Example

- $(\mathcal{R}^+_0,\leq)$ is not well-founded.
  - For example, $\{1/2^n \mid n\in\mathcal{N}\}$ does not have a minimal element.
- However, 0 is "almost" a minimal element.

José Fiadeiro – 2017/18

Upper and lower bounds

- Let ≤ be a partial ordering on a set A and $B \subseteq A$
  - a is said to be an *upper bound* of B if for all $b \in B$ $b \leq a$;
    - a is the *least upper bound* of B if it's smaller than every other upper bound
  - a is said to be a *lower bound* of B if for all $b \in B$ $a \leq b$;
    - a is the *greatest lower bound* of B if it's bigger than every other lower bound
- For example, 0 is the greatest lower bound of $\{1/2^n \mid n \in \mathcal{N}\}$ within $(\mathcal{R}+0, \leq)$.

Lattices

- A poset $(A, \leq)$ is a *lattice* if every two elements a and b have:
  - a least upper bound, denoted $a \vee b$
  - a greatest lower bound, denoted $a \wedge b$
- For example, if A is a set, $(2^A, \subseteq)$ is a lattice
  - the least upper bound of any two subsets is their union
  - and the greatest lower bound is their intersection

Example

- A software company supports four text editors (*objects*), the set of which is $\mathbf{G}=\{e_1, e_2, e_3, e_4\}$, which share eight features (*attributes*), the set of which is $\mathbf{M}=\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$.
- Whenever a change needs to be made to one of those features, it is important for the company to know which editors use it, which it does through the following relation (*incidence*) **I**

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
e_1	1	1	1	1	0	0	0	0
e_2	1	1	1	1	0	0	1	1
e_3	1	1	1	1	1	1	0	0
e_4	1	1	1	1	1	1	1	1

Example

- Given a set of objects $A \subseteq G$, we define the set A' of all the attributes that are common to the objects in A, i.e.

$$A' = \{m \in M \mid \textit{for all } g \in A,\ g\ I\ m\}$$

- Given a set of attributes $B \subseteq M$, we define the set B' of all the objects have all the attributes in B, i.e.

$$B' = \{g \in G \mid \textit{for all } m \in B,\ g\ I\ m\}$$

Example

- A *formal concept* is a pair (A,B) where A ⊆ G and B ⊆ M such that $A' = B$ and $B' = A$. The set A is called the *extent* of the concept, and the B is called its *intent*.
- For example, $A=\{e_2, e_4\}$ and $B=\{f_1, f_2, f_3, f_4, f_7, f_8\}$ define a formal concept.

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
e_1	1	1	1	1	0	0	0	0
e_2	1	1	1	1	0	0	1	1
e_3	1	1	1	1	1	1	0	0
e_4	1	1	1	1	1	1	1	1

Concept lattice

- Given a set G (of objects), a set M (of attributed), and a relation I (incidence) between G and M, we define the following partial ordering on the formal concepts

$$(A_1,B_1) \leq (A_2,B_2) \;\; iff \;\; A_1 \subseteq A_2 \;\; (iff \;\; B_2 \subseteq B_1)$$

- This partial order defines a lattice.

José Fiadeiro – 2017/18

Example

- The formal concepts are:
  - c_1 : $(\{e_1, e_2, e_3, e_4\}, \{f_1, f_2, f_3, f_4\})$
  - c_2 : $(\{e_2, e_4\}, \{f_1, f_2, f_3, f_4, f_7, f_8\})$
  - c_3 : $(\{e_3, e_4\}, \{f_1, f_2, f_3, f_4, f_5, f_6\})$
  - c_4 : $(\{e_4\}, \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\})$

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
e_1	1	1	1	1	0	0	0	0
e_2	1	1	1	1	0	0	1	1
e_3	1	1	1	1	1	1	0	0
e_4	1	1	1	1	1	1	1	1

- The ordering is as follows: