Overview

Nonlinear Control Seminar

Linear Matrix Inequalities:

History, Techniques, and Applications to Control Theory

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07 February 2000

- Introduction
- History of LMIs
- LMI Techniques
- Methods to solve
  - Ellipsoid algorithm
  - Interior-point methods
- Control applications
  - Analysis
  - Synthesis
  - $\mathcal{S}$-procedure
  - Inverse optimality
- Summary
- References

Introduction

- Most material for this presentation originates from S. Boyd, his colleagues and students [1–5, 7]
- A linear matrix inequality (LMI) has the form

$$F(x) := F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{1}$$

  where $x \in \mathbb{R}^m$ is the variable and symmetric matrices $F_i = F_i^T \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, m$, are given
- Combine LMI with optimization to get semidefinite programming (SDP) problem:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & F(x) \geq 0 \end{aligned} \tag{2}$$

  where $c \in \mathbb{R}^m$ is given
- LMI (1) is equivalent to n polynomial inequalities in x, using principal minors of $F(x)$
- LMI (1) is a convex constraint: If $F(x) \geq 0$ and $F(y) \geq 0$, then for all λ, $0 \leq \lambda \leq 1$,

$$F(\lambda x + (1 - \lambda)y) = \lambda F(x) + (1 - \lambda)F(y) \geq 0$$

  and can be used to represent:
  - linear inequalities
  - (convex) quadratic inequalities
  - matrix norm inequalities
  - Lyapunov and convex quadratic matrix inequalities

History

- Lyapunov made the first steps in 1890's. Showed differential equation

  $$\dot{x}(t) = Ax(t)$$

  is stable if and only if there exists $P > 0$ such that

  $$A^T P + PA < 0$$

  This is a special form of LMI

- Lyapunov showed how to solve using $Q = Q^T > 0$ and solving for P in

  $$A^T P + PA = -Q$$

- So, LMI used to analyze stability and could be solved analytically

- In 1940's, Lur'e and others applied Lyapunov's methods to solve control problems
  - Considered stability of system with actuator nonlinearity
  - Did not form matrix inequalities, but stability criteria have form of LMIs
  - Reduced equations to polynomial inequalities
  - Solved by hand
  - First to apply Lyapunov's methods to practical problems

History

- In the 1960's, we received contributions from Yakubovich, Popov and Kalman.
  - Reduced LMIs from the problem of Lur'e to simple graphical criteria using positive-real lemma
  - Yielded Popov criterion, circle criterion and variations
  - Applied to higher order systems
  - Did not extend to systems with multiple nonlinearities
  - Showed how to solve a class of LMIs with graphical methods
- Details on Yakubovich's work:
  - [8]: Related existence of Lyapunov function to existence of solution to matrix inequalities:
    1. Given A, a, b, with A Hurwitz, find $H = H^T$ with
    $$G = -(A^T H + HA), \qquad g = -(Ha + b)$$
    such that
    $$G - gg^T > 0$$
    2. Given B, $c \neq 0$, $d \neq 0$, B Hurwitz, find $X = X^T$ satisfying
    $$-Y \equiv B^T X + XB < 0, \qquad Xc + d = 0$$
  - [9]: Considers systems of the form
  $$\dot{x} = Px + q\phi(\sigma), \qquad \sigma = r^T x$$
  where P is Hurwitz, $\phi(\sigma)$ is a continuous function satisfying
  $$0 \leq \sigma\phi(\sigma) \leq \mu_0\sigma^2, \qquad \mu_0 \leq +\infty$$

History

Used matrix inequalities to find a new proof for conditions to solve problem and existence of Lyapunov function of form

$$\Omega(x) = x^T H x + \vartheta \int_0^\sigma \phi(\sigma)\, d\sigma$$

- [10]: Considers absolute stability of systems with forced vibrations like

$$\dot{x} = Px + q\phi(\sigma) + f(t), \qquad \sigma = r^T x$$

where $f(t)$ is a bounded vector function and $\phi(\sigma)$ is a generally discontinuous function with isolated points of discontinuity

