Math 325-001 — Problem Set #5
Due: Monday, March 15 by midnight

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Read Section 1.5 of our text (especially the part about proof by induction). Then do #12 on page 51 (of the 2nd Edition).

(2) Define a sequence $\{a_n\}_{n=1}^{\infty}$ recursively by $a_1 = 2$ and $a_n = \frac{a_{n-1}}{2} + \frac{1}{a_{n-1}}$ for all $n \geq 2$.
   (a) Prove $a_n > 0$ for all $n \in \mathbb{N}$ by induction on n.
   (b) Prove $a_n^2 \geq 2$ for all $n \in \mathbb{N}$ by induction on n.
   (c) Prove the sequence is decreasing. *Tip*: Use part (b).
   (d) Since the sequence is decreasing and bounded below (from part (a)), it necessarily converges (as you showed on the previous problem set). Determine what the sequence converges to. *Tip*: Use that

$$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} a_{n-1}}{2} + \frac{1}{\lim_{n\to\infty} a_{n-1}}$$

   so that if we set $L = \lim_{n\to\infty} a_n$ then we have $L = \frac{L}{2} + \frac{1}{L}$.

(3) Prove the sequence $\left\{\frac{-n-1}{\sqrt{n}}\right\}_{n=1}^{\infty}$ diverges to $-\infty$.

(4) Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence such that $a_n > 0$ for all n. Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ diverges to infinity if and only if the sequence $\{\frac{1}{a_n}\}_{n=1}^{\infty}$ converges to 0.

(5) For each of the following, give an explicit example as indicated. No proofs are necessary.
   (a) A sequence that has a subsequence that converge to 1, another subsequence that converges to 2, and a third subsequence that converges to 3.
   (b) A sequence that has one subsequence that is monotone and converges to 0 and another subsequence that is monotone and diverges to $+\infty$.
   (c) A sequence of natural numbers such that for each $j \in \mathbb{N}$, it has a subsequence that converges to j. (Feel free to just describe the pattern – no formulas needed. As a hint, recall that the constant sequence j converges to j.)

(6) Suppose that $\{d_n\}_{n=1}^{\infty}$ is a sequence with $d_n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for all n. Prove that the sequence $\{q_n\}_{n=1}^{\infty}$, where

$$q_n = \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

   converges. (The point of this problem is that when we write a decimal expansion $.d_1 d_2 d_3 d_4 \cdots$, we mean the number given by $\lim_{n\to\infty} q_n$; you are justifying that every decimal expansion corresponds to a real number. For a hint, use the Monotone Convergence Theorem, and consider the case with all $d_n = 9$ to get an upper bound.)

Bonus: Show that for every real number $r \in [0, 1]$, there is a sequence $\{d_n\}_{n=1}^{\infty}$ with $d_n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for all n such that the sequence $\{q_n\}_{n=1}^{\infty}$, where

$$q_n = \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

converges to r. (In short, every real number has a decimal expansion.)