A Brief Recap

- Characterize nominal quadratic performance
- How extend to robust performance specifications?
- Nominal and robust linear matrix inequalities

Recap: Nominal System Description

Consider LTI system with state-space description

$$\begin{cases} \dot{x} &= Ax + Bw, \quad x(0) = 0 \\ z &= Cx + Dw \end{cases}$$

or input-output description $z = Tw$ and transfer matrix

$$T(s) = C(sI - A)^{-1}B + D.$$

Interpretation: w is disturbance. Analyze effect onto output z.

Cosidered following manners to quantify effect of w onto z:

- w and z deterministic signals: system gain.
- w white noise: asymptotic output variance.
- w impulses: energy of outputs.

Recap: Quadratic Performance Analysis

Quadratic performance with symmetric P_p: Exists $\epsilon > 0$ such that

$$\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T P_p \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq -\epsilon \int_0^\infty \|w(t)\|^2 \, dt \ \text{ for } \ w \in L_2[0,\infty).$$

Let A be Hurwitz. System satisfies quadratic P_p-performance iff

$$\begin{pmatrix} I \\ T(i\omega) \end{pmatrix}^* P_p \begin{pmatrix} I \\ T(i\omega) \end{pmatrix} \prec 0 \ \text{ for } \ \omega \in \mathbb{R} \cup \{\infty\}$$

iff there exists symmetric solution X of LMI

$$\left(\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ A & B \\ \hline 0 & I \\ C & D \end{array}\right) \prec 0 \ \text{ with } \ \begin{pmatrix} Q_p & S_p \\ S_p^T & R_p \end{pmatrix} = P_p.$$

Remark: Characterizing Performance AND Stability

We assumed A to be Hurwitz and X was not required to be positive definite. In many practical cases the considered performance specification implicitly involves the inequality $A^T X + XA \prec 0$.

Example: Quadratic performance specification with $R_p \succcurlyeq 0$.

If the performance LMI's are satisfied we can then conclude that

A is Hurwitz if and only if $X \succ 0$.

Therefore we can characterize stability **and** performance with the same LMI's by just including the extra condition $X \succ 0$. This will be essential for controller synthesis.

From Nominal to Robust Performance

Recall how we obtained from nominal stability characterizations the corresponding robust stability tests against time-varying rate-bounded parametric uncertainties.

As a major beauty of the dissipation approach, this generalization works without any technical delicacies for performance as well!

Only for time-invariant uncertainties the frequency-domain characterizations make sense. For time-varying uncertainties (and time-varying systems) we have to rely on the time-domain interpretations.

We provide an illustration for quadratic performance!

Robust Quadratic Performance

Uncertain input-output system described as

$$\begin{aligned} \dot{x}(t) &= A(\delta(t))x(t) + B(\delta(t))w(t) \\ z(t) &= C(\delta(t))x(t) + D(\delta(t))w(t). \end{aligned}$$

with continuously differential parameter-curves $\delta(.)$ that satisfy

$$\delta(t) \in \boldsymbol{\delta} \ \text{ and } \ \dot{\delta}(t) \in \boldsymbol{v} \quad (\boldsymbol{\delta},\ \boldsymbol{v} \subset \mathbb{R}^p \ \text{ compact}).$$

Robust quadratic performance: Exponential stability and existence of $\epsilon > 0$ such that for $x(0) = 0$, for all parameter curves and for all trajectories

$$\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T \begin{pmatrix} Q_p & S_p \\ S_p^T & R_p \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq -\epsilon \|w\|_2^2.$$

L_2-gain, passivity, ...

Characterization of Robust Quadratic Performance

Let $R_p \succcurlyeq 0$. Moreover suppose there exists a continuously differentiable Hermitian-valued $X(\delta)$ such that $X(\delta) \succ 0$ and

$$\begin{pmatrix} \sum_{k=1}^{p} \partial_k X(\delta) v_k + A(\delta)^T X(\delta) + X(\delta) A(\delta) & X(\delta) B(\delta) \\ B(\delta)^T X(\delta) & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & I \\ C(\delta) & D(\delta) \end{pmatrix}^T P_p \begin{pmatrix} 0 & I \\ C(\delta) & D(\delta) \end{pmatrix} \prec 0$$

for all $\delta \in \boldsymbol{\delta}$, $v \in \boldsymbol{v}$. Then the uncertain system satisfies the robust quadratic performance specification.

