Multiplier Relaxations

- Linear fractional representations
- Robust stability tests with multipliers
- Relations to the structured singular value
- Extension to dynamic uncertainties: Integral quadratic constraints

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Rational Parameter Dependence

Consider uncertain system

$$\dot{x} = F(\delta)x \quad \text{with} \quad F(\delta) = \begin{pmatrix} -1 & 2\delta_1 \\ -\frac{1}{\delta_1+1} & -4+3\delta_2 \end{pmatrix}$$

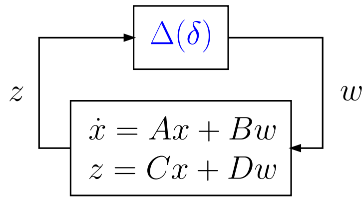
where $\delta_1 \in [-0.5, 1]$ and $\delta_2 \in [-1, 1]$.

How can we handle **rational** parameter dependence?

$\dot{x} = A(\delta)x$ can be represented as

$$\left.\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cw + Dw \end{aligned}\right\} \; w = \Delta(\delta)z$$

with $\Delta(\delta)$ depending **linearly** on the parameters δ.



Derivation

Rewrite $\xi = F(\delta)x$ as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1+\delta_1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -1 & 2\delta_1 \\ -1 & (-4+3\delta_2)(1+\delta_1) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or as

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w_1 = \begin{pmatrix} -1 & 0 \\ -1 & -4+3\delta_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4+3\delta_2 \end{pmatrix} w_2$$

$$w_1 = \delta_1 z_1,\ z_1 = \xi_2, \qquad w_2 = \delta_1 z_2,\ z_2 = x_2$$

or as

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w_1 = \begin{pmatrix} -1 & 0 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} w_2 + \begin{pmatrix} 0 \\ 3 \end{pmatrix} w_3$$

$$w_1 = \delta_1 z_1,\ z_1 = \xi_2, \qquad w_2 = \delta_1 z_2,\ z_2 = x_2, \qquad w_3 = \delta_2 z_3,\ z_3 = x_2 + w_2.$$

Derivation

Hence $\xi = F(\delta)x$ can be written as

$$\xi = \begin{pmatrix} -1 & 0 \\ -1 & -4 \end{pmatrix} x + \begin{pmatrix} 0 & 2 & 0 \\ -1 & -4 & 3 \end{pmatrix} w$$

$$z = \begin{pmatrix} -1 & -4 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} -1 & -4 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} w, \quad w = \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \end{pmatrix} z.$$

Therefore we can choose

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|ccc} -1 & 0 & 0 & 2 & 0 \\ -1 & -4 & -1 & -4 & 3 \\ \hline -1 & -4 & -1 & -4 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right), \; \Delta(\delta) = \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \end{pmatrix}.$$

Linear Fractional Representation (LFR)

Suppose $F(\delta_1, \ldots, \delta_p)$ is a matrix-valued function that is **rational** in $\delta = (\delta_1, \ldots, \delta_p)$ and that has no pole at zero. The one can construct matrices A, B, C, D such that

$$F(\delta) = A + B\Delta(\delta)\left(I - D\Delta(\delta)\right)^{-1} C =: \begin{pmatrix} A & B \\ C & D \end{pmatrix} \star \Delta(\delta)$$

where

$$\Delta(\delta) = \begin{pmatrix} \delta_1 I_{d_1} & & 0 \\ & \ddots & \\ 0 & & \delta_p I_{d_p} \end{pmatrix} \quad \text{and} \quad I_{d_k} \quad \text{is identity of size } d_k.$$

If $F(\delta)$ depends **affinely** on δ one can choose $D = 0$.

The operation $\star$ is often called lower linear fractional transformation or **star-product**. Our example shows how to actually prove this result.

Linear Fractional Representation (LFR)

- The LFR is called well-posed on the set $\boldsymbol{\delta} \subset \mathbb{R}^p$ if $I - D\Delta(\delta)$ is non-singular for all $\delta \in \boldsymbol{\delta}$.

