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Robust Control

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Preamble

This note has been written for a basic course in robust and optimal control at 9th term of the System Construction Line, Institute of Electronic Systems, Aalborg University. Originally, the note was intended for a course consisting of six modules of four hours each. Currently, the courses at Aalborg University has been reduced to five modules, and hence, the note contains material, that can not be included in the course. The note has been adapted to a level, which can be expected of 9th term students at the System Construction Line. The students are expected to be acquainted with classical feedback control theory.

The purposes of the note is to provide an introduction to modern robust and optimal control, especially in $\mathcal{H}_\infty$ and μ theory. In Chapter 1, a short introduction to the concept of robust control is given; In Chapter 2 nominal and robust stability for single variable (SISO) systems is described. In Chapter 3 nominal and robust performance for SISO systems is analyzed, and the concepts $\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimal control are introduced. Chapter 4 gives an introduction to the analysis of multi variable systems, and in Chapter 5, stability and performance of multi variable systems are studied. In Chapter 6, a solution to the $\mathcal{H}_\infty$ control problem is presented. Finally, in Chapter 7, an introduction to the structured singular value μ is given, and controller design with μ is treated.

Key Words

Robust optimal control; robust stability; robust performance; $\mathcal{H}_\infty$ optimal control; μ analysis; μ synthesis;

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Chapter 1

Robust Feedback Control

Chapter 1 contains a general presentation of classical feedback control, with an emphasis on robustness to model uncertainty of feedback systems. The need to develop design methods, that explicitly can handle model uncertainty is demonstrated.

1.1 Feedback and Uncertainty

Control of a dynamical process by feedback of a measured output is a well-known principle, typically with the primary objective of keeping the output of the process close to a given reference.

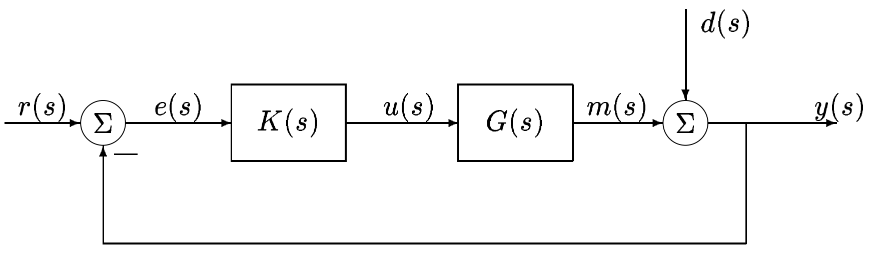


Figure 1.1: *Standard feedback configuration.*

In Figure 1.1, a standard feedback configuration is shown, consisting of a process $G(s)$ and a compensator $K(s)$. For systems with feedback the following important properties can often be established:

- Systems that are unstable from nature, can be stabilized
- The effect of external disturbances can be attenuated
- The time constants or characteristic frequencies of the system can be shifted
- Finally, often the above properties can be obtained even with an incomplete knowledge of the properties of the process itself

Hence, feedback controlled systems have attractive properties, that can be made robust with some care.

The advantages of feedback can be illustrated by considering the uncertain elements in a process. There are two reasons why, an output from a process is not completely known a priori. First, the dynamical properties of the system are not entirely known, i.e., a model can be considered only as an approximate description of a physical process. Second, unknown disturbances can influence the process. An output y is not only a function of the input u, controlled by the compensator, but also of a disturbance d, which often can not be measured.

The basic properties of feedback (closed loop control) is exposed by a comparison with a feedforward configuration (open loop control). By open loop (OL) control the result depend completely on the accuracy by which the process has been modeled, since the compensator determines the input based only on the model and the reference. Deviations caused by the disturbance d and model uncertainties have a full impact as a discrepancy between the actual and the expected output.

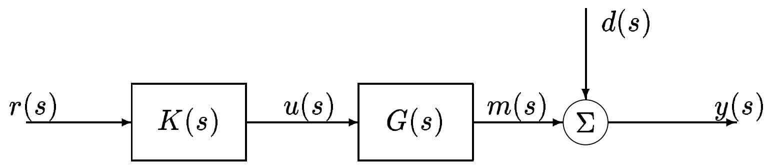


Figure 1.2: *Feedforward configuration.*

In contrast to this, the feedback compensator compares the actual value of the output y with the reference r, and determines the input u based on the error e. On top of the a priori knowledge available on the system in terms of a mathematical model, the feedback compensator exploits the knowledge of the actual behavior of the process and the simultaneous disturbances, which are implicitly given by the measurement y. Hence, it is possible to reduce the effect of the disturbance d as well as the effect of the imperfect modeling.

Design of feedback compensators embarks from requirements concerning the static and dynamical behavior of the controlled system. These requirements could include the following, [Lun89]:

1. Stability: the operation point of the controlled systems must be stable.
2. Asymptotic control: for a given class of reference inputs $r(t)$ and disturbances $d(t)$ the error $e(t)$ must tend to zero, as the time t tends to infinity.
3. Dynamical requirements: the controlled system must fulfill a set of specifications, such as bounds on the step response, and requirements to some degree of decoupling between the various signals, etc.
4. Requirements on robustness: The properties that are specified for the controlled system above, must be preserved under a given class of variations in the process dynamics.

In process control, the requirements of item 3 are often formulated as specifications for the output $y(t)$ and for the control signal $u(t)$ by step or ramp shaped variations in reference or disturbance. The specifications, e.g. can be bounds on overshoot M_p, rise time t_r and settling time t_s for a step response, see Figure 1.3.

The requirements can also be formulated in the frequency domain as specifications for the open loop or closed loop transfer function, for example as conditions on a closed loop resonance peak M_r or bandwidth f_0, see Figure 1.4.

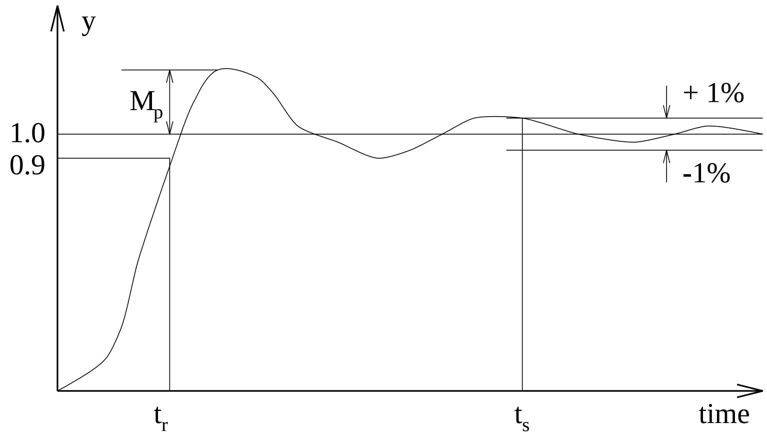


Figure 1.3: *Time domain design specifications: conditions on M_p, t_r and t_s for output y by a step in the reference signal r.*

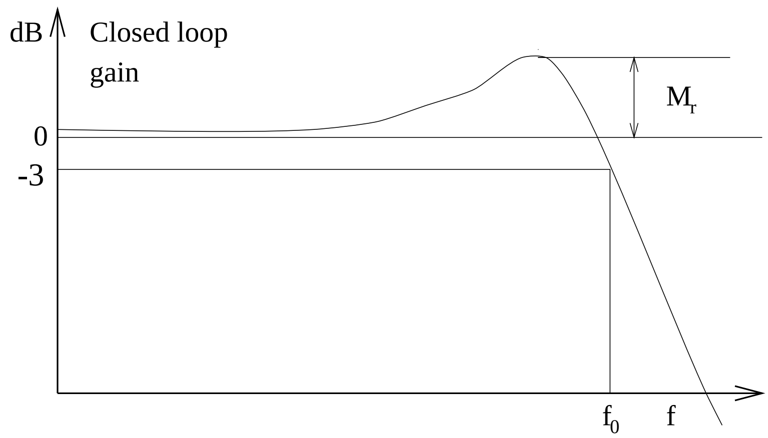


Figure 1.4: *Frequency domain design specifications: conditions on M_r and f_0.*

1.2 Classical Compensator Design Methods

A majority of the methods, that are applied to the design of compensators, presume an exact model of the process with an emphasis on items 1-3.

Stability analysis in the frequency plane was developed by Nyquist and Bode. Design of the compensators were carried out by methods based on 'trial and error'.

Design rules such as Ziegler and Nichols' are based on simple models, but they have succesfully been applied to processes with a higher complexity.

Root locus analysis was developed by W.R. Evans, and aims at obtaining prescribed dynamical properties by determining a compensator, which provides a satisfactory closed loop pole placement.

Moreover, methods have been developed that provide explicit formulae for a compensator with a prescribed pole placement.

Methods, that aim at satisfying performance specifications formulated as conditions on the integral of the square error, have been introduced by Newton, Gould & Kaiser and further developed by Kalman et al.

Even though one of the objectives of feedback control is to to reduce the effects of model deviations, all these methods presume a perfect model, and only indirectly accounts for the fact, that a model can never be perfect.

Since several of the methods have been applied succesfully for many years due to the inherent robustness of feedback compensators, it can be questioned whether it is reasonable to include model variations as an explicit design condition. There are, however, many examples where the introduction of feedback not automatically imply the robustness required to actual model variations. The following example originates from [Lun89].

Example 1.1 (Robust Stability)
A process is described by the model

$$G(s) = \frac{1}{1+\tau_1 s}$$

which experimentally has been shown to provide approximately the same step response as the real process. The process can, e.g., be controlled by a proportional compensator with the gain K. Theoretically, it should be possible without problems to select the gain K arbitrarily large.

It turns out in reality, however, that a more accurate model of the process is given by

$$\hat{G}(s) = \frac{1-\tau_0 s}{(1+\tau_1 s)(1+\tau_2 s)}$$

Now, it is apparent that the control system will become unstable, if K is chosen too large. To ensure stability, K must be chosen below the bound

$$K < \frac{\tau_1+\tau_2}{\tau_0}$$

If for example $\tau_0 = \tau_2 = 0.1 \cdot \tau_1$, K must satisfy $K < 11$ to ensure stability.

The example illustrate, that even if simple step response experiments lead to a model, that approximately describe the process, such a model can not be used for compensator design without evaluating the region of validity of the model.

The example further illustrate the main problem in the design of compensators for systems with model uncertainties: the desire of good performance lead to a need for high gains. This is in contrast to robustness, since model deviations easily can lead to instability in systems with high gains.

The fact that practical compensators have been designed in the past, is due to experienced engineers who design compensators with a certain conservatism, by not taking the gains to the theoretical limit. For compensators that are derived by minimizing a cost function, some robustness can be achieved by incorporating control signal increments in the cost function.

The development of methods that directly incorporate model uncertainties in compensator design, has accelerated within the past 20 years.

1.3 Model Uncertainty.

Normally, the first step in a compensator design would be the derivation of a model of the process to be controlled. Design of compensators based on the model derived must take into consideration under which conditions the model is valid. If certainty is required that the compensator is going to work under all conditions, it is necessary to augment the model of the process with a model, which expresses the possible deviations from the nominal model.

There are to reasons why, the output of the process can not be predicted exactly by the derived model of the preocess. The process can be influenced by disturbances, and the dynamics of the model can deviate from that of the model, see Figure 1.5

Disturbances are external signals, that are independent of the process inputs. The effect of disturbances can be aggregated at the output of the process as an exogenous input d, added to the model output, as shown in Figure 1.5a.

Deviations between the dynamics of the process and its model lead to discrepancies that in contrast to disturbances are not caused by unknown disturbances, but highly depend on the input u. Hence, model uncertainties can be represented by an error model with input z and output w, see Figure 1.5b.

Rejection of disturbances is an integral part of compensator design related to items 1-3. Thus, in this context model uncertainties are emphasized. There are three main sources for model uncertainty:

1. Incomplete knowledge on the process; this type of uncertainty might be due to the fact, that the model has been derived from the laws of physics, although the exact parameters of the process can not be determined from the available knowledge on the process. If the model is determined experimentally, the accuracy of the model depends on whether the process has been excited by inputs, that are suited for determining a model, and to what extent the process has been influenced by disturbances during the experiment.

2. Model simplification: even though the original system might be known in great detail, the model might have been reduced in order to simplify the design task.

3. Incomplete model structure: in general it is desirable to design compensators based on a linear model. Hence, nonlinearities in actuators or sensors must be omitted. Other types of nonlinearities is caused by nonlinear dynamics of the process itself. This type of nonlinearities often result in parameters that depend on the operating point.

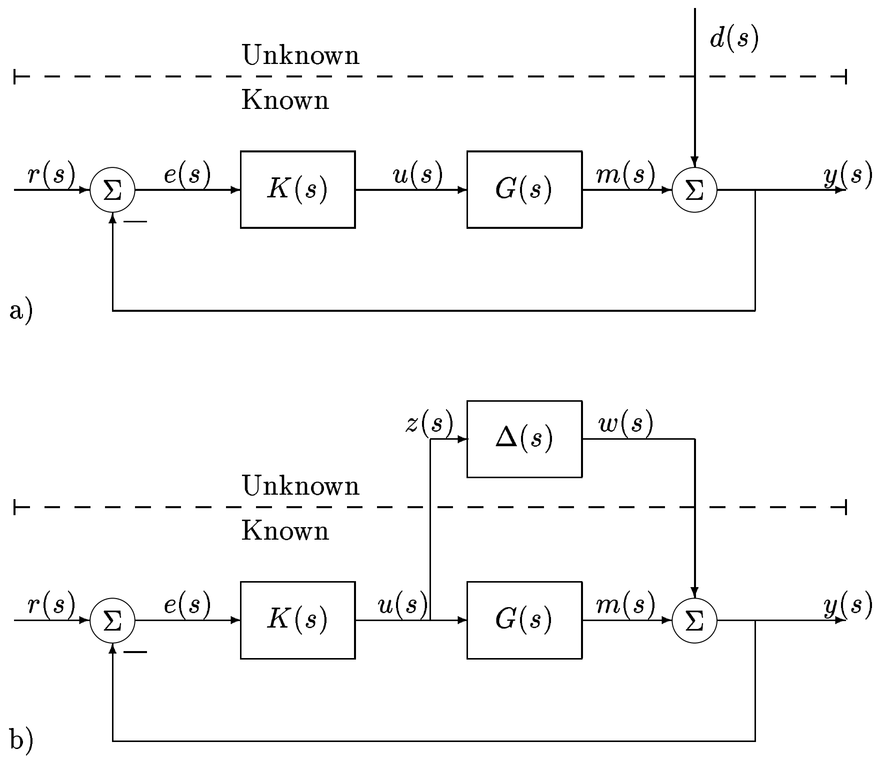


Figure 1.5: *Disturbances vs. model uncertainty in the closed loop system. a) Closed loop system exposed to disturbances. b) Closed loop system with model uncertainty.*

4. Time varying parameters: also result in variation of the parameters of the model.

In any circumstance, the model will only be an approximate representation of the physical process.

Systems with significant model uncertainties are simply refered to as *uncertain systems* irrespective of whether the uncertainties are caused by lack of information, model simplifications, nonlinearities, or time varying parameters.

In the literature, various concepts are associated with special types of model uncertainties. *Parametric uncertainty* is the type of uncertainty that can be compensated for by adjusting the parameters of the model. *Structural uncertainty* is the type of uncertainty that relate to an incomplete or incorrect model structure, e.g. by applying a linear model for process that exhibits nonlinear behavior, or by omitting dynamical elements in the process. Moreover, model uncertainty is classified subject to whether only the overall level of uncertainty is known, or whether also the character of the uncertainty is known. To this end *structured uncertainty* is distinguished from *unstructured uncertainty.*

Model uncertainty can be described by defining a set of possible models. In this set or family of models, each member can represent the original process, but it is unknown which specific member that actually does. Dependent on the character of the model uncertainty, the set of possible models appear in different ways. By mapping all possible models in the Nyquist plane, a bounded region in the plane is associated with each frequency rather than an isolated point.

Norm bounded model deviations were suggested by Doyle in connection with compensator design. The absolute value of the model deviation was limited to a certain frequency dependent quantity. This corresponds to circular bounds in the Nyquist plane.

1.4 Feedback Systems with Model Uncertainties.

Model uncertainty can not in general be treated as disturbances, although this is often done in practice, especially if the uncertainty is relatively small. Exogenous disturbances can not influence the stability of a linear closed loop system. However, as it was seen from Example 1.1, a model deviation can influence stability. In the example, the proportional gain had to be within a certain upper bound, depending on the parameters of the 'real' model.

In the sequel, a series of concepts are introduced which are significant for uncertain feedback systems:

- *Sensitivity* is a measure of how a certain property (e.g. the closed loop transfer function) depend on differential deviations of a parameter or on the entire vector of parameters. Sensitivity analysis deal with small deviations in the neighborhood of a nominal value, and does not refer to the admissible size of a deviation.
- *Robustness* describe the capability of a system to exhibit satisfactory properties for all models in a given family. Hence, robustness refer to an acceptable region for the desired properties and for given bounds of model deviations.
- Neither sensitivity nor robustness indicate the likelihood of a certain model deviation to arise in a certain time interval. *Reilability* is a measure for the capability of a system to function satisfactorily within a certain time range. In reliability analysis, statistical methods are applied. Thus, quantitative measures for reliability are probabilities, that a system functions satisfactorily at a certain time instant, or within a certain time range.
- *Adaptivity* is the capability of a control system to adjust to changes in the process. Some time after the changes in the process have occured, an adaptive system will have adjustet itself, and will again function satisfactorily.

  In feedback control, sensitivity, robustness, and adaptivity play crucial roles. Designing robust controllers, involves looking for linear compensators that ensure satisfactory performance for all possible model variations. As an alternative, adaptive control methods try to adjust the compensator to the model that provides the best fit at any time instance.

Reliability analysis is mainly used in connection with implementation of compensators and other process control equipment, and mainly to answer questions like: "How large is the probability that the process control equipment performs satisfactorily?"

1.5 Robust Control.

These lecture notes addresses the analysis of robustness of control systems and the design of robust control systems. This has been the subject of intense research during the past years. In this context, the following research results are crucial:

- In 1976, Youla et al. [YJB76a], [YJB76b] demonstrated, that it is possible to parameterize all compensators that stabilize a given system. This greatly simplifies the design of a stabilizing controller.
- In 1979, Doyle and Stein [DS79] pointed out that the good phase and amplitude margins of the LQ controller, easily are ruined by an observer, and also suggested a method to recover these appealing properties.
- In 1981, Zames, [Zam81], suggested to apply the $\mathcal{H}_\infty$ norm rather than the traditonal $\mathcal{H}_2$ norm in order to evaluate the performance of a compensator.
- In 1981, Doyle and Stein, [DS81], showed that model uncertainties can be described well as norm bounded deviations. Along with the $\mathcal{H}_\infty$ norm, norm bounded deviations constitute an excellent tool to describe the robustness of a compensator. In 1982, Doyle [Doy82] suggested to generalize this to structured uncertainties by introducing the structured singular value, μ.
- In 1989, Doyle et al. [DGKF89] derived explicit design formulae for compensator design that minimized the $\mathcal{H}_\infty$ norm based on state space models.

Chapter 2

Nominal and Robust Stability

In order to be able to design a robust compensator to control a given process, it is necessary not only to specify a nominal model of the process, but also the model uncertainty to which the control system has to be robust. The compensator is required to make the output follow variations in the reference signal and to attenuate disturbances. Hence, to design the compensator, it is also necessary to know the anticipated character of the reference signal and of the disturbances. Given these, performance specifications can be expressed as requirements of a certain level of error reduction in the presence of these inputs.

Design of a robust compensator is based on:

- A model of the process
- A description of the model uncertainty
- Knowledge of the character of the inputs (for reference and disturbances)
- Performance specifications

A compensator is said to be robust, if the performance specifications are satisfied both for the nominal model of the process as for any other model contained in the admissible set of models as specified by the model uncertainty.

In the following sections, a framework for specification of the model, of the model uncertainty, of the character of the inputs and of performance specifications for single variable (SISO) systems are formulated. For multi variable (MIMO) systems, see Chapter 5 on page 43.

2.1 A Model of the Proces

The concepts and methods used in the sequel are based on linear time invariant models.

2.2 Model Uncertainty

Model uncertainty is often specified in the frequency domain. This can lead to a region for each frequency ω in the Nyquist plane in which the 'real' model of the process $G_\Delta(j\omega)$ is known to be contained in. The shape of this region $g(\omega)$ is determined by the way in which the uncertainty is specified. Thus, specification of uncertainties in amplitude and phase will lead to a sector bound region as shown in Figure 2.1a. Several of the results and methods described in the following are based on norm bounded model deviations. For SISO systems, this means that the uncertainty at each frequency is bounded by a circle of radius $\ell_a(\omega)$ in the Nyquist plane, see Figure 2.1b.

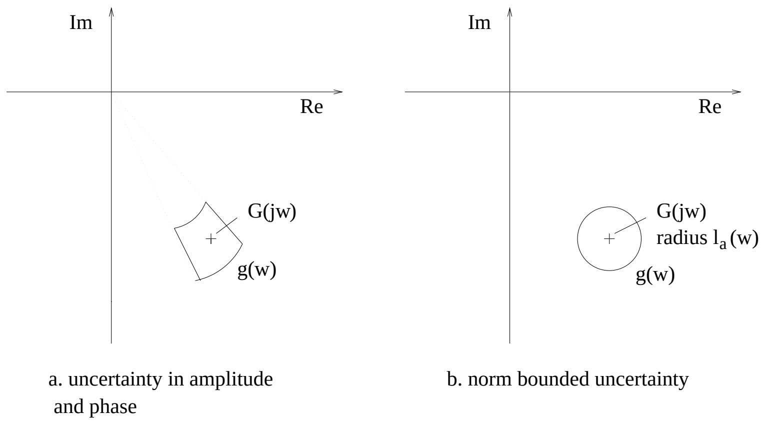


Figure 2.1: *Uncertainty regions by specification of (a) amplitude and phase uncertainty (b) norm bounded uncertainty.*

Based on this, a family of models can be defined:

$$\mathcal{G} = \{G_\Delta : |G_\Delta(j\omega) - G(j\omega)| \le \ell_a(\omega)\} \quad (2.1)$$
$$= \{G_\Delta : |G_\Delta(j\omega) - G(j\omega)| \le \ell_m(\omega)|G(j\omega)|\} \quad (2.2)$$

where $G(j\omega)$ denote the nominal model, and $G_\Delta(j\omega)$ denote possilbe models of the process. $\ell_a(\omega)$ is the maximal additive model uncertainty, and $\ell_m(\omega)$ is the maximal multiplicative (relative) uncertainty.

An arbitrary member of $\mathcal{G}$ can be described as:

$$G_\Delta(j\omega) = G(j\omega)(1 + \Delta_m(j\omega)) = G(j\omega) + \Delta_a(j\omega) \quad (2.3)$$

where the actual multiplicative or additive model deviation (Δ_m or Δ_a) is bounded by:

$$|\Delta_m(j\omega)| \le \ell_m(\omega) \quad (2.4)$$
$$|\Delta_a(j\omega)| \le \ell_a(\omega) \quad (2.5)$$

A multiplicative model uncertainty description often increase with increasing frequences, as the models applied for compensator design are derived with emphasis on the description of the dominating dynamics.

If $\ell_m(\omega) > 1$, the norm bounded model deviation allow the models to have a different number of zeros in the right half plane. A zero on the $j\omega$ axis is faciliated e.g. by $\Delta_m(j\omega) = -1$, see (2.3).

This uncertainty description, however, is not very suited for describing model deviations, where the number of poles in the right half plane might vary, since this means that $\ell_m(\omega)$ must assume the value ∞.

Note, that ℓ_a and ℓ_m are frequency dependent scalars and, hence, that they are functions of ω (rather than $j\omega$). Often, however, $\ell_a(\omega)$ and $\ell_m(\omega)$ will be represented by normal transfer functions, where mainly the amplitude will be of significance.

2.3 Nominal Stability

A control system is internally stable, if an excitation by a bounded signal anywhere in the system can not stimulate an unbounded signal somewhere. In Figure 2.2 a controlled process is shown with three inputs (r, u', d) and three outputs (e, u, y) to the overall system.

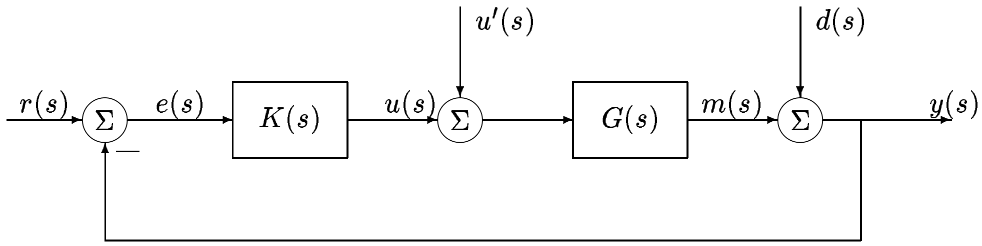


Figure 2.2: *Controlled system with inputs to the analysis of internal stability.*

Analyzing internal stability does not necessarily imply testing a 3×3 matrix, since several of the signals contain the same information from a stability point of view. The signals d and r, for instance, have the same influence on the output u with respect to stability. Choosing r and u' as inputs and y and u as outputs, the transfer matrix below can be determined:

$$\begin{bmatrix} y(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} \frac{GK(s)}{1+GK(s)} & \frac{G(s)}{1+GK(s)} \\ \frac{K(s)}{1+GK(s)} & \frac{-GK(s)}{1+GK(s)} \end{bmatrix} \begin{bmatrix} r(s) \\ u'(s) \end{bmatrix} \tag{2.6}$$

From (2.6) it can be seen, that the controlled system is internally stable, only if none of the four elements in the matrix have poles in the right half plane. In a similar fashion, if both $G(s)$ and $K(s)$ are stable, it suffices to analyze the characteristic equation $1 + GK(s) = 0$.

Note that if $G(s)$ has an unstable pole, it does not suffice to apply a compensator which exactly cancels the unstable pole in order to achieve internal stability. This can be seen fromg (2.6), since a bounded input u' imply an unbounded output y.

2.4 Robust Stability.

In the sequel, conditions for the controlled system to be robustly stable will be studied. This means that the system is stable for all models of the process contained in $\mathcal{G}$. It is assumed that all models in $\mathcal{G}$ has the same number of poles n in the right half plane.

Under these conditions, the controlled system is stable, only if the Nyquist curve for $GK(j\omega)$ encompasses the Nyquist point $(-1, 0)$ exactly n times counter-clockwise.

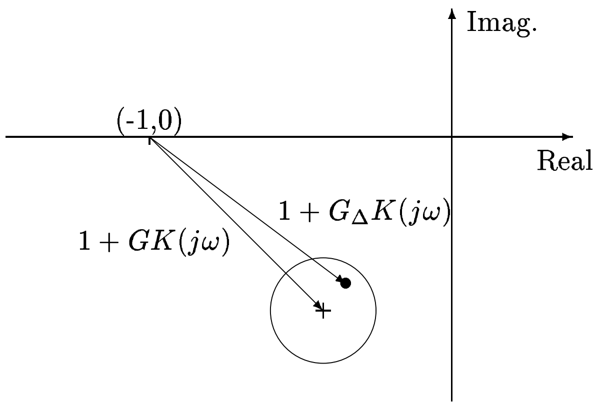


Figure 2.3: *Distance to the Nyquist point* $(-1, 0)$ *for the nominal system* $G(s)$ *and for an arbitrary model in* $\mathcal{G}$.

If $K(s)$ stabilizes the nominal system $G(s)$, the condition for $K(s)$ to stabilize all models in $\mathcal{G}$, is that the number of encirclements of $(-1, 0)$ does not change. This is equivalent to the region in the Nyquist plane, covered by all possible $G_\Delta K(j\omega)$ not to include $(-1, 0)$, or that the distance from $G(j\omega)$ to $(-1, 0)$ is larger than $GK(j\omega)\ell_m(\omega)$:

$$|1 + GK(j\omega)| > |GK(j\omega)\ell_m(\omega)| \ , \qquad \forall\omega \tag{2.7}$$

This expression can be rewritten as:

$$\frac{|GK(j\omega)|}{|1 + GK(j\omega)|}\ell_m(\omega) < 1 \ , \qquad \forall\omega \tag{2.8}$$

or

$$|T(j\omega)\ell_m(\omega)| < 1 \ , \qquad \forall\omega \tag{2.9}$$

Here $T(s)$ is the closed loop transfer function from reference to output, which is also called the complementary sensitivity function. Hence, if the model uncertainty is specified by bounding the norm of the model deviation, an analytical experession is obtained for robust stability, which offer the possibility to apply the condition directly in design methods that, e.g., involve minimization. If the model uncertainty is given in other ways, for example by specifying the uncertainties for amplitude and phase, it is more obvious to apply design methods, where a graphical inspection is used to make sure that the uncertainty region for the open loop transfer function does not contain the Nyquist point $(-1, 0)$.

Chapter 3

Nominal and Robust Performance

This chapter presents approaches to formulate performance specifications for a control system.

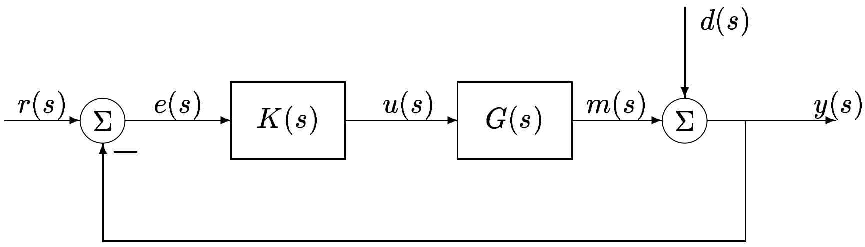


Figure 3.1: *Standard feedback configuration.*

Figure 3.1 shows a standard system with feedback control. The controlled system has an input reference r and a disturbance signal d. Since the two inputs have the same transfer function to the error signal e (except for the sign difference), they are treated collectively, denoted by the signal v.

A compensator is traditionally designed for a specific input. This is true for classical design methods, where the design often aims at achieving certain characteristics for the closed loop response to a step or ramp input as mentioned in Chapter 1. Likewise, linear quadratic control aims at minimizing the error for a given input signal.

In practice, it is often more relevant to design the compensator for a class of related inputs with the same characteristics. The exposition below aims at assessing the input error for different compensators exposed to exogenous signals of the same 'size' interpreted in a norm sense.

3.1 Signal Norms.

To measure the 'size' of a time domain signal, a norm will be applied. Predominantly, the *2 norm* will be applied, which is defined by:

$$\|v\|_2 := \left(\int_{-\infty}^{\infty} v(t)^2 dt \right)^{\frac{1}{2}} \tag{3.1}$$

Other signals norm are possible, however. For a map $\|\cdot\|$ as e.g. 3.1 to be a norm of a time signal $v(t)$, it must possess the following four properties:

1. $\|v\| \geq 0 \quad \forall v$
2. $\|v\| = 0 \Leftrightarrow v(t) = 0\ , \quad \forall t$
3. $\|av\| = |a|\ \|v\|\ , \quad \forall a \in R$
4. $\|v + u\| \leq \|v\| + \|u\|$

It is left as an exercise to the reader to check that the 2 norm as well as the norms listed below actually possess these properties.

The 1 norm

$$\|v\|_1 := \int_{-\infty}^{\infty} |v(t)| dt \tag{3.2}$$

The ∞ norm

$$\|v\|_\infty := \sup_t |v(t)| \tag{3.3}$$

The 1 norm of a signal might represent a consumption of some ressource. The ∞ norm might be of relevance when checking boundedness of a signal, for instance for a system with physical limitations.

The 2 norm which will be used mostly in these notes, can be interpreted as the energy of a signal.

For each of the norms above we can define the linear space of signals $v(t)$ which has a bounded value for either the 1 norm, the 2 norm or the ∞ norm. These function spaces are called $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_\infty$ (Lebesgue spaces). In these notes, these spaces will not be treated further. The reader is refered to [ZDG96] or [TC96] for a more thorough introduction.

In addition to the norms listed above, it is relevant to introduce a quantity, represent the mean power:

$$\|v\|_{\mathcal{P}} := \lim_{T \to \infty} \left(\frac{1}{2T} \int_{-T}^{T} v(t)^2 dt \right)^{\frac{1}{2}} \tag{3.4}$$

$\|\cdot\|_{\mathcal{P}}$ is not a norm, since it does not possess property 2. It is usefull, however, because it is defined for a class of signals for which the 2 norm does not exist, and it does possess all the other properties which characterize norms. This class involves signals with non vanishing power. If, however, $\|v\|_2 < \infty$, then $\|v\|_{\mathcal{P}} = 0$. $\|v\|_{\mathcal{P}}$ is also known as the RMS (Root Mean Square) value of $v(t)$.

In the same way as in time domain, norms can be defined in frequency domain. In these notes, lower case letters will be used for time domain signals as well as for frequency domain signals, so the relevant norm will appear from the context. In frequency domain, the most relevant norms are the 2 norm and the ∞ norm.

