DISC Course on Linear Matrix Inequalities in Control

Siep Weiland
Department of Electrical Engineering
Eindhoven University of Technology

Course 2004 - Class 2

Linear Matrix Inequalities (LMI's)

A **linear matrix inequality (LMI)** is an expression

$$F(x) = F_0 + x_1 F_1 + \ldots + x_n F_n \prec 0$$

where

- $x = \mathrm{col}(x_1, \ldots, x_n)$ is a vector of reals, the **decision variables**,
- $F_i = F_i^\top$ are real symmetric matrices and
- $\prec 0$ means negative definite, i.e.,

$$\begin{aligned} F(x) \prec 0 \quad &\Leftrightarrow \quad z^\top F(x) z < 0 \text{ for all } z \neq 0 \\ &\Leftrightarrow \quad \text{all eigenvalues of } F(x) \text{ are negative} \\ &\Leftrightarrow \quad \lambda_{\max}(F(x)) < 0 \end{aligned}$$

Note that F is an **affine function** of the decision variables.

Simple examples

- $1 + x < 0$
- $1 + x_1 + 2x_2 < 0$
- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \prec 0.$

Comments

- Only very simple cases can be treated analytically.
- Need to resort to numerical techniques!
- The LMI's $F(x) \preceq 0$, $F(x) \succ 0$ and $F(x) \succeq 0$ are similarly defined.
- If $F(x)$ is **linear** in x, then

  $$F(x) \prec 0 \text{ implies } F(\alpha x) \prec 0 \text{ for all } \alpha > 0.$$

  may cause numerical trouble.
- Similar if F_j are **complex** with $F_j = \bar{F}_j^\top =: F_j^*$ (that is **Hermitian**).

Main LMI problems

The LMI feasibility problem:
Test whether there exists $x_1, \ldots, x_n$ such that $F(x) \prec 0$.

The LMI optimization problem:
Minimize $c_1 x_1 + \ldots + c_n x_n$ over all $x_1, \ldots, x_n$ that satisfy $F(x) \prec 0$.

How is this solved?
$F(x) \prec 0$ is feasible if and only if $\min_x \lambda_{\max}(F(x)) < 0$ and therefore involves minimizing the function

$$f : x \mapsto \lambda_{\max}(F(x))$$

Possible because this function is convex!

There exist efficient algorithms for this (interior point, ellipsoid).

More general definition of LMI

More general:

A **linear matrix inequality** is an inequality

$$F(x) \prec 0$$

where $F : \mathcal{X} \to \mathbb{S}$ is an **affine function**, $\mathcal{X}$ finite dimensional, $\mathbb{S}$ set of real symmetric matrices.

- Allows defining **matrix valued LMI's**.
- F affine means $F(x) = F_0 + T(x)$ with T a linear map (a matrix).
- With $\{e_j\}_{j=1}^n$ basis of $\mathcal{X}$, any $x \in \mathcal{X}$ can be expanded as $x = \sum_{j=1}^n x_j e_j$ so that

$$F(x) = F_0 + T(x) = F_0 + \sum_{j=1}^{n} x_j F_j \qquad \text{with } F_j = T(e_j)$$

used with $\mathcal{X} = \mathbb{R}^{m_1 \times m_2}$

Matrix valued LMI's

- $\mathcal{X} = \mathbb{R}^{m_1 \times m_2}$ with standard basis $E_1, \ldots, E_{m_1 m_2}$ of $m_1 \times m_2$ matrices having one 1 at one of its entries. Then

$$X = \sum_{j=1}^{n} x_j E_j; \qquad F(X) = F_0 + \sum_{j=1}^{n} x_j F_j; \qquad n = m_1 m_2$$

- $\mathcal{X} = \mathbb{S}^m$ set of $m \times m$ real symmetric matrices with basis E_j, $j = 1, \ldots n$. Then

$$X = \sum_{j=1}^{n} x_j E_j; \qquad F(X) = F_0 + \sum_{j=1}^{n} x_j F_j; \qquad n = m(m+1)/2$$

- **Example:** $F : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ defines LMI

$$\boxed{F(X) = A^\top X + XA + Q \prec 0}$$

for any matrix A and $Q \in \mathbb{S}^n$.

Why are LMI's interesting?

Reason 1:

LMI's define **convex constraints** on x, i.e., $\mathcal{S} := \{x \mid F(x) \prec 0\}$ is convex.

Indeed, $F(\alpha x_1 + (1-\alpha)x_2) = \alpha F(x_1) + (1-\alpha)F(x_2) \prec 0.$

Why are LMI's interesting?

Reason 1:

LMI's define **convex constraints** on x, i.e., $\mathcal{S} := \{x \mid F(x) \prec 0\}$ is convex.