- There exists a simple calculus of LFR's. For example, sums, products and LFR's of LFR's are LFR's.

- There exists a nice toolbox for working with LFR's ...

  ... developed by J.F. Magni, ONERA-CERT, Toulouse.

Here is a simple **recipe** of how to derive LFR's and how to work with them, even if the uncertainties are not parametric.

View uncertainties as **systems** processing signals and just employ the usual techniques for manipulating system interconnections.

Testing Well-Posedness

Given $\boldsymbol{\delta} \subset \mathbb{R}^p$ how can we numerically verify well-posedness of the LFR?

Is it true that $\det(I - D\Delta(\delta)) \neq 0$ for all $\delta \in \boldsymbol{\delta}$?

Remember that this is precisely the fundamental question in structured singular-value (SSV) theory.

Once we understand the details around this problem, it is rather straightforward to extend to all kinds of robust stability and robust performance characterizations. The key for developing such extension is dissipativity.

We can only hint at the real power of this approach!

Testing Well-Posedness

Suppose there exists a **multiplier** P which satisfies

$$\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} \succcurlyeq 0 \quad \text{for all} \quad \delta \in \boldsymbol{\delta}$$

and at the same time

$$\begin{pmatrix} I \\ D \end{pmatrix}^T P \begin{pmatrix} I \\ D \end{pmatrix} \prec 0.$$

Then $\det(I - D\Delta(\delta)) \neq 0$ for all $\delta \in \boldsymbol{\delta}$.

Is a triviality! How to use in computations? Extensions? Interpretation?

If $\boldsymbol{\delta}$ is compact it can be shown that **iff** holds (full block S-procedure).

Proof

Suppose there exists some $\delta_0 \in \boldsymbol{\delta}$ such that $\Delta_0 := \Delta(\delta_0)$ renders $I - D\Delta_0$ **singular**. Then there exists a vector $w \neq 0$ such that

$$(I - D\Delta_0)w = 0.$$

For $z = \Delta_0 w$ we infer $z \neq 0$ and

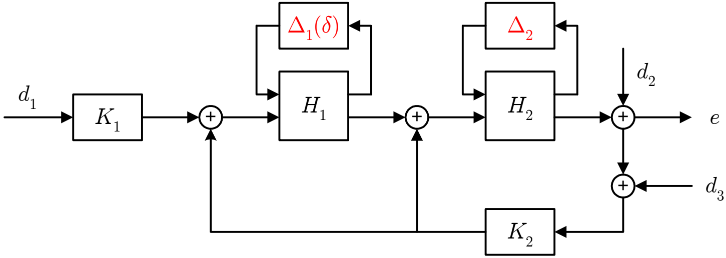
$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} I \\ D \end{pmatrix} z \text{ and } \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \Delta_0 \\ I \end{pmatrix} w.$$

Since $z \neq 0$ we can infer

$$0 > z^T \begin{pmatrix} I \\ D \end{pmatrix}^T P \begin{pmatrix} I \\ D \end{pmatrix} z = \begin{pmatrix} z \\ w \end{pmatrix}^T P \begin{pmatrix} z \\ w \end{pmatrix} = w^T \begin{pmatrix} \Delta_0 \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta_0 \\ I \end{pmatrix} w \geq 0$$

which is a **contradiction**. Same proof for **complex** matrices!

