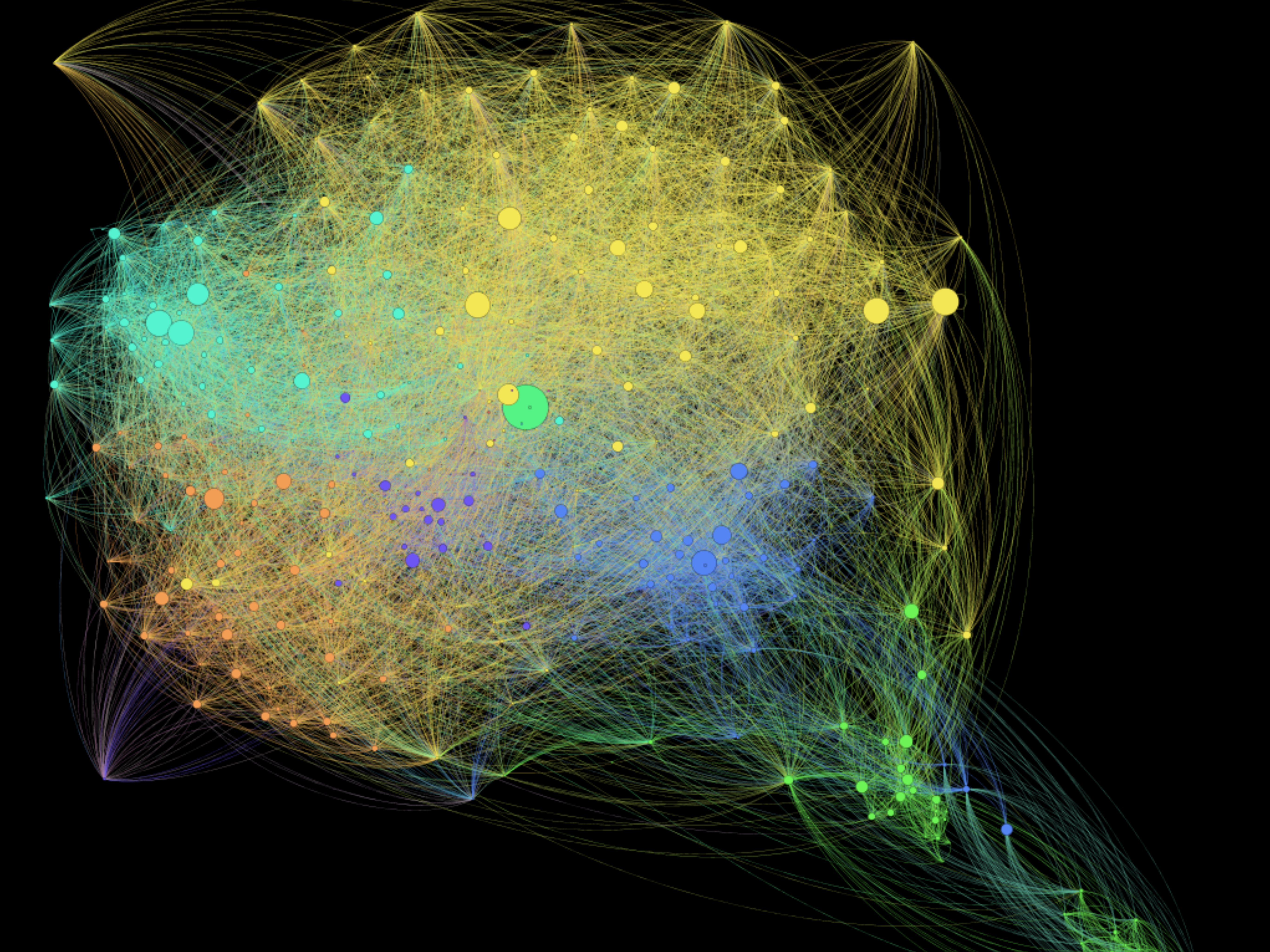
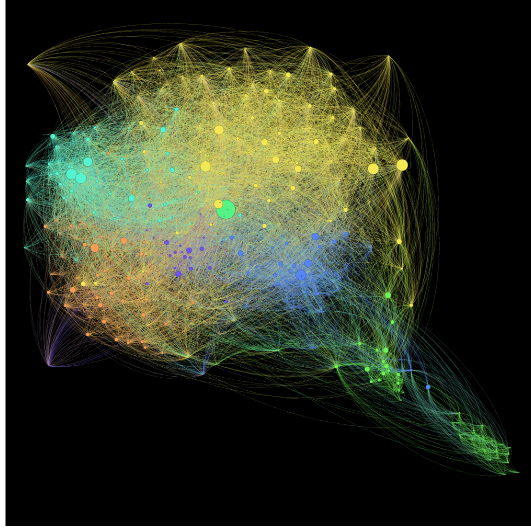
Relations

- Sets and set membership give us a way of classifying and organising information in classes (or types).
- Yet, we are often more interest in looking at objects in terms of how they relate to other objects, i.e., in their *social lives*.



Relations



- In a project of CS4234 (Large-scale Data Storage and Processing), students last year:
  - Analysed the Enron Corpus – a database of over 600,000 emails generated by 158 employees of Enron Corporation.
  - Used MapReduce and Hadoop for: (1) cleaning the dataset, (2) extracting a social network graph induced by the individual emails, and (3) analysing its structural properties.
  - Used Gephi – a social network visualisation software – to glean insight into social relationships between the individual employees within the organisation.

