DISC Course on Linear Matrix Inequalities in Control

Siep Weiland
Department of Electrical Engineering
Eindhoven University of Technology

Course 2004 - Class 1

Control and optimization

Theory of H_2, LQG and H_∞-control synthesizes optimal controllers. However, the control paradigm is often restricted:

- Performance specs in terms of **complete closed-loop transfer**. Often (always?) only particular channels are relevant.
- Performance measure does not allow to impose particular time specs.
- Structured time-varying/nonlinear uncertainties can not be incorporated.
- Can only design LTI controllers.

For this course,

Controller is viewed as decision variable of optimization problem.
Specifications are constraints on controlled closed-loop system.

Optimization

Casting optimization problems in mathematics:

- $\mathcal{X}$: **decision set**
- $\mathcal{S}$: **feasible decisions**
- $f : \mathcal{S} \to \mathbb{R}$: **cost function** or **objective function**

f assigns to each decision $x \in \mathcal{S}$ a **cost** $f(x) \in \mathbb{R}$.

Goal is to select the decision $x \in \mathcal{S}$ that minimizes the cost $f(x)$.

This abstract formulation is hopelessly general. Requires the introduction of structural properties on $\mathcal{S}$ and f to convert to numerically efficient solutions.

Some classifications

Concrete features of problem formulation:

- $\mathcal{X}$ is a real vector space: **continuous problem**
  - ◇ $\dim \mathcal{X} < \infty$: **finite dimensional** problem
  - ◇ $\dim \mathcal{X} = \infty$: **infinite dimensional** problem
- $\mathcal{X}$ is a finite/discrete set: **combinatorial problem**
- Set of feasible decisions often described by equations and inequalities:

$$\mathcal{S} = \{x \in \mathcal{X} \mid g_k(x) \leq 0 \text{ for } k \in K, \quad h_\ell(x) = 0 \text{ for } \ell \in L\}$$

  - ◇ case K and L finite: **nonlinear program**
  - ◇ case K or L infinite: **semi-infinite** optimization.

Questions in optimization problems

Minimize f over $\mathcal{S}$ means:

- What is least possible cost? Compute **optimal value**

$$\boxed{f_{\text{opt}} := \inf_{x \in \mathcal{S}} f(x) \geq -\infty}$$

Convention: If $\mathcal{S} = \emptyset$ then $f_{\text{opt}} = +\infty$.
Convention: If $f_{\text{opt}} = -\infty$ then problem is said to be **unbounded**.

- Can we find, for arbitrary $\varepsilon > 0$, the **almost optimal solutions**

$$\boxed{x_\varepsilon \in \mathcal{S} \text{ with } f_{\text{opt}} \leq f(x_\varepsilon) \leq f_{\text{opt}} + \varepsilon. ?}$$

By definition of the infimum, almost optimal solutions always exist.

Solutions in optimization problems

- Does there exist an **optimal solution**? That is, does there exist

$$\boxed{x_{\text{opt}} \in \mathcal{S} \text{ with } f_{\text{opt}} = f(x_{\text{opt}}) \text{ ?}}$$

  If exists, x_{opt} is called a **minimizer** of f, and we write

$$f(x_{\text{opt}}) = \min_{x \in \mathcal{S}} f(x).$$

- **Set of all optimal solutions** is

$$\boxed{\arg\min_{x \in \mathcal{S}} f(x) := \{x \in \mathcal{S} \mid f_{\text{opt}} = f(x)\}}$$

- Is the optimal solution **unique**? When is it?

Recap: infimum and minimum of functions

Any $f : \mathcal{S} \to \mathbb{R}$ has an **infimum** $f_- \in \mathbb{R} \cup -\infty$ denoted $\inf_{x\in\mathcal{S}} f(x)$.

