Math 314. Week 11 worksheet (§6.1, §6.2, §6.3).

The **dot product** of two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ is

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

We use the dot product to define:

- two vectors are **orthogonal** if $\mathbf{v} \cdot \mathbf{w} = 0$;
- the **length** of a vector is $\sqrt{\mathbf{v} \cdot \mathbf{v}}$. We write $||\mathbf{v}||$ for the length of $\mathbf{v}$.

Idea: orthogonal vectors are perpendicular/form a right angle

A set of vectors $\{\mathbf{u_1}, \ldots, \mathbf{u_t}\}$ is

- an **orthogonal set** if each pair of vectors $\mathbf{u_i}, \mathbf{u_j}$, $i \neq j$ is orthogonal;
- an **orthonormal set** if each pair of vectors $\mathbf{u_i}, \mathbf{u_j}$, $i \neq j$ is orthogonal and each vector $\mathbf{u_i}$ has length one.

Every orthonormal set is an orthogonal set.

THEOREM: Every orthogonal set is a linearly independent set.

A. SETS OF VECTORS IN $\mathbb{R}^2$. For each of the following, either draw a picture of a set of vectors in $\mathbb{R}^2$ that fits the description, or explain why no such set exists.

(1) A set of two vectors that is an orthonormal set.
(2) A set of two vectors that is an orthogonal set, but not orthonormal.
(3) A set of two vectors that is linearly dependent.
(4) A set of two vectors that is linearly independent, but not an orthogonal set.
(5) A set of three vectors that is an orthogonal set.

B. A SET OF VECTORS IN $\mathbb{R}^3$. Consider the vectors $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$.

(1) Compute the dot products $\mathbf{u} \cdot \mathbf{u}$, $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{w}$, $\mathbf{v} \cdot \mathbf{v}$, $\mathbf{v} \cdot \mathbf{w}$, $\mathbf{w} \cdot \mathbf{w}$.
(2) Compute $||\mathbf{u}||$, $||\mathbf{v}||$, and $||\mathbf{w}||$.
(3) Is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ an orthogonal set?
(4) Is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ an orthonormal set?
(5) Find a scalar c such that $c\mathbf{u}$ is a *unit vector*—a vector of length one.
(6) Let $U = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$. Compute $U^T U$, and compare it to part (1).

FACT: If $U = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \cdots & \mathbf{u_n} \end{bmatrix}$, then the (i, j)-entry of $U^T U$ is the dot product $\mathbf{u_i} \cdot \mathbf{u_j}$. This means that $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ is an orthonormal set if and only if $U^T U = I_n$.

A set of vectors $\{\mathbf{u_1}, \dots, \mathbf{u_t}\}$ in a subspace W of $\mathbb{R}^n$ is

- an **orthogonal basis** if it is an orthogonal set that is a basis for W
- an **orthonormal basis** if it is an orthonormal set that is a basis for W.

One reason orthonormal bases are useful is because it is easy to find the weights/coordinates in such a basis: if $\mathcal{U} = \{\mathbf{u_1}, \dots, \mathbf{u_t}\}$ is an *orthonormal* basis[1] for W, and $\mathbf{w} = c_1\mathbf{u_1} + \cdots + c_t\mathbf{u_t} \in W$, then $c_i = \mathbf{u_i} \cdot \mathbf{w}$. Put another way,

$$[\mathbf{w}]_{\mathcal{U}} = \begin{bmatrix} \mathbf{u_1} \cdot \mathbf{w} \\ \mathbf{u_2} \cdot \mathbf{w} \\ \vdots \\ \mathbf{u_t} \cdot \mathbf{w} \end{bmatrix} \quad \text{for } \mathbf{w} \in W.$$

C. Consider the plane H in $\mathbb{R}^3$ consisting of points that satisfy the equation $4x + y + z = 0$.

(1) Is H a subspace of $\mathbb{R}^3$?
(2) Find a basis[2] for H. Is it an orthonormal basis?
(3) Consider the vectors $\mathbf{u} = [\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}]^T$, $\mathbf{v} = [0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}]^T$. Are $\mathbf{u}, \mathbf{v} \in H$?
(4) Is $\{\mathbf{u}, \mathbf{v}\}$ an orthonormal set?
(5) Explain why $\mathcal{U} = \{\mathbf{u}, \mathbf{v}\}$ is an orthonormal basis for H.
(6) Find the $\mathcal{U}$-coordinates of the point $[-1, 1, 3]^T$.

