Introduction to Numerical Solution of Fractional Systems

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INTRODUCTION TO NUMERICAL SOLUTION OF FRACTIONAL SYSTEMS

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OUTLINE

1. INTRODUCTION

1.1 Fractional Derivatives

Pros

- Generalization of ordinary derivatives.
- Nonlocal operators $\longrightarrow$ Memory and heritage.
- More accuracy and robustness.
- Unexplored areas and applications.

Cons

- There are multiple fractional derivatives definitions.
- The derivatives are not always in terms of elementary functions.
- Some definitions require strong conditions on the functions to differentiate.

1.2 Caputo Definition

The Caputo definition of fractional derivative is

$$\mathcal{D}_C^{\alpha} y(t) = J^{m-\alpha} y^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{y^{(m)}(\lambda)}{(t-\lambda)^{1-m+\alpha}} d\lambda \tag{1}$$

where $\alpha \geq 0$, $m = \lceil \alpha \rceil$, $\Gamma(\cdot)$ is the gamma function and $J^{m-\alpha}$ is the Riemann-Liouville integral. From now on, the Caputo-type fractional derivative will be denoted as

$$\mathcal{D}_C^{\alpha} y(t) = \frac{d^{\alpha}}{dt^{\alpha}} y(t) \tag{2}$$

For example

$$\begin{aligned}
\frac{d^{0.5}}{dt^{0.5}}[t] &= \frac{1}{\Gamma(1-0.5)} \int_0^t \frac{d}{d\lambda}(\lambda) \cdot \frac{d\lambda}{(t-\lambda)^{1-0.5+1}} \\
&= \frac{1}{\sqrt{\pi}} \int_0^t \frac{d\lambda}{(t-\lambda)^{1/2}} \quad \text{, let } u = t-\lambda \rightarrow du = -d\lambda. \\
&= \frac{1}{\sqrt{\pi}} \int_0^t \frac{du}{u^{1/2}} \\
&= 2\sqrt{\frac{t}{\pi}}
\end{aligned}$$

1.3 Riemann-Liouville Integral

The Riemann-Liouville integral of order α is defined as

$$J^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{y(\lambda)}{(t-\lambda)^{1-\alpha}}d\lambda \tag{3}$$

Properties

- $$J^{\alpha}\left[f(t)+g(t)\right] = J^{\alpha}f(t) + J^{\alpha}g(t) \tag{4}$$
- $$J^{\alpha}J^{\beta}f(t) = J^{\alpha+\beta}f(t) \tag{5}$$
- $$\frac{d^{\alpha}}{dt^{\alpha}}\left[J^{\alpha}y(t)\right] = y(t) \tag{6}$$
- $$J^{\alpha}\left[\frac{d^{\alpha}}{dt^{\alpha}}y(t)\right] = y(t) - \sum_{r=0}^{m-1}\frac{y_r t^r}{r!} \tag{7}$$

1.4 The Tautochrone Problem

It is desired to find a curve such that, if an object starts on any point along this curve, the time that it requires to slide down to the origin is the same.

- Tauto $\rightarrow$ equal.
- Chrono $\rightarrow$ time.
- Abel, XVIII century.

Assumptions:

- The object moves only by the force of gravity.
- It moves without friction.

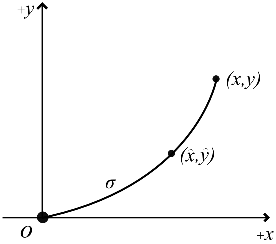


Figure 1. Tautochrone problem.

Click for GIF

1.4 The Tautochrone Problem

Using the energy conservation law,

$$mgy = \frac{1}{2} m \left(\frac{d\sigma}{dt}\right)^2 + mg\hat{y}$$

$$\vdots$$

$$\frac{d^{0.5}}{dy^{0.5}}\sigma(y) = \frac{\sqrt{2g}}{\Gamma\left(\frac{1}{2}\right)} T$$

Through some analytical procedures, we obtain

$$x = \frac{gT^2}{\pi^2}[t + \sin(t)]$$

$$y = \frac{gT^2}{\pi^2}[1 - \cos(t)]$$

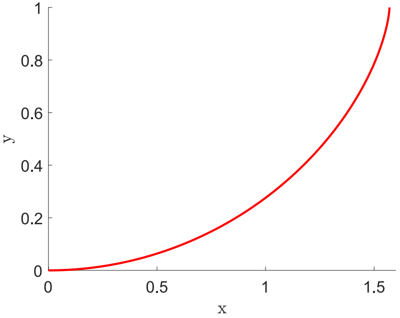


Figure 2. Tautochrone curve.