Relations

- The mathematical notion that captures relationships between objects is that of a *relation*. For example,
  - José *teaches* CS1860
  - Adrian *teaches* CS1801
  - Dave *teaches* CS1820
  - Carlos *teaches* CS1890
  - Carlos *teaches* CS1840
  - Elizabeth *teaches* CS1870
  - …

Relations

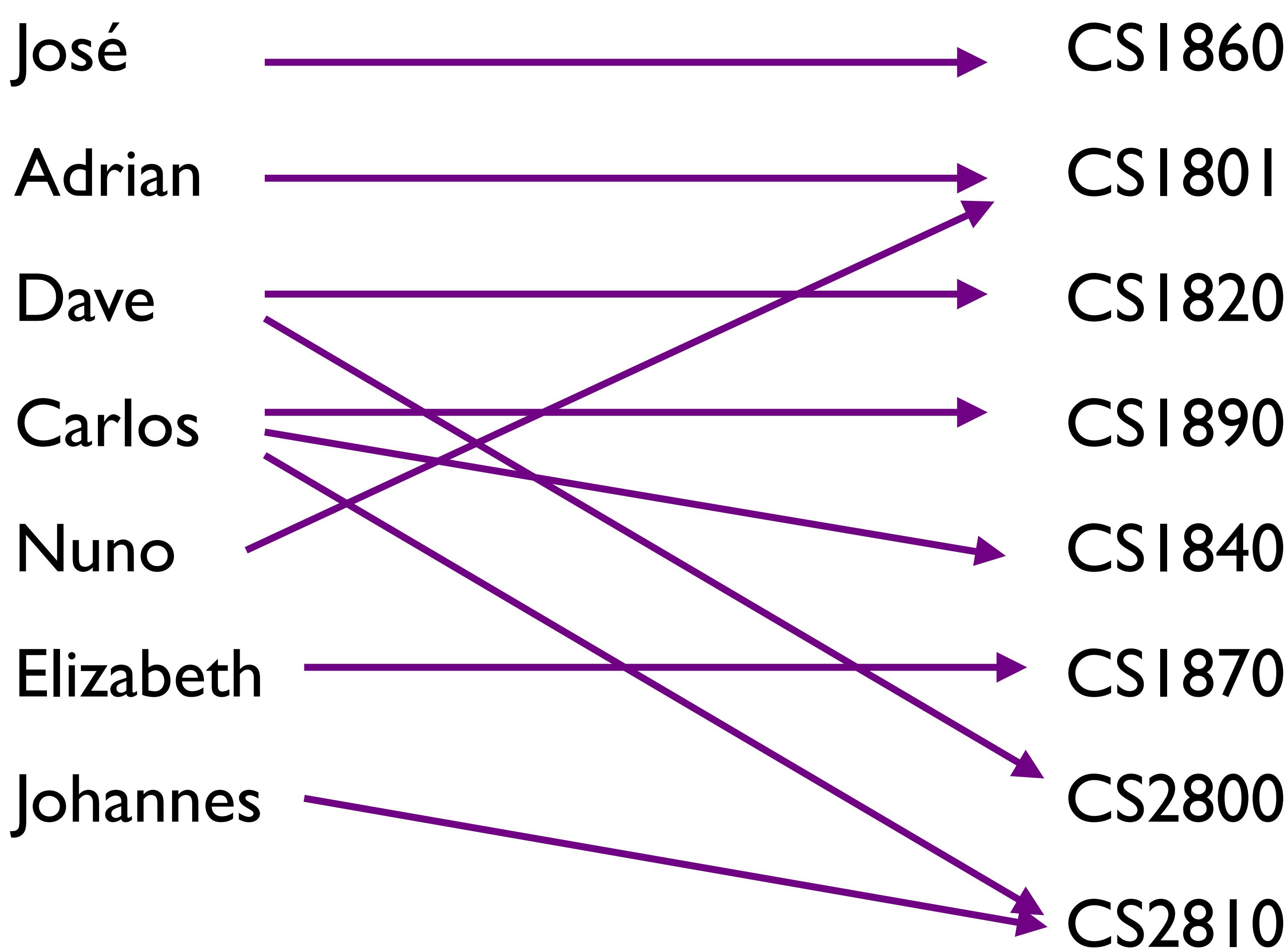
- Binary relations such as *teaches* can be described through the pairs of objects that are in relation to each other
  - (José, CS1860)
  - (Adrian, CS1801)
  - (Dave, CS1820)
  - (Carlos, CS1890)
  - (Carlos, CS1840)
  - (Elizabeth, CS1870)
  - …

Relations

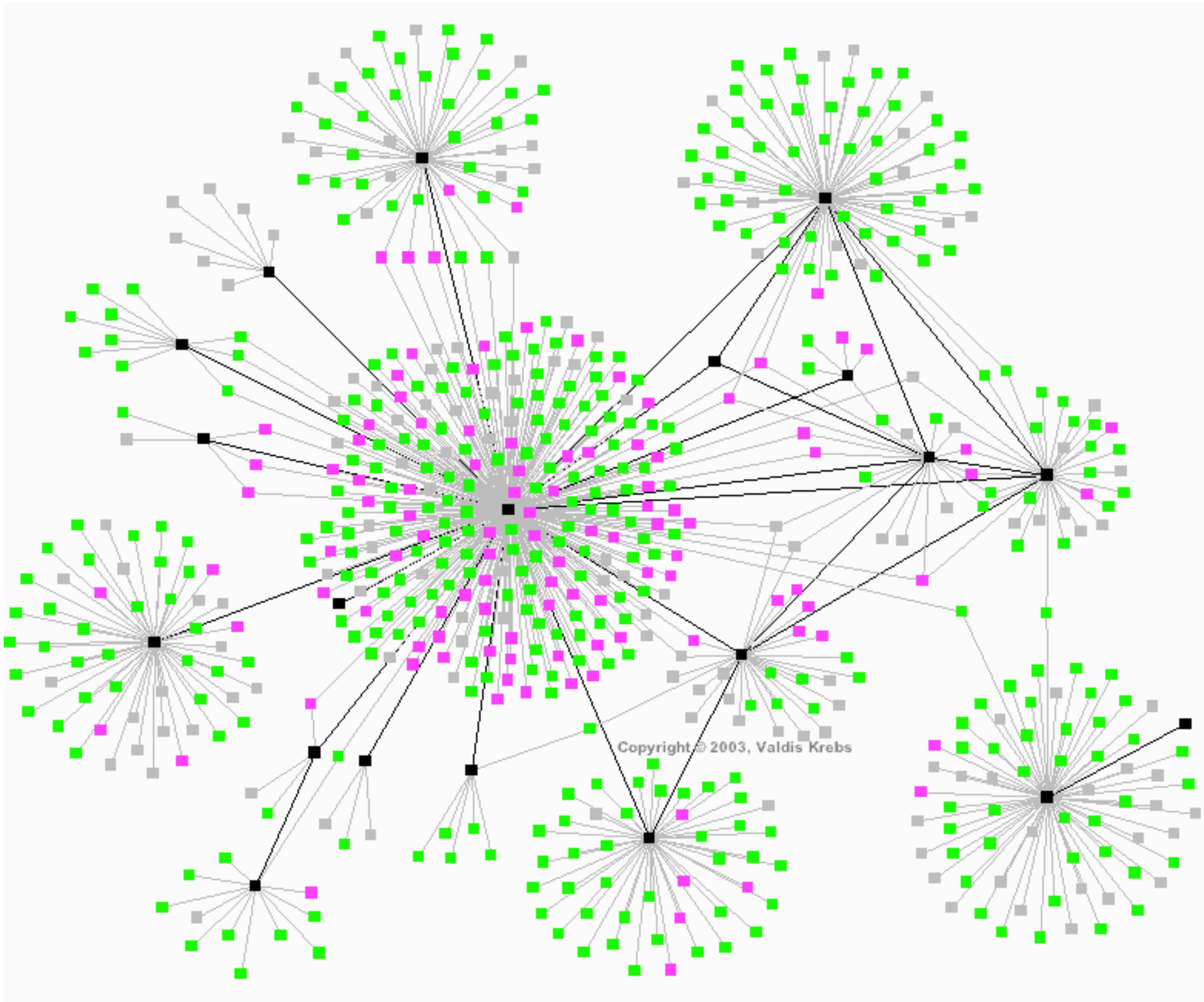
- A binary relation R is thus a subset of the cartesian product $S \times T$ of two sets, each set representing a class of objects.
  - $teaches \subseteq Lecturer \times Course$
    - (José, CS1860) ∈ teaches
    - (Adrian, CS1801) ∈ teaches
    - (Dave, CS1820) ∈ teaches
    - (Carlos, CS1850) ∈ teaches
    - …

