Math 325. Quiz #1

(1) State the *Well Ordering Axiom* for $\mathbb{N}$.

Every nonempty subset of $\mathbb{N}$ has a minimum element.

(2) Write the contrapositive of the following statement:
"Let a and b be integers. If a and b are both odd, then ab is odd."

Let a, b be integers. If ab is even, then a is even or b is even.

(3) Consider the statement:
"There exists a natural number x such that for every natural number y, $x^2 > y$."

(a) Write the negation of the statement above.

For every natural number x, there exists a natural number y such that $x^2 \leq y$.

(b) Either prove the statement above or prove its negation.

Let $x \in \mathbb{N}$ be given. Take $y = x^2$. Since $y = x^2$, $y \geq x^2$, as required. (proof of negation)

Math 325. Quiz #2

(1) State the *Completeness Axiom* of the real numbers.

Every nonempty bounded above subset of $\mathbb{R}$ has a supremum.

(2) True or false, and justify: There exists an open interval (a, b) with $a < b$ that contains only irrational numbers.

False: by density of ~~real~~ rational numbers, there is a rational number between a and b.

(3) True or false, and justify: If a set S of real numbers has a supremum $\ell = \sup(S)$, then $\ell \in S$.

False: $\sup((0,1)) = 1$, but $1 \notin (0,1)$.

(4) Let S be a set of real numbers. Which of the following is statements is equivalent to "S is bounded above" ? For which sets S is the other statement true?

(a) $\forall s \in S, \exists r \in \mathbb{R} : s \leq r$

(b) $\exists r \in \mathbb{R} : \forall s \in S, s \leq r$

(b) is bounded above.

(a) is true of every set of real numbers.

Math 325. Quiz #3

(1) State the definition for a sequence $\{a_n\}_{n=1}^{\infty}$ to *converge* to a real number L.

For every $\varepsilon > 0$, there is some $N \in \mathbb{R}$ such that for all natural numbers n with $n > N$, we have $|a_n - L| < \varepsilon$.

(2) Let $S \subseteq \mathbb{R}$ be nonempty and bounded above. Let $T = \{2x \mid x \in S\}$. Prove that $\sup(T) = 2\sup(S)$.

Let $\ell = \sup(S)$. Given $t \in T$, write $t = 2x$ for some $x \in S$. Then $t/2 = x \leq \ell$ since ℓ is an upper bound for S, hence $t \leq 2\ell$. Since $t \in T$ is arbitrary, 2ℓ is an upper bound for T.

Let b be an upper bound for T, and suppose that $b < 2\ell$. Then $b/2 < \ell$, so $b/2$ is not an upper bound for S, by definition of supremum. Thus, there is some $x \in S$ such that $x > b/2$. Then $t = 2x > b$, with $t \in T$. This contradicts that b is an upper bound for T. We conclude that any upper bound b for T is at least 2ℓ, and hence 2ℓ is the supremum of T.

Math 325. Quiz #4

(1) State the definition for a sequence $\{a_n\}_{n=1}^{\infty}$ to be *strictly increasing*.

For all $n \in \mathbb{N}$, $a_n < a_{n+1}$.

(2) Determine if TRUE or FALSE, and justify with a short argument or counterexample:

(a) If $\{a_n\}_{n=1}^{\infty}$ converges to 1, then there is some $N \in \mathbb{R}$ such that $a_n < 3/2$ for all natural numbers $n > N$.

True: Take $\varepsilon = 1/2$; $\exists N$: for all $n > N$, $|a_n - 1| < 1/2$
$\Rightarrow a_n < 1 + 1/2 = 3/2$.

(b) If $\{a_n\}_{n=1}^{\infty}$ converges to 1, then there is some $N \in \mathbb{R}$ such that $a_n = 1$ for all natural numbers $n > N$.

False: Take $\{1 + 1/n\}_{n=1}^{\infty}$.

(c) If $\{a_n\}_{n=1}^{\infty}$ is a sequence with $0 < a_{n+1} - a_n < a_n < 25$ for all $n \in \mathbb{N}$, then $\{a_n\}_{n=1}^{\infty}$ converges.

True: $a_{n+1} - a_n > 0 \Rightarrow$ (strictly) increasing
$a_n < 25 \Rightarrow$ bounded above
Any increasing bounded above sequence converges.