2. INTEGER ORDER METHODS

The following initial value problem (IVP) will be treated:

$$\begin{cases} y' = f(t, y) \\ y(0) = y_0 , \quad t \in [0, T] \end{cases} \tag{8}$$

Example:

$$\begin{cases} y' - y = 0 \\ y(0) = -3 \end{cases} \tag{9}$$

The exact solution to this problem is given by

$$y(t) = -3e^{-t} \tag{10}$$

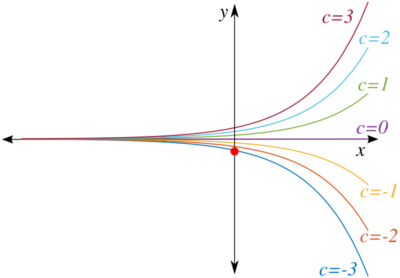


Figure 3. Solution for problem 9.

2. INTEGER ORDER METHODS

2.1 Fourth-Order Runge-Kutta Method (RK4)

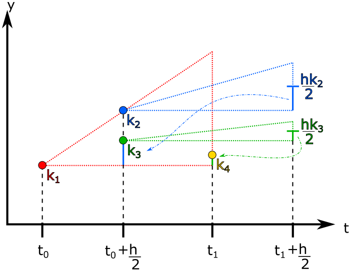
- Improved Euler with more accuracy.
- 4^{th}*order* $\rightarrow$ optimal.

This method approximates the solution to the IVP as follows:

$$y_{i+1} = y_i + h\left(\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\right) \tag{11}$$

where

$$\begin{aligned}
k_1 &= f\left(t_i, y_i\right) \\
k_2 &= f\left(t_i + h/2, y_i + hk_1/2\right) \\
k_3 &= f\left(t_i + h/2, y_i + hk_2/2\right) \\
k_4 &= f\left(t_i + h, y_i + hk_3\right)
\end{aligned}$$



The algorithm works as

```
y = runge_kutta(f,y0,T,N)
```

2.2 Comparison Euler - RK4

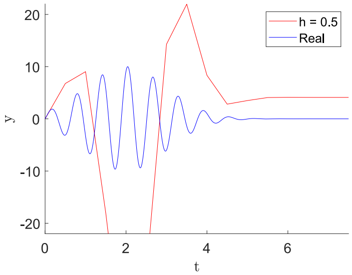
Example: approximate a solution to the IVP

$$\begin{cases} y' = 10e^{-\frac{(t-2)^2}{2}} \left(10\cos(10t) - (t-2)\sin(10t)\right) \\ y(0) = 0 \end{cases} \tag{12}$$

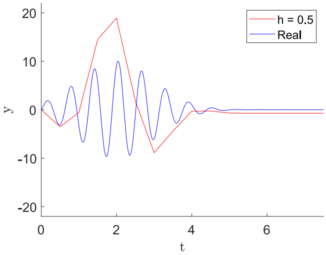
The exact solution to this problem is

$$y(t) = 10e^{-\frac{(t-2)^2}{2}} \sin(10t) \tag{13}$$

Euler



RK4



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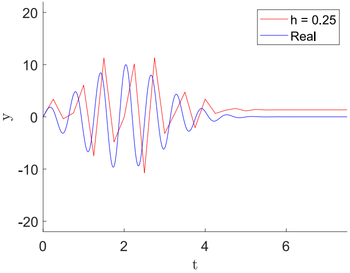

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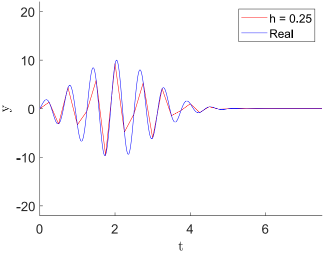

