DISC Course on Linear Matrix Inequalities in Control

Siep Weiland
Department of Electrical Engineering
Eindhoven University of Technology

Course 2004 - Class 3

Dissipative systems

Given a system Σ described by

$$\begin{cases} \dot{x} = f(x,w) \\ z = g(x,w) \end{cases} \qquad x(t) \in X, \quad w(t) \in W, \quad z(t) \in Z$$

We call the system Σ **dissipative** with respect to the **supply function** $s : W \times Z \longrightarrow \mathbb{R}$ if there exists a **storage function** $V : X \longrightarrow \mathbb{R}$ such that

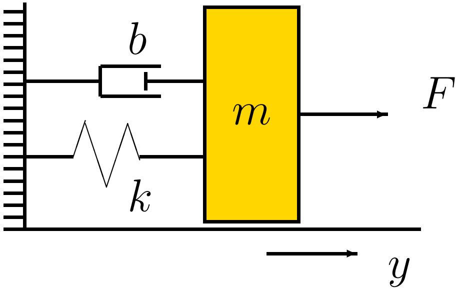
$$V(x(t_0)) + \int_{t_0}^{t_1} s(w(t), z(t))dt \;\geq\; V(x(t_1)) \tag{1}$$

for all $t_0 \leq t_1$ and (w, x, z) in system behavior.

Interpretation:
s: supply delivered to system; $V(x)$: internally stored "energy".
In any experiment, final storage can be at most initial storage plus supplied energy over a time interval

– system is **conservative** if $=$ in (1)
– system is **strictly dissipative** if "$>$" in (1)

Example: mechanical systems



Modelled by differential equation

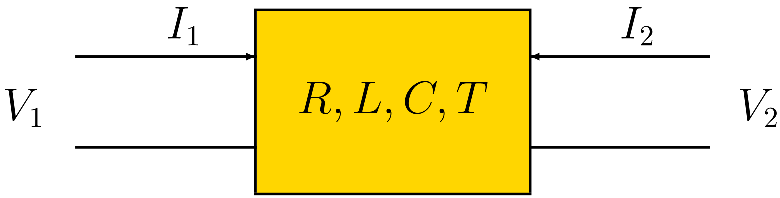
$$\boxed{ky + b\dot{y} + m\ddot{y} = F}$$

Let force F be input, velocity $v = \dot{y}$ output.

This defines a dissipative system with supply $s(F, v) = Fv$ and storage function:

$$\boxed{V(y, v) = \frac{1}{2}mv^2 + \frac{1}{2}ky^2}$$

Example: electrical networks



- Electrical network with voltages w as input, currents z as output.
- State $x = \mathrm{col}_i(V_{C_i}, I_{L_i})$, stacked capacitor voltages and inductance currents of components.
- External ports $w = \mathrm{col}(V_1, V_2)$, $z = \mathrm{col}(I_1, I_2)$.
- Supply $s(w, z) = w^\top z$ (power).
- Storage function is electrical energy

$$\boxed{V(x) = \sum_i C_i V_{C_i}^2 + \sum_j L_j I_{L_j}^2}$$

Example: thermodynamics

Variables: T temperature, Q heating rate, W rate of mechanical work.

- **First law of thermodynamics**

$$E(x(t_0)) + \int_{t_0}^{t_1} (W(t) + Q(t))dt \ = \ E(x(t_1))$$

- **Second law of thermodynamics**

$$S(x(t_0)) + \int_{t_0}^{t_1} \frac{Q(t)}{T(t)}dt \ \geq \ S(x(t_1))$$

Other common supply functions

- $s(w,z) = z^\top w$
- $s(w,z) = \|w\|^2 + \|z\|^2$
- $s(w,z) = \|z\|^2 - \|w\|^2$

Equivalent local characterization

Theorem: If storage function $V(\cdot)$ is differentiable, then the dissipation inequality is equivalent to the **differential dissipation inequality**

$$V_x(x)f(x,w) \leq s(w, g(x,w)), \qquad \text{for all } x \in X \text{ and } w \in W.$$