Numerical search for $X(\delta)$: Same as for stability!

Extends to other LMI performance specifications!

Sketch of Proof

Exponential stability: Left-upper block is

$$\sum_{k=1}^{p} \partial_k X(\delta) v_k + A(\delta)^T X(\delta) + X(\delta) A(\delta) + \underbrace{C(\delta)^T R_p C(\delta)}_{\succcurlyeq 0} \prec 0.$$

Can hence just apply our general result on robust exponential stability.

Performance: Add to right-lower block ϵI for some small $\epsilon > 0$ (compactness). Left- and right-multiply inequality with col$(x(t), w(t))$ to infer

$$\frac{d}{dt} x(t)^T X(\delta(t)) x(t) + \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T P_p \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} + \epsilon w(t)^T w(t) \leq 0.$$

Integrate on $[0, T]$ and use $x(0) = 0$ to obtain

$$x(T)^T X(\delta(T)) x(T) + \int_0^T \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T P_p \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq -\epsilon \int_0^T w(t)^T w(t)\, dt.$$

Take limit $T \to \infty$ to arrive at required quadratic performance inequality.

Semi-Infinite LMI-Constraints

Nominal performance could be reduced to an LMI feasibility test: Does there exist a solution $x \in \mathbb{R}^n$ of some LMI

$$F_0 + x_1 F_1 + \cdots + x_n F_n \prec 0.$$

We have seen that testing robust performance can be reduced to the following question: Does there exist some $x \in \mathbb{R}^n$ with

$$F_0(\delta) + x_1 F_1(\delta) + \cdots + x_n F_n(\delta) \prec 0 \quad \text{for all} \quad \delta \in \boldsymbol{\delta}$$

where $F_0(\delta), \ldots, F_n(\delta)$ are Hermitian-valued functions of $\delta \in \boldsymbol{\delta}$.

Is a generic formulation for the **robust counterpart** of an LMI feasibility test in which the data matrices are affected by uncertainties.

Robust Optimization

- Introduction to robust optimization and robust LMI problems
- Lagrange duality
- How to construct tractable relaxations

Recommended Literature

[1] A. Ben-Tal, A. Nemirovski, Lectures on Modern Convex Optimization. Philadelphia, SIAM Publications (2001).

[2] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, Cambridge (2004).

Example: Impulse Response Filter Design

Target impulse response $h(.)$. Given impulse responses $h_1(.), \ldots, h_n(.)$.

Find $x_1, \ldots, x_n \in \mathbb{R}$ such that $x_1 h_1(.) + \cdots + x_n h_n(.)$ optimally approximates $h(.)$ uniformly on $[0, T]$ (maximum norm approximation):

$$\gamma_{\text{opt}} := \inf_{x_1, \ldots, x_n \in \mathbb{R}} \max_{t \in [0,T]} |h(t) - \sum_{k=1}^{n} x_k h_k(t)|$$

After time-discretization $\{t_1, \ldots, t_m\} \in [0, T]$ solve instead the LP

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & -\gamma \leq h(t_l) - \sum_{k=1}^{n} x_k h_k(t_l) \leq \gamma, \ l = 1, \ldots, m. \end{array}$$

Example: Impulse Response Filter Design

Randomly generated response $h(.)$ with 20 poles.

Approximate by responses $h_k(.)$ of $1/(s+1)^k$, $k = 1, \ldots, 14$.

Use uniformly-spaced time-grid of 200 points on interval $[0, 15]$.

Nominal response

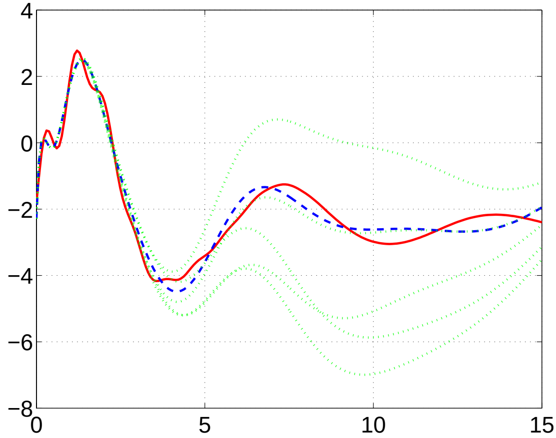
(red full)

