Controller Synthesis

- State-feedback and estimation problems
- Output feedback synthesis ...

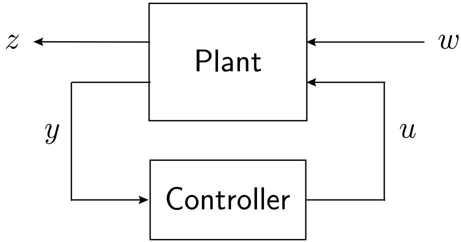
  ... convexifying controller parameter transformation
- Mixed-objective control

Controller Synthesis

Have seen several analysis specifications (stability/performance).

All formulated in terms of matrix inequalities.

Try to achieve them by designing a controller.



Generalized plant framework: System interconnection and weights.

System Descriptions and Problem Formulation

Signals: Disturbance w, controlled output z, control u, measurement y.

Open-loop system and controller described as

$$\begin{pmatrix} \dot{x} \\ \hline z \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A & B_1 & B \\ \hline C_1 & D_1 & E \\ C & F & 0 \end{array}\right) \begin{pmatrix} x \\ \hline w \\ u \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \dot{x}_c \\ u \end{pmatrix} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} x_c \\ y \end{pmatrix}.$$

Controlled closed-Loop system described with calligraphic matrices:

$$\begin{pmatrix} \dot{\xi} \\ z \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \xi \\ w \end{pmatrix}.$$

Find controller that renders $\mathcal{A}$ Hurwitz (internal stability) and achieves desired performance specification for controlled system.

Dependence on Controller Parameters

Recall the easily derived explicit formula for **general output-feedback:**

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \left(\begin{array}{cc|c} A + BD_cC & BC_c & B_1 + BD_cF \\ B_cC & A_c & B_cF \\ \hline C_1 + ED_cC & EC_c & D_1 + ED_cF \end{array}\right).$$

Special case $C = I$, $F = 0$: **Dynamic** or **static state-feedback**:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \left(\begin{array}{cc|c} A + BD_c & BC_c & B_1 \\ B_c & A_c & 0 \\ \hline C_1 + ED_c & EC_c & D_1 \end{array}\right) \text{ or } \begin{pmatrix} A + BD_c & B_1 \\ C_1 + ED_c & D_1 \end{pmatrix}.$$

Other **information structures** or **configurations** ...

... full-information feedback, estimation problems.

Design of Quadratic Performance Controllers

Given $P_p = \begin{pmatrix} Q_p & S_p \\ S_p^T & R_p \end{pmatrix}$ with $R_p \succcurlyeq 0$, find controller that renders $\mathcal{A}$ Hurwitz and quadratic performance specification satisfied.

Guaranteed for controlled system iff exists $\mathcal{X}$ with

$$\mathcal{X} \succ 0, \quad \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & Q_p & 0 & S_p \\ \hline I & 0 & 0 & 0 \\ 0 & S_p^T & 0 & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{array}\right) \prec 0.$$

The variables are $\mathcal{X}$ and A_c, B_c, C_c, D_c which do **not enter affinely**.
Essential aspect: Will construct linearizing change of variables.

Various Ways of Writing LMI

$$\begin{pmatrix} \mathcal{A}^T\mathcal{X}+\mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \mathcal{B}^T\mathcal{X} & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ \mathcal{C} & \mathcal{D} \end{pmatrix}^T \begin{pmatrix} Q_p & S_p \\ S_p^T & R_p \end{pmatrix} \begin{pmatrix} 0 & I \\ \mathcal{C} & \mathcal{D} \end{pmatrix} =$$

$$= \left(\begin{array}{cc} I & 0 \\ \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \hline 0 & I \\ \mathcal{C} & \mathcal{D} \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \hline 0 & I \\ \mathcal{C} & \mathcal{D} \end{array}\right) =$$

$$= \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & Q_p & 0 & S_p \\ \hline I & 0 & 0 & 0 \\ 0 & S_p^T & 0 & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{array}\right).$$

Static State-Feedback Synthesis

$$\mathcal{X} \succ 0, \quad \begin{pmatrix} (A+BD_c)^T\mathcal{X} + \mathcal{X}(A+BD_c) & \mathcal{X}B_1 \\ B_1^T\mathcal{X} & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & I \\ C_1 + ED_c & D_1 \end{pmatrix}^T P_p \begin{pmatrix} 0 & I \\ C_1 + ED_c & D_1 \end{pmatrix} \prec 0.$$