DEFINITION: The **orthogonal complement** of a subspace $W \subseteq \mathbb{R}^n$ is the set of vectors that are orthogonal to *every* vector in W. We write $W^\perp$ for the orthogonal complement of W. It is also a subspace of $\mathbb{R}^n$.

THEOREM: If W is a subspace of $\mathbb{R}^n$, then any vector $\mathbf{v} \in \mathbb{R}^n$ can be written as $\mathbf{v} = \hat{\mathbf{v}} + \mathbf{z}$ with $\hat{\mathbf{v}} \in W$ and $\mathbf{z} \in W^\perp$ in exactly one way. The vector $\hat{\mathbf{v}}$ is called the **projection of v onto** W, written as $\mathrm{proj}_W(\mathbf{v})$. $\mathrm{proj}_W(\mathbf{v})$ is the closest point to $\mathbf{v}$ on W.

FORMULA (IF YOU HAVE AN ORTHONORMAL BASIS): If $\mathcal{U} = \{\mathbf{u_1}, \dots, \mathbf{u_t}\}$ is an *orthonormal* basis[3] for W, then

$$\mathrm{proj}_W(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{u_1})\mathbf{u_1} + \cdots + (\mathbf{v} \cdot \mathbf{u_t})\mathbf{u_t}.$$

In terms of matrices, if $U = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \cdots & \mathbf{u_n} \end{bmatrix}$, then $\mathrm{proj}_W(\mathbf{v}) = UU^T\mathbf{v}$.

D. PROJECTION ONTO A LINE. Let W be the line through the origin and the point $[1, 3]^T$ in $\mathbb{R}^2$.

(1) Draw W and $W^\perp$.
(2) Find[4] a basis for W.
(3) A set with one element is automatically orthogonal; there's no condition. Find an orthonormal basis for W.
(4) Find the projection of the point $[0, 2]^T$ onto W. Do the same for $[-5, -5]^T$.
(5) Find a basis[5] for $W^\perp$.

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[1] Warning: This is ONLY true for an ORTHONORMAL basis. That's why we like them so much.
[2] Hint: H is the null space of a 1×3 matrix.
[3] Warning: This formula ONLY works for an ORTHONORMAL basis!
[4] Hint: Don't compute anything!
[5] Start by finding a vector in $W^\perp$.

E. PROJECTIONS. Suppose that W is a subspace of $\mathbb{R}^n$.

(1) Using the fact that $\operatorname{proj}_W(\mathbf{v})$ is the closest point to $\mathbf{v}$ on W, explain why $\operatorname{proj}_W(\mathbf{w}) = \mathbf{w}$ for any point $\mathbf{w} \in W$.
(2) Now, suppose that $\mathcal{U} = \{\mathbf{u_1}, \ldots, \mathbf{u_t}\}$ is an orthonormal basis for W. Use the formula above to show[6] that $\operatorname{proj}_W(\mathbf{w}) = \mathbf{w}$ for any point $\mathbf{w} \in W$.
(3) Explain why $\operatorname{proj}_W : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation. If $\mathcal{U} = \{\mathbf{u_1}, \ldots, \mathbf{u_t}\}$ is an orthonormal basis for W, what is the standard matrix of proj_W? What is its range?

F*. PROJECTIONS AND ORTHOGONAL COMPLEMENTS. Let W be a subspace of $\mathbb{R}^n$. For this problem, think about projection in terms of its definition.