Indeed, $F(\alpha x_1 + (1-\alpha)x_2) = \alpha F(x_1) + (1-\alpha)F(x_2) \prec 0.$

Reason 2:

Solution set of system of k individual LMI's

$$F_1(x) \prec 0, \ldots, F_k(x) \prec 0$$

is convex and representable as **one single LMI**

$$F(x) = \begin{pmatrix} F_1(x) & 0 & \ldots & 0 \\ 0 & \ddots & & 0 \\ 0 & 0 & \ldots & F_k(x) \end{pmatrix} \prec 0$$

Allows to combine LMI's!!
Single LMI constraint is equally good as multiple LMI constraint.

Reason 3: affine constraint elimination

We can lump the combined constraints

- $F(x) \prec 0$ and $Ax = a$
- $F(x) \prec 0$ and $x = By + b$ for some y
- $F(x) \prec 0$ and $x \in \mathcal{M}$ with $\mathcal{M}$ an **affine set**.

in one LMI.

How?

Reason 3: affine constraint elimination

We can lump the combined constraints

- $F(x) \prec 0$ and $Ax = a$
- $F(x) \prec 0$ and $x = By + b$ for some y
- $F(x) \prec 0$ and $x \in \mathcal{M}$ with $\mathcal{M}$ an **affine set**.

in one LMI.

How?

Write $\mathcal{M} = x_0 + \mathcal{M}_0$, $\mathcal{M}_0$ a linear subspace with basis $\{e_j\}_{j=1}^m$. Then, with $x \in \mathcal{M}$ and $F(x) = F_0 + T(x)$ with T linear:

$$F(x) = F_0 + T\left(x_0 + \sum_{j=1}^m x_j e_j\right) = \underbrace{F_0 + T(x_0)}_{\text{constant}} + \underbrace{\sum_{j=1}^m x_j T(e_j)}_{\text{linear}}$$

$$= \widetilde{F}_0 + x_1 \widetilde{F}_1 + \ldots + x_m \widetilde{F}_m = \widetilde{F}(\widetilde{x})$$

Note $\widetilde{x}$ unconstrained and $\widetilde{x}$ has lower dimension than x !!

Reason 4: the Schur lemma

Theorem: The symmetric matrix $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ is negative definite **if and only if**

$$M_{11} \prec 0 \quad \text{and} \quad M_{22} - M_{21}M_{11}^{-1}M_{12} \prec 0$$

if and only if

$$M_{22} \prec 0 \quad \text{and} \quad M_{11} - M_{12}M_{22}^{-1}M_{21} \prec 0$$

Matrices

$$S' := M_{22} - M_{21}M_{11}^{-1}M_{12}, \qquad S'' := M_{11} - M_{12}M_{22}^{-1}M_{21}$$

are called **Schur complements**.

Reason 4: the Schur lemma

Theorem: The symmetric matrix $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ is negative definite

if and only if

$$M_{11} \prec 0 \quad \text{and} \quad M_{22} - M_{21}M_{11}^{-1}M_{12} \prec 0$$

if and only if

$$M_{22} \prec 0 \quad \text{and} \quad M_{11} - M_{12}M_{22}^{-1}M_{21} \prec 0$$

Why?

$$\begin{pmatrix} I & 0 \\ -M_{11}^{-1}M_{12} & I \end{pmatrix}^\top \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ -M_{11}^{-1}M_{12} & I \end{pmatrix} = \begin{pmatrix} M_{11} & 0 \\ 0 & S' \end{pmatrix}$$

is of the form

$T^\top MT$ with T nonsingular,

a **congruence transformation** which won't change the signs of eigenvalues.

Allows to linearize **some** nonlinear constraints into LMI's

Theorem: Let F be an affine function with

$$F(x) = \begin{pmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{pmatrix}, \qquad F_{11}(x) \text{ is square.}$$

Then

$$F(x) \prec 0 \quad \Longleftrightarrow \quad \begin{cases} F_{11}(x) \prec 0 \\ F_{22}(x) - F_{21}(x)\left[F_{11}(x)\right]^{-1} F_{12}(x) \prec 0. \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} F_{22}(x) \prec 0 \\ F_{11}(x) - F_{12}(x)\left[F_{22}(x)\right]^{-1} F_{21}(x) \prec 0 \end{cases} .$$