Graphical representation

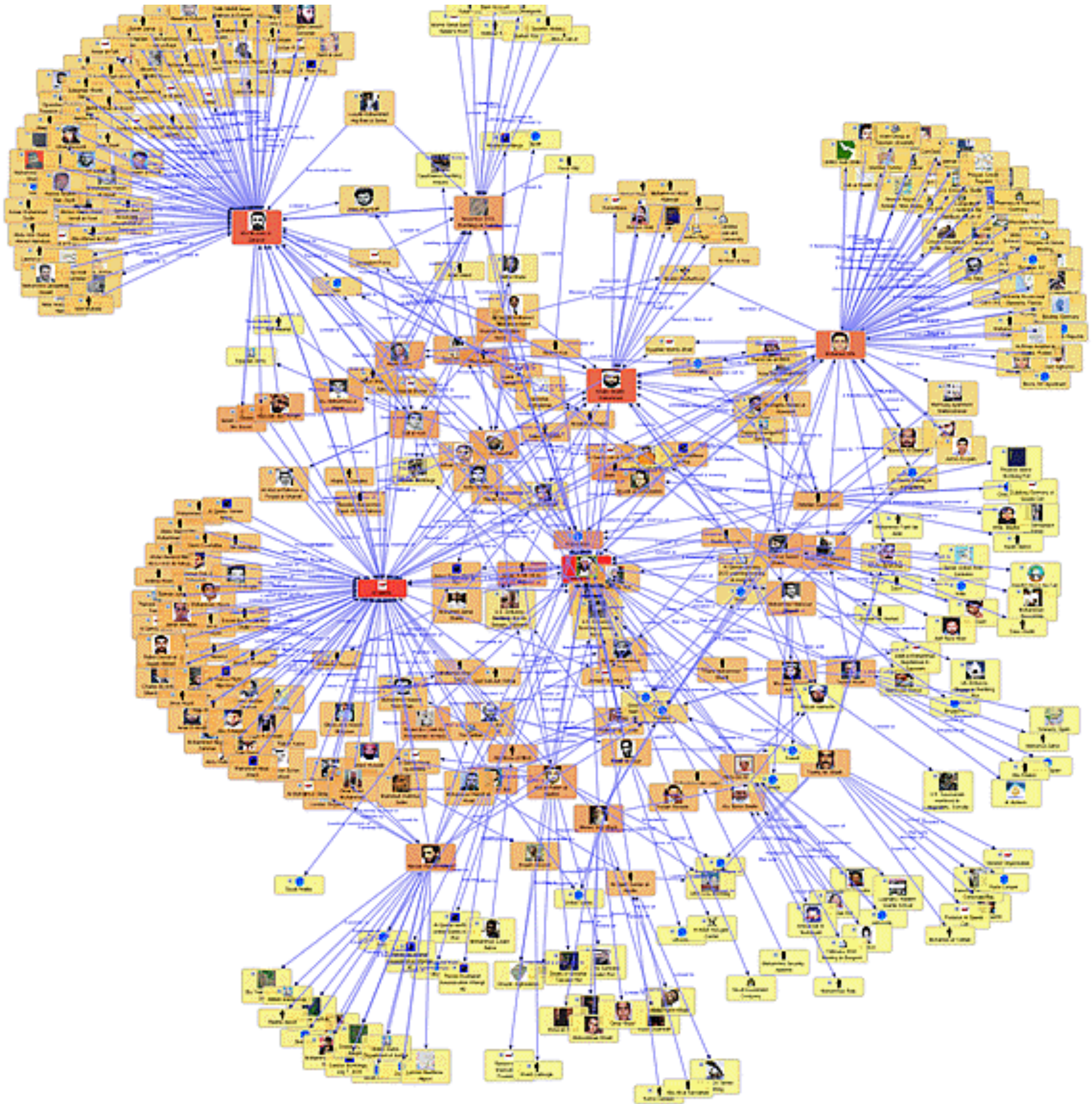
- Some relations can be represented graphically:



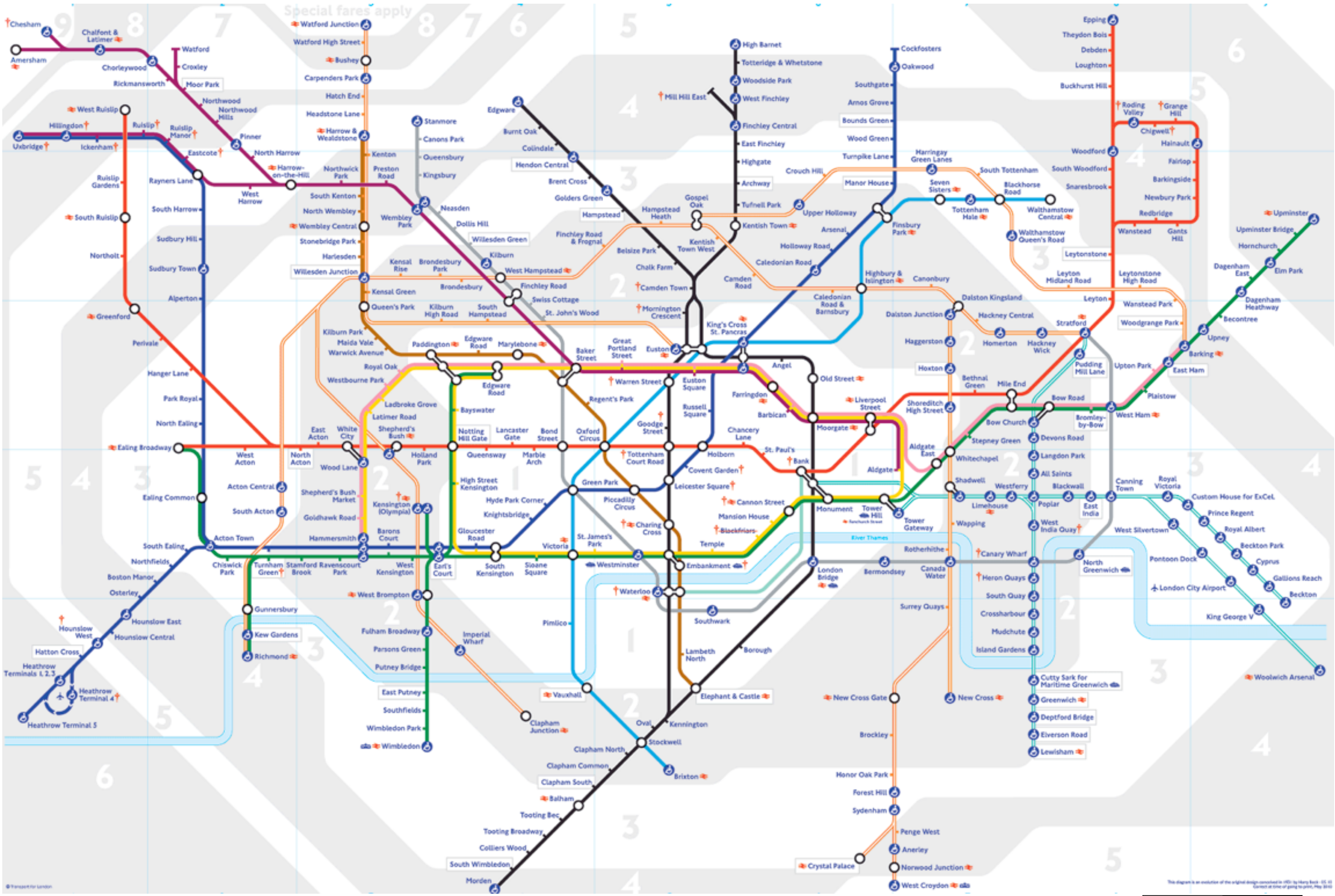
Graphical representation



Graphical representation



Graphical representation



Matrix representation

- Alternatively, we can use a matrix:

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} \text{CS1860} & \text{CS1801} & \text{CS1820} & \text{CS1890} & \text{CS1840} & \text{CS1870} & \text{CS2800} & \text{CS2810} \end{array} \\
\begin{array}{l} \text{José} \\ \text{Adrian} \\ \text{Dave} \\ \text{Carlos} \\ \text{Nuno} \\ \text{Elizabeth} \\ \text{Johannes} \end{array} &
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

Inverse relation

- For every relation $R \subseteq S \times T$, its inverse $R^{-1} \subseteq T \times S$ is defined by $a\ R^{-1}\ b$ *iff* $b\ R\ a$.
  - For example, the inverse of *teaches* $\subseteq$ *Lecturer* $\times$ *Course* is *is_taught_by* $\subseteq$ *Course* $\times$ *Lecturer*:
    - CS1860 *is_taught_by* José
    - CS1801 *is_taught_by* Adrian
    - CS1820 *is_taught_by* Dave
    - CS180 *is_taught_by* Carlos
    - CS1840 *is_taught_by* Carlos
    - …