The infimum is uniquely defined by the properties

- $f_- \leq f(x)$ for all $x \in \mathcal{S}$
- $f_- < \infty$: for all $\varepsilon > 0$ exists $x \in \mathcal{S}$ with $f(x) \leq f_- + \varepsilon$.
  $f_- = -\infty$: for all $\varepsilon > 0$ exists $x \in \mathcal{S}$ with $f(x) \leq -\varepsilon$.

If there exists $x_0 \in \mathcal{S}$ with $f(x_0) = \inf_{x\in\mathcal{S}} f(x)$ we say that f **attains its minimum on** $\mathcal{S}$ and write $f_- = \min_{x\in\mathcal{S}} f(x)$.

If it exists, the **minimum** of f is uniquely defined through the properties

- $f_- \leq f(x)$ for all $x \in \mathcal{S}$
- there exists $x_0 \in \mathcal{S}$ for which $f_- = f(x_0)$.

A first result on existence of optimal solutions

Theorem: (Weierstrass) If $f : \mathcal{S} \to \mathbb{R}$ is continuous and $\mathcal{S}$ is a compact subset of a normed linear space, then there exists $x_{\min}, x_{\max} \in \mathcal{S}$ such that for all $x \in \mathcal{S}$

$$\inf_{x \in \mathcal{S}} f(x) = f(x_{\min}) \leq f(x) \leq f(x_{\max}) = \sup_{x \in \mathcal{S}} f(x)$$

Comments:

- Answers question of existence of optimal solutions for special $\mathcal{S}$ and f.
- Gives no clue on *how* to find $x_{\min}$, $x_{\max}$.
- No answer to uniqueness issue
- $\mathcal{S}$ compact if for every sequence $x_n \in \mathcal{S}$ a subsequence x_{n_m} exists which converges to a point $x \in \mathcal{S}$.
- Conditions are restrictive!

Convex sets

A set $\mathcal{S}$ in a linear vector space $\mathcal{X}$ is **convex** if

$$\{x_1, x_2 \in \mathcal{S}\} \quad \Longrightarrow \quad \{\alpha x_1 + (1-\alpha)x_2 \in \mathcal{S} \text{ for all } \alpha \in (0,1)\}$$

Convention: the empty set and singletons are convex.

The point $\alpha x_1 + (1-\alpha)x_2$ with $\alpha \in (0,1)$ is a **convex combination** of x_1 and x_2. More generally,

The point $x \in \mathcal{S}$ is a **convex combination** of $x_1, \ldots, x_n \in \mathcal{S}$ if

$$x := \sum_{i=1}^{n} \alpha_i x_i \qquad \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1$$

- Convex combination of convex combination is convex combination.
- The set of all convex combinations of $x_1, \ldots, x_n$ is convex.

Basic properties of convex sets

Theorem: Let $\mathcal{S}$ and $\mathcal{T}$ be convex sets in a normed space $\mathcal{X}$. Then

1. $\alpha\mathcal{S} := \{x \mid x = \alpha s, s \in \mathcal{S}\}$ is convex
2. $\mathcal{S} + \mathcal{T} := \{x \mid x = s + t, s \in \mathcal{S}, t \in \mathcal{T}\}$ is convex
3. closure of $\mathcal{S}$ and interior of $\mathcal{S}$ are convex
4. the intersection of any family of convex sets is convex.

Recall:

- $x \in \mathcal{S}$ is **interior point** of $\mathcal{S}$ if there exist $\varepsilon > 0$ such that
$$\{y \mid \|x - y\| \leq \varepsilon\} \subseteq \mathcal{S}.$$
- $x \in \mathcal{X}$ is **closure point** of $\mathcal{S} \subseteq \mathcal{X}$ if for all $\varepsilon > 0$ there exists $y \in \mathcal{S}$ such that $\|x - y\| \leq \varepsilon$.
- Last property is very important!

Examples of convex sets

Theorem: The intersection of any family of convex sets is convex.

