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# Unified Design for $H_2$, $H_\infty$, and Mixed Control of Spacecraft

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**A unified approach is presented for nonlinear $H_2$, $H_\infty$, and mixed $H_2/H_\infty$ attitude control of spacecraft systems, with external disturbances and parameter perturbations considered. The design objective is to specify a controller such that the quadratic optimal control performance $H_\infty$ or mixed $H_2/H_\infty$ performances can be achieved. It is shown that these problems are special cases of the so-called two-player Nash differential game problem. An explicit solution to the coupled Riccati-like equations of the nonlinear Nash game control problem can be obtained when nonlinear minimax theory and linear-quadratic optimal control techniques are combined. Moreover, because of the skew symmetric property of the spacecraft system and adequate choice of state variable transformation, these problems can be reduced to solving two algebraic Riccati-like equations. Furthermore, a closed-form solution to these two algebraic equations can be obtained with a very simple form for the preceding control designs. Finally, three experimental simulation results based on the ROCSAT-1 spacecraft system are presented to demonstrate the effectiveness of the proposed design methods.**

## Nomenclature

| | |
|---|---|
| $(b_1, b_2, b_3)$ | = the body-fixed reference frame |
| $e$ | = tracking error vector |
| $(e_1, e_2, e_3)$ | = the inertial-fixed reference coordinates |
| $h$ | = total spacecraft angular momentum in body axes $[h_1\ h_2\ h_3]^T$; $h = J\omega$ |
| $J$ | = inertia matrix |
| $(o_1, o_2, o_3)$ | = the orbital coordinate reference |

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$P$ = weighting matrix of tracking error with respect to the initial time
$Q$ = weighting matrix of tracking error
$Q_f$ = weighting matrix of tracking error with respect to the final time
$Q_1$ = weighting matrix of tracking error for the $H_\infty$ performance
$Q_{1f}$ = weighting matrix of tracking error with respect to the final time for the $H_\infty$ performance
$Q_2$ = weighting matrix of tracking error for the $H_2$ performance
$Q_{2f}$ = weighting matrix of tracking error with respect to the final time for the $H_2$ performance
$r$ = filtered link of tracking error
$S_{ai}$ = saturation value of the $i$th actuator output torque
$T$ = state–space transformation matrix
$u$ = control torque vector
$W$ = weighting matrix of control torque
$W_1$ = weighting matrix of control torque for the $H_\infty$ performance
$W_2$ = weighting matrix of control torque for the $H_2$ performance
$w$ = compound of the system uncertainties
$\gamma$ = desired disturbance attenuation level
$\Delta J$ = perturbed part of inertia matrix
$\theta$ = attitude Euler angle $[\theta_1 \;\; \theta_2 \;\; \theta_3]^T$
$\theta_r$ = reference trajectory of attitude Euler angle $[\theta_{r1} \;\; \theta_{r2} \;\; \theta_{r3}]^T$
$[\nu\times]$ = cross-product matrix associated with the vector $\nu = [\nu_1 \;\; \nu_2 \;\; \nu_3]^T$;

$$[\nu\times] = \begin{bmatrix} 0 & -\nu_3 & \nu_2 \\ \nu_3 & 0 & -\nu_1 \\ -\nu_2 & \nu_1 & 0 \end{bmatrix}$$

$\tau_a$ = torque vector due to actuator $[\tau_{a1} \;\; \tau_{a2} \;\; \tau_{a3}]^T$
$\tau_{\text{aero}}$ = aerodynamic disturbance torque vector
$\tau_d$ = external disturbance torque vector
$\tau_g$ = gravity gradient torque vector
$\tau_{\text{solar}}$ = solar radiation pressure disturbance torque vector
$\Omega$ = the bounded region: $\{x \in R^3 \mid -\pi/2 < \theta_2 < \pi/2\}$
$\omega$ = general angular velocity vector in body axes $[\omega_1 \;\; \omega_2 \;\; \omega_3]^T$
$\omega_0$ = orbital rate

## Introduction

THE attitude control of spacecraft has received extensive attention in recent decades, and several methods of spacecraft attitude control have been developed to treat this problem. The feasibility of applying the feedback linearization technique to spacecraft attitude control and the momentum management problem has been discussed.[1,2] Based on linearization by coordinate transformation and nonlinear feedback, a controller for attitude control has also been derived.[3] A model reference adaptive control has been presented for large-angle rotational maneuvers of spacecraft systems.[4] A sliding manifold approach[5,6] as well as optimal control theory[7–9] have also been used for spacecraft attitude control. More relevant to this study, the approach of $H_\infty$ optimal control has been applied to the space station attitude and momentum control problem while the linearized equations of motion have been considered.[10]

Recently, along this line, Chen et al.[11] developed a nonlinear $H_\infty$ control design to treat the spacecraft attitude tracking problem under parameter perturbation and external noise. A minimax theory was used in this study to achieve the $H_\infty$ tracking control design so that the effect of the equivalent external disturbance on tracking error could be attenuated below a prescribed value $\gamma$. It has been shown that[11] the sufficient condition for solvability is that the weighting matrix $W$ of the control variable must satisfy the constraint $W < 2\gamma^2 I$. The solution is obviously not unique, and some degrees of freedom are available for the construction of the control law. With this in mind, the question arises as to how to handle the remaining degrees of freedom. In this work, we use these remaining degrees of freedom to achieve the $H_2$ optimal tracking control of spacecraft systems.

Over the past 10 years, mixed $H_2/H_\infty$ optimal control has been studied for linear systems.[12–18] The main purpose of this type of control is to design an $H_2$ optimal control for the worst-case external disturbance whose effects on system output must be attenuated below a desired value (i.e., to design an $H_2$ optimal control under $H_\infty$ disturbance attenuation constraint). A sufficient condition for the mixed $H_2/H_\infty$ control problem of nonlinear systems by use of the Nash games theory was obtained by Chen et al.[19] However, the general solution for the nonlinear mixed $H_2/H_\infty$ control problem has not yet been obtained.

Because nonlinear $H_2$, $H_\infty$, and mixed $H_2/H_\infty$ controls are important for the attitude tracking control of spacecraft systems under parameter variations and external disturbances, a unified approach for these controls is proposed in this study to compare the performances of these control methods and then to demonstrate the excellent performance of the mixed $H_2/H_\infty$ control method. Each of the $H_2$, $H_\infty$, and mixed $H_2/H_\infty$ control designs are cases of the so-called two-player Nash differential game problems.[20] Therefore a pair of coupled time-varying Riccati-like equations must first be solved to design any of these controllers. Next, two time-varying coupled Riccati-like equations can be transformed into two corresponding algebraic coupled Riccati-like equations by means of an adequate choice of a state transformation and by use of the skew-symmetric property of spacecraft systems. Then, through Cholesky factorization,[21] these two coupled algebraic Riccati equations can be easily solved. Finally, with the special conditions for cases of the Nash game problems, the simplified $H_2$, $H_\infty$, and mixed $H_2/H_\infty$ attitude tracking control laws for the perturbed spacecraft systems are ultimately obtained.

## Mathematical Model and Problem Formulation

### Spacecraft Model

Consider a spacecraft moving in a circular orbit. The coordinate systems used in the attitude control are shown in Fig. 1. The inertial-fixed reference coordinates $(e_1, e_2, e_3)$ with their origin at the center of the Earth are used to determine the orbital position of the spacecraft. The orbital coordinate reference $(o_1, o_2, o_3)$ is rotating about the $o_2$ axis with respect to the inertial-fixed coordinate system $(e_1, e_2, e_3)$ at the orbital rate $\omega_0$. The axes of this reference frame are chosen such that the roll axis $o_1$ is in the flight direction, the pitch axis $o_2$ is perpendicular to the orbital plane, and the yaw axis $o_3$ points toward the Earth. The last reference system used is the body-fixed reference frame $(b_1, b_2, b_3)$. The orientation of the spacecraft with respect to the reference frame $(o_1, o_2, o_3)$ is obtained, in this work, by a yaw–pitch–roll $(\theta_3–\theta_2–\theta_1)$ sequence of rotations. The

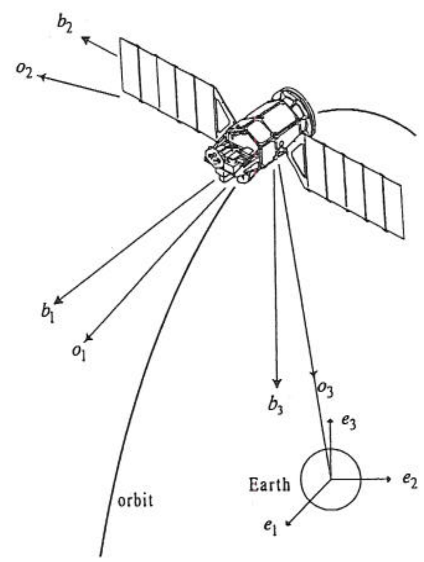


**Fig. 1 Appearance of the ROCSAT-1 spacecraft and coordinate systems used in the spacecraft attitude control.**

origins of both orbit coordinates and body-fixed coordinates are at the center of mass of the spacecraft.

The nonlinear equations of motion, in terms of components along the body-fixed control axes, are given by the attitude kinematic equation [Eq. (1)] and the spacecraft kinematic equation [Eq. (2)] and can be written as[22,23]

$$\omega = R(\theta)\dot{\theta} - \omega_c(\theta) \tag{1}$$

$$J\dot{\omega} = [h\times]\omega + \tau_g + \tau_a + \tau_d \tag{2}$$

where

$$R(\theta) = \begin{bmatrix} 1 & 0 & -\sin\theta_2 \\ 0 & \cos\theta_1 & \sin\theta_1\cos\theta_2 \\ 0 & -\sin\theta_1 & \cos\theta_1\cos\theta_2 \end{bmatrix}$$

$$\omega_c(\theta) = \omega_0 \begin{bmatrix} \cos\theta_2\sin\theta_3 \\ \cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3 \\ -\sin\theta_1\cos\theta_3 + \cos\theta_1\sin\theta_2\sin\theta_3 \end{bmatrix}$$

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

$$\tau_g = 3\omega_0^2[c\times]Jc \quad \text{with} \quad c = \begin{bmatrix} -\sin\theta_2 \\ \sin\theta_1\cos\theta_2 \\ \cos\theta_1\cos\theta_2 \end{bmatrix}$$

and $\tau_d = \tau_{\text{aero}} + \tau_{\text{solar}}$.