Best approximation

(dashed blue)

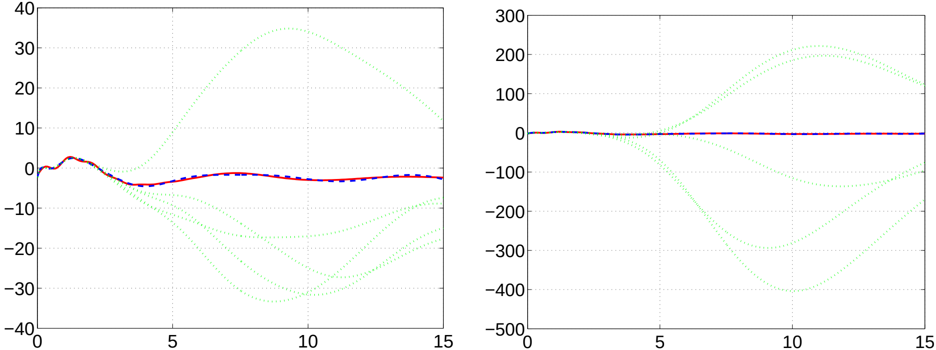
Random deviation by $\pm 0.1\%$ in optimal coeffs

(greed dotted)



Example: Analysis of Sensitivity

The observed sensitivity drastically gets worse if trying to find better approximations by increasing the number of building blocks. For example with 15 and 16 we obtain



It is not wise to only optimize, but it is most relevant to guarantee robustness against uncertainties (e.g. due to real-life implementation).

Robust Linear Programming

Consider a standard linear program (LP)

$$\text{minimize} \;\; c^T x \;\; \text{subject to} \;\; Ax \leq b.$$

To construct the data (matrices, vectors) A, b, c one has to measure/estimate real numbers which typically does not lead to exact information. We say that the data is affected by **uncertainty**.

Worst-case uncertainty model: We only know about the data that

$$(A, b, c) \in \mathcal{U} \;\; \text{with some subset} \;\; \mathcal{U} \subset \mathbb{R}^{m \times n} \times \mathbb{R}^m \times \mathbb{R}^n.$$

The **robust counterpart** of the LP is the optimization problem

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & Ax \leq b, \;\; c^T x \leq \gamma \;\; \text{for all} \;\; (A, b, c) \in \mathcal{U}. \end{array}$$

Example: Interval Uncertainty

Suppose A, c are not affected by uncertainty, but the component b_i is only known to be contained in $[\underline{b}_i, \overline{b}_i]$. Hence we have

$$\mathcal{U} = \{(A_0, b, c_0) : \ \underline{b} \leq b \leq \overline{b}\}.$$

It is obvious that

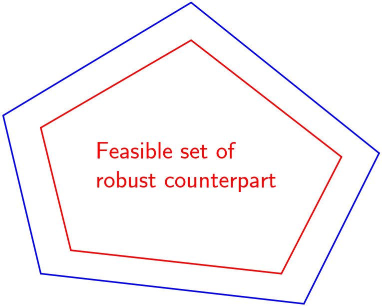
$$A_0 x \leq b \ \text{ for all } b \text{ with } \ \underline{b} \leq b \leq \overline{b} \quad \text{iff} \quad A_0 x \leq \underline{b}.$$

Therefore the robust counterpart is

$$\begin{array}{ll} \text{minimize} & c_0^T x \\ \text{subject to} & A_0 x \leq \underline{b} \end{array}$$

and therefore again an LP.

Other perturbations?



Polytopic Uncertainty

Suppose that the uncertainty is a convex hull with known generators:

$$\mathcal{U} = \mathrm{co}\{(A^1, b^1, c^1), \ldots, (A^N, b^N, c^N)\}.$$

It is obvious that $Ax \leq b,\ c^T x \leq \gamma$ for all $(A, b, c) \in \mathcal{U}$ if and only if

$$A^1 x \leq b^1,\ (c^1)^T x \leq \gamma,\ \ldots,\ A^N x \leq b^N,\ (c^N)^T x \leq \gamma.$$

Hence the robust counterpart is to minimize γ over these latter constraints. This is again a (possibly large sized) LP.

