

Math 325-001 — Problem Set #5
Due: Monday, March 15 by midnight

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like “I collaborated with Steven Smale on problems 1 and 3”. If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

- (1) Read Section 1.5 of our text (especially the part about proof by induction). Then do #12 on page 51 (of the 2nd Edition).
- (2) Define a sequence $\{a_n\}_{n=1}^{\infty}$ recursively by $a_1 = 2$ and $a_n = \frac{a_{n-1}}{2} + \frac{1}{a_{n-1}}$ for all $n \geq 2$.
 - (a) Prove $a_n > 0$ for all $n \in \mathbb{N}$ by induction on n .
 - (b) Prove $a_n^2 \geq 2$ for all $n \in \mathbb{N}$. (Hint: Write $a_n - 2$ in terms of a_{n-1} and factor.)
 - (c) Prove the sequence is decreasing. *Tip:* Use part (b).
 - (d) Since the sequence is decreasing and bounded below (from part (a)), it necessarily converges (as you showed on the previous problem set). Determine what the sequence converges to. *Tip:* Use that

$$\lim_{n \rightarrow \infty} a_n = \frac{\lim_{n \rightarrow \infty} a_{n-1}}{2} + \frac{1}{\lim_{n \rightarrow \infty} a_{n-1}}$$

so that if we set $L = \lim_{n \rightarrow \infty} a_n$ then we have $L = \frac{L}{2} + \frac{1}{L}$.

- (3) Prove the sequence $\left\{ \frac{-n-1}{\sqrt{n}} \right\}_{n=1}^{\infty}$ diverges to $-\infty$.
- (4) Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence such that $a_n > 0$ for all n . Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ diverges to infinity if and only if the sequence $\left\{ \frac{1}{a_n} \right\}_{n=1}^{\infty}$ converges to 0.
- (5) For each of the following, give an explicit example as indicated. No proofs are necessary.
 - (a) A sequence that has a subsequence that converge to 1, another subsequence that converges to 2, and a third subsequence that converges to 3.
 - (b) A sequence that has one subsequence that is monotone and converges to 0 and another subsequence that is monotone and diverges to $+\infty$.
 - (c) A sequence of natural numbers such that for each $j \in \mathbb{N}$, it has a subsequence that converges to j . (Feel free to just describe the pattern – no formulas needed. As a hint, recall that the constant sequence j converges to j .)
- (6) Suppose that $\{d_n\}_{n=1}^{\infty}$ is a sequence with $d_n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for all n . Prove that the sequence $\{q_n\}_{n=1}^{\infty}$, where

$$q_n = \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

converges. (The point of this problem is that when we write a decimal expansion $.d_1d_2d_3d_4\cdots$, we mean the number given by $\lim_{n \rightarrow \infty} q_n$; you are justifying that every decimal expansion corresponds to a real number. For a hint, use the Monotone Convergence Theorem, and consider the case with all $d_n = 9$ to get an upper bound.)

Bonus: Show that for every real number $r \in [0, 1]$, there is a sequence $\{d_n\}_{n=1}^{\infty}$ with $d_n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for all n such that the sequence $\{q_n\}_{n=1}^{\infty}$, where

$$q_n = \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

converges to r . (In short, every real number has a decimal expansion.)