Define the new variables

$$Y = \mathcal{X}^{-1} \quad \text{and} \quad M := D_c\mathcal{X}^{-1}.$$

Left-multiplying first row and right-multiplying first column of second inequality with $\mathcal{X}^{-1}$ (congruence!) leads to inequalities in Y and M on next slide. Note that the parameter change is invertible:

$$\mathcal{X} = Y^{-1} \quad \text{and} \quad D_c = MY^{-1}.$$

Static State-Feedback Synthesis

Synthesis inequalities for static state-feedback design:

$$Y \succ 0,\quad \begin{pmatrix} (AY + BM)^T + (AY + BM) & B_1 \\ B_1^T & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & I \\ C_1 Y + EM & D_1 \end{pmatrix}^T P_p \begin{pmatrix} 0 & I \\ C_1 Y + EM & D_1 \end{pmatrix} \prec 0.$$

Since $R_p \succcurlyeq 0$ this is easily turned into a genuine LMI (as seen below).

- Check whether the synthesis inequalities have solution Y, M.
- If **no** we are sure that quadratic performance cannot be achieved.
- If **yes** then $D_c = MY^{-1}$ achieves quadratic performance.

Output-Feedback: Controller Parameter Change

Recall that $\mathcal{A}$ is partitioned. Partition accordingly:

$$\mathcal{X} = \begin{pmatrix} X & U \\ U^T & * \end{pmatrix}, \quad \mathcal{X}^{-1} = \begin{pmatrix} Y & V \\ V^T & * \end{pmatrix}.$$

Observe that $YX + VU^T = I$ (used later) and that

$$\mathcal{Y} = \begin{pmatrix} Y & I \\ V^T & 0 \end{pmatrix}, \ \mathcal{Z} = \begin{pmatrix} I & 0 \\ X & U \end{pmatrix} \ \text{ satisfy } \ \mathcal{Y}^T\mathcal{X} = \mathcal{Z}.$$

Transform controller parameters as

$$\begin{pmatrix} K & L \\ M & N \end{pmatrix} = \begin{pmatrix} XAY & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} U & XB \\ 0 & I \end{pmatrix} \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} V^T & 0 \\ CY & I \end{pmatrix}.$$

Output-Feedback: Block Transformation

Why? Short computation reveals

$$\mathcal{Y}^T \mathcal{X} \mathcal{Y} = \begin{pmatrix} Y & I \\ I & X \end{pmatrix},$$

$$\left(\begin{array}{c|c} \mathcal{Y}^T(\mathcal{X}\mathcal{A})\mathcal{Y} & \mathcal{Y}^T(\mathcal{X}\mathcal{B}) \\ \hline \mathcal{C}\mathcal{Y} & \mathcal{D} \end{array}\right) =$$

$$= \left(\begin{array}{cc|c} AY + BM & A + BNC & B_1 + BNF \\ K & XA + LC & XB_1 + LF \\ \hline C_1 Y + EM & C_1 + ENC & D_1 + ENF \end{array}\right).$$

Achieve **affine** dependence on X, Y and K, L, M, N!

Output-Feedback: Congruence Transformation

For necessity: Can assume w.l.o.g. that $\mathcal{Y}$ has full column rank.

For sufficiency: We will make sure that $\mathcal{Y}$ is square and non-singular.

Transform

$$\mathcal{X} \succ 0, \quad \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & Q_p & 0 & S_p \\ \hline I & 0 & 0 & 0 \\ 0 & S_p^T & 0 & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{array}\right) \prec 0$$

by congruence with matrix $\mathrm{diag}(\mathcal{Y}, I)$ into

$$\mathcal{Y}^T\mathcal{X}\mathcal{Y} \succ 0, \quad \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}^T \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & Q_p & 0 & S_p \\ \hline I & 0 & 0 & 0 \\ 0 & S_p^T & 0 & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \mathcal{Y}^T(\mathcal{X}\mathcal{A})\mathcal{Y} & \mathcal{Y}^T(\mathcal{X}\mathcal{B}) \\ \mathcal{C}\mathcal{Y} & \mathcal{D} \end{array}\right) \prec 0.$$

Output-Feedback: Synthesis Inequalities

Substitute to obtain **synthesis inequalities** in X, Y, K, L, M, N:

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0,$$

$$* \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & Q_p & 0 & 0 & S_p \\ \hline I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & S_p^T & 0 & 0 & R_p \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \hline AY + BM & A + BNC & B_1 + BNF \\ K & XA + LC & XB_1 + LF \\ C_1Y + EM & C_1 + ENC & D_1 + ENF \end{array}\right) \prec 0$$

This is a quadratic matrix inequality. Since $R_p \succcurlyeq 0$ it can be turned into an LMI (Schur).