Question: Linearize

$$F(X) := A^\top X + XA + XBR^{-1}B^\top X + Q \prec 0, \qquad R \succ 0$$

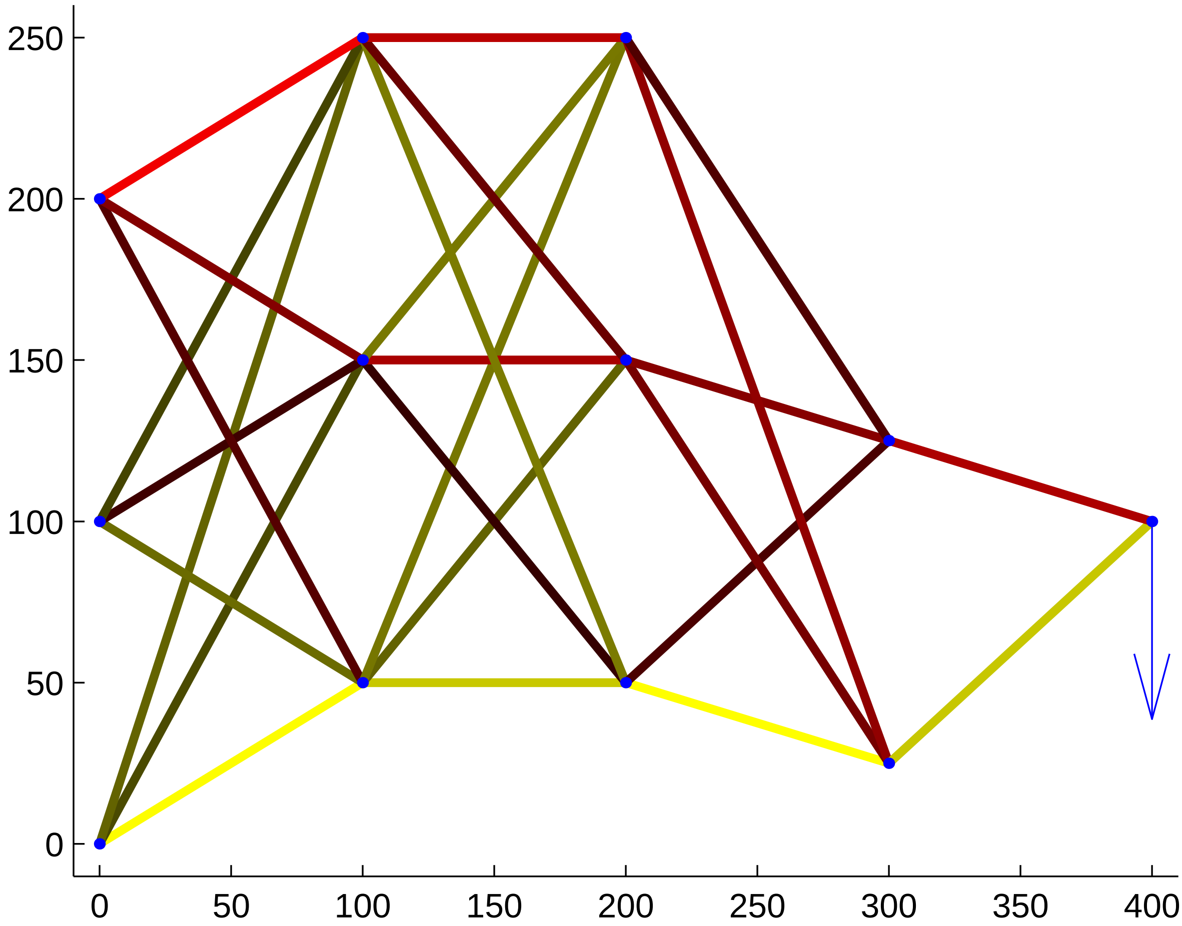
An LMI? A convex constraint on X?

What are LMI's good for?

- Many engineering optimization problems can be **translated** into LMI problems
- Various computationally difficult optimization problems can be effectively **approximated** by LMI problems
- In practice, description of data is affected by uncertainty. **Robust optimization** problems can be either translated or approximated by standard LMI problems.

Essential topic of this course:
How to translate/approximate a given (uncertain) optimization problem into/by an LMI problem?

Truss topology design



Trusses

- Trusses consist of straight members ('bars') connected at **joints**.
- One distinguishes **free** and **fixed joints**.
- Connections at the joints can rotate.
- The loads (or the weights) are assumed to be applied at the free joints.
- This implies that all internal forces are directed **along the members**, (so no bending forces occur).
- Construction reacts based on principle of statics: the sum of the forces in any direction, or the moments of the forces about any joint, are zero.
- This results in a **displacement** of the joints and a new **tension distribution** in the truss.

More on trusses

Many applications (roofs, cranes, bridges, space structures, ...)!!

Design your own bridge

Truss topology design

Problem features:

- Connect nodes by N bars of length $\ell = \mathrm{col}(\ell_1, \ldots, \ell_N)$ (fixed) and cross sections $s = \mathrm{col}(s_1, \ldots, s_N)$ (to be designed)
- Impose **bounds** on cross sections $a_k \leq s_k \leq b_k$ and total volume $\ell^\top s \leq v$ (and hence an upperbound on total weight of the truss). Let $a = \mathrm{col}(a_1, \ldots, a_N)$ and $b = \mathrm{col}(b_1, \ldots, b_N)$.
- Distinguish **fixed** and **free** nodes.
- Apply external forces $f = \mathrm{col}(f_1, \ldots f_M)$ to some free nodes. These result in a node displacements $d = \mathrm{col}(d_1, \ldots, d_M)$.

  Mechanical model defines relation $A(s)d = f$ where $A(s) \succeq 0$ is the **stiffness matrix** which depends linearly on s.

