Research Statement: Jack Jeffries

1. Introduction

My research is centered in Commutative Algebra, the study of commutative rings and modules, especially Noetherian rings. This field has natural connections to Algebraic Geometry, Singularity Theory, and Number Theory. Indeed, to a solution set of a system of polynomial equations (a *variety*) or an analytic singularity, there is a natural commutative ring associated, its *coordinate ring*, the ring of polynomial functions on the set. The ring-theoretic properties of the coordinate ring contain much information about the geometry of the original space. Geometric properties of complex varieties (e.g., being locally diffeomorphic to $\mathbb{C}^n$, being a finite flat cover of some $\mathbb{C}^n$) correspond to algebraic properties that are defined in greater generality (regularity, Cohen-Macaulayness, respectively), including for rings that are less directly geometric in nature, such as $\mathbb{F}_p$-algebras or $\mathbb{Z}$-algebras.

Such flexibility is not just an appealing abstraction. One technique that takes advantage of this is reduction to positive characteristic: one can often deduce statements about rings of characteristic zero (such as those coming from classical geometry) by proving analogous statements in positive characteristic p, where one can take great advantage of the identity $(x+y)^{p^e} = x^{p^e} + y^{p^e}$. In particular, this means that the set of p^eth powers of elements of R forms a subring $R^{p^e} \subseteq R$. Properties characterized by the R^{p^e}-module structure of R (e.g., F-purity, strong F-regularity) correspond to properties in characteristic zero characterized by resolutions of singularities (log-canonicity, log-terminality, respectively).

There are other important connections between Commutative Algebra and Combinatorics, Algebraic Topology, Representation Theory, Mathematical Physics, and numerous other fields. I am interested in Commutative Algebra and its connections to other areas—and there are connections yet to be discovered. The goal of this statement is to give a brief invitation to some questions in my research, so some precision will be left to the references.

2. Differential operators and Commutative Algebra

One area that I have been interested in recently is the use of differential operators on rings to study singularities. This is a generalization of the notion of a derivation. The (noncommutative) ring D_S of differential operators on the polynomial ring $S = \mathbb{C}[x_1, \dots, x_d]$ is the ring generated by R and $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}$. The study of the theory of modules over this ring (D-modules) is rich, culminating in a classification theorem, the Riemann-Hilbert correspondence, that generalizes the solution to Hilbert's 20th problem [8, 13, 31]. Another key result in this theory is the existence, for any $f \in S$, of a (polynomial expession of a) differential operator $\delta(s) \in D_S[s]$ and polynomial $b(s) \in \mathbb{Q}[s]$ such that $\delta(s) \cdot f^{s+1} = b(s) f^s$ for all integers s; the minimal monic such $b(s)$ is called the *Bernstein-Sato polynomial* of f, denoted $b_{f,S}$. The polynomial $b_{f,S}$ has many interesting connections to singularities of f, e.g., the Milnor fiber, monodromy, log-thresholds, Hodge theory, and zeta functions [45, 48, 63].

To any commutative ring R, there is a ring of differential operators R. When $R = S/I$, with S as above, one can characterize the differential operators on R as the quotient of the ring of operators on S that act on R by those that act trivially on R: $D_R = \{\delta \in D_S \mid \delta \cdot I \subseteq I\}/ID_S$. There is a general definition that is intrinsic [28]. These general rings of differential operators are less well-behaved than the ring D_S and do not have as well-developed of a module theory. For example, it is known that these rings can fail to be (left or right) Noetherian [5], and many authors have studied them [47, 52, 54, 59].

2.1. **Differential operators, symbolic powers, and p-derivations.** The nth symbolic power $\mathfrak{p}^{(n)}$ of a prime ideal $\mathfrak{p}$ is defined algebraically as the $\mathfrak{p}$-primary component of $\mathfrak{p}^n$ in a primary decomposition. A classical theorem of Zariski and Nagata gives a characterization of symbolic powers in $S = \mathbb{C}[x_1, \ldots, x_d]$: $\mathfrak{p}^{(n)}$ consists of the elements f that vanish to order n along the zero set of $\mathfrak{p}$, in the sense that $\delta(f) \in \mathfrak{p}$ for every differential operator δ of order at most $n-1$ [15, 53, 66]. In joint work with De Stefani and Grifo [17], we give an analogous characterization of symbolic powers of primes in $\mathbb{Z}[x_1, \ldots, x_d]$ using differential operators and p-derivations, a notion due to Buium and Joyal [10, 40]. The use of p-derivations in Commutative Algebra is new, and more applications are likely to emerge.