- With $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$, the
  - **hyperplane**: $\{x \in \mathbb{R}^n \mid a^\top x = b\}$
  - **half-space**: $\{x \in \mathbb{R}^n \mid a^\top x \leq b\}$

  are convex.
- The intersection of finitely many hyperplanes and half-spaces defines a **polyhedron**.
  Any polyhedron is convex and can be described as
  $$\{x \in \mathbb{R}^n \mid Ax \leq b, \quad Cx = d\}$$
  for suitable matrices A, C, vectors b, d.

A **polytope** is a compact polyhedron.

Convex hulls

The **convex hull**, $\mathrm{co}(\mathcal{S})$, of any subset $\mathcal{S} \subset \mathcal{X}$ is the intersection of all convex sets containing $\mathcal{S}$. That is,

$$\mathrm{co}(\mathcal{S}) := \cap\{\mathcal{T} \mid \mathcal{T} \text{ is convex }, \mathcal{S} \subseteq \mathcal{T}\}$$

- The convex hull $\mathrm{co}(\mathcal{S})$ is **convex**.
- $\mathrm{co}(\mathcal{S})$ is equal to the set of **all convex combinations** of points of $\mathcal{S}$.
- If $\mathcal{S}$ is a finite set, then the convex hull $\mathrm{co}(\mathcal{S})$ is a **polytope**.

In fact, **any polytope** is the convex hull of a finite set $\mathcal{S}$.

Example: $\{x \in \mathbb{R}^n \mid a \leq x \leq b\}$ is defined by $2n$ inequalities and is the convex hull of 2^n points.

Convex functions

A function $f : \mathcal{S} \longrightarrow \mathbb{R}$ is **convex** if

- $\mathcal{S}$ is convex and
- for all $x_1, x_2 \in \mathcal{S}$, $\alpha \in (0, 1)$ there holds

$$\boxed{f(\alpha x_1 + (1-\alpha)x_2) \quad \leq \quad \alpha f(x_1) + (1-\alpha) f(x_2)}$$

It is **strictly convex** if $<$ instead of $\leq$.

Examples:

- $f(x) = x^2$ on $\mathbb{R}$
- $f(x) = |x|$ on $\mathbb{R}$
- $f(x) = \|x\|$ on $\mathbb{R}$.
- $f(x) = \sin x$ on $[\pi, 2\pi]$.

Convex functions (more general) –optional

A Hermitian valued function $F : \mathcal{S} \to \mathbb{H}^n$ is **(strictly) convex** if $\mathcal{S}$ is convex and for all $x_1, x_2 \in \mathcal{S}$, $\alpha \in (0, 1)$ there holds

$$\boxed{F(\alpha x_1 + (1-\alpha)x_2) \quad \preceq (\prec) \quad \alpha F(x_1) + (1-\alpha)F(x_2).}$$

Here,

- $\mathbb{H}^n$ is the set of $n \times n$ Hermitian matrices.

  That is $A \in \mathbb{H}^n$ if $A = A^* = \bar{A}^\top$.
- All eigenvalues of Hermitian matrices are real.
- $A \prec 0$ means that A is **negative definite**, that is

  $x^* A x < 0$ for all complex vectors $0 \neq x \in \mathbb{R}^n$

  Equivalently, all **eigenvalues** of A are negative.

Sublevel sets and convex functions

Theorem: If $f : \mathcal{S} \to \mathbb{R}$ is convex then for any $\gamma \in \mathbb{R}$ the **sublevel set**

$$\mathcal{S}^{\gamma} := \{x \in \mathcal{S} \mid f(x) \leq \gamma\}$$

is convex.

Remarks:

- Converse is not true: f can be non-convex if all its sublevel sets are convex.
- $\mathcal{S}^{\gamma} = \emptyset$ if $\gamma < \inf_{x \in \mathcal{S}} f(x)$.
- If $\gamma' \leq \gamma''$ then $\mathcal{S}^{\gamma'} \subset \mathcal{S}^{\gamma''}$.

The above result is simple, but has **many** applications.

