Robust and LPV Control

- Robust controller design
- Parameter Elimination and Dualization
- LPV controller synthesis with multipliers
- An illustrative missile example

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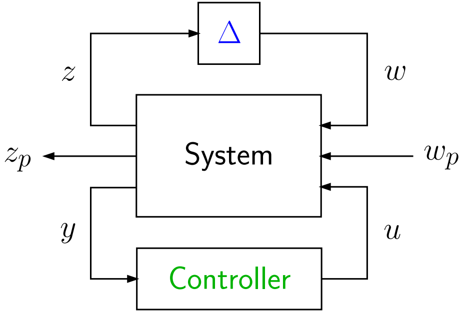
Configuration for Robust Controller Synthesis

Design controller guaranteeing:

- robust stability
- robustly desired performance specification on $w_p \to z_p$.



Consider following approach:

- Use robust performance characterization with **multipliers**
- Try to satisfy the multiplier characterization with suitable controller

Just for notational simplicity concentrate on **robust stabilization**.
Consider time-varying parametric uncertainty and quadratic stability.

System Descriptions

Uncontrolled LTI part:

$$\begin{aligned} \dot{x} &= Ax + B_1 w + Bu \\ z &= C_1 x + D_1 w + Eu \\ y &= Cx + Fw \end{aligned}$$

Controller:

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y \end{aligned}$$

Controlled LTI part:

$$\begin{aligned} \dot{\xi} &= \mathcal{A}\xi + \mathcal{B}w \\ z &= \mathcal{C}\xi + \mathcal{D}w \end{aligned}$$

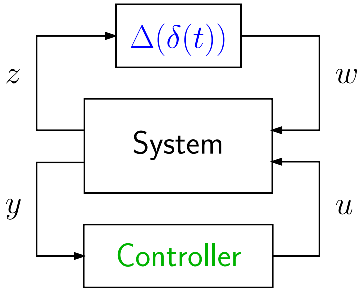
Uncertainty: $w(t) = \Delta(\delta(t))z(t)$.

w: uncertainty input

z: uncertainty output

u: control input

y: measured output



Robust Stability Analysis Inequalities

Assume $\delta(t) \in \boldsymbol{\delta} = \mathrm{co}\{\delta^1, \ldots, \delta^N\}$ (polytope) containing zero.

Robust stability guaranteed if exist $\mathcal{X}$ and Q, R, S with

$$Q \prec 0, \quad \begin{pmatrix} \Delta(\delta^k) \\ I \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta(\delta^k) \\ I \end{pmatrix} \succ 0, \quad k = 1, \ldots, N$$

$$\mathcal{X} \succ 0, \quad \left(\begin{array}{cc} I & 0 \\ \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \hline 0 & I \\ \mathcal{C} & \mathcal{D} \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^T & R \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \hline 0 & I \\ \mathcal{C} & \mathcal{D} \end{array}\right) \prec 0.$$

Apply standard procedure to step from analysis to synthesis.

Robust Synthesis Inequalities

Exists controller guaranteeing robust stability if exist v, Q, R, S:

$$Q \prec 0, \quad \begin{pmatrix} \Delta(\delta^k) \\ I \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta(\delta^k) \\ I \end{pmatrix} \succ 0, \quad k = 1, \ldots, N$$

$$\boldsymbol{X}(v) \succ 0, \quad \left(\begin{array}{cc} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \hline 0 & I \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^T & R \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \hline 0 & I \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{array}\right) \prec 0.$$

Unfortunately **not convex** in all variables v and Q, R, S!

No technique known how to convexify in general!

Dualization Lemma

Suppose that $R \succ 0$ and $Q - S^T R S \prec 0$. Then

$$\begin{pmatrix} I \\ W \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} I \\ W \end{pmatrix} \prec 0$$

is equivalent to

$$\begin{pmatrix} W^T \\ -I \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}^{-1} \begin{pmatrix} W^T \\ -I \end{pmatrix} \succ 0.$$

Note that $\operatorname{im} \begin{pmatrix} W^T \\ -I \end{pmatrix}$ equals orthogonal complement of $\operatorname{im} \begin{pmatrix} I \\ W \end{pmatrix}$.