Rough Recap: LTI Robust Stability Analysis



Parametric Uncertainty

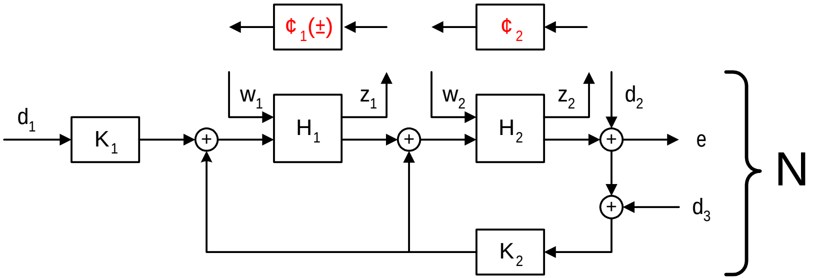
$$\Delta_1(\delta) = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}, \quad |\delta_1| \leq 1, \ |\delta_2| \leq 1.$$

Dynamic Uncertainty - Plant-Model Mismatch

real-rational proper and stable Δ_2 with $\|\Delta_2\|_\infty \leq 1$.

Rough Recap: LTI Robust Stability Analysis

Name input and output signals of uncertainties and disconnect.



Collect signals in w, z, d, e and determine transfer matrix

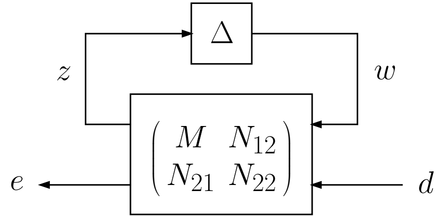
$$\begin{pmatrix} z \\ e \end{pmatrix} = N \begin{pmatrix} w \\ d \end{pmatrix} = \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} w \\ d \end{pmatrix}$$

With $\Delta(s) = \mathrm{diag}(\Delta_1(\delta), \Delta_2(s))$ reconnect uncertainty as $w = \Delta z$.

Rough Recap: LTI Robust Stability Analysis

Interconnection of

$$\begin{pmatrix} z \\ e \end{pmatrix} = \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} w \\ d \end{pmatrix}$$



with the uncertainty

$$w = \Delta z$$

leads to following description of uncertain controlled interconnection:

$$e \longleftarrow \boxed{N_{22}+N_{21}\Delta(I-M\Delta)^{-1}N_{12}} \longleftarrow d$$

Robust stability guaranteed if $I - M\Delta$ has proper and stable inverse.

Main Result on Robust LTI Stability

Suppose $M(s)$ is a real-rational proper and stable transfer matrix. Let $\Delta(s)$ be a proper and stable dynamic uncertainty which is block-diagonal and whose norm is bounded by one.

Is $(I - M(s)\Delta(s))^{-1}$ proper and stable for all these uncertainties?

The information on structure (block-diagonal) and size (bounded by one) can be captured in terms of the frequency response of $\Delta(s)$ as

$$\Delta(i\omega) \in \mathbf{\Delta} \quad \text{for all} \quad \omega \in \mathbb{R} \cup \{\infty\}$$

if we define the set of block-diagonal complex matrices

$$\mathbf{\Delta} = \{\text{diag}(\Delta_1, \ldots, \Delta_p) : \quad \Delta_k \in \mathbb{C}^{m_k \times n_k}, \ \sigma_{\max}(\Delta_k) \leq 1\}.$$

Main Result on Robust LTI Stability

Suppose for each $\omega \in \mathbb{R} \cup \{\infty\}$ there exists a $P(\omega)$ such that

$$\begin{pmatrix} \Delta \\ I \end{pmatrix}^* P(\omega) \begin{pmatrix} \Delta \\ I \end{pmatrix} \succcurlyeq 0 \quad \text{for all} \quad \Delta \in \boldsymbol{\Delta}$$

and at the same time

$$\begin{pmatrix} I \\ M(i\omega) \end{pmatrix}^* P(\omega) \begin{pmatrix} I \\ M(i\omega) \end{pmatrix} \prec 0.$$

Then $I - M\Delta$ has a proper and stable inverse for all proper and stable Δ whose frequency response takes values in $\boldsymbol{\Delta}$.

This is actually true for **arbitrary** subsets of complex matrices $\boldsymbol{\Delta}$ which contain zero and are **convex** (or just star-shaped with respect to zero)!