The 2 norm

$$\|v\|_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |v(j\omega)|^2 d\omega\right)^{\frac{1}{2}} \tag{3.5}$$

Note the following important relationship (Parseval's Theorem):

$$\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{v}(j\omega)|^2 d\omega\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} v(t)^2 dt\right)^{\frac{1}{2}} \tag{3.6}$$

where $\hat{v}$ is the Fourier transform of v. This means that the 2 norm of a time domain signal equals the 2 norm of the associated frequency domain signal.

∞-*normen*

$$\|v\|_\infty := \sup_\omega |v(j\omega)| \tag{3.7}$$

For the ∞ norm there is no equivalent relationship to Parseval's Theorem (3.6).

3.2 Norms for Systems and Transfer Functions.

Since signals at different places in a process are related by the dynamics of the system, it is relevant to define norms related to the system itself. Since a system is charaterized by its impulse response $g(t)$ or its transfer function $G(s) = \mathcal{L}(g(t))$, the 2 norm for a system is defined by:

$$\|G\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 dt\right)^{\frac{1}{2}} \tag{3.8}$$

$$= \left(\int_{-\infty}^{\infty} |g(t)|^2 dt\right)^{\frac{1}{2}} \tag{3.9}$$

$$:= \|g\|_2 \tag{3.10}$$

where Parseval's Theorem was applied for the second equal sign. For transfer functions, the $\mathcal{H}_\infty$ norm is defined in a similar fashion:

$$\|G\|_{\mathcal{H}_\infty} := \sup_\omega |G(j\omega)| \tag{3.11}$$

3.3 Specification of inputs.

Some knowledge on the potential inputs to a system is important in order to be able to specify the performance of a control loop in a reasonable way.

The approach below involves describing a class of inputs, which are bounded by a norm. The next step is to specify performance by bounding the allowable norm of the output signal, often the control error e or a filtered version of the error. For these specification the 2 norm or possibly the RMS value will be used. [1]

[1] The use of other norms for compensator specifications can be very relevant and is the subject of research presently. For example, the use of the 1-norm in time domain can be highly relevant, and is currently the subject of intensive research. For example, the use of the 1 norm in time domain can be relevant, if the objective is to reduce consumption of some ressource.

A class of signals can be defined as those signals v' having a 2 norm bounded by 1:

$$V' = \left\{ v'(s) : \|v'\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} v'(j\omega)^* v'(j\omega) d\omega} \leq 1 \right\} \tag{3.12}$$

By filtering v', a new class of signals v can be established, characteristic as input to the process either as disturbances or as reference signals.

$$V = \{ v(s) = W(s)v'(s) : \|v'\|_2 \leq 1 \} \tag{3.13}$$

By designing the control system such that it minimizes the possible error given a certain class of normbounded input signals, all potential inputs are treated equally. This can be an advantage, as it is rarely known in the design phase for certain, which reference changes or disturbances, the system will encounter under real operating conditions.

Hence, the reference r and the disturbance d can be characterized by prepending input filters to our standard feedback loop, see Figure 3.2.

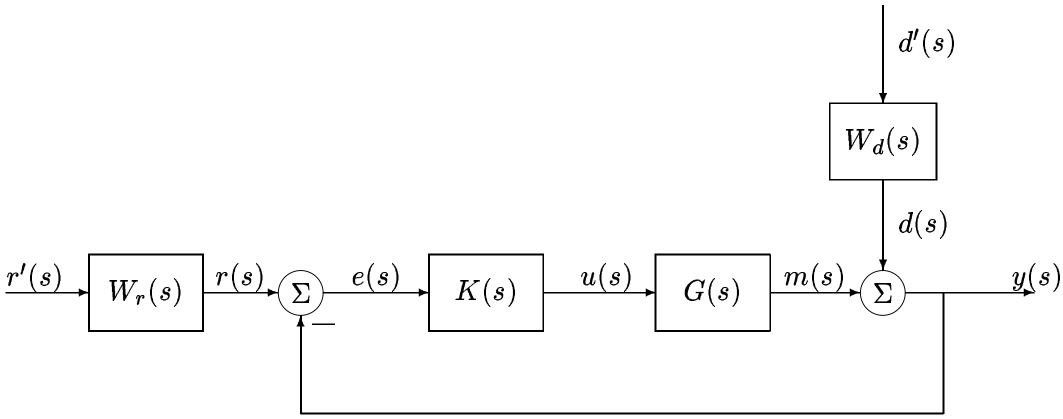


Figure 3.2: *Control loop with unit norm bounded signals as inputs.*

Example 3.1 (Input Specification)
This example has been taken from [MZ89]. Assume that the reference is known only to change in step, and that only small disturbances are present such that $d(s) \approx 0$ and $v(s) = r(s)$. Then an input weight $W_r(s)$ must be chosen such that $v(s) = s^{-1}$ and $v'(s) \in V'$. An obvious choice would seem to be $W_r(s) = 1/s$ and to introduce $v'(s)$ as an impulse ($v'(s) = 1$). However, $v'(s) = 1$ does not belong to the set V' since the integral in (3.12) become infinite (an impulse does note have finite 2 norm). Hence, the weight $W_r(s) = 1/s$ does not provide the desired characteristics for the reference $r(s)$. Instead, the following weight can be used:

$$W_r(s) = \frac{s+\beta}{s\sqrt{2\beta}}, \qquad \beta > 0 \tag{3.14}$$

With this filter, a step input is contained in V. For instance, assume that $v'(s)$ is given by:

$$v'(s) = \frac{\sqrt{2\alpha}}{s+\alpha} \tag{3.15}$$

then $v'(s) \in V'$ *since*

$$\|v'\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} v'(j\omega)^* v'(j\omega) d\omega} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\alpha}{\omega^2+\alpha^2} d\omega} = 1 \tag{3.16}$$

For $\alpha = \beta$, $v(s)$ *is then given by:*

$$v(s) = W_r(s)v'(s) = \frac{s+\beta}{s\sqrt{2\beta}}\frac{\sqrt{2\beta}}{s+\beta} = \frac{1}{s} \tag{3.17}$$

Choosing α *unequal to* β, $v(s)$ *can be obtained as a lead/lag modified step.*

$$v(s) = \sqrt{\frac{\alpha}{\beta}}\frac{s+\beta}{s(s+\alpha)} \tag{3.18}$$

This implies that a control which has been designed for the norm bounded input set given by (3.13) *using the weight* (3.14) *will work not only for step input but also for similar inputs. Moreover,* V *will contain the whole spectrum of signals where* v' *is norm bounded, for example input signals with damped oscillations.*

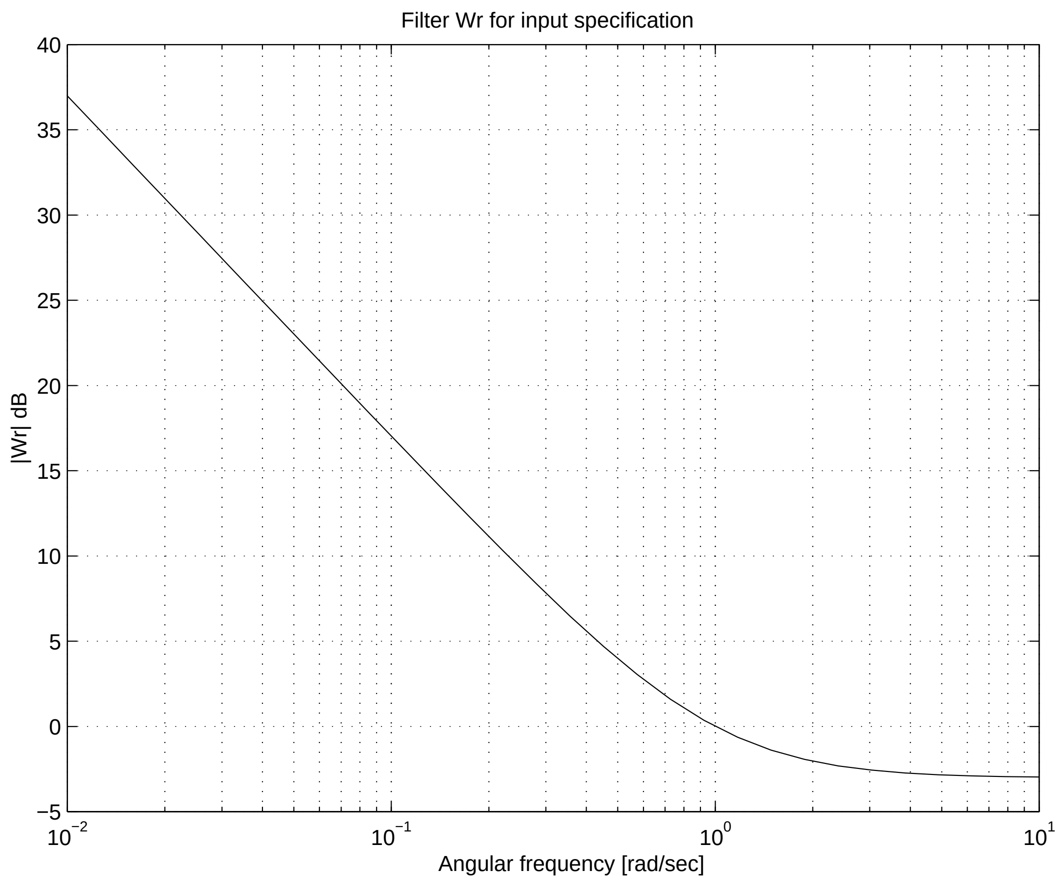


Figure 3.3: *Example of an input specification* ($\beta = 1$)

In Figure 3.3, a Bode plot is shown for the amplitude of this input specification with $\beta = 1$. *A model of this type complies well with many control problems, since the large amplitude at* *low frequences is consistent with anticipated low frequent reference changes or disturbances, whereas the control system is not expected to be able to attenuate disturbances above a certain frequency.*

The configuration shown in Figure 3.2 can be used also if the controller should be designed for a specific input. v' can also be chosen as an impulse, $v'(s) = 1$. $v(s)$ then becomes a signal with the same Laplace transform as $W(s)$.

For a sinusoidal input v', the 2 norm does no longer apply, since the integral in (3.1) becomes infinite. In this case it is more appropriate to use the RMS value $\|\cdot\|_{\mathcal{P}}$. If a filter $W(s)$ is applied in this case, the frequency response for $W(jw)$ specifies the amplitudes of v that are anticipated in (or specified for) the control system at each frequency.

3.4 Requirements for Performance.

The ultimate requirement to the compensator is, that it works 'well' for the real system. This requirements can be subdivided into the following four categories:

1. Nominal stability: the compensator must ensure internal stability in the controlled system, provided the model is correct.
2. Nominal performance: the compensator must minimize the error e. (E.g. expressed as the 2 norm of the error for given inputs.)
3. Robust stability: for all models in $\mathcal{G}$ (see (2.2)) the compensator must ensure internal stability.
4. Robust performance: for all models in $\mathcal{G}$ the compensator must ensure that the error is within a specified bound.

The requirements (1) and (3) has been described above. The performance specifications will be further described below.

The main objective of the compensator is to minimize the error e occuring in the face of a reference r and/or a disturbance d. The most important transfer functions, see Figure 3.1 are

$$\frac{e(s)}{r(s)} = -\frac{e(s)}{d(s)} = \frac{y(s)}{d(s)} = \frac{1}{1+G(s)K(s)} = S(s) \tag{3.19}$$

$$\frac{y(s)}{r(s)} = \frac{G(s)K(s)}{1+G(S)K(s)} = T(s) \tag{3.20}$$

The quantity $S(s)$ is called the *sensitivity* function. Usually, it is desired to make $S(s)$ small, due to the wish to minimize the error e. For physical systems $G(s)K(s)$ is proper (the order of the numerator is not larger than the order of the denominator). In practice, it will even be strictly proper (the order of the numerator is smaller than the order of the denominator). This corresponds to the fact that usually very high frequencies will not pass through the system.

$$\lim_{s\to\infty} G(s)K(s) = 0 \tag{3.21}$$

This implies that

$$\lim_{s\to\infty} S(s) = 1 \tag{3.22}$$

Hence, the sensitivity $S(s)$ can only be made 'small' in a bounded frequency region. The quantity $T(s)$ is called the *complementary sensitivity* function, due to the relation

$$S(s) + T(s) = 1 \tag{3.23}$$

For a strictly proper system, a similar limiting behavior is observed:

$$\lim_{s\to\infty} T(s) = 0 \tag{3.24}$$

3.5 $\mathcal{H}_2$ Optimal Control (LQ).

As a measure of the quality of a compensator, the integral of the square of the error for a given reference signal or a given disturbance is often used. By $\mathcal{H}_2$ optimal or linear quadratic (LQ) control, the input $v'(s)$ is assumed to be equal to an impulse $v'(s) = 1$. In other words, the optimal compensator minimizes the 2 norm of the error for a specific input given by $v(s) = W(s)1$. Hence, K is the compensator that solves the following minimization problem:

$$\begin{aligned}
\min_{K\in\mathcal{K}} \|e\|_2 &= \min_{K\in\mathcal{K}} \left(\int_{-\infty}^{\infty} |e(t)|^2 dt\right)^{\frac{1}{2}} && (3.25)\\
&= \min_{K\in\mathcal{K}} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |e(j\omega)|^2 d\omega\right)^{\frac{1}{2}} && (3.26)\\
&= \min_{K\in\mathcal{K}} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |S(j\omega)v(j\omega)|^2 d\omega\right)^{\frac{1}{2}} && (3.27)\\
&= \min_{K\in\mathcal{K}} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |S(j\omega)W(j\omega)|^2 d\omega\right)^{\frac{1}{2}} && (3.28)\\
&= \min_{K\in\mathcal{K}} \|SW\|_{\mathcal{H}_2} && (3.29)
\end{aligned}$$

where $\mathcal{K}$ is the set of all stabilizing compensators.

An $\mathcal{H}_2$ optimal compensator thus minimizes the $\mathcal{H}_2$ norm of the sensitivity S weighted by W.

3.6 $\mathcal{H}_\infty$ Optimal Control.

Rather than determining a compensator to minimze the error for a specific input, a compensator can be chosen to minimize the maximal error of all inputs contained in V, see (3.13), i.e. for all norm bounded signals weighed with a frequency dependent weight W. With this input specification the compensator is designed for a class of inputs with a better potential match with the actual input compared to the $\mathcal{H}_2$ design which uses a specific signal, especially if the inputs are now well known.

One approach to minimize the error for all inputs in a set V is to look for a compensator, which minimizes the largest value assumed by the 2 norm of the error e for any input v in V. This can be expressed as

$$K(s) = \arg\min_{K\in\mathcal{K}} \sup_{v\in V} \|e\|_2 = \min_{K\in\mathcal{K}} \sup_{v'\in V'} \|SWv'\|_2 \tag{3.30}$$

In order to find the optimal compensator K, for a given K it must first be established for which input the largest value of the 2 norm is achieved:

$$\sup_{v \in V} \|e\|_2 = \sup_{v' \in V'} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |S(j\omega)W(j\omega)v'(j\omega)|^2 d\omega \right)^{\frac{1}{2}} \tag{3.31}$$

It can be shown, that this integral assumes its largest value if v' is a sinusoid, modified in such a way that the 2 norm exists. This could for example be a signal equal to a constant times a sinusoid for a long time interval and zero otherwise. The constant must be chosen such that the signal has unit norm. The angular frequency of the sinusoid has to be the value for which $|S(j\omega)W(j\omega)|$ assumes its maximal value. Then

$$\sup_{v} \|e\|_2 = \sup_{\omega} |S(j\omega)W(j\omega)| = \|SW\|_{\mathcal{H}_\infty} \tag{3.32}$$

Hence, the largest value of the error assumed for any $v \in \mathcal{V}$ simply equals the $\mathcal{H}_\infty$ norm of SW. The desired compensator is the solution to the following minimization problem

$$\min_{K \in \mathcal{K}} \sup_{v \in V} \|e\|_2 = \min_{K \in \mathcal{K}} \|SW\|_{\mathcal{H}_\infty} = \min_{K \in \mathcal{K}} \sup_{\omega} |S(j\omega)W(j\omega)| \tag{3.33}$$

In other words, the $\mathcal{H}_\infty$ optimal control minimizes the $\mathcal{H}_\infty$ norm of the sensitivity function S weighted by W.

In comparison, the $\mathcal{H}_2$ optimal compensator minimizes the mean value over all frequencies of the square of $|S(j\omega)W(j\omega)|^2$, whereas the $\mathcal{H}_\infty$ optimal compensator minimizes the maximal value of $|S(j\omega)W(j\omega)|$.

By scaling $W(j\omega)$ it is possible to formulate the $\mathcal{H}_\infty$ nominal performance condition in the form

$$|S(j\omega)W(j\omega)| < 1 , \qquad \forall \omega \tag{3.34}$$

or

$$\|SW\|_{\mathcal{H}_\infty} < 1 \tag{3.35}$$

The advantages by applying the $\mathcal{H}_\infty$ norm for specification of the performance requirements above the $\mathcal{H}_2$ norm are:

- The designer can bound the peak value of $S(j\omega)$ directly by chosing the input weight $W(j\omega)$ appropriately.
- Also using the input weight $W(j\omega)$, the designer can specify the desired bandwidth of the sensitivity function $S(j\omega)$, defined as the value of ω where $S(jw)$ stays beyond $\frac{1}{\sqrt{2}}$ (-3 dB).

Moreover, the $\mathcal{H}_\infty$ norm facilitates a tool for specifying robust performance as we shall see in the next section.

3.7 Robust Performance.

If a compensator is designed only based on requirements for nominal performance and robust stability, $\mathcal{G}$ might contain a model which is close to instability for the closed loop system. This is likely to give a very poor performance. To ensure the compensator to work well for all models in $\mathcal{G}$, robust performance should is required for the models in $\mathcal{G}$.

3.8 $\mathcal{H}_2$ Robust Performance

The $\mathcal{H}_2$ optimal robust compensator is defined as the compensator, which achieves the smallest performance for the worst proces model in $\mathcal{G}$. This requires the model with the poorest performance to be found, i.e. the model which for a given input v obtains the largest 2 norm of the error.

The maximal value for the 2 norm of the error can be determined by for each frequency to find the maximal value of the perturbed sensitivity $S_\Delta(j\omega)$ via geometrical considerations

$$\begin{aligned} S_\Delta(j\omega) &= \frac{1}{|1+G_\Delta(j\omega)K(j\omega)|} && (3.36)\\ &\leq \frac{1}{|1+G(j\omega)K(j\omega)|-|G(j\omega)K(j\omega)\ell_m(\omega)|} \qquad \forall G_\Delta \in \mathcal{G} && (3.37)\\ &= \frac{|S(j\omega)|}{1-|T(j\omega)\ell(\omega)|} G_\Delta \in \mathcal{G} && (3.38)\end{aligned}$$

The equation (3.38) describes an upper bound for the sensitivity function $S_\Delta(j\omega)$ which in turn can be used to compute the largest value of the 2 norm for a given compensator $K(j\omega)$

$$\max_{G_\Delta \in \mathcal{G}} \|e\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|S(j\omega)W(j\omega)|^2}{|1-|T(j\omega)\ell_m(\omega)|\,|^2} d\omega} \qquad (3.39)$$

The numerator in the expression under the integral sign is due to uncertainty in the model. Now, a compensator must be found which achieves the minimal value of the worst performance given by the above expression. This is not immediately possible, since the integral depends in a complicated way on K. The expression, however, can be used to check whether a given compensator satisfies a set of given performance specifications.

3.9 $\mathcal{H}_\infty$ Robust Performance.

The $\mathcal{H}_\infty$ optimal robust compensator is characterized by minimizing the largest value of $|S_\Delta(j\omega)|$ weighted with $W(j\omega)$ for the worst proces model.

According to the requirement for $\mathcal{H}_\infty$ nominal performance in (3.33), the compensator for all proces models must satisfy:

$$\|S_\Delta W\|_{\mathcal{H}_\infty} = \sup_\omega |S_\Delta(j\omega)W(j\omega)| < 1 \qquad \forall G_\Delta \in \mathcal{G} \qquad (3.40)$$

The model G_Δ in $\mathcal{G}$ which achieves the largest value of $S_\Delta(j\omega)$ is determined by 3.38. Inserting into (3.40) results in the requirement:

$$\frac{|S(j\omega)W(j\omega)|}{|1-|T(j\omega)\ell_m(\omega)|\,|} < 1 \qquad \forall\omega \qquad (3.41)$$

Rewriting gives:

$$|S(j\omega)W(j\omega)| + |T(j\omega)\ell_m(\omega)| < 1 \qquad \forall\omega \qquad (3.42)$$

Hereby, conditions have been formulated both for robust performance and for robust stability for SISO systems.

An example of such a specification is the control of a water supply pump with a maximal performance of 3 m^3/h at 2.5 bar. The requirements for the specific system in mind was originally formulated in time domain as conditions on the the error of the pressure by a step in water consumptions at 0.67 m^3/h

- maksimal transient error: 0.4 bar
- settling time for error of 0.1 bar: 2 sec
- maksimal stationary error: 0.1 bar

The water consumption influences the pressure in a way which resembles a first order linear transfer function. Thus, the disturbance can be modeled as

$$d(s) = \frac{K}{s\tau + 1}\frac{K_s}{s} \tag{3.43}$$

where $KK_s = 1.3\ bar/(m^3/h)$ and $\tau = 0.75\ sec$. For the sensitivity furnction, the corresponding response to a disturbance can now be computed. A first order sensitivity bound S_p with a high frequency gain slightly above 1 was chosen:

$$S_p(s) = \frac{1.4(s + 0.01)}{s + 2.5} \tag{3.44}$$

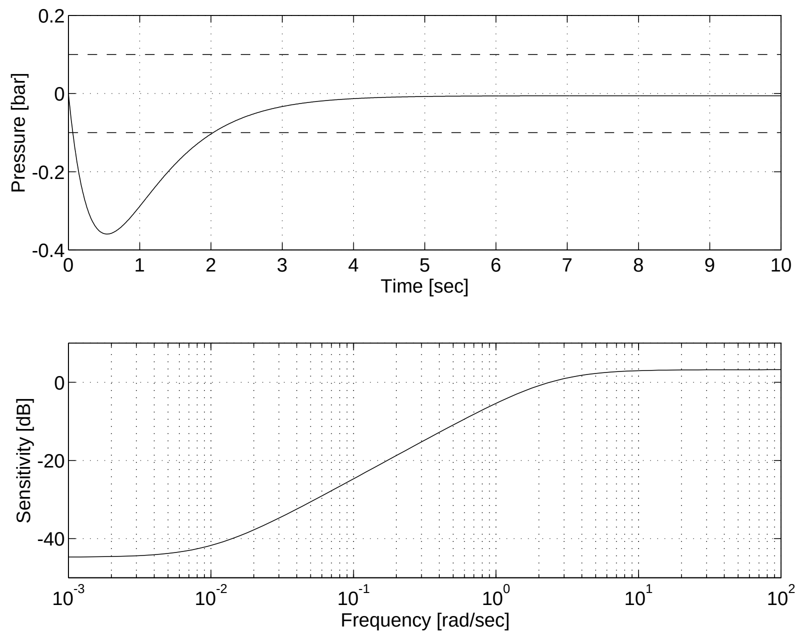


Figure 3.4: *Time domain repsonse to disturbance and corresponding frequency reponse of the sensitivity specification $S_p(s)$.*

In Figure 3.4 time response and frequency response are shown for this specification. The time response is seen to satisfy the original requirements. It is therefore assumed that a sensitivity which is below this specification at all frequencies also will satisfy the time domain requirements. Note, that this assumption is not justified in all cases from a theoretical point of view, but has shown to be applicable in practice. Now, a weight function corresponding to this bound can be formulated

$$W_p(z) = \frac{1}{S_p(z)} \tag{3.45}$$

Often, the uncertainties in physical systems are larger at high frequencies. For the pump the transfer function has been determined by system identification where the input signal has been a square wave with the frequency 0.4 Hz, corresponding to 2.51 rad/sec. Hence, the multiplicative model uncertainty that was estimated by system identification is smallest at this frequency, and is largest at high frequencies. In Figure 3.5 the sensitivity specifications given by W and the complementary sensitivity specifications given by ℓ_m for the pump are given

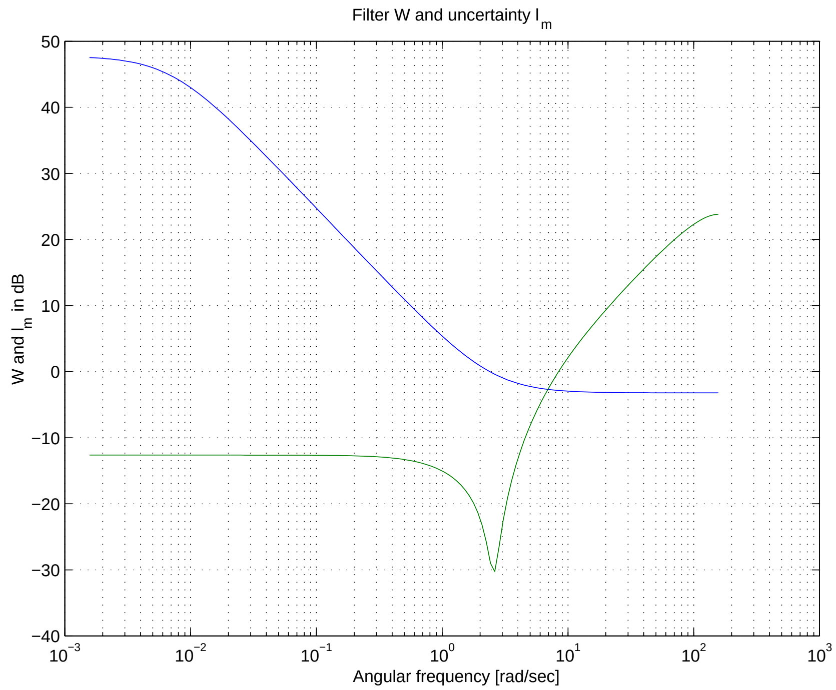


Figure 3.5: *Combination of input specification and uncertainty*

In Chapter 6 and Chapter 7, design algorithms aimed at satifying requirements of this type are presented.

3.10 Loop Shaping

A classical approach to obtain a good design, is to fit the open loop transfer function in frequency domain. It is therefore interesting to assess whether the requirements to robust stability, nominal performance, and robust performance can be reformulated as open loop conditions.

In frequency regions, where $|W(j\omega)|$ is much larger than 1, the requirement for nominal performance is that the sensitivity must be much smaller than 1. Hence, the condition for sensitivity can be approximated well by reformulating the condition below on the open loop gain

$$|K(j\omega)G(j\omega)| > |W(j\omega)| \tag{3.46}$$

Similarly, in frequency regions where $\ell_m(\omega)$ is much larger than 1, the requirements for robust stability can be reformulated as

$$|K(j\omega)G(j\omega)| < |\frac{1}{\ell_m(j\omega)}| \tag{3.47}$$

In Figure 3.6, such requirements are shown for the pump system along with an open loop candidate, which satisfies the requirements.

Such a design method suggests an inversion of the process dynamics. It is therefore important to note that pole-zero cancellation in the right half plane must be avoided, as it would cause the closed loop system to become unstable.

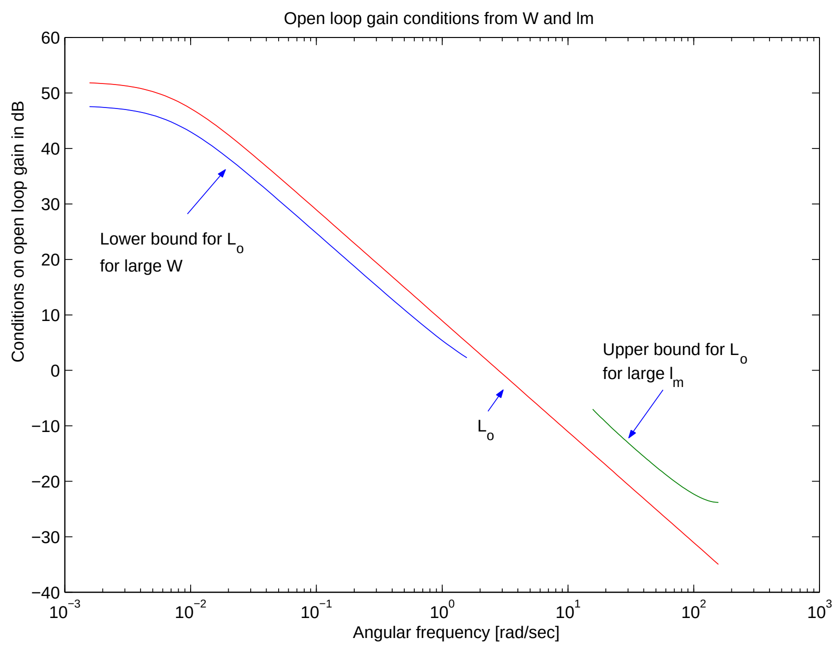


Figure 3.6: *Requirements for the open loop gain from W and ℓ_m in regions where each of them are large*

In this fashion, specification of the weight functions are seen to be a trade-off between a good disturbance attenuation and robustness, and the choice becomes interesting, where the two curves intersect. At this frequency, it must be ensured that both weightings are below 1 for the requirements to be feasible. In the frequency region where $W(j\omega)$ and $\ell_m(\omega)$ cross 1, unfortunately, the requirement for robust performance does not lead to any requirement, which can directly be translated into open loop conditions. Thus, robust performance requirements by this type of design must be satisfied by trial and error.

Chapter 4

An Introduction to Multivariable Systems

In this chapter a number of tools for the analysis of multivariable systems will be given. In particular, this will include poles and zeros of multivariable systems, the generalized Nyquist Theorem for stability analysis of multivariable systems, and frequency responses for multivariable systems using singular values.

4.1 Poles and zeros of Multivariable Systems

For a single input single output (SISO) system with the transfer function

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} \tag{4.1}$$

the poles are defined as the values of the complex values s for which $G(s) = \infty$, and the zeros as the values of s for which $G(s) = 0$.

For a multiple input multiple output (MIMO) system with the *transfer matrix*:

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1n_u}(s) \\ \vdots & \vdots & \dots & \vdots \\ G_{n_y1}(s) & G_{n_y2}(s) & \cdots & G_{n_yn_u}(s) \end{bmatrix} \tag{4.2}$$

the poles can be defined in analogy with the definition for SISO system. I.e., as the poles of every scalar transfer function $G_{11}(s) \cdots G_{n_yn_u}(s)$ in $G(s)$. This definition is reasonable, as at least one of the scalar functions $G_{11}(s) \cdots G_{n_yn_u}(s)$ equals ∞, when s assumes the value of a pole. Hence, at least one of the entries in the transfer matrix equals ∞, when s is a pole.

Zeros for multivariable systems, however, can *not* in a reasonable way be defined simply as the values for which one of the entries of the transfer matrix equals zero, so the multivariable zeros are *not* directly related to the zeros of the indiviual transfer functions $G_{11}(s) \cdots G_{n_yn_u}(s)$. Instead, the zeros of a multivariable system are defined as the values of the complex variable s for which $G(s)$ loses rank. The rank of a matrix A is defined as the number of linearly independent columns of A and denoted by

$$\operatorname{rank} A \tag{4.3}$$

Thus, the definition of zeros for multivariable systems becomes:

Definition 4.1 (Zeros for Multivariable Systems) *The zeros of a transfer matrix $G(s)$ are defined as those values of the complex variable s for which rank $G(s)$ is less than its maximal value.*

Zeros defined in this way are called *transmission zeros*. The reason is the following. If the transfer function $G(s)$ loses rank for $s = z_0$, it can be shown, see Section 4.3.3, that there exists an input vector $u_0 \neq 0$, such that

$$G(z_0)u_0 = 0 \tag{4.4}$$

Hence, the transmission of certain input signals is blocked for $s = z_0$.

4.1.1 Smith-McMillan Form of a Transfer Matrix

Poles and transmission zeros of a transfer matrix $G(s)$ can be found, e.g. by transforming $G(s)$ to its *Smith-McMillan form*. It can be shown that any proper[1] transfer matrix $G(s)$ can be written in its Smith-McMillan form:

$$\begin{aligned} G(s) &= U_1(s)M(s)U_2(s) && (4.5)\\ &= U_1(s)\,\mathrm{diag}\left\{\frac{\chi_1(s)}{\phi_1(s)},\cdots,\frac{\chi_r(s)}{\phi_r(s)},0,\cdots,0\right\}U_2(s) && (4.6)\end{aligned}$$

where $U_1(s)$ and $U_2(s)$ are *unimodular* matrices. $U(s)$ is said to be a unimodular matrix, if and only if its determinant $\det U(s)$ is independent of s, i.e. if $\det U(s)$ is constant. $M(s)$ is a pseudo-diagonal matrix, and is called the Smith-McMillan form of $G(s)$. $G(s)$ and $M(s)$ are said to be *similar*, denoted as $G(s) \sim M(s)$. The polynomials $\{\chi_i(s), \phi_i(s)\}$ have to be monic (i.e. having the coefficient 1 for the highest power of s) and coprime (i.e. having no common factors or - equivalently - no common roots). Finally, $\{\chi_i(s), \phi_i(s)\}$ have to possess the following divisibility properties:

$$\left.\begin{array}{c}\chi_i(s)|\chi_{i+1}(s)\\ \phi_{i+1}(s)|\phi_i(s)\end{array}\right\} \quad i = 1,\cdots,r-1 \tag{4.7}$$

The notation $\chi_i(s)|\chi_{i+1}(s)$ above means that the polynomial $\chi_{i+1}(s)$ is a factor of the polynomial $\chi_i(s)$ (with no remainder). Next, the following pole and zero polynomials are defined:

$$\begin{aligned} p(s) &= \phi_1(s)\phi_2(s)\cdots\phi_r(s) && (4.8)\\ z(s) &= \chi_1(s)\chi_2(s)\cdots\chi_r(s) && (4.9)\end{aligned}$$

Now, it can be shown, that the poles and transmission zeros of the transfer function $G(s)$ can be founds as the roots of $p(s)$ and $z(s)$, respectively. The degree of the pole polynomial $p(s)$ is called the *McMillan* degree of $G(s)$.