In general: Let $P = P^*$ be nonsingular with k negative eigenvalues.
If the subspace $\mathcal{S}$ with dimension k is P-negative then $\mathcal{S}^\perp$ is P-positive.

Dual Robust Synthesis Inequalities

Exists controller guaranteeing robust stability if exist v, $\tilde{Q}$, $\tilde{R}$, $\tilde{S}$:

$$\tilde{Q} \succ 0, \quad \begin{pmatrix} -I \\ \Delta(\delta^k)^T \end{pmatrix}^T \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix} \begin{pmatrix} -I \\ \Delta(\delta^k)^T \end{pmatrix} \prec 0, \quad k = 1, \ldots, N$$

$$\boldsymbol{X}(v) \succ 0, \left(\begin{array}{cc} \boldsymbol{A}(v)^T & \boldsymbol{C}(v)^T \\ -I & 0 \\ \hline \boldsymbol{B}(v)^T & \boldsymbol{D}(v)^T \\ 0 & -I \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} \\ 0 & 0 & \tilde{S}^T & \tilde{R} \end{array}\right) \left(\begin{array}{cc} \boldsymbol{A}(v)^T & \boldsymbol{C}(v)^T \\ -I & 0 \\ \hline \boldsymbol{B}(v)^T & \boldsymbol{D}(v)^T \\ 0 & -I \end{array}\right) \succ 0.$$

Note that multipliers are related as $\begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix} = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}^{-1}$.

No progress in general. However it helps for **state-feedback synthesis**

Static State-Feedback Synthesis - Lucky Case!

Recall block substitution:

$$\left(\begin{array}{cc} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{array}\right) = \left(\begin{array}{cc} AY + BM & B_1 \\ C_1 Y + EM & D_1 \end{array}\right).$$

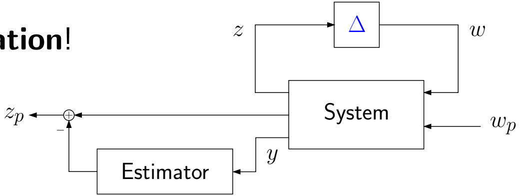
Last column does not depend on v ...

... dual inequalities are **affine** in all variables ...

... robust state-feedback synthesis possible with LMI's!

Similar results for **robust estimation**!

Very good exercise.



Elimination of Transformed Controller Parameters

Unstructured matrix variables in **one** LMI can often be eliminated.

For example let us recall the particular structure

$$\begin{pmatrix} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix} = \left(\begin{array}{cc|c} AY & A & B_1 \\ 0 & XA & XB_1 \\ \hline C_1Y & C_1 & D_1 \end{array}\right) +$$

$$+ \left(\begin{array}{cc} 0 & B \\ I & 0 \\ \hline 0 & E \end{array}\right) \begin{pmatrix} K & L \\ M & N \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C & F \end{array}\right).$$

Can eliminate $\begin{pmatrix} K & L \\ M & N \end{pmatrix}$ in synthesis inequalities. How?

Elimination Lemma

Consider the following quadratic matrix inequality in Z:

$$\begin{pmatrix} I \\ U^T Z V + W \end{pmatrix}^T \underbrace{\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}}_{P} \begin{pmatrix} I \\ U^T Z V + W \end{pmatrix} \prec 0$$

Let $U_\perp$, $V_\perp$ be basis matrices of $\ker(U)$, $\ker(V)$.

Suppose that $R \succ 0$ and $Q - S^T R S \prec 0$. Then the quadratic matrix inequality has a solution Z iff

$$V_\perp^T \begin{pmatrix} I \\ W \end{pmatrix}^T P \begin{pmatrix} I \\ W \end{pmatrix} V_\perp \prec 0 \;\; \& \;\; U_\perp^T \begin{pmatrix} W^T \\ -I \end{pmatrix}^T P^{-1} \begin{pmatrix} W^T \\ -I \end{pmatrix} U_\perp \succ 0.$$

Can explicitly construct solution Z if solvability conditions satisfied.

