Chapter 1

Glossary

circle power Given a circle C and a point P, let L be the line through the point P that passes through the center of the circle. The A and B be the intersection points of the L with the circle. Define a signed distance $\left\|\overline{PX}\right\|$ for a line segment $\overline{PX}$ to be negative if the line segment lies entirely within the circle, and positive otherwise. The *circle power* of P relative to C, denoted by $pow_C(P)$, is given by:

$$pow_C(P) = \left\|\overline{PA}\right\| \left\|\overline{PB}\right\|.$$

Alternatively, if C is defined implicitly by $(x - x_c)^2 + (y - y_c)^2 = r_c^2$, then the circle power can be expressed as:

$$pow_C(P) = (x - x_c)^2 + (y - y_c)^2 - r_c^2.$$

common power point The point of the plane (or $\mathbb{R}^3$) containing a decorated triangle (or tetrahedron) that has the same circle power with respect to each of the weight circles.

decorated simplex (edge, triangle, tetrahedron,...) A simplex is called *decorated* when weights are assigned to the vertices of the simplex and then actualized by embedding the simplex into Euclidean space together with spheres centered at the vertices with radii determined by the weights. The orthocircle (if one exists) is sometimes considered part of the decorated simplex when appropriate.

edge curvature Given a three-dimensional piecewise flat manifold $(M, \mathcal{T}, d)$, the *edge curvature* along an edge $\{i, j\}$, measures how much that edge differs from Euclidean space. Specifically, the edge curvature K_{ij} is given by

$$K_{ij} = \left(2\pi - \sum_{\substack{k,l, \text{ such that} \\ \{i,j,k,l\} \in \mathcal{T}}} \beta_{ij,kl}\right) l_{ij},$$

where l_{ij} is the edge length, and $\beta_{ij,kl}$ is the dihedral angle of the edge $\{i,j\}$ of the tetrahedron $\{i,j,k,l\}$. In a triangulation of three-dimensional Euclidean space $K_{ij}=0$ for all edges.

Einstein constant For a piecewise flat manifold $(M,\mathcal{T},d)$, the *Einstein constant* λ is given by

$$\lambda = \frac{EHR\left(M,\mathcal{T},d\right)}{3\mathcal{V}},$$

where $EHR\left(M,\mathcal{T},d\right)$ is the Einstein-Hilbert-Regge functional and $\mathcal{V}$ is the total volume.

Einstein metric Given a piecewise flat manifold $(M,\mathcal{T},d)$, we say that d is an *Einstein metric* provided that for all edges $\{i,j\}$ in the triangulation we have:

$$K_{ij} = \lambda l_{ij}\frac{\partial \mathcal{V}}{\partial l_{ij}},$$

where K_{ij} is the edge curvature, l_{ij} is the edge length, $\mathcal{V}$ is the total volume, and λ is the Einstein constant.

hinge A hinge consists of a pair of triangles that share a single edge. Often, two adjacent triangles in a simplicial surface are identified as a hinge while studying the edge they share. Any hinge can be isometrically embedding $\mathbb{R}^2$. There is a natural generalization to three (and higher) dimensions for two tetrahedra sharing a face. Similarly, this generalized hinge can be isometrically embedded in $\mathbb{R}^3$.

inversive distance Start with a decorated edge e_{ij}, that is, an edge of length l_{ij} with weight circles of radius r_i, r_j centered on its vertices. The inversive distance η_{ij} of the edge e_{ij} can be calculated with the formula:

$$\eta_{ij} = \frac{l_{ij}^2 - r_i^2 - r_j^2}{2r_ir_j}.$$

When the two weight circles intersect with angle θ_{ij}, we have the simple formula:

$$\eta_{ij} = \cos\left(\theta_{ij}\right).$$

The former formula was obtained by using the law of cosines and solving for the $\cos\left(\theta_{ij}\right)$ term. When the weight circle do not intersect and do not contain one or the other $\eta_{ij} > 1$. When the weight circles intersect at some angle then $-1 \leq \eta_{ij} \leq 1$. If one weight circle contains the other we have $\eta_{ij} < 1$.