Testing Feasibility is LMI Problem

Constant V, S, T and affine $Q(v)$, $U(v)$, $W(v)$ in variables v.

Linearization Lemma. Testing existence of v with $U(v) \succ 0$ and

$$\begin{pmatrix} V \\ W(v) \end{pmatrix}^T \begin{pmatrix} Q(v) & S \\ S^T & TU(v)^{-1}T^T \end{pmatrix} \begin{pmatrix} V \\ W(v) \end{pmatrix} \prec 0$$

is LMI problem.

Proof. Second inequality reads as

$$V^TQ(v)V + V^TSW(v) + W(v)^TS^TV + W(v)^TTU(v)^{-1}T^TW(v) \prec 0.$$

Hence first and second inequality equivalent to LMI

$$\begin{pmatrix} V^TQ(v)V + V^TSW(v) + W(v)^TS^TV & W(v)^TT \\ T^TW(v) & -U(v) \end{pmatrix} \prec 0.$$

Output-Feedback: Controller Construction

Solve synthesis inequalities to determine X, Y and K, L, M, N.

Determine non-singular U, V with $VU^T = I - YX$.

Analysis inequalities are satisfied for

$$\mathcal{X} = \begin{pmatrix} Y & V \\ I & 0 \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ X & U \end{pmatrix}$$

$$\begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} = \begin{pmatrix} U & XB \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} K - XAY & L \\ M & N \end{pmatrix} \begin{pmatrix} V^T & 0 \\ CY & I \end{pmatrix}^{-1}$$

- A_c has the same size as A. Construct full order controller.
- Freedom in choice of U, V: Controller state-coordinate change.

General Procedure

- Rewrite analysis inequalities in terms of blocks $\mathcal{X}$, $\mathcal{X}\mathcal{A}$, $\mathcal{X}\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$.
- Find formal congruence transformation involving $\mathcal{Y}$ to transform into inequalities in terms of blocks $\mathcal{Y}^T\mathcal{X}\mathcal{Y}$, $\mathcal{Y}^T(\mathcal{X}\mathcal{A})\mathcal{Y}$, $\mathcal{Y}^T(\mathcal{X}\mathcal{B})$, $\mathcal{C}\mathcal{Y}$, $\mathcal{D}$.
- Obtain synthesis inequalities by substitution
$$\mathcal{Y}^T\mathcal{X}\mathcal{Y} \to \boldsymbol{X}(v), \quad \begin{pmatrix} \mathcal{Y}^T[\mathcal{X}\mathcal{A}]\mathcal{Y} & \mathcal{Y}^T[\mathcal{X}\mathcal{B}] \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \to \begin{pmatrix} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix}$$
with affine $\boldsymbol{A}(v)$, $\boldsymbol{B}(v)$, $\boldsymbol{C}(v)$, $\boldsymbol{D}(v)$ in new variables v.
- Controller construction independent of particular analysis inequalities! Construction leads to controller of **same order** as plant.
- Works **both** in continuous-time and discrete-time in identical fashion.

Variables and Blocks

State-Feedback: $v = \left(\; Y, \quad M \;\right)$ and

$$\boldsymbol{X}(v) = Y, \quad \begin{pmatrix} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix} = \begin{pmatrix} AY + BM & B_1 \\ C_1Y + EM & D_1 \end{pmatrix}$$

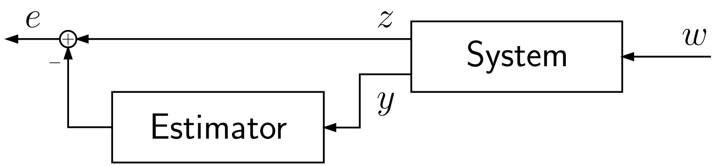
Output-Feedback: $v = \left(\; X, \quad Y, \quad K, \quad L, \quad M, \quad N \;\right)$ and

$$\boldsymbol{X}(v) = \begin{pmatrix} Y & I \\ I & X \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{pmatrix} = \left(\begin{array}{cc|c} AY + BM & A + BNC & B_1 + BNF \\ K & XA + LC & XB_1 + LF \\ \hline C_1Y + EM & C_1 + ENC & D_1 + ENF \end{array}\right)$$

Estimation Problems

Interconnection for estimation:



Derivation of convexifying parameter transformation as exercise.

