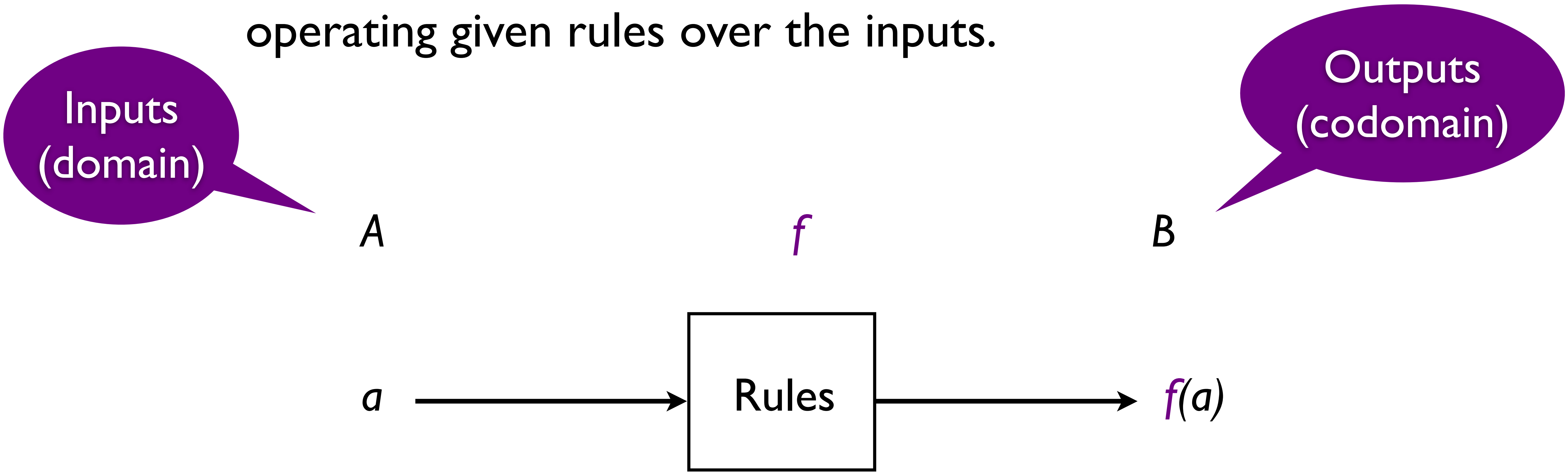
Functions

- Some relations relate inputs with outputs produced by operating given rules over the inputs.