Here, V_x is the gradient of V:

$$V_x = \begin{pmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{pmatrix}$$

This is easily seen: for all $t_1 > t_0$:

$$\frac{1}{t_1 - t_0}\left(V(x(t_1)) - V(x(t_0))\right) \leq \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} s(w(t), z(t))dt$$

Let $t = t_1 \to t_0$. Then

- left-hand side converges to $\frac{d}{dt}V(x(t)) = V_x(x(t))\dot{x}(t) = V_x(x(t))f(x(t), w(t))$.
- right-hand side converges to $s(w(t), z(t)) = s(w(t), g(x(t), w(t)))$.

Strict dissipativity

The system $\dot{x} = f(x, w)$, $z = g(x, w)$ with supply rate s is said to be **strictly dissipative** if there exists a storage function $V : X \to \mathbb{R}$ and an $\varepsilon > 0$ such that

$$V(x(t_0)) + \int_{t_0}^{t_1} s(w(t), z(t))\, dt - \varepsilon^2 \int_{t_0}^{t_1} \|w(t)\|^2\, dt \;\geq\; V(x(t_1))$$

for all $t_0 \leq t_1$ and all trajectories (w, x, z).

This implies that the dissipation inequality holds with $>$.

Other refinements:

- time-varying systems
- uncertain systems
- discrete time systems (???)
- conditions on periodic trajectories only.

Classifying all storage functions

Assume $x^* \in X$ point of neutral storage $V(x^*) = 0$. Defines set of **normalized storage functions**

$$\mathcal{V}(x^*) := \{V \mid V(x^*) = 0, \quad V \text{ satisfies dissipation inequality}\}.$$

What can $\mathcal{V}(x^*)$ be ???

Classifying all storage functions

Assume $x^* \in X$ point of neutral storage $V(x^*) = 0$. Defines set of **normalized storage functions**

$$\mathcal{V}(x^*) := \{V \mid V(x^*) = 0, \quad V \text{ satisfies dissipation inequality}\}.$$

What can $\mathcal{V}(x^*)$ be ???

Introduce **available storage** $V_{\text{av}} : X \to \mathbb{R} \cup \{\infty\}$ and **required supply** $V_{\text{req}} : X \to \mathbb{R} \cup \{-\infty\}$:

$$V_{\text{av}}(x_0) := \sup\left\{-\int_0^{t_1} s(w(t), z(t))\, dt \;\middle|\; t_1 \geq 0;\; x(0) = x_0\right\} \qquad \text{(AV)}$$

$$V_{\text{req}}(x_0) := \inf\left\{\int_{t_{-1}}^{0} s(w(t), z(t))\, dt \;\middle|\; t_{-1} \leq 0;\; x(0) = x_0\right\} \qquad \text{(RQ)}$$

Classifying all storage functions

Theorem: Suppose that Σ is controllable. Then (Σ, s) is dissipative
if and only if

$$-\infty < V_{\text{av}}(x) < \infty \quad \text{for all } x \in X$$

if and only if

$$-\infty < V_{\text{req}}(x) < \infty \quad \text{for all } x \in X$$

In that case:

- $V_{\text{av}}, V_{\text{req}} \in \mathcal{V}(x^*)$.
- $V \in \mathcal{V}(x^*)$ implies $V_{\text{av}}(x) \leq V(x) \leq V_{\text{req}}(x)$ for all x.
- $\mathcal{V}(x^*)$ is convex set.
  Thus: $V_\alpha := \alpha V_{\text{av}} + (1-\alpha)V_{\text{req}} \in \mathcal{V}(x^*)$ for all $\alpha \in (0,1)$.

Interpretation:
A dissipative system can neither supply nor store an infinite amount of energy.