For example, $f_k : \mathcal{S} \to \mathbb{R}$ are convex, $\gamma_k \in \mathbb{R}$. What can we say about

$$\{x \in \mathcal{S} \mid f_k(x) \leq \gamma_k, \quad k = 1, \ldots, K\} \qquad ???$$

Example: multi-objective control

Quantification of design specs by **functional inequalities** $f : \mathcal{S} \to \mathbb{R} \cup \{\infty\}$:

$$\boxed{\mathcal{S}^\gamma = \{x \in \mathcal{S} \mid f(x) \leq \gamma\}}$$

- Natural ordering: $\mathcal{S}^{\gamma_1} \subseteq \mathcal{S}^{\gamma_2}$ whenever $\gamma_1 \leq \gamma_2$.
- Allows multi-criterion specification

$$\mathcal{S}_\gamma = \mathcal{S}_1^{\gamma_1} \cap \mathcal{S}_2^{\gamma_2} \cap \ldots \cap \mathcal{S}_K^{\gamma_K}$$

for some multi-index $\gamma = (\gamma_1, \ldots, \gamma_K)$.

Example:

$$\mathcal{S}^{(\gamma_1,\gamma_2)} = \underbrace{\{x \in \mathcal{S} \mid f_1(x) = \|T\|_{H_\infty} < \gamma_1\}}_{\mathcal{S}_1^{\gamma_1}} \cap \underbrace{\{x \in \mathcal{S} \mid f_2(x) = \|T\|_{H_2} < \gamma_2\}}_{\mathcal{S}_2^{\gamma_2}}$$

where T is the 'closed-loop' transfer associated with the decision ('controller') $x \in \mathcal{S}$.

But what's a suitable design now ??

Pareto optimal solutions

Consider the multi-criterion specification

$$\mathcal{S}^{\gamma} = \mathcal{S}_1^{\gamma_1} \cap \mathcal{S}_2^{\gamma_2} \cap \ldots \cap \mathcal{S}_K^{\gamma_K}$$

for some multi-index $\gamma = \mathrm{col}(\gamma_1, \gamma_2, \ldots, \gamma_K) \in \mathbb{R}^K$.

A specification $\gamma^* \in \mathbb{R}^K$ is called **Pareto optimal** if $\mathcal{S}^{\gamma}$ is feasible for $\gamma > \gamma^*$ and infeasible for $\gamma < \gamma^*$. A point $x^* \in \mathcal{S}^{\gamma^*}$ (if exists) is called a **Pareto optimal solution**.

Interpretations:

- Every relaxation of γ^* is feasible; every tightening of γ^* is infeasible. Defines a **partial ordering** on design specifications.
- γ feasible but not Pareto optimal $\Longrightarrow$ γ can be tightened.
- γ infeasible but not Pareto optimal $\Longrightarrow$ γ should be relaxed.
- Set of all Pareto optimal specifications is **trade-off surface** in $\mathbb{R}^K$.

How to find Pareto optimal solutions in multi-objective control designs?

Example: quadratic functions

Consider the **quadratic function**

$$f(x) = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} q & s^\top \\ s & R \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = q + 2s^\top x + x^\top R x$$

When is it convex ??.

Its sublevel set $\mathcal{S}_0 := \{x \mid f(x) \leq 0\}$ is a

- **half-space** if $R = 0$
- **ellipsoid** if $s = 0$ and $R \succ 0$.

Affine sets

A subset $\mathcal{S}$ of a linear vector space is **affine** if $x = \alpha x_1 + (1 - \alpha) x_2$ belongs to $\mathcal{S}$ for every $x_1, x_2 \in \mathcal{S}$ and $\alpha \in \mathbb{R}$.

- Geometric idea: line through any two points belongs to set.
- Every affine set is convex.
- $\mathcal{S}$ affine if and only if

$$\boxed{\mathcal{S} = \{x \mid x = x_0 + m, m \in \mathcal{M}\}}$$

with $\mathcal{M}$ a linear subspace.