- [11]: Absolute stability for a class of nonlinear systems with a bound on derivative

$$\dot{x} = Px + q\phi(\sigma), \qquad \sigma = r^T x$$

where $\phi(\sigma)$ is differentiable

$$0 \le \sigma\phi(\sigma) \le \mu_0 \sigma^2, \qquad \mu_0 \le +\infty$$

and one of conditions:

$$\phi'(\sigma) \le \alpha_1, \qquad \phi'(\sigma) \ge \alpha_2$$

- [12]: Absolute stability of systems with hysteresis nonlinearities

$$\dot{x} = Px + q\phi[\sigma, \phi_0]_t, \qquad \sigma = r^T x$$

where $\phi[\sigma, \phi_0]_t$ is a hysteresis function. Paper derives conditions for stability based on matrix inequalities

History

- In 1970's, could solve the LMI appearing in positive-real lemma by solving a certain algebraic Riccati equation. Can solve
$$\begin{bmatrix} A^TP + PA + Q & PB + C^T \\ B^TP + C & R \end{bmatrix} \geq 0$$
by studying symmetric solutions to ARE
$$A^TP + PA - (PB + C^T)R^{-1}(B^TP + C) + Q = 0$$
- By 1971, several methods for solving LMIs:
  - Directly, for small systems
  - Graphically
  - Solving Lyapunov or Riccati equations
- In 1971, J. C. Willems wonders if there are computational algorithms for solving
- Observation: LMIs in system and control theory can be formulated as convex optimization problems and solved numerically
  - Pyatnitskii and Skorodinskii first elaborated in 1982
  - Reduced general version of problem of Lur'e to convex optimization problem using LMIs
  - Solved using ellipsoid algorithm
- In 1984, N. Karmarkar introduced linear programming algorithm that was efficient in practice — led to intense study of interior-point methods for linear programming
- In 1988, Nesterov and Nemirovskii developed interior-point methods that apply directly to convex optimization problems with LMIs

LMI Techniques

- Can express multiple LMIs $F^{(1)}(x) > 0, \ldots, F^{(p)}(x) > 0$ as a single LMI

$$\mathrm{diag}[F^{(1)}(x), \ldots, F^{(p)}(x)] > 0$$

- When F_i are diagonal, the LMI $F(x) > 0$ is a set of linear inequalities
- The nonlinear inequalities

$$R(x) > 0, \qquad Q(x) - S(x)R^{-1}(x)S^T(x) > 0 \tag{3}$$

with $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, and $S(x)$ affine in x, can be represented as the LMI:

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0 \tag{4}$$

- Examples:

1. Matrix norm constraint (maximum singular value) $\|Z(x)\| < 1$, where $Z(x) \in \mathbb{R}^{p \times q}$ and depends affinely on x, is equivalent to LMI

$$\begin{bmatrix} I & Z(x) \\ Z^T(x) & I \end{bmatrix} > 0$$

since $\|Z(x)\| < 1$ is equivalent to $I - ZZ^T > 0$

2. The constraint $c^T(t)P^{-1}(x)c(x) < 1$, $P(x) > 0$, where $c(x) \in \mathbb{R}^n$ and $P(x) = P^T(x) \in \mathbb{R}^{n \times n}$ depend affinely on x, is expressed as LMI:

$$\begin{bmatrix} P(x) & c(x) \\ c^T(x) & 1 \end{bmatrix} > 0$$

LMI Techniques

- Lyapunov inequality

$$A^T P + PA < 0$$

  where $A \in \mathbb{R}^{n \times n}$ is given and $P = P^T$ is the variable can be converted to LMI form by letting $P_1, \ldots, P_m$ be a basis for symmetric $n \times n$ matrices ($m = \frac{1}{2}n(n+1)$) and then take $F_0 = 0$ and $F_i = -A^T P_i - P_i A$.