Sketch of proof

(only if) Run a loop: let $x_0 \in X$, $t_{-1} \leq 0 \leq t_1$ and (w, x, z) a trajectory with

$$x(t_{-1}) = x^* \qquad x(0) = x_0 \qquad x(t_1) = x^*.$$

Then

$$-\infty < -\int_0^{t_1} s(t)dt \leq \int_{t_{-1}}^{0} s(t)dt < +\infty.$$

Now take

- supremum over $t_1 > 0$:

$$-\infty < V_{\text{av}}(x_0) < \infty$$

- infimum over $t_{-1} < 0$:

$$-\infty < V_{\text{req}}(x_0) < \infty$$

(if) Available storage at $x(t_0)$ is more than available storage at $x_(t_1)$ while passing from $x(t_0)$ to $x(t_1)$ in arbitrary way. Hence $V_{\text{av}}(x_0)$ is a storage function.

Linear dissipative systems

Consider the **linear system** Σ:

$$\dot{x} = f(x, w) = Ax + Bw; \quad z = g(x, w) = Cx + Dw$$

with **quadratic supply function**

$$s(w, z) = \begin{pmatrix} w \\ z \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} =$$
$$= \begin{pmatrix} x \\ w \end{pmatrix}^\top \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

Then for **quadratic storge functions** $V(x) = x^\top K x$ the differential dissipation inequality reads

$$2x^\top K(Ax + Bw) \leq s(w, Cx + Dw) \qquad \text{for all } x \text{ and } w$$

This is equivalent to an LMI condition!!

Main result for dissipative systems

Theorem: Suppose Σ is controllable, s quadratic. Equivalent are

1. (Σ, s) is dissipative.
2. (Σ, s) admits a **quadratic** storage function $V(x) := x^\top K x$
3. There exists a symmetric K such that

$$\underbrace{\begin{pmatrix} A^\top K + KA & KB \\ B^\top K & 0 \end{pmatrix} - \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}}_{F(K)} \preceq 0.$$

Moreover, $V(x) = x^\top K x$ is a quadratic storage function in $\mathcal{V}(0)$ if and only if

$$F(K) \preceq 0.$$

- $F(K)$ is called the **dissipation matrix**.
- Characterizes **all** normalized storage functions as LMI feasibility set.

Relation to frequency domain inequality

Associate with state space system its **transfer function**

$$\dot{x} = Ax + Bw, \quad z = Cx + Dw \quad \longleftrightarrow \quad T(s) = C(Is - A)^{-1}B + D$$

Theorem: Suppose Σ is controllable. Then there exists $K = K^\top$ with

$$\begin{pmatrix} A^\top K + KA & KB \\ B^\top K & 0 \end{pmatrix} - \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \preceq 0$$

if and only if
T satisfies the **frequency domain inequality** (FDI)

$$0 \preceq \begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \text{ for all } \omega \in \mathbb{R} \cup \{\infty\},\ i\omega \notin \lambda(A)$$

Provides a frequency domain test.

Trajectory based proof: LMI implies FDI

Take a $\tau = 2\pi/\omega$ periodic input $w(t) = \exp(i\omega t)w_0$. Then

$$\begin{aligned} x(t) &= \exp(i\omega t)(i\omega I - A)^{-1}Bw_0 \\ z(t) &= \exp(i\omega t)T(i\omega)w_0 \end{aligned}$$

are τ periodic trajectories of the system and

$$s(w(t), z(t)) = \exp(-i\omega t)w_0^* \begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* P \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} w_0 \exp(i\omega t)$$

is **constant** for all t.

But then, integrating over k periods:

$$\int_0^{k\tau} s(w(t), z(t))dt = k\tau\, w_0^* \begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* P \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} w_0 \geq 0$$

which holds for any w_0 and any $k\tau > 0$.

This gives

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* P \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \succeq 0.$$

Main result for strictly dissipative systems

Theorem: Suppose A no eigenvalues $i\omega$ and s is quadratic. Equivalent are

1. (Σ, s) is **strictly dissipative**.
2. (Σ, s) admits a **quadratic** storage function $V(x) := x^\top K x$.
3. There exists a symmetric K such that

$$F(K) \prec 0.$$

4. For all $\omega \in \mathbb{R} \cup \{\infty\}$ with $i\omega \notin \lambda(A)$:

$$0 \prec \begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix}$$

Moreover, $V(x) = x^\top K x$ is a quadratic storage function in $\mathcal{V}(0)$
if and only if

$$F(K) \prec 0.$$

The pair (A, B) may be non-controllable here!