Proof

Take any proper and stable Δ whose frequency response has its values in $\boldsymbol{\Delta}$. Fix $\omega \in \mathbb{R} \cup \{\infty\}$. Then $\Delta(i\omega) \in \boldsymbol{\Delta}$ implies that

$$\begin{pmatrix} \Delta(i\omega) \\ I \end{pmatrix}^* P(\omega) \begin{pmatrix} \Delta(i\omega) \\ I \end{pmatrix} \succcurlyeq 0.$$

The second inequality

$$\begin{pmatrix} I \\ M(i\omega) \end{pmatrix}^* P(\omega) \begin{pmatrix} I \\ M(i\omega) \end{pmatrix} \prec 0$$

guarantees that $I - M(i\omega)\Delta(i\omega)$ is non-singular.

This implies that $I - M(s)\Delta(s)$ is non-singular for all s in the closed right-half plane or at infinity. (Nontrivial! Exploit that $\boldsymbol{\Delta}$ is convex.) Hence $I - M(s)\Delta(s)$ has a proper and stable inverse.

Example I: Full Block Relaxation

Uncertainties: p norm-bounded full blocks:

$$\boldsymbol{\Delta} = \{\text{diag}(\Delta_1, \ldots, \Delta_p) : \quad \sigma_{\max}(\Delta_k) \leq 1, \quad k = 1, \ldots, p\}.$$

Choose family of multipliers:

$$P(y_1, \ldots, y_p) = \left(\begin{array}{c|c} \text{diag}(y_1 I, \ldots, y_p I) & 0 \\ \hline 0 & -\text{diag}(y_1 I, \ldots, y_p I) \end{array}\right)$$

where the free real numbers $y_1, \ldots, y_p$ are taken such that

$$T(y_1, \ldots, y_p) = \text{diag}(y_1, \ldots, y_p) \preccurlyeq 0.$$

Reason? Simply since then

$$\begin{pmatrix} \Delta \\ I \end{pmatrix}^* P(y) \begin{pmatrix} \Delta \\ I \end{pmatrix} = \text{diag}(y_1(\Delta_1^* \Delta_1 - I), \ldots, y_p(\Delta_p^* \Delta_p) - I) \succcurlyeq 0.$$

Example I: Full Block Relaxation

At each frequency $\omega \in \mathbb{R} \cup \{\infty\}$ check feasibility of LMI problem

$$\begin{pmatrix} I \\ M(i\omega) \end{pmatrix}^* P(y) \begin{pmatrix} I \\ M(i\omega) \end{pmatrix} \prec 0, \quad T(y) \preccurlyeq 0.$$

Feasibility for all frequencies is sufficient for robust stability.

- If $p \leq 3$ the test is also **necessary** for robust stability!
- In practice only checked for finitely many frequencies.
- $\mathrm{diag}(\sqrt{-y_1}I, \ldots, \sqrt{-y_p}I)$ are usual D-scalings in SSV theory.

Example II: Repeated Block Relaxation

Uncertainties: One repeated block in disk (intersected with circle):

$$\boldsymbol{\Delta} = \left\{ \delta I : \ \delta \in \mathbb{C}, \ \begin{pmatrix} \delta \\ 1 \end{pmatrix}^* \begin{pmatrix} q & s \\ s^* & r \end{pmatrix} \begin{pmatrix} \delta \\ 1 \end{pmatrix} = 0, \ \begin{pmatrix} \delta \\ 1 \end{pmatrix}^* \begin{pmatrix} \hat{q} & \hat{s} \\ \hat{s}^* & \hat{r} \end{pmatrix} \begin{pmatrix} \delta \\ 1 \end{pmatrix} \geq 0 \right\}$$

Multipliers: Use Hermitian **matrices** y and $\hat{y}$ to define

$$P(y, \hat{y}) = \begin{pmatrix} qy & sy \\ s^* y & ry \end{pmatrix} + \begin{pmatrix} \hat{q}\hat{y} & \hat{s}\hat{y} \\ \hat{s}^*\hat{y} & \hat{r}\hat{y} \end{pmatrix} \ \text{ with } \ T(\hat{y}) = -\hat{y} \preccurlyeq 0.$$