*Remark 1:* If the control torque is limited by the saturation of actuator, the spacecraft dynamics equation (2) is of the following form:

$$J\dot{\omega} = [h\times]\omega + \tau_g + \tau_{a,\text{sat}} + \tau_{\text{aero}} + \tau_{\text{solar}} \tag{3}$$

where $\tau_{a,\text{sat}} = [\text{sat}(\tau_{a1})\ \text{sat}(\tau_{a2})\ \text{sat}(\tau_{a3})]^T$ and

$$\text{sat}(\tau_{\text{ai}}) = \begin{cases} S_{\text{ai}}, \tau_{\text{ai}} \geq S_{\text{ai}} \\ \tau_{\text{ai}}, |\tau_{\text{ai}}| < S_{\text{ai}}, & \text{for} \quad i = 1, 2, 3 \\ -S_{\text{ai}}, \tau_{\text{ai}} \leq -S_{\text{ai}} \end{cases}$$

To apply the proposed design method, dynamic equation (3) can be put into the form of Eq. (2) with $\tau_d = \tau_{\text{aero}} + \tau_{\text{solar}} + \tau_{a,\text{sat}} - \tau_a$. In this situation, the deviation $\tau_{a,\text{sat}} - \tau_a$ can be considered as a part of the external disturbance. □

*Remark 2:* This description of Eq. (1) is defined for all $(\theta_1, \theta_2, \theta_3)$ except $\theta_2 = \pm(2n+1)\pi/2$ for any integer $n$. This singularity [i.e., the determinant of matrix $R(\theta)$ becomes zero at $\theta_2 = \pm(2n+1)\pi/2$] arises owing to the choice of the set of rotations that define the orientation of the spacecraft relative to the orbital frame. However, when the orientation corresponding to the singularity $\theta_2 = \pm\pi/2$ lies in the control region of attitude angles, another set of rotations can be defined to eliminate this singularity.[24] In this paper, we are interested in the trajectories in the bounded region $\Omega$. □

Differentiating Eq. (1) gives

$$\dot{\omega} = R(\theta)\ddot{\theta} + \left[\frac{\mathrm{d}}{\mathrm{d}t}R(\theta)\right]\dot{\theta} - \left[\frac{\mathrm{d}}{\mathrm{d}t}\omega_c(\theta)\right] \tag{4}$$

Substituting $\omega$ and $\dot{\omega}$ from Eqs. (1) and (4) into Eq. (2) and premultiplying Eq. (4) by matrix $R^T(\theta)$, we obtain

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta,\dot{\theta}) = f + f_d \tag{5}$$

where

$$M(\theta) = R^T(\theta)JR(\theta)$$

$$C(\theta,\dot{\theta}) = R^T(\theta)J\left[\frac{\mathrm{d}}{\mathrm{d}t}R(\theta)\right] - R^T(\theta)[h\times]R(\theta)$$

$$G(\theta,\dot{\theta}) = -R^T(\theta)J\left[\frac{\mathrm{d}}{\mathrm{d}t}\omega_c(\theta)\right] + R^T(\theta)[h\times]\omega_c(\theta) - 3\omega_0^2R^T(\theta)[c\times]Jc$$

$$f = R^T(\theta)\tau_a, \qquad f_d = R^T(\theta)\tau_d$$

Note that Eq. (5) is linear in the inertia matrix $J$.

In practical spacecraft systems, however, perturbations in system parameters that are due to the flexible structure, unmodeled dynamics, and the change of the orientation of solar arrays on the spacecraft are inevitable. Thus, the inertia matrix can be rewritten as[10]

$$J = J_0 + \Delta J \tag{6}$$

Therefore the parameter matrices in spacecraft model equation (5) can be divided into nominal parts and perturbed parts, i.e.,

$$[h\times] = [h_0\times] + [\Delta h\times] \tag{7a}$$

$$M(\theta) = M_0(\theta) + \Delta M(\theta) \tag{7b}$$

$$C(\theta,\dot{\theta}) = C_0(\theta,\dot{\theta}) + \Delta C(\theta,\dot{\theta}) \tag{7c}$$

$$G(\theta,\dot{\theta}) = G_0(\theta,\dot{\theta}) + \Delta G(\theta,\dot{\theta}) \tag{7d}$$

where

$$h_0 = J_0\omega, \qquad M_0 = R^TJ_0R$$

$$C_0 = R^TJ_0\left(\frac{\mathrm{d}}{\mathrm{d}t}R\right) - R^T[h_0\times]R$$

$$G_0 = -R^TJ_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\omega_c\right) + R^T[h_0\times]\omega_c - 3\omega_0^2R^T[c\times]J_0c$$

and

$$\Delta h = (\Delta J)\omega, \qquad \Delta M = R^T(\Delta J)R$$

$$\Delta C = R^T(\Delta J)\left(\frac{\mathrm{d}}{\mathrm{d}t}R\right) - R^T[\Delta h\times]R$$

$$\Delta G = -R^T(\Delta J)\left(\frac{\mathrm{d}}{\mathrm{d}t}\omega_c\right) + R^T[\Delta h\times]\omega_c - 3\omega_0^2R^T[c\times](\Delta J)c$$

*Assumption:* The nominal inertia matrix $J_0$ is an exactly known constant and is symmetric positive definite. Moreover, the perturbed inertia matrix $\Delta J$ and external disturbance $\tau_d$ are both bounded but unknown. □

With Eq. (7), differential equation (5) can be rewritten as

$$M_0(\theta)\ddot{\theta} + C_0(\theta,\dot{\theta})\dot{\theta} + G_0(\theta,\dot{\theta}) = f + w \tag{8}$$

where

$$w = f_d - [\Delta M(\theta)\ddot{\theta} + \Delta C(\theta,\dot{\theta})\dot{\theta} + \Delta G(\theta,\dot{\theta})]$$

is the compound of the system uncertainties caused by the effects of the parameter variation of $J$ and the equivalent external disturbance $f_d$. Under the above assumption, spacecraft system equation (8) has the following properties:

*Property 1:* The matrix $M_0(\theta)$ is symmetric positive definite.

*Property 2:* The matrix

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}M_0(\theta) - C_0(\theta,\dot{\theta})$$

is skew symmetric,[25] that is,

$$x^T\left[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}M_0(\theta) - C_0(\theta,\dot{\theta})\right]x = 0 \qquad \forall x \in R^3 \tag{9}$$

□

**Problem Formulation**

In this paper, we develop an attitude tracking control design. The desired attitude reference trajectory is assumed to be available as bounded functions of time in terms of angle vector $\theta_r \in C^2$ (the class of twice continuously differentiable functions), its corresponding velocity vector $\dot{\theta}_r$, and acceleration vector $\ddot{\theta}_r$.

Define the tracking error as follows:

$$e =: \begin{bmatrix} \dot{\tilde{\theta}} \\ \tilde{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} - \dot{\theta}_r \\ \theta - \theta_r \end{bmatrix} \tag{10}$$

Then, by Eqs. (8) and (10), the tracking error dynamic equation is obtained as

$$\dot{e} = \begin{bmatrix} -M_0^{-1}(\theta)C_0(\theta,\dot{\theta}) & 0_{3\times 3} \\ I_{3\times 3} & 0_{3\times 3} \end{bmatrix} e + \begin{bmatrix} -\ddot{\theta}_r - M_0^{-1}(\theta)[C_0(\theta,\dot{\theta})\dot{\theta}_r + G_0(\theta,\dot{\theta})] \\ 0_{3\times 3} \end{bmatrix} + \begin{bmatrix} M_0^{-1}(\theta)(f+w) \\ 0_{3\times 3} \end{bmatrix} \tag{11}$$

Since error dynamic equation (11) is complicated, it is not directly applied in this study to the tracking control design. To simplify the control formulation and the stability analysis, we define the filtered link of tracking error $r(t)$ and the state–space transformation matrix $T$ as[26]

$$r(t) = \lambda\dot{\tilde{\theta}} + \Lambda\tilde{\theta} \tag{12}$$

$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0_{3\times 3} & I_{3\times 3} \end{bmatrix} = \begin{bmatrix} \lambda I_{3\times 3} & \Lambda \\ 0_{3\times 3} & I_{3\times 3} \end{bmatrix} \tag{13}$$

for some positive scale $\lambda$ and positive definite matrix $\Lambda \in R^{3\times 3}$, which are constants and should be adequately determined later. Then error dynamic equation (11) can be modified as a compact form:

$$\dot{e} = T^{-1}\begin{bmatrix} \dot{r}(t) \\ \dot{\tilde{\theta}}(t) \end{bmatrix} = A_T(e,t)e + B_T(e,t)\{\lambda[-F(e,t) + f]\} + B_T(e,t)d \tag{14}$$

where

$$A_T(e,t) = T^{-1}\begin{bmatrix} -M_0^{-1}(\theta)C_0(\theta,\dot{\theta}) & 0_{3\times 3} \\ (1/\lambda)I_{3\times 3} & -(1/\lambda)\Lambda \end{bmatrix} T$$

$$B_T(e,t) = T^{-1}BM_0^{-1}(\theta)$$

$$F(e,t) = M_0(\theta)[\ddot{\theta}_r - (1/\lambda)\Lambda\dot{\tilde{\theta}}] + C_0(\theta,\dot{\theta})[\dot{\theta}_r - (1/\lambda)\Lambda\tilde{\theta}] + G_0(\theta,\dot{\theta}), \qquad d = \lambda w$$

with

$$B = \begin{bmatrix} I_{3\times 3} \\ 0_{3\times 3} \end{bmatrix} \tag{15}$$

If the following applied torque is selected,[26]

$$f = F(e,t) + (1/\lambda)u \tag{16}$$

then the tracking error dynamic equation, driven by the control torque $u$, becomes

$$\dot{e} = A_T(e,t)e + B_T(e,t)u + B_T(e,t)d \tag{17}$$

Formally, three problems that are considered in this paper for nonlinear spacecraft system (17) can be stated as follows.

*Problem 1 (the nonlinear quadratic optimal attitude tracking control problem):* Consider the nonlinear spacecraft system of the form of Eq. (17) without considering disturbance $d$ (i.e., let $d = 0$ in our design procedure). Given some weighting matrices $Q$ and $W$, the nonlinear quadratic optimal attitude tracking control problem is said to be solved if there exists an optimal control law $u^*$ that satisfies[26]

$$J(u^*, 0) = \min_u J(u, d = 0) = \min_u \left\{ e^T(t_f)Q_f e(t_f) + \int_0^{t_f} [e^T(t)Qe(t) + u^T(t)Wu(t)]\,dt \right\} \tag{18}$$

for all $t_f \in [0,\infty)$ and for some positive definite matrix $Q_f = Q_f^T > 0$.

*Problem 2 (the nonlinear $H_\infty$ attitude tracking control problem):* Consider the nonlinear spacecraft system (17). Given a desired disturbance attenuation level $\gamma > 0$ and weighting matrices $Q$ and $W$, the nonlinear $H_\infty$ attitude tracking control problem is said to be solved if there exists an $H_\infty$ control law $u$ that satisfies[27,28]

$$e^T(t_f)Q_f e(t_f) + \int_0^{t_f} [e^T(t)Qe(t) + u^T(t)Wu(t)]\,dt \le e^T(0)Pe(0) + \gamma^2 \int_0^{t_f} \Gamma d^T(t)d(t)]\,dt \tag{19}$$

for all $d(t) \in L_2[0, t_f]$, $t_f \in [0,\infty)$, and for some positive definite matrices $Q_f = Q_f^T > 0$ and $P = P^T > 0$, i.e., the attenuation level of combined disturbances is guaranteed to be below a specified value $\gamma$.

*Problem 3 (the nonlinear mixed $H_2/H_\infty$ attitude tracking control problem):* Consider the nonlinear spacecraft system (17). Given a desired disturbance attenuation level $\gamma > 0$ and weighting matrices $Q_1$, $Q_2$, $W_1$, and $W_2$, the nonlinear mixed $H_2/H_\infty$ attitude tracking control problem is said to be solved if there exists a control law $u$ such that the following $H_2$ (quadratic) optimal tracking performance,[12–17]

$$\min_{u(t)} \left\{ e^T(t_f)Q_{2f}e(t_f) + \int_0^{t_f} \left[e^T(t)Q_2e(t) + u^T(t)W_2u(t)\right] dt \right\} \tag{20}$$

can be achieved for all $t_f \in [0,\infty)$ and for some positive definite matrix $Q_{2f} = Q_{2f}^T > 0$ under the following $H_\infty$ optimal disturbance rejection constraint:

$$e^T(t_f)Q_{1f}e(t_f) + \int_0^{t_f} \left[e^T(t)Q_1e(t) + u^T(t)W_1u(t)\right] dt \le e^T(0)Pe(0) + \gamma^2 \int_0^{t_f} d^T(t)d(t)\,dt \tag{21}$$

for some positive definite matrices $Q_{1f} = Q_{1f}^T > 0$ and $P = P^T > 0$.