Application to Quadratic Performance Synthesis

Exists controller that renders $\mathcal{A}$ Hurwitz and the QP spec for

$$P_p = \begin{pmatrix} Q_p & S_p \\ S_p^T & R_p \end{pmatrix} \quad \text{with} \quad R_p \succcurlyeq 0$$

satisfied iff there exists v with

$$\boldsymbol{X}(v) \succ 0, \quad \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & Q_p & 0 & S_p \\ \hline I & 0 & 0 & 0 \\ 0 & S_p^T & 0 & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{array}\right) \prec 0.$$

Suppose P is non-singular and partition $\tilde{P} := P^{-1}$ as P.

Let Φ, Ψ be basis matrices of $\ker \begin{pmatrix} B^T & E^T \end{pmatrix}$, $\ker \begin{pmatrix} C & F \end{pmatrix}$.

QP Synthesis Inequalities after Elimination

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0,$$

$$\Psi^T \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline XA & XB_1 \\ C_1 & D_1 \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & Q_p & 0 & S_p \\ \hline I & 0 & 0 & 0 \\ 0 & S_p^T & 0 & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline XA & XB_1 \\ C_1 & D_1 \end{array}\right) \Psi \prec 0,$$

$$\Phi^T \left(\begin{array}{cc} YA^T & YC_1^T \\ B_1^T & D_1^T \\ \hline -I & 0 \\ 0 & -I \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & \tilde{Q}_p & 0 & \tilde{S}_p \\ \hline I & 0 & 0 & 0 \\ 0 & \tilde{S}_p^T & 0 & \tilde{R}_p \end{array}\right) \left(\begin{array}{cc} YA^T & YC_1^T \\ B_1^T & D_1^T \\ \hline -I & 0 \\ 0 & -I \end{array}\right) \Phi \succ 0.$$

Much fewer variables! Nice system theoretic interpretation!

Alternative Robust Synthesis Inequalities

Synthesis inequalities with non-convex coupling $(k = 1, \ldots, N)$:

$$\begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix} = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}^{-1}, \quad \begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0, \quad Q \prec 0, \quad \tilde{R} \succ 0$$

$$[*] \left(\begin{array}{cc|cc} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^T & R \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A & B_1 \\ \hline 0 & I \\ C_1 & D_1 \end{array}\right) \Psi \prec 0, \quad [*] \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta(\delta^k) \\ I \end{pmatrix} \succ 0$$

$$[*] \left(\begin{array}{cc|cc} 0 & Y & 0 & 0 \\ Y & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} \\ 0 & 0 & \tilde{S}^T & \tilde{R} \end{array}\right) \left(\begin{array}{cc} A^T & C_1^T \\ -I & 0 \\ \hline B_1^T & D_1^T \\ 0 & -I \end{array}\right) \Phi \succ 0, \quad [*] \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix} \begin{pmatrix} -I \\ \Delta(\delta^k)^T \end{pmatrix} \prec 0$$

Main Points

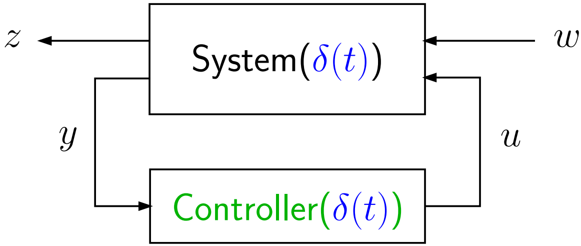
- General procedure: Genuine multi-objective synthesis with Youla
- General difficulties in robust controller synthesis

  Illustrated for specific uncertainty/multiplier class

  Show specific information structure and remedy
- Useful Technical Lemmas:

  Linearization, Dualization, Elimination

Gain-Scheduling Control for LPV Systems



Given a parameter-dependent system, design a controller that stabilizes and achieves optimal performance, with the extra advantage (in contrast to a robust controller) that it can take on-line measurements of the parameters as information into account.