- Standard LMI problems:

  1. **LMI problems (LMIP):** given LMI $F(x) > 0$, find x^{feas} such that $F(x^{feas}) > 0$ or determine that the LMI is infeasible
  2. **Eigenvalue problems (EVP):** minimize the maximum eigenvalue of a matrix that depends affinely on a variable, subject to an LMI constraint or determine constraint is infeasible:

$$\begin{aligned} \text{minimize} \quad & \lambda \\ \text{subject to} \quad & \lambda I - A(x) > 0, \qquad B(x) > 0 \end{aligned}$$

     where A and B are symmetric and depend affinely on x. Can convert to semidefinite programming problem (2)

LMI Techniques

3. **Generalize eigenvalue problem (GEVP):** minimize the maximum generalized eigenvalue of a pair of matrices that depend affinely on a variable, subject to an LMI constraint:

$$\begin{aligned} &\text{minimize} \quad \lambda \\ &\text{subject to} \quad \lambda B(x) - A(x) > 0, \quad B(x) > 0, \quad C(x) > 0 \end{aligned}$$

   where A, B, and C are symmetric and depend affinely on x

4. **Convex problem (CP):**

$$\begin{aligned} &\text{minimize} \quad \log\det A^{-1}(x) \\ &\text{subject to} \quad A(x) > 0, \qquad B(x) > 0 \end{aligned}$$

   where A and B are symmetric matrices that depend affinely on x. Note: $A > 0 \;\Rightarrow\; \log\det A^{-1}$ is a convex function of A.

   – Can transform CP into EVP
   – Can use to solve this problem: Given a set of points in $\mathbb{R}^n$ find the minimum volume ellipsoid, centered at the origin, that contains all of the points

- LMI Problems with analytical solutions:
  – Lyapunov's inequality
  – Positive-real lemma
  – Bounded-real lemma
  – Synthesis of state-feedback for linear systems
  – Synthesis of estimator gains for observer

Ellipsoid Algorithm

- Simple approach guaranteed to solve problem
- Relatively efficient, but interior-point methods are better
- Assume problem has at least one optimal point (i.e., the constraints are feasible)
- Start with an ellipsoid $\mathcal{E}^{(0)}$ that is guaranteed to contain the optimal point
- Find a cutting plane that passes through center $x^{(0)}$ of $\mathcal{E}^{(0)}$
- Find a vector $g^{(0)}$ such that optimal point is in $\{z \ : \ {g^{(0)}}^T(z - x^{(0)}) \leq 0\}$
- Then
$$\mathcal{E}^{(0)} \cap \left\{ z \ : \ {g^{(0)}}^T(z - x^{(0)}) \leq 0 \right\}$$
contains an optimal point
- Now find ellipsoid $\mathcal{E}^{(1)}$ of minimum volume that contains the sliced ellipsoid
- $\mathcal{E}^{(1)}$ contains an optimal point
- Repeat the slicing process
- Volume of ellipsoids decreases geometrically
- Boyd *et al.* [4] provides details on constructing ellipsoids and cutting planes for different versions of LMI problems

Interior-Point Methods

- Have used interior-point methods since 1988
- Can be very efficient for LMI problems
- Barrier technique, a.k.a. sequential unconstrained minimization
- Problem:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0,\ i = 1, \ldots, m \\ & Ax = b \end{aligned} \tag{5}$$

  where f_i are convex and have continuous second derivatives. Assume a strictly feasible point $x^{(0)}$ is given, such that $Ax^{(0)} = b$ and $f_i(x^{(0)}) < 0$, $i = 1, \ldots, m$.