Reason? Simply since then

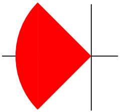
$$\begin{pmatrix} \delta I \\ I \end{pmatrix}^* P(y, \hat{y}) \begin{pmatrix} \delta I \\ I \end{pmatrix} = y \begin{pmatrix} \delta \\ 1 \end{pmatrix}^* \begin{pmatrix} q & s \\ s^* & r \end{pmatrix} \begin{pmatrix} \delta \\ 1 \end{pmatrix} + \hat{y} \begin{pmatrix} \delta \\ 1 \end{pmatrix} \begin{pmatrix} \hat{q} & \hat{s} \\ \hat{s}^* & \hat{r} \end{pmatrix} \begin{pmatrix} \delta \\ 1 \end{pmatrix} \succcurlyeq 0.$$

Numerical procedure as before.

Example II: Repeated Block Relaxation

Repeated block $\Delta = \delta I$ with δ

constrained in amplitude AND phase



$$\begin{pmatrix} \delta I \\ I \end{pmatrix}^* P(y) \begin{pmatrix} \delta I \\ I \end{pmatrix} \succcurlyeq 0 \ \text{ if } \ 1 - \delta^*\delta \geq 0, \quad \pm\mathrm{Im}(\delta) - \mathrm{Re}(\delta) \geq 0$$

$$\Uparrow$$

$$y_j \preccurlyeq 0, \ \begin{pmatrix} \delta I \\ I \end{pmatrix}^* P(y) \begin{pmatrix} \delta I \\ I \end{pmatrix} + y_1[1-\delta^*\delta] + \sum_{k=2}^{3} y_k\left[(-1)^k \mathrm{Im}(\delta) - \mathrm{Re}(\delta)\right] \succcurlyeq 0 \ \ \forall\ \delta$$

$$\Uparrow$$

$$T(y) = \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{pmatrix} \preccurlyeq 0, \ \ P(y) = \begin{bmatrix} y_1 & 0 \\ 0 & -y_1 \end{bmatrix} + \begin{bmatrix} 0 & (i-1)y_2 \\ * & 0 \end{bmatrix} + \begin{bmatrix} 0 & (i+1)y_3 \\ * & 0 \end{bmatrix}$$

Numerical procedure as before.

Example III: Convex Hull Relaxation

Structured real parametric uncertainty $\Delta(\delta)$ with $\delta \in \text{co}\{\delta^1, \ldots, \delta^N\}$:

$$\begin{bmatrix} \Delta(\delta) \\ I \end{bmatrix}^T P(y) \begin{bmatrix} \Delta(\delta) \\ I \end{bmatrix} \succcurlyeq 0 \quad \text{for} \quad \delta \in \text{co}\{\delta^1, \ldots, \delta^N\}$$

$$\Uparrow$$

$$P(y) = y,\ \underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix}^T y \begin{bmatrix} I \\ 0 \end{bmatrix} \preccurlyeq 0, \quad \begin{bmatrix} \Delta(\delta^j) \\ I \end{bmatrix}^T y \begin{bmatrix} \Delta(\delta^j) \\ I \end{bmatrix} \succcurlyeq 0, \quad j = 1, \ldots, N}_{T(y) \preccurlyeq 0}$$

Just put blocks on the diagonal of $T(y)$.

Same numerical procedure. Better but more costly than standard SSV!

Comments

For three types of blocks have seen conceptually standard construction of relaxations which are all described in the same generic fashion.

- For diagonal uncertainties it is easy to combine these relaxations by diagonal augmentation.
- Computational scheme leads to much more flexibility in the choice of uncertainty sets than standard SSV.
- Generic technique extends to much broader problem class.