Equivalent representations

Note:

$$F(K) = \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^\top \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} - \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}$$

$$= \left(\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right)^\top \left(\begin{array}{cc|cc} 0 & K & 0 & 0 \\ K & 0 & 0 & 0 \\ \hline 0 & 0 & -Q & -S \\ 0 & 0 & -S^\top & -R \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right)$$

$$= \begin{pmatrix} A^\top K + KA - C^\top RC & KB - (SC)^\top - C^\top RD \\ B^\top K - SC - D^\top RC & -Q - SD - (SD)^\top - D^\top RD \end{pmatrix}$$

Let $T := Q + SD + (SD)^\top + D^\top RD$. Using a Schur complement: $F(K) \prec 0$ **if and only if** $T \succ 0$ and

$$A^\top K + KA + C^\top RC \\ + \left(KB + (SC)^\top + C^\top RD\right) T^{-1} \left(B^\top K + SC + D^\top RC\right) \prec 0$$

The KYP Lemma (continuous time)

Theorem: There exists symmetric K such that

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^\top \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + M \prec 0$$

if and only if south-east block of M is negative definite and for all $\omega \in \mathbb{R}$

$$\begin{pmatrix} A - i\omega I & B \end{pmatrix} \underbrace{\begin{pmatrix} x \\ w \end{pmatrix}}_{\neq 0} = 0 \quad \text{implies} \quad \begin{pmatrix} x \\ w \end{pmatrix}^* M \begin{pmatrix} x \\ w \end{pmatrix} < 0.$$

If A has no eigenvalues on imaginary axis, the latter condition is equivalent to

$$\begin{pmatrix} (i\omega I - A)^{-1} B \\ I \end{pmatrix}^* M \begin{pmatrix} (i\omega I - A)^{-1} B \\ I \end{pmatrix} \prec 0 \quad \text{for all } \omega \in \mathbb{R}.$$

The KYP Lemma (discrete time)

Theorem: There exists symmetric K such that

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^{\top} \begin{pmatrix} -K & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + M \prec 0$$

if and only if for all $z \in \mathbb{C}$ with $|z| = 1$,

$$\begin{pmatrix} A - zI & B \end{pmatrix} \underbrace{\begin{pmatrix} x \\ w \end{pmatrix}}_{\neq 0} = 0 \quad \text{implies} \quad \begin{pmatrix} x \\ w \end{pmatrix}^{*} M \begin{pmatrix} x \\ w \end{pmatrix} < 0.$$

If A has no eigenvalues on unit circle, the latter condition is equivalent to

$$\begin{pmatrix} (zI - A)^{-1}B \\ I \end{pmatrix}^{*} M \begin{pmatrix} (zI - A)^{-1}B \\ I \end{pmatrix} \prec 0 \quad \text{for all } z \in \mathbb{C}, |z| = 1.$$

Proof only if part

Let $w \in \mathbb{C}^m$ and $x \in \mathbb{C}^n$ be such that

$$\begin{pmatrix} A - i\omega I & B \end{pmatrix} \underbrace{\begin{pmatrix} x \\ w \end{pmatrix}}_{\neq 0} = 0$$

Then

$$\begin{pmatrix} x \\ w \end{pmatrix}^* \left[\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^\top \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + M \right] \begin{pmatrix} x \\ w \end{pmatrix} \quad \begin{pmatrix} x \\ w \end{pmatrix}^* \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^\top \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x \\ w \end{pmatrix}^* M \begin{pmatrix} x \\ w \end{pmatrix}.$$

will not depend on K.

