Contents

1. Wednesday, January 27

What is a number? Certainly the things used to count sheep, money, etc. are numbers: $1, 2, 3, \dots$. We will call these the *natural numbers* and write $\mathbb{N}$ for the set of all natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}.$$

Since we like to keep track of debts too, we'll allow negatives and 0, which gives us the *integers*:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}.$$

(The symbol $\mathbb{Z}$ is used since the German word for number is *zahlen*.)

Fractions should count as numbers also, so that we can talk about eating one and two-thirds of a pizza last night. We define a *rational number* to be a number expressible as a quotient of two integers: $\frac{m}{n}$ for $m, n \in \mathbb{Z}$ with $n \neq 0$. For example

$$\frac{5}{3}, \frac{2}{7}, \frac{2019}{2020}$$

are rational numbers. Of course, we often talk about numbers such as "two and a fourth", but that the same as $\frac{9}{4}$. Every integer is a rational number just by taking 1 for the denominator; for example, $7 = \frac{7}{1}$. The set of all rational numbers is written as $\mathbb{Q}$ (for "quotient").

You might not have thought about it before, but an expression of the form $\frac{m}{n}$ is really an "equivalence class": the two numbers $\frac{m}{n}$ and $\frac{a}{b}$ are deemed equal if $mb = na$. For example $\frac{6}{9} = \frac{2}{3}$ because $6 \cdot 3 = 9 \cdot 2$.

We'll talk more about decimals later on, but recall for now that a decimal that terminates is just another way of representing a rational number. For example, 1.9881 is equal to $\frac{19881}{10000}$. Less obvious is the fact that a decimal that repeats also represents a rational number: For example, $1.333\ldots$ is rational (it's equal to $\frac{4}{3}$) and so is $23.91278278278\ldots$. We'll see why this is true later in the semester.

Are these all the numbers there are? Maybe no one in this class would answer "yes", but the ancient Greeks believed for a time that every number was rational. Let's convince ourselves, as the Greeks did eventually, that there must be numbers that are not rational. Imagine a square of side length 1. By the Pythagorean Theorem, the length of its diagonal, call this number c, must satisfy

$$c^2 = 1^2 + 1^2 = 2.$$

That is, there must be a some number whose square is 2 since certainly the length of the diagonal in such a square is representable as a number. Now, let's convince ourselves that there is no *rational number* with this property. In fact, I'll make this a theorem.

Theorem 1.1. *There is no rational number whose square is* 2*.*

Preproof Discussion 1. *Before launching a formal proof, let's philosophize about how one shows something does not exist. To show something does not exist, one proves that its existence is not possible. For example, I know that there must not be large clump of plutonium sewn into the mattress of my bed. I know this since, if such a clump existed, I'd be dead by now, and yet here I am, alive and well!*

More generally and formally, one way to prove the falsity of a statement P is to argue that if we assume P to be true then we can deduce from that assumption something that is known to be false. If you can do this, then you have proven P is false. In symbols: If one can prove

$$P \Longrightarrow \textit{Contradiction}$$

then the statement P must in fact be false.

In the case at hand, letting P be the statement "there is a rational number whose square is 2", the Theorem is asserting that P is false. We will prove this by assuming P is true and deriving an impossibility.

This is known as a proof by contradiction. (Some mathematicians would actually not consider this to be a proof by contradiction. For some, a proof by contradiction refers to when the truth of a statement P is established by assuming the statement "not P" and deducing from that a falsity.)

Proof. By way of contradiction, assume there were a rational number q such that $q^2 = 2$. By definition of "rational number", we know that q can be written as $\frac{m}{n}$ for some integers m and n such that $n \neq 0$. Moreover, we may assume that we have written q is reduced form so that m and n have no prime factors in common. In particular, we may assume that not both of m and n are even. (If they were both even, then we could simplify the fraction by factoring out common factors of 2's.) Since $q^2 = 2$, $\frac{m^2}{n^2} = 2$ and hence $m^2 = 2n^2$. In particular, this shows m^2 is even and, since the square of an odd number is odd, it must be that m itself is even. So, $m = 2a$ for some integer a. But then $(2a)^2 = 2n^2$ and hence $4a^2 = 2n^2$ whence $2a^2 = n^2$. For the same reason as before, this implies that n must be even. But this contradicts the fact that m and n are not both even.

We have reached a contradiction, and so the assumption that there is a rational number q such that $q^2 = 2$ must be false. $\square$

A version of the previous proof was known even to the ancient Greeks.

Our first major mathematical goal in the class is to make a formal definition of the real numbers. Before we do this, let's record some basic properties of the rational numbers. I'll state this as a Proposition (which is something like a minor version of a Theorem), but we won't prove them; instead, we'll take it for granted to be true based on our own past experience with numbers.

For the rational numbers, we can do arithmetic $(+, -, \times, \div)$ and we also have a notion of size $(<, >)$. The first seven observations below describe the arithmetic, and the last three describe the notion of size.

Proposition 1.2. *The set of rational numbers form an "ordered field". This means that the following ten properties hold:*

1. *There are operations $+$ and $\cdot$ defined on $\mathbb{Q}$, so that if p, q are in $\mathbb{Q}$, then so are $p + q$ and $p \cdot q$.*
2. *Each of $+$ and $\cdot$ is a commutative operation (i.e., $p + q = q + p$ and $p \cdot q = q \cdot p$ hold for all rational numbers p and q).*

(3) *Each of $+$ and $\cdot$ is an associative operation (i.e., $(p+q)+r = p+(q+r)$ and $(p\cdot q)\cdot r = p\cdot(q\cdot r)$ hold for all rational numbers p, q, and r).*
(4) *The number 0 is an identity element for addition and the number 1 is an identity element for multiplication. This means that $0+q=q$ and $1\cdot q = q$ for all $q \in \mathbb{Q}$.*
(5) *The distributive law holds: $p\cdot(q+r) = p\cdot q + p\cdot r$ for all $p,q,r \in \mathbb{Q}$.*
(6) *Every number has an additive inverse: For any $p \in \mathbb{Q}$, there is a number $-p$ satisfying $p+(-p)=0$.*
(7) *Every nonzero number has a multiplicative inverse: For any $p \in \mathbb{Q}$ such that $p \neq 0$, there is a number p^{-1} satisfying $p\cdot p^{-1} = 1$.*
(8) *There is a "total ordering" $\leq$ on $\mathbb{Q}$. This means that*
  (a) *For all $p,q \in \mathbb{Q}$, either $p \leq q$ or $q \leq p$.*
  (b) *If $p \leq q$ and $q \leq p$, then $p = q$.*
  (c) *For all $p,q,r \in \mathbb{Q}$, if $p \leq q$ and $q \leq r$, then $p \leq r$.*
(9) *The total ordering $\leq$ is compatible with addition: If $p \leq q$ then $p+r \leq q+r$.*
(10) *The total ordering $\leq$ is compatible with multiplication by non-negative numbers: If $p \leq q$ and $r \geq 0$ then $pr \leq qr$.*

2. Friday, January 29

Which of the properties from Proposition 1.2 does $\mathbb{N}$ satisfy?

The commutativity, associativity, distributive law, multiplicative identity, and all of the ordering properties are true for $\mathbb{N}$. There is one other important property of $\mathbb{N}$, which we accept to be true without proof. Such a property is called an axiom.

Axiom 2.1 (Well-ordering axiom)**.** *Every nonempty subset of $\mathbb{N}$ has a smallest element (which we call its minimum).*

As we will discuss later, the well-ordering axiom is closely related to the principle of induction.

Example 2.2. For the set of all even multiples of 7, $S = \{7\cdot(2n) \mid n \in \mathbb{N}\}$, we have $\min(S) = 14$.

We expect everything from Proposition 1.2 to be true for the real numbers. We will build them into our definition. To define the real numbers $\mathbb{R}$, we take the ten properties listed in the Proposition to be axioms. It turns out the set of real numbers satisfies one key additional property, called the *completeness axiom*, which I cannot state yet.

Axioms. *The set of all real numbers, written $\mathbb{R}$, satisfies the following eleven properties:*

(Axiom 1) *There are operations + and $\cdot$ defined on $\mathbb{R}$, so that if $x, y \in \mathbb{R}$, then so are $x + y$ and $x \cdot y$.*

(Axiom 2) *Each of + and $\cdot$ is a commutative operation.*

(Axiom 3) *Each of + and $\cdot$ is an associative operation.*

(Axiom 4) *The real number 0 is an identity element for addition and the real number 1 is an identity element for multiplication. This means that $0 + x = x$ and $1 \cdot x = x$ for all $x \in \mathbb{R}$.*

(Axiom 5) *The distributive law holds: $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathbb{R}$.*

(Axiom 6) *Every real number has an additive inverse: For any $x \in \mathbb{R}$, there is a number $-x$ satisfying $x + (-x) = 0$.*

(Axiom 7) *Every nonzero real number has a multiplicative inverse: For any $x \in \mathbb{R}$ such that $x \neq 0$, there is a real number x^{-1} satisfying $x^{-1} \cdot x = 1$.*

(Axiom 8) *There is a "total ordering" $\leq$ on $\mathbb{R}$. This means that*
- (a) *For all $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$.*
- (b) *If $x \leq y$ and $y \leq z$, then $x \leq z$.*
- (c) *For all $x, y, z \in \mathbb{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.*

(Axiom 9) *The total ordering $\leq$ is compatible with addition: If $x \leq y$ then $x + z \leq y + z$ for all z.*

(Axiom 10) *The total ordering $\leq$ is compatible with multiplication by non-negative real numbers: If $x \leq y$ and $z \geq 0$ then $zx \leq zy$.*

(Axiom 11) *The completeness axiom holds. (I will say what this means later.)*

There are many other familiar properties that are consequences of this list of axioms. As an example we can deduce the following property:

"Cancellation of Addition": If $x + y = z + y$ then $x = z$.

Let's prove this carefully, using just the list of axioms: If $x + y = z + y$ then we can add $-y$ (which exists by Axiom 6) to both sides to get $(x+y)+(-y) = (z+y)+(-y)$. This can be rewritten as $x+(y+(-y)) = z + (y + (-y))$ (Axiom 3) and hence as $x + 0 = z + 0$ (Axiom 6), which gives $x = z$ (Axiom 4 and Axiom 2).

For another example, we can deduce the following fact from the axioms:

$$r \cdot 0 = 0 \text{ for any real number } r.$$

Let's prove this carefully: Let r be any real number. We have $0+0 = 0$ (Axiom 4) and hence $r\cdot(0+0) = r\cdot 0$. But $r\cdot(0+0) = r\cdot 0+r\cdot 0$ (Axiom 5) and so $r \cdot 0 = r \cdot 0 + r \cdot 0$. We can rewrite this as $0 + r \cdot 0 = r \cdot 0 + r \cdot 0$ (Axiom 4). Now apply the Cancellation of Addition property (which we previously deduced from the axioms) to obtain $0 = r \cdot 0$.

As I said, there are many other familiar properties of the real numbers that follow from these axioms, but I will not list them all. The great news is that all of these familiar properties follow from this short list of axioms. We will prove a couple, but for the most part, I'll rely on your innate knowledge that facts such as $r \cdot 0 = 0$ hold.

I owe you a description of the very important Completeness Axiom, and it will take a bit of time to do so. Before we get to this, it will be helpful to review set notation, and some basics of proof-writing.

Often, sets are described as subsets of other larger sets, by specifying properties. For example, when I write

$$S = \{m \in \mathbb{Z} \mid m = a^2 \text{ for some } a \in \mathbb{Z}\}$$

I am specifying a subset of the set of all integers $\mathbb{Z}$. In words, S is: "the set of those integers that are equal to the square of some integer". We could also write this set out by listing its elements:

$$S = \{0, 1, 4, 9, 16, 25, 36, \dots\}.$$

It's safer in general to use the former description, since you don't have to worry about the reader getting the pattern.

The previous is an example of a subset of $\mathbb{Z}$, but we will mostly be concerned with subsets of $\mathbb{R}$. For example, we might consider the set

$$\{x \in \mathbb{R} \mid x^2 < 2\}.$$

We will also deal with "intervals" a lot. When I write $(0,1)$ I mean the set $\{x \in \mathbb{R} \mid 0 < x < 1\}$. That is, it is all real numbers strictly between 0 and 1.