For uncertainty in few parameters (small N) this approach is perfect. Even for simple convex sets the explicit description might require to use many generators (large N) - then this approach is impractical.

There is a beautiful alternative based on **Lagrange dualization**.

General Convex Programming

Let $\mathcal{S} \subset \mathcal{X} = \mathbb{R}^{n \times k}$ be a set of real matrices, and suppose $f : \mathcal{S} \to \mathbb{R}$, $G : \mathcal{S} \to \mathbb{R}^{m \times m}$ (symmetric-valued) and $H : \mathcal{S} \to \mathbb{R}^{p \times q}$ are given.

Consider **primal** optimization problem with optimal value p_{opt}:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{S},\ G(x) \preccurlyeq 0,\ H(x) = 0 \end{array}$$

Is **convex** if $\mathcal{S}$, f, G are convex and H is defined on $\mathcal{X}$ and affine.

It is trivial that the availability of any feasible point allows to compute an **upper bound** on the optimal value: If $x_0 \in \mathcal{S}$ satisfies $G(x_0) \preccurlyeq 0$ and $H(x_0) = 0$ then $f(x_0) \geq p_{\text{opt}}$.

Lagrange duality is often a powerful tool to compute **lower bounds**.

Recap

For real matrices $p \times q$ matrices A and B the standard inner product is defined and denoted as

$$\langle A, B \rangle = \mathrm{trace}(A^T B)$$

The corresponding (Frobenius) norm is $\|A\|_F = \sqrt{\mathrm{trace}(A^T A)}$.

Recall that any linear transformation L from $\mathbb{C}^{p\times q}$ into $\mathbb{C}$ can be represented as $L(X) = \langle Y, X \rangle$ for some matrix $Y \in \mathbb{C}^{p\times q}$.

Observation for real symmetric matrices G and Y:

$$G \preccurlyeq 0 \quad \Longleftrightarrow \quad \langle Y, G \rangle \leq 0 \ \text{ for all } \ Y \succcurlyeq 0.$$

Lower Bounds

Let $x \in \mathcal{S}$ satisfy $G(x) \preccurlyeq 0$ and $H(x) = 0$ (feasible point).

For arbitrary $Y \succcurlyeq 0$ and Z infer $\langle Y, G(x)\rangle \leq 0$ and $\langle Z, H(x)\rangle = 0$. Therefore $f(x) \geq f(x) + \langle Y, G(x)\rangle + \langle Z, H(x)\rangle$ and hence

$$\inf_{x\in\mathcal{S},\, G(x)\preccurlyeq 0,\, H(x)=0} f(x) \geq$$

$$\geq \inf_{x\in\mathcal{S},\, G(x)\preccurlyeq 0,\, H(x)=0} f(x) + \langle Y, G(x)\rangle + \langle Z, H(x)\rangle \geq$$

$$\geq \inf_{x\in\mathcal{S}} f(x) + \langle Y, G(x)\rangle + \langle Z, H(x)\rangle.$$

The best lower bound is obtained by maximization over $Y \succcurlyeq 0$ and Z.

$$\inf_{x\in\mathcal{S},\, G(x)\preccurlyeq 0,\, H(x)=0} f(x) \geq \sup_{Y\succcurlyeq 0,\, Z} \left[\inf_{x\in\mathcal{S}} f(x) + \langle Y, G(x)\rangle + \langle Z, H(x)\rangle\right].$$

Terminology and Observations

- Lagrange function: $L(x, Y, Z) := f(x) + \langle Y, G(x) \rangle + \langle Z, H(x) \rangle$.
- Lagrange dual cost: $l(Y, Z) := \inf_{x \in \mathcal{S}} L(x, Y, Z) \in [-\infty, \infty]$.
- Lagrange dual problem: $d_{\text{opt}} = \sup_{Y \succcurlyeq 0,\, Z} l(Y, Z)$.

Elementary:

- $l(Y, Z)$ computed by solving **unconstrained** problem. Is always **concave** function. Only pairs (Y, Z) with $l(Y, Z) > -\infty$ are interesting.
- Dual problem is **concave maximization** problem. Constraints are often simpler than in primal problem.
- Have **weak duality**: Dual optimal value $\leq$ Primal optimal value.