The positive real lemma

Theorem: Equivalent statements are:

1. System is strictly dissipative with respect to $s(w,z) = z^\top w$
2. There exists a symmetric K such that

$$\begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & K & 0 & 0 \\ K & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -I \\ 0 & 0 & -I & 0 \end{array}\right) \begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix} \prec 0$$

3. For all $\omega \in \mathbb{R}$ with $\det(i\omega I - A) \neq 0$ one has

$$T(i\omega)^* + T(i\omega) \succ 0.$$

4. All system trajectories (w,z) satisfy $\int_0^\infty z^\top(t) w(t) dt > 0$.

Towards nominal performance

Consider the stable system

$$\dot{x} = Ax + Bw, \quad z = Cx + Dw, \quad \longleftrightarrow \quad T(s) = C(Is - A)^{-1}B + D$$

With w a disturbance, minimize effect of w on output z.

With suitable signal norms, the system gain is defined as

$$\|T\| := \sup_{0 \neq \|w\| < \infty} \frac{\|z\|}{\|w\|}$$

or, equivalently, system gain is at most γ if

$$\|z\| \ \leq \ \gamma \|w\| \qquad \text{for all } w \neq 0$$

- Reflects maximal amplification of disturbance.
- small gain means good disturbance attenuation.

H_∞ nominal performance

With $\mathcal{L}_2$ norm $\|w\|_2^2 := \int_0^\infty \|w(t)\|^2 dt$ this yields

$$\|T\|_{2,2} < \gamma$$

H_∞ nominal performance

With $\mathcal{L}_2$ norm $\|w\|_2^2 := \int_0^\infty \|w(t)\|^2 dt$ this yields

$$\|T\|_{2,2} < \gamma$$

if and only if for some $\varepsilon > 0$

$$\|z\|_2^2 \leq (\gamma^2 - \varepsilon^2)\|w\|_2^2 \text{ for all } w \neq 0$$

H_∞ nominal performance

With $\mathcal{L}_2$ norm $\|w\|_2^2 := \int_0^\infty \|w(t)\|^2 dt$ this yields

$$\|T\|_{2,2} < \gamma$$

if and only if for some $\varepsilon > 0$

$$\|z\|_2^2 \leq (\gamma^2 - \varepsilon^2)\|w\|_2^2 \text{ for all } w \neq 0$$

if and only if

$$\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^\top \begin{pmatrix} -(\gamma^2 - \varepsilon^2) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq 0 \quad \text{for all } w \neq 0$$

H_∞ nominal performance

With $\mathcal{L}_2$ norm $\|w\|_2^2 := \int_0^\infty \|w(t)\|^2 dt$ this yields

$$\|T\|_{2,2} < \gamma$$

if and only if for some $\varepsilon > 0$

$$\|z\|_2^2 \leq (\gamma^2 - \varepsilon^2)\|w\|_2^2 \text{ for all } w \neq 0$$

if and only if

$$\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^\top \begin{pmatrix} -(\gamma^2 - \varepsilon^2) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq 0 \quad \text{for all } w \neq 0$$

if and only if for all $\hat{w}(i\omega)$

$$\int_{-\infty}^\infty \hat{w}(i\omega)^* \begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} -(\gamma^2 - \varepsilon^2)I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \hat{w}(i\omega) d\omega \leq 0$$

H_∞ nominal performance

With $\mathcal{L}_2$ norm $\|w\|_2^2 := \int_0^\infty \|w(t)\|^2 dt$ this yields

$$\|T\|_{2,2} < \gamma$$

if and only if for some $\varepsilon > 0$

$$\|z\|_2^2 \leq (\gamma^2 - \varepsilon^2)\|w\|_2^2 \text{ for all } w \neq 0$$

if and only if

$$\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^\top \begin{pmatrix} -(\gamma^2 - \varepsilon^2) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq 0 \quad \text{for all } w \neq 0$$

if and only if for all $\hat{w}(i\omega)$

$$\int_{-\infty}^\infty \hat{w}(i\omega)^* \begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} -(\gamma^2 - \varepsilon^2)I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \hat{w}(i\omega) d\omega \leq 0$$