2.2. **Reduction to positive characteristic.** A natural question is when differential operators behave well under reduction to positive characteristic: if R is a $\mathbb{Z}$-algebra, when does $D_{R/pR} \cong D_R \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})$? A positive answer allows one to study D_R by reduction to positive characteristic. Also, in light of the isomorphism $D_{R/pR} \cong \bigcup_{e \in \mathbb{N}} \mathrm{End}_{R^{p^e}}(R)$ [65], a positive answer (for all p for a fixed R) gives a uniformity statement of the Frobenius structure of R as p varies.

To study this question, in [36], I give new formulas for the rings of differential operators in terms of local cohomology modules. These formulas allow us to show that the isomorphism above holds for large classes of rings R, giving a positive answer to a question of Smith and Van den Bergh [58] in a substantial case. These formulas also give new insight into local cohomology, including new conceptual examples of local cohomology modules with infinitely many associated primes. The study of finiteness of associated primes in local cohomology is itself a question of major interest [6, 32, 34, 49, 57]. In work in progress with Singh, we show that there are clasical invariant rings for which the answer to the question above is negative. This answers the aforementioned question of Smith and Van den Bergh.

2.3. **Bernstein-Sato polynomials on non-regular rings.** In [4], it was shown that Bernstein-Sato polynomials exist on certain singular rings, namely, direct summands of polynomial rings. Moreover, if $f \in R \subseteq S$, with R a direct summand of S, then $b_{f,R}$ divides $b_{f,S}$. In joint work with Brenner and Núñez-Betancourt [9], we give sufficient conditions for equality $b_{f,R} = b_{f,S}$, that suffice to compute $b_{f,R}$ whenever R is a ring of invariants of a finite group of invertible order (orbifold singularity) or a toric ring.

With Àlvarez Montaner, Hernández, Núñez-Betancourt, Teixeira, and Witt, [51] we develop the theory of the Bernstein-Sato polynomial as a formal identity (rather than just for integer values of s), as well as the theory of V-filtrations and its connection with multiplier ideals to wider classes of rings.

2.4. **Subsequent directions.** In work in progress with Singh, we aim to determine, for all classical invariant rings that are hypersurfaces, whether differential operators lift modulo p. Furthermore, we aim to determine for these rings in positive characteristic when every differential operator on the invariant ring lifts to an invariant differential operator on the polynomial ring; this would yield the positive characteristic analgoue of a theorem of Levasseur and Stafford [47]. This would give explicit insight into the behavior of differential operators as characteristic varies in an interesting example.

3. Separating sets, in the sense of Derksen and Kemper

One recent development in invariant theory is the study of separating sets, see [20, 21, 25]. Given an action of a group G on an affine variety V over a field $\Bbbk$, one obtains an action of G on the coordinate ring R of V. When $\Bbbk$ is algebraically closed, the ring of invariants $R^G = \{f \in R \mid g(f) = f \text{ for all } g \in G\}$ can be interpreted as the set of algebraic functions that are constant on G-orbits of V. When the group G is finite, the invariants can be used to distinguish orbits in V: that is, orbits $G \cdot v$ and $G \cdot w$ coincide if and only if $f(v) = f(w)$ for all $f \in R^G$; if G is infinite, there may be distinct orbits on which all invariants agree.

For an action of an algebraic group on an affine variety, a *separating set* is a collection of invariants $\Lambda \subseteq R^G$ such that for any two points $x, y \in V$, if there exists an invariant $f \in R^G$ with $f(x) \neq f(y)$, then there exists $g \in \Lambda$ with $g(x) \neq g(y)$ [19]. In particular, if G is finite, a separating set is a set of invariants that suffices to distinguish orbits. The ring of invariants itself, or any generating set for R^G, is a separating set. However, there may be much smaller separating sets [43].

Separating sets are often easier to compute and work with than generating sets for the ring of invariants: there always exists a finite separating set for an action, even if the ring of invariants is not finitely generated [19, 24]. In general, a separating set of size m is equivalent to an embedding of the quotient space $V /\!\!/ G$ into $\Bbbk^m$. This motivates the following question of Kemper [43]: for a given action, what is the smallest size of a separating set?

