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Robust stabilization of the angular velocity of a rigid body

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Abstract

The problem of robust control for the angular velocity of a rigid body subject to external disturbances is addressed. It is shown that if the disturbances are matched there exists a Lipschitz continuous control law attenuating the effect of the disturbances; whereas in the case of non-matched disturbances no such a feedback law exists. Hence, a new concept of disturbance attenuation is introduced and it is proved that the aforementioned problem is solvable in this weaker sense.

1. Introduction and preliminaries

The problem of asymptotic stabilization of the angular velocity of a rigid body has been the subject of several investigations. In the papers [1, 2] it was shown that the zero solution of Euler's angular velocity equations can be made asymptotically stable by means of two control torques, whereas in the works [3, 4, 5] the same problem has been addressed and solved in the case of only one control torque. The starting point of all these works are the equations describing the angular velocity of a rigid body (with respect to a body-fixed reference frame) assuming that no external disturbance is present. In this work we relax this assumption: we consider the equations describing the angular velocity of a rigid body perturbed by external disturbances and we try to derive a control law attenuating the effect of these disturbances (in a sense to be specified).

1.1. The system

Consider a rigid body in an inertial reference frame and let ω_1, ω_2 and ω_3 denote the angular velocity components along a body fixed reference frame having the origin at the center of gravity and consisting of the three principal axes. The Euler equations for the rigid body, subject to three external disturbances acting in the direction of the principal axes and to two independent control aligned with two principal axes, are

$$\begin{array}{rcl} I_1\dot{\omega}_1 &=& (I_2-I_3)\omega_2\omega_3 + v_1 + P_1 w_1 \\ I_2\dot{\omega}_2 &=& (I_3-I_1)\omega_3\omega_1 + v_2 + P_2 w_2 \\ I_3\dot{\omega}_3 &=& (I_1-I_3)\omega_1\omega_2 + P_3 w_3, \end{array} \tag{1}$$

where $I_1 > 0$, $I_2 > 0$ and $I_3 > 0$ denote the principal moments of inertia, $v = \mathrm{col}(v_1, v_2)$ the control torques, $w = \mathrm{col}(w_1, w_2, w_3)$ the external disturbances assumed to be in L_2^e, and P_1, P_2 and P_3 three real numbers. In what follows we assume that the rigid body is not symmetric *i.e.* $I_1 \neq I_2 \neq I_3$ and, without lack of generality, that $I_1 > I_2 > I_3$.

System (1) can be re-written as

$$\begin{array}{rcl} \dot{\omega}_1 &=& A_1\omega_2\omega_3 + g_1 v_1 + p_1 w_1 \\ \dot{\omega}_2 &=& A_2\omega_3\omega_1 + g_2 v_2 + p_2 w_2 \\ \dot{\omega}_3 &=& A_3\omega_1\omega_2 + p_3 w_3, \end{array} \tag{2}$$

where $A_1 = \dfrac{(I_2-I_3)}{I_1} > 0$, $A_2 = \dfrac{(I_3-I_1)}{I_2} < 0$, $A_3 = \dfrac{(I_1-I_2)}{I_3} > 0$, $g_1 = \dfrac{1}{I_1} > 0$, $g_2 = \dfrac{1}{I_2} > 0$ and $p_1 = \dfrac{P_1}{I_1}$, $p_2 = \dfrac{P_2}{I_2}$, $p_3 = \dfrac{P_3}{I_3}$.

Finally, after the feedback transformation

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \dfrac{u_1 - A_1\omega_2\omega_3}{g_1} \\ \dfrac{u_2 - A_2\omega_3\omega_1}{g_2} \end{bmatrix},$$

we obtain a control system with state $\Omega = (\omega_1, \omega_2, \omega_3)$, *new* control variables $u = (u_1, u_2)$ and exogenous inputs $w = (w_1, w_2, w_3)$, described by equations of the form

$$\dot{\Omega} = f(\Omega) + Bu + Pw \tag{3}$$

where

$$f(\Omega) = \begin{bmatrix} 0 & 0 & A\omega_1\omega_2 \end{bmatrix}^T,$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}$$

and $A = A_3$.