## Unified Design

In this section, we solve the spacecraft attitude tracking control problems that are formulated in the preceding section. For this purpose, we first consider the following nonzero-sum, two-player Nash differential game problem,[28] which includes the nonlinear quadratic optimal control, $H_\infty$ control, and mixed $H_2/H_\infty$ control problems in it.[19]

*The nonzero-sum, two-player Nash differential game problem:* Consider the nonlinear spacecraft attitude tracking control system of the form in Eq. (17). Given some prescribed $\gamma_1 > 0$ and $\gamma_2 > 0$ and weighting matrices $Q_1$, $Q_2$, $W_1$, and $W_2$, then the nonzero-sum, two-player Nash differential game problem is said to be solved if there are Nash equilibrium strategies $u^*(e,t)$ such that the following inequalities are satisfied[20]:

$$J_1[u^*(e,t), d^*(e,t)] \ge J_1[u^*(e,t), d] \qquad \forall d \in L_2[0,t_f] \tag{22}$$

$$J_2[u^*(e,t), d^*(e,t)] \le J_2[u, d^*(e,t)] \qquad \forall u \in L_2[0,t_f] \tag{23}$$

where

$$J_1(\boldsymbol{u},\boldsymbol{d}) = \boldsymbol{e}^T(t_f)Q_{1f}\boldsymbol{e}(t_f) + \int_0^{t_f}\left[\boldsymbol{e}^T(t)Q_1\boldsymbol{e}(t) + \boldsymbol{u}^T(t)W_1\boldsymbol{u}(t) - \gamma_1^2\boldsymbol{d}^T(t)\boldsymbol{d}(t)\right]\mathrm{d}t \quad (24)$$

$$J_2(\boldsymbol{u},\boldsymbol{d}) = \boldsymbol{e}^T(t_f)Q_{2f}\boldsymbol{e}(t_f) + \int_0^{t_f}\left[\boldsymbol{e}^T(t)Q_2\boldsymbol{e}(t) + \boldsymbol{u}^T(t)W_2\boldsymbol{u}(t) - \gamma_2^2\boldsymbol{d}^T(t)\boldsymbol{d}(t)\right]\mathrm{d}t \quad (25)$$

for all $t_f \in [0,\infty)$ and for some positive definite matrices $Q_{1f} = Q_{1f}^T > 0$ and $Q_{2f} = Q_{2f}^T > 0$.

For the convenience of design, we take $W_1 = W_2 = W$ throughout this study. By using the standard technique of completing the squares, we have the following useful lemma.

*Lemma 1:* For the nonlinear spacecraft attitude tracking control system in Eq. (17), if the optimal control law and the worst-case disturbance are chosen, respectively, as

$$\boldsymbol{u}^*(\boldsymbol{e},t) = -W^{-1}B_T^T(\boldsymbol{e},t)P_2(\boldsymbol{e},t)\boldsymbol{e}(t) \quad (26a)$$

$$\boldsymbol{d}^*(\boldsymbol{e},t) = \left(1/\gamma_1^2\right)B_T^T(\boldsymbol{e},t)P_1(\boldsymbol{e},t)\boldsymbol{e}(t) \quad (26b)$$

where $P_1(\boldsymbol{e},t)$ and $P_2(\boldsymbol{e},t)$ are the solutions of the following coupled time-varying Riccati-like equations,

$$\dot{P}_1(\boldsymbol{e},t) + P_1(\boldsymbol{e},t)A_T(\boldsymbol{e},t) + A_T(\boldsymbol{e},t)^T P_1(\boldsymbol{e},t) + Q_1 - [P_1(\boldsymbol{e},t)B_T(\boldsymbol{e},t) \quad P_2(\boldsymbol{e},t)B_T(\boldsymbol{e},t)] \times \begin{bmatrix}\left(-1/\gamma_1^2\right)I_{3\times3} & W^{-1}\\ W^{-1} & -W^{-1}\end{bmatrix}\begin{bmatrix}B_T(\boldsymbol{e},t)^T P_1(\boldsymbol{e},t)\\ B_T(\boldsymbol{e},t)^T P_2(\boldsymbol{e},t)\end{bmatrix} = 0 \quad (27)$$

$$\dot{P}_2(\boldsymbol{e},t) + P_2(\boldsymbol{e},t)A_T(\boldsymbol{e},t) + A_T(\boldsymbol{e},t)^T P_2(\boldsymbol{e},t) + Q_2 - [P_1(\boldsymbol{e},t)B_T(\boldsymbol{e},t) \quad P_2(\boldsymbol{e},t)B_T(\boldsymbol{e},t)] \times \begin{bmatrix}\left(\gamma_2^2/\gamma_1^4\right)I_{3\times3} & \left(-1/\gamma_1^2\right)I_{3\times3}\\ \left(-1/\gamma_1^2\right)I_{3\times3} & W^{-1}\end{bmatrix}\begin{bmatrix}B_T(\boldsymbol{e},t)^T P_1(\boldsymbol{e},t)\\ B_T(\boldsymbol{e},t)^T P_2(\boldsymbol{e},t)\end{bmatrix} = 0 \quad (28)$$

$$P_1[\boldsymbol{e}(t_f),t_f] = Q_{1f}, \qquad P_2[\boldsymbol{e}(t_f),t_f] = Q_{2f}$$

with $P_1(\boldsymbol{e},t) = P_1^T(\boldsymbol{e},t) > 0$ and $P_2(\boldsymbol{e},t) = P_2^T(\boldsymbol{e},t) > 0$, then Eqs. (26a) and (26b) are such that inequalities (22) and (23) hold. □

*Proof:* Let us first consider the cost function $J_2(\boldsymbol{u},\boldsymbol{d})$. It is obvious that Eq. (25) can be rewritten as

$$J_2(\boldsymbol{u},\boldsymbol{d}) = \boldsymbol{e}^T(t_f)Q_{2f}\boldsymbol{e}(t_f) + \int_0^{t_f}\left\{\boldsymbol{e}^T(t)Q_2\boldsymbol{e}(t) + \boldsymbol{u}^T(t)W\boldsymbol{u}(t) - \gamma_2^2\boldsymbol{d}^T(t)\boldsymbol{d}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{e}^T(t)P_2(\boldsymbol{e},t)\,\boldsymbol{e}(t)\right]\right\}\mathrm{d}t - \boldsymbol{e}^T(t_f)P_2[\boldsymbol{e}(t_f),t_f]\boldsymbol{e}(t_f) + \boldsymbol{e}^T(0)P_2[\boldsymbol{e}(0),0]\boldsymbol{e}(0) \quad (29)$$

By the terminal condition in Eq. (28), we obtain

$$J_2(\boldsymbol{u},\boldsymbol{d}) = \boldsymbol{e}^T(0)P_2[\boldsymbol{e}(0),0]\boldsymbol{e}(0) + \int_0^{t_f}\left[\boldsymbol{e}^T(t)Q_2\boldsymbol{e}(t) + \boldsymbol{u}^T(t)W\boldsymbol{u}(t) - \gamma_2^2\boldsymbol{d}^T(t)\boldsymbol{d}(t) + \dot{\boldsymbol{e}}^T(t)P_2(\boldsymbol{e},t)\boldsymbol{e}(t) + \boldsymbol{e}^T(t)\dot{P}_2(\boldsymbol{e},t)\boldsymbol{e}(t) + \boldsymbol{e}^T(t)P_2(\boldsymbol{e},t)\dot{\boldsymbol{e}}(t)\right]\mathrm{d}t \quad (30)$$

Substituting attitude tracking error dynamic equation (17) into Eq. (30) leads to

$$J_2(\boldsymbol{u},\boldsymbol{d}) = \boldsymbol{e}^T(0)P_2[\boldsymbol{e}(0),0]\boldsymbol{e}(0) + \int_0^{t_f}\left\{\boldsymbol{e}^T(t)\left[\dot{P}_2(\boldsymbol{e},t) + P_2(\boldsymbol{e},t)A_T(\boldsymbol{e},t) + A_T^T(\boldsymbol{e},t)P_2(\boldsymbol{e},t) + Q_2\right]\boldsymbol{e}(t) + \boldsymbol{u}^T(t)W\boldsymbol{u}(t) + \boldsymbol{u}^T(t)B_T^T(\boldsymbol{e},t)P_2(\boldsymbol{e},t)\boldsymbol{e}(t) + \boldsymbol{e}^T(t)P_2(\boldsymbol{e},t)B_T(\boldsymbol{e},t)\boldsymbol{u}(t) - \gamma_2^2\boldsymbol{d}^T(t)\boldsymbol{d}(t) + \boldsymbol{d}^T(t)B_T^T(\boldsymbol{e},t)P_2(\boldsymbol{e},t)\boldsymbol{e}(t) + \boldsymbol{e}^T(t)P_2(\boldsymbol{e},t)B_T(\boldsymbol{e},t)\boldsymbol{d}(t)\right\}\mathrm{d}t \quad (31)$$

Thus, by the worst-case disturbance $\boldsymbol{d}^*(\boldsymbol{e},t)$ in Eq. (26b) and Riccati-like equation (28), we have

$$J_2[\boldsymbol{u},\boldsymbol{d}^*(\boldsymbol{e},t)] = \boldsymbol{e}^T(0)P_2[\boldsymbol{e}(0),0]\boldsymbol{e}(0) + \int_0^{t_f}\left\{\left[\boldsymbol{u}(t) + W^{-1}B_T^T(\boldsymbol{e},t)P_2(\boldsymbol{e},t)\boldsymbol{e}(t)\right]^T \times W\left[\boldsymbol{u}(t) + W^{-1}B_T^T(\boldsymbol{e},t)P_2(\boldsymbol{e},t)\boldsymbol{e}(t)\right]\right\}\mathrm{d}t \quad (32)$$

which results in

$$J_2[\boldsymbol{u}^*(\boldsymbol{e},t),\boldsymbol{d}^*(\boldsymbol{e},t)] = \boldsymbol{e}^T(0)P_2[\boldsymbol{e}(0),0]\boldsymbol{e}(0) \quad (33)$$

Then we have

$$J_2[\boldsymbol{u}^*(\boldsymbol{e},t),\boldsymbol{d}^*(\boldsymbol{e},t)] \le J_2[\boldsymbol{u},\boldsymbol{d}^*(\boldsymbol{e},t)] \qquad \forall \boldsymbol{u}(t) \in L_2[0,t_f]$$

Similarly, the cost function $J_1(\boldsymbol{u},\boldsymbol{d})$ can be rewritten as

$$J_1(\boldsymbol{u},\boldsymbol{d}) = \boldsymbol{e}^T(0)P_1[\boldsymbol{e}(0),0]\boldsymbol{e}(0) + \int_0^{t_f}\left\{\boldsymbol{e}^T(t)\left[\dot{P}_1(\boldsymbol{e},t) + P_1(\boldsymbol{e},t)A_T(\boldsymbol{e},t) + A_T^T(\boldsymbol{e},t)P_1(\boldsymbol{e},t) + Q_1\right]\boldsymbol{e}(t) + \boldsymbol{u}^T(t)W\boldsymbol{u}(t) + \boldsymbol{u}^T(t)B_T^T(\boldsymbol{e},t)P_1(\boldsymbol{e},t)\boldsymbol{e}(t) + \boldsymbol{e}^T(t)P_1(\boldsymbol{e},t)B_T(\boldsymbol{e},t)\boldsymbol{u}(t) - \gamma_1^2\boldsymbol{d}^T(t)\boldsymbol{d}(t) + \boldsymbol{d}^T(t)B_T^T(\boldsymbol{e},t)P_1(\boldsymbol{e},t)\boldsymbol{e}(t) + \boldsymbol{e}^T(t)P_1(\boldsymbol{e},t)B_T(\boldsymbol{e},t)\boldsymbol{d}(t)\right\}\mathrm{d}t \quad (34)$$

It can be deduced from Riccati-like equation (27) and the optimal control $\boldsymbol{u}^*(\boldsymbol{e},t)$ in Eq. (26a) that

$$J_1[\boldsymbol{u}^*(\boldsymbol{e},t),\boldsymbol{d}] = \boldsymbol{e}^T(0)P[\boldsymbol{e}(0),0]\boldsymbol{e}(0) - \int_0^{t_f}\left\{\left[\gamma_1\boldsymbol{d}(t) - \frac{1}{\gamma_1}B_T^T(\boldsymbol{e},t)P_1(\boldsymbol{e},t)\boldsymbol{e}(t)\right]^T \times \left[\gamma_1\boldsymbol{d}(t) - \frac{1}{\gamma_1}B_T^T(\boldsymbol{e},t)P_1(\boldsymbol{e},t)\boldsymbol{e}(t)\right]\right\}\mathrm{d}t \quad (35)$$

Then we can conclude that

$$J_1[\boldsymbol{u}^*(\boldsymbol{e},t),\boldsymbol{d}^*(\boldsymbol{e},t)] = \boldsymbol{e}^T(0)P_1[\boldsymbol{e}(0),0]\boldsymbol{e}(0) \ge J_1[\boldsymbol{u}^*(\boldsymbol{e},t),\boldsymbol{d}] \qquad \forall \boldsymbol{d}(t) \in L_2[0,t_f]$$

QED

As a result of Lemma 1, three cases can be obtained in the following theorems.

*Theorem 1 (the nonlinear quadratic optimal attitude tracking control):* For the nonlinear spacecraft attitude tracking control system in Eq. (17) with $\boldsymbol{d} = 0$, the optimal control law,

$$\boldsymbol{u}^*(\boldsymbol{e},t) = -W^{-1}B_T^T(\boldsymbol{e},t)P(\boldsymbol{e},t)\boldsymbol{e}(t) \quad (36)$$

can solve nonlinear quadratic optimal attitude tracking control problem (18) if $P(e,t)$ satisfies the following time-varying Riccati-like equation:

$$\dot{P}(e,t) + P(e,t)A_T(e,t) + A_T(e,t)^T P(e,t) + Q - P(e,t)B_T(e,t)W^{-1}B_T^T(e,t)P(e,t) = 0 \tag{37}$$

with $P(e,t) = P^T(e,t) \geq 0$ and $Q_f = P[e(t_f), t_f]$. □

*Proof:* Let $\gamma_1 \to \infty$, $\gamma_2 = 0$, $Q_{1f} = Q_{2f} = Q_f$, and $Q_1 = Q_2 = Q$ in Lemma 1. In this case, we have $d^* \to 0$ and $\gamma_1^2 d^{*T} d^* \to 0$. Then, from Eqs. (27) and (28), we have $P_1(e,t) = P_2(e,t) = P(e,t)$, which satisfies Eq. (37). Then the cost function in Problem 1 is obtained as the following:

$$\begin{aligned} J(u,0) &= J_1(u,0) = J_2(u,0) \\ &= e^T(0)P[e(0),0]e(0) + \int_0^{t_f} \{e^T(t)[\dot{P}(e,t) \\ &\quad + P(e,t)A_T(e,t) + A_T^T(e,t)P(e,t) + Q]e(t) \\ &\quad + u^T(t)Wu(t) + u^T(t)B_T^T(e,t)P(e,t)e(t) \\ &\quad + e^T(t)P(e,t)B_T(e,t)u(t)\}\,dt \end{aligned} \tag{38}$$

By Riccati-like equation (37), we obtain

$$\begin{aligned} J(u,0) &= e^T(0)P[e(0),0]e(0) \\ &\quad + \int_0^{t_f} \{[u(t) + W^{-1}B_T^T(e,t)P(e,t)e(t)]^T \\ &\quad \times W[u(t) + W^{-1}B_T^T(e,t)P(e,t)e(t)]\}\,dt \end{aligned} \tag{39}$$

Then we can conclude that

$$J[u^*(e,t),0] = \min_{u(t)\in L_2[0,t_f]} J(u,0) = e^T(0)P[e(0),0]e(0) \tag{40}$$

This is the $H_2$ control performance in Eq. (18). QED

*Theorem 2 (the nonlinear $H_\infty$ attitude tracking control):* For the nonlinear spacecraft attitude tracking control system in Eq. (17), the $H_\infty$ control law

$$u(e,t) = -W^{-1}B_T^T(e,t)P(e,t)e(t) \tag{41}$$

solves nonlinear $H_\infty$ attitude tracking control problem (19) if $P(e,t)$ satisfies the following time-varying Riccati-like equation:

$$\dot{P}(e,t) + P(e,t)A_T(e,t) + A_T(e,t)^T P(e,t) + Q - P(e,t)B_T(e,t)[W^{-1} - (1/\gamma^2)I]B_T^T(e,t)P(e,t) = 0 \tag{42}$$

with $P(e,t) = P^T(e,t) \geq 0$ and $Q_f = P[e(t_f), t_f]$. □

*Proof:* Let $\gamma_1 = \gamma_2 = \gamma$, $Q_{1f} = Q_{2f} = Q_f$, and $Q_1 = Q_2 = Q$ in Lemma 1. Then we get $P_1(e,t) = P_2(e,t) = P(e,t)$, which satisfies Eq. (42). Moreover, we have the following performance:

$$J[u^*(e,t),d] \leq J[u^*(e,t),d^*(e,t)] \leq J[u,d^*(e,t)] \tag{43}$$

where

$$J(u,d) = e^T(t_f)Q_f e(t_f) + \int_0^{t_f} [e^T(t)Qe(t) + u^T(t)Wu(t) - \gamma^2 d^T(t)d(t)]\,dt \tag{44}$$

Then, by the fact that

$$J[u^*(e,t),d^*(e,t)] = e^T(0)P[e(0),0]e(0) \tag{45}$$

inequality (43) results in the $H_\infty$ attitude tracking performance of expression (19). QED

*Theorem 3 (the nonlinear mixed $H_2/H_\infty$ attitude tracking control):* For the nonlinear spacecraft attitude tracking control system in Eq. (17), if the optimal control law $u^*(e,t)$ and the worst case disturbance $d^*(e,t)$ are chosen, respectively, as

$$u^*(e,t) = -W^{-1}B_T^T(e,t)P_2(e,t)e(t) \tag{46a}$$

$$d^*(e,t) = (1/\gamma^2)B_T^T(e,t)P_1(e,t)e(t) \tag{46b}$$

where $P_1(e,t)$ and $P_2(e,t)$ are the solutions of the following coupled time-varying Riccati-like equations,

$$\begin{aligned} &\dot{P}_1(e,t) + P_1(e,t)A_T(e,t) + A_T(e,t)^T P_1(e,t) + Q_1 \\ &- [P_1(e,t)B_T(e,t) \quad P_2(e,t)B_T(e,t)] \\ &\times \begin{bmatrix} (-1/\gamma^2)I & W^{-1} \\ W^{-1} & -W^{-1} \end{bmatrix} \begin{bmatrix} B_T(e,t)^T P_1(e,t) \\ B_T(e,t)^T P_2(e,t) \end{bmatrix} = 0 \end{aligned} \tag{47}$$

$$\begin{aligned} &\dot{P}_2(e,t) + P_2(e,t)A_T(e,t) + A_T(e,t)^T P_2(e,t) + Q_2 \\ &- [P_1(e,t)B_T(e,t) \quad P_2(e,t)B_T(e,t)] \\ &\times \begin{bmatrix} 0_{3\times 3} & (-1/\gamma^2)I_{3\times 3} \\ (-1/\gamma^2)I_{3\times 3} & W^{-1} \end{bmatrix} \begin{bmatrix} B_T(e,t)^T P_1(e,t) \\ B_T(e,t)^T P_2(e,t) \end{bmatrix} = 0 \\ &P_1[e(t_f),t_f] = Q_{1f}, \qquad P_2[e(t_f),t_f] = Q_{2f} \end{aligned} \tag{48}$$

with $P_1(e,t) = P_1^T(e,t) \geq 0$ and $P_2(e,t) = P_2^T(e,t) \geq 0$, the nonlinear mixed $H_2/H_\infty$ attitude tracking control problem (20) under relation (21) is solved by Eqs. (46a) and (46b). □

*Proof:* Take $\gamma_1 = \gamma$ and $\gamma_2 = 0$ in Lemma 1. Then it follows that inequality (22) results in the performance of relation (21) and inequality (23) results in the performance of expression (20). Therefore we conclude that the nonlinear mixed $H_2/H_\infty$ attitude tracking control problem is solved by $u^*(e,t)$ and $d^*(e,t)$ in Eqs. (46a) and (46b), in which $P_1(e,t)$ and $P_2(e,t)$ are simultaneously the solutions of Eqs. (47) and (48). QED

*Remark 3:* From the proofs of Theorems 1, 2, and 3, the nonlinear quadratic optimal control, $H_\infty$ control, and mixed $H_2/H_\infty$ attitude tracking control problems of spacecraft dynamic systems are special cases of the nonzero-sum, two-player Nash differential game problem, as shown in the following.

*Case 1:* The nonlinear quadratic optimal attitude tracking control problem:

$$\gamma_1 \to \infty, \quad \gamma_2 = 0, \quad Q_{1f} = Q_{2f} = Q_f, \quad Q_1 = Q_2 = Q$$

*Case 2:* The nonlinear $H_\infty$ attitude tracking control problem:

$$\gamma_1 = \gamma_2 = \gamma, \quad Q_{1f} = Q_{2f} = Q_f, \quad Q_1 = Q_2 = Q$$

*Case 3:* The nonlinear mixed $H_2/H_\infty$ attitude tracking control problem:

$$\gamma_1 = \gamma, \qquad \gamma_2 = 0$$

□

Therefore the nonlinear $H_2$, $H_\infty$, and mixed $H_2/H_\infty$ attitude tracking control of spacecraft dynamic systems can have a unified designed procedure by means of the Nash game approach. In this situation, the key design issue lies in how to solve simultaneous Riccati-like equations (27) and (28) or their special cases, Eq. (37) in $H_2$ control, Eq. (42) in $H_\infty$ control, and Eqs. (47) and (48) in mixed $H_2/H_\infty$ control.

## Solution of the Nonlinear Time-Varying Riccati Equations

Based on the analysis in the preceding section, our design problem is reduced to solving time-varying Riccati-like equation (37) for $H_2$ tracking control, Eq. (42) for $H_\infty$ tracking control, and Eqs. (47) and (48) for mixed $H_2/H_\infty$ tracking control. For a unified approach, we first consider original coupled time-varying Riccati-like equations (27) and (28) in Lemma 1. In general, however, it is difficult to solve coupled time-varying Riccati-like equations (27) and (28). Fortunately, in a spacecraft system, we can further simplify Riccati-like equations (27) and (28) to algebraic matrix equations by adequately selecting the nonlinear function matrices $P_1(e,t)$ and $P_2(e,t)$ and by using the skew symmetric property of Eq. (9).

Because state transformation equation (13) has been involved in the process of design, without loss of generality, we suggest that the solutions $P_1(e,t)$ and $P_2(e,t)$ of coupled Riccati-like equations (27) and (28) can be put in more explicit forms, such as the following:

$$P_1(e,t) = T^T \begin{bmatrix} M_0(e,t) & 0_{3\times3} \\ 0_{3\times3} & K_1 \end{bmatrix} T \tag{49a}$$

$$P_2(e,t) = T^T \begin{bmatrix} M_0(e,t) & 0_{3\times3} \\ 0_{3\times3} & K_2 \end{bmatrix} T \tag{49b}$$

where $K_1$ and $K_2$ are some positive definite symmetric constant matrices. In the following paragraphs, it is demonstrated that, under some conditions, these suggested matrices $P_1(e,t)$ and $P_2(e,t)$ are the solutions of Riccati-like equations (27) and (28). Furthermore, the constant matrices $T$, $K_1$, and $K_2$ can be solved from a pair of coupled algebraic Riccati-like equations.

