Math 325-001 — Problem Set #3
Due: Monday, February 22 by midnight

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Given any two real numbers x and y, $\max\{x, y\}$ refers to the larger of the two numbers x and y; that is, $\max\{x, y\}$ is x if $x \geq y$ and otherwise it is y. Similarly, $\min\{x, y\}$ refers to the smaller of the two numbers x and y; that is, $\min\{x, y\}$ is x if $x \leq y$ and otherwise it is y.

  (a) Prove that for all real numbers x and y

$$\max\{x, y\} = \frac{x + y + |x - y|}{2}.$$

  (b) Find a similar formula for $\min\{x, y\}$ and prove that your formula is correct.

(2) Let x and y be real numbers.

  (a) Show that, if $|x - 3| < \epsilon$ and $|y - 3| < \epsilon$ for some $\epsilon > 0$, then $|x - y| < 2\epsilon$.

  (b) Show that, if $|x - 5| < \epsilon$ and $|y - 6| < \epsilon$ for some $\epsilon > 0$, then $1 - 2\epsilon < |x - y| < 1 + 2\epsilon$.

(3) Let r be any real number. Consider the set

$$S_r = \{q \in \mathbb{Q} \mid q < r\}.$$

In words, S_r is the set of those *rational* numbers that are strictly less than r. Prove that the supremum of S_r is r.

(4) Prove, using the formal definition of convergence, that the sequence $\{\frac{1}{\sqrt{n}}\}_{n=1}^{\infty}$ converges to 0. (At the risk of being overly pedantic, for $n \in \mathbb{N}$, by $\sqrt{n}$ we mean the unique positive real number whose square is n. Such a number exists by the Completeness Axiom, which proved in detail in the case $n = 2$.)

(5) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that converges to some positive number $L > 0$. Prove that the set $M = \{n \in \mathbb{N} \mid a_n \leq 0\}$ is finite (or empty).