Fundamental Nontrivial Result: LDT

Lagrange Duality Theorem LDT

Weak duality is always true: $p_{\text{opt}} \geq l(Y, Z)$ for all $Y \succcurlyeq 0$, Z.

Let primal be convex and satisfy **Slater's constraint qualification**:
Exists a relative interior point $\hat{x}$ of $\mathcal{S}$ with $G(\hat{x}) \prec 0$, $H(\hat{x}) = 0$.

Then **strong duality** holds: Exist $Y \succcurlyeq 0$, Z with $l(Y, Z) = p_{\text{opt}}$.

Strong duality can be compactly expressed as

$$\inf_{x \in \mathcal{S},\, G(x) \preccurlyeq 0,\, H(x)=0} f(x) = \max_{Y \succcurlyeq 0,\, Z} \left[\inf_{x \in \mathcal{S}} f(x) + \langle Y, G(x) \rangle + \langle Z, H(x) \rangle \right]$$

Slater without equality constraint: Exists $\hat{x} \in \mathcal{S}$ with $G(\hat{x}) \prec 0$.

Remarks

Can only sketch a few of the very many consequences ...

- $G_0 + G(x) \prec 0$ is often called the strict version of $G_0 + G(x) \preccurlyeq 0$, and $\hat{x}$ as appearing in Slater's constraint qualification is said to satisfy the strict inequality, or to be a strictly feasible point.

  Strict feasibility is essential for LMI algorithms to run well!

- $\hat{x}$ is a relative interior point of $\mathcal{S}$ if there exists some $\epsilon > 0$ with

  $$\{x \in \mathcal{L} : \; \|x - \hat{x}\| < \epsilon\} \subset \mathcal{S},$$

  where $\mathcal{L}$ denotes the affine hull of $\mathcal{S}$ (smallest affine manifold which contains $\mathcal{S}$). If $\mathcal{S}$ has interior points then $\mathcal{L} = \mathcal{X}$, and the definition reduces to the usual one of interior points.

  In practice often have $\mathcal{S} = \mathcal{X}$ ... then all points of $\mathcal{S}$ are interior.

Dualize Standard LMI

If primal strictly feasible then

$$\inf_{F_0 + x_1 F_1 + \cdots + x_n F_n \prec 0} c_1 x_1 + \cdots + c_n x_n =$$

$$= \max_{Y \succcurlyeq 0} \left[\inf_{x_1, \ldots, x_n} \sum_{j=1}^{n} c_j x_j + \langle Y, F_0 + \sum_{j=1}^{n} x_j F_j \rangle \right] =$$

$$= \max_{Y \succcurlyeq 0} \left[\inf_{x_1, \ldots, x_n} \sum_{j=1}^{n} (c_j + \langle Y, F_j \rangle) x_j + \langle Y, F_0 \rangle \right] =$$

$$= \max_{Y \succcurlyeq 0,\ c_j + \langle Y, F_j \rangle = 0,\ j = 1, \ldots, n} \langle Y, F_0 \rangle.$$

Includes extreme case: Primal has value $-\infty$ iff dual is infeasible.

LDT Application: Optimality Conditions

- **Sufficiency:** Suppose there exist $Y_0 \succcurlyeq 0$, Z_0 such that $x_0 \in \mathcal{S}$ satisfies $G(x_0) \preccurlyeq 0$, $H(x_0) = 0$ and is an optimal solution of the **unconstrained problem**

$$\min_{x \in \mathcal{S}} f(x) + \langle Y_0, G(x) \rangle + \langle Z_0, H(x) \rangle$$

  with **complementary slackness**

$$\langle Y_0, G(x_0) \rangle = 0.$$

  Then x_0 is an optimal solution of the primal problem.

- **Necessity:** Suppose x_0 is an optimal solution of the primal problem. If the primal is convex and satisfies Slater, then there exist $Y_0 \succcurlyeq 0$, Z_0 such that x_0 is a solution of the **unconstrained problem** and satisfies **complementary slackness**.

LDT Application: Verifying Set Inclusions

Is the set $\{x \in \mathcal{S} : \ G(x) \preccurlyeq 0\}$ contained in $\{x \in \mathcal{S} : \ f(x) \geq 0\}$?