Goal:

Maximize stiffness or, equivalently, minimize elastic energy $f^\top d$

Truss topology design

Problem: Find $s \in \mathbb{R}^N$ which minimizes elastic energy $f^\top d$ subject to the constraints

$$A(s) \succ 0, \qquad A(s)d = f, \qquad a \leq s \leq b, \qquad \ell^\top s \leq v$$

- **Data:** Total volume $v > 0$, node forces f, bounds a, b, lengths ℓ and symmetric matrices $A_1, \ldots, A_N$ that define the linear stiffness matrix $A(s) = s_1 A_1 + \ldots + s_N A_N$.
- **Decision variables:** Cross sections s and displacements d (both vectors).
- **Cost function:** stored elastic energy $d \mapsto f^\top d$.
- **Constraints:**
  - ◇ **Semi-definite constraint:** $A(s) \succ 0$
  - ◇ **Non-linear equality constraint:** $A(s)d = f$
  - ◇ **Linear inequality constraints:** $a \leq s \leq b$ and $\ell^\top s \leq v$.

From truss topology design to LMI's

- First **eliminate affine equality constraint** $A(s)d = f$:

$$
\begin{array}{ll}
\text{minimize} & f^\top \left(A(s)\right)^{-1} f \\
\text{subject to} & A(s) \succ 0, \quad \ell^\top s \leq v, \quad a \leq s \leq b
\end{array}
$$

From truss topology design to LMI's

- First **eliminate affine equality constraint** $A(s)d = f$:

$$\begin{array}{ll} \text{minimize} & f^\top (A(s))^{-1} f \\ \text{subject to} & A(s) \succ 0, \quad \ell^\top s \le v, \quad a \le s \le b \end{array}$$

- **Push objective to constraints** with auxiliary variable γ:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & \gamma > f^\top (A(s))^{-1} f, \quad A(s) \succ 0, \quad \ell^\top s \le v, \quad a \le s \le b \end{array}$$

From truss topology design to LMI's

- First **eliminate affine equality constraint** $A(s)d = f$:

$$\begin{array}{ll} \text{minimize} & f^\top \left(A(s)\right)^{-1} f \\ \text{subject to} & A(s) \succ 0, \quad \ell^\top s \le v, \quad a \le s \le b \end{array}$$

- **Push objective to constraints** with auxiliary variable γ:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & \gamma > f^\top \left(A(s)\right)^{-1} f, \quad A(s) \succ 0, \quad \ell^\top s \le v, \quad a \le s \le b \end{array}$$

- **Apply Schur lemma** to linearize

From truss topology design to LMI's

- First **eliminate affine equality constraint** $A(s)d = f$:

$$\begin{array}{ll} \text{minimize} & f^\top (A(s))^{-1} f \\ \text{subject to} & A(s) \succ 0, \quad \ell^\top s \le v, \quad a \le s \le b \end{array}$$

- **Push objective to constraints** with auxiliary variable γ:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & \gamma > f^\top (A(s))^{-1} f, \quad A(s) \succ 0, \quad \ell^\top s \le v, \quad a \le s \le b \end{array}$$

- **Apply Schur lemma** to linearize:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & \begin{pmatrix} \gamma & f^\top \\ f & A(s) \end{pmatrix} \succ 0, \quad \ell^\top s \le v, \quad a \le s \le b \end{array}$$

Note that the latter is an **LMI optimization problem** as all constraints on s are formulated as LMI's!!