- Introduce a barrier

$$\begin{aligned} \text{minimize} \quad & f_0(x) + I_{feas}(x) \\ \text{subject to} \quad & Ax = b \end{aligned} \tag{6}$$

  where

$$I_{feas}(x) = \begin{cases} 0 & f_i(x) \leq 0,\ i = 1, \ldots, m \\ +\infty & \text{otherwise} \end{cases}$$

- I_{feas} is not differentiable, so approximate with $\widehat{I}$ where
  - $\widehat{I}$ is convex and smooth
  - $\text{Domain}(\widehat{I}) = \{x\ :\ f_i(x) < 0,\ i = 1, \ldots, m\}$
  - $\widehat{I}(x)$ grows without bound as x approaches boundary

Interior-Point Methods

- Now

$$\begin{aligned} \text{minimize} \quad & f_0(x) + \widehat{I}(x) \\ \text{subject to} \quad & Ax = b \end{aligned} \tag{7}$$

- Use a descent method to solve
- Scale barrier function to better approximate I_{feas}

$$\begin{aligned} \text{minimize} \quad & f_0(x) + (1/t)\widehat{I}(x) \\ \text{subject to} \quad & Ax = b \end{aligned} \tag{8}$$

  For large t, we get a good solution
- Example

$$\begin{aligned} \text{minimize} \quad & x^2 + 1 \\ \text{subject to} \quad & 2 \leq x \leq 4 \end{aligned}$$

  with barrier function

$$\widehat{I}(x) = \begin{cases} -\log(x-2) - \log(4-x) & \text{if } 2 < x < 4 \\ +\infty & \text{otherwise} \end{cases}$$

  Figures 1 and 2 show the results
- Many variations and more technical details
- Can tailor interior-point methods based on problem structure
- Efficient for standard problems

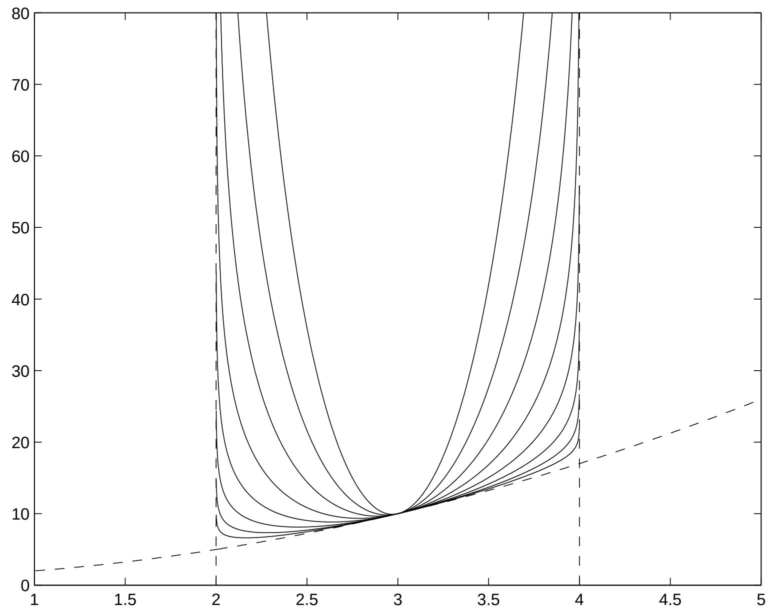

Figure 1: The function $f_0(x) + (1/t)\widehat{I}(x)$ for $f_0(x) = x^2 + 1$, with barrier function $\widehat{I}(x) = -\log(x-2) - \log(4-x)$ and different values of t.