- Find estimator which minimizes H_∞-norm of $w \to e$ ...

  ... worst case error as small as possible.

- Find estimator which minimizes H_2-norm of $w \to e$ ...

  ... optimally reduce asymptotic variance against white noise.

Illustration for H_∞ Control Problem

Theorem. There exists controller that renders $\mathcal{A}$ Hurwitz and

$$\|\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{B}+\mathcal{D}\|_\infty < \gamma$$

satisfied iff there exists v with

$$\boldsymbol{X}(v) \succ 0$$

$$\left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & -\gamma I & 0 & 0 \\ \hline I & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma}I \end{array}\right) \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \boldsymbol{A}(v) & \boldsymbol{B}(v) \\ \boldsymbol{C}(v) & \boldsymbol{D}(v) \end{array}\right) \prec 0.$$

Linearization Lemma: Can directly compute minimal achievable γ since

$$\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\gamma}I \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} (\gamma I)^{-1} \begin{pmatrix} 0 & I \end{pmatrix}.$$

Illustration for H_2-Synthesis

Recall: $\mathcal{A}$ Hurwitz and $\|\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{B}+\mathcal{D}\|_2<\gamma$ iff exists $\mathcal{X}\succ 0$ with

$$\mathcal{D}=0,\quad \mathcal{A}\mathcal{X}+\mathcal{X}\mathcal{A}^T+\gamma^{-1}\mathcal{B}\mathcal{B}^T\prec 0,\quad \mathrm{trace}(\mathcal{C}\mathcal{X}\mathcal{C}^T)<\gamma$$

iff (Schur) exist $\mathcal{X}$ and Z with $\mathrm{trace}(Z)<\gamma$ and

$$\mathcal{D}=0,\quad \begin{pmatrix}\mathcal{A}^T\mathcal{X}+\mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}\\ \mathcal{B}^T\mathcal{X} & -\gamma I\end{pmatrix}\prec 0,\quad \begin{pmatrix}\mathcal{X} & \mathcal{C}^T\\ \mathcal{C} & Z\end{pmatrix}\succ 0.$$

Formal congruence trafo with diag$(\mathcal{Y},I)$.

Exists a controller which renders $\mathcal{A}$ Hurwitz and closed-loop H_2-norm smaller than γ iff exist v and Z with $\mathrm{trace}(Z)<\gamma$ and

$$\boldsymbol{D}(v)=0,\quad \begin{pmatrix}\boldsymbol{A}(v)^T+\boldsymbol{A}(v) & \boldsymbol{B}(v)\\ \boldsymbol{B}(v)^T & -\gamma I\end{pmatrix}\prec 0,\quad \begin{pmatrix}\boldsymbol{X}(v) & \boldsymbol{C}(v)^T\\ \boldsymbol{C}(v) & Z\end{pmatrix}\succ 0.$$

Allowed: Affine equality constraints & auxiliary variables.

Remarks

- No hypotheses on system required.
- Take precautions to render controller construction well-conditioned.
- Can directly optimize affine functional of involved variables.

  For example it is possible to directly compute

$$\inf_{\mathcal{A}\ \text{stable}} \|\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{B}+\mathcal{D}\|_k, \quad k=2,\infty.$$

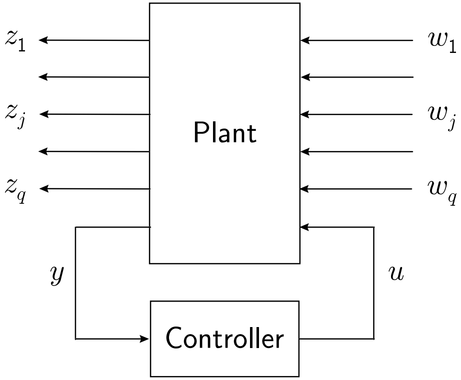
- **Warning:** Optimal controllers do in general not exist.

  If γ approaches optimum for H_∞ or H_2 problem, the poles of closed-loop system move to imaginary axis and/or the controller parameters blow up.

  Stay away from optimality! Remove fast poles by residualization!