Yalmip coding for LMI optimization problem

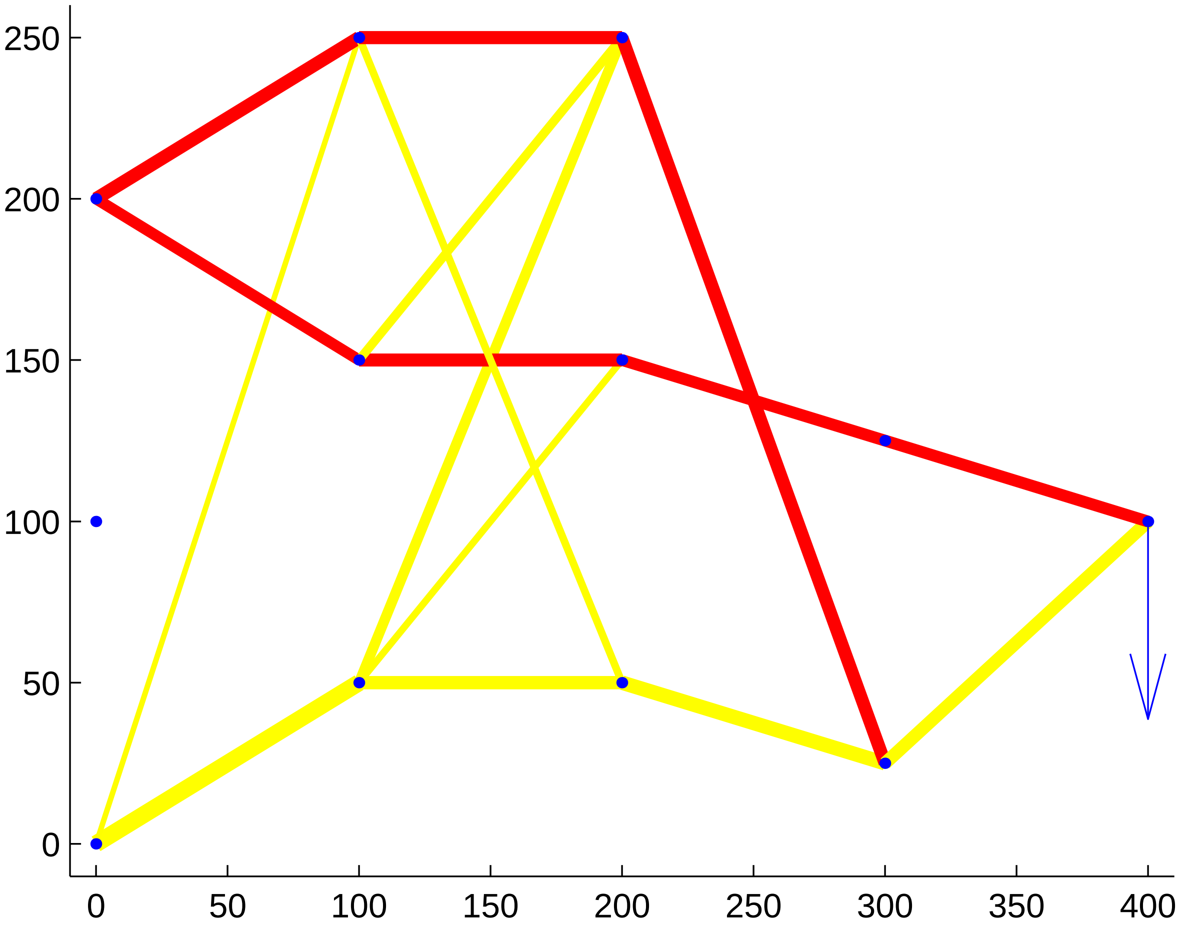
Equivalent LMI optimization problem:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & \begin{pmatrix} \gamma & f^\top \\ f & A(s) \end{pmatrix} \succ 0, \quad \ell^\top s \le v, \quad a \le s \le b \end{array}$$

The following YALMIP code solves this problem:

```
gamma=sdpvar(1,1); x=sdpvar(N,1,'full');
lmi=set([gamma f'; f A*diag(x)*A']);
lmi=lmi+set(l'*x<=v);
lmi=lmi+set(a<=x<=b);
options=sdpsettings('solver','csdp');
solvesdp(lmi,gamma,options); s=double(x);
```

Result: optimal truss



Useful software:

General purpose MATLAB interface **Yalmip**

- Free code developed by J. Löfberg accessible here

  Get Yalmip now

  Run `yalmipdemo.m` for a comprehensive introduction.
  Run `yalmiptest.m` to test settings.

- Yalmip uses the usual Matlab syntax to define optimization problems.
  Basic commands `sdpvar`, `set`, `sdpsettings` and `solvesdp`.
  Very easy to use!!!

- Yalmip needs to be connected optimization solver for semi-definite programming.
  There exist many solvers:

CSDP	GLPK	OOQP
DSDP	SeDuMi	MOSEK

- Alternative Matlab's **LMI toolbox** for dedicated control applications.

Stability

Consider stability of dynamical systems in state space form

$$\dot{x}(t) = f(x(t)), \qquad x(t_0) = x_0$$

A point x^* is a **fixed point** if $x(t) = x^*$ for all $t > t_0$ whenever $x_0 = x^*$.

A fixed point x^* is **exponentially stable** if there exists $M > 0$ and $\alpha > 0$ such that

$$\|x(t) - x^*\| \leq \|x_0 - x^*\| M e^{-\alpha(t-t_0)} \text{ for all } x_0 \text{ and } t \geq t_0$$

- constant M and the decay rate α do not depend on t_0 and x_0.
- There exist **many** refinements to this definition.
- **Lyapunov theory** provides well established tools to verify stability.

Stability of linear systems

Theorem: The origin of the linear system

$$\dot{x}(t) = Ax(t)$$

is exponentially stable **if and only if** there exists X such that

$$X \succ 0 \qquad A^\top X + XA \prec 0.$$

Leads to combined LMI

$$F(X) := \begin{pmatrix} -X & 0 \\ 0 & A^\top X + XA \end{pmatrix} \prec 0$$

Notes:

- Stability verified as an **LMI feasibility problem**
- Stay with us to see generalization to different stability notions.
- $V(x) := x^\top X x$ will qualify as Lyapunov function.