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1.2. More on the model (3)

This paper takes inspiration from a real control problem, namely the problem of controlling the angular velocity of a rigid body, *e.g.* a satellite, in presence of external disturbances. Hence, before going any further we briefly discuss whether the mathematical model (3) does indeed describe a *real object*.

We assume throughout this work that the principal moments of inertia I_1, I_2 and I_3 are precisely known. This is the case for deep space mission satellites, where the mechanical parameters are generally determined quite accurately. On the other hand, deep space mission satellites are subjects to the action of gravitational fields and external disturbances, like the solar wind. Finally, there is one main feature which is common to all satellites controlled with gas jet actuators. The control action is not continuous, as the actuators deliver their action in a sequence of pulses (see Figure 1 for a pictorial description). Hence, the difference between the *ideal* control action delivered by a state feedback control law and the actual control action significantly contributes to the external disturbances. We also note that, for this reason, the two disturbance signals w_1 and w_2 have in general larger amplitudes than the signal w_3.

We conclude that the model (3) can be considered a truthful model if we address the problem of controlling the angular velocity of a deep space mission satellite controlled with gas jet actuators and operating in failure configuration, *i.e.* with only two controls. One may argue that deep space mission satellites are always equipped with redundant actuators. Nevertheless, for such satellites, which typically travel for several years, the possibility of recovering from the failure of one actuator is of high practical, scientific and economic interest.

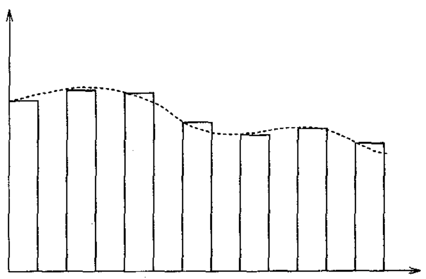

Figure 1: Ideal continuous control (dashed) and actual control action (solid) delivered by a gas jet actuator.

1.3. Problem statement

For the system (3) we study the problem of disturbance attenuation with internal stability. A rather standard way of handling such a problem is as follows (see [6, 7, 8] for further detail). Given a $\gamma > 0$ and a penalty variable

$$z = H(\Omega) = H(\omega_1, \omega_2, \omega_3),$$

where $H(\Omega)$ is a differentiable mapping, such that $H(0) = 0$, find, if possible, a pair $(u^\star(\Omega), V(\Omega))$, where $u^\star(\Omega) = u^\star(\omega_1, \omega_2, \omega_3)$ is a state feedback control law, with $u^\star(0) = 0$, and $V(\Omega) = V(\omega_1, \omega_2, \omega_3)$ is a positive definite and proper function, such that, along the trajectories of the closed loop system, one has

$$HJI_\gamma^{(u^\star,V)}(\Omega) =$$
$$V_\Omega\Big[f(\Omega) + Bu^\star(\Omega)\Big] + V_\Omega \frac{PP^T}{4\gamma^2} V_\Omega^T + H^T(\Omega)H(\Omega) < 0. \tag{4}$$

Remark 1 It must be pointed out that in general it is not possible to find a pair $(u^\star(\Omega), V(\Omega))$ such that condition (4) holds true for all nonzero $\Omega \in I\!R^3$. Hence, we have to content ourselves with weaker (in a sense to be specified) results.

Till now we have not made any assumption on the differentiability of the function V. In what follows we assume that V is Lipschitz continuous, as the assumption of differentiability, *i.e.* $V \in C^1$, is too restrictive (see Proposition 1 hereafter). Hence, we need to make precise the way in which equation (4) must be understood.

In what follows we say that a pair $(u^\star(\Omega), V(\Omega))$ is differentiable if both $u^\star(\Omega)$ and $V(\Omega)$ are differentiable; and that a pair $(u^\star(\Omega), V(\Omega))$ is Lipschitz continuous if either $u^\star(\Omega)$ or $V(\Omega)$ is non-differentiable, but only Lipschitz continuous.

1.4. A short digression

In this section we briefly recall some facts and definitions from non-smooth analysis, for further detail the reader is referred to [9], where the mathematical machinery of non-smooth analysis is introduced; [10], where the role of non-smooth Lyapunov functions in stability theory is discussed; and [11], where non-smooth control Lyapunov functions are introduced.

