Homework #2 volunteered solutions

Problem 1. *Compute a k-basis for $\mathrm{H}^1_{(x)}(k[x,y])$ and show that this module is neither noetherian nor artinian.*

Solution 1 (Eamon Quinlian). We compute $H^1_{(x)}(k[x,y]) =: H$ via the Čech complex

$$0 \to k[x,y] \to k[x,y]_x \to 0,$$

from which we see that

$$H = \frac{k[x,y]_x}{k[x,y]}.$$

We conclude that a k-basis is given by the following set of monomials:

$$\{x^i y^j : i < 0, j \geq 0\}.$$

The two following chains of ideals contradict the noetherian and artinian properties respectively:

$$(x^{-1}) \subseteq (x^{-2}) \subseteq (x^{-3}) \subseteq \cdots$$
$$(x^{-1}y) \supseteq (x^{-1}y^2) \supseteq (x^{-1}y^3) \supseteq \cdots$$

Problem 2. *Let (V, pV, K) be a complete DVR with uniformizer $p \in \mathbb{Z}$. Let $R = V[\![x_1, \ldots, x_t]\!]$. Note that R is local of dimension $t+1$ with residue field K.*

(1) Use the Čech complex to show that $\mathrm{H}^{t+1}_{(p,\underline{x})}(R) \cong \bigoplus_{\underline{\alpha}\,:\,\alpha_i<0} \frac{V[1/p]}{V} \cdot \underline{x}^{\underline{\alpha}}$.

(2) Show that $\mathrm{Hom}_V^{(\underline{x})-\mathrm{cts}}(R, E_V(K)) \cong \mathrm{Hom}_V^{(\underline{x})-\mathrm{cts}}\left(R, \frac{V[1/p]}{V}\right)$ is an injective hull for K as an R-module. Use this description to show that $E_R(K) \cong \mathrm{H}^{t+1}_{(p,\underline{x})}(R)$.

Solution 2 (Eamon Quinlian). **(a)**

The tail of the Čech complex for R looks like

$$\cdots \to R_{x_1\cdots x_n} \oplus \bigoplus_i R_{px_1\cdots\widehat{x_i}\cdots x_n} \to R_{px_1\cdots x_n} \to 0$$

and thus

$$H^{t+1}_{(p,x)}(R) \cong \frac{R_{px_1\cdots x_n}}{\mathrm{im}\, R_{x_1\cdots x_n} \oplus \bigoplus_i R_{px_1\cdots\widehat{x_i}\cdots x_n}}.$$

We have

$$R_{px_1\cdots x_n} \simeq \bigoplus_{\alpha\in\mathbb{Z}^n} V[1/p]x^\alpha,$$
$$R_{x_1\cdots x_n} \simeq \bigoplus_{\alpha\in\mathbb{Z}^n} Vx^\alpha,$$
$$R_{px_1\cdots\widehat{x_i}\cdots x_n} \simeq \bigoplus_{\alpha\in\mathbb{Z}^n,\alpha_i\geq 0} V[1/p]x^\alpha,$$

from which we conclude

$$H^{t+1}_{(p,x)}(R) = \bigoplus_{\alpha\in\mathbb{Z}^n_{<0}} \frac{V[1/p]}{V}[x^\alpha]$$

as required. (We write $[x^\alpha]$ for the class of the monomial x^α in the quotient).

For part (b) we will also need to observe the R-module structure on this module. The elements of V act on the elements of $V[1/p]/V$ as usual, whereas a monomial x^β acts on (the class of) a monomial x^α by

$$x^\beta \cdot [x^\alpha] = \begin{cases} 0 \text{ if } \alpha_i + \beta_i \geq 0 \text{ for some i} \\ [x^{\beta+\alpha}] \text{ otherwise.} \end{cases}$$

(b)

We claim that $E_R(K) \cong V[1/p]/V$. Recall first that V is a PID and that $V[1/p]$ is its fraction field. Therefore $V[1/p]$ is a divisible V-module and, since divisibility descends to quotients, so is $V[1/p]/V$ – and thus it is injective. The fact that $V[1/p]/V$ is an essential extension of K – where K embeds as the multiples of $1/p$ – follows because $V[1/p]/V$ is p-torsion with a one-dimensional socle (namely, the multiples of $1/p$) – then use Theorem 2.44.