More generally, if a, b are real numbers and $a < b$, then

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

(What if $b \leq a$?) The set (a, b) is called an *open interval*.

We also have $[a, b]$, known as a *closed interval* and defined to be

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

We also have $[a, b)$, $(a, b]$, (a, ∞), $[a, \infty)$, $(-\infty, b)$, and $(-\infty, b]$, all of which you probably have seen before.

We will also have need to consider sets defined in more complicated ways such as

$$S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}.$$

The latter is a bit different than the previous examples. The previous ones had form { element of a set | property holds }, but this one has the form { expression involving symbols | allowable values of these symbols }. Explicitly, this example is the set $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$.

Recall also a few ways of making sets from others:

- union : $S \cup T = \{x \mid x \in S \text{ or } x \in T\}$
- intersection : $S \cup T = \{x \mid x \in S \text{ and } x \in T\}$
- set difference : $S \smallsetminus T = \{x \mid x \in S \text{ and } x \notin T\}$.

Let's now talk a bit more about rules of logic, methods of proof, quantification, etc. Our book has a very nice treatment of these topics in Sections 1.4 and 1.5. Part of the next problem set will involve your reading these sections on your own and doing some of the exercises. Here, I'll just give some highlights.

Let me start with some rules of logic, and how that affects proofs. First, a *statement* is a sentence (or sometimes sequence of sentences) that is either true or false. Things like "Jack's shirt is ugly" is not a statement, nor is "Go Huskers!". But "All odd numbers are prime" is a statement — it happens to be false. The sentence

> "The digit 9 occurs infinitely often in the decimal expansion of π."

is a statement, as it is surely either true or false. But, no one knows which!

An odder example is "This sentence is false". Is it a statement? (Is it true? Is it false?) No!

If P and Q are any two statements, then we can form compound statements from them such as

- P and Q.
- P or Q.
- Not P.
- If P then Q.

The "truth values" for the first three are pretty clear, but be careful about the last.

- "P and Q" is a true statement when both P and Q are true statements.
- "P or Q" is a true statement when either P or Q is a true statement.
- "Not P" is true when P is a false statement.
- "If P then Q" is true when P is false or Q is true. In other words "If P then Q" is logically equivalent to "not P or Q".

Which of the following are true?

(1) If $1 + 1 = 1$, then I am the pope.
(2) If 8 is prime then every real number is an integer.
(3) If my name is Jack then I am the pope.

(4) If it had been raining this morning then I would have brought an umbrella with me to class.

All but the third are true.

Most of the statements that we consider are, or can be framed as if-then statements: anything with hypotheses and a conclusion is an if-then statement. How do we prove such a statement? To give a "direct proof" of "if P then Q" we:

(1) Assume P,
(2) Do some stuff, then
(3) Conclude Q.

For example, the Goldbach Conjecture posits that if n is an even integer greater than 2, then n is a sum of two primes. (A conjecture is a statement that people believe to be true based on some evidence, but is not proven.) I can't prove this conjecture, but I can tell you the first and last sentence of a proof: "Assume that n is an even integer. ... Thus, n is a sum of two primes."

3. Monday, February 1

As I said earlier, "If P then Q" is the same as "not P or Q". It follows that "If not Q then not P" is the same as "not not Q or not P" and hence is the same as "not P or Q". That is:

> "If P then Q" is logically equivalent to "If not Q then not P".

"If not Q then not P" is known as the *contrapositive* of "If P then Q". So, an if-then statement and its contrapositive are logically equivalent.

Often when proving an if-then statement, it works a bit better to give a "direct" proof of the contrapositive. That is, in a proof of "If P then Q" *by contraposition* we:

(1) Assume not Q,
(2) Do some stuff, then
(3) Conclude not P.

Example 3.1. An *irrational number* is a real number that is not rational. Consider the following assertion:

> Let r be any rational number and let x be any real number. If x is irrational then $x + r$ is irrational.

This is logically equivalent to:

> Let r be any rational number and let x be any real number. If $x + r$ is rational then x is rational.

Let us prove the latter statement "directly": Let r be any rational number and let x be any real number. Suppose $x+r$ is rational. Then since r is rational, $-r$ is also rational (by Proposition 1.2, part (6)). It follows that $(x+r)+(-r)$ is also rational (by Proposition 1.2, part (1)) and hence $(x+r)+(-r) = x+(r+(-r)) = x+0 = x$ is rational.

Never, ever, ever, ever confuse the contrapositive of an if-then statement with its *converse*. The converse of "If P then Q" is "If Q then P".

Example 3.2. Give examples of statements that are true whose converses are false.

Recall that when we say "P if and only if Q" we mean "If P then Q, and if Q then P". In other words, an "if and only if" statement includes both an if-then statement and its converse. The statement "P if and only if Q" is true when either P and Q are both true or P and Q are both false, and it is false in the other two cases, when one is true and the other is false. A proof of such a statement generally has two parts, one where we prove P implies Q (either directly or by contraposition) and one where we prove Q implies P (again either directly or by contraposition).

Let me also say a bit about quantification: This refers to usage of "for every" or "there exists". For example, "For every real number x, x^2 is strictly positive" and "There exists an even integer that is prime".

"For every" statements are sometimes better cast as if-then statements. For example, the first one above is equivalent to "If x is a real number, then x^2 is strictly positive". So, be aware that sometimes, as in this example, there is an implicit "For every" clause lurking about even if you don't see those words written.

The negation of a "for every" clause usually involves "there exists". For example the negation of "For every real number x, x^2 is strictly positive." is "There is a real number x such that x^2 is not strictly positive".

The negation of a "there exists" statement usually involves "for every". For example, the negation of "There is an even integer that is prime" is "For every even integer n, n is not prime" or better "If n is an even integer, then n is not prime".

In general,

- the negation of "For every $x \in S$, P" is "There exists $x \in S$ such that not P";
- the negation of "There exists $x \in S$ such that P" for some statement P is "For every $x \in S$, not P".

How do we prove statements with quantifiers? To prove "For every $x \in S$, P" , we

(1) Take an arbitrary $x \in S$,
(2) Do some stuff, then
(3) Conclude that P holds for x.

In the first step we specify one element of S, but we *don't get to decide which one*. In particular, its name should be a variable, rather than the name of any specific element in S.

To *disprove* "For every $x \in S$, P" we can give a counterexample. That means that we get to choose an element of S, and show that P fails for our choice.

To prove "There exists $x \in S$ such that P" , we just need to give an example: we can choose any element of S and show that P holds for that element.

Things get harder when we combine "for every" and "there exist" clauses in one statement. One very important point here is that order matters a lot. For example,

"For every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $n < m$"

and

"There is an $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n < m$"

have very different meanings. In fact, the first is clearly true (since given an n one could, for example, take $m = n + 1$) and the second is false.

Never interchange the positions of "for every" and "there exist" unless you intend to change the meaning!

When we combine "for every" and "there exist" clauses with a negation things can also get confusing. For example: the negation of "For every integer m there is an integer n such that $n > m$" is "There exists an integer m such that for every integer n, $n \leq m$."

Using symbols sometimes helps focus attention on the underlying logic. We write $\forall$ and $\exists$ in place of "for every" and "there exists", sometimes. For example the negation of "$\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > x$" is "$\exists x \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, n \leq x$"

Discussion Questions, February 3

- Write the contrapositive, and the converse of each statement. Is the statement true or false? Is the converse true or false? Explain why (but don't write a full proof). For each statement below, a, b are real numbers.

◊ If a is irrational, then $1/a$ is irrational.

Contrapositive: "If $1/a$ is rational, then a is rational. [True]
Converse: "If $1/a$ is irrational, then a is irrational. [True]

◊ If a and b are both irrational, then ab is irrational.

Contrapositive: "If ab is rational, then either a or b is rational. [False]
Converse: "If ab is irrational, then a and b are both irrational. [False]

◊ If $x > 3$ then $x^2 > 9$.

Contrapositive: "If $x^2 \leq 9$, then $x \leq 3$. [True]
Converse: "If $x^2 > 9$, then $x > 3$. [False]

- Write the negation of each statement. Is the statement true or false? Explain why (but don't write a full proof).

◊ $\exists x \in \mathbb{Q}$: $x^2 = 2$.

Negation: $\forall x \in \mathbb{Q}$, $x^2 \neq 2$. [The original is false.]

◊ $\forall x \in \mathbb{Q}$, $x^2 > 0$.

Negation: $\exists x \in \mathbb{Q}$, $x^2 \leq 0$. [The original is false.]

◊ $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$: $xy = 1$.

Negation: $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, xy \neq 1$. [The original is false.]

◊ $\exists x \in \mathbb{R}$: $\forall y \in \mathbb{R}$, $e^y < x$.

Negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : e^y \geq x$. [The original is false.]

◊ $\exists x \in \mathbb{R}$: $\forall y \in \mathbb{R}$, $\sin(y) < x$.

> Negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : \sin(y) \geq x$. [The original is true.]

- Prove the following statements using the axioms of $\mathbb{R}$ and facts we have proven in class.
  - ♢ Let x be a real number. If there is a real number y such that $xy = 1$, then x is nonzero.[1]

> We argue the contrapositive. Let x be zero. Then, for any $y \in \mathbb{R}$, we have $xy = 0$, by a fact we proved in class. In particular, we have $xy \neq 0$, as required.

  - ♢ If x is a nonzero real number, then x^2 is also nonzero.[2]

> Let x be a nonzero real number. By Axiom 7, there is an element $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$. Then $x^2 \cdot (x^{-1})^2 = (xx^{-1})(xx^{-1}) = 1$, using Axioms 2 and 3 in the first equality and Axiom 5 in the second. By the previous fact (using $y = (x^{-1})^2$) we conclude that $x^2 \neq 0$.

  - ♢ For any real number x, $x \geq 0$ if and only if $-x \leq 0$.[3]

> Let $x \geq 0$. Adding $(-x)$ to both sides (which exists by Axiom 6), we obtain $0 = x + (-x) \geq 0 + (-x) = -x$ (by Axiom 9 and Axiom 5). Conversely, let $-x \leq 0$. Adding x to both sides, we obtain $0 = x + (-x) \leq x + 0 = x$ (by Axiom 9 and Axiom 5).

  - ♢ $0 \leq 1$. [4]

> To obtain a contradiction, suppose that $1 < 0$. Then $0 = -1 + 1 < -1 + 0 = -1$. By the previous fact, we then have $1 > 0$, which contradicts the hypothesis.

  - ♢ For any real number x, $(-1) \cdot x = -x$.

[1]Hint: Consider the contrapositive of this statement.
[2]Hint: Use x^{-1} and the previous statement.
[3]Hint: Add something to both sides.
[4]Hint: Try a proof by contradiction and use the previous fact.

Observe that

$$x + (-1) \cdot x = 1 \cdot x + (-1) \cdot x = (1 + -1) \cdot x = 0 \cdot x = 0.$$

We also have $x+(-x) = 0$, so $x+(-1)x = x+(-x)$. By cancellation of addition, we conclude that $(-1)x = -x$.

$\diamondsuit$ The product of two negative real numbers is nonnegative.

4. Friday, February 5

Definition 4.1. Let S be any subset of $\mathbb{R}$. A real number b is called an *upper bound* of S provided that for every $s \in S$, we have $s \leq b$.

For example, the number 1 is an upper bound for the set $(0, 1)$. The number 182 is also an upper bound of this set and so is π. It is pretty clear that 1 is the "best" (i.e., smallest) upper bound for this set, in the sense that every other upper bound of $(0, 1)$ must be at least as big as 1. Let's make this official:

Proposition 4.2. *If b is an upper bound of the set $(0, 1)$, then $b \geq 1$.*

I will prove this claim using just the axioms of the real numbers (in fact, I will only use the first 10 axioms):

Proof. Suppose b is an upper bound of the set $(0, 1)$. By way of contradiction, suppose $b < 1$. (Our goal is to derive a contradiction from this.)

Consider the number $y = \frac{b+1}{2}$ (the average of b and 1). I will argue that $b < y$ and $b \geq y$, which is not possible.