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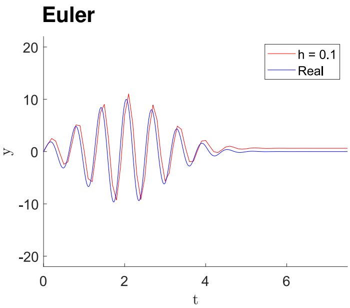

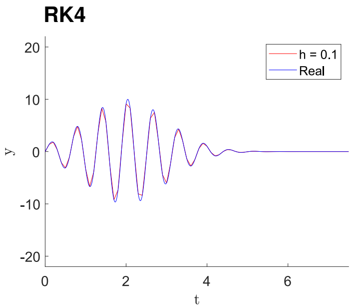

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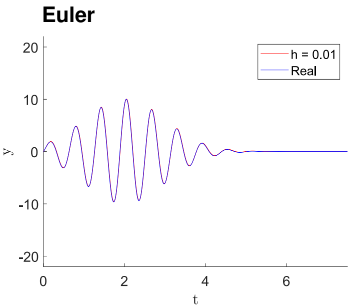

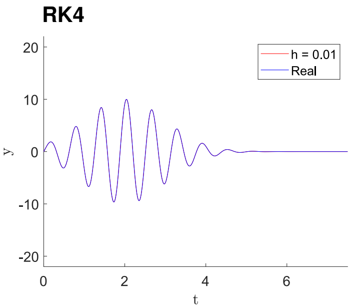

2. INTEGER ORDER METHODS

2.3 Systems of ODEs

Consider the system of ordinary differential equations

$$\begin{cases} y_1' = f_1(t, y_1, y_2, \ldots, y_n) & y_1(0) = y_1 \\ y_2' = f_2(t, y_1, y_2, \ldots, y_n) & y_2(0) = y_2 \\ \quad \vdots \\ y_n' = f_n(t, y_1, y_2, \ldots, y_n) & y_n(0) = y_n \end{cases} \tag{14}$$

which can be synthesized as

$$\begin{cases} \mathbf{y}' = F(t, \mathbf{y}) \\ \mathbf{y}(0) = \mathbf{y}_0 \end{cases} \tag{15}$$

For example, the Lotka-Volterra equations (predator-prey model) is a system of ODEs as follows:

$$\begin{cases} y_1' = \alpha y_1 - \beta y_1 y_2 \\ y_2' = \delta y_1 y_2 - \gamma y_2 \end{cases} \tag{16}$$

where y_1 represents the number of preys and y_2 is the amount of predators.

2.3 Systems of ODEs

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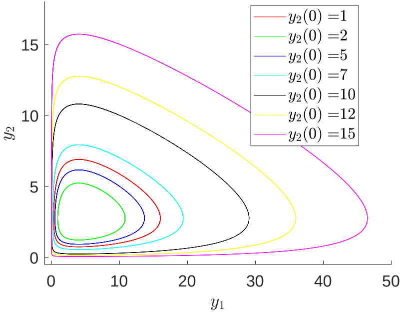


Figure 4. Phase portrait of Lotka-Volterra equations.

2.4 Multi-Term ODEs

In case of higher order ODEs, they can be transformed into a sytem of first order ODEs, using phase variables x_i. Suppose we have an equation as the following:

$$\begin{cases} y^{(n)} = f\left(t, y, y', ..., y^{(n-1)}\right) \\ y(0) = y_0, \ \ldots, \ y^{(n-1)}(0) = y_{(n-1)}, \ \ t \in [0, T] \end{cases} \tag{18}$$

with the following substitution

$$\begin{cases} x_1 = y \\ x_2 = y' \\ \qquad \vdots \\ x_n = y^{(n-1)} \end{cases} \longrightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \qquad \vdots \\ x_{n-1}' = x_n \\ x_n' = f\left(t, x_1, x_2, ..., x_{n-1}, x_n\right) \end{cases} \tag{19}$$