It can be shown that the Smith-McMillan form $M(s)$ of a transfer matrix $G(s)$ can be determined by a series of elementary row and column operations on $G(s)$. Let $d(s)$ be the smallest

[1] A transfer matrix $G(s)$ is said to be *proper* if all its entries satisfy $|G_{ij}(s)| \to c_{ij} < \infty$ for $s \to \infty$ and *strictly proper* if $|G_{ij}(s)| \to 0$ for $s \to \infty$.

common denominator for the entries $G_{ij}(s)$ of $G(s)$. Then, $G(s)$ can be written as:

$$G(s) = \frac{1}{d(s)}N(s) = \frac{1}{d(s)}\begin{bmatrix} N_{11}(s) & \cdots & N_{1n_u}(s) \\ \vdots & \cdots & \vdots \\ N_{n_y1}(s) & \cdots & N_{n_yn_u}(s) \end{bmatrix} \tag{4.10}$$

where $N(s)$ is a polynomial matrix (not a transfer matrix). There are three elementary types of row or column operations to perform on $N(s)$:

- Interchange of two rows or columns.
- Multiplication of a row or a column with a constant.
- Addition of one row or column multiplied with a polynomial to another.

A common property of these operations is, that they do not change the rank of the matrix $N(s)$. Each of these elementary operations can be represented as a pre- or postmultiplication of $N(s)$ by a suitable matrix $L(s)$ called an *elementary matrix*. It can be shown, that all elementary matrices are unimodular. Now, $N(s)$ can be rewritten as a sequence of row and column operations:

$$\begin{aligned} N(s) &= L_1(s)S(s)L_2(s) &(4.11)\\ &= L_1(s)\mathrm{diag}\left\{\epsilon_1(s), \epsilon_2(s), \cdots, \epsilon_r(s), 0, 0, \cdots, 0\right\} L_2(s) &(4.12)\end{aligned}$$

Here, $S(s)$ is a pseudo-diagonal polynomial matrix. $S(s)$ is called the *Smith form* of $N(s)$. The polynomials $\epsilon_i(s)$ are monic and have the following divisibility properties:

$$\epsilon_i(s)|\epsilon_{i+1}(s), \quad i = 1, \cdots, r-1 \tag{4.13}$$

Hence, the Smith form of a polynomial matrix is equivalent to the Smith-McMillan form of a transfer matrix. Now, the point is that the polynomials $\epsilon_i(s)$ can be determined from the *determinant divisors*:

$$\begin{aligned} D_0(s) &= 1 &(4.14)\\ D_i(s) &= \text{greatest common divisor for all } i \times i \text{ subdeterminants of } N(s) &(4.15)\end{aligned}$$

where every of the greatest common divisors are normalized to a monic polynomial. It can be shown, see [Mac89, Pages 40–43], that the polynomials $\epsilon_i(s)$ are given by:

$$\epsilon_i(s) = \frac{D_i(s)}{D_{i-1}(s)}, \quad i = 1, \cdots, r \tag{4.16}$$

Hence, the Smith-McMillan form of $G(s)$ is given by:

$$M(s) = \frac{1}{d(s)}S(s) \tag{4.17}$$

In summary, this establishes the following procedure for determining the Smith-McMillan form of a transfer matrix $G(s)$:

Procedure 4.1 (Smith-McMillan form of a transfer matrix)

1. *Let $G(s)$ be a proper transfer matrix. Find the smallest common denominator $d(s)$ for all entries in $G(s)$ and rewrite $G(s)$ as:*

$$G(s) = \frac{1}{d(s)} N(s) \tag{4.18}$$

2. *Determine the Smith form of $N(s)$:*

$$N(s) \sim S(s) = diag\,\{\epsilon_1(s), \epsilon_2(s), \cdots, \epsilon_r(s), 0, 0, \cdots, 0\} \tag{4.19}$$

where $\epsilon_i(s)$ is determined from the determinant divisors $D(s)$:

$$\epsilon_i(s) = \frac{D_i(s)}{D_{i-1}(s)}, \quad i = 1, \cdots, r \tag{4.20}$$

3. *Then, the Smith-McMillan form of $G(s)$ is given by:*

$$G(s) \sim M(s) = \frac{1}{d(s)} S(s) \tag{4.21}$$

The following example has been taken from [Mac89, Ex. 2.2].

Example 4.1 (The Smith-McMillan form of a transfer matrix)
Let $G(s)$ be given by:

$$G(s) = \begin{bmatrix} \dfrac{1}{s^2+3s+2} & \dfrac{-1}{s^2+3s+2} \\[2ex] \dfrac{s^2+s-4}{s^2+3s+2} & \dfrac{2s^2-s-8}{s^2+3s+2} \\[2ex] \dfrac{s-2}{s+1} & \dfrac{2s-4}{s+1} \end{bmatrix} \tag{4.22}$$

The smalles common denominator for $G_{ij}(s)$ is $d(s) = s^2 + 3s + 2$ and $G(s)$ can be written as

$$G(s) = \frac{1}{s^2+3s+2} \begin{bmatrix} 1 & -1 \\ s^2+s-4 & 2s^2-s-8 \\ (s-2)(s+2) & (2s-4)(s+2) \end{bmatrix} \tag{4.23}$$

$$= \frac{1}{d(s)} N(s) \tag{4.24}$$

The Smith form of the polynomial matrix $N(s)$ is given by

$$N(s) \sim S(s) = \mathrm{diag}\,\{\epsilon_1(s), \epsilon_2(s)\} \tag{4.25}$$

where $\epsilon_i(s)$ is determined from the determinant divisors $D(s)$:

$$\epsilon_i(s) = \frac{D_i(s)}{D_{i-1}(s)}, \quad i = 1, 2 \tag{4.26}$$

Now,

$$D_0(s) = 1 \tag{4.27}$$
$$D_1(s) = \gcd\left\{1, -1, s^2+s-4, 2s^2-s-8, s^2-4, 2s^2-8\right\} = 1 \tag{4.28}$$
$$D_2(s) = \gcd\left\{\left|\begin{array}{cc} 1 & -1 \\ s^2+s-4 & 2s^2-s-8 \end{array}\right|, \left|\begin{array}{cc} 1 & -1 \\ s^2-4 & 2s^2-8 \end{array}\right|, \left|\begin{array}{cc} s^2+s-4 & 2s^2-s-8 \\ s^2-4 & 2s^2-8 \end{array}\right|\right\} \tag{4.29}$$
$$= \gcd\left\{3(s^2-4), 3(s^2-4), 3s(s^2-4)\right\} = s^2-4 = (s+2)(s-2) \tag{4.30}$$

where gcd *denotes the **greatest common divisor**. Further,*

$$\epsilon_1(s) = \frac{D_1(s)}{D_0(s)} = 1 \tag{4.31}$$
$$\epsilon_2(s) = \frac{D_2(s)}{D_1(s)} = (s+2)(s-2) \tag{4.32}$$

Thus, the Smith-McMillan form of $G(s)$ is given by:

$$G(s) \sim M(s) = \frac{1}{d(s)}S(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & 0 \\ 0 & \frac{s-2}{s+1} \\ 0 & 0 \end{bmatrix} \tag{4.33}$$

where the common factors in numerator and denominator have been cancelled. Hence, the pole and zero polynomials for $G(s)$ are given by

$$p(s) = (s+1)^2(s+2) \tag{4.34}$$
$$z(s) = s-2 \tag{4.35}$$

Thus, $G(s)$ has the poles $s = -1, -1, -2$ and the transmission zero $s = 2$.

4.1.2 State Space Descriptions

A system described by the state space representation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4.36}$$
$$y(t) = Cx(t) + Du(t) \tag{4.37}$$

can be depicted as in Figure 4.1. Since an integration in time domain corresponds to s^{-1} in the s domain, the transfer matrix $G(s)$ corresponding to the state space description above, can be written as

$$G(s) = C(sI-A)^{-1}B + D \tag{4.38}$$

where I is an identity matrix of the same dimensions as A. The inverse of a square nonsingular matrix X is given by

$$X^{-1} = \frac{1}{\det X}\text{adj } X \tag{4.39}$$

where $\text{adj}\, X$ is the adjoint of X. Hence, if $sI - A$ is nonsingular, $G(s)$ can be written as

$$G(s) = \frac{1}{\det(sI-A)}C\text{adj}(sI-A)B + D \tag{4.40}$$

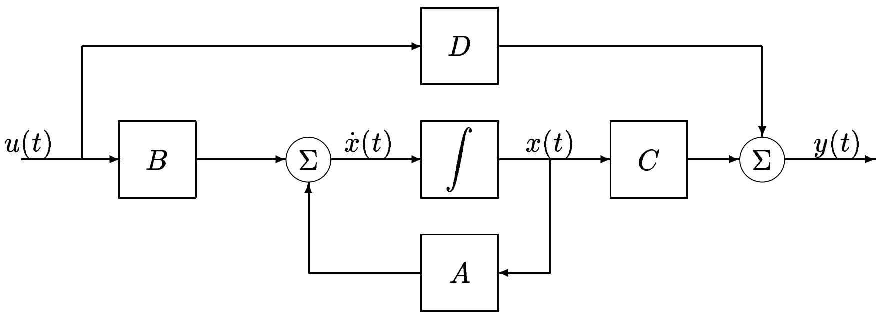


Figure 4.1: *State space representation of a system.*

Since $C\,\mathrm{adj}(sI - A)B$ is a polynomial matrix, obviously all poles of $G(s)$ (i.e. the values of s for which $G(s) = \infty$) have to be roots of the polynomial $\det(sI - A)$. The roots of $\det(sI - A)$ coincide with the eigenvalues of A. Hence, all poles of $G(s)$ have to be eigenvalues of A. The opposite needs not be the case always, since roots of $\det(sI - A)$ might be cancelled in (4.38), and consequently they will not appear as poles of $G(s)$. This is the case, when the realization (A, B, C, D) is uncontrollable, unobservable, or both. On the other hand, if the realization (A, B, C, D) is both controllable and observable, the roots of $\det(sI - A)$ equals the poles of $G(s)$ and the pole polynomial $p(s)$ will be given by

$$p(s) = \det(sI - A) \tag{4.41}$$

This means, that the dimension of A can not be smaller than the McMillan degree of $G(s)$. Hence, a state space realization which is both controllable and observable is called a *minimal realization*. These results can be summarized as the following theorem.

Theorem 4.1 (Poles for systems with state space descriptions) *Let $G(s)$ be a transfer matrix with a minimal realization (A, B, C, D) and let $p(s)$ be the Smith-McMillan pole polynomial of $G(s)$. Then*

$$\dim A = \deg p(s) \tag{4.42}$$

Hence, the McMillan degree of $G(s)$ equals the dimension of a minimal realization. Moreover, the eigenvalues of A equal the poles of $G(s)$.

Note, that if (A, B, C, D) is a *non-minimal* realization, then the poles of $G(s)$ constitute a proper subset of the eigenvalues of A.

4.1.2.1 From Transfer Matrix to State Space Description

The transformation of a state space description to a transfer function description is unique, given by (4.38). In contrast, there are several ways in which a transfer function can be transformed into a state space description. A straightforward approach would be to derive separate state space descriptions for each column in $G(s)$, i.e. for each input, and then collect these separate state space descriptions into an overall state space model. Let $G_i(s)$ be the ith column in $G(s)$ such that

$$G(s) = [G_1(s), G_2(s), \cdots, G_{n_u}(s)] \tag{4.43}$$

Write each column as

$$G_i(s) = \frac{1}{d_i(s)} n_i(s) + \delta_i \tag{4.44}$$

where $d_i(s)$ is the common denominator polynomial of $G_i(s)$:

$$d_i(s) = s^{k_i} + d_i^1 s^{k_i-1} + \cdots + d_i^{k_i} \tag{4.45}$$

Note, that $d_i(s)$ is monic. $n_i(s)$ is a vector of polynomials, each having a degree strictly less than k_i. The jth entry of $n_i(s)$ can be written as the polynomial:

$$n_{ji}(s) = n_{ji}^1 s^{k_i-1} + n_{ji}^2 s^{k_i-2} + \cdots + n_{ji}^{k_i} \tag{4.46}$$

δ_i is a vector consisting entirely of constants.

A state space description in controllable canonical form of the column $G_i(s)$ is then given by the realization $(A_i, B_i, C_i, \delta_i)$ where

$$A_i = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -d_i^{k_i} & -d_i^{k_i-1} & -d_i^{k_i-2} & -d_i^{k_i-3} & \cdots & -d_i^1 \end{bmatrix} \tag{4.47}$$

$$B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad C_i = \begin{bmatrix} n_{1i}^{k_i} & n_{1i}^{k_i-1} & \cdots & n_{1i}^1 \\ n_{2i}^{k_i} & n_{2i}^{k_i-1} & \cdots & n_{2i}^1 \\ \vdots & \vdots & & \vdots \\ n_{n_y i}^{k_i} & n_{n_y i}^{k_i-1} & \cdots & n_{n_y i}^1 \end{bmatrix} \tag{4.48}$$

Finally, a realization (A, B, C, D) of $G(s)$ can be found as

$$A = \mathrm{diag}\{A_1, A_2, \cdots, A_{n_u}\} \quad B = \mathrm{diag}\{B_1, B_2, \cdots, B_{n_u}\} \tag{4.49}$$

$$C = [C_1, C_2, \cdots, C_{n_u}] \qquad D = [\delta_1, \delta_2, \cdots, \delta_{n_u}] \tag{4.50}$$

This realization is controllable, but not necessarily observable. If a minimal is required, there exist algorithms to remove the unobservable modes, see e.g. [Mac89, Section 8.3.5]. The MATLAB™ function `tfm2ss.m` from MATLAB™'s Robust Control Toolbox produces a similar state space realization and `minreal.m` from MATLAB™'s Control Toolbox can extract a minimal realization from a non-minimal one.

Remark

In computer aided design, and especially in MATLAB™, it is easier in general to work with state space descriptions, since it is difficult to represent transfer matrices, as this requires three dimensional structures. Robust control, however, is mainly based on frequency response analysis of a number of transfer matrices, such as the sensitivity function $S(s)$ and the complementary sensitivity function $T(s)$. It is no problem, however, to compute the frequency response based on a state space description of the system. Hence, a multivariable system will often be represented in state space representation, although the analysis is performed in the frequency domain. This mixture of time and frequency domain is actually quite typical for modern robust control theory. The most popular solution to the $\mathcal{H}_\infty$ control problem is formulated in terms of state space matrices, even though the problem is formulated in frequency domain. To facilitate this dual representation, the following notation for a transfer function of a system based on its state space matrices has been introduced:

$$G(s) = C(sI - A)^{-1}B + D \triangleq \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{4.51}$$

4.2 Nominal Stability for Multivariable Systems

In the sequel, the conditions for nominal statibility for multivariable systems will be presented. These conditions are entirely analogous to those for scalar (SISO) systems. Just as for SISO systems, a multivariable system is stable, if all its (Smith-McMillan) poles are located in the left half plane.

4.2.1 Internal Stability

Consider the control system in Figure 4.2. In contrast to previously, it is assumed that all signals are vectors, and that all transfer 'functions' are matrices of compatible dimensions.

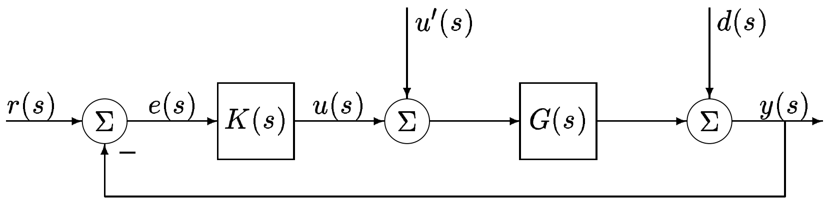


Figure 4.2: *Multivariable control system.*

Again, internal stability will be understood as the property, that a bounded signal introduced anywhere in the loop does not stimulate an unbounded signal anywhere else in the loop. Just as for SISO systems, the condition for internal stability reduces to stability of the composite transfer matrix from $r(s)$ (or $d(s)$) and $u'(s)$ to $y(s)$ and $u(s)$. Hence, internal stability for the closed loop system is equivalent to the system

$$\begin{bmatrix} y(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} (I + G(s)K(s))^{-1}G(s)K(s) & (I + G(s)K(s))^{-1}G(s) \\ K(s)(I + G(s)K(s))^{-1} & -K(s)(I + G(s)K(s))^{-1}G(s) \end{bmatrix} \begin{bmatrix} r(s) \\ u'(s) \end{bmatrix} \tag{4.52}$$

having all its poles in the complex left half plane. Note, the expressions with inverses rather than denominators in (4.52), as denominators do not make sense for matrices. Moreover, it is (very) important to remember that matrix multiplication is *non-commutative*, i.e. that the order of the factors are significant, as $G(s)K(s) \neq K(s)G(s)$ in general.

For the system in (4.52) to be stable, all four transfer matrices must be stable.

Example 4.2 (Internal stability)
If the controller $K(s)$ is stable, internal stability can be checked by checking stability of the transfer function $(I+G(s)K(s))^{-1}G(s)$. Indeed, assume that $K(s)$ and $(I+G(s)K(s))^{-1}G(s)$

are actually stable transfer matrices. In that case any combination (product) of these two transfer matrices will be stable. Thus, $(I+G(s)K(s))^{-1}G(s)K(s)$ *and* $-K(s)(I+G(s)K(s))^{-1}G(s)$ *will be stable. Stability of* $(I+G(s)K(s))^{-1}$ *still needs to be established, but this can be seen from*

$$(I+G(s)K(s))^{-1} = I - (I+G(s)K(s))^{-1}G(s)K(s) \tag{4.53}$$

and from the stability of $(I+G(s)K(s))^{-1}G(s)K(s)$*. In the same fashion, it is easily shown, that if the system* $G(s)$ *is stable, a necessary and sufficient condition for internal stability of the closed loop in Figure 4.2 is stability of* $K(s)(I+G(s)K(s))^{-1}$*. If both* $K(s)$ *and* $G(s)$ *are stable, internal stability can be checked via stability analysis of* $(I+G(s)K(s))^{-1}$ *or simply by assessing the closed loop.*

4.2.2 The Generalized Nyquist Theorem

The Generalized Nyquist Theorem plays a central role in robust control theory. Just like the classical Nyquist Theorem for singlevariable systems, the Generalized Nyquist Theorem facilitates an assesment of the stability of a control system by analyzing *open loop* properties, typically in terms of $G(s)K(s)$.

Consider the system in Figure 4.2 for $u'(s) = 0$ and $d(s) = 0$. Let the open loop transfer matrix $G(s)K(s)$ have the following state space description

$$\dot{x}(t) = A_o x(t) + B_o e(t) \tag{4.54}$$
$$y(t) = C_o x(t) + D_o e(t) \tag{4.55}$$

When closing the loop, $e(t) = r(t) - y(t)$ is established, and hence

$$y(t) = C_o x(t) + D_o(r(t) - y(t)) \tag{4.56}$$
$$\Leftrightarrow \quad (I + D_o)y(t) = C_o x(t) + D_o r(t) \tag{4.57}$$
$$\Leftrightarrow \quad y(t) = (I + D_o)^{-1}C_o x(t) + (I + D_o)^{-1}D_o r(t) \tag{4.58}$$
$$\Leftrightarrow \quad y(t) = C_c x(t) + D_c r(t) \tag{4.59}$$

where $C_c = (I + D_o)^{-1}C_o$ and $D_c = (I + D_o)^{-1}D_o$. Moreover:

$$\dot{x}(t) = A_o x(t) + B_o(r(t) - y(t)) \tag{4.60}$$
$$\Leftrightarrow \quad \dot{x}(t) = A_o x(t) + B_o \left(r(t) - (I + D_o)^{-1}C_o x(t) - (I + D_o)^{-1}D_o r(t)\right) \tag{4.61}$$
$$\Leftrightarrow \quad \dot{x}(t) = \left(A_o - B_o(I + D_o)^{-1}C_o\right) x(t) + B_o \left(I - (I + D_o)^{-1}D_o\right) r(t) \tag{4.62}$$
$$\Leftrightarrow \quad \dot{x}(t) = A_c x(t) + B_c r(t) \tag{4.63}$$

where

$$A_c = A_o - B_o(I + D_o)^{-1}C_o \tag{4.64}$$
$$B_c = B_o \left(I - (I + D_o)^{-1}D_o\right) \tag{4.65}$$
$$= B_o \left((I + D_o)^{-1}(I + D_o - D_o)\right) = B_o(I + D_o)^{-1} \tag{4.66}$$

Now, the *open loop characteristic polynomial* $\phi_{OL}(s)$ is defined:

$$\phi_{OL}(s) = \det(sI - A_o) \tag{4.67}$$

and the *closed loop characteristic polynomial* $\phi_{CL}(s)$:

$$\phi_{CL}(s) = \det(sI - A_c) \tag{4.68}$$

As described in Section 4.1.2, the stability of the closed loop system is determined by the roots of the characteristic polynomial $\phi_{CL}(s)$. To reach a 'Nyquist-similar' expression, $\phi_{CL}(s)$ must be expressed in terms of the open loop transfer matrix $G(s)K(s)$.

To that end, the *return difference* matrix $F(s)$ is introduced as

$$F(s) = I + G(s)K(s) \tag{4.69}$$

Moreover, the following lemma, which is presented without a proof, will be instrumental.

Lemma 4.1 (Schur's formula for block partitioned determinants) *Let a square matrix P be partitioned as*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{4.70}$$

Then, the determinant of P can be expressed as

$$\det P = \det P_{11} \det(P_{22} - P_{21}P_{11}^{-1}P_{12}), \qquad \textit{provided } \det P_{11} \neq 0 \tag{4.71}$$

or

$$\det P = \det P_{22} \det(P_{11} - P_{12}P_{22}^{-1}P_{21}), \qquad \textit{provided } \det P_{22} \neq 0 \tag{4.72}$$

Now, write the determinant of the return difference matrix $F(s)$ as

$$\det F(s) = \det(I + C_o(sI - A_o)^{-1}B_o + D_o) \tag{4.73}$$

With the following variable substitutions

$$P_{11} = sI - A_o \qquad P_{12} = -C_o \tag{4.74}$$

$$P_{21} = B_o \qquad P_{22} = I + D_o \tag{4.75}$$

the right hand side of (4.73) corresponds to the last term of the right hand side of (4.71). Thus, multiplying by $\det P_{11} = \det(sI - A_o)$ gives

$$\det(sI - A_o)\det(I + D_o + C_o(sI - A_o)^{-1}B_o) = \det\left(\begin{bmatrix} sI - A_o & B_o \\ -C_o & I + D_o \end{bmatrix}\right) \tag{4.76}$$

$$\Leftrightarrow \qquad \det F(s) = \frac{1}{\det(sI - A_o)} \det\left(\begin{bmatrix} sI - A_o & B_o \\ -C_o & I + D_o \end{bmatrix}\right) \tag{4.77}$$

Once again, applying Schur's formula, it can be shown that

$$\det\left(\begin{bmatrix} I_r & -B_o(I + D_o)^{-1} \\ 0 & I_n \end{bmatrix}\right) = \det I_r \det\left(I_n + 0I_rB_o(I + D_o)^{-1}\right) \tag{4.78}$$

$$= \det I_r \det I_n = 1 \tag{4.79}$$

Combining (4.79) with (4.77) results in

$$\det F(s) = \frac{1}{\det(sI - A_o)} \det\left(\begin{bmatrix} I_r & -B_o(I + D_o)^{-1} \\ 0 & I_n \end{bmatrix}\right) \det\left(\begin{bmatrix} sI - A_o & B_o \\ -C_o & I + D_o \end{bmatrix}\right) \tag{4.80}$$

$$\Rightarrow \qquad \det F(s) = \frac{1}{\det(sI - A_o)} \det\left(\begin{bmatrix} sI - A_o + B_o(I + D_o)^{-1}C_o & 0 \\ -C_o & I + D_o \end{bmatrix}\right) \tag{4.81}$$

$$\Rightarrow \qquad \det F(s) = \frac{1}{\det(sI - A_o)} \det(sI - A_c) \det(I + D_o + c_o(sI - A_c)^{-1}0) \tag{4.82}$$

$$\Leftrightarrow \qquad \det F(s) = \frac{1}{\det(sI - A_o)} \det(sI - A_c) \det(I + D_o) \tag{4.83}$$

Since $\lim_{s\to\infty} F(s) = I + D_o$, eventually the following is obtained:

$$\det F(s) = \frac{\phi_{CL}}{\phi_{OL}} \det F(\infty) \tag{4.84}$$

$$\Leftrightarrow \qquad \phi_{CL} = \frac{\det F(s)}{\det F(\infty)} \phi_{OL} \tag{4.85}$$

Now, the closed loop characteristic polynomial ϕ_{CL} has been expressed in terms of the return diffence matrix $F(s)$ and of the open loop characteristic polynomial ϕ_{OL}. Note, that $\det F(\infty)$ is merely a constant scaling factor. For physical systems (in continuous time), the open loop transfer matrix is usually strictly proper, and so is $\det F(\infty) = 1$. The zeros and poles of $\det F(s)$ can now be related to the poles of the open loop system and of the closed loop system, respectively. From (4.84) it is seen that

- Closed loop poles (i.e. roots of ϕ_{CL}) appear as zeros of $\det F(s)$.
- Open loop poles (i.e. roots of ϕ_{OL}) appear as poles of $\det F(s)$.

Now, assume that $\det F(s)$ has n_p poles and n_z zeros in the right half plane. In exactly the same fashion as for SISO systems, it can be seen from the *argument principle* that:

$$\Delta \arg \det F(s) = -2\pi(n_z - n_p) \tag{4.86}$$

hvor $\Delta \arg$ denote the variation of the angle (the argument) of $\det F(s)$ when s traverses the Nyquist $\mathcal{D}$ contour[2] such that $\Delta \arg /(2\pi)$ indicates the number of counter-clockwise encirclements of the origin.

In order for the closed loop to be stable, $n_z = 0$ must be satisfied, as n_z indicate the number of RHP poles for the closed loop system. Hence,

$$n_z = 0 \Rightarrow \Delta \arg \det F(s) = 2\pi n_p \tag{4.87}$$

This leads directly to the Generalized Nyquist Theorem:

[2]The Nyquist $\mathcal{D}$ contour proceeds along the imaginary axis from the origin to infinity, along a half circle in the right half plane until it meets the negative imaginary axis, and finally again up to the origin. If the contour meets poles on the imaginary axis, it is indented such that these poles are omitted.

Theorem 4.2 (The Generalized Nyquist Theorem I) *If the open loop transfer matrix $G(s)K(s)$ has n_p poles in the right half plan, then the the system is internally stable, if and only if the Nyquist curve for $\det F(s) = \det(I + G(s)K(s))$ encircles the origin n_p times counter-clockwise, when s traverses the Nyquist $\mathcal{D}$ contour, assuming that no RHP pole-zero cancellations have taken place in forming the product $G(s)K(s)$.*

An equivalent stability criterion can be derived by virtue of the *characteristic loci.* If $\lambda_i(\omega)$ denotes an eigenvalue of $G(j\omega)K(j\omega)$, then the charateristic loci for $G(s)K(s)$ are defined as the graphs of $\lambda_i(\omega)$ for $1 \leq i \leq n$, where n is the dimension of the product $G(s)K(s)$, as s traverses the Nyquist $\mathcal{D}$ contour. Since the determinant of a matrix X equals the products of the eigenvalues of X, the following is obtained

$$\Delta \arg \det(I + G(s)K(s)) = \Delta \arg \prod_i \lambda_i(I + G(j\omega)K(j\omega)) \tag{4.88}$$

$$= \Delta \arg \prod_i (1 + \lambda_i(G(j\omega)K(j\omega))) \tag{4.89}$$

$$= \sum_i \Delta \arg(1 + \lambda_i(G(j\omega)K(j\omega))) \tag{4.90}$$

As the number of encirclements of the origin by $(1 + \lambda_i(G(j\omega)K(j\omega)))$ equals the number of encirclements by $\lambda_i(G(j\omega)K(j\omega))$ of the Nyquist point $(-1 + 0j)$, an alternative Nyquist stability condition can be formulated:

Theorem 4.3 (The Generalized Nyquist Theorem II) *If the open loop transfer matrix $G(s)K(s)$ has n_p poles in the right half plane, the system is internally stable if and only the characteristic loci of $G(s)K(s)$ encircles the Nyquist point $(-1 + 0j)$ counter-clockwise n_p times, as s traverses the Nyquist $\mathcal{D}$ contour, assuming that no RHP pole-zero cancellations have taken place in forming the product $G(s)K(s)$.*

Note, that the two Nyquist theorems addresses *internal* stability of the closed loop system, i.e. the stability of all the transfer matrices in (4.52). The reason is that the open loop transfer matrix is the same for all these four systems, namely $G(s)K(s)$.

4.3 Frequency Responses for Multivariable Systems

The analysis of frequency responses play a central role in classical SISO control as well as in modern multivariable robust control. For singlevariable systems, the frequency response for a transfer function $G(s)$ is naturally given as the absolute value plus the argument of the complex variable $G(j\omega)$ for $0 \leq \omega \leq \infty$. For multivariable systems, however, $G(j\omega)$ is a complex matrix so it is not obvious how to associate a single scalar value (the gain) with a full matrix.

As the Generalized Nyquist Theorem is baed on eigenvalues, it would seem tempting to use the *spectral radius*:

$$\rho(G(j\omega)) = \max_i |\lambda_i(G(j\omega))| \tag{4.91}$$

as a measure for the gain of $G(j\omega)$. However, it turns out that eigenvalues do not necessarily capture a good picture of the gain of a transfer matrix. Instead, the concept of norms introduced in Chapter 3 will be extended to include vectors and matrices.

4.3.1 Vector Norms

If $x \in \mathbf{C}^n$ is an n dimensional complex vector, the Hölder or p norm of x is defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \tag{4.92}$$

In control theory, mainly the 1-, 2-, and ∞-norms are important, as they have a reasonable physical interpretation:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \text{absolute sum} \tag{4.93}$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{x^* x} \qquad \text{Euklidean length} \tag{4.94}$$

$$\|x\|_\infty = \max_i |x_i| \qquad \text{max absolute value} \tag{4.95}$$

In (4.94), x^* denote the complex conjugate transpose of x. Note, moreover, that the 2 norm $\|x\|_2$ is the usual Euclidean length of a complex vector x.

4.3.2 Induced Norms

Now, matrix p-norms can be defined from the corresponding vector p-norms. Indeed, let $A \in \mathbf{C}^{m \times n}$ be an $m \times n$ dimensional complex matrix. Then the p-norm of A is defined as:

$$\|A\|_p = \sup_{x \in \mathbf{C}^n, x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}, \qquad \forall A \in \mathbf{C}^{m \times n} \tag{4.96}$$

The matrix p-norm is said to be an *induced norm*. It is induced by the corresponding vector p-norm. $\|A\|_p$ can be interpreted as the maximal gain of the matrix A measured as the p-norm ratio between the vectors before and after multiplication by A. It is difficult in general to compute matrix norms. For $p = 1, 2, \infty$, however, there exist quite simple algorithms to compute $\|A\|_p$. For $A = [a_{ij}] \in \mathbf{C}^{m \times n}$ it can be shown that:

$$\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}| \qquad \text{max column sum} \tag{4.97}$$

$$\|A\|_2 = \max_i \sqrt{\lambda_i(A^* A)} \qquad \text{largest singular value - see below} \tag{4.98}$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \qquad \text{max row sum} \tag{4.99}$$

4.3.3 Singular Values

The matrix norms defined above, can now be used as measures for the gain of a frequency response $G(j\omega)$. Consider the input-output relationship:

$$y = G(j\omega) u \tag{4.100}$$

Thus, the gain of the frequency response $G(j\omega)$ will be described as the ratio between the output y and the input u measured by the vector p-norm. This seems to be a reasonable way to measure the gain. The problem is, however, that any norm $\|G(j\omega)u\|_p$ depends not only of the magnitude of the input $\|u\|_p$ but also of the *direction* of the input vector. This fact is one of the fundamental differences between SISO and MIMO systems, since the input for a SISO system is always a scalar. Using the matrix p-norm to evaluate the frequency response $G(j\omega)$, the *maximal gain for all directions of the input vector* u is found. Hence, an upper bound for the gain of $G(j\omega)$ is achieved.