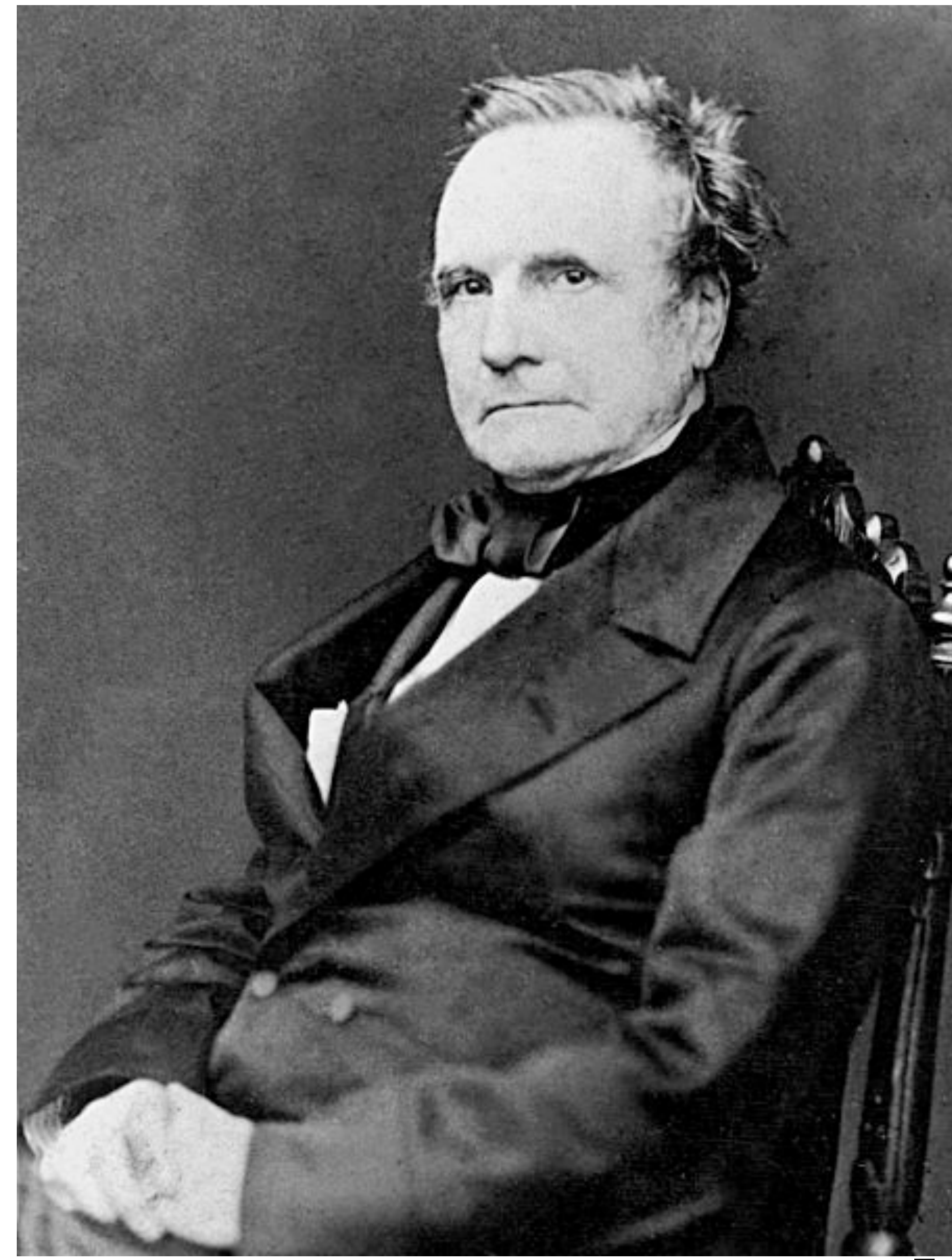


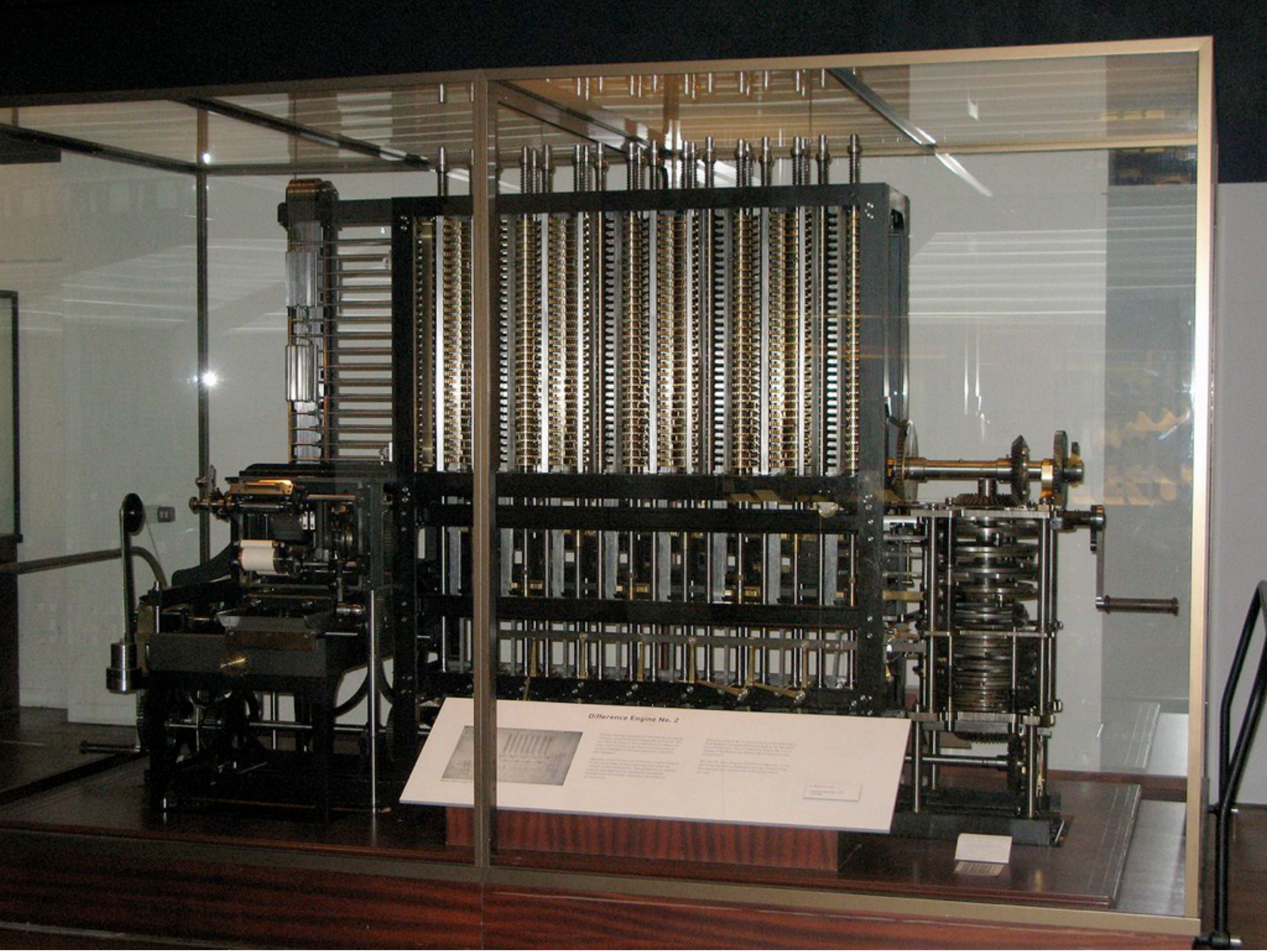
Functions

- Some relations relate inputs with outputs produced by operating given rules over the inputs.

Charles Babbage

- Charles Babbage,1791 – 1871) was an English mathematician, philosopher, inventor and mechanical engineer, who is best remembered now for originating the concept of a programmable computer.
- Babbage began working in 1822 on what he called the difference engine, made to compute values of polynomial functions. It was created to calculate a series of values automatically.




Difference Engine No. 2

Functions