Consider the second and the third terms on the left-hand side of Riccati-like equations (27) and (28). Using the skew-symmetric property in Eq. (9) and the relations (49a) and (49b), we get

$$P_1(e,t)A_T(e,t) + A_T^T(e,t)P_1(e,t) = \begin{bmatrix} 0_{3\times3} & K_1 \\ K_1 & 0_{3\times3} \end{bmatrix} + T^T \begin{bmatrix} -\dot{M}_0(e,t) & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix} T \tag{50}$$

$$P_2(e,t)A_T(e,t) + A_T^T(e,t)P_2(e,t) = \begin{bmatrix} 0_{3\times3} & K_2 \\ K_2 & 0_{3\times3} \end{bmatrix} + T^T \begin{bmatrix} -\dot{M}_0(e,t) & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix} T \tag{51}$$

It can also be easily checked that

$$B_T^T(e,t)P_1(e,t) = B^T T, \qquad B_T^T(e,t)P_2(e,t) = B^T T \tag{52}$$

By using the results of Eqs. (50–52), we can reduce coupled Riccati-like equations (27) and (28) to the following coupled algebraic Riccati-like equations:

$$\begin{bmatrix} 0_{3\times3} & K_1 \\ K_1 & 0_{3\times3} \end{bmatrix} + Q_1 - T^T B\left(W^{-1} - \frac{1}{\gamma_1^2} I_{3\times3}\right) B^T T = 0 \tag{53}$$

$$\begin{bmatrix} 0_{3\times3} & K_2 \\ K_2 & 0_{3\times3} \end{bmatrix} + Q_2 - T^T B\left(W^{-1} - \frac{2}{\gamma_1^2} I_{3\times3} + \frac{\gamma_2^2}{\gamma_1^4} I_{3\times3}\right) B^T T = 0 \tag{54}$$

In addition, the optimal control law and the worst-case disturbance $\boldsymbol{u}^*(e,t)$ and $\boldsymbol{d}^*(e,t)$, respectively, can be rewritten as

$$\boldsymbol{u}^*(e,t) = -W^{-1}B^T T e, \qquad \boldsymbol{d}^*(e,t) = \left(1/\gamma_1^2\right) B^T T e \tag{55}$$

From the preceding analysis, matrices $P_1(e,t)$ and $P_2(e,t)$ in Eqs. (49a) and (49b) will be the solutions of coupled Riccati-like equations (27) and (28) if matrices $K_1$, $K_2$, and $T$ satisfy coupled algebraic Riccati-like Eqs. (53) and (54) simultaneously. Furthermore, the positive definite symmetric property of $K_1$ and $K_2$ must be satisfied. To guarantee the solvability, further assumptions and constraints on the weighting matrices $Q_1$, $Q_2$, and $W$ are required.

For the simplicity of design, let

$$W = a^2 I_{3\times3}, \qquad Q_2 = \alpha Q_1 \tag{56}$$

where $\alpha > 0$, $a > 0$, and the positive definite symmetric matrix $Q_1$ can be factorized by Cholesky factorization[21] as

$$Q_1 = \begin{bmatrix} Q_{11}^T Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22}^T Q_{22} \end{bmatrix} \tag{57}$$

Using the definitions of $T$ and $B$ in Eqs. (13) and (15) together with assumptions (56) and (57), we can split coupled Riccati-like equations (53) and (54) into the following equations:

$$Q_{11}^T Q_{11} - \frac{\gamma_1^2 - a^2}{a^2\gamma_1^2} T_{11}^T T_{11} = 0 \tag{58}$$

$$K_1 + Q_{12} - \frac{\gamma_1^2 - a^2}{a^2\gamma_1^2} T_{11}^T T_{12} = 0 \tag{59}$$

$$K_1 + Q_{12}^T - \frac{\gamma_1^2 - a^2}{a^2\gamma_1^2} T_{12}^T T_{11} = 0 \tag{60}$$

$$Q_{22}^T Q_{22} - \frac{\gamma_1^2 - a^2}{a^2\gamma_1^2} T_{12}^T T_{12} = 0 \tag{61}$$

$$\alpha Q_{11}^T Q_{11} - \frac{\gamma_1^4 - a^2\left(2\gamma_1^2 - \gamma_2^2\right)}{a^2\gamma_1^4} T_{11}^T T_{11} = 0 \tag{62}$$

$$K_2 + \alpha Q_{12} - \frac{\gamma_1^4 - a^2\left(2\gamma_1^2 - \gamma_2^2\right)}{a^2\gamma_1^4} T_{11}^T T_{12} = 0 \tag{63}$$

$$K_2 + \alpha Q_{12}^T - \frac{\gamma_1^4 - a^2\left(2\gamma_1^2 - \gamma_2^2\right)}{a^2\gamma_1^4} T_{12}^T T_{11} = 0 \tag{64}$$

$$\alpha Q_{22}^T Q_{22} - \frac{\gamma_1^4 - a^2\left(2\gamma_1^2 - \gamma_2^2\right)}{a^2\gamma_1^4} T_{12}^T T_{12} = 0 \tag{65}$$

From Eqs. (58) and (61), the submatrices $T_{11}$ and $T_{12}$ can be obtained as

$$T_{11} = \frac{a\gamma_1}{\sqrt{\gamma_1^2 - a^2}} Q_{11}, \qquad T_{12} = \frac{a\gamma_1}{\sqrt{\gamma_1^2 - a^2}} Q_{22} \tag{66}$$

But, from Eqs. (62) and (65), matrices $T_{11}$ and $T_{12}$ are obtained as

$$T_{11} = \frac{a\sqrt{\alpha}\gamma_1^2}{\sqrt{\gamma_1^4 - a^2\left(2\gamma_1^2 - \gamma_2^2\right)}} Q_{11} \tag{67a}$$

$$T_{12} = \frac{a\sqrt{\alpha}\gamma_1^2}{\sqrt{\gamma_1^4 - a^2\left(2\gamma_1^2 - \gamma_2^2\right)}} Q_{22} \tag{67b}$$

To guarantee the solvability of this problem, the following constraints must be satisfied:

$$0 < a < \gamma_1, \qquad 0 < a < \frac{\gamma_1^2}{\sqrt{2\gamma_1^2 - \gamma_2^2}} \quad \text{if} \quad \gamma_1 > \frac{\gamma_2}{\sqrt{2}} \tag{68}$$

Moreover, to guarantee the unique solutions of $T_{11}$ and $T_{12}$, from Eqs. (66) and (67) we have the following relation:

$$\alpha = \frac{\gamma_1^4 - a^2\left(2\gamma_1^2 - \gamma_2^2\right)}{\gamma_1^4 - a^2\gamma_1^2} \quad \text{or} \quad a = \sqrt{\frac{1-\alpha}{(2-\alpha)\gamma_1^2 - \gamma_2^2}}\,\gamma_1^2 \tag{69}$$

Hence we have

$$T = \begin{bmatrix} \dfrac{a\gamma_1}{\sqrt{\gamma_1^2 - a^2}} Q_{11} & \dfrac{a\gamma_1}{\sqrt{\gamma_1^2 - a^2}} Q_{22} \\ 0_{3\times3} & I_{3\times3} \end{bmatrix} \tag{70}$$

To satisfy $T_{11} = \lambda I_{3\times3}$ in Eq. (13), the weighting matrix $Q_{11}$ in Eq. (57) must be a diagonal form, that is,

$$Q_{11} = q_{11} I_{3\times3} \tag{71}$$

for some positive scale $q_{11}$, and the scale $\lambda$ can be represented as

$$\lambda = \frac{a\gamma_1 q_{11}}{\sqrt{\gamma_1^2 - a^2}} \tag{72}$$

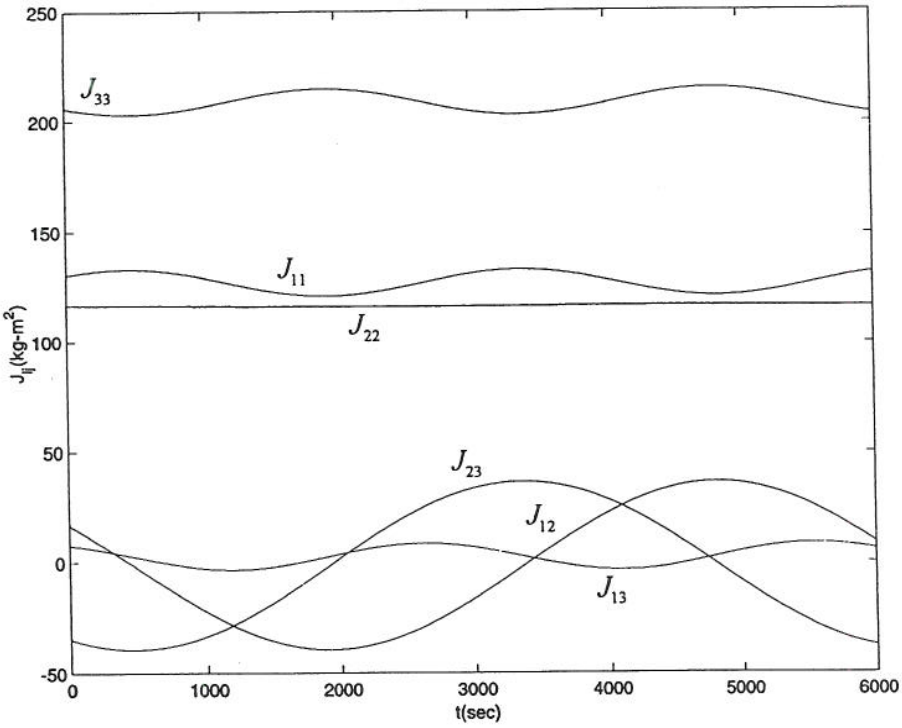


**Fig. 2 Moments of inertia and the products of inertia of the ROCSAT-1 spacecraft at its nominal circular orbit environment.**

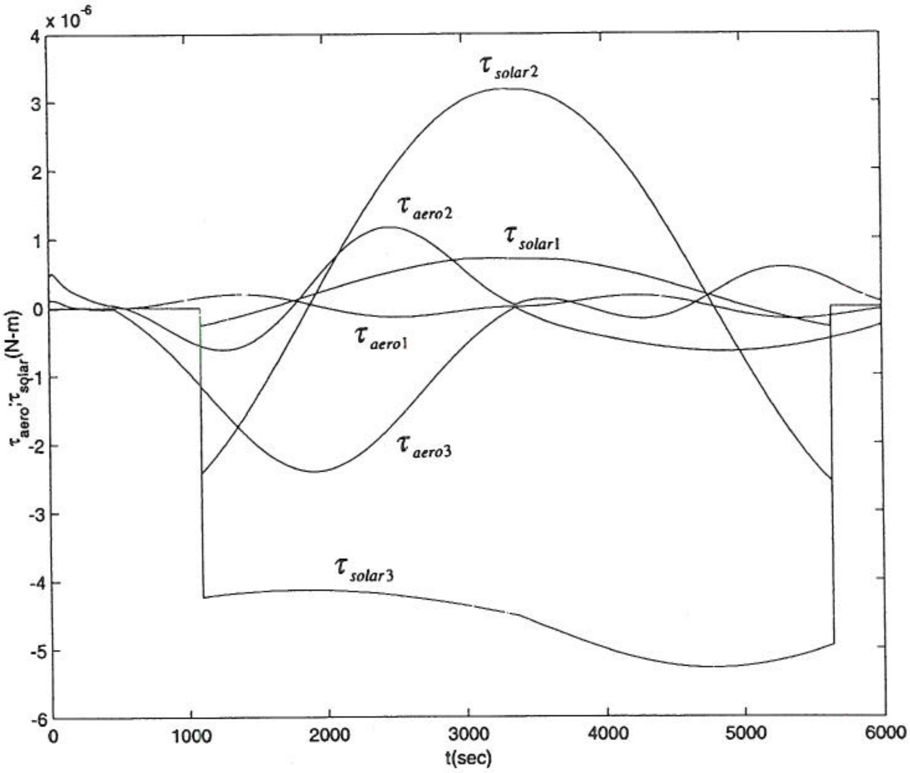


**Fig. 3 External disturbances $\tau_{\text{aero}}$ and $\tau_{\text{solar}}$ of the ROCSAT-1 spacecraft at its nominal circular orbit environment.**

Furthermore, substituting Eq. (70) into Eqs. (59) and (60), we obtain

$$K_1 = Q_{11}^T Q_{22} - Q_{12} > 0_{3\times 3} \tag{73}$$

which must be a positive definite symmetric matrix. Similarly, from Eqs. (63) and (64), we have

$$K_2 = \alpha K_1 \tag{74}$$

From the preceding analysis, a solution to the nonzero-sum, two-player Nash differential game problem is concluded in Corollary 1.