Both the 1-, 2-, and the ∞-norm have potential applications in control theory. The theoretical basis, however, has only been well developed for the 2-norm, although control theory based on the other norms are intensively researched at the moment. Here, the frequency response for $G(s)$ will be evaluated by

$$\|G(j\omega)\|_2 = \sup_{u \in \mathbf{C}^n, u \neq 0} \frac{\|G(j\omega)u\|_2}{\|u\|_2} = \max_i \sqrt{\lambda_i(G^*(j\omega)G(j\omega))} \tag{4.101}$$

$\max_i \sqrt{\lambda_i(G^*(j\omega)G(j\omega))}$ is also known as the maximal singular value $\bar{\sigma}(G(j\omega))$. If ω is varied from 0 to ∞, the matrix 2-norm of $G(j\omega)$ can be computed for all values of ω. This provides a frequency dependent upper bound for the gain of the transfer matrix $G(s)$. It is desired, though, also to know a lower bound for the gain. Such a lower bound is given by the *smallest singular value*[3] defined as

$$\underline{\sigma}(G(j\omega)) = \inf_{u \in \mathbf{C}^n, u \neq 0} \frac{\|G(j\omega)u(j\omega)\|_2}{\|u(j\omega)\|_2} \tag{4.102}$$

In summary:

> If the gain of a transfer matrix $G(s)$ is measured as the ratio between the 2-norms of the output y and the input u, the largest and the smallest singular values of $G(j\omega)$ determine an upper and a lower bound, respectively, for this gain.

The analysis of the gain of $G(s)$ can be made even more specific in terms of a *singular value decomposition* of $G(j\omega)$. To that end, the singular values of a complex matrix A are introduced by the following lemma.

Lemma 4.2 (Singular values and eigenvalues) *The singular values of a complex matrix $A \in \mathbf{C}^{m\times n}$, denoted $\sigma_i(A)$, equals the k largest (non-negative) square roots of the eigenvalues of A^*A, where $k = \min n, m$. Hence*

$$\sigma_i(A) = \sqrt{\lambda_i(A^*A)}, \qquad i = 1, 2, \cdots, k \tag{4.103}$$

Usually, the singular values are ordered such that $\sigma_i \geq \sigma_{i+1}$.

Hence:

$$\bar{\sigma}(A) = \sigma_1(A) = \sup_{x \in \mathbf{C}^n, x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \|A\|_2 \tag{4.104}$$

$$\underline{\sigma}(A) = \sigma_k(A) = \inf_{x \in \mathbf{C}^n, x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \tag{4.105}$$

[3]provided $G(s)$ has at least as many rows as columns. Otherwise, the smallest gain will always be 0

The ratio κ between the largest and the smallest singular value is known as the *condition number* for A:

$$\kappa(A) = \frac{\bar{\sigma}(A)}{\underline{\sigma}(A)} \tag{4.106}$$

Now, the singular value decomposition of a complex matrix A can be introduced:

Lemma 4.3 (Singular value decomposition) *Let $A \in \mathbf{C}^{m\times n}$ be a complex matrix. Then there exist two unitary matrices[4] $U \in \mathbf{C}^{m\times m}, V \in \mathbf{C}^{n\times n}$ and a diagonal matrix $\Sigma \in \mathbf{R}^{m\times n}$ such that:*

$$\begin{aligned} A &= U\Sigma V^* && (4.107)\\ &= \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_n^* \end{bmatrix} && (4.108)\\ &= \sum_{i=1}^{k} u_i \sigma_i v_i^* && (4.109) \end{aligned}$$

where

Σ_k	:	$diag(\sigma_1, \sigma_2, \cdots, \sigma_k)$.
$u_1 \to u_m$	:	*the m columns of U*
$v_1^* \to v_n^*$	:	*the n rows of V^**

This is known as the singular value decomposition (SVD) of the matrix A.

The singular value decomposition for a *real* matrix A can be interpreted in the following way. Any real matrix A geometrically maps a hyper-sphere of unit radius into a hyper-ellipsoid. The singular values $\sigma_i(A)$ specify the length of the main axis for the ellipsoid. *The singular vectors* u_i specify the orthogonal directions of these main axes, and the singular vectors v_i are mapped to the u_i vectors with a gain of σ_i, i.e., $Av_i = \sigma_i u_i$.

Example 4.3 (SVD for a real matrix)
This example has been taken from [MZ89]. Let A be given by

$$A = \begin{bmatrix} 0.8712 & -1.3195 \\ 1.5783 & -0.0947 \end{bmatrix} \tag{4.110}$$

The singular value decomposition of A is given by $A = U\Sigma V^*$ with:

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, V = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ -1 & -\sqrt{3} \end{bmatrix} \tag{4.111}$$

A geometrical interpretation is given in Figure 4.3 on the next page with $V = [v_1, v_2]$ and $U = [u_1, u_2]$.

[4] A *unitary matrix U* satisfies the equation $U^*U = I$.

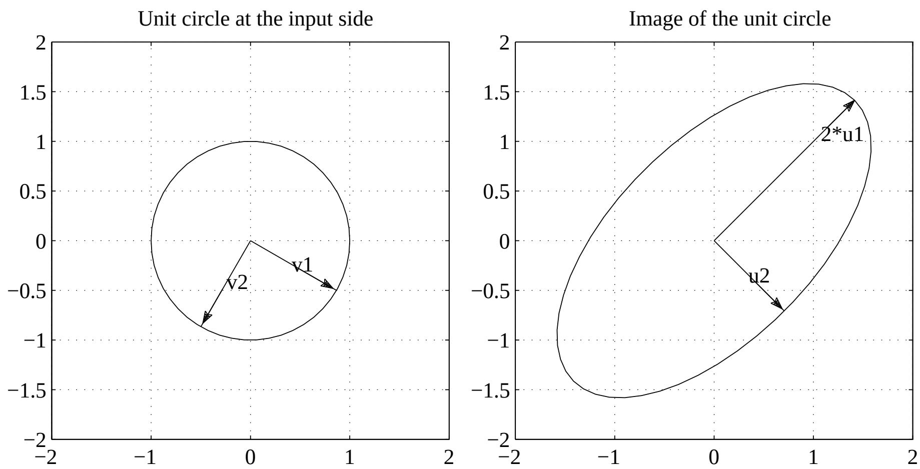


Figure 4.3: Singular value decomposition of a real matrix.

In the sequel, some of the important properties of singular values are listed:

$$\bar{\sigma}(A) = \sup_{u \in \mathbf{C}^n, u \neq 0} \frac{\|Au\|_2}{\|u\|_2} \tag{4.112}$$

$$\underline{\sigma}(A) = \inf_{u \in \mathbf{C}^n, u \neq 0} \frac{\|Au\|_2}{\|u\|_2} \tag{4.113}$$

$$\underline{\sigma}(A) \leq |\lambda_i(A)| \leq \bar{\sigma}(A) \tag{4.114}$$

$$\bar{\sigma}(A) = \frac{1}{\underline{\sigma}(A^{-1})} \tag{4.115}$$

$$\underline{\sigma}(A) = \frac{1}{\bar{\sigma}(A^{-1})} \tag{4.116}$$

$$\bar{\sigma}(\alpha A) = |\alpha|\bar{\sigma}(A) \tag{4.117}$$

$$\bar{\sigma}(A+B) \leq \bar{\sigma}(A) + \bar{\sigma}(B) \tag{4.118}$$

$$\bar{\sigma}(AB) \leq \bar{\sigma}(A)\bar{\sigma}(B) \tag{4.119}$$

$$\max\{\bar{\sigma}(A), \bar{\sigma}(B)\} \leq \bar{\sigma}([A\ B]) \leq \sqrt{2}\max\{\bar{\sigma}(A), \bar{\sigma}(B)\} \tag{4.120}$$

$$\sum_{i=1}^{n} \sigma_i^2 = \text{tr } \{A^* A\} \tag{4.121}$$

where

$\lambda_i(A)$	: The i'th eigenvalue of A.
Properties (4.115) and (4.116)	: Require the existence of G^{-1}.
α	: A constant (complex) scalar.
tr $\{A^* A\}$	: The trace of $A^* A$.

Reconsider the input-output relationship:

$$y = G(j\omega)u \tag{4.122}$$

By virtue of (4.109) on page 40 this can be formulated as

$$y = \sum_{i=1}^{k} u_i \sigma_i v_i^* u \tag{4.123}$$

Since V is unitary, v_i^* and v_j will be orthogonal, such that $v_i^* v_j = 0$, for $i \neq j$ and $v_i^* v_i = 1$. Now, assume that $u = \alpha v_j$. The input-output equation becomes:

$$y = \sum_{i=1}^{k} u_i \sigma_i v_i^* v_j \alpha \tag{4.124}$$

$$= \sigma_j \alpha u_j \tag{4.125}$$

This illustrates, that if the input vector u is in the direction of v_j, the gain of the system will be exactly σ_j, and the output vector y will be in the direction of u_j. Not surprisingly, the vector sets $\{v_1, v_2, \cdots, v_n\}$ and $\{u_1, u_2, \cdots, u_m\}$ are known as the *main input directions* and the *main output directions*, respectively. The singular values σ_i are also known as the *directional gains* for the transfer matrix $G(s)$.

Hence, if $G(s)$ is a transfer matrix, the singular values $\sigma_i(G(j\omega))$ for $i = 1, \cdots, k$ can be plotted as functions of the frequency ω. This family of curves constitutes the multivariable generalization of the Bode amplitude curve for SISO systems. For multivariable systems, the gain of a sinusoidal input $ue^{j\omega t}$ depends on the directions of the complex vector u as illustrated above. The gain is at least $\underline{\sigma}(G(j\omega))$ and at most $\bar{\sigma}(G(j\omega))$. A plot of the condition number $\kappa(G(j\omega))$, see (4.106) on page 40, as a function of the frequency ω illustrate the sensitivity of the gain to the direction of the input vector u. If $\kappa(G(j\omega)) \gg 1$, the gain of the system will depend strongly on the direction of the input vector, and $G(s)$ is said to be ill-conditioned. In contrast, if $\kappa(G(j\omega)) \approx 1$, $\forall \omega \geq 0$, the gain of the system will be insensitive to the direction of the input vector, and the system is said to be well-conditioned. A well-conditioned multivariable system largely behaves as a SISO system, and compensator design for such systems is fairly straightforward. For ill-conditioned systems, however, much more care must be taken in the design and analysis of compensators.

Chapter 5

Robustness Analysis for Multivariable Systems

In this chapter, stability and performance for multivariable systems with uncertainty will be considered. Consider a general multivariable system as depicted in Figure 5.1. All signals will in general be vectors, and $G(s)$ and $K(s)$ will be transfer matrices. $d(s)$ is an output disturbance signal and $n(s)$ represents measurement noise. Disturbances related to the input $u(s)$ have been neglected, but can easily be included if necessary. In Section 5.1.2, the performance problem is formulated as a 2×2 problem. To facilitate this reformulation, a sign convention is introduced such that the minus usually included in the loop, instead is incorporated in the compensator $K(s)$. The error is defined as $e(s) = y(s) - r(s)$. Due to the presence of measurement noise, however, the error can not be seen directly in the block diagram.

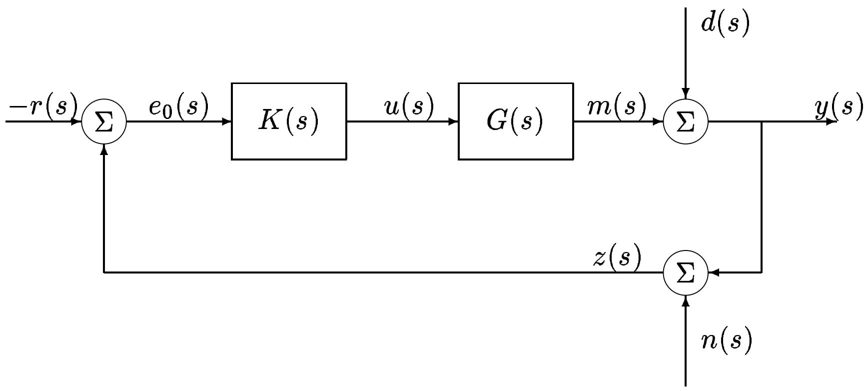


Figure 5.1: *General feedback configuration.*

The design of the compensator $K(s)$ in general has to satisfy the following four objectives:

- Nominal closed loop stability.
- Nominal Performance.
- Robust stability.
- Robust performance.

Robust stability means that the system is stable not only for the nominal model $G(s)$, but also for a family of models $\mathcal{G}$, containing all possible models of the process under the uncertainty assumptions made. Obviously, the nominal performance can be degraded signficantly before actual instability occurs. Hence, it is natural also to demand robust performance, i.e. the performance specifications have to be satisfied for all $G_\Delta(s) \in \mathcal{G}$.

Nominal stability for MIMO systems was treated in Chapter 4, where the Generalized Nyquist Theorem was derived. In the sequel, the requirement for nominal permance will be considered.

5.1 Nominal Performance

Based on Figure 5.1, it can easily be shown that

$$y(s) = T_o(s)(n(s) - r(s)) + S_o(s)d(s) \tag{5.1}$$
$$e_0(s) = S_o(s)\,(n(s) + d(s) - r(s)) \tag{5.2}$$
$$e(s) = y(s) - r(s) = S_o(s)\,(d(s) - r(s)) + T_o(s)n(s) \tag{5.3}$$
$$u(s) = M_o(s)\,(n(s) + d(s) - r(s)) \tag{5.4}$$

where

$$T_o(s) = (I - G(s)K(s))^{-1}\,G(s)K(s) = G(s)K(s)\,(I - G(s)K(s))^{-1} \tag{5.5}$$
$$S_o(s) = (I - G(s)K(s))^{-1} \tag{5.6}$$
$$M_o(s) = K(s)\,(I - G(s)K(s))^{-1} = (I - K(s)G(s))^{-1}\,K(s) \tag{5.7}$$

is, respectively, the complementary sensitivity, the sensitivity, and the control sensitivity. A subscript $(\cdot)_o$ has been used to emphasize that these sensitivity functions all have been derived by cutting the loop at the output y. As matrix multiplication is not commutative, sensitivity functions defined at the input $u(s)$ have to be distinguished from sensitivities defined at the output $y(s)$. The sensitivity functions evaluated at the input $u(s)$ are given by:

$$T_i(s) = K(s)G(s)\,(I - K(s)G(s))^{-1} = (I - K(s)G(s))^{-1}\,K(s)G(s) \tag{5.8}$$
$$S_i(s) = (I - K(s)G(s))^{-1} \tag{5.9}$$
$$M_i(s) = (I - K(s)G(s))^{-1}\,K(s) \tag{5.10}$$

Note that since $M_i(s) = M_o(s) = M(s)$, the control sensitivity is independent of whether the loop is broken at the input or at the output. The input sensitivities are not highly relevant for performance analysis, but their significance will become clear in Section 5.2 in connection with robust stability.

Now, let $v(s) = d(s) - r(s)$ denote the 'generic' external disturbance. Then the following observations can be made based on (5.1)-(5.4) :

- To achieve a good disturbance attenuation, i.e. for $v(s)$ to influence $e(s)$ as little as possible, it can be seen from (5.3), that the output sensitivity $S_o(s)$ has to be small.
- To achieve a good suppression of measurement noise, i.e. for $n(s)$ to influence $e(s)$ as little as possible, it is seen from (5.3) that the complementary sensitivity $T_o(s)$ has to be small.

- In order for disturbances $v(s)$ and measurement noise $n(s)$ not to deteriorate the input $u(s)$, it is seen from (5.4), that the control sensitivity $M(s)$ has to be small.

It still needs to be quantified, however, what 'small' means in this context. To do this in a systematic manner, the norms for signals and transfer functions introduced in Chapter 3 have to be extended to multivariable systems.

5.1.1 Norms of Signals for Multivariable Systems

In Chapter 3, the following norms for scalar signals $v(t)$ were introduced:

$$\text{The 1-norm:} \qquad \|v\|_1 = \int_{-\infty}^{\infty} |v(t)| dt \tag{5.11}$$

$$\text{The 2-norm:} \qquad \|v\|_2 = \left(\int_{-\infty}^{\infty} v^2(t) dt \right)^{\frac{1}{2}} \tag{5.12}$$

$$\text{The } \infty\text{-norm:} \qquad \|v\|_\infty = \sup_t |v(t)| \tag{5.13}$$

$$\text{The power 'norm':} \qquad \|v\|_\mathcal{P} = \text{pow}(v) = \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} v^2(t) dt \right)^{\frac{1}{2}} \tag{5.14}$$

The power 'norm' $\|v\|_\mathcal{P}$ is actually not a norm, but rather a semi-norm. With some abuse of terms it will refered to in these notes, nevertheless, as the power norm. Now, assuming $v(t)$ is a vector rather than a scalar, the power (semi-) norm is defined as:

$$\|v\|_\mathcal{P} = \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} \|v(t)\|_2^2 dt \right)^{\frac{1}{2}} \tag{5.15}$$

where the integrand $\|v(t)\|_2$ is the vector 2-norm, introduced in Section 4.3.1 on page 38:

$$\|v(t)\|_2 = \sqrt{\sum_{i=1}^{n} |v_i(t)|^2} = \sqrt{v^T(t) v(t)} \tag{5.16}$$

Here, $v^T(t)$ denotes the transpose of $v(t)$. Note, that the notation $\|v\|$ denotes a signal norm for the signal $v(t)$, whereas $\|v(t)\|$ denotes a vector norm for the *value* of the signal at time t. In addition to the vector 2-norm, in Section 4.3.1 on page 38 also the vector 1- and the vector ∞-norms were introduced:

$$\text{1-normen:} \qquad \|v(t)\|_1 = \sum_{i=1}^{n} |v_i(t)| \tag{5.17}$$

$$\infty\text{-normen:} \qquad \|v(t)\|_\infty = \max_i |v_i(t)| \tag{5.18}$$

Now, combining the scalar $1, 2, \infty$ signal norms with the $1, 2, \infty$ vector norms leads to nine different multivariable signal norms:

$$\|v\|_{1,1} = \int_{-\infty}^{\infty} \|v(t)\|_1 dt = \int_{-\infty}^{\infty} \sum_{i=1}^{n} |v_i(t)| dt \tag{5.19}$$

$$\|v\|_{1,2} = \int_{-\infty}^{\infty} \|v(t)\|_2 dt = \int_{-\infty}^{\infty} \sqrt{v^T(t)v(t)} dt \tag{5.20}$$

$$\|v\|_{1,\infty} = \int_{-\infty}^{\infty} \|v(t)\|_\infty dt = \int_{-\infty}^{\infty} \max_i |v_i(t)| dt \tag{5.21}$$

$$\|v\|_{2,1} = \left(\int_{-\infty}^{\infty} \|v(t)\|_1^2 dt\right) = \sqrt{\int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} |v_i(t)|\right)^2 dt} \tag{5.22}$$

$$\|v\|_{2,2} = \left(\int_{-\infty}^{\infty} \|v(t)\|_2^2 dt\right) = \sqrt{\int_{-\infty}^{\infty} v^T(t)v(t) dt} \tag{5.23}$$

$$\|v\|_{2,\infty} = \left(\int_{-\infty}^{\infty} \|v(t)\|_\infty^2 dt\right) = \sqrt{\int_{-\infty}^{\infty} \left(\max_i |v_i(t)|\right)^2 dt} \tag{5.24}$$

$$\|v\|_{\infty,1} = \sup_t \|v(t)\|_1 = \sup_t \sum_{i=1}^{n} |v_i(t)| \tag{5.25}$$

$$\|v\|_{\infty,2} = \sup_t \|v(t)\|_2 = \sup_t \sqrt{v^T(t)v(t)} \tag{5.26}$$

$$\|v\|_{\infty,\infty} = \sup_t \|v(t)\|_\infty = \sup_t \max_i |v_i(t)| \tag{5.27}$$

i.e., $\|v\|_{x,y}$ denotes the norm obtained by applying the vector y-norm at every time instance t and by applying the signal norm $\|\cdot\|_x$ as the temporal norm.

These signal norms can now be used with the power norm $\|v\|_{\mathcal{P}}$ to define a number of induced norms for multivariable transfer functions. To that end, let $g(t)$ be the impulse response matrix associated with the transfer matrix $G(s)$. The output $y(t)$ is then given by:

$$y(t) = g(t) * u(t) \tag{5.28}$$

where $g(t) * u(t)$ denotes the convolution integral of $g(t)$ and $u(t)$. Now, an induced norm for $g(t)$ can be defined as:

$$\|g(t)\|_{(u_x,u_y)\to(y_x,y_y)} = \sup_{u\neq 0} \frac{\|y\|_{y_x,y_y}}{\|u\|_{u_x,u_y}} = \sup_{u\neq 0} \frac{\|g * u\|_{y_x,y_y}}{\|u\|_{u_x,u_y}} \tag{5.29}$$

where u_x, u_y, y_x, y_y can be chosen as any combination among 1, 2, ∞ and $\mathcal{P}$. Hence, the induced norm $\|g(t)\|_{(u_x,u_y)\to(y_x,y_y)}$ measure the maximal 'gain' of the system when the input is measured by the norm $\|u\|_{u_x,u_y}$ and the output by the norm $\|y\|_{y_x,y_y}$. In this way, more than a hundred different induced norms can be defined for $g(t)$. Roughly this corresponds to more than a hundred different ways to measure the gain of multivariable system, although only a few of these have any practical significance.

Example 5.1 (Nominal Performance)
Consider a control system with the objective of minimizing the consumption of a certain ressource. Thus, it would be reasonable to measure the control error $e(t)$ by the norm $\|e\|_{1,1}$.

Assume that characterizing the disturbances $d(t)$ by their energy is reasonable as well. Then $\|d\|_{2,2}$ is a good measure for $d(t)$. This means that the control objective can be translated into a minimization of the gain interpreted as the following norm:

$$\|s\|_{(1,1)\to(2,2)} = \sup_{d\neq 0} \frac{\|e\|_{1,1}}{\|d\|_{2,2}} \tag{5.30}$$

where $s(t)$ is the impulse response matrix for the transfer function $S(s)$ from $d(s)$ to $e(s)$.

At first glance, all these induced norms might seem overwhelming. It can be shown, however, that several of them always equal either zero or infinity, and therefore are without any practical significance. Current research activities include the development of compensator design procedures related to a number of the norms mentioned above.

In the 'classical' robust robust control theory, only one single induced norm has been considered: the 'pure' 2-norm, i.e., the norm induced by $\|\cdot\|_{2,2}$ on $\|\cdot\|_{2,2}$:

$$\|g\|_{(2,2)\to(2,2)} = \sup_{u\neq 0} \frac{\|g*u\|_{2,2}}{\|u\|_{2,2}} = \sup_{u\neq 0} \frac{\sqrt{\int_{-\infty}^{\infty} y^T(t)y(t)dt}}{\sqrt{\int_{-\infty}^{\infty} u^T(t)u(t)dt}} \tag{5.31}$$

As the $\|\cdot\|_{2,2}$ norm measures the energy of a signal, $\|g\|_{(2,2)\to(2,2)}$ represent the largest possible energy gain for a system. The reason why this norm has been of particular interest is in part that it is applicable in practice in many cases, and that it leads to simple conditions both for robust stability and for nominal performance.

An important property for $\|\cdot\|_{2,2}$ is obtained from Parseval's Theorem:

$$\sqrt{\int_{-\infty}^{\infty} u^T(t)u(t)dt} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} u^*(j\omega)u(j\omega)d\omega} \tag{5.32}$$

where $u(j\omega)$ denotes the Laplace transform of $u(t)$. Hence, also: også

$$\begin{aligned}
\|g\|_{(2,2)\to(2,2)} &= \sup_{u\neq 0} \frac{\|g*u\|_{2,2}}{\|u\|_{2,2}} && (5.33)\\
&= \sup_{u\neq 0} \frac{\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} y^*(j\omega)y(j\omega)d\omega}}{\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} u^*(j\omega)u(j\omega)d\omega}} && (5.34)\\
&= \sup_{u\neq 0} \frac{\sqrt{\int_{-\infty}^{\infty} u(j\omega)^*G(j\omega)^*G(j\omega)u(j\omega)d\omega}}{\sqrt{\int_{-\infty}^{\infty} u^*(j\omega)u(j\omega)d\omega}} && (5.35)\\
&= \sup_{\omega} \bar{\sigma}(G(j\omega)) && (5.36)\\
&= \|G\|_{\mathcal{H}_\infty} && (5.37)
\end{aligned}$$

To recapitulate:

> *Let a multivariable system be described by its transfer matrix $G(s)$. If the input $u(t)$ to the system and the output $y(t)$ are both measured by the 2-norm $\|\cdot\|_{2,2}$, the maximal gain of the system is given by the $\mathcal{H}_\infty$ norm of $G(s)$:*
>
> $$\|G\|_{\mathcal{H}_\infty} = \sup_{\omega} \bar{\sigma}(G(j\omega)) = \sup_{u\neq 0} \frac{\|y\|_{2,2}}{\|u\|_{2,2}} \tag{5.38}$$

The $\mathcal{H}_\infty$ norm is said to be induced by the signal 2-norm.

Thus, the singular value concept introduced in Chapter 4 can be used to evaluate the $\mathcal{H}_\infty$ norm of a transfer matrix. If $\bar{\sigma}(G(j\omega))$ is plotted versus frequency, $\|G\|_{\mathcal{H}_\infty}$ is given as the maximal value.

Returning to the observations concerning performance given in Page 44, it would be tempting to formulate the following requirement for good disturbance attenuation:

$$\bar{\sigma}(S_o(j\omega)) \ll 1 \qquad \forall\omega \geq 0 \tag{5.39}$$

Likewise, for good suppression of measurement, a requirement could be:

$$\bar{\sigma}(T_o(j\omega)) \ll 1 \qquad \forall\omega \geq 0 \tag{5.40}$$

For $v(s)$ and $n(s)$ not to impact on $u(s)$, it could be required that:

$$\bar{\sigma}(M(j\omega)) \ll 1 \qquad \forall\omega \geq 0 \tag{5.41}$$

Since

$$S_o(j\omega) - T_o(j\omega) = I$$

the complementary sensitivity $T_o(j\omega)$ and the sensitivity $S_o(j\omega)$ can not both be small in the same frequency range. Hence, optimal disturbance attenuation and suppression of measurement noise can not be obtained in the same frequency interval. This is a well-known result from classical control theory. Fortunately, disturbances $v(s)$ are normally concentrated in the low frequency range, whereas measurement noise often appears in the high frequency range. Thus, the complementary sensitivity $T_o(j\omega)$ and the sensitivity $S_o(j\omega)$ can be shaped such that $\bar{\sigma}(S_o(j\omega))$ is small for low frequencies while $\bar{\sigma}(T_o(j\omega))$ is small for high frequencies. In general, measurement noise will limit the achievable bandwidth for a control system. If the measurement noise is significant below the desired bandwidth, it might be necessary to replace some of the sensors by better products. In Section 5.2 it will be seen, however, that robustness to model uncertainties also limits the sensitivity functions.

Often, a performance specification will be given as a weighted sensitivity specification:

$$\bar{\sigma}(W_{p2}(j\omega)S_o(j\omega)W_{p1}(j\omega)) \leq 1\ , \qquad \forall\omega \geq 0 \tag{5.42}$$

where $W_{p1}(s)$ and $W_{p2}(s)$, respectively, denote an input and an output weight, see Figure 5.2. As a precondition, the weights must be scaled, such that the '1' on the right hand side of (5.42) makes sense.

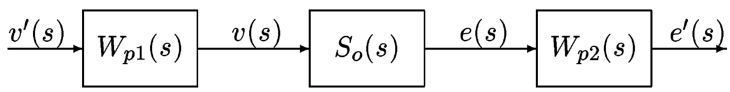


Figure 5.2: *Nominal performance specification. Output sensitivity with input weight $W_{p1}(s)$ and output weight $W_{p2}(s)$.*

The *normalized input vector* $v'(t)$ is assumed to be belong to the following set of norm bounded signals:

$$\mathcal{D}'_t = \left\{ v'(t) \,\middle|\, \|v'\|_{2,2}^2 \leq 1 \right\} \tag{5.43}$$

Hence, the Laplace transform of $v'(t)$ belongs to the following set:

$$\mathcal{D}' = \left\{ v'(s) \,\middle|\, \|v'\|_{2,2}^2 \leq 1 \right\} \tag{5.44}$$

The input weight $W_{p1}(s)$ is used to transform the normalized input $v'(s)$ to the physical input $v(s)$. Thus, the physical input is assumed to belong to:

$$\mathcal{D} = \left\{ v(s) \,\middle|\, \left\| W_{p1}^{-1} v \right\|_{2,2}^2 \leq 1 \right\} \tag{5.45}$$

As the frequency content of v is usually concentrated at low frequencies, $W_{p1}(s)$ would usually be large at low frequencies and small at high frequencies.

The output weight $W_{p2}(s)$ is primarily used to trade off the importance of the individual signals in $e(t)$ and possibly to specify the frequency range in which the errors are most significant.

The $\mathcal{H}_\infty$ norm of $W_{p2}(s)S_o(s)W_{p1}(s)$ is thus given by:

$$\|W_{p2}(s)S_o(s)W_{p1}(s)\|_{\mathcal{H}_\infty} = \sup_{v' \neq 0} \frac{\|e'\|_{2,2}}{\|v'\|_{2,2}} \tag{5.46}$$

$$\Rightarrow \quad \|W_{p2}(s)S_o(s)W_{p1}(s)\|_{\mathcal{H}_\infty} = \sup_{v' \in \mathcal{D}'} \|e'\|_{2,2} \tag{5.47}$$

$$\Leftrightarrow \quad \|W_{p2}(s)S_o(s)W_{p1}(s)\|_{\mathcal{H}_\infty} = \sup_{v' \in \mathcal{D}'} \sqrt{\int_{-\infty}^{\infty} e'(t)^T e'(t) dt} \tag{5.48}$$

Hence, the 2-norm of the normalized error $\|e'\|_{2,2}$ will at most be equal to the $\mathcal{H}_\infty$ norm of $W_{p2}(s)S_o(s)W_{p1}(s)$.

This enables the formulation of a nominal performance criterion. Given the weight functions $W_{p1}(s)$ and $W_{p2}(s)$, a stabilizing compensator should be designed such that the cost

$$\mathcal{J}_{np} = \|W_{p2}(s)S_o(s)W_{p1}(s)\|_{\mathcal{H}_\infty} \tag{5.49}$$

is minimized. This can also be formulated as:

$$K(s) = \arg \min_{K(s) \in \mathcal{K}_S} \|W_{p2}(s)S_o(s)W_{p1}(s)\|_{\mathcal{H}_\infty} \tag{5.50}$$

where $\mathcal{K}_S$ denotes the set of all stabilizing compensators. If a stabilizing compensator satisfies $\|W_{p2}(s)S_o(s)W_{p1}(s)\|_{\mathcal{H}_\infty} < 1$ the controlled system is said to have nominal performance.

5.1.2 2×2 Block Problem

A convenient way to formulate the nominal performance problem is in terms of the 2×2 block problem. In Figure 5.3 it is shown how to include $W_{p1}(s)$ and $W_{p2}(s)$ in the closed loop. The extended closed loop system can then be represented in the so-called 2×2 block form, see Figure 5.3, where the signals $r(s)$, $n(s)$ and $e(s)$ from Figure 5.1 have been omitted, as only the transfer function from $d'(s)$ to $e'(s)$ is included in the performance. If it is required to include reference changes or measurement noise in the performance cost function, this can be done analogously, as the $d'(s)$ vector just has to be extended with $r(s)$ and $n(s)$. In addition to the nominal model, the generalized system $N(s)$ also contains the weight functions for nominal performance.

$$N(s) = \begin{bmatrix} W_2(s)W_1(s) & W_2(s)G(s) \\ W_1(s) & G(s) \end{bmatrix} \tag{5.51}$$

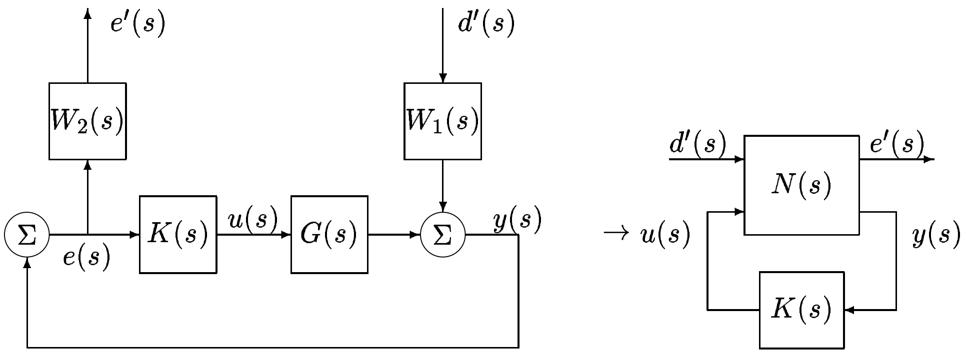


Figure 5.3: *Nominal performance problem. Extended closed loop system and corresponding* 2×2 *block form.*

The transfer function from $d'(s)$ to $e'(s)$ is now given by:

$$e'(s) = \left(N_{11}(s) + N_{12}(s)K(s)\left(I - N_{22}(s)K(s)\right)^{-1} N_{21}(s)\right) d'(s) \tag{5.52}$$

$$= W_{p2}(s)S_o(s)W_{p1}(s)d'(s) \tag{5.53}$$

Using the notation:

$$F_l(N(s), K(s)) = \left(N_{11}(s) + N_{12}(s)K(s)\left(I - N_{22}(s)K(s)\right)^{-1} N_{21}(s)\right) \tag{5.54}$$

the nominal performance problem can be written as:

$$K(s) = \arg \min_{K(s)\in\mathcal{K}_S} \|F_l(N(s), K(s))\|_{\mathcal{H}_\infty} \tag{5.55}$$

This is a so-called standard $\mathcal{H}_\infty$ problem, which can be solved by known methods. These methods will be described in Chapter 6 on page 63.