- A function between two sets A and B is a relation $f \subseteq A \times B$ such that:
  - for every $a \in A$, there is some $b \in B$ such that $(a,b) \in f$
  - for every $a \in A$, if $(a,b_1) \in f$ and $(a,b_2) \in f$ then $b_1 = b_2$
- In other words, for every $a \in A$, there is one and only one $b \in B$ such that $(a,b) \in f$ – therefore we write $b = f(a)$ and $f\colon A \to B$

Functions

- Is this relation a function $\{0, 1\} \rightarrow \{0, 1\}$?

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ 1 & \longrightarrow & 1 \end{array}$$

Yes

Functions

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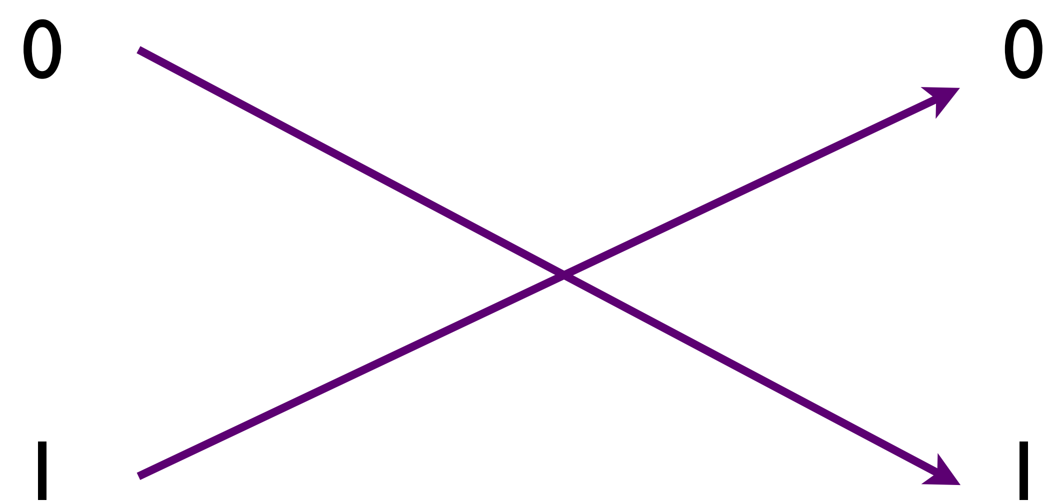