Let $W(x)$ be a Lipschitz continuous function. The generalized gradient $\partial W(x)$ is the set-valued map defined by

$$\partial W(x) = \{\xi \in (I\!R^3)^\star \ : \ \xi \cdot v \leq W^0(x, v) \ \forall v \in I\!R^3\},$$

where $W^0(x, v)$ is the generalized directional derivative of Clarke [9], given by

$$W^0(x, v) = \limsup_{y \to x,\, h \to 0+} \frac{W(y + hv) - W(y)}{h}.$$

Observe that, although the generalized gradient $\partial W(x)$ is defined via the generalized directional derivative of Clarke, it can be computed directly from the function $W(\cdot)$: it is, roughly speaking, the convex hull of all gradients of $W(\cdot)$ near the point x. Using the notion of generalized gradient, it is possible to give a mathematically precise meaning to the equation (4) when the function $V(\Omega)$ is not differentiable, but merely Lipschitz continuous, as follows

$$HJI_\gamma^{(u^\star,V)}(\Omega) =$$

$$\max_{\xi\in\partial V(\Omega)}\left[\xi f(\Omega)+\xi Bu^\star(\Omega)+\xi\frac{PP^T}{4\gamma^2}\xi^T+H^T(\Omega)H(\Omega)\right].$$

2. Main results

In this section we present the main results of this work, which will be discussed in Section 3, and proved in Section 4.

Proposition 1 *Consider the system (3). Let N be a neighborhood of the origin in $I\!R^3$ and $\gamma>0$.*

Then, the set of all differentiable pairs $(u^\star(\Omega),V(\Omega))$, such that $HJI_\gamma^{(u^\star,V)}<0$ for all nonzero $\Omega\in N$ is generically empty, i.e. *for a generic triple (p_1,p_2,p_3) and a generic differentiable penalty function $H(\Omega)$ there exists no differentiable pair $(u^\star(\Omega),V(\Omega))$ such that $HJI_\gamma^{(u^\star,V)}<0$ for all nonzero $\Omega\in N$.*

Moreover, the same holds true even if some p_i are zero.

Proposition 2 *Consider the system (3). Let*

$$H(\Omega)=\begin{bmatrix}c_1\omega_1\\c_2\omega_2\\c_3\omega_3\end{bmatrix}$$

and $\gamma>0$. Suppose, moreover, $p_3=0$.

Then there exists a pair $(u^\star(\Omega),V(\Omega))$, which is Lipschitz continuous, such that $HJI_\gamma^{(u^\star,V)}<0$ for all nonzero $\Omega\in I\!R^3$.

Proposition 3 *Consider the system (3). Let*

$$H(\Omega)=\begin{bmatrix}c_1\omega_1\\c_2\omega_2\\c_3\omega_3\end{bmatrix},$$

$K_\epsilon\subset I\!R^3$ be a compact set of measure (volume) smaller or equal to ϵ and such that $(0,0,0)\in K_\epsilon$, and $\gamma>0$.

Then, for any $\epsilon>0$ there exists a differentiable pair $(u^\star(\Omega),V(\Omega))$, perhaps depending on ϵ, such that $HJI_\gamma^{(u^\star,V)}<0$ for all $\Omega\in I\!R^3/K_\epsilon$.

Proposition 4 *Consider the system (3). Let*

$$H(\Omega)=\begin{bmatrix}c_1\omega_1\\c_2\omega_2\\c_3\omega_3\end{bmatrix},$$

and $\gamma>0$. Suppose, moreover, $p_3=0$, $c_3=0$ and $c_1c_2=0$, but $|c_1|+|c_2|>0$.

Then there exists a differentiable pair $(u^\star(\Omega),V(\Omega))$ such that $HJI_\gamma^{(u^\star,V)}<0$ for all nonzero $\Omega\in I\!R^3$.

Table 1 summarizes the achieved results.

3. Discussion of the main results

Proposition 1 shows that, for a generic disturbance matrix P and a generic (differentiable) penalty function $H(\Omega)$, the problem of disturbance attenuation for the Euler equations (3) is not solvable in any neighborhood of the origin if we restrict the search of the solution to the set of differentiable pairs $(u^\star(\Omega),V(\Omega))$. Motivated by this negative result we investigate the solvability of this disturbance attenuation problem in two different directions.