Show that $(R^{-1})^{-1} = R$

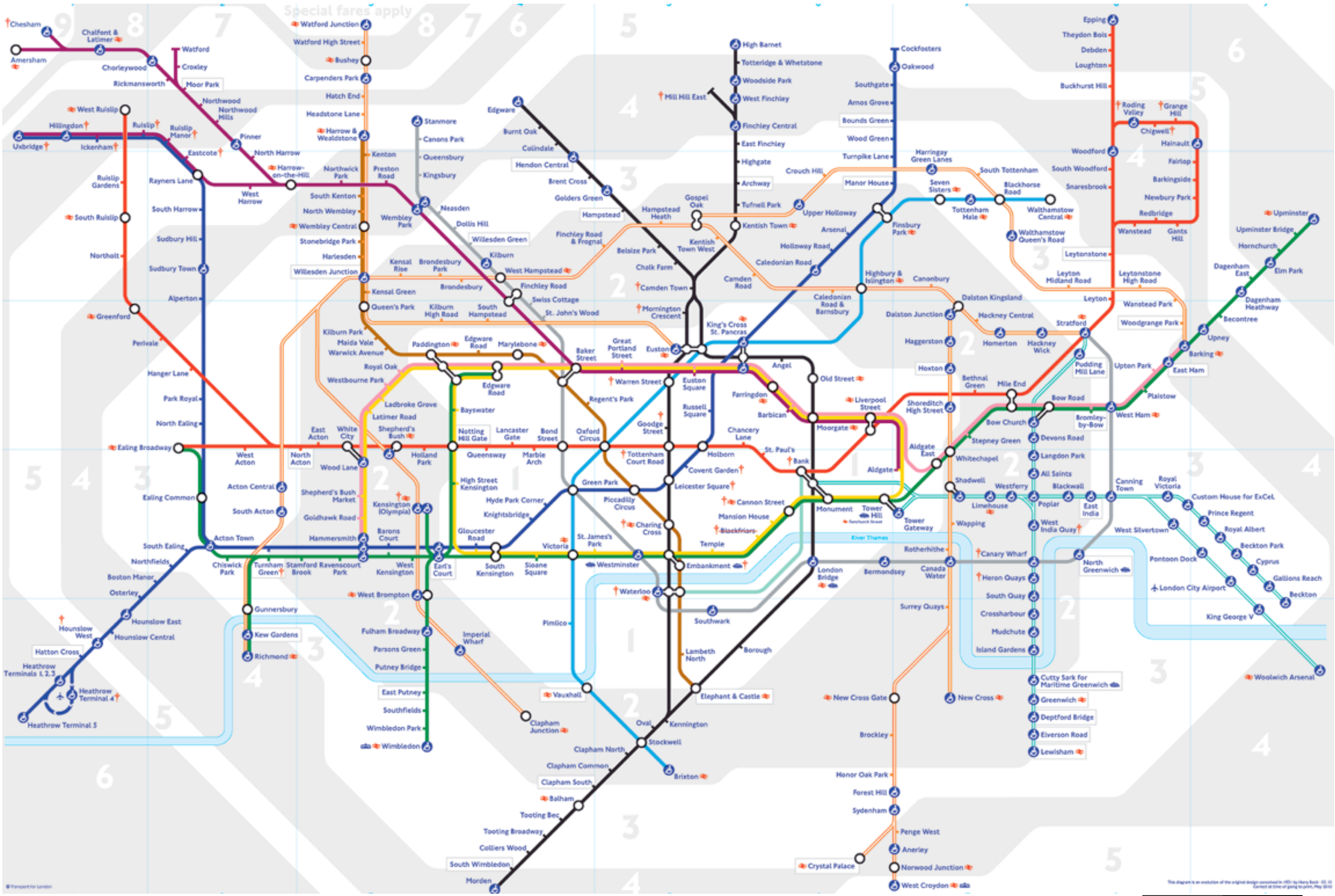
Complement relation

- For every relation $R \subseteq S \times T$, its *complement* $\overline{R} \subseteq S \times T$ is defined by $(a,b) \in \overline{R}$ *iff* $(a,b) \notin R$.
  - For example, the complement of *teaches* ⊆ *Lecturer* × *Course* is *does_not_teach* ⊆ *Lecturer* × *Course*:
    - José *does_not_teach* CS1801
    - Adrian *does_not_teach* CS1860
    - Dave *does_not_teach* CS1860
    - Carlos *does_not_teach* CS1820
    - …

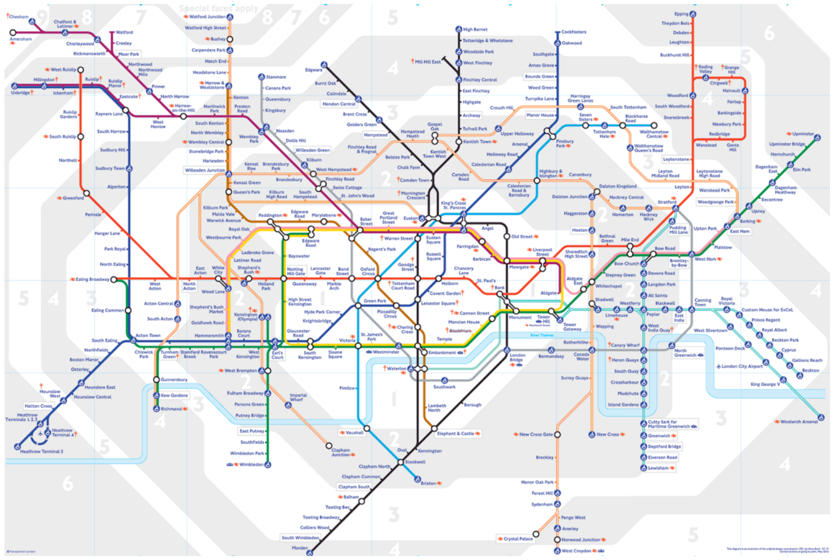
Relation on a set

- If $R \subseteq A \times A$, we say that R is a relation on A.

Properties of relations

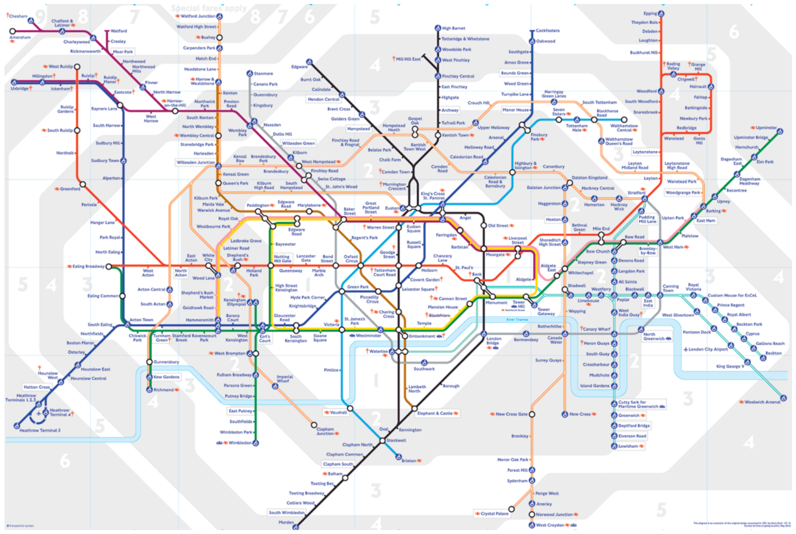


Properties of relations



- The relation defined by the graph is
  - *is_one_stop_from* $\subseteq$ *Station* × *Station*
- Another interesting relation is
  - *is_connected_to* $\subseteq$ *Station* × *Station* defined by:
    - a *is_connected_to* *b iff there exist* $a_0, \ldots, a_n$ *such that* $a_0 = a$, $a_n = b$ *and, for every* $0 \leq i < n$, a_i *is_one_stop_from* a_{i+1}
  - What interesting properties does this relation have?