$$x_1(0) = y_0, \ldots, \ x_{n-1}(0) = y_{(n-1)}$$

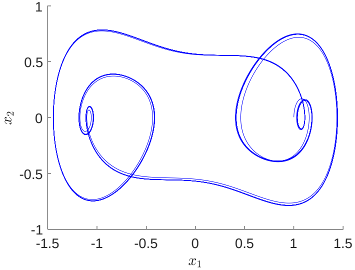
2.4 Multi-Term ODEs

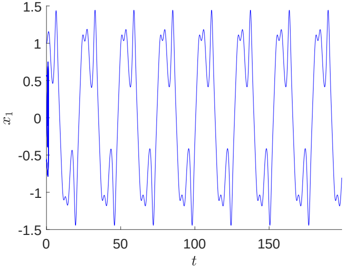
Example: consider the Duffing oscillator (Click for GIF)

$$y'' + \delta y' + \alpha y + \beta y^3 = \gamma \cos(\omega t) \tag{20}$$

the equivalent system of first order ODEs is

$$\begin{cases} x_1 = y \\ x_2 = y' \end{cases} \longrightarrow \begin{cases} x_1' = x_2 \\ x_2' = -\delta x_2 - \alpha x_1 - \beta x_1^3 + \gamma \cos(\omega t) \end{cases} \tag{21}$$





$\delta = 0.3$, $\alpha = -1$, $\beta = 1$,$\gamma = 0.37$, $\omega = 1.2$ and initial conditions $x_1(0) = 1$, $x_2(0) = 0$.

3. FRACTIONAL ORDER METHODS

Consider the fractional IVP

$$\begin{cases} \dfrac{d^\alpha}{dt^\alpha} y(t) = f(t, y) \\ y(0) = y_0, \ldots, y^{(m-1)}(0) = y_{(m-1)} \quad \alpha \in \mathbb{R}^+ \quad t \in [0, T] \end{cases} \tag{22}$$

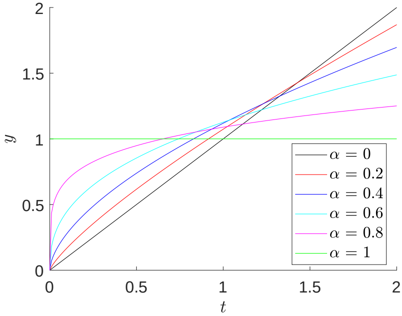
where $m = \lceil \alpha \rceil$.

Example

$$\frac{d^{0.5}}{dt^{0.5}} y(t) = 2\sqrt{\frac{y}{\pi}} \tag{23}$$

whose solution is

$$y(t) = t \tag{24}$$



3. FRACTIONAL ORDER METHODS

3.1 Adams-Bashforth-Moulton Predictor-Corrector (ABM)

Using quadrature theory, the solution can be approximated as

$$y_h(t_{n+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+2)} f\left(t_{n+1}, y_h^{\mathrm{p}}(t_{n+1})\right) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f\left(t_j, y_h(t_j)\right) \tag{25}$$

- $y_h^{\mathrm{p}}(t_{n+1})$ is a predicted value.
- $a_{j,n+1}$ is a quadrature coefficient.

The algorithm works as

```
y = abm(f,alpha,y0,T,N)
```

- `f` is the right-hand side of the differential equation.
- `alpha` is the order of the differential equation.
- `y0` is the initial conditions.
- `T` is the simulation time.
- `N` is the number of partitions on the interval $[0, \mathrm{T}]$.

3. FRACTIONAL ORDER METHODS

3.1 Adams-Bashforth-Moulton Predictor-Corrector (ABM)

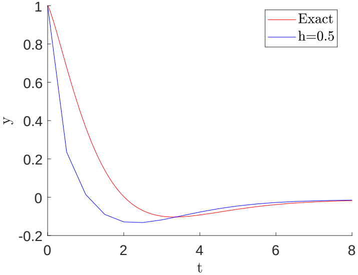
Example

Give an approximate solution to

$$\begin{cases} \dfrac{d^{1.25}}{dt^{1.25}} y(t) = -y(t) \\ y'(0) = 0,\ y(0) = 1 \end{cases} \tag{26}$$

The exact solution is given by

$$y(t) = E_{1.25,\,1}(-t^{1.25}) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{1.25k}}{\Gamma(1.25k+1)} \tag{27}$$