Since we are assuming $b < 1$, we have $\frac{b}{2} < \frac{1}{2}$ and hence

$$b = \frac{2b}{2} = \frac{b}{2} + \frac{b}{2} < \frac{b}{2} + \frac{1}{2} = \frac{b+1}{2} = y.$$

So, $b < y$.

Similarly,

$$1 = \frac{1+1}{2} > \frac{b+1}{2} = y$$

so that

$$y < 1.$$

Since $\frac{1}{2} \in S$ and b is an upper bound of S, we have $\frac{1}{2} \leq b$. Since we already know that $b < y$, it follows that $\frac{1}{2} < y$ and hence $0 < y$. We have proven that $y \in (0, 1)$. But, remember that b is an upper bound of $(0, 1)$, and so we get $y \leq b$ by definition.

To summarize: given an upper bound b of $(0, 1)$, starting with the assumption that $b < 1$, we have deduced the existence of a number y

such that both $b < y$ and $y \leq b$ hold. As this is not possible, it must be that $b < 1$ is false, and hence $b \geq 1$. $\square$

This claim proves the (intuitively obvious) fact that 1 is "least upper bound" of the set $(0, 1)$. The notion of "least upper bound" will be an extremely important one in this class.

Definition 4.3. A subset S of $\mathbb{R}$ is called *bounded above* if there exists at least one upper bound for S. That is, S is bounded above provided there is a real number b such that $s \leq b$ for all $s \in S$.

For example, $(0, 1)$ is bounded above, by for example 50.

The subset $\mathbb{N}$ of $\mathbb{R}$ is not bounded above — there is no real number that is larger than every natural number. This fact is surprisingly non-trivial to deduce just using the axioms; in fact, one needs the Completeness Axiom to show it. But of course our intuition tells us that it is obviously true.

Let's give a more interesting example of a subset of $\mathbb{R}$ that is bounded above.

Example 4.4. Define S to be those real numbers whose squares are less than 2:

$$S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

I claim S is bounded above. In fact, I'll prove 2 is an upper bound: Suppose $x \in S$. If $x > 2$, then $x \cdot x > x \cdot 2$ and $x \cdot 2 > 2 \cdot 2$, and hence $x^2 > 4 > 2$. This contradicts the fact that $x \in S$. So, we must have $x \leq 2$.

A nearly identical argument shows that 1.5 is also an upper bound (since $1.5^2 = 2.25 > 2$) and similarly one can show 1.42 is an upper bound. But 1.41 is not an upper bound. For note that $1.411^2 = 1.99091$ and so $1.41 \in S$ but $1.411 > 1.41$.

Question: What is the smallest (or least) upper bound for this set S? Clearly, it ought to be $\sqrt{2}$ (i.e., the positive number whose square is equal to exactly 2), but there's a catch: how do we know that such real number exists?

Definition 4.5. Suppose S is subset of $\mathbb{R}$ that is bounded above. A *supremum* (also known as a *least upper bound*) of S is a number ℓ such that

(1) ℓ is an upper bound of S (i.e., $s \leq \ell$ for all $s \in S$) and
(2) if b is any upper bound of S, then $\ell \leq b$.

Example 4.6. 1 is a supremum of $(0, 1)$. Indeed, it is clearly an upper bound, and in the "Claim" above, we proved that if b is any upper bound of $(0,1)$ then $b \geq 1$. Note that this example shows that a supremum of S does not necessarily belong to S.

Example 4.7. I claim 1 is a supremum of $(0,1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$. It is by definition an upper bound. If b is any upper bound of $(0,1]$ then, since $1 \in (0,1]$, by definition we have $1 \leq b$. So 1 is the supremum of $(0,1]$.

The subset $\mathbb{N}$ does not have a supremum since, indeed, it does not have any upper bounds at all.

Can you think of an example of a set that is bounded above but has no supremum? There is only one such example and it is rather silly: the empty set is bounded above. Indeed, every real number is an upper bound for the empty set. So, there is no least upper bound.

Having explained the meaning of the term "supremum", I can finally state the all-important completeness axiom:

Axiom (Completeness Axiom)**.** *Every nonempty, bounded-above subset of $\mathbb{R}$ has a supremum.*

5. Monday, February 8

Note that I keep saying *a* supremum of a set, but in fact, when they exist, there is only one possible supremum of a given set.

Proposition 5.1. *If a subset of $\mathbb{R}$ has a supremum, then it is unique.*

Preproof Discussion 2. *The proposition has the general form "If a thing with property P exists, then it is unique".*

How do we prove a statement such as "If a thing with property P exists, then it is unique"? We argue that if two things x and y both have property P, then x and y must be the same thing.

Proof. Suppose both x and y are both suprema of the same subset S of $\mathbb{R}$. Then, since y is an upper bound of S and x is a supremum of S, by part (2) of the definition of "supremum" we have $y \geq x$. Likewise, since x is an upper bound of S and y is a supremum of S, we have $x \geq y$ by definiton. Since $x \leq y$ and $y \leq x$, we conclude $x = y$. $\square$

From now on we will speak of *the* supremum of a set (when it exists).

Let us now explore consequences of the completeness axiom. First up, we show that it implies that $\sqrt{2}$ really exists:

Proposition 5.2. *There is a positive real number whose square is* 2.

Proof. Define S to be the subset

$$S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

S is nonempty since, for example, $1 \in S$, and it is bounded above, since, for example, 2 is an upper bound for S, as we showed earlier. So, by the Completeness Axiom, S has a least upper bound, and we know it is unique from the proposition above. Let us call it ℓ. I will prove $\ell^2 = 2$.

We know one of $\ell^2 > 2$, $\ell^2 < 2$ or $\ell^2 = 2$ must hold. We prove $\ell^2 = 2$ by showing that both $\ell^2 > 2$ and $\ell^2 < 2$ are impossible.

We start by observing that $1 \leq \ell \leq 2$. The inequality $1 \leq \ell$ holds since $1 \in S$ and ℓ is an upper bound of S, and the inequality $\ell \leq 2$ holds since 2 is an upper bound of S and ℓ is the least upper bound of S.

Suppose $\ell^2 < 2$. We show this leads to a contradiction by showing that ℓ is not an upper bound of S in this case. We will do this by constructing a number that is ever so slightly bigger than ℓ and belongs to S. Let $\varepsilon = 2 - \ell^2$. Then $0 < \varepsilon \leq 1$ (since $\ell^2 < 2$ and $\ell^2 \geq 1$). We will now show that $\ell + \varepsilon/5$ is in S: We have

$$(\ell + \varepsilon/5)^2 = \ell^2 + \frac{2}{5}\ell\varepsilon + \frac{\varepsilon^2}{25} = \ell^2 + \varepsilon(\frac{2\ell}{5} + \frac{\varepsilon}{25}).$$

Now, using $\ell \leq 2$ and $0 < \varepsilon \leq 1$, we deduce

$$0 < \frac{2\ell}{5} + \frac{\varepsilon}{25} \leq \frac{4}{5} + \frac{\varepsilon}{25} < 1.$$

Putting these equations and inequalities together yields

$$(\ell + \frac{\varepsilon}{5})^2 < \ell^2 + \varepsilon = 2.$$

So, $\ell + \frac{\varepsilon}{5} \in S$ and yet $\ell + \frac{\varepsilon}{5} > \ell$, contradicting the fact that l is an upper bound of S. We conclude $\ell^2 < 2$ is not possible.

Assume now that $\ell^2 > 2$. Our strategy will be to construct a number ever so slightly smaller than ℓ, which therefore cannot be an upper bound of S, and use this to arrive at a contradiction. Let $\delta = \ell^2 - 2$. Then $0 < \delta \leq 2$ (since $\ell \leq 2$ and hence $\ell^2 - 2 \leq 2$). Since $\delta > 0$, we have $\ell - \frac{\delta}{5} < \ell$. Since ℓ is the least upper bound of S, $\ell - \frac{\delta}{5}$ must not be an upper bound of S. By definition, this means that there is $r \in S$ such that $\ell - \frac{\delta}{5} < r$. Since $\delta \leq 2$ and $\ell \geq 1$, it follows that $\ell - \frac{\delta}{5}$ is positive and hence so is r. We may thus square both sides of $\ell - \frac{\delta}{5} < r$ to obtain

$$(\ell - \frac{\delta}{5})^2 < r^2.$$

Now

$$(\ell - \frac{\delta}{5})^2 = \ell^2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} = \delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25}$$

since $\ell^2 = \delta + 2$. Moreover,

$$\delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} = 2 + \delta(1 - \frac{2\ell}{5} + \frac{\delta}{25}) \geq 2 + \delta(1 - \frac{4}{5} + \frac{\delta}{25})$$

since $\ell \leq 2$. We deduce that

$$\delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} \geq 2 + \delta(\frac{1}{5}) \geq 2.$$

Putting these inequalities together gives $r^2 > 2$, contrary to the fact that $r \in S$. We conclude that $\ell^2 > 2$ is also not possible.

Since $\ell^2 < 2$ and $\ell^2 > 2$ are impossible, we must have $\ell^2 = 2$. $\square$

The collection of rational numbers does not satisfy the completeness axiom and indeed it is precisely the completeness axiom that differentiates $\mathbb{R}$ from $\mathbb{Q}$.

Example 5.3. Within the set $\mathbb{Q}$ the subset $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ does not have a supremum. That is, no matter which rational number you pick that is an upper bound for S, you may always find an even smaller one that is also an upper bound of S.

It is precisely the completeness axiom that assures us that everything that ought to be a number (like the length of the hypotenuse in an isosceles right triangles of side length 1) really is a number. It gives us that there are "no holes" in the real number line — the real numbers are *complete*.

For example, we can use it to prove that $\sqrt[8]{147}$ exists: Let $S = \{x \in \mathbb{R} \mid x^8 < 147\}$. Then S is nonempty (e.g., $0 \in S$) and bounded above (e.g., 50 is an upper bound) and so it must have a supremum ℓ. A proof similar to (but even messier than) the proof of Proposition 5.2 above shows that ℓ satisfies $\ell^8 = 147$.

The completeness axiom is also at the core of the Intermediate Value Theorem and many of the other major theorems we will cover in this class.

6. Wednesday, February 10

We now discuss a few consequence of the completeness axiom.

Theorem 6.1. *If x is any real number, then there exists a natural number n such that $n > x$.*

This looks really stupid at first. How could it be false? But consider: there are examples of ordered fields, i.e. situations in which Axioms 1–10 hold, in which this Theorem is not true! So, its proof must rely on the Completeness Axiom.

Proof. Let x be any real number. By way of contradiction, suppose there is no natural number n such that $n > x$. That is, suppose that for all $n \in \mathbb{N}$, $n \leq x$. Then $\mathbb{N}$ is a bounded above (by x). Since it is also clearly nonempty, by the Completeness Axiom, $\mathbb{N}$ has a supremum, call it ℓ. Consider the number $y := \ell - 1$. Since $y < \ell$ and ℓ is the supremum of $\mathbb{N}$, y cannot be an upper bound of $\mathbb{N}$. So, there must be some $m \in \mathbb{N}$ such that such that $\ell - 1 < m$. But then by adding 1 to both sides of this inequality we get $\ell < m + 1$ and, since $m + 1 \in \mathbb{N}$, this contradicts that assumption that ℓ is the supremum of $\mathbb{N}$.

We conclude that, given any real number x, there must exist a natural number n such that $n > x$. □

Corollary 6.2 (Archimedean Principle)**.** *If $a \in R$, $a > 0$ and $b \in \mathbb{R}$, then for some natural number n we have $na > b$.*

"No matter how small a is and how large b is, if we add a to itself enough times, we can overtake b."

Proof. We apply Theorem 6.1 to the real number $x = \frac{b}{a}$. It gives that there is a natural number n such that $n > x = \frac{b}{a}$. Since $a > 0$, upon multiplying both sides by a we get $n \cdot a > b$. □

Corollary 6.3 (Density of the Rational Numbers)**.** *Between any two distinct real numbers there is a rational number; more precisely, if $x, y \in \mathbb{R}$ and $x < y$, then there exists $q \in \mathbb{Q}$ such that $x < q < y$.*

Proof. We will prove this by consider two cases: $x \geq 0$ and $x < 0$.

