Math 314. Week 4 worksheet (§2.1, §2.2, §2.3).

A. REVIEW. Decide if the following statements are TRUE or FALSE.

(1) It's possible for $A\mathbf{x} = \mathbf{b}$ to have a solution and $A\mathbf{x} = \mathbf{c}$ to have no solution (for the same matrix A, different vectors $\mathbf{b}, \mathbf{c}$).
(2) Whether the system $A\mathbf{x} = \mathbf{b}$ is consistent or not depends only on A and not on $\mathbf{b}$.
(3) It's possible for $A\mathbf{x} = \mathbf{b}$ to have exactly one solution and $A\mathbf{x} = \mathbf{c}$ to have infinitely many solutions (for the same matrix A, different vectors $\mathbf{b}, \mathbf{c}$).
(4) Whether the system $A\mathbf{x} = \mathbf{b}$ has a free variable or not depends only on A and not on $\mathbf{b}$.
(5) It's possible for the system $A\mathbf{x} = \mathbf{b}$ to have a free variable and to not have a solution.
(6) No matter what A is, there's always some $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has a solution.
(7) If $\mathbf{p}$ is a solution to $A\mathbf{x} = \mathbf{b}$, then the solution set of $A\mathbf{x} = \mathbf{b}$ is $\{\mathbf{p} + \mathbf{q} \mid \mathbf{q} \text{ is a solution of } A\mathbf{x} = \mathbf{0}\}$
(8) There is a 5×3 matrix A such that $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.
(9) If A is any 5×3 matrix, then $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.
(10) There is a 3×5 matrix A such that $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.

B. REVIEW. Without computing anything, fill in the blanks: for some numbers a, b, c,

$$\begin{cases} 7x - 6y + z &= a \\ 3x + 6y - 5z &= b \\ 6x - 2y - 3z &= c \end{cases} \text{ is consistent } \iff \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ is in } \underline{\hspace{2cm}} \left\{ \begin{bmatrix} 7 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} -6 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix} \right\}$$

$$\iff \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ is in the } \underline{\hspace{3cm}} \text{ of the function } T(\mathbf{x}) = \begin{bmatrix} 7 & -6 & 1 \\ 3 & 6 & -5 \\ 6 & -2 & -3 \end{bmatrix} \mathbf{x}.$$

DEFINITION: The product of the matrices A and $B = \begin{bmatrix} \mathbf{b_1} & \cdots & \mathbf{b_n} \end{bmatrix}$, is $AB = \begin{bmatrix} A\mathbf{b_1} & \cdots & A\mathbf{b_n} \end{bmatrix}$, whenever $A\mathbf{b_1}, \ldots, A\mathbf{b_n}$ are valid products. Otherwise, we cannot take the product AB.

C. MATRIX MULTIPLICATION. Let

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 1 & -2 \end{bmatrix}.$$

(1) Is AB a valid product? What size is it? Compute AB by computing $A\mathbf{b_1}, A\mathbf{b_2}, A\mathbf{b_3}$ as in the definition above.
(2) What are the domain and codomain of the transformation $T(\mathbf{x}) = A\mathbf{x}$? What are the domain and codomain of the transformation $U(\mathbf{x}) = B\mathbf{x}$?
(3) The composition $T \circ U$ is the function "first apply U, then apply T." What is the domain of this composition?[1] What is the codomain of this composition?[2]
(4) Explain why $(T \circ U)(\mathbf{x}) = AB\mathbf{x}$. Explain why the standard matrix of $(T \circ U)$ is AB.

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[1]Hint: To input into $T \circ U$, you start by inputting into U.
[2]Hint: The outputs of $T \circ U$ come out as outputs of T.

D. TRANSFORMATIONS IN $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation "rotate 90 degrees counterclockwise," and $U : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation "reflect over the x-axis."

(1) Draw the image of the picture from the first page under the composition $T \circ U$. Can you describe the transformation $T \circ U$?[3]
(2) Based on your description of $T \circ U$ from the previous part, compute its standard matrix.
(3) Compute the standard matrix of T and of U. Call then A and B respectively.
(4) Compute AB. Compare to part (2).
(5) Now draw the image of the picture from the first page under the composition $U \circ T$. Describe this map as a single reflection, and find its standard matrix.
(6) Compute BA.

DEFINITION: The $n \times n$ **identity matrix** is the $n \times n$ matrix with 1's on the diagonal, and 0's in every other entry:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

DEFINITION: Let A be an $n \times n$ matrix. A matrix B is the **inverse matrix** of A if $AB = BA = I_n$. If B is the inverse of A, we write A^{-1} for B.

DEFINITION: An $n \times n$ matrix is **invertible** if it has an inverse. Otherwise, it is **singular**.

E. INVERSE MATRICES AND TRANSFORMATIONS.

(1) Use the definition to check that $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.
(2) Explain geometrically what the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\mathbf{x}) = A\mathbf{x}$ does when $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for some numbers a, b. What would you do to undo this transformation? When is A invertible, and what is its inverse?
(3) Explain geometrically what the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$? for some numbers a, b, c. What would you do to undo this transformation? When is A invertible, and what is its inverse?

F. INVERTING 2×2 MATRICES. Use the formula to determine if the following 2×2 matrices are invertible, and find their inverses:

(1) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
(2) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
(3) $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$
(4) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

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[3] Hint: You can consider it as a single reflection.

G. Computing inverses. Compute the inverse of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

H. Invertible matrices. Determine if each of the following matrices are invertible:

(1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(2) $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ -1 & -3 & -6 & -9 \end{bmatrix}$

(3) $C =$ some 5×7 matrix.

(4) $D =$ some 7×7 matrix where the last column is the sum of the two before it.

I*. Inverse functions.

(1) Show that if A is invertible, then $(A^{-1})^T$ is the inverse of A^T.

(2) Show that A is invertible if and only if A^T is invertible.

(3) If A, B, C are $n \times n$ matrices, when is ABC invertible (in terms of A, B, C)? If so, find a formula for its inverse.

J*. Inverse functions.

(1) Explain why, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is *any* function, then T has an inverse function if and only if $T(\mathbf{x}) = \mathbf{y}$ has a unique solution for every $\mathbf{y}$.

(2) Explain why, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, and T has an inverse function, then the inverse function to T is a linear transformation.[4]

(3) Explain why if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and T is invertible, then $m = n$.

(4) Can there be an invertible function $T : \mathbb{R}^n \to \mathbb{R}^m$ for $m \neq n$?[5]

[4] Challenge: do this without using the inverse matrix theorem!

[5] This function must NOT be a linear transformation, based on the previous part.