It follows that $\operatorname{Hom}_V^{x-cts}(R, E_V(K)) = \operatorname{Hom}_V^{x-cts}(R, V[1/p]/V)$ and, by HW1#7, these are in turn isomorphic to $E_R(K)$.

Now observe or recall that $\operatorname{Hom}_V^{(x)-cts}(R, V[1/p]/V)$ consists of those V-linear maps from R to $E_R(K)$ that are zero on monomials of sufficiently high degree. Therefore one has

$$\operatorname{Hom}_V^{x-cts}(R, E_V(K)) = \bigoplus_{\alpha \in \mathbb{Z}^n_{\geq 0}} \frac{V[1/p]}{V} \delta_\alpha$$

where δ_α is the dual element of x^α.

We note that the R-module action is given by multiplying the input, and thus a monomial x^β acts on δ_α via

$$x^\beta \delta_\alpha = \delta_\alpha(x^\beta \bullet) = \begin{cases} 0 \text{ if } \beta_i > \alpha_i \text{ for some } i \\ \delta_{\alpha-\beta} \text{ otherwise.} \end{cases}$$

We conclude that $H^{t+1}_{(p,x)}(R) \cong E_R(K)$ via the map that exchanges δ_α with $[x^{-\alpha-(1,1,\ldots,1)}]$.

Problem 3. *Let $(R, \mathfrak{m}, k)$ be a complete local ring.*

(1) Show that if $R \hookrightarrow M$ splits, then $R \otimes_R N \longrightarrow M \otimes_R N$ is injective for all R-modules N.
(2) Show that if $R \otimes_R E_R(k) \longrightarrow M \otimes_R E_R(k)$ is injective, then $R \hookrightarrow M$ splits.
(Moral: $E_R(k)$ is the "least flat" module.)

Solution 3 (Takumi Murayama). *Proof of (a).* Since $f \colon R \to M$ splits, we have a factorization

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{id}_R"'] & M \arrow[r, "g"] & R \end{tikzcd}$$

of the identity on R. Applying $- \otimes_R N$, we have a factorization

$$\begin{tikzcd} R \otimes_R N \arrow[r, "f \otimes \mathrm{id}_N"] \arrow[rr, bend right, "\mathrm{id}_{R \otimes_R N}"'] & M \otimes_R N \arrow[r, "g \otimes \mathrm{id}_N"] & R \otimes_R N \end{tikzcd}$$

of the identity on $R \otimes_R N$. The composition is injective, hence $f \otimes \mathrm{id}_N$ is injective as well. □

Proof of (b). Denote $f \colon R \to M$. We claim we have the following commutative diagram with exact rows:

$$\begin{tikzcd} \operatorname{Hom}_R\big(M \otimes_R E_R(k), E_R(k)\big) \arrow[r, "(f \otimes \mathrm{id}_{E_R(k)})^*"] \arrow[d, "\wr"] & \operatorname{Hom}_R\big(R \otimes_R E_R(k), E_R(k)\big) \arrow[r] \arrow[d, "\wr"] & 0 \\ \operatorname{Hom}_R\big(M, \operatorname{Hom}_R(E_R(k), E_R(k))\big) \arrow[r, "f^*"] & \operatorname{Hom}_R\big(R, \operatorname{Hom}_R(E_R(k), E_R(k))\big) \arrow[r] & 0 \\ \operatorname{Hom}_R(M, R) \arrow[r, "f^*"] \arrow[u, "\wr"] & \operatorname{Hom}_R(R, R) \arrow[r] \arrow[u, "\wr"] & 0 \end{tikzcd}$$

The top row is the Matlis dual of the map $f \otimes \mathrm{id}_{E_R(k)} \colon R \otimes_R E_R(K) \to M \otimes_R E_R(k)$, and the second row is obtained from the first by Hom-tensor adjunction. The last row is obtained from the isomorphism [**?**, Thm. 2.40], where we use the fact that R is complete. Since the last row is surjective, we can choose

$g \in \operatorname{Hom}_R(M, R)$ such that $f^*(g) = g \circ f = \mathrm{id}_R$, and so this map $g \colon M \to R$ is exactly a splitting for $f \colon R \to M$. $\square$