Transitive



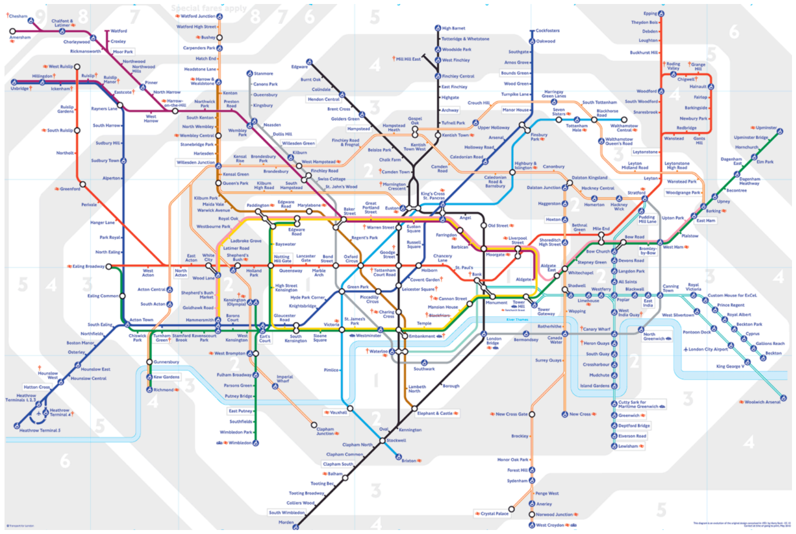
- The relation *is_connected_to* satisfies:
  - if a *is_connected_to* b and b *is_connected_to* c then a *is_connected_to* c

- A relation $R \subseteq A \times A$ is *transitive* iff, for all $a, b, c \in A$, $a\ R\ b$ and $b\ R\ c$ implies $a\ R\ c$.

- The *transitive closure* of $R \subseteq A \times A$ is $R^+ \subseteq A \times A$ defined by:
  - a R^+ b *iff there exist* $a_0, \ldots, a_{n+1}$ *such that* $a_0 = a$, $a_{n+1} = b$ *and, for every* $0 \leq i \leq n$, $a_i\ R\ a_{i+1}$

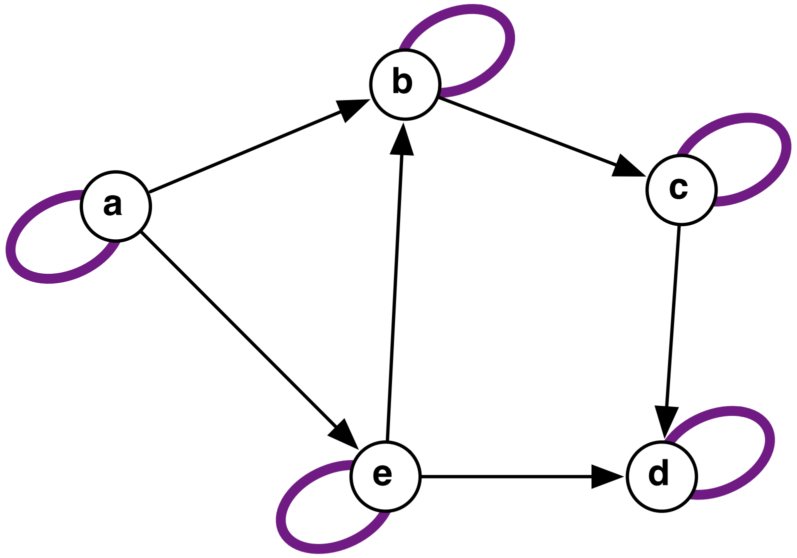
Is it transitive?

Reflexive

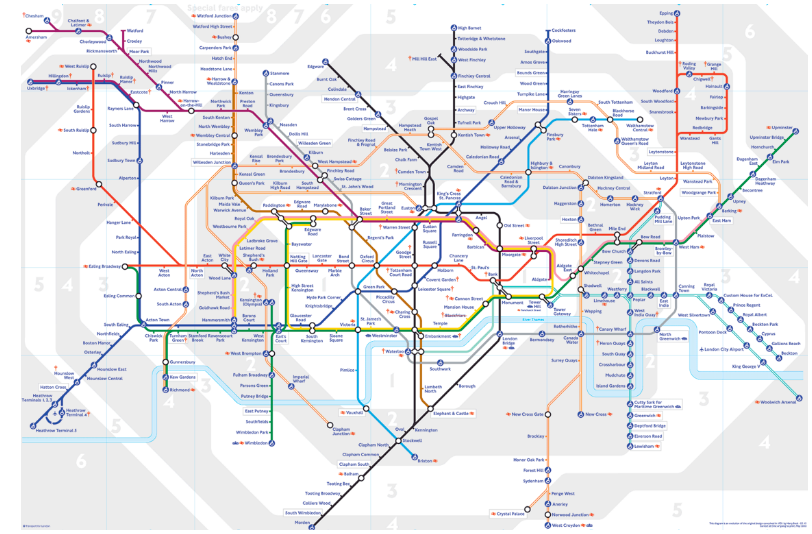


- The relation *is_connected_to* satisfies:
  - for all a, a *is_connected_to* a
- A relation $R \subseteq A \times A$ is *reflexive* iff, for all $a \in A$, $a\ R\ a$.
- It is normally easy to detect that a relation is reflexive:

$$\begin{bmatrix} 1 & ? & ? \\ ? & 1 & ? \\ ? & ? & 1 \end{bmatrix}$$



R*

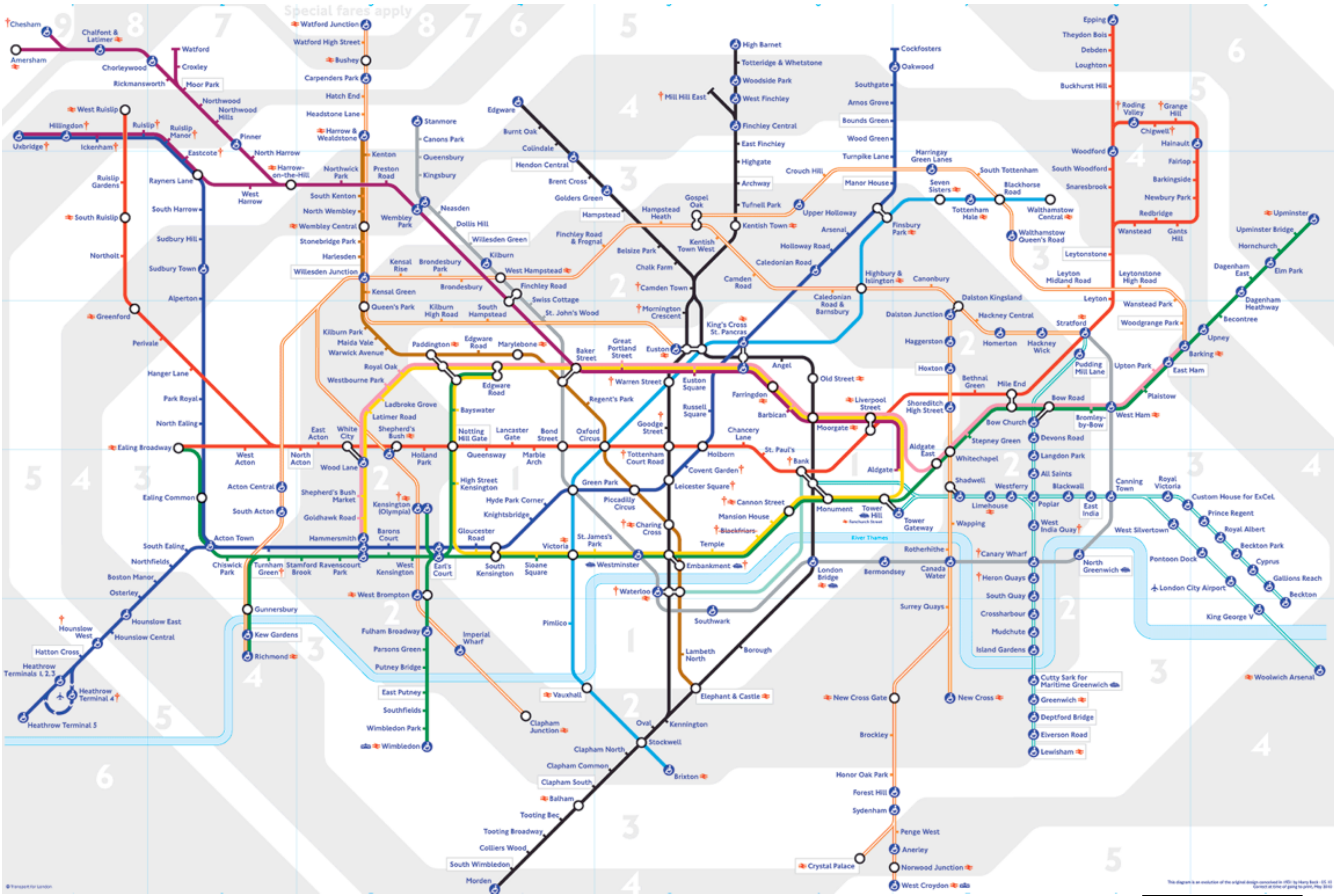


- The *transitive and reflexive closure* of $R \subseteq A \times A$ is $R^* \subseteq A \times A$ defined by:
  - a R^* b *iff there exist* $a_0, \ldots, a_n$ *such that* $a_0 = a$, $a_n = b$ *and, for every* $0 \leq i < n$, $a_i \; R \; a_{i+1}$
- *is_connected_to* = *is_one_stop_from**

Is it transitive?

Is it reflexive?

Question



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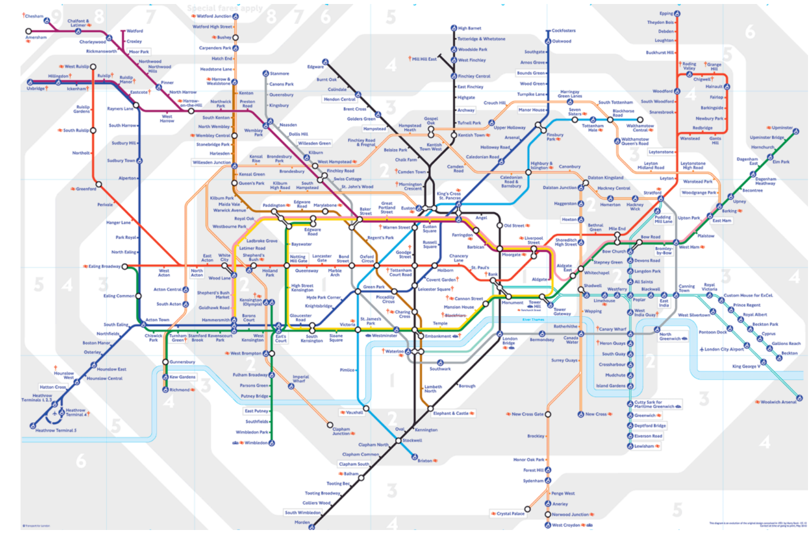
Question

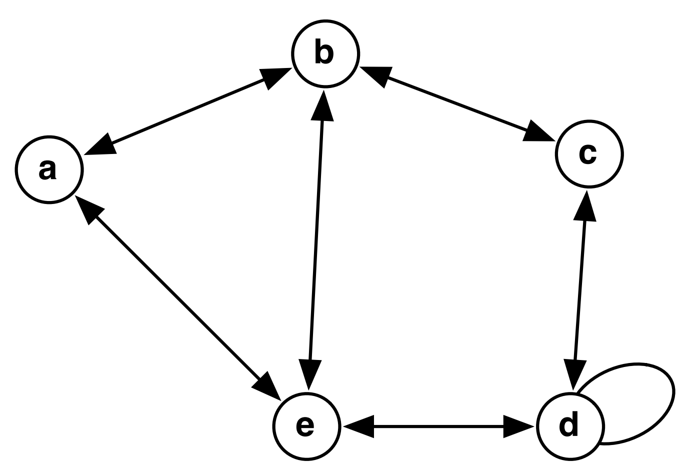
- Why are there no arrows on the map?