Equivalent question: Is the value of $\inf_{x \in \mathcal{S},\, G(x) \preccurlyeq 0} f(x)$ nonnegative?

Trivial consequence of weak and strong duality:

- The answer is yes **if** there exists some matrix $Y \succcurlyeq 0$ for which $f(x) + \langle Y, G(x) \rangle \geq 0$ for all $x \in \mathcal{S}$.
- **Iff** holds in above statement in case that $\mathcal{S}$, f, G are convex and there exists $\hat{x} \in \mathcal{S}$ with $G(\hat{x}) \prec 0$.

In control this is often called S-procedure/S-lemma.

SDP Uncertainty in Uncertain LP's

Suppose that LP uncertainty set $\mathcal{U}$ has an LMI-representation: There exist **linear** $A(\delta)$, $b(\delta)$, $c(\delta)$ and $F(\delta)$ (Hermitian-valued) such that

$$\mathcal{U} = \left\{(A^0, b^0, c^0) + (A(\delta), b(\delta), c(\delta)) : \ \delta \in \mathbb{R}^p, \ F_0 + F(\delta) \preccurlyeq 0\right\}.$$

Technical hypothesis: Exists $\hat{\delta}$ with $F_0 + F(\hat{\delta}) \prec 0$.

Main task is to reformulate $Ax \leq b, \ c^T x \leq \gamma$ for all $(A, b, c) \in \mathcal{U}$ or

$$\left.\begin{array}{l} 0 \leq [b^0 + b(\delta)] - [A^0 + A(\delta)]x \\ 0 \leq \gamma - [c^0 + c(\delta)]^T x \end{array}\right\} \text{ for all } \delta \text{ with } F_0 + F(\delta) \preccurlyeq 0.$$

Robust Counterpart with SDP Uncertainty

We investigate the inequalities row-by-row. Represent the ν-th row of $A^0 + A(\delta)$, $b^0 + b(\delta)$, the column $c^0 + c(\delta)$, and $F_0 + F(\delta)$ as

$$a_\nu^0 + \sum_{k=1}^p \delta_k a_\nu^k, \quad b_\nu^0 + \sum_{k=1}^p \delta_k b_\nu^k, \quad c^0 + \sum_{k=1}^p \delta_k c^k, \quad F_0 + \sum_{k=1}^p \delta_k F_k.$$

Now fix the vector x and the index ν. We have to guarantee

$$0 \leq \left[b_\nu^0 + \sum_{k=1}^p \delta_k b_\nu^k\right] - \left[a_\nu^0 + \sum_{k=1}^p \delta_k a_\nu^k\right] x \text{ for all } \delta \text{ with } F_0 + \sum_{k=1}^p \delta_k F_k \preccurlyeq 0$$

or (just reorder)

$$0 \leq \left[b_\nu^0 - a_\nu^0 x\right] + \sum_{k=1}^p \delta_k \left[b_\nu^k - a_\nu^k x\right] \text{ for all } \delta \text{ with } F_0 + \sum_{\nu=1}^p \delta_k F_k \preccurlyeq 0.$$

Robust Counterpart with SDP Uncertainty

A perfect situation to apply convex version of S-procedure (slide 25)!

Condition is true iff there exists $Y_\nu \succcurlyeq 0$ such that

$$0 \leq \left[b_\nu^0 - a_\nu^0 x\right] + \sum_{k=1}^{p} \delta_k \left[b_\nu^k - a_\nu^k x\right] + \operatorname{trace}\left(Y_\nu \left[F_0 + \sum_{k=1}^{p} \delta_k F_k\right]\right)$$

for all $\delta \in \mathbb{R}^p$. After reordering equivalent to

$$0 \leq \left[b_\nu^0 - a_\nu^0 x + \operatorname{trace}(Y_\nu F_0)\right] + \sum_{k=1}^{p} \delta_k \left[b_\nu^k - a_\nu^k x + \operatorname{trace}(Y_\nu F_k)\right].$$

for all $\delta \in \mathbb{R}^p$. Obviously true if and only if

$$0 \leq \left[b_\nu^0 - a_\nu^0 x + \operatorname{trace}(Y_\nu F_0)\right] \quad \text{and}$$

$$\left[b_\nu^k - a_\nu^k x + \operatorname{trace}(Y_\nu F_k)\right] = 0 \quad \text{for} \quad k = 1, \ldots, p.$$

