{tikzcd}$$

Once we showed the existence of the dotted line, it's quite straight forward to check that $R \to R/J^N \to \mathrm{Hom}_A(R/J^N, E_A(k))$ is the lift of our origin map and hence the injectivity is proved by Baer's criterion. The existence is guarenteed by the fact that R/J^n is module-finite over A, hence $\mathrm{Hom}_A(R/J^n, E_A(k)) \cong E_{R/J^n}(k)$. Since both $I/(I \cap J^N)$ and R/J^N are R/J^n modules, the map lifts.

Now we start to compute $E_R(R/m)$ where $R = \mathbb{Z}_p[[\underline{x}]]$ and $m = (p, \underline{x})$. Let $A = \mathbb{Z}_p$ and $\nu = (p)$. Then $J = (\underline{x})$ and we have

$$\begin{aligned}
E_R(R/m) &= \mathrm{Hom}_{\mathbb{Z}_p}^{\text{J-cts}}(\mathbb{Z}_p[[\underline{x}]], E_{\mathbb{Z}_p}(\mathbb{Z}/p\mathbb{Z})) \\
&= \varinjlim \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[\underline{x}]]/(\underline{x})^n, E_{\mathbb{Z}_p}(\mathbb{Z}/p\mathbb{Z}))
\end{aligned}$$

Since $E_{\mathbb{Z}_p}(\mathbb{Z}/p\mathbb{Z})$ is the same as the injective hull of $\mathbb{Z}/p\mathbb{Z}$ over $\mathbb{Z}$, we conclude that $E_{\mathbb{Z}_p}(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ as well as its completion $\mathbb{Z}_p[\frac{1}{p}]/\mathbb{Z}_p$. So

$$E_R(R/m) = \varinjlim \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[\underline{x}]]/(\underline{x})^n, \mathbb{Z}_p[\frac{1}{p}]/\mathbb{Z}_p)$$

Since $\mathbb{Z}_p[[\underline{x}]]/(\underline{x})^n$ is generated by monomials in $\underline{x}$ of total degree smaller than n over $\mathbb{Z}_p$, it is a free $\mathbb{Z}_p$ module of finite rank. Hence an element in $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[\underline{x}]]/(\underline{x})^n, \mathbb{Z}_p[\frac{1}{p}]/\mathbb{Z}_p)$ is determined by specifying its behaviour on the set of monomials, hence if we define a map $\phi \mapsto \phi(\underline{x}^{\underline{\beta}})\underline{x}^{\underline{\beta}}$, then we get an isomorphism

$$E_R(R/m) \cong \bigoplus_{\beta : \beta_i \geq 0} \frac{\mathbb{Z}_p[1/p]}{\mathbb{Z}_p} \underline{x}^{\underline{\beta}}$$

Problem 8. *Problem #1 from the worksheet on Koszul homology and CM rings*

Solution 8. (Eamon Quinlian)

(a) We have

$$S/(x_1, \ldots, x_{m-1}) \cong R[x_m, \ldots, x_t]$$

for each $1 \leq m \leq t$. As x is a nonzerodivisor on $T[x]$ – regardless of whether T is a domain or not – we conclude the x's form a regular sequence on S.

Since they form a regular sequence, the Koszul complex is exact and thus $K_\bullet(x)$ is a resolution of $S/(x_1, \ldots, x_n)$ over S. Finally, observe that ${}_\phi R \cong S/(x_1, \ldots, x_n)$ as S-modules.

(b) We compute $\mathrm{Tor}_i^S({}_\phi R, {}_\psi M)$ via the free resolution $K_\bullet(x)$ of ${}_\phi R$:

$$\begin{aligned}
\mathrm{Tor}_i^S({}_\phi R, {}_\psi M) &\cong H_i(K_\bullet(x) \otimes_S {}_\psi M) \\
&\cong H_i(K_\bullet(x; {}_\psi M)) \\
&\cong_\psi H_i(K_\bullet(f; M)) \text{ [Rmk. 1.46]} \\
&=_\psi H_i(f; M).
\end{aligned}$$

(c) We have

$$f_j H_i(f;M) = x_j \ {}_\psi H_i(f;M) = x_j \operatorname{Tor}_i^S({}_\phi R, {}_\psi M) = 0$$

where the last follows because x_j acts by zero on ${}_\phi R$ – thus by zero on a projective resolution.

(d) There is a short exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0.$$

We tensor this with $K_\bullet(f)$, which is a complex of free (thus flat) R-modules, to produce the desired exact sequence (recall $K_\bullet(f;M) := K_\bullet(f) \otimes_R M$.