Relation to Lyapunov functions

Suppose $F(X) \prec 0$. Then there exists $\varepsilon > 0$ such that

$$A^\top X + XA + \varepsilon X \prec 0.$$

Define

$$V(x) := x^\top X x$$

Then, for all $t \in \mathbb{R}$

$$\begin{aligned}\frac{d}{dt}V(x(t)) + \varepsilon V(x(t)) &= x^\top(t)[A^\top X + XA]x(t) + \varepsilon x^\top(t)Xx(t) \\ & x^\top(t)[A^\top X + XA + \varepsilon X]x(t) \leq 0\end{aligned}$$

After integration, this yields for all $t \geq t_0$,

$$x^\top(t)Xx(t) \quad \leq \quad x^\top(t_0)Xx(t_0)e^{-\varepsilon t}$$

Now use that $\lambda_{\min}(X)\|x\|^2 \leq x^\top X x \leq \lambda_{\max}(X)\|x\|^2$ to infer

$$\boxed{\|x(t)\|^2 \quad \leq \quad \|x(t_0)\|^2 \frac{\lambda_{\max}(X)}{\lambda_{\min}(X)} e^{-\varepsilon t}}$$

Algebraic proof

Sufficiency: Suppose $F(X) \prec 0$. Let λ be an eigenvalue of A, and $Ax = \lambda x$ with $x \neq 0$. Then

$$x^* \left[A^\top X + XA\right] x = (\lambda + \lambda^*) x^* X x = 2\,\mathrm{Re}(\lambda) x^* X x < 0$$

and hence $\mathrm{Re}(\lambda) < 0$. Conclude $\lambda(A) \subset \mathbb{C}^-$.

Necessity: Suppose $\lambda(A) \subset \mathbb{C}^-$ and assume (first) that for some (complex) nonsingular basis transformation T we have that $TAT^{-1} = \Lambda$ is diagonal. Then

$$\Lambda^* + \Lambda = (T^*)^{-1} A^\top T^* + TAT^{-1} \prec 0$$

Now apply a congruence transformation $T^*[\ldots]T$ to obtain

$$\boxed{A^\top T^* T + T^* T A \prec 0}$$

Hence $X = T^* T \succ 0$ will do!

Joint stabilization

Given $(A_1, B_1), \ldots, (A_k, B_k)$, find F such that $(A_1+B_1F), \ldots, (A_k+B_kF)$ asymptotically stable.

Equivalent to finding $F, X_1, \ldots, X_k$ such that

$$\begin{cases} X_j \succ 0 \\ (A_j + B_jF)X_j + X_j(A_j + B_jF)^\top \prec 0 \end{cases}$$

Not an LMI!!

Sufficient condition: $X = X_1 = \ldots = X_k$ yields

$$X \succ 0 \qquad \text{and} \qquad (A_j + B_jF)X + X(A_j + B_jF)^\top \prec 0$$

which is equivalent to

$$\begin{aligned} X \succ 0 \qquad &\text{and} \qquad A_jX + XA_j^\top + B_jK + K^\top B_j^\top \prec 0 \\ &\text{and} \qquad K = FX \end{aligned}$$

μ analysis

Problem: Given a matrix M, find a diagonal matrix D such that

$$\|DMD^{-1}\| < 1$$

For the insiders: problem occurs in μ analysis.

Since

$$\begin{aligned} \|DMD^{-1}\| < 1 \quad &\iff \quad D^{-\top}M^\top D^\top DMD^{-1} \prec I \\ &\iff \quad M^\top \underbrace{D^\top D}_{X} M \prec D^\top D \\ &\iff \quad M^\top X M - X \prec 0 \end{aligned}$$

with $X = D^\top D$. This is an LMI feasibility problem

$$F(X) = \begin{pmatrix} -X & 0 \\ 0 & M^\top X M - X \end{pmatrix} \prec 0$$

where $F : \mathcal{X} \to \mathbb{S}$ is affine and $\mathcal{X}$ the set of real diagonal matrices.

Eigenvalue problem

Problem: Given an affine function $F : \mathcal{X} \to \mathbb{S}$, minimize over all $x \in \mathcal{X}$

$$f(x) = \lambda_{\max}\left(F(x)\right).$$

Push objective to constraints:

$$\begin{aligned} f(x) < \gamma \quad &\iff \quad \lambda_{\max}(F^\top(x)F(x)) < \gamma^2 \quad \iff \quad F^\top(x)F(x) - \gamma^2 I < 0 \\ &\iff \quad \begin{pmatrix} \gamma I & F(x) \\ F^\top(x) & \gamma I \end{pmatrix} \succ 0 \end{aligned}$$

Define

$$y := \begin{pmatrix} \gamma \\ x \end{pmatrix}; \quad G(y) := -\begin{pmatrix} \gamma I & F(x) \\ F^\top(x) & \gamma I \end{pmatrix}; \quad g(y) := \gamma$$

then G affine in y and $\min_x f(x) = \min_{G(y) \prec 0} g(y)$.

Solvable by ellipsoid algorithm!