Robust Counterpart with SDP Uncertainty

$$\left.\begin{array}{l} 0 \leq [b^0 + b(\delta)] - [A^0 + A(\delta)]x \\ 0 \leq \gamma - [c^0 + c(\delta)]^T x \end{array}\right\} \text{ for all } \delta \text{ with } F_0 + F(\delta) \preccurlyeq 0$$

true **iff** following LMI system ($\nu = 1, \ldots, m$, $k = 1, \ldots, p$) is feasible:

$$Y_0 \succcurlyeq 0, \quad Y_1 \succcurlyeq 0, \quad \ldots, Y_m \succcurlyeq 0,$$

$$0 \leq \gamma - (c^0)^T x + \mathrm{trace}(Y_0 F_0) \text{ and } -(c^k)^T x + \mathrm{trace}(Y_0 F_k) = 0,$$

$$0 \leq b_\nu^0 - a_\nu^0 x + \mathrm{trace}(Y_\nu F_0) \text{ and } b_\nu^k - a_\nu^k x + \mathrm{trace}(Y_\nu F_k) = 0.$$

Robust LP counterpart

Minimize γ over $x, Y_0, \ldots, Y_m$ satisfying the above LMI system.

Robust counterparts of LP's with SDP uncertainty are SDP's.

Remark

Statement is true in general, and we showed a proof for a **specific** representation of both the LP and the LMI uncertainty. Often however (such as for matrix variables in control), both are formulated differently. Then there are two options:

- Translate your problem into the description that formed our starting point. This involves introducing bases for the underlying vector spaces, and rewriting all affine mappings into matrix-vector multiplications (for LP's) or constructing the coefficients F_k (for SDP's). This procedure is **cumbersome and hides structural properties**.

- Apply duality directly in the vector spaces under consideration. This is **much easier and allows structural insight**. Structure often allows to eliminate variables to reduce the problem size.

Example: Robust Impulse Response Filter Design

With relative implementation errors δ_k solve robust counterpart:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & -\gamma \le h(t_l) - \displaystyle\sum_{k=1}^{n} x_k(1+\delta_k)h_k(t_l) \le \gamma,\ l = 1,\ldots,m \\ & \text{for all } \delta_k \text{ with } \sqrt{\displaystyle\sum_{k=1}^{n} {\delta_k}^2} \le 0.001\, n. \end{array}$$

This perfectly falls into our general problem formulation since

$$\sqrt{\sum_{k=1}^{n} {\delta_k}^2} \le 0.001\, n \quad \text{iff} \quad \begin{pmatrix} 0.001\, nI & \delta \\ \delta^T & 0.001\, n \end{pmatrix} \succcurlyeq 0$$

with $\delta = (\delta_1 \ \cdots \ \delta_n)^T$ is an SDP representation of the uncertainty.

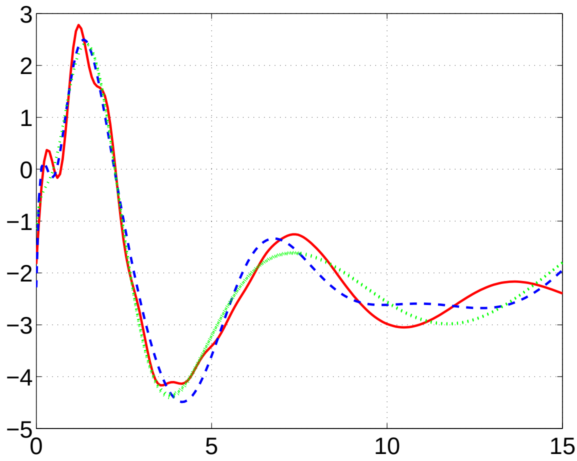
Example: Robust Impulse Response Filter Design

For the same data as in nominal case we performed a robust design with the following result:

Nominal response (red full)

Best approximation (dashed blue)

Robust filter and random deviation by $\pm 0.1\%$ in coeffs (green dotted)