In the former, we assume that the disturbances are matched and that the penalty function is linear in the state Ω. Under this assumptions we prove (Proposition 2) that, for any level of attenuation $\gamma>0$, there exists a Lipschitz continuous pair $(u^\star(\Omega),V(\Omega))$ such that $HJI_\gamma^{(u^\star,V)}(\Omega)<0$ for all nonzero Ω in $I\!R^3$. We conclude that in the case of matched disturbances the disturbance attenuation problem with internal stability is globally solvable.

In the latter, we still consider a penalty function linear in the state Ω, but we remove the assumption of matched disturbances. As a result, the region where disturbance attenuation is achieved is the whole state space without compact set, the size of which can be arbitrarily reduced by a proper choice of the design parameters. This implies that the control law is able to attenuate the effect of unmatched disturbances on the penalty variable as long as the state of the system does not evolve close to the origin. Although a bit unnatural, this property is indeed useful: disturbance attenuation is achieved away from the equilibrium, whereas nothing can be said if the state is close to the equilibrium. However, in this last case, the effect of the disturbances is already small. In [12], the property outlined in Proposition 3 is discussed in detail for general nonlinear systems. We can say that it stays to the usual definition of disturbance attenuation [7] as the notion of practical stability [13] stays to the notion of asymptotic stability [10].

The proof of Proposition 3 can be also used to determine sufficient conditions for the existence of a differentiable pair $(u^\star(\Omega),V(\Omega))$ such that $HJI_\gamma^{(u^\star,V)}(\Omega)<0$ for all nonzero Ω in $I\!R^3$. Such conditions are summarized in Proposition 4, which lends itself to the following interesting physical interpretation. If the disturbances are matched and the penalty variable is either $z=H(\Omega)=\omega_1$ either $z=H(\Omega)=\omega_2$ then the disturbance attenuation problem is solvable in the usual sense [7], *i.e.* using a smooth control law.

At the present stage we do not know whether the assumptions in Proposition 4, namely $p_3=0$, $c_3=0$ and $c_1c_2=0$, are also necessary to solve the disturbance attenuation problem with a differentiable pair

(p_1,p_2,p_3)	$H(\Omega)$	Type of solution	Region of solvability
$(p_1,p_2,0)$	$\mathrm{col}(c_1\omega_1,c_2\omega_2,c_3\omega_3)$	Lipschitz continuous	$I\!\!R^3$
$(p_1,p_2,0)$	$c_1\omega_1$	Differentiable	$I\!\!R^3$
$(p_1,p_2,0)$	$c_2\omega_2$	Differentiable	$I\!\!R^3$
(p_1,p_2,p_3)	$\mathrm{col}(c_1\omega_1,c_2\omega_2,c_3\omega_3)$	Differentiable	$I\!\!R^3/K_\epsilon$

Table 1: Summary of the achieved results.

$(u^\star(\Omega),V(\Omega))$. Nevertheless, we believe that they are not too restrictive, at least if linear penalty functions are considered, as can be deduced from the following statement.

Proposition 5 *Consider the system (3). Let N be a neighborhood of the origin in $I\!\!R^3$ and $\gamma>0$.*

Suppose $p_3c_3\neq 0$.

Then, there exists no differentiable pair $(u^\star(\Omega),V(\Omega))$, such that $HJI_\gamma^{(u^\star,V)}<0$ for all nonzero $\Omega\in N$.

4. Proof of the main results

Proof of Proposition 1. Suppose there exists a differentiable pair $(u^\star(\Omega),V(\Omega))$ such that $HJI_\gamma^{(u^\star,V)}(\Omega)<0$ for all nonzero Ω in a neighborhood of the origin of $I\!\!R^3$. Let

$$\begin{array}{rcl} u^\star(\Omega) &=& F\Omega+\text{h.o.t.},\\ V(\Omega) &=& \Omega^TX\Omega+\text{h.o.t.},\\ H(\Omega) &=& C\Omega+\text{h.o.t.}\end{array}$$