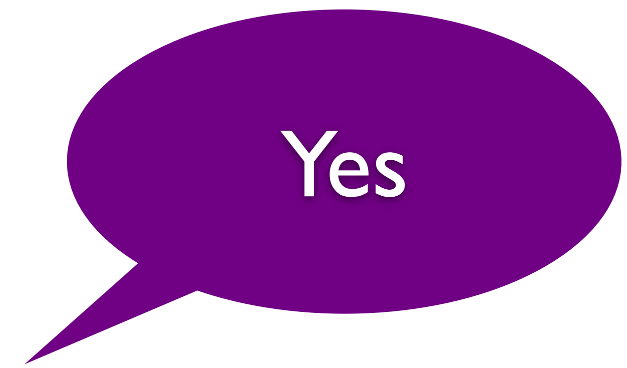

Functions

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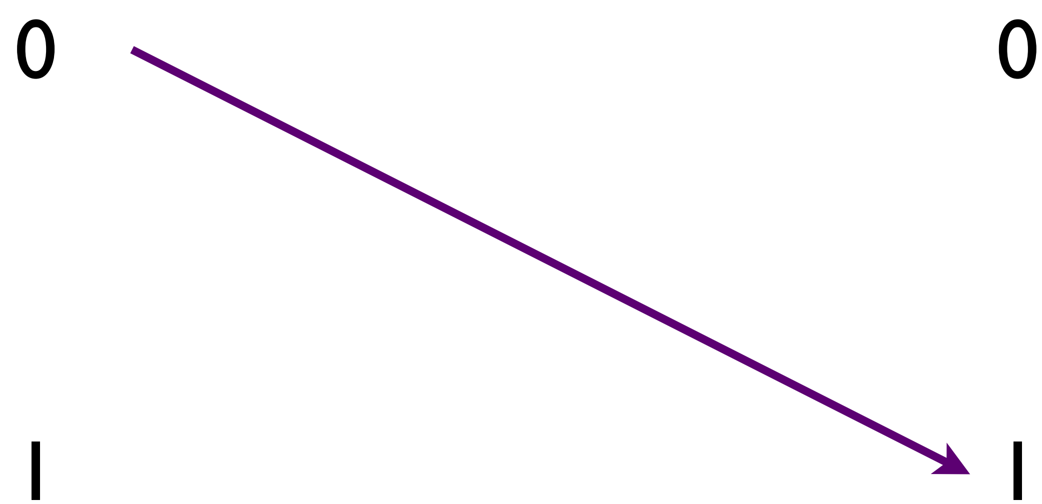