Robust Semidefinite Programming

Consider a standard semi-definite program (SDP)

$$\inf_{A_0+A_1x_1+\cdots+A_nx_n\preccurlyeq 0} c_1x_1+\cdots+c_nx_n$$

with symmetric $A_0,\ A_1,\ \ldots,\ A_n\in\mathbb{R}^{m\times m}$. Collect the data as

$$A=(A_0,A_1,\ldots,A_n)\in\mathbb{R}^{m\times m}\times\cdots\times\mathbb{R}^{m\times m},\quad c=(c_1,\ldots,c_n)\in\mathbb{R}^n.$$

Worst-case uncertainty model: Let us assume that we only know

$$(A,c)\in\mathcal{U}\ \text{ with some subset }\ \mathcal{U}\subset\mathbb{R}^{m\times m}\times\cdots\times\mathbb{R}^{m\times m}\times\mathbb{R}^n.$$

Robust counterpart of SDP: Minimize γ subject to

$$A_0+A_1x_1+\cdots+A_nx_n\preccurlyeq 0,\ \ c_1x_1+\cdots+c_nx_n\leq\gamma\ \text{ for all }\ (A,c)\in\mathcal{U}.$$

Polytopic Uncertainty

Suppose that the uncertainty is a convex hull with known generators:

$$\mathcal{U} = \mathrm{co}\{(A^1, c^1), \ldots, (A^N, c^N)\}.$$

It is as simple to prove as we have seen it for LP's that the robust counterpart of the SDP is the following SDP:

Robust SDP counterpart for polytopic uncertainty

Minimize γ subject to

$$A_0^j + A_1^j x_1 + \cdots + A_n^j x_n \preccurlyeq 0, \;\; c_1^j x_1 + \cdots + c_n^j x_n \leq \gamma, \;\; j = 1, \ldots, N.$$

General approach for finitely generated uncertainty sets.

Practically applicable for moderate number of generators.

Robust counterpart is often intractable $\rightarrow$ Requires relaxation!

Example: Interval Matrices

Consider uncertain time-varying interval system

$$\dot{x}(t) = A(t)x(t) \ \text{ with } \ -D \leq A_0 + A(t) \leq D \ \text{ for all } \ t \in \mathbb{R}.$$

The inequalities are understood elementwise.

Uniformly exponentially stable if there exists a symmetric X with

$$X \succcurlyeq I \ \text{ and } \ A^T X + XA \preccurlyeq -I \ \text{ for all } \ -D \leq A_0 + A \leq D.$$

Set described by $2\,n^2$ inequalities with 2^{n^2} generators:

n	2	4	6	8	10
$2\,n^2$	8	32	72	128	200
2^{n^2}	16	65536	68719476736	$1.8447e+019$	$1.2677e+030$

Direct approach even for small matrices untractable.

Matrix Cube Theorem (Ben-Tal & Nemirovski)

$\mathcal{D} = \{(i,j) : \quad D_{ij} > 0\}$ and $E_{ij} = e_i e_j^T$ with standard unit vectors.

Consider the LMI system in variables X and X_{ij}:

$$X \succcurlyeq I, \quad A_0^T X + X A_0 + \sum_{(i,j)\in\mathcal{D}} X_{ij} \preccurlyeq -I$$

$$X_{ij} \succcurlyeq D_{ij}[E_{ij}^T X + X E_{ij}], \; X_{ij} \succcurlyeq -D_{ij}[E_{ij}^T X + X E_{ij}], \; (i,j) \in \mathcal{D}.$$

- **Obvious:** If the LMI is feasible then $X \succcurlyeq I$ satisfies

$$A^T X + X A \preccurlyeq -I \;\text{ for all } A \text{ with } \; -D \leq A - A_0 \leq D.$$

- If LMI is **not feasible** then **there does not exist** any $X \succcurlyeq I$ with

$$A^T X + X A \preccurlyeq -I \;\text{ for all } A \text{ with } \; -\frac{\pi}{2} D \leq A - A_0 \leq \frac{\pi}{2} D.$$

Relaxations

What was done?

- A computationally complex problem is replaced by a tractable problem that tests a **sufficient** condition.
- If the tractable problem is not solvable, the original problem is not solvable for uncertainties increased by the factor $\pi/2$.

Interpretation: Have constructed a **relaxation** of the untractable problem with a priori guarantees on the **relaxation gap**.

The whole field of robust optimization and robust control lives from constructing suitable relaxations with good approximation properties, even in more general scenarios.