Affine functions

A function $f : \mathcal{S} \to \mathcal{T}$ is **affine** if

$$f(\alpha x_1 + (1-\alpha)x_2) = \alpha f(x_1) + (1-\alpha)f(x_2)$$

for all $x_1, x_2 \in \mathcal{S}$.

Theorem: If $\mathcal{S}$ and $\mathcal{T}$ are finite dimensional, then $f : \mathcal{S} \to \mathcal{T}$ is affine if and only if

$$f(x) = f_0 + T(x).$$

where $f_0 \in \mathcal{T}$ and $T : \mathcal{T} \to \mathcal{T}$ a linear map (a matrix).

Hence, affine functions are translates of linear functions.

How to check convexity of functions??

Theorem: All affine functions are convex.

Not easy to verify convexity of non-affine functions. The following is a classical result:

Theorem: Let f be twice continuously differentiable on the interior of $\mathcal{S}$. Then $f : \mathcal{S} \to \mathbb{R}$ is convex if and only if

$$\boxed{\partial^2 f(x) \succeq 0}$$

for all $x \in \mathcal{S}$.

Theorem: $f : \mathcal{S} \to \mathbb{R}$ is convex if and only if its **epigraph**

$$\boxed{\{(x, y) \mid x \in \mathcal{S}, y \geq f(x)\}}$$

is a convex set.

General convex programming

Let $\mathcal{S} = \{x \in \mathbb{R}^n \mid g(x) \leq 0, h(x) = 0\}$ be a given feasible set with $g : \mathbb{R}^n \to \mathbb{R}^k$ and $h : \mathbb{R}^n \to \mathbb{R}^\ell$.

The optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{S}, \quad g(x) \leq 0, \quad h(x) = 0 \end{array}$$

is said to be a

- **convex program** if f and g are convex and h is affine.
- **linear program** if f, g and h are all affine.
- **quadratic program** if f is quadratic and g and h are affine.

These are probably the only tractable instances of this nonlinear optimization problem.

Why is convexity important ???

Reason 1: absence of local minima

Solvers for nonlinear optimizations typically determine **local minima**.

Let $f : \mathcal{S} \to \mathbb{R}$. An element $x_0 \in \mathcal{S}$ is said to be a

- **local optimal solution** of f if there exists $\varepsilon > 0$ such that
$$f(x_0) \leq f(x) \text{ for all } x \in \mathcal{S}, \|x - x_0\| \leq \varepsilon.$$
- **global optimal solution** of f if $f(x_0) \leq f(x)$ for all $x \in \mathcal{S}$.

Main feature of convex optimizations:

Theorem: Suppose $f : \mathcal{S} \to \mathbb{R}$ is convex. Every local optimal solution of f is a global optimal solution. If f is strictly convex, then the global optimal solution is moreover unique.

Doesn't say anything about existence of optimal solutions.

Why is convexity important ???

Reason 2: uniform bounds

Trivial result:

Theorem: Suppose $\mathcal{S} = \mathrm{co}(\mathcal{S}_0)$ and $f : \mathcal{S} \to \mathbb{R}$ is convex. Then equivalent statements are

1. $f(x) \leq \gamma$ for all $x \in \mathcal{S}$
2. $f(x) \leq \gamma$ for all $x \in \mathcal{S}_0$.

Very interesting if $\mathcal{S}_0$ consists of **finite number of points**, i.e,

$$\mathcal{S}_0 = \{x_1, \ldots, x_n\}.$$

Implies **finite** test!!

Why is convexity important ???

Reason 3: subgradients

A vector $g = g(x_0) \in \mathbb{R}^n$ is called a **subgradient** of $f : \mathcal{S} \longrightarrow \mathbb{R}$ at $x_0 \in \mathcal{S}$ if

$$f(x) \geq f(x_0) + \langle g, x - x_0 \rangle \qquad \text{for all } x \in \mathcal{S}.$$

Set of all subgradients is called **subdifferential** and denoted as $\partial f(x_0)$.