Let us first assume $x \geq 0$. We apply the Archimedean Principle using $a = y - x$ and $b = 1$. (The Principle applies as $a > 0$ since $y > x$.) This gives us that there is a natural number $n \in \mathbb{N}$ such that

$$n \cdot (y - x) > 1$$

and thus

$$0 < \frac{1}{n} < y - x.$$

Since $\frac{1}{n} > 0$, using the Archimedean principle again, there is at least one natural number p such that $p \cdot \frac{1}{n} > x$. By the Well Ordering Axiom, there is a smallest natural number such that $m \cdot \frac{1}{n} > x$; call it m.

We claim that $\frac{m-1}{n} \leq x$. Indeed, if $m > 1$, then $m - 1 \in \mathbb{N} \setminus S$ (because $m - 1$ is less than the minimum), so $\frac{m-1}{n} \leq x$; if $m = 1$, then $m - 1 = 0$, so $\frac{m-1}{n} = 0 \leq x$.

So, we have

$$\frac{m-1}{n} \leq x < \frac{m}{n}$$

By adding $\frac{1}{n}$ to both sides of $\frac{m-1}{n} \leq x$ and using that $\frac{1}{n} < y - x$, we get

$$\frac{m}{n} \leq x + \frac{1}{n} < x + (y - x) = y$$

and hence

$$x < \frac{m}{n} < y.$$

Since $\frac{m}{n}$ is clearly a rational number, this proves the result in this case (when $x > 0$).

We now consider the case $x < 0$. The idea here is to simply "shift" up to the case we've already proven. By Theorem 6.1, we can find a natural number j such that $j > -x$ and thus $0 < x + j < y + j$. Using the first case, which we have already proven, applied to the number $x+j$ (which is positive), there is a rational number q such that $x + j < q < y + j$. We deduce that $x < q - j < y$, and, since $q - j$ is also rational, this proves the corollary in this case. □

Let me say a bit more about the density of the rationals: it is a consequence of this result that between any two distinct real numbers there are *infinitely many* rational numbers: For if $x, y \in \mathbb{R}$ and $x < y$, them by the Corollary there is a rational number q_1 with $x < q_1 < y$. But then we can apply the Corollary again using x and q_1, to obtain the existence of a rational number q_2 with $x < q_2 < q_1$, and yet again using x and q_2 to obtain $q_3 \in \mathbb{Q}$ with $x < q_3 < q_2$, and so on forever.

A real number is called *irrational* if it is not rational. For example, $\sqrt{2}$ is irrational, a fact we have proven: we proved it exists as a real number, using the axioms, and earlier we showed that it cannot be rational.

Corollary 6.4 (Density of the Irrational Numbers)**.** *Between any two distinct real numbers there is an irrational number; more precisely, if $x, y \in \mathbb{R}$ and $x < y$, then there exists an irrational number z such that $x < z < y$.*

Proof. We will prove this by using the Density of the Rational Numbers along with the fact that we know of at least one irrational number: $\sqrt{2}$.

Suppose $x, y \in \mathbb{R}$ and $x < y$. Then $x - \sqrt{2} < y - \sqrt{2}$ and, by the Density of Rational Numbers, there is a rational number q such that $x - \sqrt{2} < q < y - \sqrt{2}$. By adding though by $\sqrt{2}$ we obtain

$$x < q + \sqrt{2} < y.$$

As we showed earlier, the sum of a rational and an irrational number is always irrational. In particular, $q + \sqrt{2}$ is irrational. By letting $z = q + \sqrt{2}$ we have proven the Corollary. □

As with rational numbers, between any two distinct real numbers there are in fact infinitely many irrational numbers. (In particular, there are infinitely many irrational numbers, which is not something we've proven up until this point.)

Discussion Questions, February 12

(1) Let $S \subseteq \mathbb{R}$ be bounded above. Prove that there are infinitely many distinct upper bounds for S.

> Let $S \subseteq \mathbb{R}$ be bounded above. Let b be an upper bound for S. For any $n \in \mathbb{N}$, $b+n$ is an upper bound for S, since, given $s \in S$, we have $s \leq b \leq b+n$. As the set of numbers of the form $b+n$ for $n \in \mathbb{N}$ is infinite, we have exhibited infinitely many upper bounds.

(2) Let $S \subseteq \mathbb{R}$ be nonempty and bounded above. Let $T = \{3x \mid x \in S\}$. Prove that $\sup(T) = 3\sup(S)$.

(3) Compute the supremum of the set $S = \{\frac{2n-1}{n+1} \mid n \in \mathbb{N}\}$, and prove your answer is correct.

> We will show that 2 is the supremum of this set.
>
> First we show that 2 is an upper bound. Let $s \in S$. We can write $s = \frac{2n-1}{n+1}$ for some $n \in \mathbb{N}$. We then have $s = \frac{2n-1}{n+1} \leq \frac{2n+2}{n+1} = 2$, as required.
>
> Now, let b be an upper bound for S. We need to show that $b \geq 2$. To obtain a contradiction, suppose that $b < 2$, and let $\varepsilon = 2 - b$. We will show that there exists some element of s that is greater than b, which will be the desired contradiction. Since $\varepsilon > 0$, by Theorem 5.1, there exists $n \in \mathbb{N}$ such that $n > \frac{3}{\varepsilon}$. Multiplying both sides by $\frac{\varepsilon}{n}$, this implies that $\varepsilon > \frac{3}{n}$. Then, $\frac{2n-1}{n+1} = 2 - \frac{3}{n+1} > 2 - \frac{3}{n} > 2 - \varepsilon = b$. This contradicts that b is an upper bound for S, so we must have $b \geq 2$, as required.

7. Monday, February 15

We will move on to next main topic of this class soon: sequences. But first, it is useful to talk a bit about absolute values.

Definition 7.1. If x is any real number we define the *absolute value* of x, written $|x|$, to be the real number

$$|x| = \begin{cases} x & \text{if } x \geq 0 \text{ and} \\ -x & \text{if } x < 0. \end{cases}$$

Proposition 7.2. *Let x and y be arbitrary real numbers and let k be a positive real number ($k > 0$). Then*

(1) $-|x| \leq x \leq |x|$,
(2) $|-x| = |x|$ *and* $|x-y| = |y-x|$,
(3) $|x| \leq k$ *if and only if* $-k \leq x \leq k$,
(4) $|x| \geq k$ *if and only if* $x \leq -k$ *or* $x \geq k$, *and*
(5) $|x \cdot y| = |x| \cdot |y|$.

We won't prove this proposition in full since each statement is just an easy application of the definiton. But, to get a feeling for how each part is proven, let's prove one of them, part (3):

Proof of Part (3) of the Proposition. ($\Rightarrow$) Suppose $|x| \leq k$. We consider two cases:

Case I: If $x \geq 0$ then by definition $|x| = x$ and hence by assumption $x = |x| \leq k$. The inequality $-k \leq x$ also holds, since $k \geq 0$ and hence $-k \leq 0$ and we are assuming $x \geq 0$. Thus $-k \leq x \leq k$ in this case.

For the other case, assume now that $x < 0$. Then $|x| = -x$ and so by assumption we have $-x < k$. Multiplying through by -1 gives $-k < x$. The inequality $x < k$ also holds since we are assuming $x < 0$ for this case and $0 \leq k$. So $-k < x < k$ holds in this case too.

($\Leftarrow$) Suppose $-k \leq x \leq k$. We again consider two cases: If $x \geq 0$, then $|x| = x$ and so $|x| \leq k$ is immediate. If $x < 0$, then $|x| = -x$. From $-k \leq x$ we get $-x \leq k$ and thus $|x| \leq k$. □

It will be important for us to interpret absolute values in terms of distance. For any two real numbers x and y, the number $|x-y|$ is the distance between them. By the Proposition $|x-y| = |y-x|$, which in geometric language says that the distance from x to y is the same as the distance from y to x.

Example 7.3. The set of all real numbers x such that $|x-7| \leq 2$ is the closed interval $[5, 9]$. To see this using the Proposition, note that $|x-7| \leq 2$ if and only if $-2 \leq x-7 \leq 2$ by Part (3). Now add through by 7 to get $5 \leq x \leq 9$. So $\{x \in \mathbb{R} \mid |x-7| \leq 2\} = \{x \in \mathbb{R} \mid 5 \leq x \leq 9\} = [5, 9]$.

Similarly, the set of all real numbers x such that $|x-7| < 2$ is the open interval $(5, 9)$.

Theorem 7.4 (The Triangle Inequality)**.** *For any real numbers a and b we have*

$$|a+b| \leq |a| + |b|.$$

Remark 7.5. You might recall that vector form of the triangle inequality says $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$, where $\| - \|$ refers to the length of vectors. This version has a nice interpretation in terms of lengths of the sides of a triangle.

For ordinary numbers x, y, z, there is a version of this coming from the triangle inequality: Since $(x-y)+(y-z) = (x-z)$, taking $a = x-y$ and $b = y-z$ in the Triangle Inequality gives

$$|x-z| \leq |x-y| + |y-x|$$

which can be interpreted as "this distance from x to z is at most the sum of the distances from x to y and from y to z".

Proof. Let a and b be any real numbers. Part (1) of the previous Proposition gives

$$-|a| \leq a \leq |a| \text{ and } -|b| \leq b \leq |b|.$$

We add these to get

$$-(|a|+|b|) = -|a| - |b| \leq a+b \leq |a| + |b|$$

Applying part (3) of the previous Proposition (with $k = |x| + |y|$ and the x in that Proposition replaced by $a+b$), we get

$$|a+b| \leq |a| + |b|.$$

$\square$

Remark 7.6. You can also prove the Triangle Inequality just by considering all possible cases for the signs of a, b and $a+b$. But there are (nearly) 8 such cases and so that proof is rather tedious.

Corollary 7.7 (The Reverse Triangle Inequality)**.** *For any real numbers x and y we have*

$$|x-y| \geq \left||x| - |y|\right|.$$

Proof. Since $x = (x-y)+y$, by the ordinary Triangle Inequality we get

$$|x| = |(x-y)+y| \leq |x-y| + |y|$$

and thus

$$|x-y| \geq |x| - |y|.$$

By interchanging the roles of x and y in the preceding argument we get

$$|y-x| \geq |y| - |x|$$

and since $|x-y| = |y-x|$ and $|y|-|x| = -(|x|-|y|)$, we get

$$|x-y| \geq -(|x|-|y|).$$

Since $||x|-|y||$ is either $|x|-|y|$ or $-(|x|-|y|)$, this proves the statement. $\square$

We now turn our attention to the next major topic of this class: sequences of real numbers. We will spend the next few weeks developing their properties carefully and rigorously. Sequences form the foundation for much of what we will cover for the rest of the semester.

Definition 7.8. A *sequence* is an infinite list of real numbers indexed by $\mathbb{N}$:

$$a_1, a_2, a_3, \ldots.$$

(Equivalently, a sequence is a function from $\mathbb{N}$ to $\mathbb{R}$: the value of the function at $n \in \mathbb{N}$ is written as a_n.)

We will usually write $\{a_n\}_{n=1}^{\infty}$ for a sequence.

Example 7.9. To describe sequences, we will typically give a formula for the n-th term, a_n, either an explicit one or a recursive one. On rare occasion we'll just list enough terms to make the pattern clear. Here are some examples:

(1) $\{5+(-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ is the sequence that starts

$$4, \frac{11}{2}, \frac{14}{3}, \frac{21}{4}, \frac{24}{5}, \ldots.$$

(2) Let $\{a_n\}_{n=1}^{\infty}$ be defined by $a_1 = 1, a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. This gives the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

This is an example of a recursively defined sequence. It is the famed *Fibonacci sequence.*

(3) Let $\{c_n\}_{n=1}^{\infty}$ be the sequence whose n-th term is the n-th smallest positive prime integer:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots.$$

Note that here I have not really given an explicit formula for the terms of the sequence, but it is possible to describe an algorithm that lists every term of the sequence in order.

You have all probably seen an "intuitive" definition of the limit of a sequence before. For example, you probably believe that

$$\lim_{n\to\infty} 5 + (-1)^n \frac{1}{n} = 5.$$

Let's give the rigorous definition.