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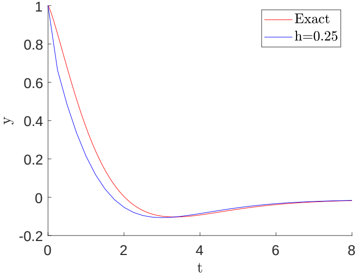

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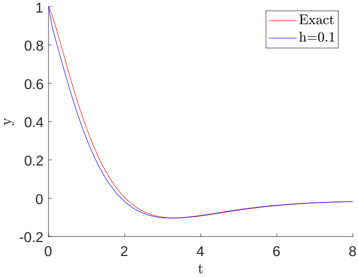

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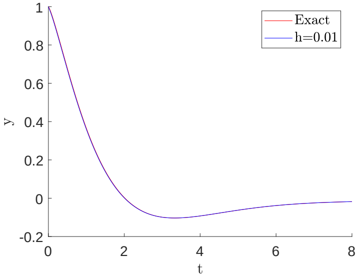

3.2 Decomposition Method

Based on decomposing f as follows

$$f(t,\mathbf{y}) = g(t) + \mathbf{A}\mathbf{y} + h(t,\mathbf{y}) \tag{28}$$

Applying the inverse operation to the Caputo fractional derivative

$$\mathbf{y}(t) = \sum_{r=0}^{m-1} \frac{\mathbf{y}_r t^r}{r!} + J^{\alpha} g(t) + J^{\alpha}\mathbf{A}\mathbf{y} + J^{\alpha} h(t,\mathbf{y}) \tag{29}$$

and supposing a solution in series, we obtain the recursive scheme

$$\begin{aligned} \mathbf{x}_0 &= \sum_{r=0}^{m-1} \frac{\mathbf{y}_r t^r}{r!} + J^{\alpha} g(t) \\ \mathbf{x}_{k+1} &= J^{\alpha}\mathbf{A}\mathbf{x}_k + J^{\alpha}\tilde{h}_k\left(t, \sum_{r=0}^{k} \mathbf{x}_j(t)\right) \end{aligned} \tag{30}$$

Where $\tilde{h}_k$ is the Adomian polynomial

$$\tilde{h}_k\left(t, \sum_{r=0}^{k} \mathbf{x}_j(t)\right) = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k} h\left(t, \sum_{j=0}^{k} \lambda^j \mathbf{x}_j(t)\right)\Bigg|_{\lambda=0}\right] \tag{31}$$

3.2 Decomposition Method

The algorithm works as

```
ySim = decomposition(f,alpha,y0,N)
```

Example

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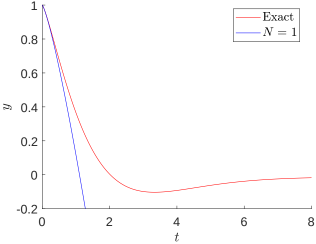

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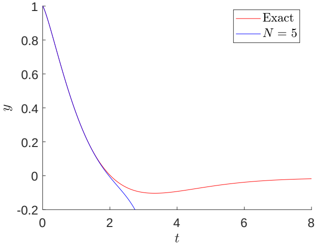

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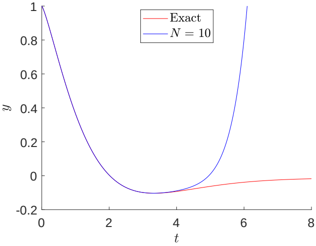

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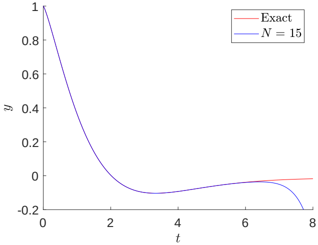

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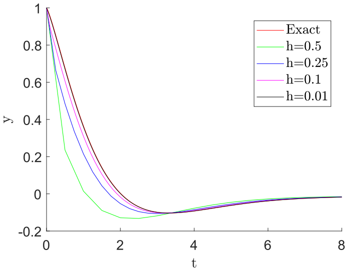
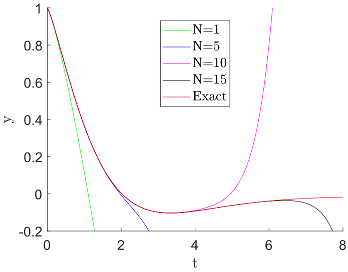