(e) We induct on $d := \operatorname{depth}_{(\underline{f})} M$. For $d = 0$, by lemma 1.24 we have

$$0 \neq \operatorname{Ann}_M(\underline{f}) = \ker(M \xrightarrow{f} M^t) = H^0(\underline{f};M).$$

Now for the inductive step, let $(x_1, \ldots, x_d)$ be a maximal regular sequence for M on $(\underline{f})$. Then $(x_2, \ldots, x_d)$ is a maximal regular sequence on M/x_1M. From (an exact argument as in) (d), we have a short exact sequence

$$0 \to K^\bullet(\underline{f};M)) \xrightarrow{x_1} K^\bullet(\underline{f};M) \to K^\bullet(\underline{f};M/x_1M) \to 0,$$

whose long exact sequence yields

$$\cdots \to H^i(f;M) \xrightarrow{x_1} H^i(f;M) \to H^i(f;M/x_1M) \to \cdots$$

Observe that the x_1-map becomes zero by part (c). This yields short exact sequences of the form

$$0 \to H^i(\underline{f};M) \to H^i(\underline{f};M/x_1M) \to H^{i+1}(\underline{f};M) \to 0$$

If $i < d-1$ then $H^i(\underline{f};M/x_1M) = 0$ and thus $H^i(\underline{f};M) = 0$ for $i \leq d-1$. At the $i = d-1$ spot we get

$$0 \to 0 \to H^{d-1}(\underline{f};M/x_1M) \to H^d(\underline{f};M) \to 0,$$

i.e. $0 \neq H^{d-1}(\underline{f};M/x_1M) \cong H^d(\underline{f};M)$.

Problem 9. *Problem #3 from the worksheet on Matlis duality with coefficient fields*

Solution 9. (David Schwein)

For (a), use the fact that $\operatorname{Hom}_k^{\mathfrak{m}-\mathrm{cts}}(A,k)$ is an injective hull of k together with the isomorphism

$$\operatorname{Hom}_k^{\mathfrak{m}-\mathrm{cts}}(M,k) \cong \operatorname{Hom}_A(M, \operatorname{Hom}_k^{\mathfrak{m}-\mathrm{cts}}(A,k)) = \operatorname{Hom}_A(M, E_A(k)),$$

which holds for M finitely generated.

For (b), since M is finite length there is some integer $n > 0$ such that $\mathfrak{m}^n M = 0$. It follows that the natural map

$$\operatorname{Hom}_k(M/\mathfrak{m}^n M, k) \to \operatorname{Hom}(M,k)$$

is an isomorphism. Therefore the natural inclusion

$$\operatorname{Hom}_k^{\mathfrak{m}-\mathrm{cts}}(M,k) \to \operatorname{Hom}_k(M,k)$$

is an isomorphism.

For (c), use the enriched isomorphism between module maps out of a direct limit and the inverse limit of module maps out of the pieces to conclude that in general, Matlis duality turns direct limits into inverse limits:

$$(\varinjlim M_i)^\vee = \operatorname{Hom}_A(\varinjlim M_i, E_k(A)) \cong \varprojlim \operatorname{Hom}_A(M_i, E_k(A)) = \varprojlim M_i^\vee.$$

In our case, $M_i^\vee \cong \operatorname{Hom}_k(M,k)$ by (c). Therefore $M^\vee$ is isomorphic to

$$\varprojlim M_i^\vee = \varprojlim \operatorname{Hom}_k(M_i,k) \cong \operatorname{Hom}_k(\varinjlim M_i, k) = \operatorname{Hom}_k(M,k).$$

For (d), finite generation of M implies that

$$M^\vee = \mathrm{Hom}_k^{\mathfrak{m}-\mathrm{cts}}(M,k) = \bigcup_n \mathrm{Hom}_k(M/\mathfrak{m}^n M, k),$$

so it suffices to prove that $\mathrm{Hom}_k(M/\mathfrak{m}^n M, k)$ has finite length. This module has finite length because it is finitely generated and annihilated by $\mathfrak{m}^n$.

For (e), finite generation of M together with (c) and (d) implies that

$$M^{\vee\vee} \cong \mathrm{Hom}_k(M^\vee, k).$$

Part (a) then implies that

$$\begin{aligned} M^{\vee\vee} &\cong \mathrm{Hom}_k(\mathrm{Hom}_k^{\mathfrak{m}-\mathrm{cts}}(M,k)) \\ &= \mathrm{Hom}_k(\varinjlim \mathrm{Hom}_k(M/\mathfrak{m}^n M, k), k) \\ &\cong \varprojlim \mathrm{Hom}_k(\mathrm{Hom}_k(M/\mathfrak{m}^n M, k), k) \\ &\cong \varprojlim M/\mathfrak{m}^n M. \end{aligned}$$

On the second-to-last line, we used the fact that the vector-space isomorphism

$$N \hookrightarrow \mathrm{Hom}_k(\mathrm{Hom}_k(N,k))$$

is compatible with the A-module structure, i.e., is an isomorphism of A-modules.