3.1. Separating sets for finite group actions.

A classical result of Chevalley, Shephard, Todd, and Serre [12, 55, 56] characterizes when a ring of invariants of a representation $V = \Bbbk^n$ of a finite group is itself a polynomial ring: when $|G|$ is a unit in $\Bbbk$, this occurs if and only if G is generated by elements that each fix a hyperplane; in the case the order of G is divisible by the characteristic of $\Bbbk$ (the *modular* case), only the "only if" implication holds.

In joint work with Dufresne [22], we give a broad generalization of the "only if" direction of this theorem, bounding the smallest size of a separating set below by $n - 1$ plus the least j such that G is generated by elements that fix a subspace of codimension j. Our key insight is to find an obstruction to existence of injective maps to $\Bbbk^m$ in terms of local cohomology. We also conjecture a necessary and sufficient condition for there to exist a separating set that generates a polynomial ring, and show that the aforementioned obstruction vanishes under this hypothesis.

3.2. Toric rings.

In another project with Dufresne [23], we study the question of the minimal number of elements in a separating set for actions of tori. Existence of injective maps from toric varieties to affine spaces is of independent interest for its connections to identifiability questions of statistical models [3]. We show that there exist separating sets consisting of invariants that involve few variables. We also calculate local cohomology and étale cohomology to provide bounds on separating sets for actions defining Segre-Veronese varieties.

3.3. Derived functors of differential operators, à la Smith and Van den Bergh.

In [36], I show that these local cohomology obstructions to injective maps to $\Bbbk^m$ in the aforementioned work on separating sets can also be interpreted as *derived functors of differential operators*. These derived functors were defined and studied by Smith and Van den Bergh [58] in connection with the question discussed in §2.2. Some further insight on the structure of these modules comes from recent joint work with Hochster on faithfulness of local cohomology modules [30].

3.4. **Subsequent directions.** The conjecture mentioned in §3.1 remains a target. This seems to be a difficult problem at this point, as invariant theory in the modular case is much more difficult than the nonmodular case. Nonetheless, the conjectured sufficient condition has strong consequences in terms of cohomology and differential operators: namely, all of the derived functors of differential operators vanish. This is a very strong and natural condition, and its study should be fruitful in any case.

We also aim to extend the results of §3.2 using the results in §3.3 and §2.2. In particular, one can reduce to positive characteristic, and use the results of [26] on differential operators on toric rings in positive characteristic to recover some known results in an independent way; by extending these methods we expect stronger results.

4. Numerical limits and convexity

Another area in which I have worked is numerical limits in local algebra and their connection to convex bodies. Many geometrically meaningful quantities can be realized as limits of sequences of algebraic quantities. For example, the Hilbert-Samuel multiplicity of an $\mathfrak{m}$-primary ideal I in a local ring $(R, \mathfrak{m})$ of Krull dimension d is given as $e(I) := \lim \frac{\text{length}(R/I^n)}{n^d/d!}$; for an ideal $I = (f_1, \dots, f_t)$ in the ring of functions analytic near the origin in $\mathbb{C}^d$, this measures the number of solutions of the equations $f_1 = \varepsilon_1, \dots, f_t = \varepsilon_t$ for almost all choices of small $\varepsilon_1, \dots, \varepsilon_t$.

Another numerical limit of interest is the F-signature of a local ring of positive characteristic [33, 58, 62]. This number quantifies the asymptotic rate of growth of the largest rank of a free R^{p^e}-module summand of R. The F-signature detects regularity and strong F-regularity, and has been used to bound the local fundamental group of singularities [1, 11, 64]. However, it is not understood how this invariant behaves as the characteristic varies, or what its characteristic zero analogue is.

It has been known that numerical limits associated to monomial ideals can be interpreted in terms of aspects of the Newton polyhedron. For example, the Hilbert-Samuel multiplicity of a monomial ideal equals the (normalized) volume of the complement of its Newton polyhedron [60]. Recently, the connection between numerical limits and convex geometry has been increasingly developed. The theory of Okounkov bodies shows that, in great generality, numerical invariants can be realized as volumes of complements of convex bodies, yielding convergence results as well as generalizations of classical inequalities [14, 42, 46].