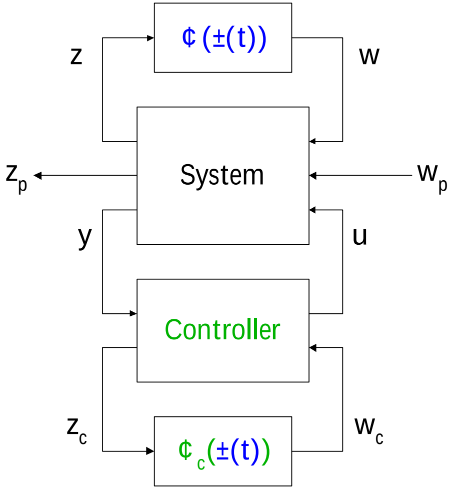
Example application: **Gain-scheduling**

Configuration for Multiplier LPV Synthesis

Design parameter-dependent controller guaranteeing

- exponential stability
- quadratic performance

If some parameters coincide with state-components procedure leads to nonlinear controller!



Again concentrate on **stability** - no performance channel.

System Descriptions

Uncontrolled LTI part:

$$\begin{aligned}\dot{x} &= Ax + B_1 w + Bu\\ z &= C_1 x + D_1 w + Eu\\ y &= Cx + Fw\end{aligned}$$

w: uncertainty input

z: uncertainty output

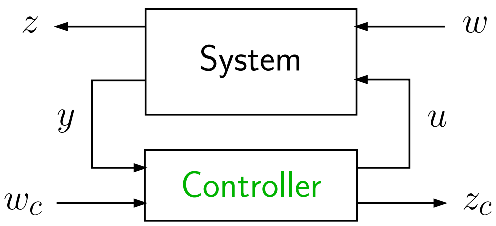
u: control input

y: measured output

Controller LTI part:

$$\dot{x}_c = A_c x_c + B_c \begin{pmatrix} y \\ w_c \end{pmatrix}$$

$$\begin{pmatrix} u \\ z_c \end{pmatrix} = C_c x_c + D_c \begin{pmatrix} y \\ w_c \end{pmatrix}$$



Fundamental Trick to Solve LPV Problem

The interconnection can be seen to result from

$$\begin{pmatrix} \dot{x} \\ \hline z \\ z_c \\ \hline y \\ w_c \end{pmatrix} = \left(\begin{array}{c|cc|cc} A & B_1 & 0 & B_2 & 0 \\ \hline C_1 & D_1 & 0 & D_{12} & 0 \\ 0 & 0 & 0 & 0 & I \\ \hline C_2 & D_{21} & 0 & D_2 & 0 \\ 0 & 0 & I & 0 & 0 \end{array}\right) \begin{pmatrix} x \\ \hline w \\ w_c \\ \hline u \\ z_c \end{pmatrix}$$

controlled with

$$\dot{x}_c = A_c x_c + B_c \begin{pmatrix} y \\ w_c \end{pmatrix}, \quad \begin{pmatrix} u \\ z_c \end{pmatrix} = C_c x_c + D_c \begin{pmatrix} y \\ w_c \end{pmatrix}$$

Can solve the **robust control problem** for this interconnection.

LPV Synthesis: Step I

Synthesis inequalities **without non-convex coupling**: $(k = 1, \ldots, N)$:

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0, \quad Q \prec 0, \quad \tilde{R} \succ 0$$

$$[*] \left(\begin{array}{cc|cc} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S \\ 0 & 0 & S^T & R \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A & B_1 \\ \hline 0 & I \\ C_1 & D_1 \end{array}\right) \Psi \prec 0, \quad [*] \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \Delta(\delta^k) \\ I \end{pmatrix} \succ 0$$

$$[*] \left(\begin{array}{cc|cc} 0 & Y & 0 & 0 \\ Y & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} \\ 0 & 0 & \tilde{S}^T & \tilde{R} \end{array}\right) \left(\begin{array}{cc} A^T & C_1^T \\ -I & 0 \\ \hline B_1^T & D_1^T \\ 0 & -I \end{array}\right) \Phi \succ 0, \quad [*] \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix} \begin{pmatrix} -I \\ \Delta(\delta^k)^T \end{pmatrix} \prec 0$$

LPV Synthesis: Step II

Find extension

$$\begin{pmatrix} Q_e & S_e \\ S_e^T & R_e \end{pmatrix} = \left(\begin{array}{cc|cc} Q & * & S & * \\ * & * & * & * \\ \hline S^T & * & R & * \\ * & * & * & * \end{array}\right) \text{ with inverse } \left(\begin{array}{cc|cc} \tilde{Q} & * & \tilde{S} & * \\ * & * & * & * \\ \hline \tilde{S}^T & * & \tilde{R} & * \\ * & * & * & * \end{array}\right)$$

such that

$$\begin{pmatrix} Q & * \\ * & * \end{pmatrix} \prec 0 \text{ and } \begin{pmatrix} \tilde{R} & * \\ * & * \end{pmatrix} \succ 0.$$

This is **always** possible.