The latter point is illustrated for time-varying parametric uncertainties, for nonlinear uncertainties and for general robust LMI problems.

Example: Quadratic Stability

Assume that $F(\delta) = A + B\Delta(\delta)\left(I - D\Delta(\delta)\right)^{-1}C$. Recall definition of quadratic stability: Exists $X \succ 0$ with

$$F(\delta)^T X + X F(\delta) \prec 0 \;\text{ for all }\; \delta \in \boldsymbol{\delta} \subset \mathbb{R}^p. \tag{1}$$

LFR is well-posed and (1) holds **if** there exists a multiplier P with

$$\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} \succcurlyeq 0 \;\text{ for all }\; \delta \in \boldsymbol{\delta}$$

that also satisfies

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^T \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^T P \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0.$$

If $\boldsymbol{\delta}$ is compact then have **iff**.

Proof of if Straightforward

Right-lower block of second inequality is $\begin{pmatrix} I \\ D \end{pmatrix}^T P \begin{pmatrix} I \\ D \end{pmatrix} \prec 0$. Infer well-posedness from slide 8. Left- and right-multiply second LMI with

$$\begin{pmatrix} I \\ \Delta(\delta)(I - D\Delta(\delta))^{-1}C \end{pmatrix}^T \text{ and } \begin{pmatrix} I \\ \Delta(\delta)(I - D\Delta(\delta))^{-1}C \end{pmatrix}.$$

Direct computation (see proof of KYP!) leads to

$$\begin{pmatrix} I \\ F(\delta) \end{pmatrix}^T \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I \\ F(\delta) \end{pmatrix} +$$

$$+[(I - D\Delta(\delta))^{-1}C]^T \underbrace{\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^T P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}}_{\succcurlyeq 0} (I - D\Delta(\delta))^{-1}C \prec 0.$$

Comments

- Numerical implementation as before. For example if $\boldsymbol{\delta}$ is a finitely generated convex set just use convex hull relaxation.

- It was totally irrelevant that we considered the particular specification
$$\begin{pmatrix} I \\ F(\delta) \end{pmatrix}^T \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I \\ F(\delta) \end{pmatrix} = F(\delta)^T X + X F(\delta) \prec 0.$$
For example we can address discrete-time robust stability or robust pole-locations in LMI region.

- It is straightforward to extend to any of the multiple performance specification considered so far. The robust LMI framework discussed below covers it all.

Example: Nonlinearity in the Loop

System with linear fractional representation

$$\dot{x} = Ax + Bw, \quad z = Cx + Dw, \quad w = \Delta(z).$$

is **exponentially stable** if ...

... exists $X \succ 0$ and **Hermitian multiplier** P with

$$\begin{pmatrix} A^T X + XA & XB \\ B^T X & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^T P \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0$$

and such that

$$\begin{pmatrix} \Delta(z) \\ z \end{pmatrix}^T P \begin{pmatrix} \Delta(z) \\ z \end{pmatrix} \succcurlyeq 0 \ \text{ for all } \ z.$$

Proof

Exists $\epsilon > 0$ with

$$\begin{pmatrix} A^T X + XA & XB \\ B^T X & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^T P \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} + \begin{pmatrix} \epsilon X & 0 \\ 0 & 0 \end{pmatrix} \prec 0.$$

Choose arbitrary system trajectory. Then left-multiply this inequality with col$(x(t), w(t))^T$ and right-multiply with col$(x(t), w(t))$. Implies

$$\frac{d}{dt} x(t)^T X x(t) + \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T P \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} + \epsilon x(t)^T X x(t) \leq 0.$$

Now note that $w(t) = \Delta(z(t))$ and the second inequality imply

$$\begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T P \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \Delta(z(t)) \\ z(t) \end{pmatrix}^T P \begin{pmatrix} \Delta(z(t)) \\ z(t) \end{pmatrix} \succcurlyeq 0.$$

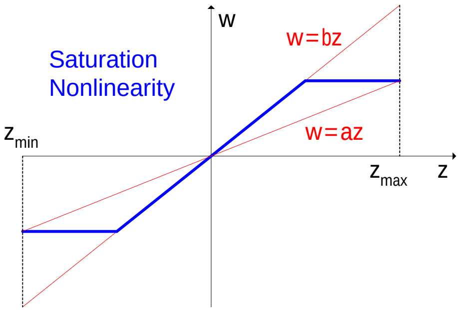
Hence $\frac{d}{dt} x(t)^T X x(t) + \epsilon x(t)^T X x(t) \leq 0$. Done!