4.1. **Polytopal and integral formulas.** There are generalizations of the Hilbert-Samuel multiplicity ideals that are not necessarily $\mathfrak{m}$-primary. Two important generalizations are the j-multiplicity and ε-multiplicity [2, 14, 41]. These multiplicities can be used to detect integral closure, form a basis for the Stückrad-Vogel intersection theory, and have important applications to equisingularity theory [27, 44]. In joint work with Montaño [37], we give formulas for the j- and ε-multiplicity of monomial ideals as volumes of explicit polytopes, generalizing a theorem of Teissier [60]. In later joint work with Montaño and Varbaro [38], we compute these generalized multiplicities for determinantal ideals, finding an interesting connection with random matrix theory and log-gases.

In my thesis [35], I use a description of the F-signature of toric rings in terms of volumes of polytopes to study the possible values of the F-signatures of these rings. In particular, we find that there are only two isomorphism classes of nonregular toric rings with F-signature greater than 1/2; F-signature greater than 1/2 is of independent interest due to purity of the branch locus and Gorensteinness [11, 18].

4.2. **Okounkov bodies for positive characteristic limits.** In joint work with Hernández [29], we generalize the methods of local Okounkov bodies [14, 42] to deal with classes of limits arising in positive characeristic. Namely, rather than the condition of a graded family of ideals in *ibid.*, we deal with families of ideals $\{I_e\}$ satisfying the condition that pth powers of elements in I_e are contained in I_{e+1}. For such families, we show that the lengths of R/I^e grow as $\mathrm{vol}(\mathcal{P}_{I_\bullet})p^{e\dim(R)}$ for some specific region $\mathcal{P}_{I_\bullet} \subseteq \mathbb{R}^{\dim(R)}$. This method gives new cases for the existence of interesting numerical limits (e.g., generalized Hilbert-Kunz multiplicity [16]), and new proofs for existence of other limits (Hilbert-Kunz multiplicity [50] and F-signature [62]). We also obtain Brunn-Minkowski type inequalities for these multiplicities.

4.3. **Differential signature.** In joint work with Brenner and Núñez-Betancourt [9], we propose and study a characteristic-free numerical limit that serves an analogue to F-signature. There are natural modules P^n_R such that $D^n_R = \mathrm{Hom}_R(P^n_R, R)$; the *differential signature* quantifies the asymptotic rate of growth of the largest rank of a free R-module summand of P^n_R. This invariant also is 1 for regular rings, positive for direct summands of regular rings, and its positivity implies log-terminality/strong F-regularity under some hypotheses. Our work also strongly motivates the analogy: differential signature is to Hilbert-Samuel multiplicity as F-signature is to Hilbert-Kunz multiplicity. We have also been able to compute differential signature in large classes of examples, including some where the F-signature is unknown.

In joint work with Smirnov [39], we use differential signature to give a characteristic free version of the bound on the local fundamental group of singularities [11]. In particular, for rings of positive differential signature (e.g., rings of invariants of linearly reductive groups), this gives a simplified proof of a finiteness theorem of Tian and Xu [7, 61].

4.4. **Subsequent directions.** An important question is to determine more general conditions for positivity of differential signature. One motivation for this is the bounds on the local fundamental group mentioned in §4.3. Another motivation is the connection with symbolic powers: positivity of differential signature is closely related to statements comparing the symbolic power topology with the "differential power" topology, another generalization of the Zariski-Nagata theorem discussed in §2.1.

Another problem is to determine the behavior of differential signature as the characteristic varies. This will give new insights into the notions of log-terminality and F-regularity, as well as the arithmetic dependence of these qualities. This is likely to provide insight into properties of singularities in mixed characteristic as well.

5. Future directions

Many of the "subsequent directions" above largely fit into a few themes that are likely to continue to be fruitful sources of meaningful questions in the next few years. One of these is the program of extending aspects of D-module theory to singularities, especially rings with nice (direct summand, log-terminal, or strongly F-regular) singularities.

Another is the program of interpreting Frobenius map aspect of positive characteristic behavior of singularities in ways that make sense over fields of characteristic zero or p, or in mixed characteristic. Methods to this effect so far include resolutions of singularities and perfectoid spaces, but the use of differential operators and p-derivations to this pursuit opens many possibilities for interesting discoveries.

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