Orderings

- The ability to order data is one of the revolutions introduced by computers.
  - search cannot be performed effectively if data is not ordered;
  - program analysis, for example for proving termination, requires an ordering to measure progress towards a final state.

Partial and total orderings

- A *partial order* on a set A is a relation $R \subseteq A \times A$ such that:
  - R is *reflexive*
  - R is *antisymmetric*:

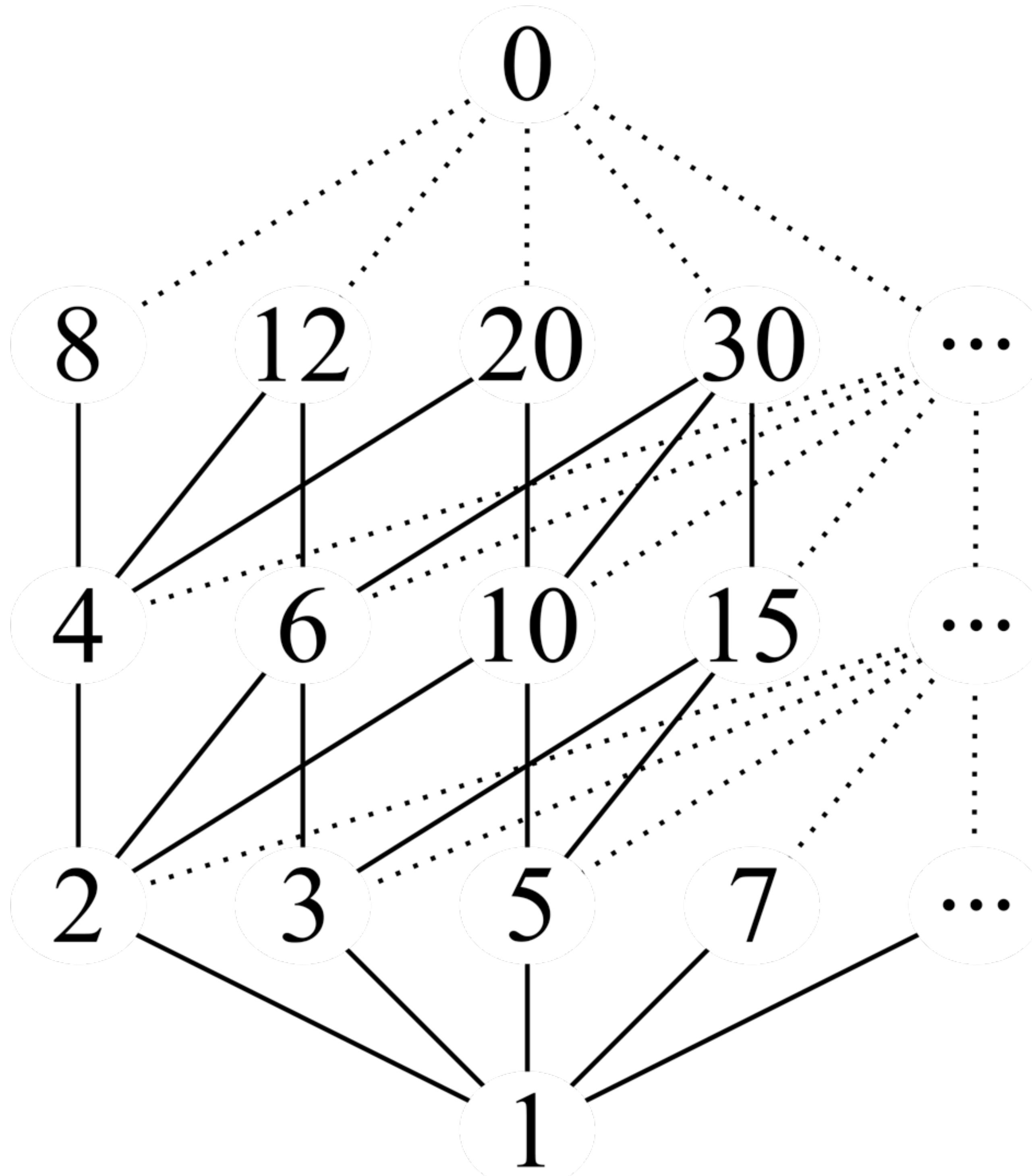
    for all $a, b \in A$, if $a \mathrel{R} b$ and $b \mathrel{R} a$ *then* $a = b$.
  - *R is transitive*.
- A *total order* on a set A is a partial order such that, for all $a, b \in A$, either $a \mathrel{R} b$ or $b \mathrel{R} a$.