Geometric idea:
Graph of affine function $x \mapsto f(x_0) + \langle g, x - x_0 \rangle$ is tangent to graph of f at x_0.

Main result of convex analysis:

Theorem: A convex function $f : \mathcal{S} \longrightarrow \mathbb{R}$ has a subgradient at every interior point x_0 of $\mathcal{S}$.

Examples and properties of subgradients

A vector $g \in \mathbb{R}^n$ is **subgradient of f at x_0** if

$$f(x) \geq f(x_0) + \langle g, x - x_0 \rangle$$

for all $x \in \mathcal{S}$.

- Example: $f(x) = |x|$ has any real number $g \in [-1, 1]$ as its subgradient at 0.
- if f is differentiable, then the gradient $g = g(x_0) = \nabla f(x_0)$ will do.
- $f(x_0)$ is global minimum if and only if 0 is subgradient of f.
- Since

$$\boxed{\langle g, x - x_0 \rangle > 0 \quad \Longrightarrow \quad f(x) > f(x_0),}$$

  all points in half-space $\{x \mid \langle g, x - x_0 \rangle > 0\}$ can be discarded in searching for minimum of f.

Ellipsoid algorithm: main ideas

Aim: Minimize the convex function $f : \mathbb{R}^n \to \mathbb{R}$.

Suppose some minimizer lies inside the ellipsoid

$$\mathcal{E}_0 := \{x \in \mathbb{R}^n \mid (x - x_0)^\top P_0^{-1}(x - x_0) \leq 1\}$$

where $P_0 \succ 0$.

Problem: Can we compute a smaller ellipsoid containing all the minimizers of f?

First point:
Compute **one subgradient** $g_0 \in \partial f(x_0)$.

If $\langle g, x - x_0 \rangle > 0$ then $f(x) > f(x_0)$.

Hence all minimizers must be contained in

$$\mathcal{E}_0 \cap \{x \in \mathbb{R}^n \mid \langle g, x - x_0 \rangle \leq 0\}$$

Ellipsoid algorithm: main ideas

Covering ellipsoid:
For $x_k \in \mathbb{R}^n$ and $P_k \succ 0$ suppose

$$\mathcal{E}_k := \{x_k \mid (x - x_k)^\top P_k^{-1}(x - x_k) \leq 1\}$$

For any nonzero $g_k \in \mathbb{R}^n$, the ellipsoid $\mathcal{E}_{k+1}$ covers

$$\mathcal{H}_k := \mathcal{E}_k \cap \{x \in \mathbb{R}^n \mid \langle g_k, x - x_k\rangle \leq 0\}$$

if we set $\lambda_k = \sqrt{g_k^\top P_k g_k}$ and $v_k = P_k g_k / \lambda_k$ and

$$x_{k+1} = x_k - \frac{1}{n+1} v_k, \qquad P_{k+1} = \frac{n^2}{n^2 - 1}\left(P_k - \frac{2}{(n+1)} v_k v_k^\top\right).$$

The volume decreases as $\operatorname{vol}(\mathcal{E}_{k+1}) \leq e^{-\frac{1}{2n}} \operatorname{vol}(\mathcal{E}_k)$.

One can prove that $\mathcal{E}_{k+1}$ is the smallest covering ellipsoid.

Ellipsoid algorithm

1. Let f, P_0, $\mathcal{E}_0$ be given. Set $k = 0$.
2. Compute a **subgradient** g_k of f at x_k. If $g_k = 0$ then stop, otherwise proceed to Step 2.
3. All minimizers are contained in

$$\mathcal{H}_k := \mathcal{E}_k \cap \{x \in \mathbb{R}^n \mid \langle g_k, x - x_k \rangle \leq 0\}.$$

4. Compute the **covering ellipsoid**

$$\mathcal{E}_{k+1} := \{x \in \mathbb{R}^n \mid (x - x_{k+1})^\top P_{k+1}^{-1}(x - x_{k+1}) \leq 1\}$$

   that contains $\mathcal{H}_k$.