Problem 4 (Problem #4 from worksheet #3)**.** *Let* $T = \dfrac{k[x,y,u,v]}{(xu - yv)}$*. Note that* T *admits an* $\mathbb{N}^2$*-grading via*

$$\deg(x) = \begin{bmatrix}1\\0\end{bmatrix}, \quad \deg(y) = \begin{bmatrix}0\\1\end{bmatrix}, \quad \deg(u) = \begin{bmatrix}0\\1\end{bmatrix}, \quad \deg(v) = \begin{bmatrix}1\\0\end{bmatrix};$$

since the defining equation is homogeneous with respect to this grading, we get a well-defined grading on T*.*

(*a*) *Show that* $\left(\dfrac{v}{x}, \dfrac{u}{y}\right)$ *is a cocyle in the Čech complex* $\check{C}^1(x, y; T)$*.*

(*b*) *Show that the class* $\left[\dfrac{v}{x}, \dfrac{u}{y}\right]$ *of the cocycle in the previous part gives a nonzero class in* $H^1_{(x,y)}(T)$*.*

(*c*) *Let* $\eta_a = \left[\dfrac{v^{a-1}y^{a-1}}{x^a y^a}\right] \in \check{C}^2(x, y; T)$*. Use the grading defined above to show that* $\eta_a \neq 0$ *in* $H^2_{(x,y)}(T)$*.*

(*d*) *Show that each of the elements* η_a *is killed by the ideal* $\mathfrak{m} = (x, y, u, v)$*. Conclude that the socle of this local cohomology module (the submodule annihilated by the maximal ideal* $\mathfrak{m}$*) is infinite-dimensional.*

(*e*) *Congratulate yourself; you have disproven a conjecture of Grothendieck!*

Solution 4 (Takumi Murayama)*. Proof of* (*a*)*.* The Čech complex is

$$\check{C}^\bullet(x, y; T) = \left\{0 \longrightarrow T \xrightarrow{\binom{1}{1}} T_x \oplus T_y \xrightarrow{(1\ -1)} T_{xy} \longrightarrow 0\right\}$$

hence $\left(\frac{v}{x}, \frac{u}{y}\right) \mapsto \frac{v}{x} - \frac{u}{y} \in T_{xy}$. But $xu - yv = 0$ in T, hence $\frac{v}{x} - \frac{u}{y} = 0 \in T_{xy}$, and so

$$\left(\frac{v}{x}, \frac{u}{y}\right) \in \check{C}^1(x, y; T). \qquad \square$$

Proof of (*b*)*.* We note that both $\frac{v}{x}$ and $\frac{u}{y}$ have bidegree $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. On the other hand, the only elements in T with bidegree $(0,0)$ are constants, hence $\left(\frac{v}{x}, \frac{u}{y}\right)$ cannot be in the image of $T \to T_x \oplus T_y$. Thus,

$$0 \neq \left[\frac{v}{x}, \frac{u}{y}\right] \in H^1_{(x,y)}(T). \qquad \square$$

Proof of (*c*)*.* Note that $\deg \eta_a = \left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]$. On the other hand, T_x only has elements in bidegree $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ for $b \geq 0$, and similarly, T_y only has elements in bidegree $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ for $a \geq 0$. Thus, there is no way for a homogeneous element $f \in T_x$ and a homogeneous element $g \in T_y$ to have difference $f - g$ of bidegree equal to $\deg \eta_a = \left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]$, hence

$$0 \neq \eta_a = \left[\frac{v^{a-1}y^{a-1}}{x^a y^a}\right] \in H^2_{(x,y)}(T). \qquad \square$$

Proof of (*d*)*.* It suffices to show that η_a is annihilated by each of the generators x, y, u, v for $\mathfrak{m}$; to do so, it suffices to show they are in the image of the may $T_x \oplus T_y \to T_{xy}$. First, $x \cdot \eta_a$ is the image of $\left(0, \frac{u^{a-1}}{y^a}\right)$ since

$$x \cdot \eta_a = x \cdot \frac{v^{a-1}y^{a-1}}{x^a y^a} = \frac{v^{a-1}y^{a-1}}{x^{a-1}y^a} = \frac{x^{a-1}u^{a-1}}{x^{a-1}y^a} = \frac{u^{a-1}}{y^a}$$

using the relation $xu = yv$ in the third equality. Next, $y \cdot \eta_a$ is the image of $\left(\frac{v^{a-1}}{x^a}, 0\right)$ since

$$y \cdot \eta_a = y \cdot \frac{v^{a-1}y^{a-1}}{x^a y^a} = \frac{v^{a-1}y^a}{x^a y^a} = \frac{v^{a-1}}{x^a}.$$

Now $u \cdot \eta_a$ is the image of $\left(\frac{v^a}{x^{a+1}}, 0\right)$ since

$$u \cdot \eta_a = u \cdot \frac{v^{a-1}y^{a-1}}{x^a y^a} = \frac{uv^{a-1}y^{a-1}}{x^a y^a} = \frac{x^{a-1}u^a}{x^a y^a} = \frac{u^a}{xy^a} = \frac{v^a}{x^{a+1}}$$

using the relation $xu = yv$ in the third equality and the relation $v/x = u/y$ verified in (a) in the last equality. Finally, $v \cdot \eta_a$ is the image of $\left(0, \frac{u^a}{x^{a+1}}\right)$ since

$$v \cdot \eta_a = v \cdot \frac{v^{a-1}y^{a-1}}{x^a y^a} = \frac{v^a}{x^a y} = \frac{u^a}{x^{a+1}}$$

again using the relation $v/x = u/y$ verified in (a) in the last equality. □

Problem 5. *(1) Give an example of a sequence of elements $f_1, \ldots, f_t$ in a ring R and an integer i such that $\mathrm{H}^i(f_1^n, \ldots, f_t^n; R) \neq 0$ for all n, but $\mathrm{H}^i_{(f_1,\ldots,f_t)}(R) = 0$.*

(2) Give an example of an ideal I in a ring R, and R-module M, and an integer j such that $\mathrm{Ext}^j_R(R/I^n, M) \neq 0$ for all n, but $\mathrm{H}^j_I(M) = 0$.

Solution 5 (Zhan Jiang). (1) Take $i = t = 2$, let $R = k[x, y]/(xy)$ and $\underline{f} = x, y$. Then $H^2(x^n, y^n; R) = R/(x^n, y^n) = k[x, y]/(x^n, y^n, xy)$. This is clearly nonzero, i.e. 1 is nonzero in it. On the other hand, the local cohomology $H^2_{(x,y)}(R) = R_{xy}/(R_x + R_y)$. But in R_{xy}, note that $1 = \frac{xy}{xy} = 0$. So it is zero.

(2) Again let $R = k[x, y]/(xy)$. Let $M = k = R/(x, y)$ and let $I = (x)$. Then

$$\cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x^n} R (\to R/(x)^n) \to 0$$

If we apply $\mathrm{Hom}_R(-, k)$, all differentials go to zero hence $\mathrm{Ext}^j_R(R/I^n, M) = \mathrm{Hom}_R(R, k) = k \neq 0$. But $H^j_I(M) = 0$ for large enough j's, i.e. $j > 1$.

Problem 6.
Show that if R is a regular ring of dimension d, the minimal injective resolution of R is of the form

$$0\left(\to R\right) \to E_R(R) \to \bigoplus_{\mathrm{ht}\,\mathfrak{p}=1} E_R(R/\mathfrak{p}) \to \bigoplus_{\mathrm{ht}\,\mathfrak{p}=2} E_R(R/\mathfrak{p}) \to \cdots \to \bigoplus_{\mathrm{ht}\,\mathfrak{p}=d} E_R(R/\mathfrak{p}) \to 0.$$

Solution 6 (Zhan Jiang). This is equivalent to asserting that the Bass number is $\mu_i(P) = \delta_{\mathrm{height}(P),i}$. Let P be a prime ideal of height h. Since $\mu_i(P) = \dim_{\kappa_P} \mathrm{Ext}^i_{R_P}(\kappa_P, R_P)$ where κ_P is the residue field R_P/PR_P and R_P is regular of dimension h. We have