José Fiadeiro – 2013/14

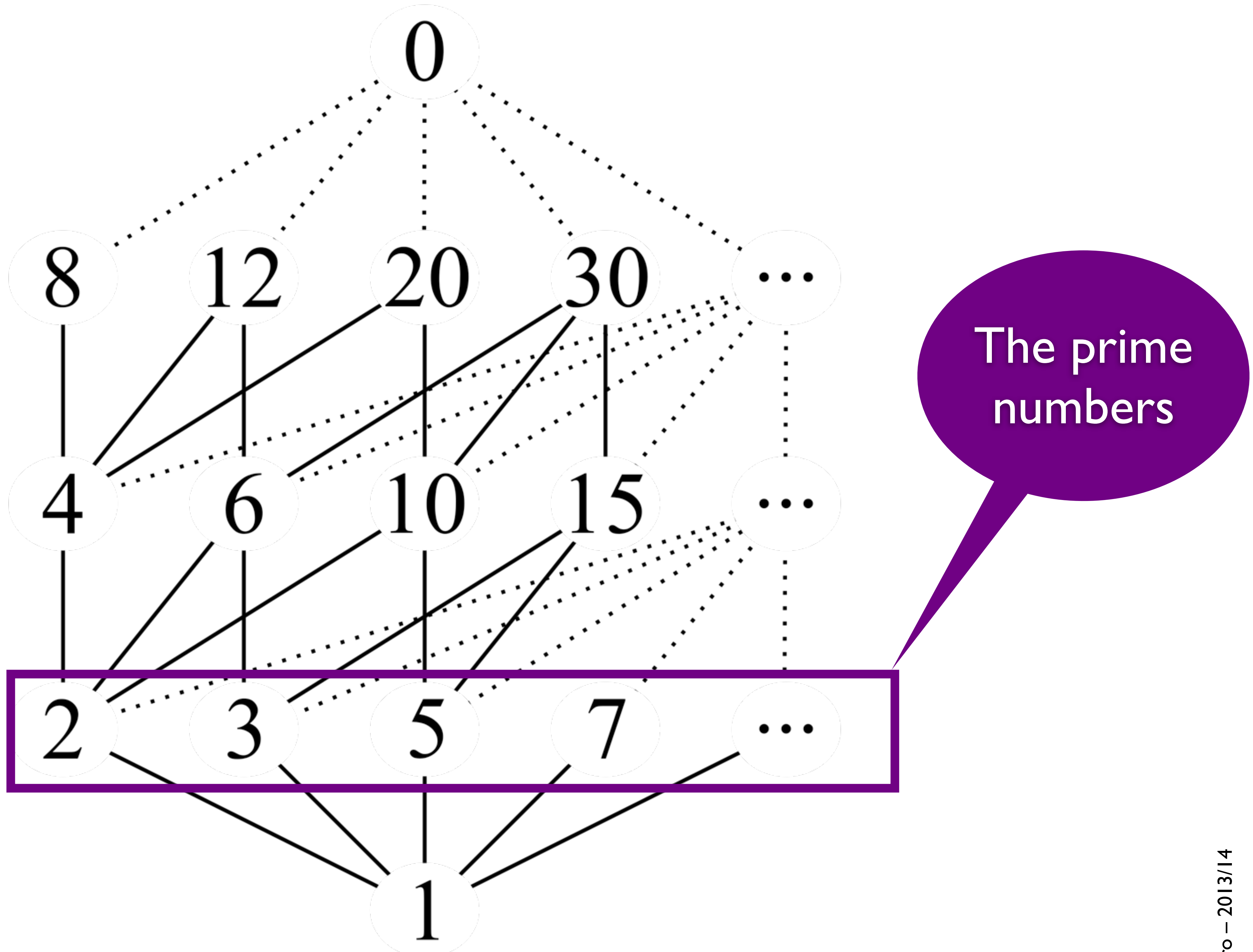
Partial and total orderings

- For example,
  - $\leq$ over the integers is a total order.
  - *x divides y* over the natural numbers is a partial order.