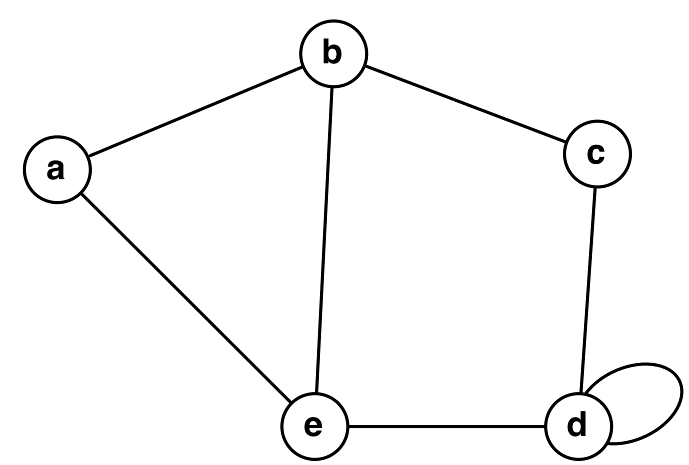
Symmetric



- The relation *is_connected_to* satisfies:
  - if *a* *is_connected_to* *b* then *b* *is_connected_to* *a*
- A relation $R \subseteq A \times A$ is *symmetric* iff, for all $a, b \in A$, $a\ R\ b$ implies $b\ R\ a$.
- It is normally easy to detect that a relation is symmetric:

$$\begin{bmatrix} ? & x & y \\ x & ? & z \\ y & z & ? \end{bmatrix}$$





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José Fiadeiro – 2014/15

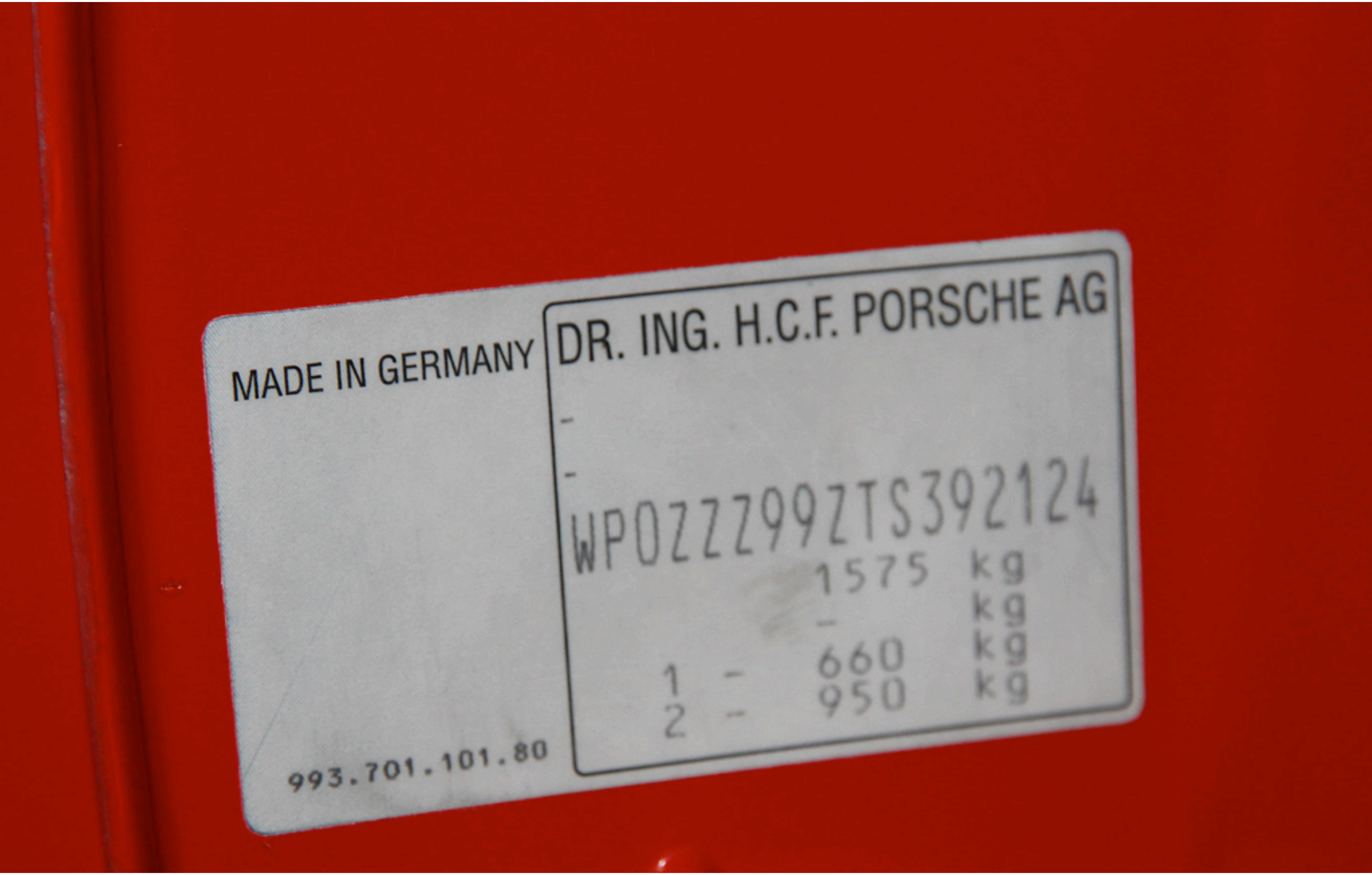
Equivalence relations

- A relation $R \subseteq A \times A$ that is reflexive, symmetric and transitive is said to be an *equivalence relation* on A.
- For example:
  - *is_connected_to*
  - $id_A = \{(a,a) \mid a \in A\}$ – the *identity relation* on A
    This is the smallest equivalence relation on A in the sense that, for every $R \subseteq A \times A$, if R is an equivalence relation then $id_A \subseteq R$
  - *neighbour* $\subseteq Person \times Person$
    $= \{(a,b) \mid a$ and b have the same postcode $\}$

Abstraction

- Equivalence relations are an *abstraction* of identity — for some reason, we want to identify objects that are equivalent.
- For example,
  - $same\_value \subseteq BankNote \times BankNote$ — any two ten pound notes are "the same" (unless you are Paddington Bear).
  - $neighbour \subseteq Person \times Person$ — sometimes we want to reason about properties of people who live in the same postcode independently of who they are.
  - $car\_I\_want\_to\_buy \subseteq Car \times Car$ — we normally abstract using certain attributes (make, colour, …) and do not insist on a particular vehicle identification number.

Abstraction



Equivalence classes

- Given an equivalence relation $R \subseteq A \times A$, we group all equivalent objects in the same *equivalence class*:
  - For every $a \in A$, $[a]_R = \{b \in A \mid (a,b) \in R\}$
  - $A/_R = \{ [a]_R \mid b \in A \}$ is called the *quotient set* of A by R.
- For example,
  - $BankNote/_{same\_value}$ has four elements corresponding to *five_pounds, ten_pounds, twenty_pounds, fifty_pounds*
  - What is $A/_{id}$?