Definition 7.10. Let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence and L a real number. We say $\{a_n\}_{n=1}^{\infty}$ *converges* to L provided the following condition is met:

> For every real number $\varepsilon > 0$, there is a real number N such that $|a_n - L| < \varepsilon$ for all natural numbers n such that $n > N$.

This is an extremely important definition for this class. Learn it by heart!

The definition of convergence can be rewritten in a number of ways to make it read better. Here is one such way:

> A sequence $\{a_n\}_{n=1}^{\infty}$ converges to L provided for every real number $\varepsilon > 0$, there is a real number N such that if $n \in \mathbb{N}$ and $n > N$, then $|a_n - L| < \varepsilon$.

In symbols, the definition is

> A sequence $\{a_n\}_{n=1}^{\infty}$ converges to L provided $\forall \varepsilon > 0, \exists N \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$ satisfying $n > N$, we have $|a_n - L| < \varepsilon$.

It's a complicated definition — three quantifiers!

Here is what the definition is saying somewhat loosely: No matter how small a number ε you pick, so long as it is positive, if you go far enough out in the sequence, all of the terms from that point on will be within a distance of ε of the limiting value L.

Example 7.11. I claim the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 5 + (-1)^n \frac{1}{n}$ converges to 5. I'll give a rigorous proof, along with some commentary and "scratch work" within the parentheses.

Proof. Let $\varepsilon > 0$ be given.

(Scratch work: Given this ε, our goal is to find N so that if $n > N$, then $|5+(-1)^n \frac{1}{n} - 5| < \varepsilon$. The latter simplifies to $\frac{1}{n} < \varepsilon$, which in turn is equivalent to $\frac{1}{\varepsilon} < n$ since ε and n are both positive. So, it seems we've found the N that "works". Back to the formal proof....)

Let $N = \frac{1}{\varepsilon}$. Then $\frac{1}{N} = \varepsilon$, since ε is positive.

(Comment: We next show that this is the N that "works" in the definition. Since this involves proving something about every natural number that is bigger than N, we start by picking one.)

Pick any $n \in \mathbb{N}$ such that $n > N$. Then $\frac{1}{n} < \frac{1}{N}$ and hence

$$|a_n - 5| = |5 + (-1)^n \frac{1}{n} - 5| = |(-1)^n \frac{1}{n}| = \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

This proves that $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ converges to 5. $\square$

8. Wednesday, February 17

Remark 8.1. A direct proof that a certain sequence converges to a certain number follows the general outline:

- Let $\varepsilon > 0$ be given. (or, if your prefer, "Pick $\varepsilon > 0$.")
- Let $N =$ [insert appropriate expression in terms of from scratch work here].
- Let $n \in \mathbb{N}$ be such that $n > N$.

- [Argument that $|a_n - L| < \varepsilon$]
- Thus $\{a_n\}_{n=1}^{\infty}$ converges to L.

Example 8.2. I claim that the sequence

$$\left\{\frac{2n-1}{5n+1}\right\}_{n=1}^{\infty}$$

congerges to $\frac{2}{5}$. Again I'll give a proof with commentary and scratch work in parentheses.

Proof. Let $\varepsilon > 0$ be given.

(Scratch work: We need n to be large enough so that

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| < \varepsilon.$$

This simplifies to $\left|\frac{-7}{25n+5}\right| < \varepsilon$ and thus to $\frac{7}{25n+5} < \varepsilon$, which we can rewrite as $\frac{7}{25\varepsilon} - \frac{1}{5} < n$.)

Let $N = \frac{7}{25\varepsilon} - \frac{1}{5}$. We solve this equation for ε: We get $\frac{7}{25\varepsilon} = \frac{5N+1}{5}$ and hence $\frac{25\varepsilon}{7} = \frac{5}{5N+1}$, which gives finally

$$\varepsilon = \frac{7}{25N+5}.$$

(Next we show this value of N works....)

Now pick any $n \in \mathbb{N}$ is such that $n > N$. Then

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| = \left|\frac{10n-5-10n-2}{25n+5}\right| = \frac{7}{25n+5}.$$

Since $n > N$, $25n + 5 > 25N + 5$ and hence

$$\frac{7}{25n+5} < \frac{7}{25N+5} = \varepsilon.$$

We have proven that if $n \in \mathbb{N}$ and $n > N$, then

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| < \varepsilon.$$

This proves $\left\{\frac{2n-1}{5n+1}\right\}_{n=1}^{\infty}$ converges to $\frac{2}{5}$. $\square$

Definition 8.3. We say a sequence $\{a_n\}_{n=1}^{\infty}$ *converges* or *is convergent* if there is (at least one) number L such that it converges to L. Otherwise, of no such L exists, we say the sequence *diverges* or *is divergent.*

(We'll show soon that if a sequence converges to a number L, then L is the *only* number to which in converges.)

Example 8.4. Let's prove the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent. This means that there is no L to which it converges.

Proof. We proceed by contradiction: Suppose the sequence did converge to some number L. Our strategy will be to derive a contradiction by showing that such an L would have to satisfy mutually exclusive conditions.

By definition, since the sequence converges to L, we have that for every $\varepsilon > 0$ there is a number N such that $|(-1)^n - L| < \varepsilon$ for all natural numbers n such that $n > N$. In particular, this statement is true for the particular value $\varepsilon = \frac{1}{2}$. That is, there is a number N such that $|(-1)^n - L| < \frac{1}{2}$ for all natural numbers n such that $n > N$. Let n be any even natural number that is bigger than N. (Certainly one exists: we know there is an integer bigger than N by Theorem 6.1. Pick one. If it is even, take that to be n. If it is odd, increase it by one to get an even integer n.) Since $(-1)^n = 1$ for an even integer n, we get

$$|1 - L| < \frac{1}{2}$$

and thus $\frac{1}{2} < L < \frac{3}{2}$.

Likewise, let n be an odd natural number bigger than N. Since $(-1)^n = -1$ for an odd integer n, we get

$$|-1 - L| < \frac{1}{2}$$

and thus $-\frac{3}{2} < L < -\frac{1}{2}$. But it cannot be that both $L > \frac{1}{2}$ and $L < -\frac{1}{2}$.

We conclude that no such L exists; that is, this sequence is divergent. $\square$

Proposition 8.5. *If a sequence converges, then there is a unique number to which it converges.*

Proof. Recall that to show something satisfying certain properties is unique, one assumes there are two such things and argues that they must be equal. So, suppose $\{a_n\}_{n=1}^\infty$ is a sequence that converges to L and that also converges to M. We will prove $L = M$.

By way of contradiction, suppose $L \neq M$. Then set $\varepsilon = \frac{|L-M|}{3}$. Since we are assuming $L \neq M$, we have $\varepsilon > 0$. According to the definition of convergence, since the sequence converges to L, there is a real number N_1 such that for $n \in \mathbb{N}$ such that $n > N_1$ we have

$$|a_n - L| < \varepsilon.$$

Also according to the definition, since the sequence converges to M, there is a real number N_2 such that for $n \in \mathbb{N}$ and $n > N_2$ we have

$$|a_n - M| < \varepsilon.$$

Pick n to be any natural number larger than $\max\{N_1, N_2\}$ (which exists by Theorem 6.1. For such an n, both $|a_n - L| < \varepsilon$ and $|a_n - M| < \varepsilon$ hold. Using the triangle inequality and these two inequalities, we get

$$|L - M| \leq |L - a_n| + |M - a_n| < \varepsilon + \varepsilon.$$

But by the choice of ε, we have $\varepsilon + \varepsilon = \frac{2}{3}|L - M|$. That is, we have deduced that $|L - M| < \frac{2}{3}|L - M|$ which is impossible. We conclude that $L = M$. $\square$

9. Friday, February 19

From now on, given a sequence $\{a_n\}_{n=1}^\infty$ and a real number L, will we use the short-hand notation

$$\lim_{n\to\infty} a_n = L$$

to mean that the given sequence converges to the given number. For example, we showed above that

$$\lim_{n\to\infty} \frac{2n-1}{5n+1} = \frac{2}{5}.$$

But, to be clear, the statement "$\lim_{n\to\infty} a_n = L$" signifies nothing more and nothing less than the statement "$\{a_n\}_{n=1}^\infty$ converges to L".

Here is some terminology we will need:

Definition 9.1. Suppose $\{a_n\}_{n=1}^\infty$ is any sequence.

(1) We say $\{a_n\}_{n=1}^\infty$ is *bounded above* if there exists at least one real number M such that $a_n \leq M$ for all $n \in \mathbb{N}$; we say $\{a_n\}_{n=1}^\infty$ is *bounded below* if there exists at least one real number m such that $a_n \geq m$ for all $n \in \mathbb{N}$; and we say $\{a_n\}_{n=1}^\infty$ is *bounded* if it is both bounded above and bounded below.

(2) We say $\{a_n\}_{n=1}^\infty$ is *increasing* if for all $n \in \mathbb{N}$, $a_n \leq a_{n+1}$; we say $\{a_n\}_{n=1}^\infty$ is *decreasing* if for all $n \in \mathbb{N}$, $a_n \geq a_{n+1}$; and we say $\{a_n\}_{n=1}^\infty$ is *monotone* if it is either decreasing or increasing.

(3) We say $\{a_n\}_{n=1}^\infty$ is *strictly increasing* if for all $n \in \mathbb{N}$, $a_n < a_{n+1}$. I leave the definition of *strictly decreasing* and *strictly monotone* to your imaginations.

Remark 9.2. Be sure to interpret "monotone" correctly. It means

$$(\forall n \in \mathbb{N}, a_n \leq a_{n+1}) \text{ or } (\forall n \in \mathbb{N}, a_n \geq a_{n+1});$$

it does *not* mean

$$\forall n \in \mathbb{N}, (a_n \leq a_{n+1}) \text{ or } (a_n \geq a_{n+1}).$$

Do you see the difference?

Example 9.3. The sequence $\{\frac{1}{n}\}_{n=1}^\infty$ is strictly increasing and bounded (above by, e.g., 1 and below by, e.g., 0).

The Fibonacci sequence $\{f_n\}_{n=1}^\infty = 1, 1, 2, 3, 5, 8, \ldots$ is strictly increasing and bounded below, but not bounded above.

The sequence $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^\infty$ is not monotone, but it is bounded (above by, e.g., 6 and below by, e.g., 4).

Is the sequence of quotients of Fibonacci numbers $\{\frac{f_{n+1}}{f_n}\}_{n=1}^\infty = \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots$ monotone? Is it bounded? Convergent?

10. Monday, February 22

Proposition 10.1. *If a sequence $\{a_n\}_{n=1}^\infty$ converges then it is bounded.*

Proof. Suppose the sequence $\{a_n\}_{n=1}^\infty$ converges to the number L. Applying the definition of "converges to L" using the particular value $\varepsilon = 1$ gives the following fact: There is a real number N such that if $n \in \mathbb{N}$ and $n > N$, then $|a_n - L| < 1$. The latter inequality is equivalent to $L - 1 < a_n < L + 1$ for all $n > N$.

Let m be any natural number such that $m > N$, and consider the finite list of numbers

$$a_1, a_2, \ldots, a_{m-1}, L + 1.$$

Let b be the largest element of this list. I claim the sequence is bounded above by b. For any $n \in \mathbb{N}$, if $1 \leq n \leq m - 1$, then $a_n \leq b$ since in this case a_n is a member of the above list and b is the largest element of this list. If $n \geq m$ then since $m > N$, we have $n > N$ and hence $a_n < L + 1$ from above. We also have $L + 1 \leq b$ (since $L + 1$ is in the list) and thus $a_n < b$. This proves $a_n \leq b$ for all n as claimed.

Now take p to be the smallest number in the list

$$a_1, a_2, \ldots, a_{m-1}, L - 1.$$

A similar argument shows that $a_n \geq p$ for all $n \in \mathbb{N}$. □

When I introduced the Completeness Axiom, I mentioned that, heuristically, it is what tells us that the real number line doesn't have any holes. The next result makes this a bit more precise:

Theorem 10.2. *Every increasing, bounded above sequence converges.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be any sequence that is both bounded above and increasing.