Duality

Multi design objective

Suppose $\mathcal{X}$ set of rational transfer functions of controlled systems.

Multiple cost functions $f_j : \mathcal{X} \to \mathbb{R}$, $j = 1, \ldots K$ define feasible systems

$$\mathcal{S}_j^{\gamma_j} := \{x \mid f_j(x) \leq \gamma_j\}, \qquad j = 1, \ldots K$$

and multi-criterion specification

$$\boxed{\mathcal{S}^{\gamma} = \mathcal{S}_1^{\gamma_1} \cap \mathcal{S}_2^{\gamma_2} \cap \ldots \cap \mathcal{S}_K^{\gamma_K}}$$

for some multi-index $\gamma = (\gamma_1, \ldots, \gamma_K)$.

Call $\gamma^* \in \mathbb{R}^K$ **Pareto optimal** if $\mathcal{S}^{\gamma}$ is feasible for $\gamma > \gamma^*$ and infeasible for $\gamma < \gamma^*$. A point $x^* \in \mathcal{S}^{\gamma^*}$ (if exists) is called a **Pareto optimal solution**.

Can we characterize Pareto optimality in multi-objective control?

From multi-objective to single objective optimization

That is, convert

$$\mathcal{S}^{(\gamma_1,\ldots,\gamma_K)} = \mathcal{S}_1^{\gamma_1} \cap \ldots \cap \mathcal{S}_K^{\gamma_K} \qquad \longrightarrow \qquad \mathcal{S}^{\gamma_0}, \quad \gamma_0 \in \mathbb{R}$$

where $\mathcal{S}^{\gamma_0} = \{x \in \mathcal{X} \mid f(x) \leq \gamma_0\}$.

But what's f??

From multi-objective to single objective optimization

That is, convert

$$\mathcal{S}^{(\gamma_1,\ldots,\gamma_K)} = \mathcal{S}_1^{\gamma_1} \cap \ldots \cap \mathcal{S}_K^{\gamma_K} \qquad \longrightarrow \qquad \mathcal{S}^{\gamma_0}, \quad \gamma_0 \in \mathbb{R}$$

where $\mathcal{S}^{\gamma_0} = \{x \in \mathcal{X} \mid f(x) \leq \gamma_0\}$.

But what's f??

Two basic strategies:

- **weighted sum objective:**

$$f_\lambda^{\text{sum}}(x) := \lambda_1 f_1(x) + \cdots + \lambda_K f_K(x)$$

- **weighted maximum objective:**

$$f_\lambda^{\max}(x) := \max\left(\lambda_1 f_1(x), \cdots, \lambda_K f_K(x)\right)$$

The weighted sum objective

For weights $\lambda_i \geq 0$ define **weighted sum objective**:

$$\boxed{f_\lambda^{\mathrm{sum}}(x) := \lambda_1 f_1(x) + \cdots + \lambda_K f_K(x)}$$

where $\lambda = \mathrm{col}(\lambda_1, \ldots, \lambda_K) \geq 0$ is **weight vector**.

Leads to **single-objective optimal value**

$$\boxed{\gamma^{\mathrm{sum}}(\lambda) = \inf_{x \in \mathcal{X}} f_\lambda^{\mathrm{sum}}(x)}$$

Notes:

- Nice geometric interpretation: See diagrams.
- $\gamma^{\mathrm{sum}}(\lambda)$ is **Lagrange dual cost function** and satisfies

  $$\gamma^{\mathrm{sum}}(\lambda) = \inf\{\lambda^\top \gamma \mid \mathcal{S}^\gamma \text{ is feasible }\}$$

  Geometry of equal cost if $\lambda^\top \gamma' = \lambda^\top \gamma''$. That is, if $\lambda^\top(\gamma' - \gamma'') = 0$ or, equivalently, if $\lambda \perp (\gamma' - \gamma'')$.

Theorem: For all weighting vectors $\lambda \geq 0$, the optimal value $\gamma^{\text{sum}}(\lambda)$ defines a hyper-plane

$$\{\gamma \mid \langle \lambda, \gamma \rangle = \lambda^\top \gamma = \gamma^{\text{sum}}(\lambda)\}$$

that is **tangent** to the set of Pareto optimal points. Stated otherwise, for all $\lambda \geq 0$ there exists a Pareto optimal multi-index $\gamma^* = \text{col}(\gamma_1^*, \ldots \gamma_K^*)$ that satisfies

$$\langle \lambda, \gamma^* \rangle = \gamma^{\text{sum}}(\lambda).$$

No converse!

That is: not every Pareto optimal specification lies on the hyperplane

$$\{\gamma \mid \langle \lambda, \gamma \rangle = \lambda^\top \gamma = \gamma^{\text{sum}}(\lambda)\}$$

When does it??