Decomposition



3.4 Systems of FDEs

Consider the system of ordinary fractional differential equations

$$\begin{cases} \dfrac{d^{\alpha_1}}{dt^{\alpha_1}} y_1 = f_1(t, y_1, y_2, \ldots, y_n) \quad y_1(0) = y_1 \\ \dfrac{d^{\alpha_2}}{dt^{\alpha_2}} y_2 = f_2(t, y_1, y_2, \ldots, y_n) \quad y_2(0) = y_2 \\ \qquad \vdots \\ \dfrac{d^{\alpha_n}}{dt^{\alpha_n}} y_n = f_n(t, y_1, y_2, \ldots, y_n) \quad y_n(0) = y_n \\ \alpha_i \in \mathbb{R}^+, \ \ t \in [0, T] \end{cases} \tag{36}$$

which can be synthesized as

$$\begin{cases} \dfrac{d^{\boldsymbol{\alpha}}}{dt^{\boldsymbol{\alpha}}} \mathbf{y} = F(t, \mathbf{y}) \\ \mathbf{y}(0) = \mathbf{y}_0, \ \boldsymbol{\alpha} \in \left(\mathbb{R}^+\right)^{n \times 1}, \ t \in [0, T] \end{cases} \tag{37}$$

3.5 Multi-Term FDEs

Suppose we have the multi-term fractional differential equation

$$\begin{cases} \frac{d^{\alpha_n}}{dt^{\alpha_n}} y(t) = f\left(t, y, \frac{d^{\alpha_1}}{dt^{\alpha_1}} y, \frac{d^{\alpha_2}}{dt^{\alpha_2}} y, \ldots, \frac{d^{\alpha_n}}{dt^{\alpha_n}} y\right) \\ y(0) = y_0, \ \ldots, y^{(m-1)}(0) = y_{(m-1)} \end{cases} \tag{38}$$

where $m = \lceil \alpha_n \rceil$ and $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n$. We select new orders $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ such that

(a) $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ must be rational numbers,
(b) $\lceil \alpha_n \rceil = \lceil \tilde{\alpha}_n \rceil$
(c) $\gcd(1, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ should be as large as possible,
(d) $\left|\alpha_j - \tilde{\alpha}_j\right|$ should be as small as possible for all j

3.5 Multi-Term FDEs

We build the approximated system of FDEs with

$$\gamma := \gcd(1, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$$

$$\tilde{N} := \frac{\tilde{\alpha}_n}{\gamma}$$

$$\begin{cases} \dfrac{d^\gamma}{dt^\gamma} x_0 = x_1 \\ \dfrac{d^\gamma}{dt^\gamma} x_1 = x_2 \\ \quad \vdots \\ \dfrac{d^\gamma}{dt^\gamma} x_{\tilde{N}-2} = x_{\tilde{N}-1} \\ \dfrac{d^\gamma}{dt^\gamma} x_{\tilde{N}-1} = f\left(t, x_0, x_{\tilde{\alpha}_1/\gamma}, \ldots, x_{\tilde{\alpha}_{n-1}/\gamma}\right) \end{cases} \tag{39}$$

$$x_j(0) = \begin{cases} y_{(j\gamma)} & \text{for } j\gamma \in \mathbb{N}_0 \\ 0 & \text{else} \end{cases} \tag{40}$$

3.5 Multi-Term FDEs

Example: Bagley-Torvik Equation (Click for GIF)

$$\begin{cases} ay'' + b\dfrac{d^{3/2}}{dt^{3/2}}y + cy = g(t) \\ y(0) = y_0,\ y'(0) = y_1 \end{cases} \tag{41}$$

Let us keep the original orders $\tilde{\alpha}_1 = 3/2$ and $\tilde{\alpha}_2 = 2$ to satisfy condition (d). Note that $\gamma = \gcd(1, 3/2, 2) = 1/2$. Then $\tilde{N} = \dfrac{\tilde{\alpha}_2}{\gamma} = 4$. Therefore, the approximated system is