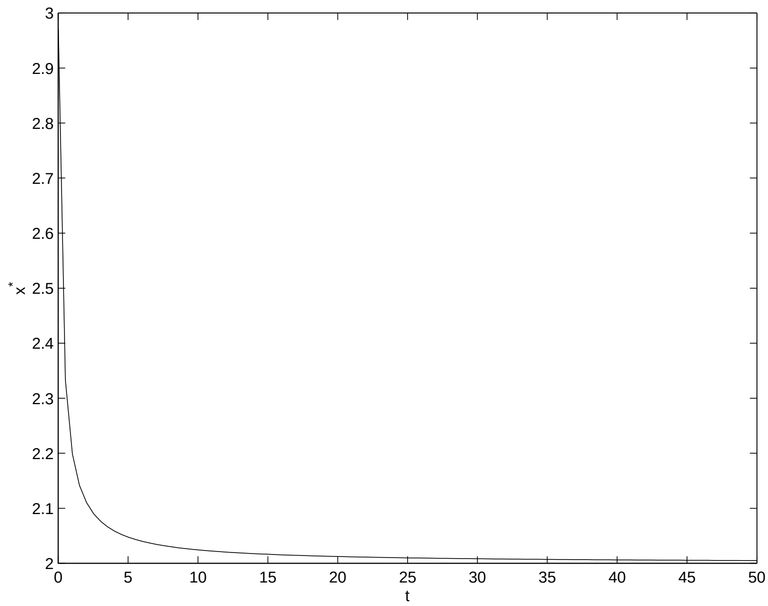

Figure 2: The optimal values for x as t increases.

Example: Analysis

- Given system

$$\dot{x}(t) = A(t)x(t), \qquad x(0) = x_0 \tag{9}$$

  where $A(t)$ can take on known values $A_1, \ldots, A_L$:

$$A(t) \in \{A_1, \ldots, A_L\}$$

  $A(t)$ may switch among the possible values

- Quadratic performance index

$$J = \int_0^\infty x^T(t)Qx(t)\,dt$$

  where $Q = Q^T \geq 0$.

- Objective: find or bound worst case value:

$$J_{wc} = \max \int_0^\infty x^T(t)Qx(t)\,dt$$

  where we take the maximum over all trajectories of (9)

- **Motivation:** One reason for studying such a linear time-varying system is suppose we have a nonlinear, time-varying system

$$\dot{z} = f(z,t), \qquad z(0) = z_0$$

  where $f(0,t) = 0$ and for all v

$$\nabla f(v) \in \mathrm{Co}\,\{A_1, \ldots, A_L\}$$

  where Co denotes the convex hull. Then J_{wc} for the linear time-varying system gives an upper bound on J_{wc} for the nonlinear system. Called *global linearization*.

Example: Analysis

- Use quadratic Lyapunov function $x^T P x$ to establish a bound on J_{wc}
- Suppose for some $P = P^T > 0$ we have
$$\frac{d}{dt}\left[x^T(t)Px(t)\right] \leq -x^T(t)Qx(t) \tag{10}$$
for all trajectories and all t (except switching times)
- Integrate from $t = 0$ to $t = T$ to get
$$x^T(T)Px(T) - x^T(0)Px(0) \leq -\int_0^T x^T(t)Qx(t)\,dt$$
- Note $x^T(T)Px(T) \geq 0$ and rearrange
$$\int_0^T x^T(t)Qx(t)\,dt \leq x_0^T P x_0$$
- Since this holds for all T, we have
$$J = \int_0^\infty x^T(t)Qx(t)\,dt \leq x_0^T P x_0$$
- This inequality holds for all trajectories, so we have
$$J_{wc} \leq x_0^T P x_0$$
for any $P > 0$ satisfying (10)
- Now we need to find a P to satisfy (10)

Example: Analysis

- Since
$$\frac{d}{dt}\left[x^T(t)Px(t)\right] = x^T(t)\left[A^T(t)P + PA(t)\right]x(t)$$
condition (10) is equivalent to
$$A_i^T P + PA_i + Q \le 0, \qquad i = 1, \ldots, L$$
which is a set of LMIs in P.