*Corollary 1:* Given some desired levels $\gamma_1 > 0$ and $\gamma_2 > 0$, let the weighting matrix $Q_1 > 0$ be taken as in Eq. (57), with $Q_{11}$, $Q_{22}$, and $Q_{12}$ satisfying the requirements in Eqs. (71) and (73) and the weighting matrix $Q_2 > 0$ be taken as in Eq. (56) with $0 < \alpha$. If the weighting matrix $W$ is of the form in Eq. (56), with $a$ satisfying Eqs. (68) and (69), then the nonzero-sum, two-player Nash differential game problem is solved by the following optimal control law and worst-case disturbance:

$$\boldsymbol{u}^* = -\frac{\gamma_1}{a\sqrt{\gamma_1^2 - a^2}}[Q_{11} \quad Q_{22}]\boldsymbol{e} \tag{75a}$$

$$\boldsymbol{d}^* = \frac{a}{\gamma_1\sqrt{\gamma_1^2 - a^2}}[Q_{11} \quad Q_{22}]\boldsymbol{e} \tag{75b}$$

□

From Corollary 1 and the results in Remark 4, three more Corollaries are obtained.

*Corollary 2 (the nonlinear quadratic optimal attitude tracking control):* Let the weighting matrix $W$ be of the form in Eq. (56) for any finite $a > 0$ and allow the weighting matrix $Q > 0$ to be further taken as

$$Q = \begin{bmatrix} Q_{11}^T Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22}^T Q_{22} \end{bmatrix} \tag{76}$$

with $Q_{11}$, $Q_{22}$, and $Q_{12}$ satisfying the requirements in Eqs. (71) and (73). Then the nonlinear quadratic optimal spacecraft attitude tracking control problem is solved by the following optimal controller:

$$u^* = -(1/a)[Q_{11} \quad Q_{22}]e \tag{77}$$

□

*Corollary 3 (the nonlinear $H_\infty$ attitude tracking control):* Given a desired disturbance attenuation level $\gamma > 0$, let the weighting matrix $W$ be of the form in Eqs. (56) with $0 < a < \gamma$ and allow the weighting matrix $Q > 0$ to be taken as in Eq. (76) with $Q_{11}$, $Q_{22}$, and $Q_{12}$ satisfying the requirements in Eqs. (71) and (73). Then the nonlinear $H_\infty$ spacecraft attitude tracking control problem is solved by the following $H_\infty$ controller:

$$u = -\frac{\gamma}{a\sqrt{\gamma^2 - a^2}}[Q_{11} \quad Q_{22}]e \tag{78}$$

□

*Corollary 4 (the nonlinear mixed $H_2/H_\infty$ attitude tracking control):* Given a desired disturbance attenuation level $\gamma > 0$, let the weighting matrix $Q_1 > 0$ be taken as in Eq. (57), with $Q_{11}$, $Q_{22}$, and $Q_{12}$ satisfying the requirements in Eqs. (71) and (73), and the weighting matrix $Q_2 > 0$ be taken as in Eq. (56) with $0 < \alpha < 1$. If the weighting matrix $W$ is of the form in Eq. (56), with $a$ satisfying

$$a = \sqrt{(1-\alpha)/(2-\alpha)}\gamma \tag{79}$$

and $0 < a < \gamma/\sqrt{2}$, then the nonlinear mixed $H_2/H_\infty$ spacecraft attitude tracking control problem is solved by the following mixed $H_2/H_\infty$ controller and the worst-case disturbance:

$$u^* = -\frac{\gamma}{a\sqrt{\gamma^2 - a^2}}[Q_{11} \quad Q_{22}]e \tag{80a}$$

$$d^* = \frac{a}{\gamma\sqrt{\gamma^2 - a^2}}[Q_{11} \quad Q_{22}]e \tag{80b}$$

□

## Design Algorithm

Based on the preceding discussion, the spacecraft attitude control design can be outlined as the following design algorithms.

**Nonlinear Quadratic Optimal Attitude Tracking Control Design Algorithm**

*Step 1:* Select the weighting matrix $W = a^2 I_{3\times 3}$ such that $a > 0$ and the weighting matrix

$$Q = \begin{bmatrix} Q_{11}^T Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22}^T Q_{22} \end{bmatrix}$$

with $Q_{11} = q_{11} I_{3\times 3} > 0$ and $Q_{11}^T Q_{22} - Q_{12} = Q_{22}^T Q_{11} - Q_{12}^T > 0_{3\times 3}$.

*Step 2:* Obtain the corresponding optimal applied torque:

$$\tau_a = R^{-T}(\theta)f = R^{-T}(\theta)[F(e,t) + (1/\lambda)u] \tag{81}$$

where

$$F(e,t) = M_0(\theta)[\ddot{\theta}_r - (Q_{22}/q_{11})\dot{\tilde{\theta}}]$$
$$+ C_0(\theta,\dot{\theta})[\dot{\theta}_r - (Q_{22}/q_{11})\tilde{\theta}] + G_0(\theta,\dot{\theta})$$
$$\lambda = aq_{11}, \qquad u = -(1/a)[Q_{11} \quad Q_{22}]e$$

□

**Nonlinear $H_\infty$ Attitude Tracking Control Design Algorithm**

*Step 1:* Choose a desired level of disturbance attenuation, $\gamma > 0$.

*Step 2:* Select the weighting matrix $W = a^2 I_{3\times 3}$ such that $0 < a < \gamma$ and the weighting matrix

$$Q = \begin{bmatrix} Q_{11}^T Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22}^T Q_{22} \end{bmatrix}$$

with $Q_{11} = q_{11} I_{3\times 3} > 0$ and $Q_{11}^T Q_{22} - Q_{12} = Q_{22}^T Q_{11} - Q_{12}^T > 0_{3\times 3}$

*Step 3:* Obtain the corresponding $H_\infty$ applied torque:

$$\tau_a = R^{-T}(\theta)f = R^{-T}(\theta)[F(e,t) + (1/\lambda)u] \tag{82}$$

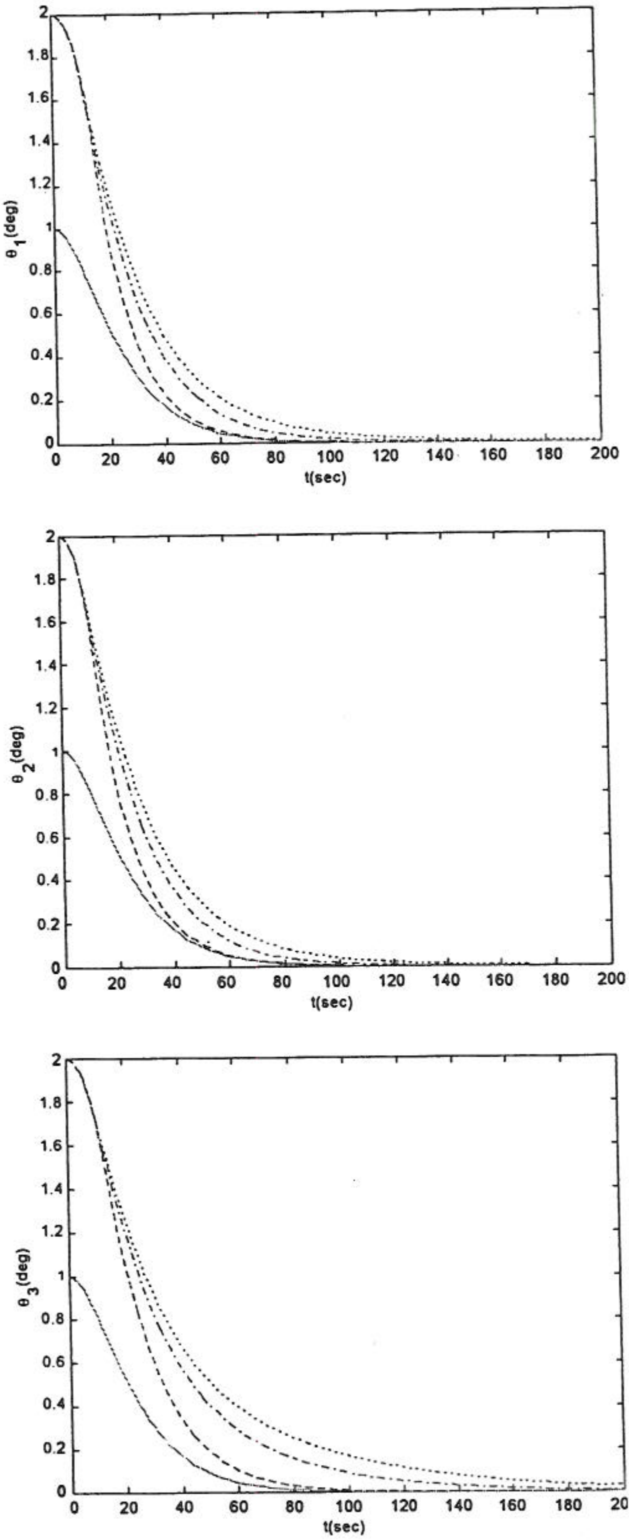


**Fig. 4 Attitude angles: $\cdots\cdot$, $H_2$, $a$ = 0.5; $-\cdot-$, $H_\infty$, $\gamma$ = 0.5, $a$ = 0.425; $---$, mixed $H_2/H_\infty$, $\gamma$ = 0.5, $\alpha$ = 0.5; and ——, desired attitude trajectory.**

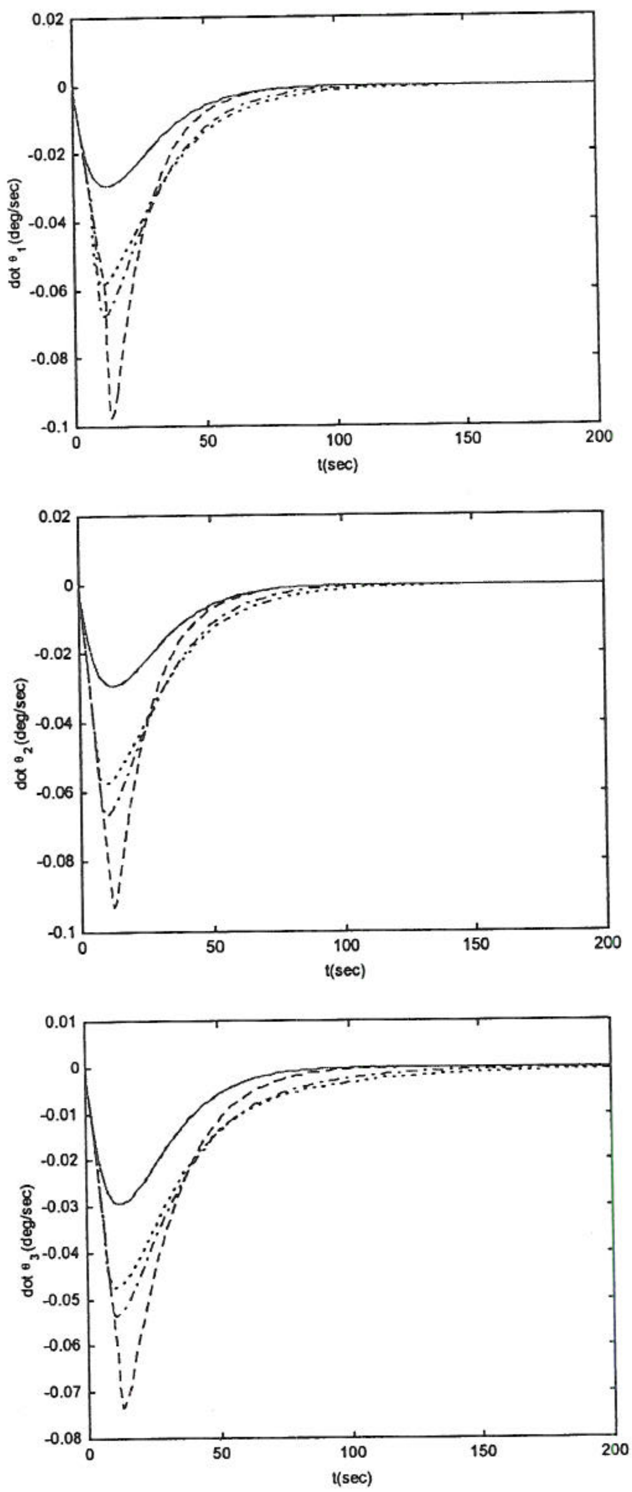


Fig. 5 Attitude angle rates: $\cdots\cdot$, $H_2$, $a = 0.5$; $-\cdot-$, $H_\infty$, $\gamma = 0.5$, $a = 0.425$; $---$, mixed $H_2/H_\infty$, $\gamma = 0.5$, $\alpha = 0.5$; and ——, desired attitude trajectory.