Size of extension determines size of scheduling function $\Delta_c(.)$.

LPV Synthesis: Step III

Find scheduling function $\Delta_c(\delta)$ such that for all $\delta \in \boldsymbol{\delta}$

$$\left(\begin{array}{cc} \Delta(\delta) & 0 \\ 0 & \Delta_c(\delta) \\ \hline I & 0 \\ 0 & I \end{array}\right)^T \left(\begin{array}{cc|cc} Q & * & S & * \\ * & * & * & * \\ \hline S^T & * & R & * \\ * & * & * & * \end{array}\right) \left(\begin{array}{cc} \Delta(\delta) & 0 \\ 0 & \Delta_c(\delta) \\ \hline I & 0 \\ 0 & I \end{array}\right) \succ 0$$

or equivalently

$$\left(\begin{array}{cc} -I & 0 \\ 0 & -I \\ \hline \Delta(\delta)^T & 0 \\ 0 & \Delta_c(\delta)^T \end{array}\right)^T \left(\begin{array}{cc|cc} \tilde{Q} & * & \tilde{S} & * \\ * & * & * & * \\ \hline \tilde{S}^T & * & \tilde{R} & * \\ * & * & * & * \end{array}\right) \left(\begin{array}{cc} -I & 0 \\ 0 & -I \\ \hline \Delta(\delta)^T & 0 \\ 0 & \Delta_c(\delta)^T \end{array}\right) \prec 0.$$

Have explicit formula for $\Delta_c(\delta)$.

LPV Synthesis: Step IV

Design LTI-part of controller

$$\dot{x}_c = A_c x_c + B_c \begin{pmatrix} y \\ w_c \end{pmatrix}, \quad \begin{pmatrix} u \\ z_c \end{pmatrix} = C_c x_c + D_c \begin{pmatrix} y \\ w_c \end{pmatrix}$$

as quadratic performance controller for extended system

$$\begin{pmatrix} \dot{x} \\ \hline z \\ z_c \\ \hline y \\ w_c \end{pmatrix} = \left(\begin{array}{c|cc|cc} A & B_1 & 0 & B_2 & 0 \\ \hline C_1 & D_1 & 0 & D_{12} & 0 \\ 0 & 0 & 0 & 0 & I \\ \hline C_2 & D_{21} & 0 & D_2 & 0 \\ 0 & 0 & I & 0 & 0 \end{array}\right) \begin{pmatrix} x \\ \hline w \\ w_c \\ \hline u \\ z_c \end{pmatrix}$$

with performance index $\begin{pmatrix} Q_e & S_e \\ S_e^T & R_e \end{pmatrix}$.

Some Remarks on Sketched Procedure

- Obtained solution of LPV problem with full block scalings.
  Can further enlarge scaling set. Requires behavioral controller [1].

- Comparison to direct implemented version in LMI-toolbox:

  - Allows rational parameter dependence

  - matrices $B(\delta)$, $E(\delta)$, $C(\delta)$, $F(\delta)$ allowed to depend on parameters

  - Care-free scheduling without determination of $\lambda_k(t)$.

- Exists extension to parameter dependent Lyapunov functions.

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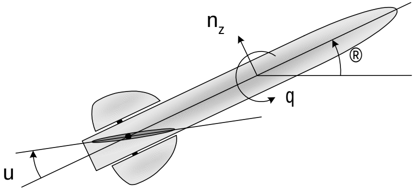
[1] C.W. Scherer, LPV control with full block multipliers, *Automatica* **37** (2001) 361-375.