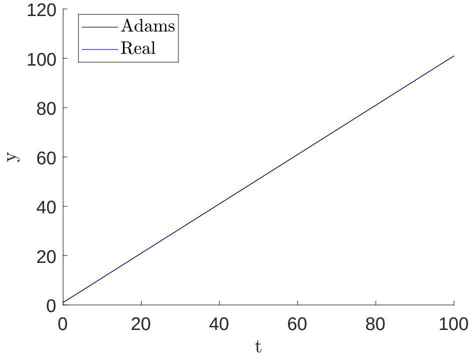
$$\begin{cases} \dfrac{d^{1/2}}{dt^{1/2}}x_0 = x_1 & x_0(0) = y_0 \\ \dfrac{d^{1/2}}{dt^{1/2}}x_1 = x_2 & x_1(0) = 0 \\ \dfrac{d^{1/2}}{dt^{1/2}}x_2 = x_3 & x_2(0) = y_1 \\ \dfrac{d^{1/2}}{dt^{1/2}}x_3 = f\left(t, x_0, x_{1.5/0.5}\right) = \dfrac{g(t) - cx_0 - bx_3}{a} & x_3(0) = 0 \end{cases} \tag{42}$$

3.5 Multi-Term FDEs

Example: Bagley-Torvik Equation

In particular, for $a = 1$, $b = c = -1$, $g(t) = t + 1$, $y_0 = 1$ and $y_1 = 1$, the exact solution to this IVP is

$$y(t) = 1 + t \tag{43}$$



4. IMPROVEMENTS

4.1 Predicted ABM

- $+$ accuracy $\implies$ $+$ execution time.
- ABM converges $\iff$ Predicted ABM converges.

The idea is to calculate different $y_h^{\mathrm{p}}(t_{n+1})$ until a tolerance is reached, for each instant of time.

```
y = pabm(f,alpha,y0,T,N,nmax,tol)
```

For example, using the same Bagley-Torvik equation with larger time step, we have

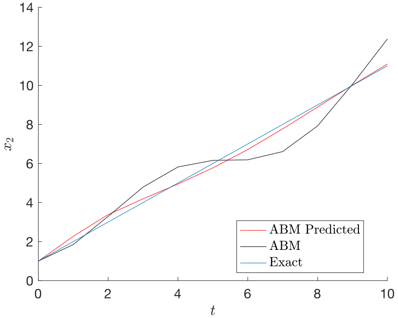


Figure 5. Comparison between the original and predicted scheme.

4. IMPROVEMENTS

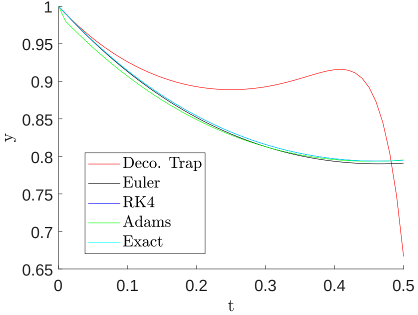
4.2 Quadrature Decomposition

The idea was to partition the time interval and, on each point, find the polynomial using approximations to the Riemann-Liouville integral.

```
y = qDecomposition(f,alpha,y0,N1,N2,T)
```

Example:

$$\begin{cases} x' = y & x(0) = 1 \\ y' = 2x - y & y(0) = -1 \end{cases} \tag{44}$$



4.3 Polynomial Decomposition

- Analytic solution.
- $-$ execution time.
- Useful for non-chaotic dynamic systems or chaotic systems for small time frames.

The main idea is based on the simple computation of

$$J^{\alpha}\left(t^{\beta}\right)=\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta} \tag{45}$$

which can be extended to non-polynomial expressions using interpolation.

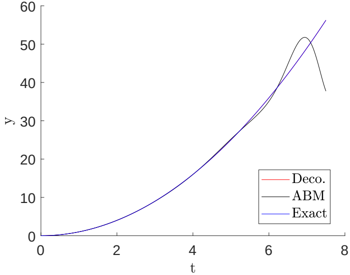
```
ySim = pDecomposition(f,alpha,y0,N)
```

4.3 Polynomial Decomposition

Example:

$$\begin{cases} \dfrac{d^3}{dt^3}y(t) + \dfrac{d^{5/2}}{dt^{5/2}}y(t) + y^2(t) = t^4 \\ y(0) = y'(0) = 0,\ y''(0) = 2 \end{cases} \tag{46}$$

The exact solution is $y(t) = t^2$. Using the procedure for multi-term FDEs, the solution can be approximated as shown in the figure.