Multi-Objective Control

Design controller which achieves multiple objectives on different channels of closed-loop system:



- Robust stabilization:

$$\|\mathcal{T}_{w_1 \to z_1}\|_\infty < \gamma_1$$

- Disturbance attenuation:

$$\|\mathcal{T}_{w_2 \to z_2}\|_2 < \gamma_2$$

No loss of generality: Relevant channels are $w_k \to z_k$, $k = 1, \ldots, q$.

Multi-Channel System Description

Open-loop system and controller:

$$\begin{pmatrix} \dot{x} \\ \hline z_1 \\ \vdots \\ z_q \\ \hline y \end{pmatrix} \begin{pmatrix} A & B_1 & \cdots & B_q & B \\ \hline C_1 & D_1 & \cdots & D_{1q} & E_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_q & D_{q1} & \cdots & D_q & E_q \\ \hline C & F_1 & \cdots & F_q & 0 \end{pmatrix} \begin{pmatrix} x \\ \hline w_1 \\ \vdots \\ w_q \\ \hline u \end{pmatrix}, \quad \begin{pmatrix} \dot{x}_c \\ u \end{pmatrix} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} x_c \\ y \end{pmatrix}.$$

Controlled closed-loop system:

$$\begin{pmatrix} \dot{\xi} \\ \hline z_1 \\ \vdots \\ z_q \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B}_1 & \cdots & \mathcal{B}_q \\ \hline \mathcal{C}_1 & \mathcal{D}_1 & \cdots & \mathcal{D}_{1q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_q & \mathcal{D}_{q1} & \cdots & \mathcal{D}_q \end{pmatrix} \begin{pmatrix} \xi \\ \hline w_1 \\ \vdots \\ w_q \end{pmatrix}.$$

Multi-Objective H_2/H_∞ Control

Find controller such that $\mathcal{A}$ is Hurwitz and

$$\|\mathcal{C}_1(sI-\mathcal{A})^{-1}\mathcal{B}_1+\mathcal{D}_1\|_\infty<\gamma_1,\quad \|\mathcal{C}_2(sI-\mathcal{A})^{-1}\mathcal{B}_2+\mathcal{D}_2\|_2<\gamma_2.$$

Related analysis inequalities:

$$\begin{pmatrix} \mathcal{A}^T\mathcal{X}_1+\mathcal{X}_1\mathcal{A} & \mathcal{X}_1\mathcal{B}_1 & \mathcal{C}_1^T \\ \mathcal{B}_1^T\mathcal{X}_1 & -\gamma_1 I & \mathcal{D}_1^T \\ \mathcal{C}_1 & \mathcal{D}_1 & -\gamma_1 I \end{pmatrix} \prec 0$$

$$\begin{pmatrix} \mathcal{A}^T\mathcal{X}_2+\mathcal{X}_2\mathcal{A} & \mathcal{X}_2\mathcal{B}_2 \\ \mathcal{B}_2^T\mathcal{X}_2 & -\gamma_2 I \end{pmatrix} \prec 0,\ \begin{pmatrix} \mathcal{X}_2 & \mathcal{C}_2^T \\ \mathcal{C}_2 & Z \end{pmatrix} \succ 0,\ \mathrm{trace}(Z)<\gamma_2,\ \mathcal{D}_2=0.$$

In general need $\mathcal{X}_1\neq\mathcal{X}_2$. Untractable in state-space.

Relaxation: Introduce **extra constraint** $\mathcal{X}_1=\mathcal{X}_2$.

Mixed-Objective H_2/H_∞ Control

Find controller such that there exists $\mathcal{X}$, Z with

$$\begin{pmatrix} \mathcal{A}^T\mathcal{X}+\mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_1 & \mathcal{C}_1^T \\ \mathcal{B}_1^T\mathcal{X} & -\gamma_1 I & \mathcal{D}_1^T \\ \mathcal{C}_1 & \mathcal{D}_1 & -\gamma_1 I \end{pmatrix} \prec 0$$

$$\begin{pmatrix} \mathcal{A}^T\mathcal{X}+\mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}_2 \\ \mathcal{B}_2^T\mathcal{X} & -\gamma_2 I \end{pmatrix} \prec 0, \ \begin{pmatrix} \mathcal{X} & \mathcal{C}_2^T \\ \mathcal{C}_2 & Z \end{pmatrix} \succ 0, \ \mathrm{trace}(Z) < \gamma_2, \ \mathcal{D}_2 = 0.$$

Solvability of mixed problem **implies** stability of $\mathcal{A}$ and the desired norm inequalities. Can hence conclude in general that

$$\text{Minimal mixed } \gamma_2 \quad \geq \quad \text{Minimal multi-objective } \gamma_2.$$

$\mathcal{X}_1 = \mathcal{X}_2$ often implies that there is a gap and the inequality is strict.