High-Performance Aircraft System



u: Control input

α: Measurable parameter

n_z: Tracked output

Nonlinear system description with aerodynamic effects:

$$
\begin{aligned}
\dot{\alpha} &= KM\left[\left(a_n\alpha^2+b_n\alpha+c_n(2-M/3)\right)\alpha+d_nu\right]+q \\
\dot{q} &= M^2\left[\left(a_m\alpha^2+b_m\alpha-c_m(7-8M/3)\right)\alpha+d_mu\right] \\
n_z &= M^2\left[\left(a_n\alpha^2+b_n\alpha+c_n(2-M/3)\right)\alpha+d_nu\right]
\end{aligned}
$$

Main Idea

Rewrite as linear parameter-varying system

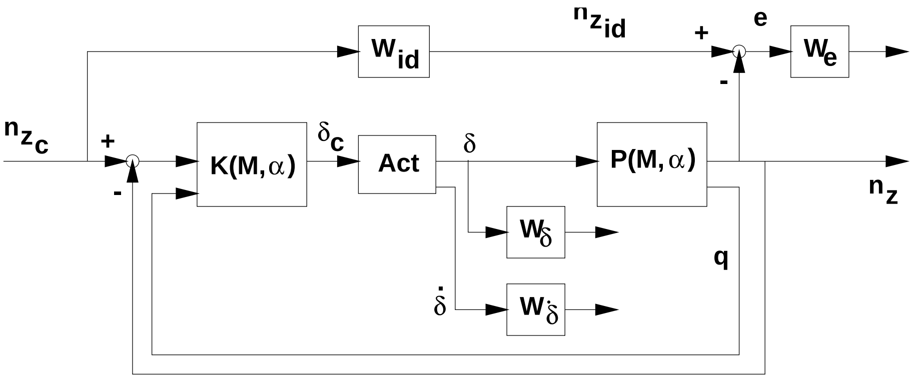
$$\begin{aligned}
\dot{\alpha} &= K\delta_1\Big[\Big(a_n{\delta_2}^2+b_n\delta_2+c_n(2-\delta_1/3)\Big)\,\alpha+d_nu\Big]+q\\
\dot{q} &= {\delta_1}^2\Big[\Big(a_m{\delta_2}^2+b_m\delta_2-c_m(7-8\delta_1/3)\Big)\,\alpha+d_mu\Big]\\
n_z &= {\delta_1}^2\Big[\Big(a_n{\delta_2}^2+b_n\delta_2+c_n(2-\delta_1/3)\Big)\,\alpha+d_nu\Big]
\end{aligned}$$

with bounds $2 \leq \delta_1(t) \leq 4$ and $-20 \leq \delta_2(t) \leq 20$.

Design good controller scheduled with $\delta_1(t)$, $\delta_2(t)$

$\rightarrow$ Is good controller for nonlinear system

Interconnection Structure



Model-Matching

Let controlled system approximately act like ideal model $\mathbf{W_{id}}$.

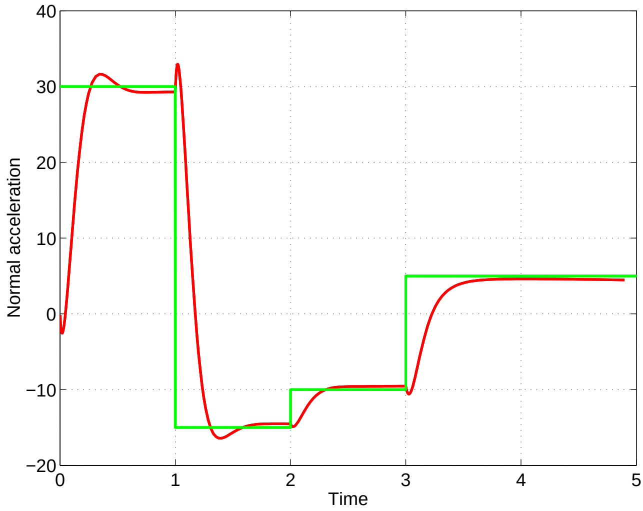
Synthesis with Convex Hull Relaxation

$M(t)$ decreases in 5 seconds from 4 to 2.

Normal acceleration

Reference

Response



Some Book References

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[2] S.P. Boyd, L. El Ghaoui et al., Linear matrix inequalities in system and control theory, Philadelphia, SIAM (1994).

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[6] Manuals for "LMI Control Toolbox" and "μ Analysis and Synthesis Toolbox"

[7] C.W. Scherer, S. Weiland, DISC Lecture Notes "LMI's in Control".