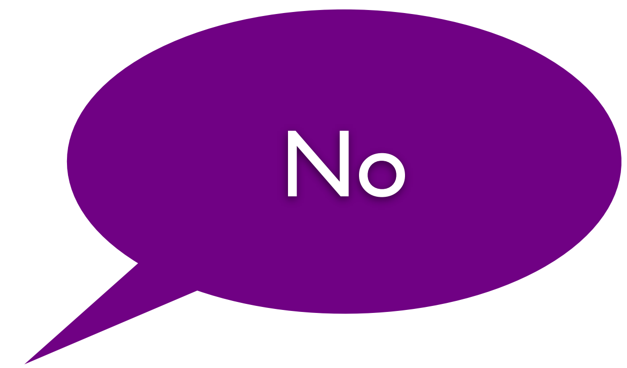

Functions

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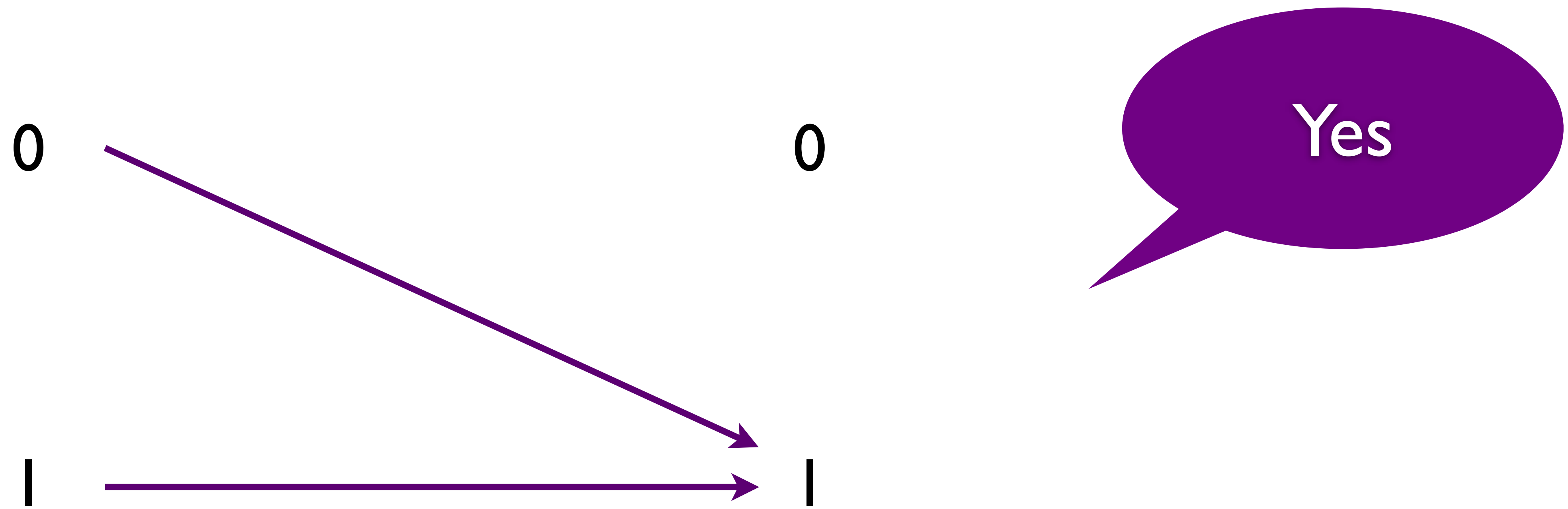
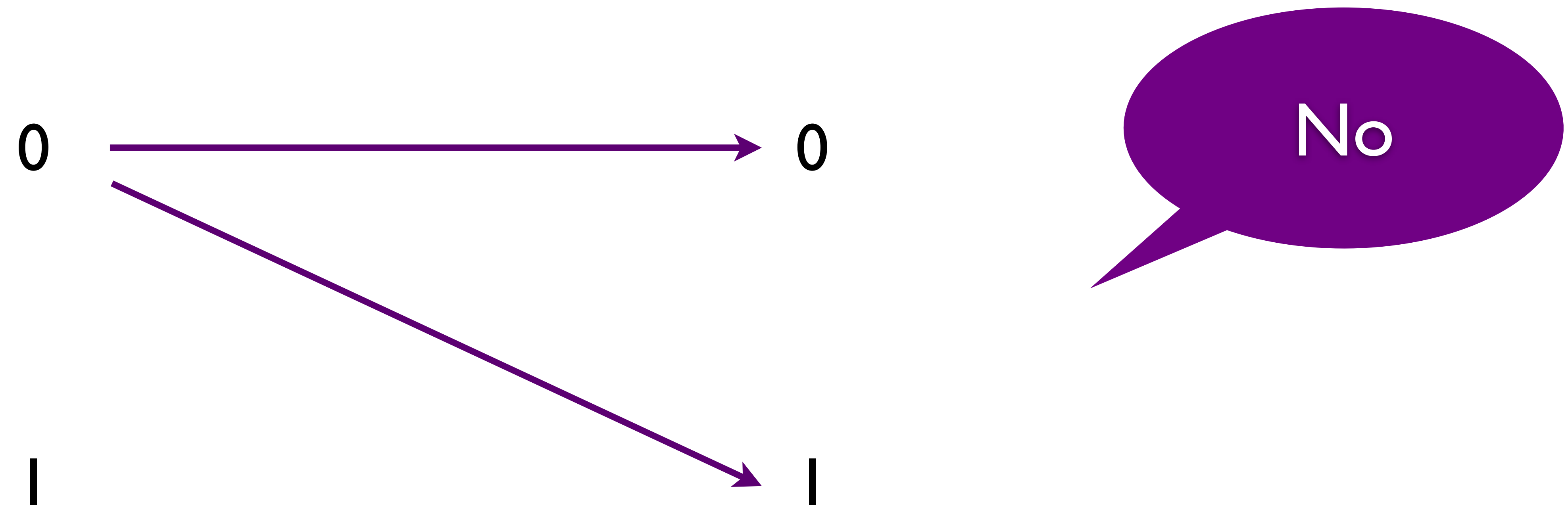


