Chapter 1

Optimizing The Geometry With Newtons Method

1.1 Introduction

This tutorial instructs the reader on how to optimize the geometry of a triangulation with respect to the Einstein-Hilbert-Regge functional using Newton's method. The reader will be guided through all necessary steps involving setup, input, execution, and output.

1.2 Setup

The first step is to identify and assign the initial geometric quantities. This step is done by the user outside of the project (usually with a pencil and paper) in order to determine the specific initial geometric quantities. This can be a daunting task due to the large number of geometric quantities in even small triangulations, however a methodically approach and consultation with the documentation of the triangulation being studied will expedite this step.

In this tutorial I will use the triangulation *3-toruspacking*, a rather complex triangulation of the three dimensional torus that admits an Einstein metric. The documentation (not yet written) for this triangulation indicates the following useful information:

Vertices	54
Edges	324
Faces	540
Tetrahedra	270

Furthermore, the documentation indicates that the vertices are arranged by

combinatorial type:

Vertices	L. Vertices	L. Edges	L. Faces	L. Tetra
1-27	18	18	48	32
28-54	6	6	12	8

.

Specifically, the documentation for this triangulation indicates that a stacked hexagonal sphere packing arrangement was used as a basis for this triangulation, creating octahedral subgraphs in the nerve of the packing. A ball was placed at the center of each octahedral subgraph, creating vertices 28-54. The documentation indicates that the zero curvature ball packing ($eta = 1$) metric should assign:

$$\begin{aligned} r_i &= R, \text{ for } 1 \leq i \leq 27 \\ r_i &= \left(\sqrt{2}-1\right) R, \text{ for } 28 \leq i \leq 54. \end{aligned}$$

Verification of the above claim will be the first sub-goal of this tutorial.

To begin, we choose the following initial geometric quantities for trial 1:

$$\begin{aligned} r_i &= 1 \text{ for } 1 \leq i \leq 54 \\ \eta_j &= 1 \text{ for } 1 \leq j \leq 324. \end{aligned}$$

We choose an output file name: trial1.txt.

For trial 2, we will stray from the ball packing metric, and alter a specific tetrahedron's eta values. For no good reason, we choose to alter tetrahedron 1. We begin by inspecting the standard triangulation file for *3-toruspacking;* this file contains the necessary combinatorial information. Once you access this file, locate the entry for Tetra: 1 as copied below:

```
Tetra: 1
8 10 16 20
1 2 3 4 5 6
1 2 3 4
60 66 86 127
```

The four rows of numbers following the heading contain the indices for the local vertices, edges, faces, and tetrahedra for tetrahedron 1. We will alter all of the edges of tetrahedron 1 to have $\eta = .8$, thus we will use the following initial geometric data for trial 2:

$$\begin{aligned} r_i &= 1 \text{ for } 1 \leq i \leq 54, \\ \eta_j &= 1 \text{ for } 1 \leq j \leq 324,\ j \notin \{1,2,3,4,5,6\}, \\ \eta_j &= .8 \text{ for } j \in \{1,2,3,4,5,6\}. \end{aligned}$$

Again we specify an output file for trial 2: trial2.txt.

1.3 Input

Now that trials 1 and 2 have been specified, we proceed by creating an input file for each trial. These input files contain all of the initial geometric quantities as well as the triangulation file, and output file. We create the input file: `trial1input.txt` for trial 1, and `trial2input.txt` for trial 2. Let N be the number of vertices, and M be the number of edges. The input file has the following format:

Line 1:

```
Triangulation: C:/Dec-Cpp/geocam/Triangulation Files/3D Manifolds/Standard Format/*triangulation
```

Line 2:

```
Output: C:/Dec-Cpp/geocam/Triangulation Files/*output file name*.txt
```

Lines 3 through N+2:

```
radius 1: *value*
radius 2: *value*
$...$
radius N+2: *value*
```

Lines N+3 through N+M+2:

```
eta 1: *value*
eta 2: *value*
...
eta M: *value*
```

Using this format, input the geometric quantities determined during setup. We will use `3-toruspacking` for `*triangulation file name*` in line 1 in both trials. For trial 1, we specify `trial1output` for `*output file name*` in line 2, and for trial 2 we use `trial2output`.

1.4 Execution

The program we will run optimizes the Einstein-Hilbert-Regge functional using Newton's method. The default (non-adjustable) accuracy for all trials is set to .00001. That is, the constant scalar curvature ratio $\frac{K_i}{V_i}$ is measured for each vertex, and the program terminates when these ratios are within .00001 of each other.

Execution is performed by executing the file: `NewtonsMethodForPoets.exe`

The user will be prompted for the file name of the input file for the trial to be run.

For this tutorial, we execute `NewtonsMethodForPoets.exe` twice, once for trial 1 and once for trial 2.

1.5 Output

The output from execution is located in the file specified in the input files for trials 1 and 2. Inspect, process and enjoy!