Example: Multipliers for Saturation Nonlinearity

As long as $z_{\text{min}} \leq z \leq z_{\text{max}}$ have

$$\begin{pmatrix} \Delta(z) \\ z \end{pmatrix}^T \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix}}_{P_1} \begin{pmatrix} \Delta(z) \\ z \end{pmatrix} \succcurlyeq 0,$$

$$\begin{pmatrix} \Delta(z) \\ z \end{pmatrix}^T \underbrace{\begin{pmatrix} 0 & -1 \\ -1 & b \end{pmatrix}}_{P_2} \begin{pmatrix} \Delta(z) \\ z \end{pmatrix} \succcurlyeq 0.$$



Choose $P(y) = y_1 P_1 + y_2 P_2$ and $T(y) = -\text{diag}(y_1, y_2)$ to infer that

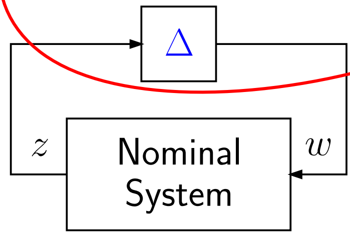
$$T(y) \preccurlyeq 0 \quad \text{implies} \quad \begin{pmatrix} \Delta(z) \\ z \end{pmatrix}^T P(y) \begin{pmatrix} \Delta(z) \\ z \end{pmatrix} \succcurlyeq 0.$$

Computations as before.

Interpretation: Separation by Dissipation

Uncertainty **anti-dissipative**

$$\begin{pmatrix} \Delta(z) \\ z \end{pmatrix}^T P \begin{pmatrix} \Delta(z) \\ z \end{pmatrix} \succeq 0$$



Separation with same Multiplier P

Nominal system **dissipative**

$$\begin{pmatrix} A^T X + XA & XB \\ B^T X & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^T P \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0$$

Comments

Linear Fractional Representations and Multipliers:

- We derived various non-trivial robust stability tests which lead to numerical tests by relaxation.
- Allows to trade-off conservatism and computation time depending on the used class of multipliers.
- Illustrated flexibility for SSV and robust quadratic stability.
- Extension to larger class of uncertainties: Separation by dissipation.

  ... Has been illustrated for static nonlinearities.

  ... Is fundamental concept behind **I**ntegral **Q**uadratic **C**onstraints.

The General Story: Robust LMI Problems

Infimize $c^T x$ over $x \in \mathbb{R}^n$ satisfying robust LMI-constraint

$$F_0(\delta) + x_1 F_1(\delta) + \cdots + x_n F_n(\delta) \prec 0 \;\; \text{for all} \;\; \delta \in \boldsymbol{\delta} \subset \mathbb{C}^p.$$

- Have seen concrete engineering examples ... there are many more!
- Is convex problem. Semi-infinite constraints render it NP-hard.
- Computationally tractable approximations: **Relaxations**

Polytope $\boldsymbol{\delta}$ and affine dependence on parameter δ ...

... easy for scalar constraints

... tough for matrix constraints !