manifold A second countable, Hausdorff topological space M is a *manifold* provided there is an integer $n > 0$ such that for each $x \in M$ there is an open set U_x containing x and a homeomorphism $h_x : U_x \to B\left(1,0\right) \subset \mathbb{R}^n$. A discrete (or piecewise flat) space is a manifold provided the sub-simplices satisfy the following conditions:

Dimension 2:

- All edges have exactly two adjacent faces.
- For a vertex v, the faces incident upon v can be arranged cyclically as $f_1, f_2, ..., f_N, f_1, ...$ so that there is an edge (containing v as an endpoint) between each pair of consecutive faces f_i, f_{i+1}, where $N+1$ is understood to be 1.

Dimension 3:

- All faces have exactly two adjacent tetrahedra.
- For each edge e, the tetrahedra incident upon e can be arranged cyclically as $\sigma_1, \sigma_2, ..., \sigma_M, \sigma_1, ...$ so that there is a face (containing e as an edge) between each pair of consecutive tetrahedra σ_i, σ_{i+1} where $M+1$ is understood to be 1.
- For each vertex v, the number of incident edges, faces and tetrahedra, E, F, T, respectively (including multiple occurrences) satisfy:

$$E - F + T = 2.$$

More generally, given a simplicial manifold M of dimension n, and a sub-simplex σ of dimension $m < n$, the sub-simplices of M of dimension greater than m have the structure of S^{n-m-1}.

orthocircle Given a decorated triangle, provided the common power point is outside all of the weight circles, there exists a circle that is orthogonal to each of the weight circles. That is, the *orthocircle* is the circle that intersects each of the weight circles orthogonally. The orthocircle does not exist when the common power point is on or inside all three circles, however if the common power point is at infinity, there is a line that is orthogonal to the three weight circles which will also be identified as the orthocircle.

piecewise flat manifold A triple $(M, \mathcal{T}, d)$ where $(M, \mathcal{T})$ is a compact triangulated manifold with triangulation $\mathcal{T}$, d is a metric on M so that the restriction of d to each simplex of $\mathcal{T}$ is isometric to a Euclidean simplex of the same dimension.

triangulation A collection of n-dimensional simplices $\mathcal{T}$ together with pairwise identifications for the $(n-1)$-dimensional faces of the simplices. More restrictions are needed to ensure that the resultant space is a manifold. Alternatively, given a space M of dimension n, a *triangulation* of M is a subdivision of M into components $\{\sigma_i\}$ by (hyper)surfaces (of dimension $n-1$) so that each component is homeomorphic to an n-dimensional ball, and each component is combinatorially (as determined by the subdivisions) equivalent to an n-simplex.

pseudo manifold A discrete *pseudo manifold* is a relaxation of the manifold conditions for a discrete space. For dimensions two and three, the bullet conditions given in the entry on manifold may no longer hold. However, the manifold condition is still (trivially) satisfied in the interior of all top dimensional simplices.

weighted triangulation (or hinge) A triangulation together with a map $w : V \to \mathbb{R}$, where V is the set of vertices of the triangulation. Each simplex becomes a decorated simplex.

dimension

metric

curvature

scalar curvature

constant scalar curvature

geometric flow

curvature flow

Yamabe flow

normalized total scalar curvature functional

Einstein-Hilbert functional

Einstein-Hilbert-Regge functional

conformal class

conformal deformation

min-max procedure

Yamabe constant

flip

Pachner move

flip algorithm

convex hinge

nonconvex hinge

bone

Delaunay triangulation (or hinge)

weighted triangulation (or hinge) A triangulation together with a map $w : V \to \mathbb{R}$, where V is the set of vertices of the triangulation. Each simplex becomes a decorated simplex.

weighted Delaunay triangulation

Voronoi diagram (or cell)

weighted Voronoi diagram (or cell)

power diagram (or cell)

negative triangles

dual, Poincare dual

dual length

dual volume