Solution of Mixed H_2/H_∞ Control

But $\mathcal{X}_1 = \mathcal{X}_2$ implies tractability: Can apply general procedure!

Mixed synthesis inequalities:

$$\begin{pmatrix} \boldsymbol{A}(v)^T + \boldsymbol{A}(v) & \boldsymbol{B}_1(v) & \boldsymbol{C}_1(v)^T \\ \boldsymbol{B}_1(v)^T & -\gamma_1 I & \boldsymbol{D}_1(v)^T \\ \boldsymbol{C}_1(v) & \boldsymbol{D}_1(v) & -\gamma_1 I \end{pmatrix} \prec 0$$

$$\begin{pmatrix} \boldsymbol{A}(v)^T + \boldsymbol{A}(v) & \boldsymbol{B}_2(v) \\ \boldsymbol{B}_2(v)^T & -\gamma_2 I \end{pmatrix} \prec 0, \quad \begin{pmatrix} \boldsymbol{X}(v) & \boldsymbol{C}_2(v)^T \\ \boldsymbol{C}_2(v) & Z \end{pmatrix} \succ 0,$$

$$\mathrm{trace}(Z) < \gamma_2, \quad \boldsymbol{D}_2(v) = 0.$$

Can be solved by standard algorithms ...

... controller construction as usual ...

... controller order **identical** to order of system!

Extensions

- For fixed α_1, α_2 and variable γ_1, γ_2, optimize $\alpha_1\gamma_1 + \alpha_2\gamma_2$.

  Analyze trade-off between specifications by playing with α_1, α_2.

- Relax constraint with tuning parameter $\alpha > 0$: $\mathcal{X}_1 = \alpha\mathcal{X}_2$.

  Line-search over α. Might reduce conservatism.

- Can include more than two LMI performance on different channels.

  Never forget conservatism.

- Possible to include other type of constraints.

  Important example: Closed-loop poles in **convex** LMI region.

Poles in Convex LMI-Region

Eigenvalues of $\mathcal{A}$ in LMI-region defined by Q, R, S iff exists $\mathcal{X}$ with

$$\mathcal{X} \succ 0, \quad \begin{pmatrix} I \\ \mathcal{A} \otimes I \end{pmatrix}^T \begin{pmatrix} \mathcal{X} \otimes Q & \mathcal{X} \otimes S \\ \mathcal{X} \otimes S^T & \mathcal{X} \otimes R \end{pmatrix} \begin{pmatrix} I \\ \mathcal{A} \otimes I \end{pmatrix} \prec 0.$$

or equivalently

$$\mathcal{X} \succ 0, \quad \mathcal{X} \otimes Q + (\mathcal{X}\mathcal{A}) \otimes S + S^T \otimes (\mathcal{X}\mathcal{A})^T + (\mathcal{A}^T\mathcal{X}\mathcal{A}) \otimes R \prec 0.$$

Assumption: $R \succcurlyeq 0$. Factorize as $R = TU^{-1}T^T$ with $U \succ 0$.

LMI's equivalent to (Schur and properties of Kronecker product):

$$\begin{pmatrix} \mathcal{X} \otimes Q + (\mathcal{X}\mathcal{A}) \otimes S + S^T \otimes (\mathcal{X}\mathcal{A}) & (\mathcal{X}\mathcal{A})^T \otimes T \\ (\mathcal{X}\mathcal{A}) \otimes T^T & -\mathcal{X} \otimes U \end{pmatrix} \prec 0.$$

Formal congruence trafo with $\mathrm{diag}(\mathcal{Y} \otimes I, \mathcal{Y} \otimes I)$. Done!

Conclusions

- View approach as Lyapunov shaping technique: Improve existing controller by adding extra specs on closed-loop system.
- Analyze designed mixed controller with **different** Lyapunov matrices.
- Exist solutions to genuine multi-objective problem:
    - Can initialize with mixed controller.
    - Can compute upper bounds and lower bounds.
    - Expensive. Controller order might be large.