5.2 Robust Stability

In the sequel, robust stability for multivariable systems will be considered. In the beginning of the 1980's, control theorists as e.g. Doyle & Stein realized that robustness with respect to unmodeled dynamics can be achieved by bounding some of the sensitivity functions which have been described in the previous section. Let $G(s)$ and $G_\Delta(s)$ describe the nominal and the real (perturbed) system, respectively. The following perturbation models will be considered.

- Additive uncertainty: $G_\Delta(s) = G(s) + \tilde{\Delta}(s)$.
- Input multiplicative uncertainty: $G_\Delta(s) = G(s)(I + \tilde{\Delta}(s))$.
- Output multiplicative uncertainty: $G_\Delta(s) = (I + \tilde{\Delta}(s))G(s)$.

- Inverse input multiplicative uncertainty: $G_\Delta(s) = G(s)(I + \tilde{\Delta}(s))^{-1}$.
- Inverse output multiplicative uncertainty: $G_\Delta(s) = (I + \tilde{\Delta}(s))^{-1}G(s)$.

In Chapter 1 similar uncertainty models were considered for scalar systems, and the size of the perturbation $\tilde{\Delta}(s)$ was bounded by the absolute value of its frequency response:

$$|\tilde{\Delta}(j\omega)| \leq \ell(\omega) \tag{5.56}$$

In this chapter, however, $\tilde{\Delta}(j\omega)$ is a matrix. Hence, it has to be decided which norm to be used to bound the size of $\tilde{\Delta}(j\omega)$. In modern robust control, mainly the matrix 2-norm is used to bound $\tilde{\Delta}(j\omega)$, for two main reasons. First, the corresponding perturbation model is appropriate to describe high frequent unmodeled dynamics, time delays, and phenomena arising from systems with distributed parameters. Second, this choice leads to mathematically simple conditions for robust stability.

Thus, it is assumed that $\tilde{\Delta}(j\omega)$ is a complex matrix, which is unknown but bounded in amplitude:

$$\|\tilde{\Delta}(j\omega)\|_2 = \bar{\sigma}(\tilde{\Delta}(j\omega)) \leq \ell(\omega), \qquad \forall\omega \geq 0 \tag{5.57}$$

Usually, two diagonal weight matrices $W_{u1}(s)$ and $W_{u2}(s)$ are introduced such that:

$$\tilde{\Delta}(s) = W_{u2}(s)\Delta(s)W_{u1}(s) \tag{5.58}$$

with

$$\|\Delta(j\omega)\|_2 = \bar{\sigma}(\Delta(j\omega)) \leq 1, \qquad \forall\omega \geq 0 \tag{5.59}$$

The input weight $W_{u1}(s)$ is used for scaling, if e.g. the signals are measured in different units. The output weight $W_{u2}(s)$ is used as a frequency dependent weight to approximate $\ell(\omega)$.

5.2.1 The Small Gain Theorem

Below, the famous *Small Gain Theorem* will be presented and applied to derive conditions for robust stability with respect to the uncertainty models introduced above. Consider the system in Figure 5.1 and let $P(s) = G(s)K(s)$ be a square transfer matrix. Then

Theorem 5.1 (The Small Gain Theorem) *Assume, that $P(s)$ is a stable transfer matrix. Then the closed loop system is stable if the spectral radius $\rho(P(j\omega)) < 1$, $\forall\omega$.*

***Proof of Theorem* 5.1 (By contradiction)** Assume that the spectral radius $\rho(P(j\omega)) < 1$, $\forall\omega$ and that the closed loop system is unstable. According to the Generalized Nyquist Theorem, instability implies that the image curve of $\det(I + P(s))$ encircles the origin as s traverses the Nyquist $\mathcal{D}$ contour. As the Nyquist $\mathcal{D}$ contour is closed, so will its image curve be also. Hence, there exists an $\epsilon \in [0, 1]$ and a frequency $\omega^\star$ such that

$$
\begin{array}{llll}
 & \det(I + \epsilon P(j\omega^\star)) = 0 & \text{i.e. the image curve crosses the origin} & (5.60)\\
\Leftrightarrow & \prod_i \lambda_i(I + \epsilon P(j\omega^\star)) = 0 & & (5.61)\\
\Leftrightarrow & 1 + \epsilon\lambda_i\left(P(j\omega^\star)\right) = 0 & \text{for some given } i & (5.62)\\
\Leftrightarrow & \lambda_i\left(P(j\omega^\star)\right) = -\dfrac{1}{\epsilon} & \text{for some given } i & (5.63)\\
\Rightarrow & |\lambda_i\left(P(j\omega^\star)\right)| \geq 1 & \text{for some given } i & (5.64)
\end{array}
$$

which is a contradiction as we assumed that $\rho(P(j\omega)) < 1$, $\forall\omega$. $\square$

The Small Gain Theorem states, that for an open loop stable system, a sufficient condition for closed loop stability is that the open loop gain measured by $\rho(P(j\omega))$ is less than unity. Fortunately, this a just a (potentially very conservative) sufficient condition for closed loop stability. Otherwise it could be impossible to satisfy the usual performance requirement for high gains in the low frequent range.

Below, the Small Gain Theorem will be applied to determine the stability of the closed loop system for the uncertainty models introduced above. This application of the Small Gain Theorem is classical in robust control. Assume, e.g., that the perturbed system can be described by an additive perturbation:

$$G_\Delta(s) = G(s) + W_{u2}(s)\Delta(s)W_{u1}(s) \tag{5.65}$$

where $\bar{\sigma}(\Delta(j\omega)) \leq 1$. This can be represented in block diagram form as shown in Figure 5.4. Now, let $P(s)$ denote the transfer matrix as 'seen' from the terminals of $\Delta(s)$. It can easily be verified that:

$$\begin{aligned} P(s) &= W_{u1}(s)K(s)\left(I - G(s)K(s)\right)^{-1} W_{u2}(s) && (5.66) \\ &= W_{u1}(s)M(s)W_{u2}(s) && (5.67) \end{aligned}$$

The following result is obtained:

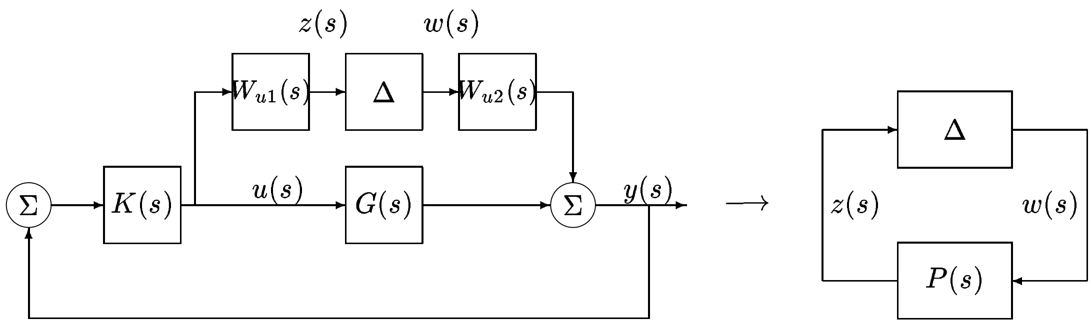


Figure 5.4: *Closed loop system with additive perturbation. $P(s)$ is the transfer matrix as seen from the terminals of $\Delta(s)$.*

Theorem 5.2 (Robust Stabilitet) *Assume that the system $P(s)$ is stable, and that the perturbation $\Delta(s)$ is such that the perturbed closed loop system is stable if and only if the image curve of $\det(I - P(s)\Delta(s))$, as s traverses the Nyquist $\mathcal{D}$ contour, does not encircle the origin. Then, the closed loop system in Figure 5.4 is stable for all perturbations $\Delta(s)$, $\bar{\sigma}(\Delta(j\omega)) \leq 1$, if and only if one of the following equivalent conditions is satisfied:*

$$\begin{aligned} & \det(I - P(j\omega)\Delta(j\omega)) \neq 0 && \forall\omega,\ \forall\Delta(j\omega) : \bar{\sigma}(\Delta(j\omega)) \leq 1 && (5.68) \\ \Leftrightarrow\quad & \rho(P(j\omega)\Delta(j\omega)) < 1 && \forall\omega,\ \forall\Delta(j\omega) : \bar{\sigma}(\Delta(j\omega)) \leq 1 && (5.69) \\ \Leftrightarrow\quad & \bar{\sigma}(P(j\omega)) < 1 && \forall\omega && (5.70) \\ \Leftrightarrow\quad & \|P(s)\|_{\mathcal{H}_\infty} < 1 && && (5.71) \end{aligned}$$

***Proof of Theorem* 5.2** Assume, that a perturbation $\Delta^\star(s)$ exists such that $\bar{\sigma}(\Delta^\star(j\omega)) \leq 1$ and such that the closed loop system is unstable. Then the image curve of $\det(I - P(s)\Delta^\star(s))$ will encircle the origin as s traverses the Nyquist $\mathcal{D}$ contour. Since the Nyquist $\mathcal{D}$ contour is closed, so is the image curve of $\det(I - P(s)\Delta^\star(s))$. Hence, an $\epsilon \in [0, 1]$ and a frequency $\omega^\star$ exist such that

$$\det(I - P(j\omega^\star)\epsilon\Delta^\star(j\omega^\star)) = 0 \tag{5.72}$$

Since

$$\bar{\sigma}(\epsilon\Delta^\star(j\omega^\star)) = \epsilon\bar{\sigma}(\Delta^\star(j\omega^\star)) \leq 1 \tag{5.73}$$

$\epsilon\Delta^\star(s)$ is just another perturbation from the set of all possible perturbation, which shows that if the closed loop system is unstable, there exists another perturbation having $\bar{\sigma}(\Delta(j\omega)) \leq 1$ for which $\det(I - P(s)\Delta(s)) = 0$. Oppositely, if there exists no perturbation for which $\det(I - P(s)\Delta(s)) = 0$, the Nyquist image curve of $\det(I - P(s)\Delta(s))$ will also not encircle the origin. Thus, necessity and sufficiency of the condition (5.68) has been established.

Sufficiency of Condition (5.69) follows directly from the Small Gain Theorem. Since

$$\rho(P(j\omega)\Delta(j\omega)) \leq \bar{\sigma}(\rho(P(j\omega)\Delta(j\omega))) \leq \bar{\sigma}(P(j\omega))\bar{\sigma}(\Delta(j\omega)) \leq \bar{\sigma}(P(j\omega)) \leq \|P(s)\|_{\mathcal{H}_\infty} \tag{5.74}$$

both Condition (5.70) and (5.71) are sufficient for robust stability. To prove necessity of (5.69), assume that there exists a perturbation $\Delta^\star(s)$ for which $\bar{\sigma}(\Delta^\star(j\omega)) \leq 1$ and a frequency $\omega^\star$ such that $\rho(P(j\omega^\star)\Delta^\star(j\omega^\star)) = 1$. Then the following holds:

$$\begin{aligned}
& |\lambda_i(P(j\omega^\star)\Delta^\star(j\omega^\star))| = 1 && \text{for some given } i && (5.75)\\
\Leftrightarrow\quad & \lambda_i(P(j\omega^\star)\Delta^\star(j\omega^\star)) = e^{j\theta} && \text{for some given } i && (5.76)\\
\Leftrightarrow\quad & \lambda_i(P(j\omega^\star)e^{-j\theta}\Delta^\star(j\omega^\star)) = +1 && \text{for some given } i && (5.77)\\
\Leftrightarrow\quad & \lambda_i(P(j\omega^\star)\tilde{\Delta}^\star(j\omega^\star)) = +1 && \text{for some given } i && (5.78)
\end{aligned}$$

where $\tilde{\Delta}^\star(s)$ is another perturbation from the set and where $\rho(P(j\omega^\star)\tilde{\Delta}^\star(j\omega^\star)) = 1$. Hence,

$$\det(I - P(j\omega^\star)\tilde{\Delta}^\star(j\omega^\star)) = 0 \tag{5.79}$$

and the necessity of Condition (5.69) has been shown.

To prove necessity of Condition (5.70), it will be shown that for each frequency ω a perturbation $\Delta^\star(s)$ exists for which $\bar{\sigma}(\Delta^\star(j\omega)) \leq 1$ such that $\det(I - P(j\omega)\Delta^\star(j\omega)) = 0$ if $\bar{\sigma}(P(j\omega)) = 1$. Indeed, let $D = \text{diag}\{1, 0, \cdots, 0\}$ and perform a singular value decomposition of $P(j\omega)$ at the frequency ω:

$$P(j\omega) = U\Sigma V^* \tag{5.80}$$

where U and V are unitary matrices. Moreover, let $\Delta^\star(j\omega) = VDU^*$. As U and V are unitary $\bar{\sigma}(\Delta^\star(j\omega)) = 1$ is such that the perturbation belongs to the admissible set. This leads to

$$\begin{aligned}
\det(I - P(j\omega)\Delta^\star(j\omega)) &= \det(I - U\Sigma V^\star VDU^\star) = \det(I - U\Sigma DU^\star)\\
&= \det\left(U(I - \Sigma D)U^\star\right) = \det(U)\det(I - \Sigma D)\det(U^\star) = \det(I - \Sigma D) = 0
\end{aligned} \tag{5.81}$$

since the first row and the first column in $I - \Sigma D$ equal zero. This establishes necessity of Condition (5.70). The necessity of (5.71) then become obvious. □

Theorem 5.2 states that if $\|P(s)\|_{\mathcal{H}_\infty} < 1$, then there exists no perturbations $\Delta(s)$ for which $\bar{\sigma}(\Delta(j\omega)) \leq 1$, causing the image curve of $\det(I - P(s)\Delta(s))$ to encircle the origin, as s traverses the Nyquist $\mathcal{D}$ contour. Note, that it is *assumed* that this is a necessary and sufficient condition for preserving stability. This assumption holds, e.g., when all perturbations $\Delta(s)$ are stable, or when $G_\Delta(s)$ and $G(s)$ have the same number of unstable poles. This assumption is standard within robust control theory.

Note also, that the $\mathcal{H}_\infty$ norm condition (5.71) is *not* conservative, when the uncertainty is bounded by the maximum singular value. Thus, if $\|P(s)\|_{\mathcal{H}_\infty} \geq 1$, a perturbation $\Delta^\star(s)$ exists, for which $\bar{\sigma}(\Delta^\star(j\omega)) \leq 1$, which will destabilize the closed loop system. If the real uncertainty is represented in a non-conservative fashion by $\Delta(s)$, then the $\mathcal{H}_\infty$ condition for robust stability is also not conservative.

Now, Theorem 5.2 will be applied to the additively perturbed system, shown in Figure 5.4. It is seen, that the perturbed system is stable, if and only if:

$$\bar{\sigma}(P(j\omega)) < 1 \qquad \forall\omega \tag{5.82}$$

$$\Leftrightarrow \quad \bar{\sigma}\left(W_{u1}(j\omega)K(j\omega)\left(I + G(j\omega)K(j\omega)\right)^{-1} W_{u2}(j\omega)\right) < 1 \qquad \forall\omega \tag{5.83}$$

$$\Leftrightarrow \quad \bar{\sigma}\left(W_{u1}(j\omega)M(j\omega)W_{u2}(j\omega)\right) < 1 \qquad \forall\omega \tag{5.84}$$

$$\Leftrightarrow \quad \|W_{u1}(j\omega)M(j\omega)W_u(j\omega)\|_{\mathcal{H}_\infty} < 1 \tag{5.85}$$

In exactly the same way as for additive uncertainties, $\mathcal{H}_\infty$ conditions for robust stability with respect to the perturbation structures formulated in Page 50 can be derived. In Table 5.1 these conditions are given.

Perturbation	Robust stability requirement	Norm bound
Additive	$M(s)$ small	$\|W_{u1}(s)M(s)W_{u2}(s)\|_{\mathcal{H}_\infty} < 1$
Input multiplicative	$T_i(s)$ small	$\|W_{u1}(s)T_i(s)W_{u2}(s)\|_{\mathcal{H}_\infty} < 1$
Output multiplicative	$T_o(s)$ small	$\|W_{u1}(s)T_o(s)W_{u2}(s)\|_{\mathcal{H}_\infty} < 1$
Inverse input mult.	$S_i(s)$ small	$\|W_{u1}(s)S_i(s)W_{u2}(s)\|_{\mathcal{H}_\infty} < 1$
Inverse output mult.	$S_o(s)$ small	$\|W_{u1}(s)S_o(s)W_{u2}(s)\|_{\mathcal{H}_\infty} < 1$

Table 5.1: *Various uncertainty descriptions and corresponding conditions for robust stability.*

Now, a design criterion for robust stability can be formulated. Given e.g. an additive uncertainty description $W_{u2}(s)\Delta(s)W_{u1}(s)$ the robust stability problem can be formulated as the design of a nominally stabilizing compensator, such that the cost function

$$\mathcal{J}_u = \|W_{u1}(s)M(s)W_{u2}(s)\|_{\mathcal{H}_\infty} \tag{5.86}$$

is minimized. Thus:

$$K(s) = \arg \min_{K(s)\in\mathcal{K}_S} \|W_{u1}(s)M(s)W_{u2}(s)\|_{\mathcal{H}_\infty} \tag{5.87}$$

where $\mathcal{K}_S$ denotes the set of all stabilizing compensators[1]. If a specific stabilizing compensator can achieve $\mathcal{J}_u < 1$, the closed loop system is said to be robustly stable.

Note, that the structure for the robust stability problem is identical to the structure of the nominal performance problem in Page 49. Thus, the robust stability problem can also be

[1]A technical subtlety: (5.87) should be understood in a conceptual way. The implied minimum might not actually be assumed. In practice, a design procedure would find a compensator 'close enough'.

formulated as a 2×2 block problem. Given an additive uncertainty specification, the corresponding 2×2 block problem can be derived as illustrated in Figure 5.5, where

$$N(s) = \begin{bmatrix} 0 & W_{u1}(s) \\ W_{u2}(s) & G(s) \end{bmatrix} \tag{5.88}$$

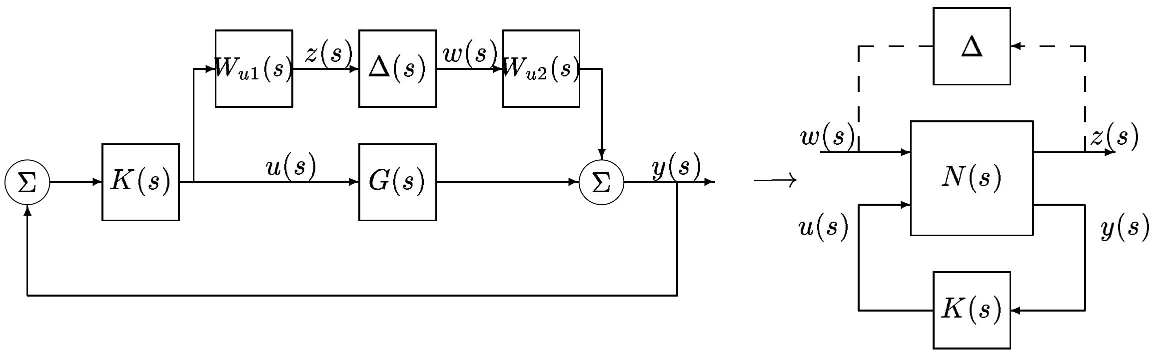


Figure 5.5: *Robust stability problem. Extended closed loop system and corresponding* 2×2 *block form.*

The transfer function from $w(s)$ to $z(s)$ is given by

$$\begin{aligned} z(s) &= W_{u1}(s)K(s)\left(I + G(s)K(s)\right)^{-1} W_{u2}(s)w(s) && (5.89)\\ &= F_l(N(s), K(s))w(s) && (5.90)\\ &= W_{u1}(s)M(s)W_{u2}(s)w(s) && (5.91) \end{aligned}$$

and, consequently, the robust stability problem can be formulated as:

$$K(s) = \arg \min_{K(s)\in\mathcal{K}} \|F_l(N(s), K(s))\|_{\mathcal{H}_\infty}. \tag{5.92}$$

which is entirely identical to the nominal performance problem. The minimization (5.92), hence, is a standard $\mathcal{H}_\infty$ problem, which can be solved by methods to be introduced in Chapter 6 on page 63.

5.3 Robust Performance

Finally, in this chapter, robust performance will be considered. The robust performance criterion can be derived from the nominal performance criterion (5.49) with the nominal sensitivity function $S_o(s)$ replaced by the perturbed sensitivity function $S_{o,\Delta}(s)$:

$$\mathcal{J}_{rp} = \|W_{p2}(s)S_{o,\Delta}(s)W_{p1}(s)\|_{\mathcal{H}_\infty} \tag{5.93}$$

To illustrate, consider an additive uncertainty model. Then, the robust performance problem can be depicted as in Figure 5.6, where

$$N(s) = \begin{bmatrix} 0 & 0 & W_{u1}(s) \\ W_{p2}W_{u2}(s) & W_{p2}W_{p1} & W_{p2}G(s) \\ W_{u2} & W_{p1} & G(s) \end{bmatrix} \tag{5.94}$$

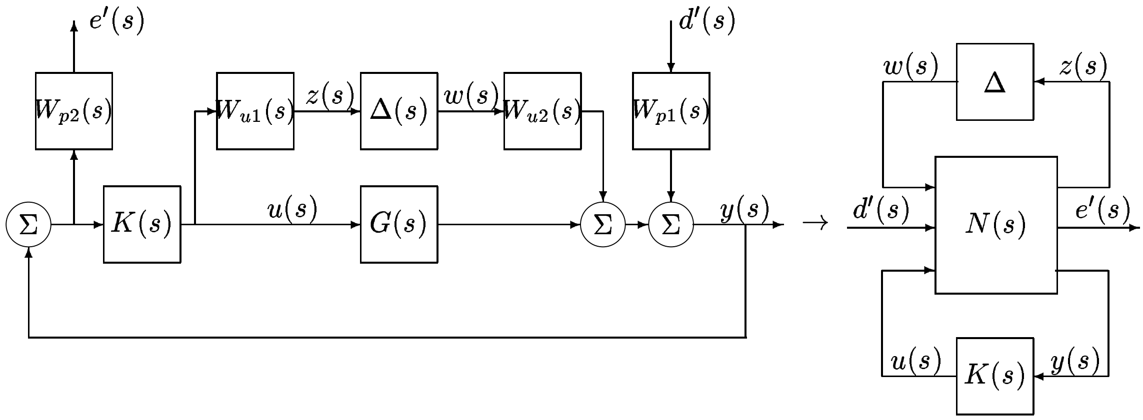


Figure 5.6: *Robust performance problem with additive uncertainty.*

Let $P(s) = F_l(N(s), K(s))$. Then, the transfer matrix from $d'(s)$ to $e'(s)$ is given by:

$$e'(s) = F_u(P(s), \Delta(s))d'(s) = \left[P_{22}(s) + P_{21}(s)\Delta(s)\left(I - P_{11}(s)\Delta(s)\right)^{-1} P_{12}(s)\right] d'(s) \quad (5.95)$$

$$= W_{p2}(s)\left(I + G(s)K(s) + G(s)W_{u2}(s)\Delta(s)W_{u1}(s)\right)^{-1} W_{p1}(s)d'(s) \quad (5.96)$$

$$= W_{p2}(s)S_{o,\Delta}(j\omega)W_{p1}(s)d'(s) \quad (5.97)$$

The robust performance design problem now becomes

$$K(s) = \arg \min_{K(s)\in\mathcal{K}_s} \sup_{\Delta(s)} \|F_u\left(F_l(N(s), K(s)), \Delta(s)\right)\|_{\mathcal{H}_\infty} \quad (5.98)$$

If

$$\|F_u(F_l(N(s), K(s)), \Delta(s))\|_{\mathcal{H}_\infty} = \|F_u(P(s), \Delta(s))\|_{\mathcal{H}_\infty} < 1 \quad (5.99)$$

for all $\Delta(s)$ for which $\|\Delta(s)\|_{\mathcal{H}_\infty} \leq 1$, the closed loop system is said to have robust performance. Note, that the $\mathcal{H}_\infty$ condition (5.99) for robust performance ressembles the $\mathcal{H}_\infty$ condition (5.71) for robust stability. Hence, it can be concluded that *the system $F_u(P(s), \Delta(s))$ satisfies robust performance if and only if it is robustly stable with respect to a norm bounded matrix perturbation $\Delta_p(s)$, for which $\bar{\sigma}(\Delta_p(j\omega)) \leq 1$, $\forall\omega \geq 0$.* This is a main motivation for using specifically the 2-norm as a measure of performance. It facilitates the equivalence of the robust performance condition with a robust stability condition by augmenting the perturbation structure by a performance block $\Delta_p(s)$, see Figure 5.7.

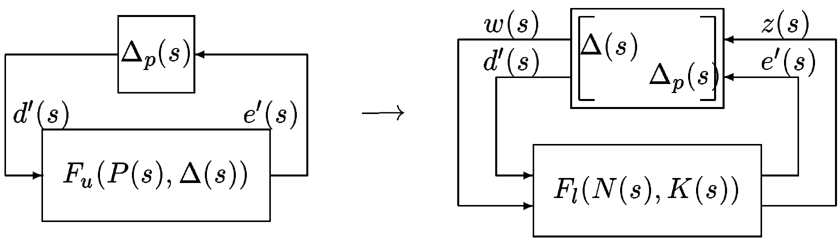


Figure 5.7: *Block diagram for the robust performance problem. The perturbation structure has been augmented with a performance block $\Delta_p(s)$.*

Let $\tilde{\Delta}(s) = \operatorname{diag}\{\Delta(s), \Delta_p(s)\}$ denote the augmented perturbation structure. Then, the following result holds:

Theorem 5.3 (Robust Performance) *Assume that the system $P(s) = F_l(N(s), K(s))$ is stable, and that the perturbation $\tilde{\Delta}(s)$ is such that the perturbed closed loop system in Figure 5.7 is stable if and only if the image curve of $\det(I - P(s)\tilde{\Delta}(s))$ does not encircle the origin as s traverses the Nyquist $\mathcal{D}$ contour. Then the system $F_u(P(s), \Delta(s))$ satisfies the robust performance condition (5.99) if and only if $P(s)$ is stable for all perturbations $\tilde{\Delta}(s)$ for which $\bar{\sigma}(\tilde{\Delta}(j\omega)) \leq 1$, $\forall \omega \geq 0$:*

$$\begin{aligned}
& \det(I - P(j\omega)\tilde{\Delta}(j\omega)) \neq 0 \qquad \forall \omega,\ \forall \tilde{\Delta}(j\omega) : \bar{\sigma}(\tilde{\Delta}(j\omega)) \leq 1 && (5.100)\\
\Leftrightarrow \quad & \rho(P(j\omega)\tilde{\Delta}(j\omega)) < 1 \qquad \forall \omega,\ \forall \tilde{\Delta}(j\omega) : \bar{\sigma}(\tilde{\Delta}(j\omega)) \leq 1 && (5.101)\\
\Leftarrow \quad & \|P(s)\|_{\mathcal{H}_\infty} < 1 && (5.102)
\end{aligned}$$

***Proof of Theorem* 5.3** Follows directly from Theorem 5.2. As the structure of $\tilde{\Delta}(s)$ is diagonal (i.e. restricted), the condition (5.102) is only a sufficient condition for robust performance.

It is intuitively clear that a necessary condition for robust performance is robust stability of the closed loop system and nominal performance for the system. Hence, for a system with additive uncertainty, a necessary condition for robust performance is given by:

$$\mathcal{J}_{np} = \|W_{p2}(s) S_o(s) W_{p1}(s)\|_{\mathcal{H}_\infty} < 1 \tag{5.103}$$

$$\mathcal{J}_u = \|W_{u1}(s) M(s) W_{u2}(s)\|_{\mathcal{H}_\infty} < 1 \tag{5.104}$$

The sufficient condition (5.102) is equivalent to the transfer function $P(s) = F_l(N(s), K(s))$ from $[w(s); d'(s)]$ to $[z(s); e'(s)]$ having $\mathcal{H}_\infty$ norm less than unity. Thus, we can reformulate the robust performance problem as an $\mathcal{H}_\infty$ problem:

$$K(s) = \arg \min_{K(s) \in \mathcal{K}_S} \|F_l(N(s), K(s))\|_{\mathcal{H}_\infty} \tag{5.105}$$

which has a known solution. If a specific compensator achieves that

$$\|F_l(N(s), K(s))\|_{\mathcal{H}_\infty} < 1 \tag{5.106}$$

then the closed loop system has robust performance. The condition (5.102), however is only sufficient (not necessary), it is potentially conservative. Hence, the closed loop system might have robust performance, although (5.106) is *not* satisfied. The reason for this is that an $\mathcal{H}_\infty$ compensator does not take into account that the off-diagonal entries in the augmented perturbation $\tilde{\Delta}(s)$ equal zero.

In general, it is difficult to determine to what extent the $\mathcal{H}_\infty$ solution is conservative, although it can be shown that:

- If the performance specification is formulated for the output sensitivity as (5.42) for a system with an additive or a multiplicative uncertainty description, and if further the weight functions $W_{p1}(s)$ and $W_{u2}(s)$ are restricted to be scalar transfer functions multiplied by an identity matrix:

$$W_{p1}(s) = w_{p1}(s) \cdot I, \qquad W_{u2}(s) = w_{u2}(s) \cdot I \tag{5.107}$$

and if the robust stability and the nominal performance criteria are strengthened by a factor of 2, the closed loop system will have robust performance. In this case, in other words, robust performance can be guaranteed, if nominal perfomance and robust stability are satisfied with a margin of 2.

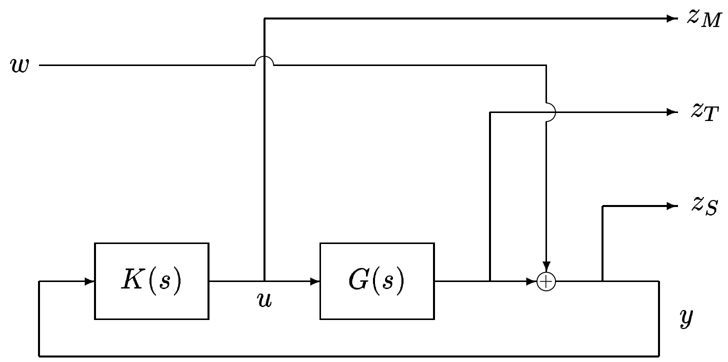
- If the performance specification is formulated for the output sensitivity in the form (5.42) as above, but for a system with an input multiplicative uncertainty (even if the weight matrices are restricted as above), and if the process is ill-conditioned, i.e.

$$\kappa(G(j\omega)) = \frac{\bar{\sigma}(G(j\omega))}{\underline{\sigma}(G(j\omega))} \gg 1 \qquad \text{in a frequency range } \Omega \tag{5.108}$$

  then robust performance can not be guaranteed *even if both nominal performance and robust stability are satisfied with a margin less than* $\sup_\omega \kappa(G(j\omega))$.

5.3.1 Specifications with Mixed Sensitivity Functions

For multivariable systems, the output sensitivity functions introduced above can be defined based on the following feedback system:



where w is a disturbance signal, z_T are the outputs, z_S are the outputs plus disturbances, and z_M is the control signal. Then, the following relation holds:

$$\begin{pmatrix} z_S \\ z_T \\ z_M \end{pmatrix} = \begin{pmatrix} S_o(s) \\ T_o(s) \\ M(s) \end{pmatrix} w$$

where
$S_o = (I - GK)^{-1}$ (sensitivity)
$M = K(I - GK)^{-1}$ (control sensitivity)
$T_o = GK(I - GK)^{-1}$ (complementary sensitivity)

In the sequel, it will be discussed how a feedback system can be designed based on specifications for the sensitivities. The approach is based on the following interpretation of the sensitivities:

Making the *sensitivity* $S_o(s)$ small implies:

- Good disturbance attenuation

- Good command-following (bandwidth)

Making the *control sensitivity* $M(s)$ small implies:

- Robustness w.r.t. additive uncertainties (see Table 5.1, Page 54)
- Moderate compensator gains
- Bounded impact of measurement noise

Making the *complementary sensitivity* $T_o(s)$ small implies:

- Robustness w.r.t. multiplicative uncertainties (see Table 5.1)
- Bounded impact of measurement noise
- Moderate compensator gains

A fundamental trade-off, which must be reflected in any compensator design, manifested as the *complementary principle* is based on the following important observation:

$$S_o(s) - T_o(s) \equiv I \tag{5.109}$$

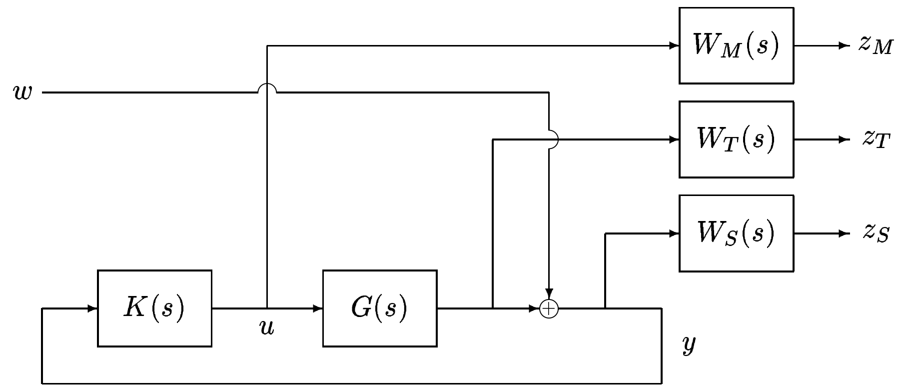
This identity shows the existence of a intrinsic trade-off between performance and robustness, since S_o and T_o can not be made small simultaneously!