(Commentary: In order to prove it converges, we need to find a candidate number L that it converges to. Since the set of numbers occurring in this sequence is nonempty and bounded above, this number is provided to us by the Completeness Axiom.)

Let S be the set of those real numbers that occur in this sequence. (This is technically different that the sequence itself, since sequences are allowed to have repetitions but sets are not. Also, sequences have an ordering to them, but sets do not.) The set S is clearly nonempty, and it is bounded above since we assume the sequence is bounded above. Therefore, by the Completeness Axiom, S has a supremum L. We will prove the sequence converges to L.

Pick $\varepsilon > 0$. Then $L - \varepsilon < L$ and, since L is the supremum, $L - \varepsilon$ is not an upper bound of S. This means that there is an element of S that is stricly bigger than $L - \varepsilon$. Every element of S is a member of the sequence, and so we get that there is an $N \in \mathbb{N}$ such that $a_N > L - \varepsilon$.

(We will next show that this is the N that "works". Note that, in the general definition of convergence of a sequence, N can be any real number, but in this proof it turns out to be a natural number.)

Let n be any natural number such that $n > N$. Since the sequence is increasing, $a_N \leq a_n$ and hence

$$L - \varepsilon < a_N \leq a_n.$$

Also, $a_n \leq L$ since L is an upper bound for the sequence, and thus we have

$$L - \varepsilon < a_n \leq L.$$

It follows that $|a_n - L| < \varepsilon$. We have proven the sequence converges to L. □

You will prove in the homework that any decreasing, bounded below sequence converges. Putting this together with the previous theorem yields the following.

Theorem 10.3 (Monotone Converge Theorem)**.** *Every bounded monotone sequence converges.*

Example 10.4. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ given by the formula

$$a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

We will use the Monotone Convergence Theorem to prove that this sequence converges.

First, we need to see that the sequence is increasing. Indeed, for every n we have that $a_{n+1} = a_n + \frac{1}{a_{n+1}^2} \geq a_n$.

Next, we need to show that it is bounded above. Observe that

$$\begin{aligned} a_n &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &\leq 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} \\ &= 1 + (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n-1} - \frac{1}{n}) \\ &= 1 + 1 - \frac{1}{n}, \end{aligned}$$

so we have $a_n \leq 2$ for all n. This means that $\{a_n\}_{n=1}^{\infty}$ is bounded above by 2.

Hence, by the Monotone Convergence Theorem, $\{a_n\}_{n=1}^{\infty}$ converges. Leonhard Euler was particularly interested in this sequence, and was able to prove that it converges to $\frac{\pi^2}{6}$. This requires some other ideas, so we won't do that here.

As we have seen, proving sequences converge using just the definition can be tedious and hard, and finding limits can be tricky. The next very long Theorem will make the task easier in some cases.

Theorem 10.5. *The following six things hold.*

(1) *For any real number c, the constant sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = c$ converges to c.*
(2) *The sequence $\{1/n\}_{n=1}^{\infty}$ converges to 0.*

For the remaining parts, assume $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are any two sequences that both converge.

(3) *The sequence $\{a_n + b_n\}_{n=1}^{\infty}$ also converges and*

$$\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} (a_n) + \lim_{n \to \infty} (b_n).$$

(4) *For any real number c, the sequence $\{c \cdot a_n\}_{n=1}^{\infty}$ also converges and*
$$\lim_{n\to\infty}(c \cdot a_n) = c \cdot \lim_{n\to\infty}(a_n).$$
(5) *The sequence $\{a_n \cdot b_n\}_{n=1}^{\infty}$ also converges and*
$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty}(a_n) \cdot \lim_{n\to\infty}(b_n).$$
(6) *If $b_n \neq 0$ for all n and $\lim_{n\to\infty}(b_n) \neq 0$, then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ also converges and*
$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}.$$

Note that in the last item of the Theorem, we have to assume $b_n \neq 0$ for all n (in order that the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ be well defined), and we also have to assume $\lim_{n\to\infty}(b_n) \neq 0$ (so that the right-hand side makes sense). The latter assumption does not follow from the former: for example, if $b_n = \frac{1}{n}$ then $b_n \neq 0$ for all n but $\lim_{n\to\infty} b_n = 0$.

Example 10.6. Before proving (parts of) the theorem, let us illustrate it by redoing our justification that the sequence $\left\{\frac{2n-1}{5n+1}\right\}_{n=1}^{\infty}$ converges to $\frac{2}{5}$. At first blush, this looks to be impossible since $\{2n-1\}_{n=1}^{\infty}$ does not converge and so the hypotheses are not met. The trick is to first rewrite the n-th term as
$$\frac{2n-1}{5n+1} = \frac{2-1/n}{5+1/n}.$$

By the Theorem, Part (2) the sequence $\{1/n\}_{n=1}^{\infty}$ converges to 0 and by Part (1) the constant sequence 5 converges to 5. So, by applying Part (3) of the theorem we deduce that $\{5+1/n\}$ converges to 5. Similarly, Parts (2) and (3) give that $\{-1/n\}_{n=1}^{\infty}$ converges to 0 and so by Parts (1) and (3), $\{2-1/n\}_{n=1}^{\infty}$ converges to 2. Finally, by applying Part (6) of Theorem we conclude that $\left\{\frac{2n-1}{5n+1}\right\}_{n=1}^{\infty}$ converges to $\frac{2}{5}$.

Discussion Questions, February 24

- Let c be a real number, and $\{c\}_{n=1}^{\infty}$ be the sequence where every term is equal to c. Prove that this sequence converges to c.

- Prove that the sequence $\{1/n\}_{n=1}^{\infty}$ converges to 0.

- Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that converges to L, and $\{b_n\}_{n=1}^{\infty}$ be a sequence that converges to M. Prove that $\{a_n+b_n\}_{n=1}^{\infty}$ converges to $L+M$.

[Hint: Given $\varepsilon > 0$, apply the definitions of "$\{a_n\}_{n=1}^{\infty}$ converges to L" and "$\{b_n\}_{n=1}^{\infty}$ converges to M" with the value $\frac{\varepsilon}{2}$.]

- Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that converges to L, and c be a real number. Prove that $\{ca_n\}_{n=1}^{\infty}$ converges to cL.

- Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that converges to L. Assume that $a_n \neq 0$ for all $n \in \mathbb{N}$ and that $L \neq 0$. Prove that $\{1/a_n\}_{n=1}^{\infty}$ converges to $1/L$.

All of these are parts of Theorem 10.5. Here is a proof of all of the parts of Theorem 10.5, with scratch work; the numbering is that of Theorem 10.5 rather than the discussion questions.

Proof. To prove (1), pick $\varepsilon > 0$. Let $N = 0$ (or, really, any number you want). If $n \in \mathbb{N}$ and $n > N$, then $|a_n - c| = |c - c| = 0 < \varepsilon$ and hence the constant sequence $\{c\}_{n=1}^{\infty}$ converges to c.

To prove (2), we pick $\varepsilon > 0$. Let $N = \frac{1}{\varepsilon}$. If $n \in \mathbb{N}$ and $n > N$, then

$$|\frac{1}{n} - 0| = \frac{1}{n} > \frac{1}{N} = \varepsilon$$

and thus $\{1/n\}_{n=1}^{\infty}$ converges to 0.

For the rest of this proof, assume $\{a_n\}_{n=1}^{\infty}$ converges to L and $\{b_n\}_{n=1}^{\infty}$ converges to M.

For Part (3), we need to prove $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$. Pick $\varepsilon > 0$.

("Scratch work": We need to figure out how big n needs to be in order that $|(a_n + b_n) - (M + L)| < \varepsilon$. Note that $|(a_n + b_n) - (M + L)| = |(a_n - M) + (b_n - L)| \leq |(a_n - M)| + |(b_n - L)|$ by the triangle inequality. Intuitively, we can make each of $|(a_n - M)|$ and $|(b_n - L)|$ as small as we like by taking n large enough. We need their sum to be smaller than ε and so if we can make each of them be smaller that $\varepsilon/2$, we're golden. Back to the proof...)

Since $\{a_n\}_{n=1}^{\infty}$ converges to L and $\frac{\varepsilon}{2}$ is positive, there is a number N_1 such that for all $n \in \mathbb{N}$ with $n > N_1$ we have

$$|a_n - L| < \frac{\varepsilon}{2}.$$

Likewise, since $\{b_n\}_{n=1}^{\infty}$ converges to M, there is a number N_2 such that for all $n \in \mathbb{N}$ with $n > N_2$ we have

$$|b_n - M| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. If $n \in \mathbb{N}$ and $n > N$, then $n > N_1$ and $n > N_2$ and hence we have

$$|a_n - L| < \frac{\varepsilon}{2} \text{ and } |b_n - M| < \frac{\varepsilon}{2}.$$

Using these inequalities and the triangle inequality we get

$$|(a_n+b_n)-(M+L)| = |(a_n-M)+(b_n-L)| \leq |(a_n-M)|+|(b_n-L)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

This proves that $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$.

Part (4) will follow from parts (1) and (5) put together, but we can also prove it on its own. We note first that if $c = 0$, then $ca_n = 0$ for all n, and hence by part (1), we have $\{ca_n\}_{n=1}^{\infty}$ converges to 0 in this case. Now, assume that $c \neq 0$, and let $\varepsilon > 0$ be given.

("Scratch work": We need n to be large enough so that $|ca_n - cL| < \varepsilon$. We can write $|ca_n - cL| = |c||a_n - L|$, so if $|a_n - L| < \frac{\varepsilon}{|c|}$, we'll be set.)

By definition of convergence, there is some $N \in \mathbb{R}$ such that $|a_n - L| < \frac{\varepsilon}{|c|}$ for all natural numbers $n > N$. (Note here that it is important that $c \neq 0$; this is why we singled out the case $c = 0$ first.) Then, for all natural numbers $n > N$, we have $|ca_n - cL| = |c||a_n - L| < |c|\varepsilon/|c| = \varepsilon$, as required.

For (5), we need to prove $\{a_n \cdot b_n\}_{n=1}^{\infty}$ converges to $L \cdot M$. Pick $\varepsilon > 0$.

("Scratch work": The goal is to make $|a_n b_n - LM|$ small and the trick is to use that

$$\begin{aligned}|a_n b_n - LM| &= |a_n(b_n - M) + (a_n - L)M| \\ &\leq |a_n(b_n - M)| + |(a_n - L)M| \\ &= |a_n||b_n - M| + |a_n - L||M|\end{aligned}$$

Our goal will be to take n to be large enough so that each of $|a_n||b_n - M|$ and $|a_n - L||M|$ is smaller than $\varepsilon/2$. We can make $|a_n - L|$ as small as we like and $|M|$ is just a fixed number. So, we can "take care" of the second term by chooseing n big enough so that $|a_n - L| < \frac{\varepsilon}{2|M|}$. A irritating technicality here is that $|M|$ could be 0, and so we will use $\frac{\varepsilon}{2|M|+1}$ instead. The other term $|a_n||b_n - M|$ is harder to deal with since each factor varies with n. Here we use that convergent sequence are bounded so that we can find a real number X so that $|a_n| \leq X$ for all n. Then we choose n large enough so that $|b_n - M| < \frac{\varepsilon}{2X}$. back to the proof.)

Since $\{a_n\}$ converges, it is bounded by Proposition 10.1, which gives that there is a strictly positive real number X so that $|a_n| \leq X$ for all $n \in \mathbb{N}$. Since $\{b_n\}$ converges to M and $\frac{\varepsilon}{2X} > 0$, there is a number N_1 so that if $n > N_1$ then $|b_n - M| < \frac{\varepsilon}{2X}$. Since $\{a_n\}$ converges to L and $\frac{\varepsilon}{2|M|+1} > 0$, there is a number N_2 so that if $n \in \mathbb{N}$ and $n > N_2$, then

$|a_n - L| < \frac{\varepsilon}{2|M|+1}$. Let $N = \max\{N_1, N_2\}$. For any $n \in \mathbb{N}$ such that $n > N$, we have

$$\begin{aligned} |a_n b_n - LM| &= |a_n(b_n - M) + (a_n - L)M| \\ &\leq |a_n(b_n - M)| + |(a_n - L)M| \\ &= |a_n||b_n - M| + |a_n - L||M| \\ &< X\frac{\varepsilon}{2X} + \frac{\varepsilon}{2|M|+1}|M| \\ &< \varepsilon. \end{aligned}$$

This proves $\{a_n \cdot b_n\}_{n=1}^{\infty}$ converges to $L \cdot M$.