Theorem: If f_j are convex $j = 1, \ldots, K$, then all design specifications $\mathcal{S}_1^{\gamma_1}, \ldots, \mathcal{S}_K^{\gamma_K}$ are convex and the feasible region

$$\Gamma := \{\gamma \in \mathbb{R}^K \mid \mathcal{S}^\gamma := \mathcal{S}_1^{\gamma_1} \cap \ldots \cap \mathcal{S}_K^{\gamma_K} \text{ is non-empty } \}$$

is convex.

Important consequence:

Theorem: Under the above conditions, **every** Pareto optimal specification is obtained by the minimization of the weighted sum

$$f_\lambda^{\text{sum}}(x) = \sum_{i=1}^{K} \lambda_i f_j(x)$$

for some $\lambda \geq 0$.

Allows duality!!

Dual optimization

Consider dual function

$$\boxed{\gamma^{\text{sum}}(\lambda) = \inf_{x\in\mathcal{X}} f_\lambda^{\text{sum}}(x) = \inf\{\lambda^\top\gamma \mid \mathcal{S}^\gamma \text{ non-empty}\}}$$

Some observations:

- $\gamma^{\text{sum}}(\lambda)$ is a **concave** function of λ.
- if multi-index γ is such that $\gamma^{\text{sum}}(\lambda) > \lambda^\top\gamma$ for some λ then the multi-objective specification

  $$\mathcal{S}^\gamma = \mathcal{S}_1^{\gamma_1} \cap \ldots \cap \mathcal{S}_K^{\gamma_K}$$

  is **infeasible** (immediate from above equality).
- if feasibility region Γ is convex and closed then the last issue characterizes **all feasible and infeasible specifications** via duality!

Duality theorem

Theorem: If the region Γ of feasible specifications is closed and convex, then the following are equivalent:

1. $\gamma \in \Gamma$.
2. there is no $\lambda \geq 0$ for which $\gamma^{\text{sum}}(\lambda) \; > \; \lambda^\top \gamma$.
3. for all $\lambda \geq 0$ we have $\gamma^{\text{sum}}(\lambda) - \lambda^\top \gamma \leq 0$.
4. the optimization problem

$$D_{\text{opt}} := \sup_{\lambda \geq 0} \left(\gamma^{\text{sum}}(\lambda) - \lambda^\top \gamma \right)$$

   has value function $D_{\text{opt}} \leq 0$.

Some important observations:

- Completely characterizes all feasible multi-indeces $\gamma = (\gamma_1, \ldots, \gamma_K)$.

Some important observations (ctd.):

- $\gamma^\top \lambda$ is **maximum cost** of $f_\lambda^{\text{sum}}(x)$ when $x \in \mathcal{S}^\gamma$.
- Dual cost criterion in item 3 can therefore be interpreted as

$$\boxed{\gamma^\top \lambda - \gamma^{\text{sum}}(\lambda) = \sup_{x \in \mathcal{S}^\gamma} f_\lambda^{\text{sum}}(x) - \inf_{x \in \mathcal{S}^\gamma} f_\lambda^{\text{sum}}(x)}$$

  which is obviously ≥ 0. Represents the **design freedom** in accepting bounds γ and accepting weights λ.

- Since $-\gamma^{\text{sum}}(\lambda)$ is convex, the optimization in item 4 is a convex optimization problem. Moreover, an optimal solution λ_{opt} always exist. (The $\sup$ in item 4 is therefore a $\max$).
- If cost functionals f_j are convex then Γ is convex.

Example: Linear Quadratic Control

Given the controllable system $\dot{x} = Ax + Bu$ with $x(0) = x_0$, find a Pareto optimal feedback $u = Fx$ such that the controlled system is stable, its state $\|x\|_2 \leq \gamma_x$ and the control effort $\|u\|_2 \leq \gamma_u$.

Let q and r denote two nonnegative weights and consider the weighted sum control problem

$$\gamma^{\text{sum}}(q, r) = \inf \int_0^\infty q\|x(t)\|^2 + r\|u(t)\|^2 = \inf q\|x\|_2^2 + r\|u\|_2^2$$

Yields optimal control and optimal value:

$$F_{q,r} = -\frac{1}{r}BX, \qquad A^\top X + XA - \frac{1}{r}XBB^\top X + qI = 0$$
$$\gamma^{\text{sum}}(q, r) = x_0^\top X_{q,r} x_0$$

Duality theorem promises that (γ_x, γ_u) is feasible **if and only if**

$$D_{\text{opt}} := \max_{q,r} \left(x_0^\top X_{q,r} x_0 - q\gamma_x - r\gamma_u \right) \leq 0$$

Summary of this class

- We introduced LMI's as inequalities on matrix valued affine functions.
- We related a feasibility and optimization problem to LMI's.
- LMI's define convex constraints on decision variables.
- We considered an elimination property of affine constraints and a linearization property by Schur complements.
- Applications on truss topology design and stability of dynamical systems.
- We derived first results on duality through geometric arguments.
- The duality theorem.

Gallery

Joseph-Louis Lagrange (1736) Aleksandr Mikhailovich Lyapunov (1857)

Next class