Functions

- Which of these relations are functions $\{0,1\} \rightarrow \{0,1\}$?



Terminology

- Let f: $A \rightarrow B$ be a function:
  - for every $a \in A$, $f(a)$ is the *image* of a.
  - A is the *domain* and B is the *codomain* of f.
  - The set $f(A) = \{ f(a) \mid a \in A \}$ is the *image* of A. We have $f(A) \subseteq B$.

Invertible functions

- A function $f: A \rightarrow B$ is *invertible* if f^{-1} is a function (which is then called its *inverse*).

- If $f: A \rightarrow B$ is invertible then, for every $a \in A$,

$$f^{-1}(f(a)) = a$$

Example

- When data is transmitted errors can be introduced.
- For example, $(1, 0, 0, 1, 1, 0, 0, 1)$ is sent and $(1, 0, 1, 1, 1, 0, 0, 1)$ is received.
- How can the receiver know that there has been an error, and ask for the data to be resent?

Example

- Instead of sending a string of length 8 send one of length 9, where the last digit on the string is the sum of the previous 8 digits modulo 2.
  - For example, instead of (1, 0, 0, 1, 1, 0, 0, 1), (1, 0, 0, 1, 1, 0, 0, 1, 0) is sent.
- The coding function is *bit_parity*: $\{0,1\}^n \rightarrow \{0,1\}^{n+1}$
  - $bit\_parity(b_1, b_2, \ldots, b_n) = (b_1, b_2, \ldots, b_n, b_{n+1})$ where

$$b_{n+1} = (b_1 + b_2 + \ldots + b_n)\%2$$

Example

But not the other way around

- The receiver checks that the last digit is correct.
If it is not, the receiver knows there has been an error.
    - If (1, 0, 1, 1, 1, 0, 0, 1, 0) is received, there was an error.
- If there is no error, the receiver applies the inverse of *bit_parity* to decode the message:
    - $bit\_parity^{-1}(b_1, b_2, \ldots, b_n, b_{n+1}) = (b_1, b_2, \ldots, b_n)$

José Fiadeiro – 2013/14

Injective functions

- A function f: $A \rightarrow B$ is *injective* if, for every $a_1, a_2 \in A$,

$$f(a_1) = f(a_2) \text{ implies } a_1 = a_2$$

- We also say that f is *one-to-one*, or *1-1*.
- Question: is $f(x) = x^2 + 2$ an injective function?
- Better question: is $f(x) = x^2 + 2$ an injective function *over* $\mathbb{Z}$?

Surjective functions

- A function $f: A \to B$ is *surjective* if

  for every $b \in B$, there exists $a \in A$ such that $b = f(a)$

- We also say that f is *onto*.

- Question: is $f(x) = x^2 + 2$ an surjective function over $\mathbb{Z}$?

Bijective functions

- A function $f: A \to B$ is *bijective* (or a bijection) if it is both injective and surjective.
- Theorem 1:
  - A function $f: A \to B$ is invertible iff it is bijective.
- Exercises:
  - If f is surjective then $f(A) = B$
  - If $f: A \to B$ is injective then f defines a bijection between A and $f(A)$.