To prove Part (6), we first prove a slightly weaker statement:

Claim: If the sequence $\{b_n\}_{n=1}^{\infty}$ converges to M, $b_n \neq 0$ for all n and $M \neq 0$, then the sequence $\{\frac{1}{b_n}\}_{n=1}^{\infty}$ converges to $\frac{1}{M}$.

To prove this claim, pick $\varepsilon > 0$.

(Scratch work: We want to show $\left|\frac{1}{b_n} - \frac{1}{M}\right| < \varepsilon$ holds for n sufficiently large. We have

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|M - b_n|}{|b_n||M|}.$$

We can make the top of this fraction as small as we like, but the problem is that the bottom might be very small too since b_n might get very close to 0. But since b_n converges to M and $M \neq 0$ if we go far enough out, it will be close to M. In particular, if b_n is within a distance of $\frac{|M|}{2}$ of M then $|b_n|$ will be at least $\frac{|M|}{2}$. So for n sufficiently large we have $\frac{|b_n - M|}{|b_n||M|} < 2\frac{|b_n - M|}{|M|^2}$. And then for n sufficiently large we also get $|b_n - M| < \frac{|M|^2}{2\varepsilon}$. Back to the formal proof....)

Since $\{b_n\}$ converges to M and $\frac{|M|}{2} > 0$, there is an N_1 such that for $n > N_1$ we have $|b_n - M| < \frac{|M|}{2}$ and hence $|b_n| > \frac{|M|}{2}$. Again using that $\{b_n\}$ converges to M and that $\frac{\varepsilon|M|^2}{2} > 0$, there is an N_2 so that for $n > N_2$ we have $|b_n - M| < \frac{\varepsilon|M|^2}{2}$. Let $N = \max\{N_1, N_2\}$. If $n > N$, then we have

$$\begin{aligned} \left|\frac{1}{b_n} - \frac{1}{M}\right| &= \frac{|b_n - M|}{|b_n||M|} \\ &< \frac{2}{|M|}\frac{|b_n - M|}{|M|} \\ &= 2\frac{|b_n - M|}{|M|^2} \end{aligned}$$

since $|b_n| > |M|/2$ and hence $\frac{1}{|b_n|} < \frac{2}{|M|}$. But then

$$2\frac{|b_n - M|}{|M|^2} < 2\frac{\frac{\varepsilon |M|^2}{2}}{|M|^2} = \varepsilon$$

since $|b_n - M| < \frac{\varepsilon |M|^2}{2}$. Putting these together gives

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| < \varepsilon$$

for all $n > N$. This proves $\{\frac{1}{b_n}\}_{n=1}^{\infty}$ converges to $\frac{1}{M}$.

We have proven the claim. To finish the proof of (6), we use the claim and apply (5) to the convergent sequences $\{a_n\}$ and $\{1/b_n\}$. $\square$

11. Friday, February 26

The following is another useful technique:

Theorem 11.1 (The "squeeze" principle)**.** *Suppose $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are three sequences such that*

- *$\{a_n\}_{n=1}^{\infty}$ converges to L,*
- *$\{c_n\}_{n=1}^{\infty}$ also converges to L (same value), and*
- *there is a real number M such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ such that $n > M$.*

Then $\{b_n\}_{n=1}^{\infty}$ also converges to L,.

The heuristic version of this theorem is:

> If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ and b_n is "eventually" between a_n and c_n, then $\lim_{n\to\infty} b_n = L$ too.

Remark 11.2. Our text assumes $a_n \leq b_n \leq c_n$ holds *for all* $n \in \mathbb{N}$ in its version of this theorem; i.e, it assumes "the b_n's are trapped between the a_n's and the c_n's all the time". The version I've stated here applies to more situations and is only slighlty harder to prove.

Example 11.3. Let us show $\{\frac{(-1)^n}{n}\}_{n=1}^{\infty}$ converges to 0 using the Squeeze Theorem. Note that Theorem 10.5 alone cannot be used in this example. But for all n we have

$$\frac{-1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$$

and Theorem 10.5 does give that

$$\lim_{n\to\infty} \frac{1}{n} = 0 \text{ and } \lim_{n\to\infty} \frac{-1}{n} = -\lim_{n\to\infty} \frac{1}{n} = 0.$$

By the Squeeze Theorem, we conclude $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Proof. Assume $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ both converge to L and that there is a real number M such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ such that $n > M$. We need to prove $\{b_n\}_{n=1}^{\infty}$ converges to L.

Pick $\varepsilon > 0$. Since $\{a_n\}_{n=1}^{\infty}$ converges to L there is a number N_1 such that if $n \in \mathbb{N}$ and $n > N_1$ then $|a_n - L| < \varepsilon$ and hence $L - \varepsilon < a_n < L + \varepsilon$. Likewise, since $\{c_n\}_{n=1}^{\infty}$ converges to L there is a number N_2 such that if $n \in \mathbb{N}$ and $n > N_2$ then $L - \varepsilon < c_n < L + \varepsilon$. Let

$$N = \max\{N_1, N_2, M\}$$

where M is defined as in the statement of the Theorem. If $n \in \mathbb{N}$ and $n > N$, then $n > N_1$ and hence $L - \varepsilon < a_n$, and $n > N_2$ and hence $c_n < L + \varepsilon$, and $n > M$ and hence $a_n < b_n < c_n$. Combining these facts gives that for $n \in \mathbb{N}$ such that $n > N$, we have

$$L - \varepsilon < b_n < L + \varepsilon$$

and hence $|b_n - L| < \varepsilon$. This proves $\{b_n\}_{n=1}^{\infty}$ converges to L. $\square$

End of material for exam 1

12. Monday, March 1

Here is a corollary of the Squeeze Theorem that is sometimes handy.

Corollary 12.1. (1) *If the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 0, then the sequence $\{|a_n|\}_{n=1}^{\infty}$ also converges to 0.*

(2) *If $\{a_n\}_{n=1}^{\infty}$ converges to 0 and $\{b_n\}_{n=1}^{\infty}$ is any bounded sequence, then $\{a_n b_n\}_{n=1}^{\infty}$ converges to 0.*

Proof. (1) Assume $\{a_n\}_{n=1}^{\infty}$ converges to 0. We need to prove $\{|a_n|\}_{n=1}^{\infty}$ converges to 0. Pick $\varepsilon > 0$. Since $\{a_n\}_{n=1}^{\infty}$ converges to 0, there is a number N such that if $n \in \mathbb{N}$ and $n > N$, then $|a_n - 0| < \varepsilon$. For this same N, if $n > N$ then

$$||a_n| - 0| = ||a_n|| = |a_n| < \varepsilon.$$

This proves $\{|a_n|\}_{n=1}^{\infty}$ converges to 0.

(2) Since $\{b_n\}$ is bounded, there is a positive real number X such that $|b_n| \leq X$ for all n. Thus $0 \leq |a_n b_n| \leq X|a_n|$ holds for all n and hence

$$-X|a_n| \leq a_n b_n \leq X|a_n|$$

holds for all n. By the Lemma, since $\{a_n\}_{n=1}^{\infty}$ converges to 0, so does $\{|a_n|\}_{n=1}^{\infty}$. Using Theorem 10.5, we get that $\{X|a_n|\}_{n=1}^{\infty}$ and $\{-X|a_n|\}_{n=1}^{\infty}$ also both converge to 0. Finally, by the Squeeze Theorem, $\{a_n b_n\}_{n=1}^{\infty}$ converges to 0 too. $\square$

Remark 12.2. More generally, if $\{a_n\}_{n=1}^{\infty}$ converges to L, then the sequence $\{|a_n|\}_{n=1}^{\infty}$ also converges to $|L|$, but I will not take the time to prove this now. The converse of this statement is false however. For example, consider the sequence $\{(-1)^n\}_{n=1}^{\infty}$. The sequence $\{|(-1)^n|\}_{n=1}^{\infty}$ is the constant sequence 1 and hence it converges to 1, but the original sequence diverges.

Example 12.3. This Corollary gives another way to prove $\{(-1)^n/n\}$ converges to 0: take $b_n = (-1)^n$ and $a_n = 1/n$.

We will discuss a bit the notion of "diverging to infinity", a concept that you might have seen before in Calculus.

It is sometimes useful to distinguish between sequences like

$$\{(-1)^n\}_{n=1}^{\infty}$$

that diverge because they "oscillate", and sequences like

$$\{n\}_{n=1}^{\infty}$$

that diverge because they "head toward infinity".

Definition 12.4. A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to* ∞ if for every real number M, there is a real number N such that if $n \in \mathbb{N}$ and $n > N$, then we have $a_n > M$.

A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to* $-\infty$ if for every real number L, there is a real number N such that if $n \in \mathbb{N}$ and $n > N$, then $a_n < L$.

Intuitively, a sequence diverges to ∞ provided that, no matter how big M is, if you go far enough along the sequence, eventually all of the terms are bigger than M. Similarly for diverges to $-\infty$.

Proposition 12.5. *If a sequence* $\{a_n\}_{n=1}^{\infty}$ *diverges to* ∞ *or diverges to* $-\infty$*, then it diverges.*

Proof. We prove the contrapositive. (What is the contrapositive? If a sequence converges, then it does not diverge to ∞ and it des not diverge to $-\infty$.) Suppose $\{a_n\}_{n=1}^{\infty}$ converges to some number L. Then since it converges, it is bounded, so that there are real numbers b and c such that $b \leq a_n \leq c$ for all n.

In particular, this means that there is no $N \in \mathbb{R}$ such that $a_n > c$ for all natural numbers $n > N$. Thus, taking "$M = c$" in the definition of diverges to ∞, we see that $\{a_n\}_{n=1}^{\infty}$ does not diverge to ∞.

Similarly, that there is no $N \in \mathbb{R}$ such that $a_n < b$ for all natural numbers $n > N$. Thus, taking "$M = b$" in the definition of diverges to $-\infty$, we see that $\{a_n\}_{n=1}^{\infty}$ does not diverge to $-\infty$. $\square$

As a matter of shorthand, we write $\lim_{n\to\infty} a_n = \infty$ to indicate that $\{a_n\}_{n=1}^{\infty}$ diverges to ∞. But note unlike when we wrote things such as $\lim_{n\to\infty} a_n = 17$, when we write $\lim_{n\to\infty} a_n = \infty$ we are asserting that $\{a_n\}_{n=1}^{\infty}$ *diverges* (in a specific way). Similarly, we write $\lim_{n\to\infty} a_n = -\infty$ to indicate that $\{a_n\}_{n=1}^{\infty}$ diverges to $-\infty$.

Example 12.6. The sequence $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to ∞. Let us prove this using the definition: Pick $M \in \mathbb{R}$. (Scratch work: I need $\sqrt{n} > M$ which will occur if $n > M^2$.) Let $N = M^2$. If $n \in \mathbb{N}$ and $n > N$, then $\sqrt{n} > \sqrt{N} = \sqrt{M^2} = |M| \geq M$. (Note that M could conceivably be negative.) This proves $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to ∞.

Example 12.7. Take the sequence $\{a_n\}_{n=1}^{\infty}$ given by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

This is known as the "harmonic series". We will show that this sequence diverges to ∞.