if and only if

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} -(\gamma^2 - \varepsilon^2)I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \preceq 0 \quad \text{for all } \omega \in \mathbb{R}$$

if and only if

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \prec 0 \quad \text{for all } \omega \in \mathbb{R}$$

if and only if

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \prec 0 \quad \text{for all } \omega \in \mathbb{R}$$

if and only if

$$T(i\omega)^* T(i\omega) \; < \gamma^2 I \text{ for all } \omega \in \mathbb{R}$$

if and only if

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \prec 0 \quad \text{for all } \omega \in \mathbb{R}$$

if and only if

$$T(i\omega)^* T(i\omega) \; < \gamma^2 I \text{ for all } \omega \in \mathbb{R}$$

if and only if

$$\|T\|_{H_\infty} := \sup_{\omega} \sigma_{\max} T(i\omega) \; < \gamma$$

if and only if

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \prec 0 \quad \text{for all } \omega \in \mathbb{R}$$

if and only if

$$T(i\omega)^* T(i\omega) \; < \gamma^2 I \text{ for all } \omega \in \mathbb{R}$$

if and only if

$$\|T\|_{H_\infty} := \sup_{\omega} \sigma_{\max} T(i\omega) \; < \gamma$$

if and only if

there exists a symmetric $K = K^\top$ such that

$$\left(\begin{array}{c|c} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right)^\top \left(\begin{array}{cc|cc} 0 & K & 0 & 0 \\ K & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & I \end{array}\right) \left(\begin{array}{c|c} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right) \prec \; 0$$

if and only if

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \prec 0 \quad \text{for all } \omega \in \mathbb{R}$$

if and only if

$$T(i\omega)^* T(i\omega) \; < \gamma^2 I \text{ for all } \omega \in \mathbb{R}$$

if and only if

$$\|T\|_{H_\infty} := \sup_\omega \sigma_{\max} T(i\omega) \; < \gamma$$

if and only if

there exists a symmetric $K = K^\top$ such that

$$\begin{pmatrix} A^\top K + KA + C^\top C & KB + C^\top D \\ B^\top K + D^\top C & D^\top D - \gamma^2 I \end{pmatrix} \prec \; 0$$

if and only if

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \prec 0 \quad \text{for all } \omega \in \mathbb{R}$$

if and only if

$$T(i\omega)^* T(i\omega) \ < \gamma^2 I \text{ for all } \omega \in \mathbb{R}$$

if and only if

$$\|T\|_{H_\infty} := \sup_\omega \sigma_{\max} T(i\omega) \ < \gamma$$

if and only if

there exists a symmetric $K = K^\top$ such that

$$\begin{pmatrix} A^\top K + KA & KB & C^\top \\ B^\top K & -\gamma^2 I & D^\top \\ C & D & -I \end{pmatrix} \prec \ 0$$

The bounded real lemma

Theorem: Take $s(w,z) = \gamma^2 w^\top w - z^\top z$ and assume A Hurwitz. Then equivalent are:

1. there exists a symmetric K such that

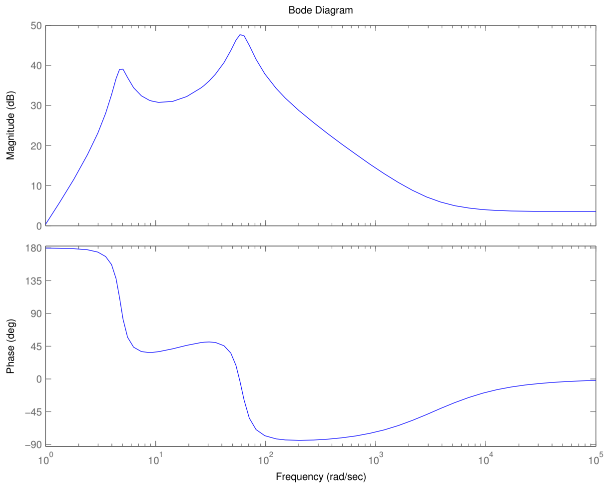
$$\begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix}^\top \left(\begin{array}{cc|cc} 0 & K & 0 & 0 \\ K & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & I \end{array}\right) \begin{pmatrix} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{pmatrix} \prec 0$$