- $\mathrm{Ext}^i_{R_P}(\kappa_P, R_P) = 0$ for all $i > h$ since κ_P has a free resolution of length h
- $\mathrm{Ext}^i_{R_P}(\kappa_P, R_P) = 0$ for all $i < h$ since $\mathrm{depth}_{PR_P}(R_P) = h$ and $\kappa_P = R_P/PR_P$
- $\mathrm{Ext}^h_{R_P}(\kappa_P, R_P) = \kappa_P$ since $\mathrm{Ext}^h_{R_P}(\kappa_P, R_P) = H^h(\underline{x}; R_P) = R_p/(\underline{x})R_P = \kappa_P$ where $\underline{x} = x_1, ..., x_h$ is a set of generators of PR_P

Problem 7.
Use the previous problem to show that if R is regular and $\mathrm{ht}(I) = h$, then

$$\mathrm{Ass}\left(\mathrm{H}^h_I(R)\right) = \{\mathfrak{p} \in \mathrm{Min}(I) \mid \mathrm{ht}\,\mathfrak{p} = h\}.$$

Solution 7 (Devlin Mallory). Let R be regular and I an ideal with height$I = h$. We may calculate $H^h_I(R)$ by applying $\Gamma_I(-)$ to the injective resolution described above: as we've seen on previous homework, $\Gamma_I(E_R(R/Q)) = 0$ if $I \subset Q$ and is 0 otherwise, and thus the above resolution becomes the complex

$$0 \to \cdots \to 0 \to \bigoplus_{\substack{\mathrm{height}\,P=h \\ P \supset I}} E_R(R/P) \to \bigoplus_{\substack{\mathrm{height}\,Q=h+1 \\ Q \supset I}} E_R(R/P) \to \ldots.$$

Of course, any height-h prime P containing I is a minimal prime of I, since height$I = h$. Thus

$$H^h_I(R) = \ker\left(\bigoplus_{\substack{\mathrm{height}\,P=h \\ P \in \mathrm{Min}\,I}} E_R(R/P) \to \bigoplus_{\substack{\mathrm{height}\,Q=h+1 \\ Q \supset I}} E_R(R/P)\right).$$

This then gives a containment

$$\operatorname{Ass}(H_I^h(R)) \subset \operatorname{Ass}\Bigg(\bigoplus_{\substack{\operatorname{height} P=h \\ P\in \operatorname{Min} I}} E_R(R/P)\Bigg).$$

Now, using the fact that $\operatorname{Ass}(\bigoplus_\alpha M_\alpha) = \bigcup \operatorname{Ass} M_\alpha$ (this is elementary in the case of a finite direct sum, and the general case follows by noting that any map $R/P \to \bigoplus_\alpha M_\alpha$ has image contained in some subproduct over finitely many M_α) we obtain the containment

$$\operatorname{Ass}(H_I^h(R)) \subset \bigcup_{\substack{\operatorname{height} P=h \\ P\in \operatorname{Min} I}} \operatorname{Ass} E_R(R/P) = \operatorname{Min} I = \bigcup_{\substack{\operatorname{height} P=h \\ P\in \operatorname{Min} I}} \{P\}\,.$$

Thus, we just need the other inclusion. Note that for any R-module M we have

$$\operatorname{Ass}_R M = \bigcup_{P\in \operatorname{Spec} R} \operatorname{Ass}_{R_P} M_P,$$

since R is noetherian, and thus to obtain the other inclusion we will just need to show that for $P \in \operatorname{Min} I$ of height h we have $PR_P = \operatorname{Ass}_{R_P}(H^h_{I_P}(R_P))$, since then $P \in \operatorname{Ass}(H_I^h(R))$. Thus, we may localize at P. But in this case, calculating the associated primes of the kernel above is trivial:

$$\bigoplus_{\substack{\operatorname{height} P=h \\ P\in \operatorname{Min} I}} E_R(R/P) \to \bigoplus_{\substack{\operatorname{height} Q=h+1 \\ Q\supset I}} E_R(R/P)$$

localizes to

$$E_R(R/P) \to 0,$$

since some $Q \supsetneq P$ and thus some element of Q becomes a unit in R_P, which means $E_R(R/Q)_P = 0$. Thus $H_I^h(R)_P = E_R(R/P)$, so $\operatorname{Ass} H_I^h(R)_P = \{P\}$ for each height-h prime P containing I, and the result follows.