Hence, any sensible synthesis method ($\mathcal{H}_\infty$ or other) must have the possibility to consider the pair (S_o, T_o) (or, alternatively, the pair (S_o, M)) with the perspective of deciding at each frequency ω whether $S_o(j\omega)$ or $T_o(j\omega)$ (alternatively: $S_o(j\omega)$ or $M(j\omega)$) should be made small.

Such compensator design algorithms are collectively called *mixed sensivitity* methods.

5.3.1.1 Formulation of Specifications for Mixed Sensitivity in $\mathcal{H}_\infty$ Norm

Consider the following diagram:



By defining:

$$z = \begin{pmatrix} z_S \\ z_T \end{pmatrix} \text{ or } z = \begin{pmatrix} z_S \\ z_M \end{pmatrix}$$

a standard problem is obtained for which the corresponding transfer functions from w to z satisfy

$$\left\| \begin{pmatrix} W_S S_o \\ W_T T_o \end{pmatrix} \right\|_\infty < 1 \qquad \text{or} \qquad \left\| \begin{pmatrix} W_S S_o \\ W_M M \end{pmatrix} \right\|_\infty < 1$$

This in turn implies that $\|W_S(s)S_o(s)\|_\infty < 1$ and that

$$\|W_T(s)T_o(s)\|_\infty < 1 \qquad \text{or} \qquad \|W_M(s)M(s)\|_\infty < 1$$

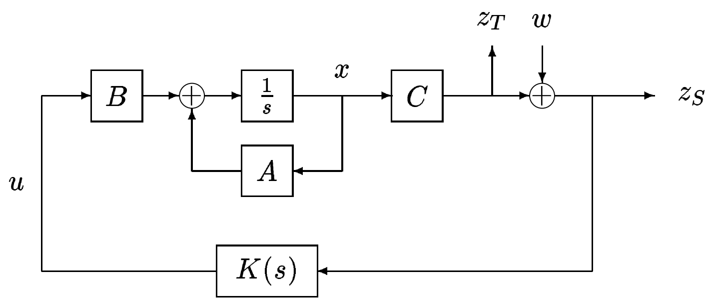
Ideally, the two relevant weight matrices (either W_S and W_T or W_S and W_M) would be given directly from the problem formulation. In practice, however, it will often turn out that to some extent the weighting matrices are 'tuning parameters', i.e. they are selected via indirect criteria. Sensible choices, though, are often in the main determined from the complementarity principle (5.109) along with other fundamental limitations for mixed sensitivity problems, which will be described in the next section.

5.3.2 The Significance of Zeros and Poles in the Right Half Plane

In formulating specifications for robust control systems, it is often important to know which inherent limitations, a given dynamical system has. There are many different aspects, that play a role in this connection. Here, the significance of possible zeros and poles in the right half plane will be emphasized.

5.3.2.1 The Significance of Zeros in the Right Half Plane

Consider a control system of the form:



where $K(s)$ is a stabilizing compensator.

If the system has a zero z in the closed right half plane, an excitation of the form

$$w = e^{zt} w_0$$

where w_0 is the input zero direction corresponding to z, will give rise to the following stationary solution:

$$u(t) = K(z)e^{zt}w_0\,, \quad z_T(t) \equiv 0\,, \quad z_S(t) = w(t)$$

from which it is concluded that

$$S_o(z)w_0 = w_0$$

which in turn implies that

$$\bar{\sigma}(S_o(z)) \geq 1$$

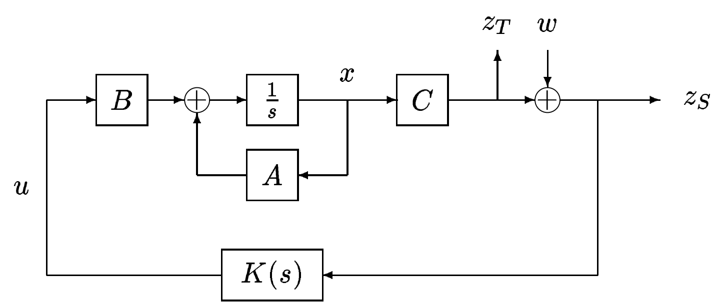
regardless of which stabilizing compensator $K(s)$ has been chosen. ($S_o(\cdot)$ denotes the output sensitivity.) Since the $\mathcal{H}_\infty$ norm of a function is at least equal to the largest singular value at any specific point in the right half plane, this leads to the inequality

$$\|W_S(\cdot)S_o(\cdot)\|_\infty \geq \|W_S(z)\|$$

In other words, every zero in the right half plane, gives rise to a lower bound for how small the $\mathcal{H}_\infty$ norm of the sensitivity for a given system can become, i.e., how good a disturbance attenuation or bandwidth can be achieved.

5.3.2.2 The significane of Poles in the Right Half Plane

Consider a control system of the form:



where $K(s)$ is a stabilizing compensator.

If the system has a pole p in the closed right half plane, an excitation of the form

$$w = e^{pt}w_0$$

gives rise to the following stationary solution:

$$x(t) = -e^{pt}x_0\,, \quad \text{hvor} \quad Cx_0 = w_0\,,$$
$$z_T(t) = -w(t)\,, \quad z_S(t) \equiv 0\,, \quad u(t) \equiv 0$$

from which it can be seen that

$$T_o(p) = -I$$

regardless of which stabilizing compensator $K(s)$ has been chosen. valgt. (Here, $T_o(s)$ is the complementary output sensitivity.) This leads to the inequlity:

$$\|W_T(\cdot)T_o(\cdot)\|_\infty \geq \|W_T(p)\|$$

In other words, the pole in the right half plane gives rise to a lower bound for how small the $\mathcal{H}_\infty$ norm of the complementary sensitivity function can become, i.e., how good a robustness, the system can achieve.

Based on the observations in this section and in Section 5.3.2.1, the following guide lines for choosing weight matrices W_S and W_T (and correspondingly for W_S and W_M) can be derived:

1. In general, the weight W_S should be chosen large at low frequencies and small at high frequencies.
2. In general, the weight W_T should be chosen small at low frequencies and large at high frequencies.
3. Choose the weight W_S to be small at frequencies, corresponding to zeros in the right half plane.
4. Choose the weight W_T to be small at frequencies, corresponding to poles in the right half plane.

These guide lines presumes that performance specifications are related to low frequencies, and that robustness problems are predominant at high frequencies. Both will be the case in most practical cases.

Chapter 6

Robust Design for Multivariable Systems

Above, analysis for multivariable control systems with respect to nominal and robust stability as well as nominal and robust performance has been assessed. It was assumed that the specifications for robustness were given in terms of weight matrices $W_{u1}(s)$ and $W_{u2}(s)$, and that the performance specifications similarly were given by weight matrices $W_{p1}(s)$ and $W_{p2}(s)$. How to derive weight matrices leading to good compensators is to some extent still an open question and possibly the most difficult in robust control. In the following, two approaches to weight matrix selection will be proposed. These approaches, though, can not be considered to be final answers to the weight selection problem in any sense.

6.1 Loop Shaping

The idea behind *loop shaping* is to find a compensator $K(s)$ which shapes the open loop system $\bar{\sigma}(L(j\omega))$, such that certain requirements for robustness and performance are satisfied.

Natural requirements for performance would be a good disturbance attenuation, resulting in a small tracking error. As seen above, the tracking error can be determined as:

$$e(s) = S_o(s)(r(s) - d(s)) + T_o(s)n(s) \tag{6.1}$$

Sometimes it can be reasonable to neglect the measurement noise $n(s)$, so the most significant requirement for performance is the output sensitivity $\bar{\sigma}(S_o(j\omega))$ to be small in the frequency range where the most dominant disturbances occur. As most disturbances are low frequent, this leads to a requirement for the output sensitivity to be small at frequencies up to a certain bound, which must be chosen by the designer depending on an evaluation of the dominating disturbances. This can be achieved by specifiying:

$$\begin{aligned} W_{p1} &= IW_p(s) && (6.2)\\ W_{p2} &= I && (6.3) \end{aligned}$$

where $W_p(s)$ is a scalar transfer function with low pass characteristics. A requirement of this type is equivalent to a condition for the smallest singular value $\underline{\sigma}(L_o(j\omega))$ of the open loop transfer function at the output to be large at low frequencies.

$$\underline{\sigma}(L_o(j\omega)) > W_p(j\omega) \tag{6.4}$$

The uncertainties of physical systems are often largest at high frequencies. If detailed knowledge of the uncertainties of the process is not available, it would be natural to specify a multiplicative uncertainty model for the output:

$$\begin{aligned} W_{u1} &= I \cdot W_u(s) && (6.5) \\ W_{u2} &= I && (6.6) \end{aligned}$$

where $W_u(s)$ is a scalar transfer function with high pass characteristics, such that $W_u(j\omega)$ has a value corresponding to the DC gain at low frequencies and a value at high frequencies of more than unity. Requirements of this type lead to the condition

$$\bar{\sigma}(T_o(j\omega)) < \frac{1}{W_u(j\omega)} \tag{6.7}$$

This is equivalent to requiring the largest singular value of the open loop transfer matrix $\bar{\sigma}(L_o(j\omega))$ to be small at high frequencies:

$$\bar{\sigma}(L_o(j\omega)) < \frac{1}{W_u(j\omega)} \tag{6.8}$$

In this way, the specification of weight functions becomes a matter of trade-off between a good disturbance attenuation and robustness, and the interesting choice is the frequency where the curves intersect. At this frequency, it should be ensured that either transfer function is smaller than unity. Otherwise, it will be impossible to meet the requirements.

Figure 6.1 shows an example of how such requirements could manifest. The method does not give an explicit answer to how the open loop singular values should proceed close to the crossover frequencies, i.e. where the family of curves for the open loop singular values intersect unity (0 dB). It can be seen, though, that it is convenient if the singular values can be made close.

Other transfer functions than $S_o(s)$ and $T_o(s)$ could be of interest to the loop shaping approach. Ensuring that the control signals $u(s)$ remain reasonably bounded, can be obtained by bounding the control sensitivity $M(s) = (I + K(s)G(s))^{-1}K(s)$. Moreover, in the case where the uncertainties can be described in terms of an additive uncertainty description, this also leads to upper bounds for the control sensitivity $M(s)$.

6.2 Modeling Individual Channels

In the loop shaping approach, the interest is mainly focused on the size of the sensitivity and the complentary sensitivity functions, which for multivariable systems leads to requirements for the largest and smallest singular values of the open loop transfer matrix. This has the

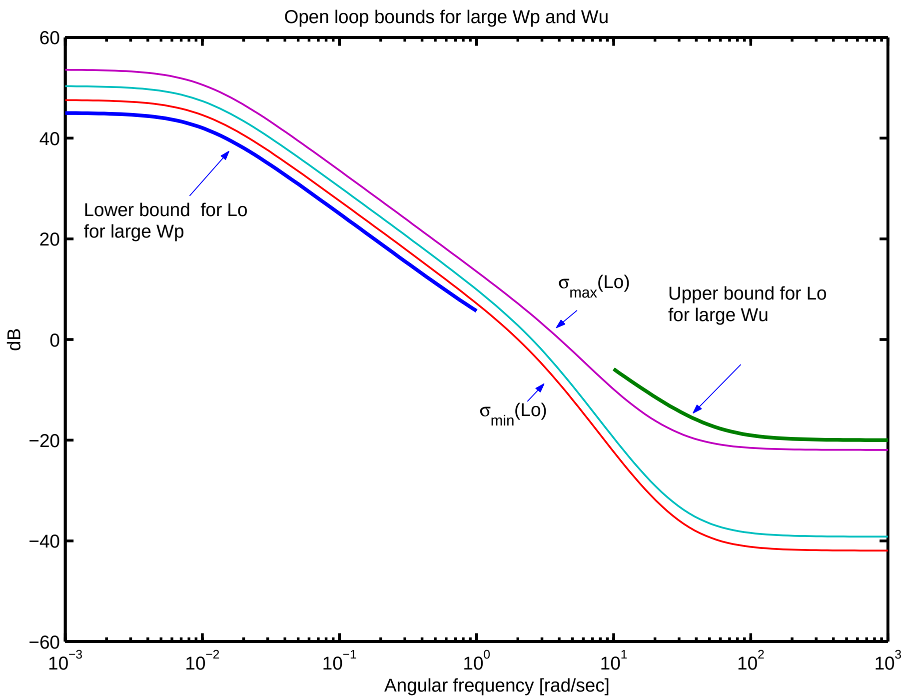


Figure 6.1: *Open loop gain requirements from W_p and W_u in frequency reqions where either is large*

limitation that problems where different requirements are related to individual channels can not be easily addressed. To illustrate this, assume that the uncertainty has been found to be:

$$\Delta = I\tilde{\Delta}W_u(s) \tag{6.9}$$

To ensure robust stability, the requirement $\bar{\sigma}(W_uT_o) < 1$ suffices. However, if only the largest singular value is bounded, the requirement becomes:

$$\bar{\sigma}(T_o) < \frac{1}{\bar{\sigma}(W_u)} \tag{6.10}$$

This condition can be far more restrictive, especially if there are large differences in the uncertainty for the individual channels of the transfer matrix.

Thus, it would be much better to evaluate the uncertainty for each individual transfer function in the transfer matrix explicitly. The evaluation can be handled, e.g. by considering parametric uncertainty descriptions or by evaluating a frequency bound for the validity of the model and introduce multiplicative uncertainties in each channel, implementing this bound.

In the same fashion, it will be of significance for the overall result, whether the actual disturbances for each channel is estimated, and whether the importance of the errors are evaluated for each output. Especially if the individual signal differ in magnitude, it is important to scale each output, such that a common bound for the norm of the output vector becomes meaningful.

6.3 $\mathcal{H}_\infty$ Control

In this section, a state space solution to the $\mathcal{H}_\infty$ control problem in the 2×2 block formulation will be presented. This solution was derived by Doyle *et al.* in 1988. The structure of this $\mathcal{H}_\infty$ solution will be compared to the well-known LQG structure, where it will be apparent that the two structures have several similarities. Actually, the LQG solution can be interpreted as a special case of the general $\mathcal{H}_\infty$ solution.

Hence, conceptually[1], a solution will be given to the problem:

$$K(s) = \arg \min_{K(s)\in\mathcal{K}_S} \|F_l(N(s), K(s))\|_{\mathcal{H}_\infty} \tag{6.11}$$

The problem of finding a solution to (6.11) was probably the most important research area within control theory during the 1980's. Initially, only algorithms that provided $\mathcal{H}_\infty$ optimal controllers of a very high order–see e.g. [Fra87]–were known, or algorithms that were only feasible for SISO systems–see e.g. [Gri86]. In 1988, however, Doyle, Glover, Khargonekar, and Francis announced a state space solution, involving only two algebraic Riccati equations, and yielding a compensator of the same order as the augmented system $N(s)$, just as for the well-known LQG solution. This was a major break-through for $\mathcal{H}_\infty$ theory. It now became evident that the LQG and the $\mathcal{H}_\infty$ problems and their solutions were related in many ways. Both compensator types have a state estimation-state feedback structure, and two algebraic Riccati equations provide the state feedback matrix K_c and the observer gain matrix K_f.

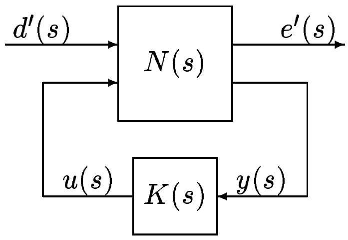


Figure 6.2: *The* 2×2 *block problem.*

Given a 2×2 block matrix $N(s)$, see Figure 6.2, and a desired upper bound γ for the $\mathcal{H}_\infty$ norm $\|F_l(N(s), K(s))\|_{\mathcal{H}_\infty}$, the $\mathcal{H}_\infty$ solution returns a compensator parametrization, often referred to as the DKGF parameterization:

$$K(s) = F_l(J(s), Q(s)) \tag{6.12}$$

of all stabilizing compensators for which $\|F_l(N(s), K(s))\|_{\mathcal{H}_\infty} < \gamma$, see Figure 6.3. Any stable transfer matrix $Q(s)$ for which $\|Q(s)\|_{\mathcal{H}_\infty} < \gamma$ will stabilize the closed loop system and make $\|F_l(N(s), K(s))\|_{\mathcal{H}_\infty} < \gamma$. Any $Q(s)$, that is unstable or has $\mathcal{H}_\infty$ norm larger than γ would either make the closed loop system unstable or imply that $\|F_l(N(s), K(s))\|_{\mathcal{H}_\infty} \geq \gamma$.

The $\mathcal{H}_\infty$ solution is given by Definition 6.1 and by Theorem 6.1.

[1]As mentioned above, the reason why this is only 'conceptually' is that the 'arg min' might not actually exist for technical reasons. To be more precise, this section will be present a method for obtaining *near optimal* solutions.

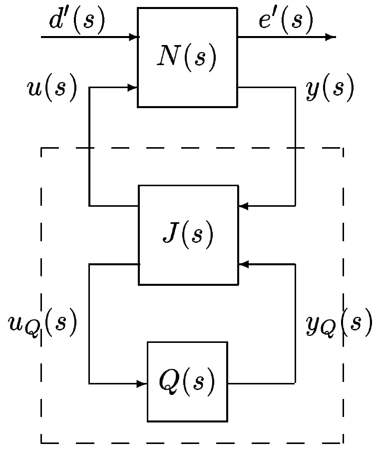


Figure 6.3: *The DGKF parameterization of all stabilizing compensators satisfying* $\|F_l(N(s), K(s))\|_{\mathcal{H}_\infty} < \gamma$.

Definition 6.1 (Riccati Solution) *Assume that the algebraic Riccati equation*

$$A^T X + XA - XRX + Q = 0 \tag{6.13}$$

has a unique stabilizing solution X, *i.e. a solution for which the eigenvalues of* $A - RX$ *are all negative. This solution will be denoted by* $X = \mathbf{Ric}(H)$ *where* H *is the associated Hamiltonian matrix*

$$H = \begin{bmatrix} A & -R \\ -Q & -A^T \end{bmatrix} \tag{6.14}$$

Theorem 6.1 (The $\mathcal{H}_\infty$ Suboptimal Control Problem) *This formulation of the solution has been taken from [Dai90]. Let* $N(s)$ *be given by its state space realization* A, B, C, D *and introduce the notation:*

$$N(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{6.15}$$

where B, C, *and* D *are partitioned consistently with* d', e', u, *and* y. *Now, make the following assumptions:*

1. (A, B_1) *and* (A, B_2) *are stabilizable (controllable).*
2. (C_1, A) *and* (C_2, A) *are detectable (observable).*
3. $D_{12}^T D_{12} = I$ *and* $D_{21} D_{21}^T = I$.
4. $D_{11} = D_{22} = 0$.

Let

$$\tilde{D}_{12} = I - D_{12} D_{12}^T, \qquad \tilde{D}_{21} = I - D_{21}^T D_{21} \tag{6.16}$$

and solve the two Riccati equations

$$X_\infty = \mathbf{Ric} \begin{bmatrix} A - B_2 D_{12}^T C_1 & \gamma^{-2} B_1 B_1^T - B_2 B_2^T \\ -C_1^T \tilde{D}_{12}^T \tilde{D}_{12} C_1 & -\left(A - B_2 D_{12}^T C_1\right)^T \end{bmatrix} \tag{6.17}$$

$$Y_\infty = \mathbf{Ric}\begin{bmatrix} \left(A - B_1 D_{21}^T C_2\right)^T & \gamma^{-2} C_1^T C_1 - C_2^T C_2 \\ -B_1 \tilde{D}_{21} \tilde{D}_{21}^T B_1^T & -\left(A - B_1 D_{21}^T C_2\right) \end{bmatrix} \tag{6.18}$$

Form the state feedback matrix K_c, the observer gain matrix K_f, and the matrix Z_∞ as

$$K_c = \left(D_{12}^T C_1 + B_2^T X_\infty\right) \tag{6.19}$$

$$K_f = \left(B_1 D_{21}^T + Y_\infty C_2^T\right) \tag{6.20}$$

$$Z_\infty = \left(I - \gamma^{-2} Y_\infty X_\infty\right)^{-1} \tag{6.21}$$

If $X_\infty \geq 0$ and $Y_\infty \geq 0$ exist, and if the spectral radius $\rho(X_\infty Y_\infty) < \gamma^2$, then the DGKF parameterization is given by:

$$J(s) = \left[\begin{array}{c|cc} A_\infty & Z_\infty K_f & Z_\infty\left(B_2 + \gamma^{-2} Y_\infty C_1^T D_{12}\right) \\ \hline -K_c & 0 & I \\ -\left(C_2 + \gamma^{-2} D_{21} B_1^T X_\infty\right) & I & 0 \end{array}\right] \tag{6.22}$$

$$= \begin{bmatrix} J_{11}(s) & J_{12}(s) \\ J_{21}(s) & J_{22}(s) \end{bmatrix} \tag{6.23}$$

where A_∞ is given by:

$$A_\infty = A - B_2 K_c + \gamma^{-2} B_1 B_1^T X_\infty - Z_\infty K_f\left(C_2 + \gamma^{-2} D_{21} B_1^T X_\infty\right) \tag{6.24}$$

Stabilizing compensators $K(s)$ satisfying $\|F_l(N(s), K(s))\|_{\mathcal{H}_\infty} < \gamma$ can then be constructed by combining $J(s)$ with any stable $Q(s)$ for which $\|Q(s)\|_{\mathcal{H}_\infty} < \gamma$:

$$K(s) = F_l(J(s), Q(s)) = J_{11}(s) + J_{12}(s) Q(s)\left(I - J_{22}(s) Q(s)\right)^{-1} J_{21}(s) \tag{6.25}$$

Then, the $\mathcal{H}_\infty$ norm of the closed loop system $F_l(N(s), F_l(J(s), Q(s)))$ satisfies:

$$\|F_l\left(N(s), F_l(J(s), Q(s))\right)\|_\infty < \gamma \tag{6.26}$$

The compensator $J_{11}(s)$ obtained for $Q(s) = 0$ is called the central $\mathcal{H}_\infty$ compensator.

6.3.1 Remarks to the $\mathcal{H}_\infty$ solution

Note, that Theorem 6.1 does *not* provide the optimal $\mathcal{H}_\infty$ compensator, but rather a compensator, satisfying $\|F_l(N(s), K(s))\|_{\mathcal{H}_\infty} < \gamma$, *after having specified* γ if a compensator achieving this exists. Thus, the designer might have to iterate on γ to approximate the optimal $\mathcal{H}_\infty$ norm γ_0. Therefore, the solution in Theorem 6.1 is called the *suboptimal* $\mathcal{H}_\infty$ compensator. This is in contrast to the LQG solution, where the optimal compensator can be found without iteration. The central compensator ($Q(s) = 0$) is never the compensator which obtains the smallest closed loop $\mathcal{H}_\infty$ norm. For a desired value of γ, however, it is always an admissible compensator and, hence, it is common to choose this particular compensator for implementation. Specifically, this compensator is returned for the MATLAB™ algorithms implemented in the Robust Control Toolbox and in the μ Analysis and Synthesis Toolbox.

Further note, that even though the $\mathcal{H}_\infty$ control problem has been formulated in frequency domain, the solution is given in state space form, i.e. in time domain. This combination of frequency domain specifications and state space computations is very symptomatic for modern control theory.

Now, consider the four assumptions in Theorem 6.1. Assumptions 1 and 2 are just requirements of stabilizability and detectability of the generalized system $N(s)$. Note, that if (A, B_2) is stabilizable or if (C_2, A) is not detectable, no stabilizing compensator ($\mathcal{H}_\infty$, PID, or other) exists! The other two conditions of Assumption 1 and 2 are included for technical reasons, but they can be alleviated. The main purpose of Assumption 3 is to ensure that the number of columns of D_{12} does not exceed the number of rows, i.e. that D_{12} is a 'tall' matrix. In the same fashion, D_{21} is not allowed to have more rows than columns, i.e. D_{21} must be a 'flat' matrix. This implies the following conditions on the dimension of the signals $e'(s)$, $d'(s)$, $u(s)$, $y(s)$:

$$\dim\, e'(s) \geq \dim\, u(s), \qquad \dim\, d'(s) \geq \dim\, y(s) \tag{6.27}$$

where dim x denotes the dimension of the vector x. Hence, the number of exogenous outputs (error signals) $[e'_1, \cdots, e'_e]$ has to be larger than or equal to the number of controllable inputs (the number of actuators) $[u_1, \cdots, u_m]$. Similarly, the number of exogenous inputs (disturbances) $[d'_1, \cdots, d'_d]$ has to be larger than or equal to the number of measured outputs (the number of sensors) $[y_1, \cdots, y_r]$. Especially for multivariable systems, it can easily happen if care is not taken, that the design problem is formulated in such a way that (6.27) is *not* satisfied. In such case, fictitious exogenous inputs or fictitious exogenous outputs have to be introduced, possibly with a small weight. Since the original result by Doyle, Glover, Khargonekar, and Francis appeared, several generalizations have been published. These results also include versions that allow for instance more control inputs than exogenous outputs. The reader should be warned, however, that in general a violation of (6.27) indicates ill-posed design conditions rather than demonstrating a flaw of the theory. Thus, it would usually be advisable to reconsider the specifications than to look for an alternative optimization procedure that could handle a violation of (6.27).

However, even if a control problem would satisfy (6.27) directly from the design specifications, it would not be anticipated to satisfy the seemingly rather restrictive algebraic condition $D_{12}^T D_{12} = I = D_{21} D_{21}^T$. Obviously, a general D matrix, motivated from physics would not satisfy this condition automatically. It can be shown, though, that a design problem with a general D matrix satisfying certain rank conditions can be transformed into a new design problem which is solvable for the same γ value and by the same controller, but such that the new design problem satisfies the following conditions:

$$D_{11} = 0 \tag{6.28}$$

$$D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{6.29}$$

$$D_{21} = \begin{bmatrix} 0 & I \end{bmatrix} \tag{6.30}$$

$$D_{22} = 0 \tag{6.31}$$

implying that both Assumption 3 and 4 are satisfied. This reformulation is obtained by a certain scaling and 'loop-shifting' procedure. It is beyond the scope of these lecture notes, however, to describe the details of this. A reference is [TC96, App. B]. The only conditions for the reformulation to be possible are that D_{12} has full column rank, and that D_{21} has full row rank, i.e. that:

$$\text{rank}\ D_{12} = \dim\, u, \qquad \text{rank}\ D_{21} = \dim\, y \tag{6.32}$$

The rank constraint for D_{12} can be interpreted as the condition that all control signals must be penalized in the optimization. This is a natural constraint, since otherwise some of the feedback gains would tend to infinity as γ would tend to its optimal value. Similarly, the rank constraint for D_{21} can be interpreted as the condition that all measurements have some noise contribution. This prevents the optimization from creating very large observer gains as γ would tend to its optimal value. In the Robust Control Toolbox, and in the μ Analysis and Synthesis Toolbox in MATLAB™, (6.32) must be satisfied in order to compute the (sub-) optimal $\mathcal{H}_\infty$ compensator.

Hence, when setting up a design problem, (6.32) must be taken into consideration. In short, this amounts to making sure that for each control input of $u(s)$, there is at least one of the transfer functions to the error signals of $e'(s)$ having equally many poles and zeros, as this corresponds to having a feed-through term (a 'D' term). Similarly, for each of the measurement signals of $y(s)$ there has to be at least one of the transfer functions from the disturbances of $d'(s)$ having equally many poles and zeros. Often, in the process of making both rank constraints satisfied, the designer would be lead to introducing additional (fictitious) signals.

In the sequel, the structure of the $\mathcal{H}_\infty$ compensator will be compared to the well-known LQG structure. To that end, Figure 6.4 illustrates the structure of the LQG compensator, and Figure 6.5 similarly illustrates the $\mathcal{H}_\infty$ compensator.