- To find a P to determine a bound on J_{wc}, solve LMI feasibility problem
$$\begin{array}{rl} \text{find} & P = P^T \\ \text{that satisfies} & P > 0, \quad A_i^T P + PA_i + Q \le 0, \quad i = 1, \ldots, L \end{array}$$

- Now optimize over Lyapunov function P to find smallest bound
$$\begin{array}{rl} \text{minimize} & x_0^T P x_0 \\ \text{subject to} & P > 0, \quad A_i^T P + PA_i + Q \le 0, \quad i = 1, \ldots, L \end{array}$$

- Final problem is an SDP in variable P, so we can solve using interior-point methods

Example: Synthesis

- Consider time-varying linear system, with input u, given by

$$\dot{x} = A(t)x(t) + B(t)u(t), \qquad x(0) = x_0 \tag{11}$$

  where

$$[A(t)\ B(t)] \in \{[A_1\ B_1], \ldots, [A_L\ B_L],\}$$

- Allow constant, linear state feedback gain K, so that $u = Kx$ and

$$\dot{x}(t) = [A(t) + B(t)K]\, x(t), \qquad x(0) = x_0$$

- Performance index:

$$J = \int_0^\infty \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]\, dt$$

  where $Q > 0$ and $R > 0$.

- Consider the worst case cost

$$\begin{aligned} J_{wc} &= \max \int_0^\infty \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]\, dt \\ &= \max \int_0^\infty \left[x^T(t)(Q + K^T RK)x(t)\right]\, dt \end{aligned}$$

  where the maximum is over all possible trajectories

- Problem: find a state feedback gain matrix K and a quadratic Lyapunov function P that minimizes the bound $x_0^T P x_0$ on J_{wc} for the closed-loop system

- Note: approach simultaneously synthesizes a controller and guarantees a performance bound

Example: Synthesis

- Pose this problem as

$$\begin{array}{ll} \text{minimize} & x_0^T P x_0 \\ \text{subject to} & (A_i + B_i K)^T P + P(A_i + B_i K) + Q + K^T R K \leq 0, \\ & i = 1, \ldots, L, \\ & P > 0 \end{array} \tag{12}$$

  where the variables are K and P

- This problem is not an SDP since the constraints are quadratic in K and there are products between P and K
- Transform to SDP using a change of variables
- Define new matrices Y and W as

$$Y = P^{-1}, \qquad W = KP^{-1}$$

  since $P > 0$ and $Y > 0$ we have

$$P = Y^{-1}, \qquad K = WY^{-1}$$

- Rewrite the inequality constraint:

$$(A_i + B_i W Y^{-1})^T Y^{-1} + Y^{-1}(A_i + B_i W Y^{-1}) + Q + K^T R K \leq 0$$

- Multiply left and right sides by Y to get

$$Y A_i^T + W^T B_i^T + A_i Y + B_i W + Y Q Y + W^T R W \leq 0$$

- Rewrite as

$$Y A_i^T + W^T B_i^T + A_i Y + B_i W + \begin{bmatrix} Y \\ W \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} Y \\ W \end{bmatrix} \leq 0 \tag{13}$$

Example: Synthesis

- Express the quadratic matrix inequality as a different linear matrix inequality using Schur complement:

$$\begin{aligned} L_i(Y,W) &:= \begin{bmatrix} -YA_i^T - W^TB_i^T - A_iY - B_iW & Y & W^T \\ Y & Q^{-1} & 0 \\ W & 0 & R^{-1} \end{bmatrix} \\ L_i(Y,W) &\geq 0 \end{aligned} \tag{14}$$

- Transformed original nonconvex matrix inequality (13) with variables P and K into a linear matrix inequality (14)
- Express $x_0^TPx_0 = x_0^TY^{-1}x_0 \leq \gamma$ as the LMI

$$\begin{bmatrix} \gamma & x_0^T \\ x_0 & Y \end{bmatrix} \geq 0$$

- Finally, we can solve original nonconvex problem (12) by solving

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & L_i(Y,W) \geq 0, \quad i = 1, \ldots, L, \\ & \begin{bmatrix} \gamma & x_0^T \\ x_0 & Y \end{bmatrix} \geq 0 \end{array} \tag{15}$$

  which is an SDP

- Shows how to synthesize state feedback gain matrix and quadratic Lyapunov function to establish a guaranteed performance bound for a time-varying system, or by global linearization, a nonlinear system