Observe that these expansions are well defined by differentiability of $u^\star(\Omega)$, $V(\Omega)$ and $H(\Omega)$. It follows that

$$ARE = XBF+F^TB^TX+X\frac{PP^T}{\gamma^2}X+C^TC\le 0.$$

We now show that for generic P and C the matrix ARE is not negative semi-definite. Let x be a vector in the kernel of F, which is not empty. Then,

$$x^TAREx = x^T\left(X\frac{PP^T}{\gamma^2}X+C^TC\right)x$$

and this is non-positive if and only if

$$x\in\text{kernel}\,\{C\}\qquad\text{and}\qquad x\in\text{kernel}\,\{P^TX\},$$

which are, obviously, non-generic conditions. $\triangleleft$

Proof of Proposition 2. Let

$$V(\Omega)=\frac{1}{2}(\omega_1+\alpha|\omega_3|)^2+\frac{1}{2}(\omega_2-\beta\omega_3)^2+\delta|\omega_3|$$

with $\alpha>0$, $\beta>0$,

$$\delta=\frac{\alpha^2c_1^2+\beta^2c_2^2+c_3^2+\sigma_3}{A\alpha\beta}$$

and $\sigma_3>0$. Note that $\delta>0$, hence $V(\Omega)$ is positive definite and proper. Let, moreover,

$$u^\star(\Omega)=\begin{bmatrix}u_1^\star(\Omega)\\ u_2^\star(\Omega)\end{bmatrix}$$

with

$$\begin{aligned} u_1^\star(\Omega) =& -A\beta\delta|\omega_3|+A\alpha^2\beta\omega_3^2+2\alpha c_1^2|\omega_3|\\ & -\left(A\alpha\beta|\omega_3|+\frac{p_1^2}{4\gamma^2}+c_1^2\right)(\omega_1+\alpha|\omega_3|)\\ & -\left(A\delta\,\mathrm{sign}(\omega_3)-A\beta^2x_3-A\alpha^2|\omega_3|\right)(\omega_2-\beta\omega_3)\\ & +A\beta(\omega_2-\beta\omega_3)^2-\sigma_1(\omega_1+\alpha|\omega_3|),\\ u_2^\star(\Omega) =& A\alpha\delta\omega_3-A\alpha\beta^2\omega_3|\omega_3|-2\beta c_2^2\omega_3\\ & -\left(A\alpha\beta\omega_3+\frac{p_2^2}{4\gamma^2}+c_2^2\right)(\omega_2-\beta\omega_3)\\ & -A\alpha\,\mathrm{sign}(\omega_3)(\omega_1+\alpha|\omega_3|)^2-\sigma_2(\omega_2-\beta\omega_3),\end{aligned}$$

$\sigma_1>0$ and $\sigma_2>0$. Then, long but straightforward calculations show that

$$HJI_\gamma^{(u^\star,V)}(\Omega)=-\sigma_1(\omega_1+\alpha|\omega_3|)^2-\sigma_2(\omega_2-\beta\omega_3)^2-\sigma_3\omega_3^2,$$

hence the claim. $\triangleleft$

Proof of Proposition 3. Let

$$V(\Omega)=\frac{1}{2}(\omega_1+\alpha\omega_3)^2+\frac{1}{2}(\omega_2-\beta\omega_3^2)^2+\frac{1}{2}\delta\omega_3^2, \quad (5)$$

with $\alpha>0$, $\beta>0$ and $\delta>0$. Simple but long calculations show that with a proper choice of

$$u^\star(\Omega)=\begin{bmatrix}u_1^\star(\Omega)\\ u_2^\star(\Omega)\end{bmatrix}$$

it is possible to obtain, for given $\sigma_1>0$, $\sigma_2>0$,

$$\begin{aligned}&HJI_\gamma^{(u^\star,V)}(\Omega)=\\ &-\sigma_1(\omega_1+\alpha\omega_3)^2-\sigma_2(\omega_2-\beta\omega_3^2)^2-S_4\omega_3^4+S_2\omega_3^2,\end{aligned} \quad (6)$$

where

$$S_4=A\alpha\beta\delta-\beta^2c_2^2 \quad (7)$$

and

$$S_2 = \alpha^2 c_1^2 + c_3^2 + \frac{\delta^2 p_3^2}{4\gamma^2}.$$

Hence, it is apparent that, regardless the choice of σ_1, σ_2, α, β and δ it is not possible to render equation (6) negative for all nonzero Ω. Nevertheless, we now show that for any $\epsilon > 0$ there exists a choice of σ_1, σ_2, α, β and δ such that equation (6) is negative for all $\Omega \in \mathbb{R}^3/K_\epsilon$, where $K_\epsilon \subset \mathbb{R}^3$ is a compact set of measure (volume) smaller or equal to ϵ and such that $(0,0,0) \in K_\epsilon$.

The set where equation (6) equals zero can be re-written as

$$\frac{(\omega_1 + \alpha\omega_3)^2}{\dfrac{S_2\omega_3^2 - S_4\omega_3^4}{\sigma_1}} + \frac{(\omega_2 - \beta\omega_3^2)^2}{\dfrac{S_2\omega_3^2 - S_4\omega_3^4}{\sigma_2}} = 1,$$

which is, for any ω_3 such that $S_2\omega_3^2 - S_4\omega_3^4 \geq 0$ an ellipse, say E_{ω_3}, of area

$$A_{\omega_3} = \pi\sqrt{\frac{S_2\omega_3^2 - S_4\omega_3^4}{\sigma_1}}\sqrt{\frac{S_2\omega_3^2 - S_4\omega_3^4}{\sigma_2}}.$$

Observe now that equation (6) is negative at a point $(\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)$ if and only if $(\bar{\omega}_1, \bar{\omega}_2)$ is an interior point of E_{ω_3} and that $\Omega = (0,0,0) \in E_0$.

Let α, β and δ be such that[1]

$$\frac{\alpha\delta}{\beta} > \frac{c_2^2}{A} > 0, \tag{8}$$

implying $S_4 > 0$. If condition (8) holds one has

$$S_2\omega_3^2 - S_4\omega_3^4 \geq 0$$

for all

$$\omega_3 \in \left[-\sqrt{\frac{S_2}{S_4}}, \sqrt{\frac{S_2}{S_4}}\right],$$

and

$$A_{\omega_3} \leq \frac{\pi}{4}\frac{S_2^2}{S_4}\sqrt{\frac{1}{\sigma_1\sigma_2}}$$

As a consequence, the region where equation (6) takes non-negative values is bounded and its volume is smaller than

$$M = \frac{\pi}{2}\frac{S_2^2}{S_4}\sqrt{\frac{1}{\sigma_1\sigma_2}}\sqrt{\frac{S_2}{S_4}}.$$

We conclude that, for any choice of $\alpha > 0$, $\beta > 0$ and $\delta > 0$ fulfilling condition (8), the volume of the region where equation (6) takes non-negative values can be reduced increasing σ_1 and σ_2. $\lhd$

Proof of Proposition 4. By assumption $p_3 = 0$, $c_3 = 0$ and either $c_1 = 0$ or $c_2 = 0$. Consider the case $c_1 = 0$ and $c_2 \neq 0$. Let $V(\Omega)$ be as in equation (5) and $u^\star(\Omega) = \mathrm{col}(u_1^\star(\Omega), u_2^\star(\Omega))$, with $u_1^\star(\Omega)$ and $u_2^\star(\Omega)$ as in the proof of Proposition 3 obviously with $p_3 = c_3 = c_1 = 0$. Then, one has

$$HJI_\gamma^{(u^\star,V)}(\Omega) = -\sigma_1(\omega_1+\alpha\omega_3)^2 - \sigma_2(\omega_2-\beta\omega_3)^2 - S_4\omega_3^4,$$

with S_4 as in equation (7). Hence, selecting $\alpha > 0$, $\beta > 0$ and $\delta > 0$ such that condition (8) holds true, we conclude the claim.

The result for $c_1 \neq 0$ and $c_2 = 0$ can be proved using

$$V(\Omega) = \frac{1}{2}(\omega_1 + \alpha\omega_3^2)^2 + \frac{1}{2}(\omega_2 - \beta\omega_3)^2 + \frac{1}{2}\delta\omega_3^2,$$

with $\alpha > 0$, $\beta > 0$ and $\delta > 0$. $\lhd$

Proof of Proposition 5. The proof is based on an argument similar to that used in the proof of Proposition 1, hence it is omitted. $\lhd$

5. Some simulations

We now present some prototype simulation results, for both cases $p_3 = 0$ and $p_3 \neq 0$, *i.e.* using the control laws resulting from the proof of Propositions 2 and 3. In both cases we take w_1 and w_2 equal to square waves of amplitude equal to one and frequency equal to one and two, respectively. For the case $p_3 \neq 0$ we take w_3 equal to a sinusoidal signal of amplitude 0.4 and frequency 0.2. This choice of disturbance inputs is coherent with the discussion in Section 1.2. The level of attenuation γ has been set equal to 0.2 in both cases.