Partial and total orderings



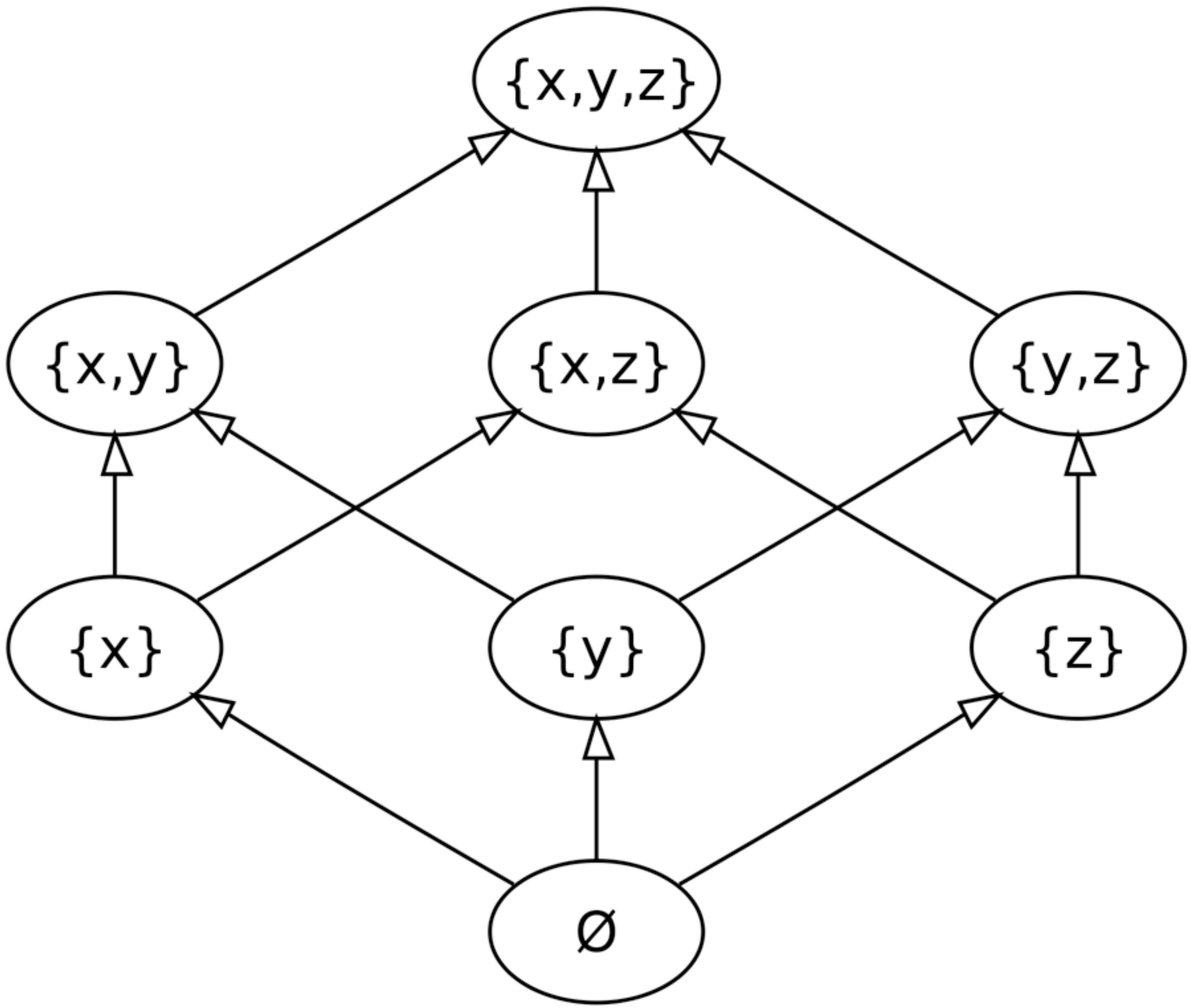
Partial and total orderings



Partial and total orderings

- For example,
  - $\leq$ over the integers is a total order.
  - x *divides* y over the natural numbers is a partial order.
  - $\subseteq$ is a partial order on 2^A.

Partial and total orderings



Strict orderings

- A *strict (partial) order* on a set A is a relation $R \subseteq A \times A$ such that:
  - R is *irreflexive* – for all $a \in A$, $(a,a) \notin R$
  - R is *antisymmetric*.
  - R *is transitive*.
- For example, < is a strict ordering of the integers.