Example: $\mathcal{S}$-Procedure

- Linear system with uncertain, time-varying, bounded feedback
- System:

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \qquad |u_i(t)| \leq |y_i(t)|, \quad i = 1, \ldots, p \end{aligned}$$

- **Objective:** find an invariant ellipsoid, $\mathcal{E}$, such that $x(T) \in \mathcal{E}$ implies $x(t) \in \mathcal{E}$ for $t \geq T$
- Ellipsoid $\mathcal{E} = \{x \;:\; x^T P x \leq 1\}$, where $P = P^T > 0$ is invariant iff $V(t) = x^T(t) P x(t)$ is nonincreasing
- Take derivative of V:

$$\frac{d}{dt} V(x(t)) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{16}$$

- Express conditions $|u_i(t)| \leq |y_i(t)|$ as quadratic inequalities:

$$u_i^2(t) - y_i^2(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} -c_i^T c_i & 0 \\ 0 & E_{ii} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \leq 0, \tag{17}$$

  for $i = 1, \ldots, p$, where c_i is the ith row of C and E_{ii} is the matrix with all entries zero except the ii entry, which is 1
- $\mathcal{E}$ is invariant iff (16) holds whenever (17) holds

Example: $\mathcal{S}$-Procedure

- The general condition is that one quadratic inequality should hold whenever some other quadratic inequalities hold:

$$\text{for all } z \in \mathbb{R}^{l+p}, \quad z^T T_i z \le 0, \;\; i = 1, \dots, p \;\Rightarrow\; z^T T_0 z \le 0 \tag{18}$$

where

$$T_0 = \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix}, \;\; T_i = \begin{bmatrix} -c_i^T c_i & 0 \\ 0 & E_{ii} \end{bmatrix}, \;\; i = 1, \dots, p$$

- Verifying (18) is difficult — consider a sufficient condition:

$$\begin{aligned} \text{there exists} \quad & \tau_1 \ge 0, \dots, \tau_p \ge 0 \\ \text{such that} \quad & T_0 \le \tau_1 T_1 + \dots + \tau_p T_p \end{aligned} \tag{19}$$

- Replacing condition (18) with (19) is known as the $\mathcal{S}$-procedure
- Now let $D = \mathrm{diag}(\tau_1, \dots, \tau_p)$, to get sufficient condition for invariance of ellipsoid $\mathcal{E}$:

$$\begin{bmatrix} A^T P + PA + C^T DC & PB \\ B^T P & -D \end{bmatrix} \le 0 \tag{20}$$

- This is an LMI in variables $P = P^T$ and diagonal D
- Solve the semidefinite feasibility problem to find invariant ellipsoid

Example: Inverse Optimality

- Given a system
$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ z(t) &= \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ x(0) &= x_0 \end{aligned}$$
with (A, B) stabilizable, (Q, A) detectable, and $R > 0$
- LQR problem finds u to minimize performance index
$$J = \int_0^\infty z^T(t) z(t)\, dt$$
- Solution is state feedback with $u = Kx$ where $K = -R^{-1}B^T P$ and P is unique positive definite solution of ARE:
$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$
- Inverse optimal control problem is given a gain K, determine if there exist $Q \geq 0$ and $R > 0$ with (Q, A) detectable such that $u(t) = Kx(t)$ is optimal control
- Formulate as an LMI
- Find $R > 0$ and $Q \geq 0$ such that there exists a positive P and a positive-definite W satisfying:
$$(A + BK)^T P + P(A + BK) + K^T RK + Q = 0$$
$$A^T W + WA < Q$$
$$B^T P + RK = 0$$
which is an LMI in P, W, R and Q

Summary

- Linear matrix inequalities can be used to represent several types of control problems
- There are efficient numerical techniques to solve LMIs
- Computational and algorithmic advances brought a solution to this historically significant problem
- Still an active area of research
- Many resources available on-line (papers, software, course notes, examples)
- First stop: `http://www.stanford.edu/~boyd/index.shtml`

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