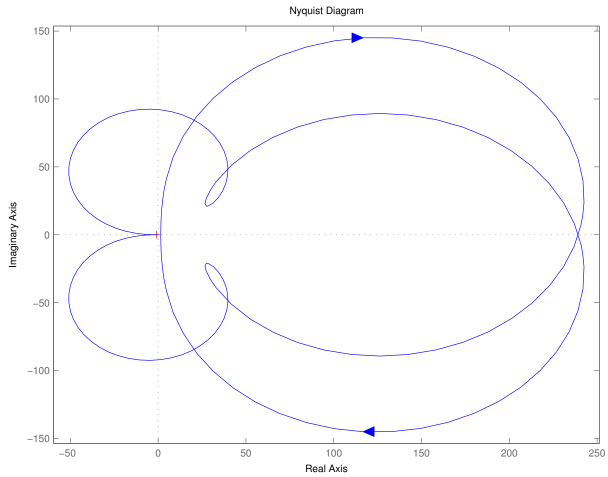
2. For all $\omega \in \mathbb{R}$ one has $T(i\omega)^* T(i\omega) < \gamma^2 I$.
3. The H_∞ norm $\|T\|_\infty < \gamma$.

Crucial result!!
Relates bound on H_∞ norm of transfer function T to an LMI feasibility test!

Frequency domain test

Bode diagram **Nyquist diagram**





$$\|T\|_{H_\infty} = \sup_\omega \sigma_{\max} T(i\omega)$$

Quadratic performance

Quadratic performance with symmetric P_p: exists $\varepsilon > 0$ such that

$$\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} P_p \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} \leq -\varepsilon \int_0^\infty w(t)^\top w(t) dt \quad \text{for all } w$$

Theorem: System satisfies quadratic P_p performance **if and only if**

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* P_p \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \prec 0 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}$$

if and only if

$$\left(\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right)^\top \left(\begin{array}{cc|cc} 0 & K & 0 & 0 \\ K & 0 & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^\top & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right) \prec 0; \quad P_p = \begin{pmatrix} Q_p & S_p \\ S_p^\top & R_p \end{pmatrix}$$

H_2 nominal performance

Suppose (A, B, C, D) system is stable. Its H_2 norm is defined as

$$\|T\|_{H_2} := \sqrt{\frac{1}{2\pi}\,\text{trace}\int_{-\infty}^{\infty} T(i\omega)^* T(i\omega)d\omega}$$

Interpretations:

- sum of L_2 norms of impulse responses in each channel

$$\|T\|_{H_2}^2 = \sum_k \|z_k\|^2 \qquad z_k \text{ impulse response from } k\text{th input}$$

- Asymptotic output variance: w white noise

$$\|T\|_{H_2} = \lim_{t\to\infty} \mathcal{E}\left(z(t)^\top z(t)\right)$$

How to compute H_2 nominal performance?

If A is stable and $D = 0$:

$$\begin{aligned} \|T\|_{H_2}^2 &= \operatorname{trace}(CP_0C^\top) < \gamma^2 \quad \text{for } AP_0 + P_0A^\top + BB^\top = 0 \\ &= \operatorname{trace}(B^\top Q_0 B) < \gamma^2 \quad \text{for } A^\top Q_0 + Q_0 A + C^\top C = 0 \end{aligned}$$

if and only if there exists $X \succ 0$ with

$$\operatorname{trace} CXC^\top < \gamma^2 \quad \text{and} \quad AX + XA^\top + BB^\top \prec 0$$