Not the same as not being reflexive

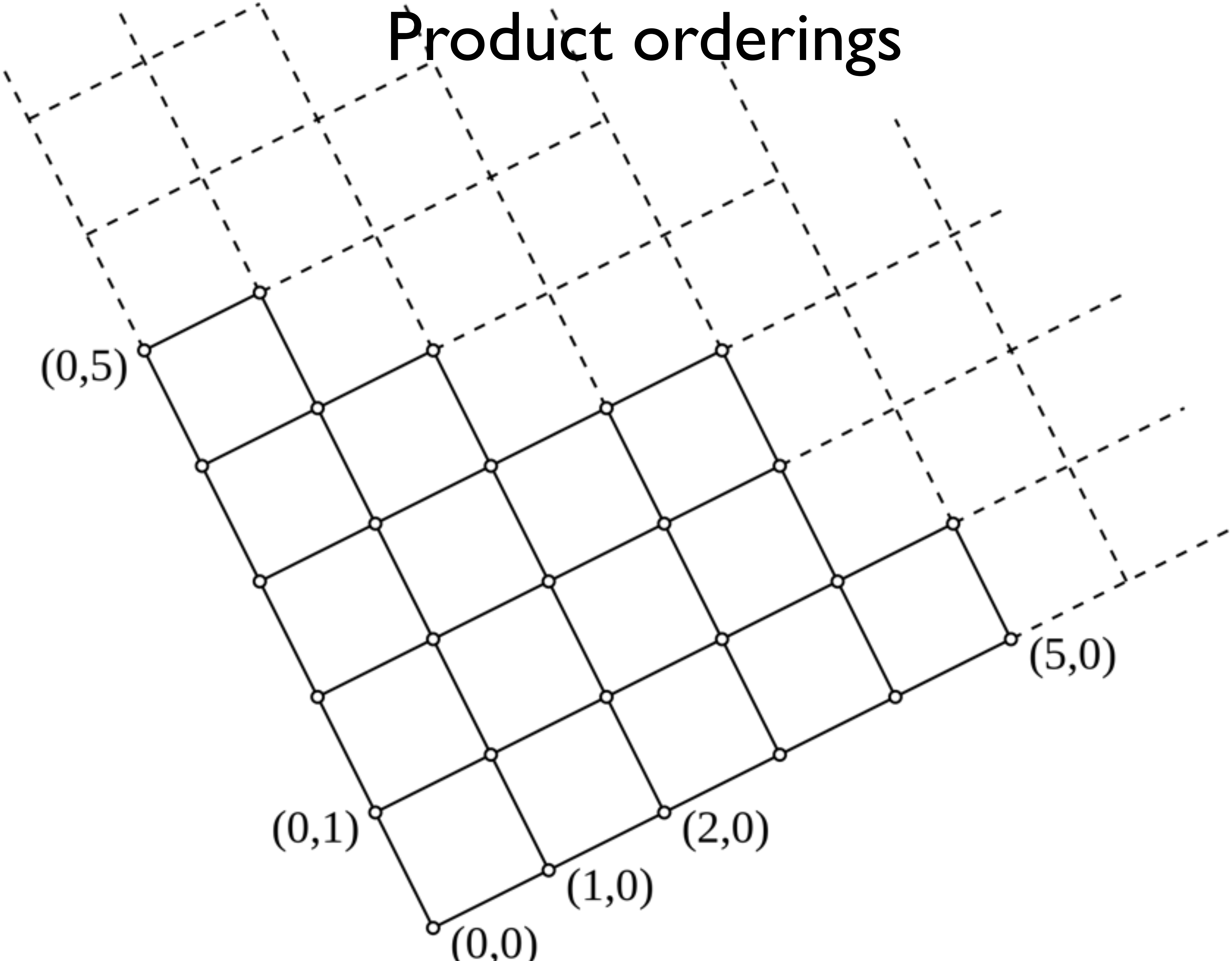
Exercise

- If $\leq$ is a partial order, then $a < b$ iff $a \leq b$ and $a \neq b$ defines a strict order.
- If $<$ is a strict order, then $a \leq b$ iff $a < b$ or $a = b$ defines a partial order.

Product orderings

- If we have a partial order ≤ on a set A, we can extend it to the *product order* on $A \times A$ as follows
  - $(a_1,a_2) \le (b_1,b_2)$ *iff* $a_1 \le b_1$ *and* $a_2 \le b_2$

Product orderings



Lexicographic orderings

- If we have a partial order $\leq$ on a set A, we can extend it to the *lexicographic order* on $A \times A \times \ldots \times A$ as follows
  - $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n)$ *iff*
    - $a_1 < b_1$, *or*
    - $a_1 = b_1$ *and* $a_2 < b_2$, *or*
    - $a_1 = b_1$ *and* $a_2 = b_2$ *and* $a_3 < b_3$, *or*
    - $\ldots$, *or*
    - $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$

Lexicographic orderings