(1) What is the kernel of the linear transformation $\operatorname{proj}_W : \mathbb{R}^n \to \mathbb{R}^n$?
(2) Explain why $\mathbf{v} = \operatorname{proj}_W(\mathbf{v}) + \operatorname{proj}_{W^\perp}(\mathbf{v})$ for every $\mathbf{v} \in \mathbb{R}^n$.

G*. PROJECTION AS CLOSEST POINT.

(1) Explain why if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, then $||\mathbf{a}+\mathbf{b}|| \geq ||\mathbf{a}||$, and if $\mathbf{b} \neq \mathbf{0}$, then $||\mathbf{a}+\mathbf{b}|| > ||\mathbf{a}||$.
(2) Explain why[7] if $\mathbf{v} = \hat{\mathbf{v}} + \mathbf{z}$ with $\hat{\mathbf{v}} \in W$ and $\mathbf{z} \in W^\perp$, then $\hat{\mathbf{v}}$ is the closest point in W to $\mathbf{v}$.

If $T : V \to W$ is a linear transformation, then the following form of the rank-nullity theorem holds:

$$\dim(\text{Range}(T)) + \dim(\text{Kernel}(T)) = \dim(V).$$

To turn T into a matrix, we need a basis for V (to turn V into stacks of numbers) and a basis for W (to turn W into stack of numbers). If $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ and $\mathcal{C} = \{\mathbf{c_1}, \ldots, \mathbf{c_m}\}$, then the matrix of T with respect to these bases is the matrix M such that $M \cdot [\mathbf{v}]_\mathcal{B} = [T(\mathbf{v})]_\mathcal{C}$. It is given by the formula

$$M = \begin{bmatrix} [T(\mathbf{b_1})]_\mathcal{C} & \cdots & [T(\mathbf{b_n})]_\mathcal{C} \end{bmatrix}.$$

H. Let P_n be the vector space of polynomials of degree at most n. Let $a_0, a_1, \ldots, a_n$ be $n+1$ distinct real numbers.

(1) Explain why the map $E : P_n \to \mathbb{R}^{n+1}$ given by $E(p(t)) = \begin{bmatrix} p(a_0) & p(a_1) & \cdots & p(a_n) \end{bmatrix}^T$ is a linear transformation.
(2) What is the kernel of E?
(3) What is dimension of the range of E?
(4) What is the range of E?
(5) Explain why, if (a_0, b_0), (a_1, b_1), ..., (a_n, b_n) are any $n+1$ points with different x-coordinates, there is a polynomial of degree at most n whose graph passes through these points.
(6) Find the matrix of E with respect to the bases $\mathcal{B} = \{1, t, t^2, \ldots, t^n\}$ and $\mathcal{E} = \{\mathbf{e_1}, \ldots, \mathbf{e_{n+1}}\}$.
(7) Explain why the matrix from the previous part is invertible.
(8) In the context of part (5), how many polynomials of degree at most n pass through these points?
(9) If (a_0, b_0), (a_1, b_1), ..., (a_n, b_n) are any $n+1$ points with different x-coordinates, and $m > n$, is there is a polynomial of degree at most m whose graph passes through these points? How many?
(10) If (a_0, b_0), (a_1, b_1), ..., (a_n, b_n) are any $n+1$ points with different x-coordinates, and $m < n$, is there is a polynomial of degree at most m whose graph passes through these points? How many?

[6]Hint: You can write $\mathbf{w} = c_1\mathbf{u_1} + \cdots + c_t\mathbf{u_t}$ for some numbers $c_1, \ldots, c_t \in \mathbb{R}$

[7]Hint: We can write any point in W as $\hat{\mathbf{v}} - \mathbf{w}$ for some other point $\mathbf{w} \in W$. Take $\mathbf{a} = \mathbf{z}$ and $\mathbf{b} = \mathbf{w}$ in the previous part.