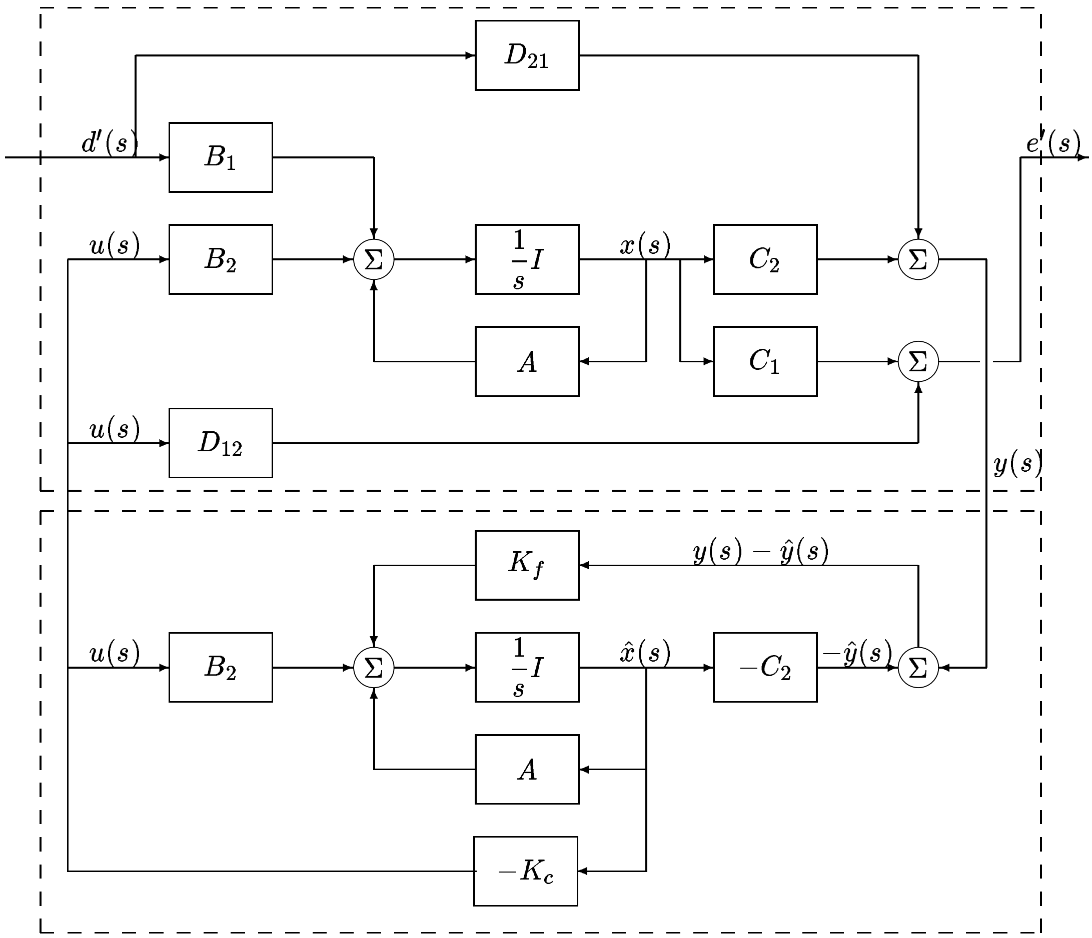


Figure 6.4: *LQG compensator structure as a 2×2 block problem. $N(s)$ is given by the upper box and the LQG compensator as the lower box.*

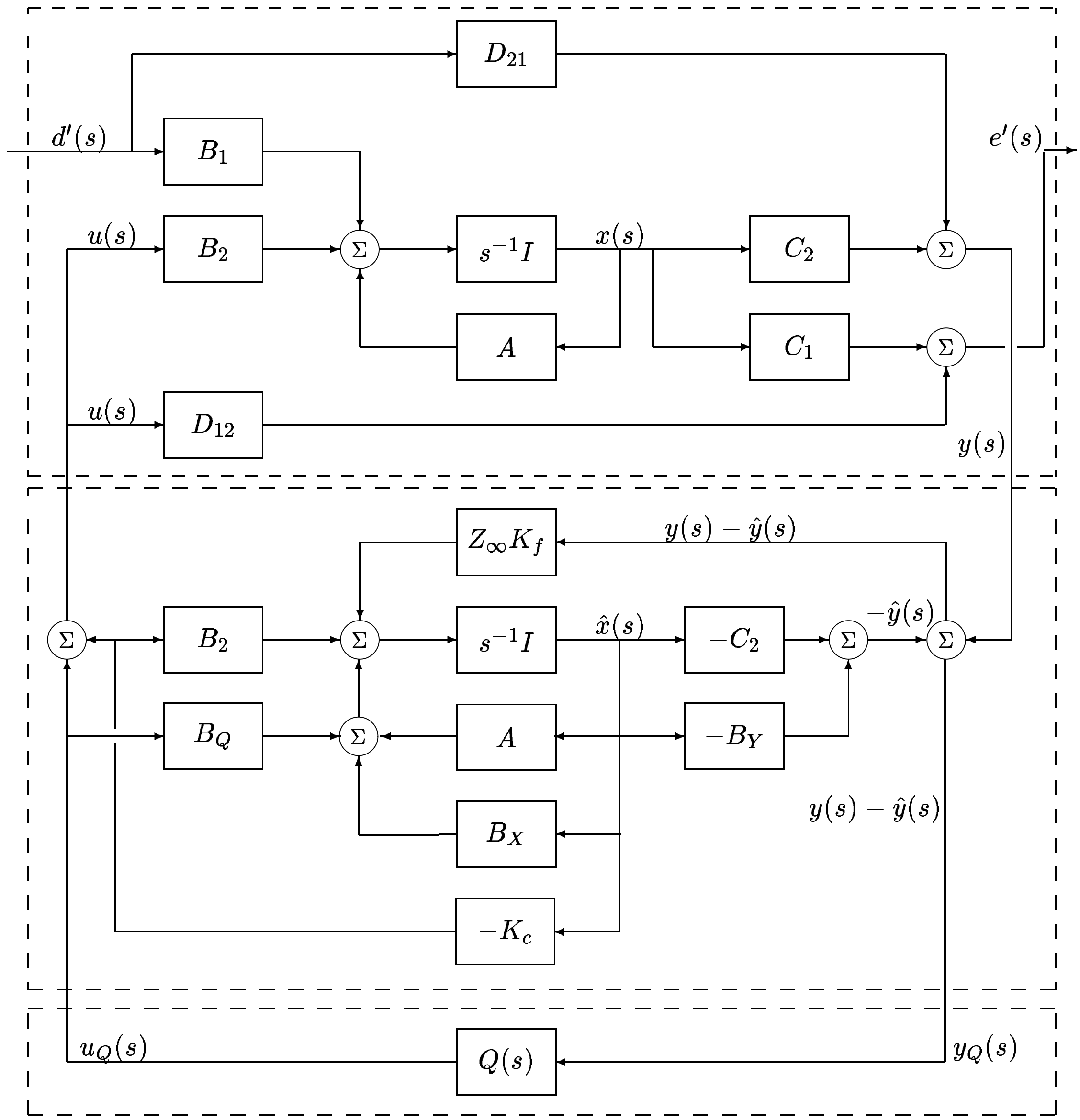


Figure 6.5: *$\mathcal{H}_\infty$ suboptimal compensator structure. $N(s)$ is given by the upper box, $J(s)$ by the middle box, and $Q(s)$ by the lower box. $B_Q = Z_\infty(B_2 + \gamma^{-2}Y_\infty C_1^T D_{12})$, $B_Y = \gamma^{-2}D_{21}B_1^T X_\infty$ and $B_X = \gamma^{-2}B_1 B_1^T X_\infty$.*

Note, that the $\mathcal{H}_\infty$ compensator $F_l(J(s), Q(s))$ like the LQG compensator is composed by a state estimator and a state feedback K_c. In contrast to the LQG compensator, a scaling matrix Z_∞ appears in series with the observer gain matrix K_f. Moreover, there is a couple of altogether new terms: $B_Y = \gamma^{-2}D_{21}B_1^T X_\infty$, $B_X = \gamma^{-2}B_1 B_1^T X_\infty$ and $B_Q = Z_\infty(B_2 + \gamma^{-2}Y_\infty C_1^T D_{12})$. For the central compensator $J_{11}(s)$ corresponding to $Q(s) = 0$, the term B_Q vanishes. Thus, a major similarity between the LQG compensator and the (central) $\mathcal{H}_\infty$ compensator becomes apparent. Actually, it can be shown, see e.g. [TCB95], that for $\gamma \to \infty$, the $\mathcal{H}_\infty$ solution tends to the corresponding LQG solution. With some right, this allows the LQG solution to be considered a special case of the $\mathcal{H}_\infty$ solution. If the Hamiltonian matrices for the LQG Riccati equations:

$$H_X = \begin{bmatrix} A - B_2 D_{12}^T C_1 & -B_2 B_2^T \\ -C_1^T \tilde{D}_{12}^T \tilde{D}_{12} C_1 & -\left(A - B_2 D_{12}^T C_1\right)^T \end{bmatrix} \tag{6.33}$$

$$H_Y = \begin{bmatrix} \left(A - B_1 D_{21}^T C_2\right)^T & -C_2^T C_2 \\ -B_1 \tilde{D}_{21} \tilde{D}_{21}^T B_1^T & -\left(A - B_1 D_{21}^T C_2\right) \end{bmatrix} \tag{6.34}$$

and for the $\mathcal{H}_\infty$ (see (6.17)-(6.18)) Riccati equations are compared, it is seen that the only difference is the additional terms $\gamma^{-2} B_1 B_1^T$ and $\gamma^{-2} C_1 C_1^T$ in the upper right corner. The LQG state feedback K_c does *not* depend on B_1, i.e. how the disturbances $d'(s)$ enter the system. Likewise, the Kalman gain K_f in the LQG state estimator does *not* depend on C_1, i.e. all states are equally weighted. In contrast, K_f in the $\mathcal{H}_\infty$ state estimator depend on C_1, i.e. on the particular linear combination of states which corresponds to the output $e'(s)$.

One of the problems with the LQG compensator is, that even though LQ (full state information) has excellent guaranteed stability margins (infinite gain margin and a phase margin of 60°), it turns out that the observer based LQG compensator often is *not* very robust. The problem is, that some states often contribute more to the gain than others, but this can not be compensated in the LQG Kalman filter as all states are weighted equally. In the $\mathcal{H}_\infty$ state estimator, however, the designer can perform such weighting by virtue of the C_1 matrix, and, hence, make the design more robust. Likewise, the disturbances $d'(s)$ can be weighted differently when computing the state feedback, and thereby the robustness of the system can be improved.

Commercially available software now exists, which support the $\mathcal{H}_\infty$ design problem, such as the MATLAB™ toolboxes [CS92, BDG+93]. Both toolboxes perform an iteration for γ in order to find a near optimal $\mathcal{H}_\infty$ compensator.

6.3.2 The MATLAB™ Toolboxes

In addition to the Control Toolbox in MATLAB™, two toolboxes are available specifically for robust control design, i.e.:

- Robust Control Toolbox.
- μ Analysis and Synthesis Toolbox.

Both toolboxes are valuable pieces of software that provide a significant help in designing robust compensators in a smooth way, and they can both be recommended. Some of the advantages of the μ toolbox are:

- Excellent utility for converting a 2×2 block strukture into a state space description (`sysic.m`).
- Very natural functions for the manipulation of systems, e.g. for interconnecting systems or closing loops (`starp.m`).
- Good approach to μ, see Chapter 7.

Unfortunately, the μ toolbox depends on an altogether new way to represent systems as well as on several new data structures which it is mandatory for a user to understand before applying the toolbox for the first time. The data structures are quite natural and intuitive, but they still require some initial time spent. As these lecture notes are intended for an ultra short course in robust control theory, this overhead for the μ toolbox makes it inconvenient in the context, and only the Robust Control Toolbox will be described below.

The most important functions from Robust Control Toolbox will be illustrated by means of a small example: the pump example introduced above.

6.3.2.1 $\mathcal{H}_\infty$ Design for the Pump

In the sequel, an $\mathcal{H}_\infty$ suboptimal design will be performed for the pump system, using the MATLAB™ Robust Control Toolbox. The design problem has been illustrated in Figure 6.6. As it can be seen, the problem has been formulated in discrete time. The output has a disturbance $d'(z)$ (with no explicit weight). An error signal $e'(z)$ has been weighted by the inverse of the upper bound for the sensitivity $S_o(z)$. Moreover, an additive uncertainty model has been formulated, which implies an upper bound for the control sensitivity. thus, the design is a mixed sensitivity design.

The first thing to be done is to rewrite the problem as a 2×2 block problem. This has also been done in Figure 6.6. Now, the transfer function $N(z)$ must be determined. From the block diagram, it can be easily be seen that:

$$\begin{bmatrix} z(z) \\ e'(z) \\ e(z) \end{bmatrix} = N(z) \begin{bmatrix} w(z) \\ d'(z) \\ u(z) \end{bmatrix} \tag{6.35}$$

$$= \begin{bmatrix} 0 & 0 & W_u(z) \\ -W_p(z) & -W_p(z) & -W_p(z)G(z) \\ -1 & -1 & -G(z) \end{bmatrix} \begin{bmatrix} w(z) \\ d'(z) \\ u(z) \end{bmatrix} \tag{6.36}$$

where

$$G(z) = \frac{0.1591z - 0.1500}{z^2 - 1.9200z + 0.9216} \tag{6.37}$$

$$W_u(z) = \frac{1.2076z^2 - 2.4013z + 1.1969}{z^2 - 1.9226z + 0.9241} \tag{6.38}$$

$$W_p(z) = \frac{z - 0.9512}{1.4125z - 1.4123} \tag{6.39}$$

Now, a state space representation for $N(z)$ has to be found. This is not completely trivial, but a common denominator can be introduced, and `tfm2ss.m` from the Robust Control Toolbox[2] can be used.

As the problem considered is a discrete time $\mathcal{H}_\infty$ problem, it would seem natural to apply a discrete time $\mathcal{H}_\infty$ algorithm. However, applying a bilinear transformation, the problem can be solved in continuous time, whereafter the compensator can be transformed back into discrete time. This is possible, as the $\mathcal{H}_\infty$ norm is invariant under a bilinear transform. Now, a γ-iteration should be applied to find a near optimal $\mathcal{H}_\infty$ compensator. Actually, the algorithm for γ iteration in the Robust Control Toolbox is only implemented for continuous time. This

[2] For this part, the μ toolbox is much easier to use.

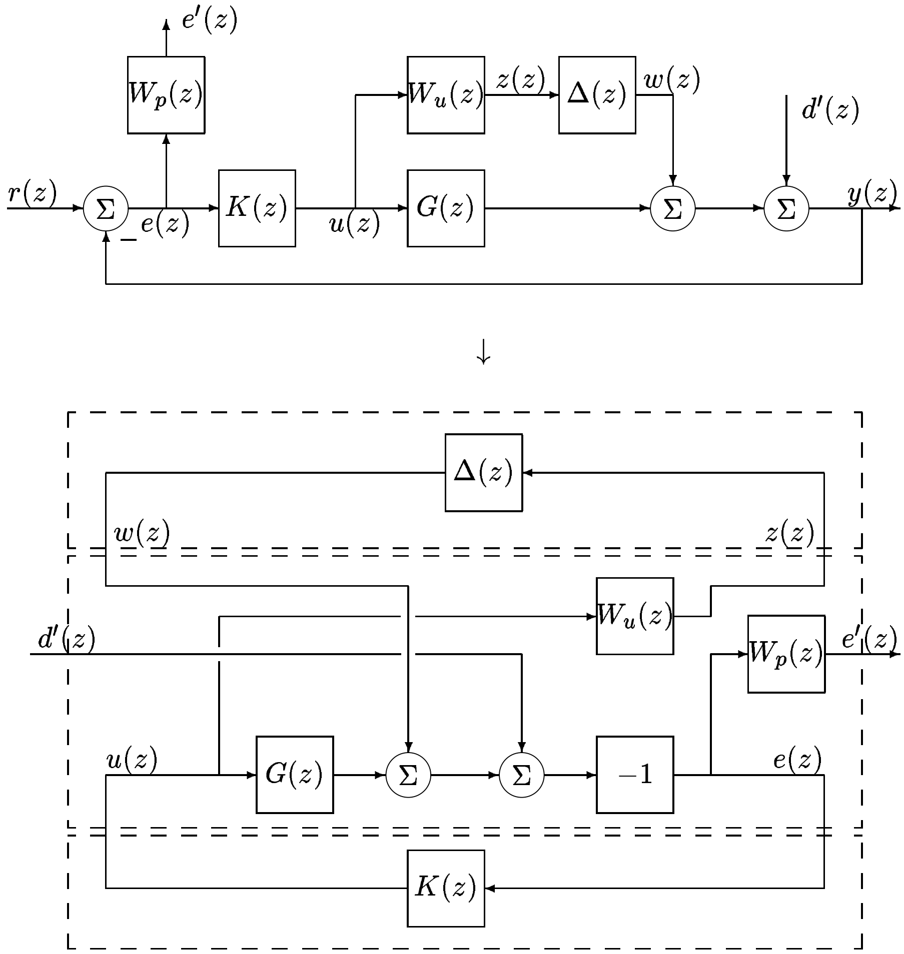


Figure 6.6: *Control design for the pump system. At the top, the weighted system is shown, and at the bottom the corresponding* 2×2 *block problem has been shown with* $N(z)$ *in the middle box.*

means that $N(z)$ must be transformed explicitly to continuous time by means of a bilinear tranform:

```
[Ac,Bc,Cc,Dc] = d2cm(A,B,C,D,Ts,'tustin');
```

The state space matrices for $N(s)$ must be partitioned subsequently, according to the various kinds of inputs and outputs. As the system has 2 exogenous inputs ($w(s)$ and $d'(s)$), 1 controllable input ($u(s)$), 2 exogenous outputs ($z(s)$ and $e'(s)$), and 1 input to the compensator ($e(s)$), the following partitioning is obtained:

```
B1 = Bc(:,1:2); B2 = Bc(:,3);
C1 = Cc(1:2,:); C2 = Cc(3,:);
D11 = Dc(1:2,1:2); D12 = Dc(1:2,3);
D21 = Dc(3,1:2); D22 = Dc(3,3);
```

Now, the command `hinfopt.m` can be applied to perform the γ-iteration:

```
[gamopt,Ak,Bk,Ck,Dk,Acl,Bcl,Ccl,Dcl] = hinfopt(Ac,B1,B2,C1,C2,D11,D12,D21,D22);
```

which produces the following screen output:

```
                   << H-Infinity Optimal Control Synthesis >>

 No     Gamma    D11<=1   P-Exist   P>=0   S-Exist   S>=0   lam(PS)<1    C.L.
------------------------------------------------------------------------------
  1  1.0000e+00   OK        OK      FAIL     OK       OK        OK       UNST
  2  5.0000e-01   OK        OK       OK      OK       OK        OK       STAB
  3  7.5000e-01   OK        OK       OK      OK       OK        OK       STAB
  4  8.7500e-01   OK        OK       OK      OK       OK        OK       STAB
  5  9.3750e-01   OK        OK      FAIL     OK       OK        OK       UNST
  6  9.0625e-01   OK        OK       OK      OK       OK        OK       STAB
  7  9.2188e-01   OK        OK      FAIL     OK       OK        OK       UNST
  8  9.1406e-01   OK        OK      FAIL     OK       OK        OK       UNST

      Iteration no. 6 is your best answer under the tolerance:   0.0100 .
>
```

At first glance, this looks excellent, as the best γ achieved seems to be 0.90625. The Robust Control Toolbox, however, for some reason uses an invers version of γ, which means that the $\mathcal{H}_\infty$ norm of the the closed loop system is:

$$\|F_l(N(s), K(s))\|_\infty = \frac{1}{0.90625} = 1.1034 \tag{6.40}$$

hence, robust performance has not been obtained, although it is pretty close. The compensator must be transformed back into discrete time:

```
[Adk,Bdk,Cdk,Ddk] = c2dm(Ak,Bk,Ck,Dk,Ts,'tustin');
```

Next, nominal performance, robust stability, and robust performance will be checked for the system. First, the closed loop system $F_l(N(z), K(z))$ is formed:

```
[Acl,Bcl,Ccl,Dcl] = feedback(A,B,C,D,Adk,Bdk,Cdk,Ddk,3,3);
```

The properties mentioned above, can be checked in the following way:

```
% Robust stability
B_rs = Bcl(:,1); C_rs = Ccl(1,:); D_rs = Dcl(1,1);
mag_rs = dbode(Acl,B_rs,C_rs,D_rs,Ts,1,w);

% Nominal performance
B_np = Bcl(:,2); C_np = Ccl(2,:); D_np = Dcl(2,2);
mag_np = dbode(Acl,B_np,C_np,D_np,Ts,1,w);

% Robust performance
```

```
B_rp = Bcl(:,1:2); C_rp = Ccl(1:2,:); D_rp = Dcl(1:2,1:2);
sv_rp = dsigma(Acl,B_rp,C_rp,D_rp,Ts,w);

figure(1);
semilogx(w,20*log10(mag_rs),'--',w,20*log10(mag_np),'-.',w,20*log10(sv_rp(1,:)),'-');
title('Nominal (-.) and robust (-) performance and robust stability (--)');
xlabel('Frequency [rad/sec]'),ylabel('Magnitude [dB]');
axis([1e-3 1e3 -30 10]);
grid;
```

which gives the result shown in Figure 6.7.

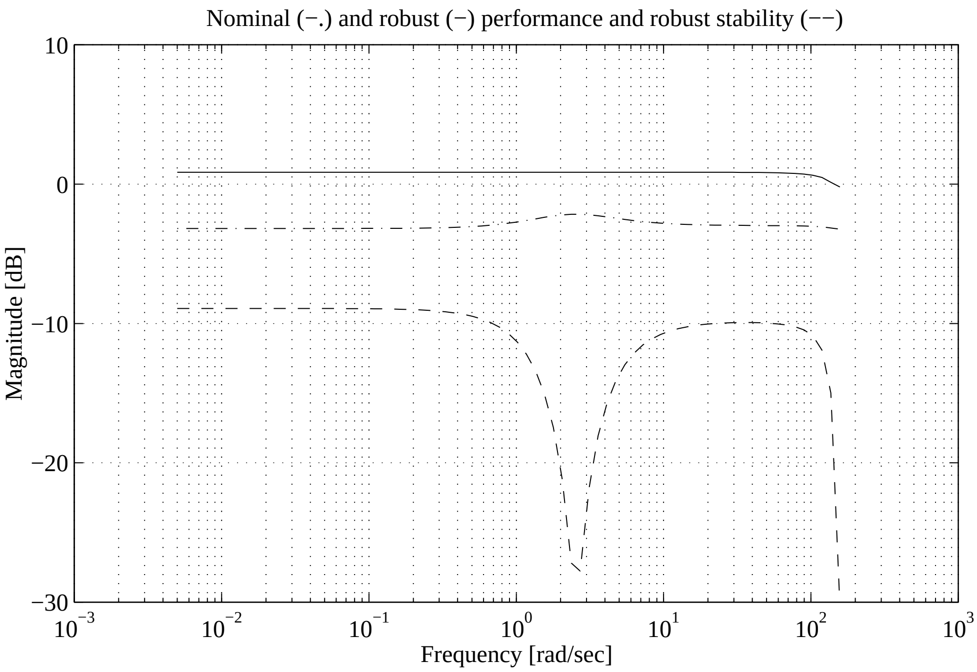


Figure 6.7: *Check for nominal performance, robust stability and robust performance with $\mathcal{H}_\infty$.*

Note, that robust stability and nominal performance have been achieved, but not guaranteed robust performance. Moreover, note that $\bar{\sigma}(F_l(N(e^{j\omega T_s}), K(e^{j\omega T_s})))$ is flat as a function of frequency, indicating that the $\mathcal{H}_\infty$ compensator is close to optimality. Finally, $\|F_l(N(z), K(z))\|_{\mathcal{H}_\infty}$ can be computed to be 1.1033, which is quite consistent with the value of γ obtained.

Chapter 7

Design of Robust Compensators based on μ Theory

There are two main limitations in the use of $\mathcal{H}_\infty$ theory for compensator design. First, only full complex perturbations $\Delta(s) \in \mathbf{C}^{n\times m}$ can be treated in a non-conservative way in an $\mathcal{H}_\infty$ robust stability test. Second, robust performance can only be handled in a conservative way *even for full complex perturbations* since stability and performance can not be separated in the $\mathcal{H}_\infty$ structure. The conservatism depends on the uncertainty structure and on the condition number κ of the system. In this chapter, it will be demonstrated, that these limitations can be avoided by using the *structured singular value* μ.

First, the analysis problem will be considered, i.e., how given a compensator $K(s)$ robust stability and robust performance is verified using μ. Then, the synthesis problem will be discussed, i.e. how to find a compensator, which is optimal with respect to μ.

7.1 μ Analysis

7.1.1 Robust Stability

In the sequel, control problems that can be represented in the block diagram structure shown in Figure 7.1 will be considered. This structure will be refered to as the $N\Delta K$ *structure.*

The similarity between the $N\Delta K$ and the 2×2 block structure is obvious. Here, however, $\Delta(s)$ will not be restricted to be a full complex block. Instead, it is assumed that $\Delta(s)$ has a certain *block diagonal* structure. Indeed, assume that $\Delta(s)$ belongs to the following bounded subset:

$$\mathbf{B\Delta} = \{\Delta(s) \in \mathbf{\Delta}\,|\bar{\sigma}(\Delta(j\omega)) < 1\} \tag{7.1}$$

where $\mathbf{\Delta}$ is defined as:

$$\begin{aligned}\mathbf{\Delta} = \big\{\mathrm{diag}\left(\delta_1^r I_{r_1}, \cdots, \delta_{m_r}^r I_{r_{m_r}}, \delta_1^c I_{r_{m_r+1}}, \cdots, \delta_{m_c}^c I_{r_{m_r+m_c}}, \Delta_1, \cdots, \Delta_{m_C}\right) |\\ \delta_i^r \in \mathbf{R}, \delta_i^c \in \mathbf{C}, \Delta_i \in \mathbf{C}^{r_{m_r+m_c+i}\times r_{m_r+m_c+i}}\big\}\end{aligned} \tag{7.2}$$

Thus, both real and complex perturbations which influences the nominal system via the $N\Delta K$ structure are considered. Very general robust stability problems can be formulated via this structure, e.g. parametric uncertainty, see Example 7.1 on the following page. Obviously, the

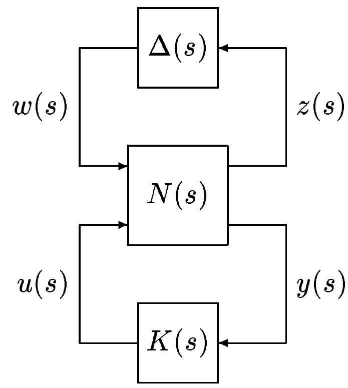


Figure 7.1: *$N\Delta K$ formulation of the robust stability problem.*

block diagonal structure of $\Delta(s)$ allow for a much more detailed uncertainty description, than if $\Delta(s)$ simply consists of one full complex block. Note, that a single full complex block of course is just a special case of the set $\mathbf{\Delta}$.

Example 7.1 (Diagonal perturbation formulation I)
This example is a slightly modified version of an example given in [Hol94]. Assume that the system $G(s)$ is given by:

$$G(s) = \frac{\alpha}{\beta s + 1} \tag{7.3}$$

where the DC gain α and the time constant β only are known with 10 % uncertainty:

$$\alpha = [27.0, 33.0], \qquad \beta = [0.9, 1.1] \tag{7.4}$$

Expressing α and β by their nominal values along with two perturbations Δ_α and Δ_β for which $|\Delta_{\alpha,\beta}| \leq 1$ can be obtained as:

$$\alpha = 30\left(1 + \frac{1}{10}\Delta_\alpha\right) \tag{7.5}$$

$$\beta = 1.0\left(1 + \frac{1}{10}\Delta_\beta\right) \tag{7.6}$$

where

$$\Delta_\alpha \in [-1, +1], \qquad \Delta_\beta \in [-1, +1] \tag{7.7}$$

Let $\mathbf{B\Delta}$ denote the set $[-1, +1]$. Then, the transfer function $G(s)$ can be written as:

$$G(s) = \frac{30\left(1 + 0.1\Delta_\alpha\right)}{\left(1 + 0.1\Delta_\beta\right)s + 1} \tag{7.8}$$

with:

$$\Delta_\alpha, \Delta_\beta \in \mathbf{B\Delta} \tag{7.9}$$

In block diagram form, $G(s)$ can be represented as shown in Figure 7.2 on the next page.

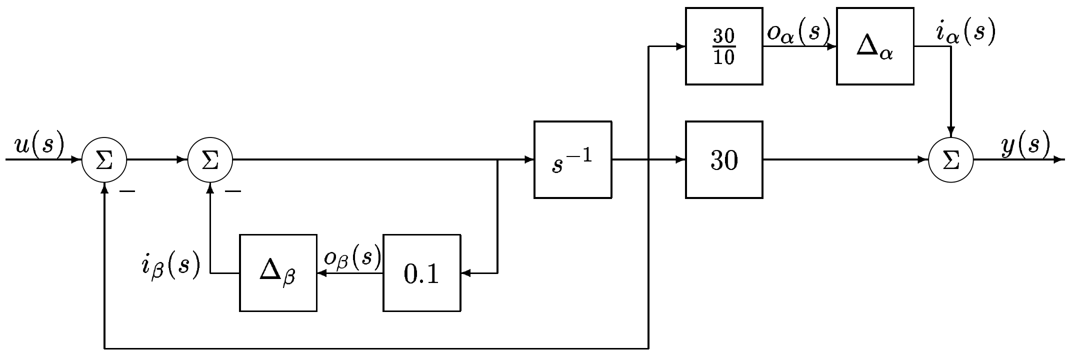


Figure 7.2: *Example 7.1: Block diagram representation of G(s).*

To determine the $N\Delta K$ formulation, the Δ block in Figure 7.2 is removed, and the transfer functions from the three inputs $i_\alpha(s)$, $i_\beta(s)$, and $u(s)$ to the three outputs $o_\alpha(s)$, $o_\beta(s)$, and $y(s)$ are determined. Standard block diagram manipulation in matrix form gives:

$$\begin{bmatrix} o_\alpha(s) \\ o_\beta(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{30}{10(s+1)} & \frac{30}{10(s+1)} \\ 0 & -\frac{s}{10(s+1)} & \frac{s}{10(s+1)} \\ 1 & -\frac{30}{s+1} & \frac{30}{s+1} \end{bmatrix} \begin{bmatrix} i_\alpha(s) \\ i_\beta(s) \\ u(s) \end{bmatrix} \tag{7.10}$$

The uncertainty blocks are given as:

$$\begin{bmatrix} i_\alpha(s) \\ i_\beta(s) \end{bmatrix} = \begin{bmatrix} \Delta_\alpha o_\alpha(s) \\ \Delta_\beta o_\beta(s) \end{bmatrix} = \begin{bmatrix} \Delta_\alpha & 0 \\ 0 & \Delta_\beta \end{bmatrix} \begin{bmatrix} o_\alpha(s) \\ o_\beta(s) \end{bmatrix} \tag{7.11}$$

Now, let $w(s)$, $z(s)$, $N(s)$, and $\Delta(s)$ be given by:

$$w(s) = \begin{bmatrix} i_\alpha(s) \\ i_\beta(s) \end{bmatrix} \tag{7.12}$$

$$z(s) = \begin{bmatrix} o_\alpha(s) \\ o_\beta(s) \end{bmatrix} \tag{7.13}$$

$$N(s) = \begin{bmatrix} 0 & -\frac{30}{10(s+1)} & \frac{30}{10(s+1)} \\ 0 & -\frac{s}{10(s+1)} & \frac{s}{10(s+1)} \\ 1 & -\frac{30}{s+1} & \frac{30}{s+1} \end{bmatrix} \tag{7.14}$$

$$\Delta(s) = diag\{\Delta_\alpha, \Delta_\beta\} = \begin{bmatrix} \Delta_\alpha & 0 \\ 0 & \Delta_\beta \end{bmatrix} \tag{7.15}$$

The perturbed system can now be described as:

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = N(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} \tag{7.16}$$

$$w(s) = \Delta(s)z(s) \tag{7.17}$$

and can immediately be put into the $N\Delta K$ structure.

Example 7.2 (Diagonal perturbation formulation II)
Now, consider a standard second order system:

$$G(s) = \frac{\alpha\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} \tag{7.18}$$

Assume that the gain α, the damping ζ, and the resonance frequency ω_n are not known exactly, but only such that:

$$\alpha = \alpha_o(1 + \delta_\alpha\Delta_\alpha) \tag{7.19}$$
$$\zeta = \zeta_o(1 + \delta_\zeta\Delta_\zeta) \tag{7.20}$$
$$\omega_n = \omega_{n_o}(1 + \delta_\omega\Delta_\omega) \tag{7.21}$$

where

$$\Delta_\alpha, \Delta_\zeta, \Delta_\omega \in \mathbf{B\Delta} \qquad \mathbf{B\Delta} = [-1; +1] \tag{7.22}$$

Hence, α_o, ζ_o, and ω_{n_o} are the nominal values, whereas δ_α, δ_ζ, and δ_ω are the relative uncertainties.

A representation of $G(s)$ in transfer function form is given in Figure 7.3 on the following page. Due to the increased complexity relative to the first order example in Example 7.1 on page 78, it is more convenient to work with state space representations. Define the states as:

$$x_1 = \dot{y}, \qquad x_2 = y \tag{7.23}$$

It can then be shown that a state space representation for $G(s)$ is given by:

$$N(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$
$$= \left[\begin{array}{cc|cccccc}
-2\zeta_o\omega_{n_o} & -\omega_{n_o}^2 & \omega_{n_o}^2 & \omega_{n_o} & 1 & 2\zeta_o\omega_{n_o}^2 & -\omega_{n_o}^2 & \alpha\omega_{n_o}^2 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha\delta_\alpha \\
-2\delta_\omega\zeta_o & -\delta_\omega\omega_{n_o} & \delta_\omega\omega_{n_o} & 0 & 0 & 2\delta_\omega\zeta_o\omega_{n_o} & -\delta_\omega\omega_{n_o} & \delta_\omega\alpha\omega_{n_o} \\
-2\delta_\omega\zeta_o\omega_{n_o} & -\delta_\omega\omega_{n_o}^2 & \delta_\omega\omega_{n_o}^2 & \delta_\omega\omega_{n_o} & 0 & 2\delta_\omega\zeta_o\omega_{n_o}^2 & -\delta_\omega\omega_{n_o}^2 & \delta_\omega\alpha\omega_{n_o}^2 \\
\delta_\omega\omega_{n_o}^{-1} & 0 & 0 & 0 & 0 & \delta_\omega & 0 & 0 \\
2\delta_\zeta\zeta_o\omega_{n_o}^{-1} & 0 & 0 & 0 & 0 & -2\delta_\zeta\zeta_o & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \tag{7.24}$$

Now, proceeding as in Example 7.1 on page 78, define:

$$w(s) = \begin{bmatrix} i_1(s) & i_2(s) & i_3(s) & i_4(s) & i_5(s) \end{bmatrix}^T \tag{7.25}$$
$$z(s) = \begin{bmatrix} o_1(s) & o_2(s) & o_3(s) & o_4(s) & o_5(s) \end{bmatrix}^T \tag{7.26}$$
$$\Delta(s) = \mathrm{diag}\,\{\Delta_\alpha, \Delta_\omega, \Delta_\omega, \Delta_\omega, \Delta_\zeta\} \tag{7.27}$$

Then the perturbed second order system is given by:

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = N(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} \tag{7.28}$$
$$w(s) = \Delta(s)z(s) \tag{7.29}$$

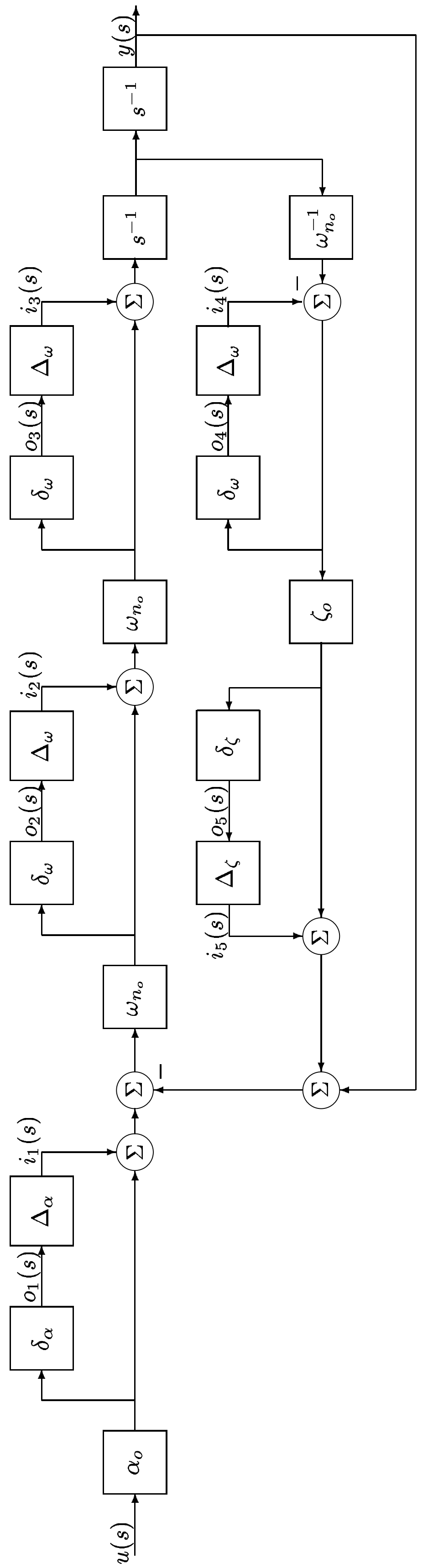


Figure 7.3: *Perturbed second order system in transfer function form.*

and can immediately be put into the $N\Delta K$ structure. Note, that in this case, the block structure for $\Delta(s)$ contains repeated scalar blocks.:

$$\mathbf{\Delta} = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 I_{3\times 3} & 0 \\ 0 & 0 & \delta_3 \end{bmatrix} \tag{7.30}$$

In general, for an arbitrary uncertain system, several equivalent $N\Delta K$ formulations will be possible, which can contain different $\Delta(s)$ structures. It might be difficult to determine a minimal *formulation, where the size of $\Delta(s)$ is the smallest possible.*

As illustrated by the above two examples, highly structured uncertainty models can be represented in the $N\Delta K$ structure. Unfortunately, extracting the uncertainty blocks can involve some tedious algebra. In the MATLAB™ μ toolbox there exists, however a very handy `m` function `sysic.m`, which facilitates an automization of this process.