Theorem: Suppose A is stable. Then $\|T\|_{H_2} < \infty$ iff $D = 0$. In that case

$$\|T\|_{H_2} < \gamma$$

if and only if there exists $X \succ 0$ such that

$$\operatorname{trace}(CXC^\top) < \gamma^2 \quad \text{and} \quad AX + XA^\top + BB^\top \prec 0.$$

if and only if there exists $Y \succ 0$ such that

$$\operatorname{trace}(B^\top YB) < \gamma^2 \quad \text{and} \quad A^\top Y + YA + C^\top C \prec 0.$$

Generalized H_2 nominal performance

With supply rate $s(w,z) = \gamma \|w\|^2$, $x(0) = 0$, $D = 0$, the dissipation inequality reads

$$\gamma \int_0^t w^\top(\tau) w(\tau) d\tau \geq V(x(t)) = x^\top(t) K x(t) \geq x^\top(t) C^\top C x(t) = z^\top(t) z(t$$

provided $K \succ C^\top C$. Hence, with **amplitude norm**

$$\boxed{\|z\|_\infty^2 := \sup_{t\geq 0} z^\top(t) z(t)}$$

we get that the **energy to peak norm**

$$\boxed{\|T\|_{2,\infty} := \sup_{0<\|w\|_2<\infty} \frac{\|z\|_\infty}{\|w\|_2} \leq \gamma}$$

For systems with scalar valued output variables

$$\|T\|_{2,\infty} = \|T\|_{H_2}$$

Theorem: Suppose A is stable. Equivalent are

1. $\|T\|_{2,\infty} := \sup_{0<\|w\|_2<\infty} \frac{\|z\|_\infty}{\|w\|_2} \;<\; \gamma$
2. $D = 0$ and there exists $K = K^\top$ such that

$$\begin{pmatrix} A^\top K + KA & KB \\ B^\top K & -\gamma I \end{pmatrix} \prec 0 \quad \text{and} \begin{pmatrix} K & C^\top \\ C & \gamma I \end{pmatrix} \succ 0$$

Remarks:
Also called the **energy-to-peak gain** of the system.

Summary

- Notion of dissipative system very useful for specifying performance
- For linear systems with quadratic supply functions, dissipativity has been characterized in terms of LMI's and FDI's
- Equivalence of LMI and FDI: the KYP lemma
- We considered nominal performance specifications:
  - ◇ H_∞ performance
  - ◇ H_2 performance
  - ◇ Quadratic perfomance
  - ◇ Generalized H_2 or energy-to-peak gain

  all characterized as LMI feasibility tests.

Discrete time systems

As for stability, we can often obtain discrete time results from continuous time counterparts by substitution

$$\begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -K & 0 \\ 0 & K \end{pmatrix}$$

Indeed, with $\dot{x} = Ax + Bw$ differentiating $V(x) = x^\top Kx$ along solutions gives

$$\frac{d}{dt}V(x(t)) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}$$

whereas with $x(t+1) = Ax(t) + Bw(t)$, $V(x) = x^\top Kx$ along solutions gives

$$V(x(t+1)) - V(x(t)) = \begin{pmatrix} x(t) \\ x(t+1) \end{pmatrix} \begin{pmatrix} -K & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}$$

Quadratic performance (discrete time)

Quadratic performance with symmetric P_p: exists $\varepsilon > 0$ such that

$$\sum_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} P_p \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} \leq -\varepsilon \sum_0^\infty w(t)^\top w(t) \quad \text{for all } w$$

Theorem: System satisfies quadratic P_p performance **if and only if**

$$\begin{pmatrix} I \\ T(z) \end{pmatrix}^* P_p \begin{pmatrix} I \\ T(z) \end{pmatrix} \prec 0 \text{ for all } z \in \mathbb{C}, \quad |z| = 1$$

if and only if

$$\left(\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right)^\top \left(\begin{array}{cc|cc} -K & 0 & 0 & 0 \\ 0 & K & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^\top & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right) \prec 0; \quad P_p = \begin{pmatrix} Q_p & S_p \\ S_p^\top & R_p \end{pmatrix}$$

Next class