5. Set k to $k + 1$ and return to Step 2.

Gives decreasing sequence of ellipsoids $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_1$, ... all guaranteed to contain minimizer of f.

Stopping criteria ellipsoid algorithm

Suppose $x^* \in \mathcal{E}_0$ is minimizer of f. The algorithm guarantees $x^* \in \mathcal{E}_k$ for all k. Hence, we have

$$\begin{aligned} f(x_k) \geq f(x^*) \geq f(x_k) + \langle g_k, x^* - x_k \rangle \geq \\ \geq f(x_k) + \inf_{\xi \in \mathcal{E}_k} \langle g_k, \xi - x_k \rangle = f(x_k) - \sqrt{g_k^\top P_k g_k} \end{aligned}$$

Therefore,

$$U_k := \min_{\ell \leq k} f(x_\ell) \ \geq \ f(x^*) \ \geq \ \max_{\ell \leq k} \left(f(x_\ell) - \sqrt{g_\ell^\top P_\ell g_\ell} \right) =: L_k$$

Stopping criterion for guaranteed accuracy:

$U_k - L_k < \varepsilon$ guarantees $|f(x_k) - f(x^*)| < \varepsilon$.

Properties of ellipsoid algorithm

- If $\mathcal{E}_0$ contains at least one minimizer of f then $f(x_k)$ converges to the minimal value of f.
- The sequence x_k is not guaranteed to converge. Certainly not to a minimizer of f.
- Exist explicit equations for x_k, P_k, $\mathcal{E}_k$ such that volume of $\mathcal{E}_k$ decreases with a factor $e^{-1/2n}$.
- Simple, robust, easy to implement.
- However, slow convergence.

Summary

- We considered general optimization problems
- Convex sets and convex functions: definitions and facts
- Convexity distinguishes easy from difficult optimization problems
- We considered subgradients and their role in optimization
- We discussed ellipsoid algorithm.

Linear Matrix Inequalities (LMI's)

A **linear matrix inequality (LMI)** is an expression

$$F(x) = F_0 + x_1 F_1 + \ldots + x_n F_n \prec 0$$

where

- $x = \mathrm{col}(x_1, \ldots, x_n)$ is a vector of reals, the **decision variables**,
- $F_i = F_i^\top$ are real symmetric matrices and
- $\prec 0$ means negative definite, i.e.,

$$\begin{aligned} F(x) \prec 0 \quad &\Leftrightarrow \quad z^\top F(x) z < 0 \text{ for all } z \neq 0 \\ &\Leftrightarrow \quad \text{all eigenvalues of } F(x) \text{ are negative} \\ &\Leftrightarrow \quad \lambda_{\max}\left(F(x)\right) < 0 \end{aligned}$$

Note that F is an **affine function** of the decision variables.

Simple examples

- $1 + x < 0$
- $1 + x_1 + 2x_2 < 0$
- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} < 0.$

All the same with $\preceq 0$, $\succeq 0$ and $\succ 0$.

Only very simple cases can be treated analytically.

Need to resort to numerical techniques!

Main LMI problems

The LMI feasibility problem:
Test whether there exists $x_1, \ldots, x_n$ such that $F(x) \prec 0$.

The LMI optimization problem:
Minimize $c_1 x_1 + \ldots + c_n x_n$ over all $x_1, \ldots, x_n$ that satisfy $F(x) \prec 0$.

How is this solved?
$F(x) \prec 0$ is feasible if and only if $\min_x \lambda_{\max}(F(x)) < 0$ and therefore involves minimizing the function

$$f : x \mapsto \lambda_{\max}(F(x))$$

This is possible because this function is convex!
There exist very efficient algorithms for this (interior point, ellipsoid).

Next class