Alternative Description of Uncertain LMI

Construct linear fractional representation

$$\frac{1}{2}\begin{bmatrix} F_0(\delta) \\ \vdots \\ F_n(\delta) \end{bmatrix} = \begin{bmatrix} C_0 \\ \vdots \\ C_n \end{bmatrix} \Delta(\delta)(I - A\Delta(\delta))^{-1}B + \begin{bmatrix} D_0 \\ \vdots \\ D_n \end{bmatrix} =$$

and define affine Hermitian-valued

$$W(x) = \begin{bmatrix} 0 & C_0^* + \sum_{j=1}^n x_j C_j^* \\ C_0 + \sum_{j=1}^n x_j C_j & (D_0 + D_0^*) + \sum_{j=1}^n x_j (D_j + D_j^*) \end{bmatrix}$$

Alternative description of LMI:

$$F_0(\delta) + \sum_{j=1}^n x_j F_j(\delta) = \begin{bmatrix} * \\ * \end{bmatrix}^* W(x) \begin{bmatrix} \Delta(\delta)(I - A\Delta(\delta))^{-1}B \\ I \end{bmatrix} \prec 0$$

Robust LMI Problem

Infimize $c^T x$ such that for all $\delta \in \boldsymbol{\delta}$:

$$\begin{bmatrix} \Delta(\delta)(I-A\Delta(\delta))^{-1}B \\ I \end{bmatrix}^* W(x) \begin{bmatrix} \Delta(\delta)(I-A\Delta(\delta))^{-1}B \\ I \end{bmatrix} \prec 0.$$

ROB

- Huge range of applications in robust optimization and control:
  Uncertain LP's, robust performance, Lyapunov design, ...
- Captures **affine** dependence on parameters: $A = 0$.
- Only few problem classes computationally tractable.

Goals: Compute **upper bounds** by efficiently solvable relaxation.

Investigate **quality** for specific problem instance.

Idea for Relaxation

The value of **ROB** is **not larger** than the infimum of $c^T x$ over all x and P which satisfy

$$\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} \succcurlyeq 0 \text{ for all } \delta \in \boldsymbol{\delta} \text{ and}$$

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^* P \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + W(x) \prec 0.$$

- Is fully elementary.
- Nontrivial: If $\boldsymbol{\delta}$ is compact have **equality** (full-block S-procedure).
- Test well-posedness separately. Is guaranteed if exist feasible x and P for which right-lower block of $W(x)$ is positive semi-definite.

Proof

Let x and P be feasible. Left- and right-multiply second inequality with

$$\begin{pmatrix} \Delta(\delta)\,(I - A\Delta(\delta))^{-1}\,B \\ I \end{pmatrix}^* \quad \text{and} \quad \begin{pmatrix} \Delta(\delta)\,(I - A\Delta(\delta))^{-1}\,B \\ I \end{pmatrix}$$

respectively. This implies

$$\begin{pmatrix} \Delta(\delta)\,(I - A\Delta(\delta))^{-1}\,B \\ I \end{pmatrix}^* W(x) \begin{pmatrix} \Delta(\delta)\,(I - A\Delta(\delta))^{-1}\,B \\ I \end{pmatrix} +$$

$$+[(I - A\Delta(\delta))^{-1}\,B]^* \underbrace{\begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}}_{\succcurlyeq 0} (I - A\Delta(\delta))^{-1}\,B \prec 0.$$

Therefore x is feasible for **ROB**.

How to Construct Relaxations?

Choose linear mappings $T(y)$ and $P(y)$ such that

$$T(y) \preccurlyeq 0 \implies \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix}^* P(y) \begin{pmatrix} \Delta(\delta) \\ I \end{pmatrix} \succcurlyeq 0 \text{ for all } \delta \in \boldsymbol{\delta}.$$

Infimum of $c^T x$ such that there exists y with

$$T(y) \prec 0, \quad \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^* P(y) \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + W(x) \prec 0$$

is **upper bound** on value of **ROB**.

REL

Nontrivial: Have **equality** for following relaxations:

- Full block relaxation for two blocks (slide 16, $p \leq 2$)
- Repeated block relaxation for one block (slide 18)
- Convex hull relaxation for one parameter (slide 20, $N = 2$)