where

$$F(e,t) = M_0(\theta)\left(\ddot{\theta}_r - \frac{Q_{22}}{q_{11}}\dot{\tilde{\theta}}\right) + C_0(\theta,\dot{\theta})\left(\dot{\theta}_r - \frac{Q_{22}}{q_{11}}\tilde{\theta}\right) + G_0(\theta,\dot{\theta})$$

$$\lambda = \frac{a\gamma q_{11}}{\sqrt{\gamma^2 - a^2}}, \qquad u = -\frac{\gamma}{a\sqrt{\gamma^2 - a^2}}[Q_{11} \quad Q_{22}]e$$

□

**Nonlinear Mixed $H_2/H_\infty$ Attitude Tracking Control Design Algorithm**

*Step 1:* Choose a desired level of disturbance attenuation, $\gamma > 0$.

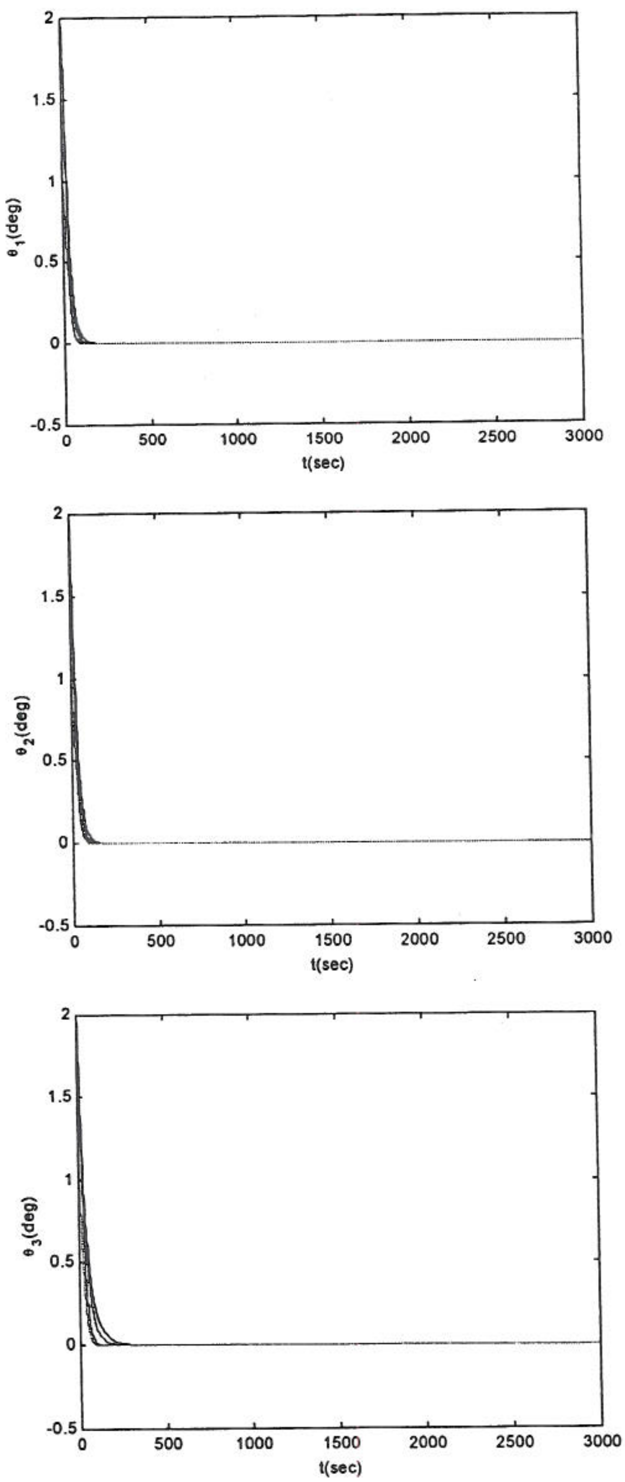


Fig. 6 Steady-state response of attitude angles: $\cdots\cdot$, $H_2$, $a = 0.5$; $-\cdot-$, $H_\infty$, $\gamma = 0.5$, $a = 0.425$; $---$, mixed $H_2/H_\infty$, $\gamma = 0.5$, $\alpha = 0.5$; and ——, desired attitude trajectory.

*Step 2:* Select the weighting matrices,

$$Q_1 = \begin{bmatrix} Q_{11}^T Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22}^T Q_{22} \end{bmatrix}$$

with $Q_{11} = q_{11} I_{3\times3} > 0_{3\times3}$, $Q_{11}^T Q_{22} - Q_{12} = Q_{22}^T Q_{11} - Q_{12}^T > 0_{3\times3}$, and $Q_2 = \alpha Q_1$, where $0 < \alpha < 1$ and $W = a^2 I_{3\times3}$ such that

$$a = \sqrt{(1-\alpha)/(2-\alpha)}\gamma, \qquad 0 < a < \gamma/\sqrt{2}$$

*Step 3:* Obtain the corresponding mixed $H_2/H_\infty$ applied torque:

$$\tau_a = R^{-T}(\theta)f = R^{-T}(\theta)[F(e,t) + (1/\lambda)u] \qquad (83)$$

where

$$F(e,t) = M_0(\theta)\left(\ddot{\theta}_r - \frac{Q_{22}}{q_{11}}\dot{\tilde{\theta}}\right) + C_0(\theta,\dot{\theta})\left(\dot{\theta}_r - \frac{Q_{22}}{q_{11}}\tilde{\theta}\right) + G_0(\theta,\dot{\theta})$$

$$\lambda = \frac{a\gamma q_{11}}{\sqrt{\gamma^2 - a^2}}, \qquad u = -\frac{\gamma}{a\sqrt{\gamma^2 - a^2}}[Q_{11} \quad Q_{22}]e$$

□

## Simulation Results

To substantiate the performance of the controller design, experimental simulations on the ROCSAT-1 spacecraft (see Fig. 1) have been made with the assistance of the National Space Program Office in Taiwan. The useful data for the ROCSAT-1 spacecraft at its nominal 600-km circular orbit environment are

$$\omega_0 = 0.0011 \text{ rad/s}$$

$$J_0 = \begin{bmatrix} 126.98 & -1.87 & 3.38 \\ -1.87 & 116.63 & -2.40 \\ 3.38 & -2.40 & 209.36 \end{bmatrix} \text{kg} \cdot \text{m}^2$$

The solar arrays are designed to point toward the sun as much as possible; the practical parameter variation of the moments of inertia $J_{ii}$ and the products of inertia $J_{ij}$ $(i \neq j, i, j = 1, 2, 3)$ when the spacecraft orbits a cycle are presented in Fig. 2. The external disturbances $\tau_{\text{aero}}$ and $\tau_{\text{solar}}$ in the body frame are presented in Fig. 3. These data are generated by a simulator according to certain practical parameter variations and external disturbances of the ROCSAT-1 spacecraft at its nominal circular orbit enviroment. We consider that the spacecraft, initially at $\theta(0) = (\pi/90, \pi/90, \pi/90)^T$ and $\dot{\theta}(0) = (0, 0, 0)^T$, is required to track a hypothetical desired attitude trajectory of the form $\ddot{\theta}_r + K_v\dot{\theta}_r + K_p\theta_r = K_r U_r$, with the initial conditions $\theta_r(0) = (\pi/180, \pi/180, \pi/180)^T$ and $\dot{\theta}_r(0) = (0, 0, 0)^T$, as well as the coefficient matrices $K_v = 0.16I_{3\times3}$, $K_p = 0.0064I_{3\times3}$, and $K_r = I_{3\times3}$, so that $\theta_r$ is a form of overdamping trajectory and is driven by the signal $U_r = (0, 0, 0)^T$ for all $t$. Moreover, for the ROCSAT-1 spacecraft, the output torque vector of the reaction wheel is limited in amplitude because of saturation. Thus we set the saturation values $S_{ai} = 0.015$ N · m, $i = 1, 2, 3$, in Eq. (3).

### Simulation 1

To compare the ability for uncertainty attenuation of these proposed methods, three cases of control are considered: quadratic optimal attitude tracking control, $H_\infty$ attitude tracking control, and mixed $H_2/H_\infty$ attitude tracking control. The simulation parameters are selected as follows:

*Case 1, quadratic optimal attitude tracking control:* Select

$$a = 0.5, \qquad Q = I_{6\times6}$$

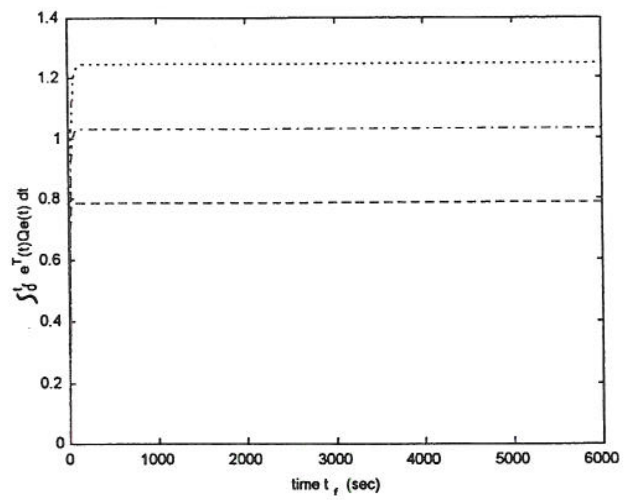


**Fig. 7 Attitude tracking performances: ····, $H_2$, $a$ = 0.5; –·–, $H_\infty$, $\gamma$ = 0.5, $a$ = 0.425; and – – –, mixed $H_2/H_\infty$, $\gamma$ = 0.5, $\alpha$ = 0.5.**

*Case 2, $H_\infty$ attitude tracking control:* Select

$$\gamma = 0.5, \qquad a = 0.425, \qquad Q = I_{6\times6}$$

*Case 3, mixed $H_2/H_\infty$ attitude tracking control:* Select

$$\gamma = 0.5, \qquad Q = I_{6\times6}, \qquad \alpha = 0.5$$

Then $a = 0.5/\sqrt{3}$.