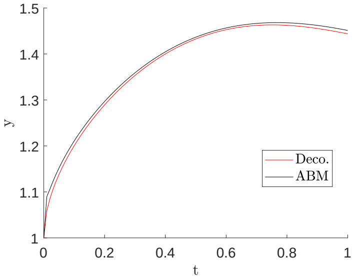


4.3 Polynomial Decomposition

Example: Fractional Lotka-Volterra model

$$\begin{cases} \dfrac{d^{\alpha_1}}{dt^{\alpha_1}} x = \alpha x - \beta xy \\ \dfrac{d^{\alpha_2}}{dt^{\alpha_2}} y = \delta xy - \gamma y \end{cases} \tag{47}$$

Using params$=[1,1,1,1]$, c.i$=[1,0.5]$ and $\alpha=[0.5,0.6]$:



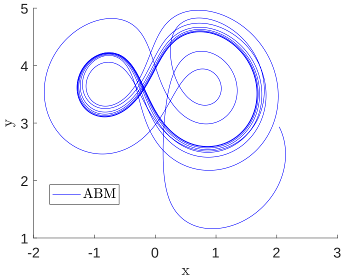
4. IMPROVEMENTS

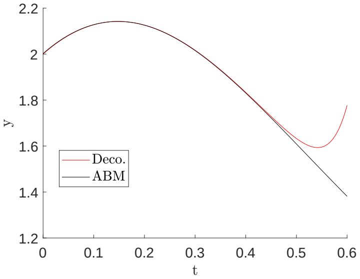
4.3 Polynomial Decomposition

Example: Financial System

$$\begin{cases} \dfrac{d^{\alpha_1}}{dt^{\alpha_1}} x = z + (y - a)x \\ \dfrac{d^{\alpha_2}}{dt^{\alpha_2}} y = 1 - by - x^2 \\ \dfrac{d^{\alpha_3}}{dt^{\alpha_3}} z = -x - cz \end{cases} \tag{48}$$

Using params$=[3, 0.1, 1]$, c.i$=[2, 3, 2]$ and $\alpha = [1, 1, 0.8]$, we obtain





4.4 Further Work

- ABM with fixed memory.

$$\frac{1}{\Gamma(1-\alpha)} \int_{t-T}^{t} \frac{y'(s)}{(t-s)^{\alpha}} ds$$

- ABM with logarithmic memory.

$$w^{p\alpha} \int_{0}^{t} \frac{f\left(w^{p} x\right)}{(t-x)^{1-\alpha}} dx$$

- Richardson extrapolation.

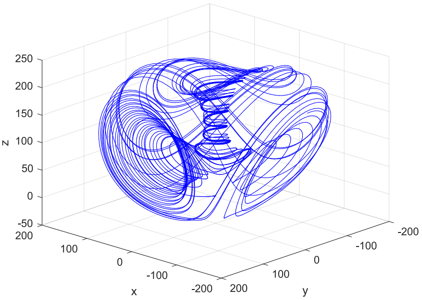
$$x_n = x\left(t_n\right) + \sum_{\mu=1}^{M_1} \gamma_{\mu} n^{-\lambda \mu}$$

- Decomposition acceleration.

$$S_n^{(k)} = \frac{S_n^{(k-1)} S_{n+2}^{(k-1)} - \left(S_{n+1}^{(k-1)}\right)^2}{S_n^{(k-1)} + S_{n+2}^{(k-1)} - 2 S_{n+1}^{(k-1)}}, \quad k \geq 1$$

5. CONCLUSIONS

- Fractional calculus $\longrightarrow$ powerful tool to model real and chaotic systems.
- Both ABM and decomposition are useful but each with pros and cons.
- Both methods can be improved, as shown.
- A summary of some essential tools to study fractional systems was successfully constructed.



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Thank you