Figures 2 and 3 display the state histories from the initial condition $\Omega = (1, 1, 2)$, whereas Figure 4 shows the corresponding phase portraits. Observe the similarity of the trajectories, despite the different control laws and the action of the disturbance w_3 in the second case.

6. Conclusions

The robust stabilization problem for the Euler equations describing the angular velocity of a rigid body subject to external disturbances has been addressed.

Sufficient conditions for the existence of differentiable or Lipschitz continuous control laws have been derived, together with some simple necessary conditions for differentiable control laws. There is still a small gap between the above conditions, which we hope could be filled in a near future.

In the case of three external disturbances we have proposed a new paradigm. It consists of solving the robust stabilization problem in a region having a *hole*. This is necessary, as the model (3) is not stabilizable in the

[1] There always exist $\alpha > 0$, $\beta > 0$ and $\delta > 0$ such that (8) holds true.

first approximation, and the robust stabilization problem for the linearization of (3) is not solvable. This new paradigm, which is studied in detail in [12], results from the observation that a controlled system must be able to reject exogenous signals when the state is away from the equilibrium; whereas when the state evolves close to the equilibrium such an ability is no more of primary importance.

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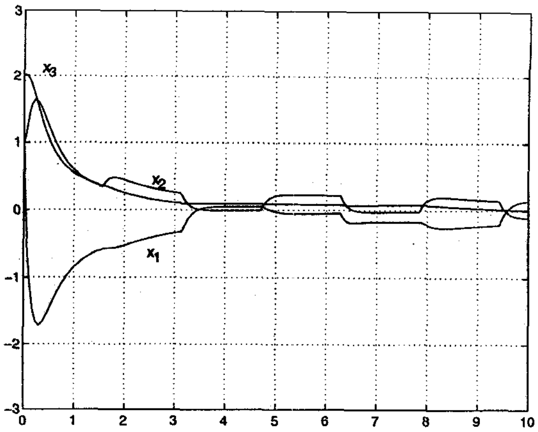

Figure 2: State histories in the case $p_3 = 0$ from the initial condition $\Omega_0 = (1, 1, 2)$.

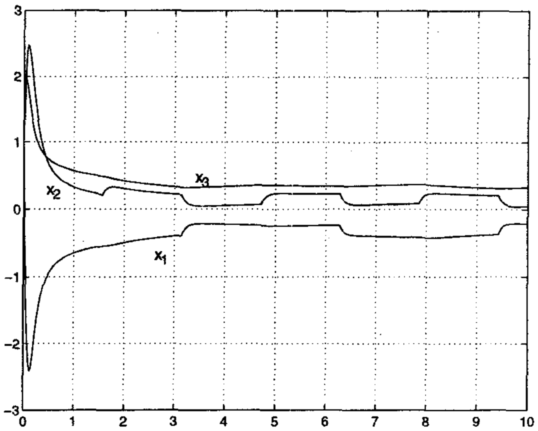

Figure 3: State histories in the case $p_3 \neq 0$ from the initial condition $\Omega_0 = (1, 1, 2)$.

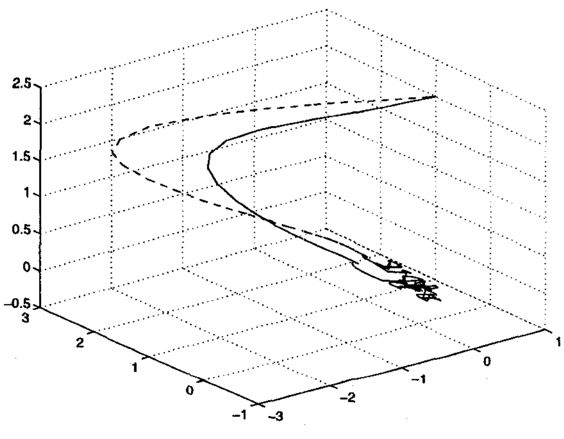

Figure 4: Phase portraits for $p_3 = 0$ (solid) and $p_3 \neq 0$ (dashed).