By means of the corresponding applied torques in Eqs. (81), (82), and (83) for the preceding three cases, respectively, the simulation results are shown in Figs. 4–8. The tracking attitude angles $\theta_1$, $\theta_2$, and $\theta_3$ are represented in Fig. 4. The tracking attitude angle rates $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ are depicted in Fig. 5. As the results of these three proposed methods reveal, the mixed $H_2/H_\infty$ attitude tracking control causes quicker decay responses and has the superior ability to diminish the effects of parameter perturbations and external disturbances. This result can be expected from the fact that the mixed $H_2/H_\infty$ attitude tracking controller is designed to achieve the $H_2$ optimal

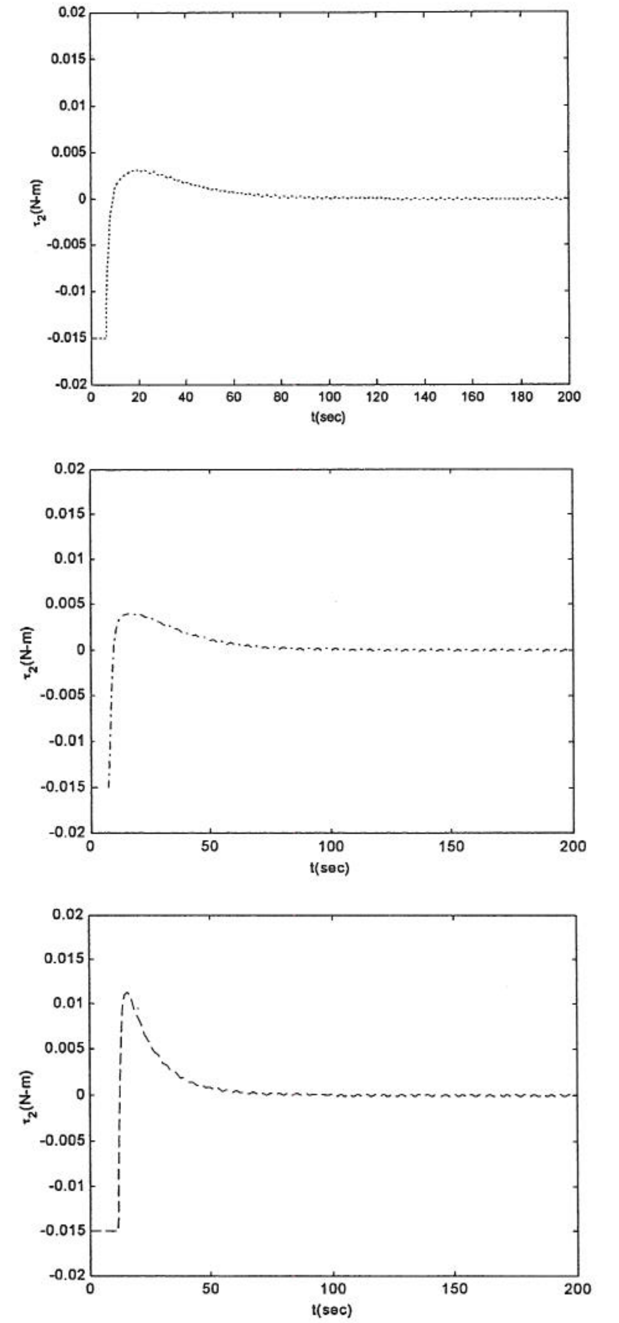


**Fig. 8 Applied torques $\tau_2(t)$: ·····, $H_2$, $a$ = 0.5; –·–, $H_\infty$, $\gamma$ = 0.5, $a$ = 0.425; and – – –, mixed $H_2/H_\infty$, $\gamma$ = 0.5, $\alpha$ = 0.5.**

control under an $H_\infty$ disturbance attenuation constraint. Furthermore, the $H_2$ controller causes the fewest disturbance attenuation abilities among the three methods (with respect to slower decay responses). This is reasonable because the $H_2$ control is designed without consideration of the combined disturbances and therefore does not, in general, guarantee any robust performance in the face of combined disturbance. The simulation result in Fig. 6 shows the steady-state response of tracking attitude angles $\theta_1$, $\theta_2$, and $\theta_3$ over $t = 3000$ s. It can be found that the effects of external disturbances

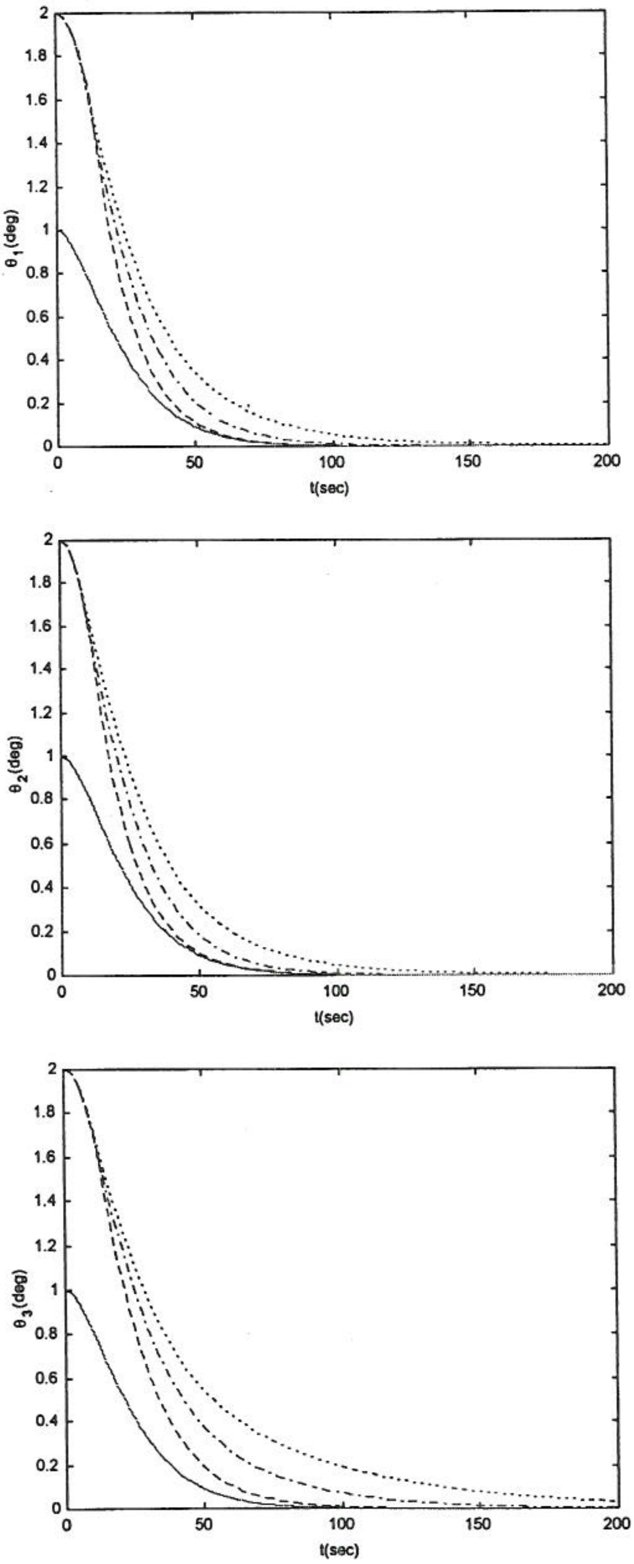


**Fig. 9 Attitude angles: · · · ·, mixed $H_2/H_\infty$, $\gamma = 0.9$, $\alpha = 0.5$; –·–, mixed $H_2/H_\infty$, $\gamma = 0.7$, $\alpha = 0.5$; – – –, mixed $H_2/H_\infty$, $\gamma = 0.5$, $\alpha = 0.5$; and ——, desired attitude trajectory.**

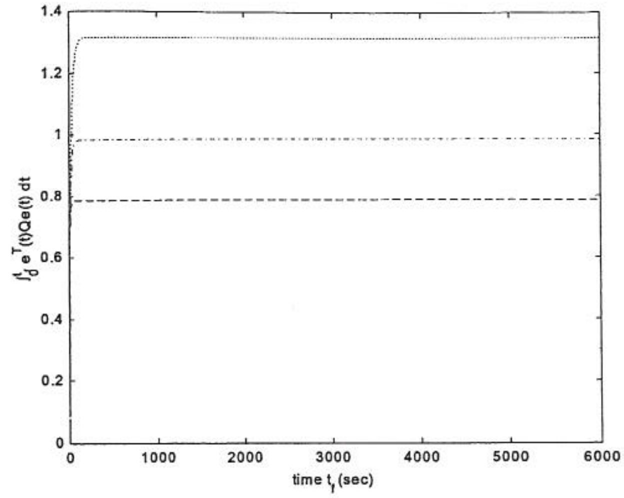


**Fig. 10 Attitude tracking performances: · · · · ·, mixed $H_2/H_\infty$, $\gamma = 0.9$, $\alpha = 0.5$; –·–, mixed $H_2/H_\infty$, $\gamma = 0.7$, $\alpha = 0.5$; and – – –, mixed $H_2/H_\infty$, $\gamma = 0.5$, $\alpha = 0.5$.**

to the tracking errors are negligible after $t = 200$ s, even when the external disturbances are at a maximum at $t = 3000$ s. The tracking performances

$$\int_0^{t_f} e^T(t) Q e(t)\, \mathrm{d}t$$

are plotted in Fig. 7. These simulation results show that the tracking performance caused by the $H_\infty$ attitude tracking controller is better than the tracking performance of $H_2$ control and worse than the tracking performance of mixed $H_2/H_\infty$ control. The applied torque $\tau_2$ is represented in Fig. 8. In these results, we can find that the chattering of control signals rises when tracking errors decay to zero. This phenomenon is inevitable because the nonlinear terms of the controllers in Eqs. (81), (82), or (83) have periodic behavior. Therefore the effect of the combined disturbance $w(t)$ turns into chattering of the control signals when tracking errors approach zero.

**Simulation 2**

To illustrate the robust capability of disturbance attenuation of the proposed mixed $H_2/H_\infty$ control design, the control law in Eq. (83) is deliberately designed to possess the following three different disturbance attenuation levels:

*Case 1:* Select $\gamma = 0.9, Q_1 = I_{6\times 6}$, and $\alpha = 0.5$. Then $a = 0.9/\sqrt{3}$.

*Case 2:* Select $\gamma = 0.7, Q_1 = I_{6\times 6}$, and $\alpha = 0.5$. Then $a = 0.7/\sqrt{3}$.

*Case 3:* Select $\gamma = 0.5, Q_1 = I_{6\times 6}$, and $\alpha = 0.5$. Then $a = 0.5/\sqrt{3}$.

The simulation results are shown in Figs. 9 and 10 for the tracking attitude angles and the tracking performances

$$\int_0^{t_f} e^T(t) Q e(t)\, \mathrm{d}t$$

respectively. It is obviously that a smaller $\gamma$ may yield a better tracking performance in attenuating the effect of combined disturbances $w(t)$.

## Conclusion

This paper presents, from a unified perspective, three nonlinear attitude control methods for spacecraft systems. They are quadratic optimal attitude tracking control, $H_\infty$ attitude tracking control, and mixed $H_2/H_\infty$ attitude tracking control. All of these control problems are shown to be special cases of the so-called two-player Nash differential game problem, for which two coupled time-varying Riccati-like equations must be solved. Unlike the conventional nonlinear $H_2$, $H_\infty$, or mixed $H_2/H_\infty$ control methods for solving the partial differential equation, general solutions can be obtained by the proposed methods by means of skew-symmetric property and state transformation techniques. According to simulation results, the mixed $H_2/H_\infty$ attitude tracking control has an excellent ability to diminish the effects of parameter perturbations and external disturbances in these three proposed methods and at the same time to achieve robust tracking performance in perturbed spacecraft attitude control systems. Moreover, a desired disturbance attenuation level can be achieved if an adequate weighting matrix in the control algorithm has been selected. From the experimental simulation results on the ROCSAT-1 spacecraft system, the proposed design algorithm exhibits significant advantages for the attitude tracking control of a spacecraft under time-varying plant perturbation and large external disturbance.

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