Dynamical uncertainty can also be included via complex blocks of appropriate dimensions.

Now, let $F_l(N(s), K(s)) = P(s)$ denote the transfer function obtained by closing the lower loop in Figure 7.1 on page 78. $P(s)$ is the *generalized closed loop transfer function* and is given by:

$$\begin{aligned} P(s) &= F_l(N(s), K(s)) && (7.31) \\ &= N_{11}(s) + N_{12}(s)K(s)\left(I - N_{22}(s)K(s)\right)^{-1} N_{21}(s) && (7.32) \end{aligned}$$

Then, given a structured uncertainty $\Delta(s) \in \mathbf{B\Delta}$, robust stability is determined through the following theorem, which is a generalization of the $\mathcal{H}_\infty$ robust stability theorem (see Theorem 5.2 on page 52).

Theorem 7.1 *Assume that the system $P(s)$ is stable, and that the perturbation $\Delta(s)$ is of such nature, that the closed loop system is stable if and only if the Nyquist curve for $\det(I - P(s)\Delta(s))$ does not encircle the origin. Then the closed loop system in Figure 7.1 on page 78 is stable for all perturbations $\Delta(s) \in \mathbf{B\Delta}$ if and only if*

$$\begin{aligned} & \det(I - P(j\omega)\Delta(j\omega)) \neq 0 && \forall\omega,\ \forall\Delta(j\omega) \in \mathbf{B\Delta} && (7.33) \\ \Leftrightarrow\quad & \rho(P(j\omega)\Delta(j\omega)) < 1 && \forall\omega,\ \forall\Delta(j\omega) \in \mathbf{B\Delta} && (7.34) \\ \Leftarrow\quad & \bar{\sigma}(P(j\omega)) < 1 && \forall\omega && (7.35) \end{aligned}$$

***Proof of Theorem* 7.1** The proof follows immediately from the proof for the $\mathcal{H}_\infty$ robust stability theorem (Theorem 5.2 on page 52) with $\Delta(s) \in \mathbf{B\Delta}$. □

Note, that (7.35) is only a sufficient condition for robust stability. Necessity of the corresponding condition for unstructured uncertainties follows from the fact, at the unstructured set contains *all* $\Delta(s)$ with $\bar{\sigma}(\Delta(j\omega)) \leq 1$. Now, however, the perturbation set is restricted to $\Delta(s) \in \mathbf{B\Delta}$ and, thus, the condition (7.35) might in general be arbitrarily conservative. Rather than a robust stability condition based on singular values, a condition is required which takes the structure of the perturbation into consideration. This is precisely the virtue of the structured singular value μ.

Given any matrix $P \in \mathbf{C}^{n\times m}$ the positive real function μ is defined by:

$$\mu_{\mathbf{\Delta}}(P) \triangleq \frac{1}{\min\{\bar{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - P\Delta) = 0\}} \tag{7.36}$$

except if no $\Delta \in \mathbf{\Delta}$ makes $I - P\Delta$ singular ($\det(I - P\Delta) = 0$); in this case, by definition $\mu_{\mathbf{\Delta}}(P) = 0$. Hence, $1/\mu_{\mathbf{\Delta}}(P)$ is the 'magnitude' of the smallest perturbation Δ measured by its singular value $\bar{\sigma}(\Delta)$ making $I - P\Delta$ singular. If $P(s)$ is a transfer matrix, $1/\mu_{\mathbf{\Delta}}(P(j\omega))$ can be interpreted as the magnitude of the smallest perturbation which moves the characteristic loci of $P(s)$ into the Nyquist point $(-1, 0)$ at the angular frequency ω.

From the definition of μ and Theorem 7.1 on the page before the following theorem for determining robust stability can be formulated (see also [DP87, PD93]:

Theorem 7.2 (Robust stability with μ) *Assume that the system $P(s)$ is stable, and that the perturbation $\Delta(s)$ is such that the closed loop system is stable if and only if the Nyquist curve for $\det(I - P(s)\Delta(s))$ does not encircle the origin. Then, the closed loop system in Figure 7.1 on page 78 is stable for all perturbations $\Delta(s) \in \mathbf{B\Delta}$ if and only if*

$$\|\mu_{\mathbf{\Delta}}(P(s))\|_\infty \leq 1 \tag{7.37}$$

where:

$$\|\mu_{\mathbf{\Delta}}(P(s))\|_\infty \stackrel{\triangle}{=} \sup_\omega \mu_{\mathbf{\Delta}}(P(j\omega)) \tag{7.38}$$

7.1.2 Robust Performance

In order to analyze a system with respect to robust performance, the normalized exogenous disturbances $d'(s)$ and the normalized error signals $e'(s)$ are included in the $N\Delta K$ formulation. Now, a general framework for the analysis and synthesis of linear systems can be formulated, see Figure 7.4. Any linear combination of control inputs u, measured outputs y, disturbances d', error signals e', perturbations w and compensator K can be described via this 'generic' system.

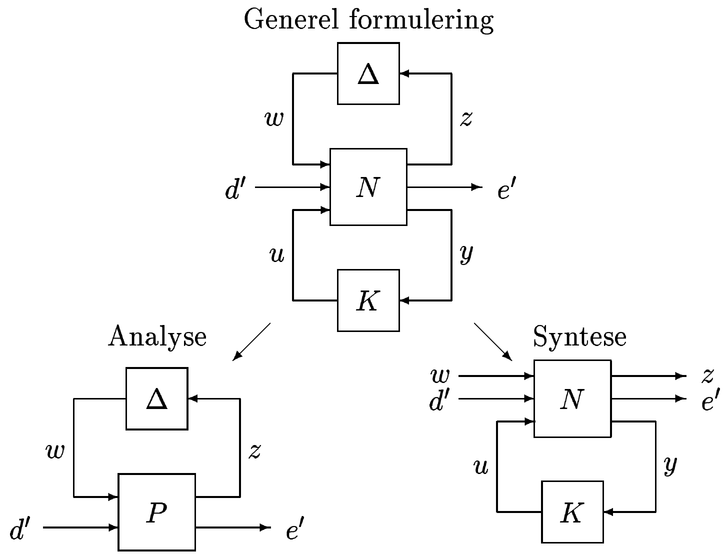


Figure 7.4: *A general framework for the analysis and for compensator design for linear systems.*

Within this framwork, analysis and design can be seen as two special cases, see Figure 7.4 on the preceding page. Like in the 2×2 block problem, scalings and weightings have been arbsorbed into the transfer matrix $N(s)$ such that $d'(s)$, $e'(s)$ and $\Delta(s)$ are normalized to one in norm. Note, that if $P(s)$ is partitioned into four blocks, consistent with the dimension of the two inputs w and d' and of the two outputs z and e', P_{11} can be identified as the transfer matrix $P(s)$ in Theorem 7.2 on the page before.

In the analysis of robust performance, the transfer matrix from $d'(s)$ to $e'(s)$ is studied. This transfer matrix is given by:

$$e'(s) = F_u(P(s), \Delta(s))d'(s) \tag{7.39}$$

$$= \left[P_{22}(s) + P_{21}(s)\Delta(s)\left(I - P_{11}(s)\Delta(s)\right)^{-1} P_{12}(s)\right] d'(s) \tag{7.40}$$

In (7.40), $P_{22}(s)$ is the weighted nominal performance transfer matrix (for example the output sensitivity) and, hence, $F_u(P(s), \Delta(s))$ is the weighted perturbed performance transfer matrix. The robust performance measure can now be formulated using (7.40) as:

$$\|F_u(P(s), \Delta(s))\|_{\mathcal{H}_\infty} = \sup_\omega \bar{\sigma}\left(F_u(P(j\omega), \Delta(j\omega))\right) < 1 \quad \forall \Delta(j\omega) \in \mathbf{B\Delta} \tag{7.41}$$

Note, that the condition for robust performance is formulated as a singular value bound, just as a robust stability condition for unstructured uncertainty. Thus, it can be concluded that *the robust performance condition* (7.41) *is satisfied if and only if the system $F_u(P(s), \Delta(s))$ is robustly stable in the face of a norm bounded perturbation $\Delta_p(s)$ with $\bar{\sigma}(\Delta_p(j\omega)) \le 1$, $\forall\omega$.* Hence, by augmenting the perturbation structure by a full complex 'performance block' $\Delta_p(s)$, robust performance can be verified via a robust stability condition. Furthermore, this augmentation of the uncertainty structure can be carried out in a quite natural way using μ, as the admissible structure for μ is precisely a block diagonal one.

This facilitates the following theorem for assessing robust performance, see also [DP87, PD93]:

Theorem 7.3 (Robust performance with μ) *Assume that performance specifications have been given as an $\mathcal{H}_\infty$ specification of the transfer matrix from $d'(s)$ to $e'(s)$ (typically a weighted sensitivity specification) of the form:*

$$\|F_u(P(s), \Delta(s))\|_{\mathcal{H}_\infty} = \sup_\omega \bar{\sigma}\left(F_u(P(j\omega), \Delta(j\omega))\right) < 1 \tag{7.42}$$

Then the perturbed closed loop system $F_u(P(s), \Delta(s))$ and the performance specification $\|F_u(P(s), \Delta(s))\|_{\mathcal{H}_\infty} < 1$, $\forall\Delta(s) \in \mathbf{B\Delta}$ if and only if:

$$\left\|\mu_{\tilde{\mathbf{\Delta}}}(P(s))\right\|_\infty \le 1 \tag{7.43}$$

where the perturbation structure has been augmented by a full complex performance block:

$$\tilde{\mathbf{\Delta}} = \left\{\mathrm{diag}\left(\Delta, \Delta_p\right) \middle| \Delta \in \mathbf{\Delta}, \Delta_p \in \mathbf{C}^{k\times k}\right\} \tag{7.44}$$

Theorem 7.3 is one of the reasons why the $\mathcal{H}_\infty$ norm is popular as a measure of performance. Indeed, if the uncertainty is bounded by the largest singular value, it is possible via μ to check for robust stability *and robust performance* in a non-conservative way. If the uncertainty is modeled in a precise way, i.e. if all systems $G_\Delta(s) \in \mathcal{G}$ could appear in practice, then the μ condition for robust performance is both necessary and sufficient. Thus, the μ theorems provide less conservative conditions for robust performance compared to the corresponding $\mathcal{H}_\infty$ conditions. Stability and performance can be separated, far more detailed uncertainty descriptions can be formulated due to the block diagonal structure of $\mathbf{\Delta}$, and non-conservative conditions for robust performance are obtained, even for ill-conditioned systems ($\kappa(G(j\omega)) \gg 1$).

(7.43) on the page before provides a simple test for robust performance. If $\mu_{\tilde{\mathbf{\Delta}}}(P(j\omega))$ is plotted against frequency, it is easy to check whether the condition (7.43) in Theorem 7.3 on the preceding page is satisfied.

Since $\Delta_1 = \text{diag}\{\Delta, 0\}$ and $\Delta_2 = \text{diag}\{0, \Delta_p\}$ are special cases of the general structure $\Delta \in \tilde{\mathbf{\Delta}}$ it is obvious that:

$$\mu_{\tilde{\mathbf{\Delta}}}(P(j\omega)) \geq \max\{\mu_{\mathbf{\Delta}}(P_{11}(j\omega)), \mu_{\mathbf{\Delta}_p}(P_{22}(j\omega)) = \bar{\sigma}(P_{22}(j\omega))\} \tag{7.45}$$

which means that a necessary condition for robust performance is that the closed loop system must be robustly stable, and that the nominal system satisfies the performance specifications.

7.1.3 Computing μ

As illustrated above, μ is a very useful tool for determining robust stability and robust performance in the face of structured as well as unstructed uncertainties. Unfortunately, the computation of μ itself is a complicated problem, which does not allow a general mathematical solution. The trouble is, that (7.36) on page 82 can not be used directly for computing μ since the optimization problem involved in general will have several local optima [DP87, FTD91].

Upper and lower bounds for μ, however, can be computed both for purely complex perturbation sets ($m_r = 0$ in (7.2) on page 77) and for mixed real and complex perturbation sets. Algorithms for computing these bounds were the subject of intense research activities in the beginning of the 1990's, see e.g. [DP87, YND91]. In the sequel some of the bounds will be presented. To avoid making the notation any more complicated than required, it will be assumed that the generalized system $P(s)$ is square, $P(s) \in \mathbf{C}^{n \times n}$.

7.1.3.1 μ for complex perturbations

First, the computation of μ will be considered in the case, where the perturbation structure consists entirely of complex blocks, i.e. when $m_r = 0$ in (7.2) on page 77. It is not difficult to show that $\mu_{\mathbf{\Delta}}(P)$ can be computed by standard functions, when $\mathbf{\Delta}$ is one of the following two sets, see e.g. [ZDG96]:

- If $\mathbf{\Delta} = \{\delta^c I_n \,|\, \delta^c \in \mathbf{C}\}$ ($m_r = 0$, $m_c = 1$, $m_C = 0$ in (7.2)), then $\mu_{\mathbf{\Delta}}(P) = \rho(P)$, the spectral radius of P (the largest absolute value of any eigenvalue of P, $\rho(P) = \max_i |\lambda_i(P)|$).
- If $\mathbf{\Delta} = \{\Delta \,|\, \Delta \in \mathbf{C}^{n \times n}\}$ ($m_r = 0$, $m_c = 0$, $m_C = 1$ in (7.2)), then $\mu_{\mathbf{\Delta}}(P) = \bar{\sigma}(P)$, the largest singular value of P.

For a general complex perturbation $\mathbf{\Delta}$ the following holds:

$$\{\delta^c I_n \,|\, \delta^c \in \mathbf{C}\} \subset \mathbf{\Delta} \subset \{\Delta \,|\, \Delta \in \mathbf{C}^{n \times n}\} \tag{7.46}$$

Therefore:

$$\rho(P) \le \mu_{\mathbf{\Delta}}(P) \le \bar{\sigma}(P) \tag{7.47}$$

These bounds, however, are yet not satisfactory, as the discrepancy between $\rho(P)$ and $\bar{\sigma}(P)$ can be arbitrarily large. Thus, the bounds of (7.47) have to be refined. This can be done through certain transformations of P which *do not affect* $\mu_{\mathbf{\Delta}}(P)$ but, nevertheless, modifies $\rho(P)$ and $\bar{\sigma}(P)$. To that end, define the following subsets of $\mathbf{C}^{n\times n}$:

$$\mathbf{Q} = \left\{ Q \in \mathbf{\Delta} \,\middle|\, m_r = 0, \delta_i^{c^*}\delta_i^c = 1, \Delta_i^*\Delta_i = I_{r_{m_c+i}} \right\} \tag{7.48}$$

$$\mathbf{D} = \Big\{ \mathrm{diag}\left(D_1, \cdots, D_{m_c}, d_1 I_{r_{m_c+1}}, \cdots, d_{m_C} I_{r_{m_c+m_C}} \right) \Big| D_i \in \mathbf{C}^{r_i\times r_i}, D_i^* = D_i > 0, d_i \in \mathbf{R}, d_i > 0 \Big\} \tag{7.49}$$

It can now be shown (see e.g. the original paper on μ by Doyle [Doy82]) that for any $\Delta \in \mathbf{\Delta}$ (for which $m_r = 0$), $Q \in \mathbf{Q}$ and $D \in \mathbf{D}$ the following holds:

$$Q^* \in \mathbf{Q}, \quad Q\Delta \in \mathbf{\Delta}, \quad \Delta Q \in \mathbf{\Delta}, \quad \bar{\sigma}(Q\Delta) = \bar{\sigma}(\Delta Q) = \bar{\sigma}(\Delta), \tag{7.50}$$

$$D\Delta = \Delta D \tag{7.51}$$

From (7.50) and (7.51), the following theorem can be derived.

Theorem 7.4 (Upper and lower bounds for $\boldsymbol{\mu}$) *For any $Q \in \mathbf{Q}$ and $D \in \mathbf{D}$ the following holds:*

$$\mu_{\mathbf{\Delta}}(PQ) = \mu_{\mathbf{\Delta}}(QP) = \mu_{\mathbf{\Delta}}(P) = \mu_{\mathbf{\Delta}}(DPD^{-1}) \tag{7.52}$$

Thus, the bounds in (7.47) *can be refined as:*

$$\max_{Q\in\mathbf{Q}} \rho(QP) \le \mu_{\mathbf{\Delta}}(P) \le \inf_{D\in\mathbf{D}} \bar{\sigma}(DPD^{-1}) \tag{7.53}$$

The lower bound $\max_{Q\in\mathbf{Q}} \rho(QP)$ is in fact an identity ($\max_{Q\in\mathbf{Q}} \rho(QP) = \mu_{\mathbf{\Delta}}(P)$), but unfortunately, the function $\rho(QP)$ is not convex, and in general it will have several local maxima. Hence, a numerical search algorithm is not guaranteed to find μ but rather just a lower bound. On the other hand, the upper bound is a convex problem, and thus, the global minimum $\inf_{D\in\mathbf{D}} \bar{\sigma}(DPD^{-1})$ can in principle always be determined. Unfortunately, the upper bound is sometimes strict, i.e. the global infimum might not be equal to μ. It can be shown, that for especially simple perturbation structure, i.e. for $m_r = 0$ and $2m_c + m_C \le 3$, the upper bound always equals μ.

However, for struktures with $2m_c + m_C > 3$, and for most matrices P, μ will be strictly less than $\inf_{D\in\mathbf{D}} \bar{\sigma}(DPD^{-1})$. On the other hand, numerical experience indicate that even for $2m_c + m_C > 3$ the upper bound is usually not highly conservative.

With the MATLAB™ μ Analysis and Synthesis Toolbox [BDG+93], commercial software is now available for computing the bounds of Theorem 7.4. For practical compensator design (at least for purely complex perturbations), the mathematical problems involved in the computation of μ seems to be of less significance.

7.1.3.2 μ with mixed perturbations

The solution to the mixed[1] μ problem has also been the subject of an intense research effort during the past ten years, see e.g. [FTD91, YND91, YND92, You93]. In these lecture notes, the computation of the bounds for the mixed μ case will not be presented in full details (one reference is [You93]), just a few of the more important results will be stated. To that end, define the following sets:

$$\mathbf{Q} = \left\{ Q \in \mathbf{\Delta} \,\middle|\, \delta_i^r \in [-1;1], \delta_i^{c^*}\delta_i^c = 1, \Delta_i^*\Delta_i = I_{r_{m_r+m_c+i}} \right\} \tag{7.54}$$

$$\begin{aligned}\mathbf{D} = \big\{ \operatorname{diag}\big(D_1, \cdots, D_{m_r+m_c}, d_1 I_{r_{m_r+m_c+1}}, \cdots, d_{m_C} I_{r_m}\big) \\ \big| D_i \in \mathbf{C}^{r_i\times r_i}, D_i^* = D_i > 0, d_i \in \mathbf{R}, d_i > 0 \big\}\end{aligned} \tag{7.55}$$

$$\mathbf{G} = \left\{ \operatorname{diag}\left(G_1, \cdots, G_{m_r}, O_{r_{m_r+1}}, \cdots, O_{r_m}\right) \middle| G_i \in \mathbf{C}^{r_i\times r_i}, G_i = G_i^* \right\} \tag{7.56}$$

$$\begin{aligned}\hat{\mathbf{D}} = \big\{ \operatorname{diag}\big(D_1, \cdots, D_{m_r+m_c}, d_1 I_{r_{m_r+m_c+1}}, \cdots, d_{m_C} I_{r_m}\big) \\ \big| D_i \in \mathbf{C}^{r_i\times r_i}, \det(D_i) \neq 0, d_i \in \mathbf{C}, d_i \neq 0 \big\}\end{aligned} \tag{7.57}$$

$$\hat{\mathbf{G}} = \{\operatorname{diag}(g_1, \cdots, g_{n_r}, O_{n_c}) \,|g_i \in \mathbf{R}\} \tag{7.58}$$

where $r_m = r_{m_r+m_c+m_C}$, $n_r = \sum_{i=1}^{m_r} r_i$ and $n_c = n - n_r$. Note, that for consistency with $P(s)$, $\sum_{i=1}^{m} r_i = n$ is required.

Then, the following upper and lower bounds for mixed μ apply:

Theorem 7.5 (Upper and lower bounds for mixed μ [FTD91]) *Let $\bar{\lambda}_R$ be the largest real eigenvalue of P and let $\rho_R(P)$ denote the spectral radius of P:*

$$\rho_R(P) \stackrel{\triangle}{=} \max\{|\lambda_R(P)| \;:\; \lambda_R(P) \text{ is a real eigenvalue of } P\} \tag{7.59}$$

If P does not have any real eigenvalues, then $\rho_R(P) = 0$. Assume further that α_* is the result of the following minimization problem:

$$\alpha_* = \inf_{D\in\mathbf{D}, G\in\mathbf{G}} \min_{\alpha\in\mathbf{R}} \left\{ \alpha \,\middle|\, \bar{\lambda}_R\left(P^*DP + j\left(GP - P^*G\right) - \alpha D\right) \leq 0 \right\} \tag{7.60}$$

Then:

$$\rho_R(P) \leq \mu_{\mathbf{\Delta}}(P) \leq \sqrt{\max(0, \alpha_*)} \tag{7.61}$$

Note, that the computation of the upper bound (7.60) involves a Linear Matrix Inequality (LMI). A number of numerical methods exists to tackle such minimizations. These require, however, even for relatively modest problems ($n \leq 100$), an optimization over scalings and $G(s)$, which can contain several thousands of parameters. Hence, dealing with such problems within reasonable computational times, require that the structure of the mixed μ problem is exploited to a wider extent, see e.g. [YND92]. Various reformulations of the upper bound problem are given in Theorem 7.6 on the next page.

[1] The μ synthesis problem in the presence of perturbations with both real and complex blocks, is often refered to simply as the mixed μ problem.

Theorem 7.6 (Reformulating the mixed μ upper bound) *Assume that a matrix $P \in \mathbf{C}^{n\times n}$ and a real positive scalar $\beta > 0$ are given. Further for any $D \in \mathbf{C}^{n\times n}$, let $P_D = DPD^{-1}$. Then the following statements are equivalent:*

1. *There exist matrices $D_1 \in \mathbf{D}$ and $G_1 \in \mathbf{G}$ such that:*

$$\bar{\lambda}_R \left(P^* D_1 P + j(G_1 P - P^* G_1) - \beta^2 D_1 \right) \leq 0 \tag{7.62}$$

2. *There exist matrices $D_2 \in \mathbf{D}$ and $G_2 \in \mathbf{G}$ (or $D_2 \in \hat{\mathbf{D}}$ and $G_2 \in \hat{\mathbf{G}}$) such that:*

$$\bar{\lambda}_R \left(P_{D_2}^* P_{D_2} + j(G_2 P_{D_2} - P_{D_2}^* G_2) \right) \leq \beta^2 \tag{7.63}$$

3. *There exist matrices $D_3 \in \mathbf{D}$ and $G_3 \in \mathbf{G}$ (or $D_3 \in \hat{\mathbf{D}}$ and $G_3 \in \hat{\mathbf{G}}$) such that:*

$$\bar{\sigma} \left[\left(\frac{P_{D_3}}{\beta} - jG_3 \right) \left(I + G_3^2 \right)^{-\frac{1}{2}} \right] \leq 1 \tag{7.64}$$

4. *There exist matrices $D_4 \in \mathbf{D}$ and $G_4 \in \mathbf{G}$ (or $D_4 \in \hat{\mathbf{D}}$ and $G_4 \in \hat{\mathbf{G}}$) such that:*

$$\bar{\sigma} \left[\left(I + G_4^2 \right)^{-\frac{1}{4}} \left(\frac{P_{D_4}}{\beta} - jG_4 \right) \left(I + G_4^2 \right)^{-\frac{1}{4}} \right] \leq 1 \tag{7.65}$$

A proof for Theorem 7.6 can be found in [You93]. Using Theorem 7.6, alternative formulations can easily be found for the mixed μ upper bound. For example, the upper bound, which is implemented in the MATLAB™ μ toolbox is derived from (7.65). Define β^* as:

$$\beta^* = \inf_{\beta \in \mathbf{R}_+, G \in \hat{\mathbf{G}}, D \in \hat{\mathbf{D}}} \left\{ \beta \,|\, \bar{\sigma}\left(P_{DG} \right) \leq 1 \right\} \tag{7.66}$$

where P_{DG} is given as:

$$P_{DG} = \left(I + G^2 \right)^{-\frac{1}{4}} \left(\frac{DPD^{-1}}{\beta} - jG \right) \left(I + G^2 \right)^{-\frac{1}{4}} \tag{7.67}$$

Then:

$$\max_{Q \in \mathbf{Q}} \rho(QP) \leq \mu_{\mathbf{\Delta}}(P) \leq \beta^* \tag{7.68}$$

7.2 μ synthesis

For compensator synthesis, it is convenient to partition the transfer matrix $F_l(N, K)$ from $[w, d']^T$ to $[z, e']^T$ as:

$$\begin{bmatrix} z(s) \\ e'(s) \end{bmatrix} = F_l(N(s), K(s)) \begin{bmatrix} w(s) \\ d'(s) \end{bmatrix} =$$

$$\left[N_{11}(s) + N_{12}(s)K(s) \left(I - N_{22}(s)K(s) \right)^{-1} N_{21}(s) \right] \begin{bmatrix} w(s) \\ d'(s) \end{bmatrix} \tag{7.69}$$

Note, that $F_l(N(s), K(s)) = P(s)$. Applying Theorem 7.3 on page 84, it is seen that a nominally stabilizing compensator $K(s)$ achieves robust performance, if and only if the structured singular value μ for every frequency $\omega \in [0, \infty]$ satisfies:

$$\mu_{\hat{\mathbf{\Delta}}}\left(F_l(N(j\omega), K(j\omega))\right) < 1 \tag{7.70}$$

Thus, the optimal robust performance problem can be formulated as:

$$K(s) = \arg \min_{K(s)\in\mathcal{K}_S} \left\| \mu_{\hat{\mathbf{\Delta}}}\left(F_l(N(s), K(s))\right) \right\|_\infty \tag{7.71}$$

where $\mathcal{K}_S$ is the set of all nominally stabilizing compensators.

7.2.1 Complex μ Synthesis – D-K iteration

Unfortunately, the optimization (7.71) can not be directly evaluated for the simple reason, that μ can not be computed exactly. Instead, an upper bound problem can be formulated as:

$$K(s) = \arg \min_{K(j\omega)\in\mathcal{K}_S} \sup_{\omega} \inf_{D(\omega)\in\mathbf{D}} \left\{ \bar{\sigma}\left(D(\omega) F_l\left(N(j\omega), K(j\omega)\right) D^{-1}(\omega) \right) \right\} \tag{7.72}$$

Unfortunately, no solution has yet been found to the minimization problem (7.72). A practical approach to is the following iterative procedure. To determine $D(\omega)$ at a number of frequencies for a given compensator $K(s)$ is equivalent to solving the complex μ upper bound problem, which has a known solution. When these scalings have been found, a stable transfer matrix $D(s)$ can be fitted, such that $D(j\omega)$ is an approximation of $D(\omega)$ for all frequencies ω. It can even be assured that $D(s)$ is minimum phase, such that D^{-1} is also stable, as the phase of $D(s)$ is absorbed into the complex perturbations. In other words: it is only necessary to fit the amplitude of $D(j\omega)$.

For given matrix scalings $D(s)$, the problem is to find a compensator $K(s)$, which minimizes the norm $\|F_l(D(s)N(s)D^{-1}(s), K(s))\|_{\mathcal{H}_\infty}$ which is a standard $\mathcal{H}_\infty$ problem, for which the solution has been given in Theorem 6.1 on page 67.

For this compensator, new D scalings can be found, and the procedure starts all over again. If this iteration (known as the D-K iteration) converges to a specific compensator, this compensator is a good candidate for a near optimal μ compensator. Even though both the computation of D scales and of the optimal $\mathcal{H}_\infty$ compensator are convex optimization problems, D-K iteration is not a *jointly convex* optimization in $D(s)$ and $K(s)$. Thus, convergence can not be guaranteed. However, numerical experience shows that D-K iteration works well in practice. The D-K procedure can be formulated as follows:

Procedure 7.1 (D-K iteration)

1. *Given an augmented system $N(s)$, let $i = 1$ and $D_i^\star(\omega) = I, \forall\omega$.*

2. *Fit a stable minimum phase transfer matrix $D_i(s)$ to the pointwise scalings $D_i^\star(\omega)$. Augment $D_i(s)$ with a identity matrix, such that $D_i(s)$ becomes compatible with $N(s)$. Construct the system $N_{D_i}(s) = D_i(s)N(s)D_i^{-1}(s)$.*

3. *Find the $\mathcal{H}_\infty$ optimal compensator $K_i(s)$:*

$$K_i(s) = \arg \min_{K(s)\in\mathcal{K}_S} \|F_l(N_{D_i}(s), K(s))\|_{\mathcal{H}_\infty} \tag{7.73}$$

4. *Find the new scalings $D^\star_{i+1}(\omega)$ as a solution to the complex μ upper bound problem:*

$$D^\star_{i+1}(\omega) = \arg \min_{D(\omega)\in\mathbf{D}} \left\{\bar{\sigma}\left(D(\omega)F_l(N(j\omega), K_i(j\omega))D^{-1}(\omega)\right)\right\} \tag{7.74}$$

for every frequency ω.

5. *Compare $D^\star_{i+1}(\omega)$ and $D^\star_i(\omega)$. Stop, if they are 'close' (in magnitude). Otherwise, let $i = i + 1$ and repeat the iteration from Step 2.*

Note, that the $\mathcal{H}_\infty$ solution is used to find the compensator in Step 3. The K step in the D-K iteration can be illustrated as shown in Figure 7.5.

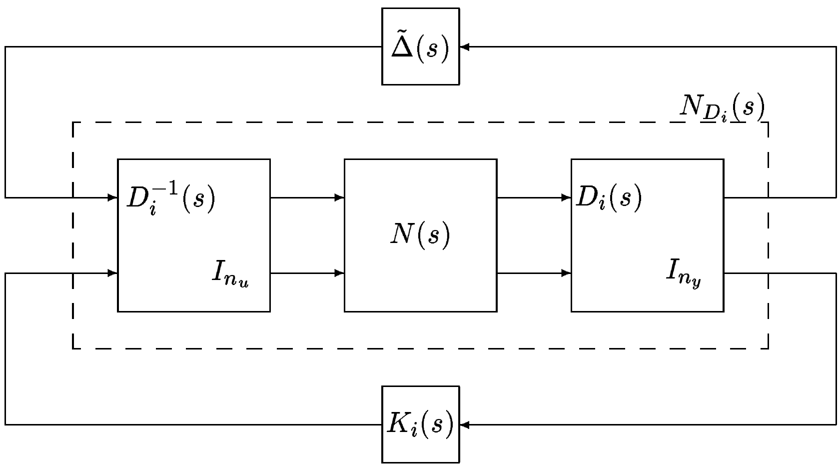


Figure 7.5: *K step in $D-K$* iteration. n_u and n_y are, respectively, the number of controllable inputs and the number of regulated outputs (error signals).

With the MATLAB™ *μ-Analysis and Synthesis Toolbox* commercially available software now exists which supports μ synthesis by *D-K* iteration. In early versions, only full complex blocks were supported. Note, that for repeated scalar blocks, the D scaling is a full matrix, and hence, the number of SISO transfer functions to be fitted grows rapidly.

Mixed μ synthesis is far more involved than the purely complex problem, and the present version of the μ toolbox does not support it fully. Recently, a couple of design methods for mixed μ synthesis has been proposed, see e.g. [You93, TC96].

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