Observe that

$$\begin{aligned} a_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\ &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\ &= 1 + \frac{1}{2} + 2\cdot\frac{1}{4} + 4\cdot\frac{1}{8} + \cdots . \end{aligned}$$

For most natural numbers n, it may be a little messy to deal with the last terms in the sum. But, if $k \in \mathbb{N}$, and $n = 2^k$, we can do this nicely:

$$\begin{aligned} a_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{2^k} \\ &\geq 1 + \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) + \underbrace{\left(\frac{1}{2^3} + \cdots + \frac{1}{2^3}\right)}_{\text{from } 2^2+1 \text{ to } 2^3} + \cdots + \underbrace{\left(\frac{1}{2^k} + \cdots + \frac{1}{2^k}\right)}_{\text{from } 2^{k-1}+1 \text{ to } 2^k} \\ &= 1 + \frac{1}{2} + 2^1\cdot\frac{1}{2^2} + 2^2\cdot\frac{1}{2^3} + \cdots + 2^{k-1}\cdot\frac{1}{2^k} = 1 + \frac{k}{2}. \end{aligned}$$

Let $M \in \mathbb{R}$ be given. Let M' be the smallest natural number greater than M (why does such a number exist?) and take $N = 2^{2M'}$. By the computation above, taking $k = 2M'$, we see that $a_N \geq 1 + \frac{2M'}{2}$. Then, for $n > N$, since $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence, we have

$$a_n \geq a_N \geq 1 + \frac{2M'}{2} = M' + 1 > M' > M,$$

which shows that $\{a_n\}_{n=1}^\infty$ diverges to ∞.

13. Monday, March 8

We will now embark on a bit of detour. I've postponed talking about proofs by induction, but we will need to use that technique on occasion. So let's talk about that idea now.

The technique of proof by induction is used to prove that an infinite sequence of statements indexed by $\mathbb{N}$

$$P_1, P_2, P_3, \dots$$

are all true. For example, for any real number x, the equation

$$(1-x)(1+x+\cdots+x^n) = 1-x^{n+1}$$

holds for all $n \in \mathbb{N}$. Fixing x, we get one statement for each natural number:

$$\begin{array}{lr}
P_1: & (1-x)(1+x) = 1-x^2 \\
P_2: & (1-x)(1+x+x^2) = 1-x^3 \\
P_3: & (1-x)(1+x+x^2+x^3) = 1-x^4 \\
\vdots & \vdots
\end{array}$$

Such a fact (for all n) is well-suited to be proven by induction.

Here is the general principle:

Theorem 13.1 (Principle of Mathematical Induction)**.** *Suppose we are given, for each $n \in \mathbb{N}$, a statment P_n. Assume that P_1 is true and that for each $k \in \mathbb{N}$, if P_k is true, then P_{k+1} is true. Then P_n is true for all $n \in \mathbb{N}$.*

"The domino analogy": Think of the statements $P_1, P_2, \dots$ as dominoes lined up in a row. The fact that $P_k \implies P_{k+1}$ is interpreted as meaning that the dominoes are arranged well enough so that if one falls, then so does the next one in the line. The fact that P_1 is true is interpreted as meaning the first one has been knocked over. Given these assumptions, for every n, the n-th domino will (eventually) fall down.

The Principle of Mathematical Induction (PMI) is indeed a theorem, which we will now prove:

Proof. Assume that P_1 is true and that for each $k \in \mathbb{N}$, if P_k is true, then P_{k+1} is true. Consider the subset

$$S = \{n \in \mathbb{N} \mid \ P_n \text{ is false}\}$$

of $\mathbb{N}$. Our goal is to show S is the empty set.

By way of contradiction, suppose S is not empty. Then by the Well-Ordering Principle, S has a smallest element, call it ℓ. (In other words, P_ℓ is the first statement in the list $P_1, P_2, \ldots,$ that is false.) Since P_1 is true, we must have $\ell > 1$. But then $\ell - 1 < \ell$ and so $\ell - 1$ is not in S. Since $\ell > 1$, we have $\ell - 1 \in \mathbb{N}$ and thus we can say that $P_{\ell-1}$ must be true. Since $P_k \Rightarrow P_{k+1}$ for any k, letting $k = \ell - 1$, we see that, since $P_{\ell-1}$ is true, P_ℓ must also by true. This contradicts the fact that $\ell \in S$. We conclude that S must be the empty set. $\square$

Remark 13.2. The above proof shows that the Principle of Mathematical Induction is a consequence of the Well-Ordering Principle. The converse is also true.

Example 13.3. For any $n \in \mathbb{N}$, if S is a set with n elements, then there are 2^n possible subset of S (including the empty set and S itself).

To prove this, we let P_n be the statement: If a set has n elements, then it has 2^n subsets.

The statement P_1 is true since a one element set has 2 subsets: itself and the empty set. Let $k \in \mathbb{N}$ and assume P_k is true. Let S be any set with $k+1$ elements, and let x be one of its elements. Let $S' = S \setminus \{x\}$. There are two types of subsets of S: those that contain x and those that don't or, equivalently, those that are contained in S' and those that are not.

Since P_k is true and S' has k elements, there are 2^k subsets of S'. That is, there are 2^k subsets of S that don't contain x. Now, every subset of S that does contain x has the form $\{x\} \cup X$ for a unique subset X of S'. Thus there are also 2^k subsets of S that do contain x. In total, there are thus $2^k + 2^k = 2^{k+1}$ subsets of S. That is, P_{k+1} is true.

By PMI, P_n is true for all n.

Example 13.4. What is wrong with the following "proof by induction":

> I claim that all horses are of the same color. To prove this, I will show that for every set of n horses, all the horses in that set have the same color.
>
> This is clearly true when $n = 1$. Let $n \in \mathbb{N}$ and assume it is true for any set of n horses. Now consider an arbitrary set of $n+1$ horses, call them $h_1, h_2, \ldots, h_{n+1}$. Divide this set into two subsets of n horses each, namely $h_1, h_2, \ldots, h_n$ and $h_2, h_3, \ldots, h_{n+1}$. By induction, each of these two sets of horses are all of the same color. But

then since h_2 belongs to both sets, it follows that all the horses in the full list $h_1, \dots, h_{n+1}$ must be all of the same color.

By PMI, for any $n \in \mathbb{N}$, all sets of n horses have the same color. Thus all horses have the same color.

14. Wednesday, March 9

I want to briefly discuss the relationship between induction and recursion, in the sense of recursively defined sequences. Recall that one way of describing a sequence is by a pair of formulas: one that gives a value for a_1, and another that gives a value for a_{n+1} in terms of a_n. The fact that such a pair of formulas yields a well-defined value of a_n for every $n \in \mathbb{N}$ is justified by induction. If we take P_n to be the statement that "the formulas determine a unique value for a_n", then P_1 is true since we have a given value for a_1, and P_n is true implies that P_{n+1} is true, since we have a formula for a_{n+1} in terms of a_n. By induction, P_n is true for all $n \in \mathbb{N}$.

The next example of a proof by induction will establish a fact that is perhaps intuitively obvious. Since it will play an important role in later proofs, we state it as a Lemma here:

Lemma 14.1. *Let $b_1, b_2, \dots$ be any strictly increasing sequence of natural numbers; that is, assume $b_k \in \mathbb{N}$ for all $k \in \mathbb{N}$ and that $b_k < b_{k+1}$ for all $k \in \mathbb{N}$. Then $b_k \geq k$ for all k.*

Proof. Suppose $b_1, b_2, \dots$ is a strictly increasing sequence of natural numbers. We prove $b_n \geq n$ for all n by induction on n. That is, for each $n \in \mathbb{N}$, let P_n be the statement that $b_n \geq n$.

P_1 is true since $b_1 \in \mathbb{N}$ and so $b_1 \geq 1$. Given $k \in \mathbb{N}$, assume P_k is true; that is, assume $b_k \geq k$. Since $b_{k+1} > b_k$ and both are natural numbers, we have $b_{k+1} \geq b_k + 1 \geq k + 1$; that is, P_{k+1} is true too. By PMI, P_n is true for all $n \in \mathbb{N}$. $\square$

We next discuss the important concept of a "subsequence".

Informally speaking, a subsequence of a given sequence is a sequence one forms by skipping some of the terms of the original sequence. In other words, it is a sequence formed by taking just some of the terms of the original sequence, but still infinitely many of them, without repetition.

We'll cover the formal definition soon, but let's give a few examples first, based on this informal definiton.

Example 14.2. Consider the sequence

$$a_n = \begin{cases} 7 & \text{if } n \text{ is divisible by 3 and} \\ \frac{1}{n} & \text{if } n \text{ is not divisible by 3.} \end{cases}$$

If we pick off every third term starting with the term a_3 we get the subsequence

$$a_3, a_6, a_9, \dots$$

which is the constant sequence

$$7, 7, 7, \dots.$$

If we pick off the other terms we form the subsequence

$$a_1, a_2, a_4, a_5, a_7, a_8, a_{10}, \dots$$

which gives the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{10}, \dots.$$

Note that it is a little tricky to find an explicit formula for this sequence.

Example 14.3. For another, simpler, example, consider the sequence $\{(-1)^n \frac{1}{n}\}_{n=1}^{\infty}$. Taking just the odd-indexed terms gives the sequence

$$-1, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, -\frac{1}{9}, \dots$$

and taking the even-indexed terms gives the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots$$

This time we can easily give a formula for each of these sequences: the first is

$$\{-\frac{1}{2n-1}\}_{n=1}^{\infty}$$

and the second is

$$\{\frac{1}{2n}\}_{n=1}^{\infty}.$$

Here is the formal definiton:

Definition 14.4. A *subsequence* of a given sequence $\{a_n\}_{n=1}^{\infty}$ is any sequence of the form

$$\{a_{n_k}\}_{k=1}^{\infty}$$

where

$$n_1, n_2, n_3, \dots$$

is any strictly increasing sequence of natural numbers — that is $n_k \in \mathbb{N}$ and $n_{k+1} > n_k$ for all $k \in \mathbb{N}$, so that

$$n_1 < n_2 < n_3 < \cdots.$$

Example 14.5. Let $\{a_n\}_{n=1}^{\infty}$ be any sequence.

Setting $n_k = 2k - 1$ for all $k \in \mathbb{N}$ gives the subsequence of just the odd-indexed terms of the original sequence.

Setting $n_k = 2k$ for all $k \in \mathbb{N}$ gives the subsequence of just the even-indexed terms of the original sequence.

Setting $n_k = 3k - 2$ for all $k \in \mathbb{N}$ gives the subsequence of consising of every third term of the original sequence, starting with the first.

Setting $n_k = 100 + k$ gives the subsequence that is that "tail end" of the original, obtained by skipping the first 100 terms:

$$a_{101}, a_{102}, a_{103}, a_{104}, \ldots.$$

Of course, there is nothing special about 100 in this example.

The following result is important:

Theorem 14.6. *If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to L, then every subsequence of this sequence also converges to L.*

Proof. Assume $\{a_n\}_{n=1}^{\infty}$ converges to L and let $n_1 < n_2 < \cdots$ be any strictly increasing sequence of natural numbers. We need to prove $\{a_{n_k}\}_{k=1}^{\infty}$ converges to L.

Pick $\varepsilon > 0$. Since $\{a_n\}_{n=1}^{\infty}$ converges to L, there is an N such that if $n \in \mathbb{N}$ and $n > N$, then $|a_n - L| < \varepsilon$. (We will show that the same N also "works" for the subsequence.)

If $k \in \mathbb{N}$ and $k > N$, then $n_k \geq k$ by Lemma 14.1, and hence $n_k > N$. It follows that $|a_{n_k} - L| < \varepsilon$. This proves $\{a_{n_k}\}_{k=1}^{\infty}$ converges to L. $\square$

Corollary 14.7. *Let $\{a_n\}_{n=1}^{\infty}$ be any sequence.*

(1) *If there is a subsequence of this sequence that diverges, then the sequence itself diverges.*
(2) *If there are two subsequence of this sequence that converge to different values, then the sequence itself diverges.*

Proof. These are both immediate consequences of the theorem. $\square$

Example 14.8. Consider the sequence

$$a_n = \begin{cases} 7 & \text{if } n \text{ is divisible by 3 and} \\ \frac{1}{n} & \text{if } n \text{ is not divisible by 3.} \end{cases}$$

Let $n_k = 3k$. Then the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ is

$$a_3, a_6, a_9, \ldots$$

which is the constant sequence

$$7, 7, 7, \ldots.$$

It converges to 7.

Now let $n_k = 3k - 2$. Then the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ is

$$a_1, a_4, a_7, \ldots$$

which is the sequence $\{\frac{1}{3k-2}\}_{k=1}^{\infty}$. It converges to 0.

Since the original sequence admits two subsequences that convege to different values, by the Corollary, the original sequence diverges.