Ασκηση 6

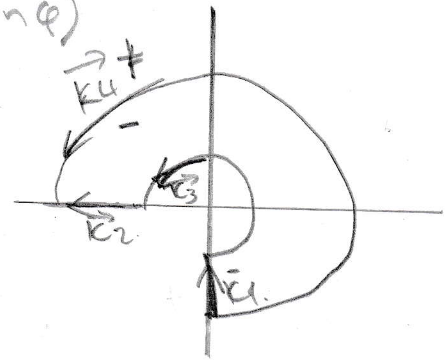
$r_T \in (b,a),\ \varphi \in (-\pi/2, \pi) \longrightarrow \vec{J} = \left(J_{r_T},\ J_0 \frac{a^2}{r_T^2}\sin\varphi\right)$

$r_T \in (b,a),\quad \varphi = -\pi/2 \longrightarrow \vec{K_1}(K_1, 0)$

$r_T \in (b,a),\quad \varphi = \pi \longrightarrow \vec{K_2}(K_2, 0)$

$r_T = b$, εσωτερικό, $\quad \varphi \in (-\pi/2, \pi) \longrightarrow \vec{K_3}(0, K_3)$

$r_T = a$, εξωτερικό, $\quad \varphi \in (-\pi/2, \pi) \longrightarrow \vec{K_4}(0, K_4)$



Για το $J(r_T, \varphi)$ εφαρμόζω ΝΔΦ στα σημεία $(r_T, \varphi) \in (b,a) \times (-\pi/2, \pi)$

ΝΔΦ: $\vec{\nabla} J + \cancel{\frac{\partial \rho}{\partial t}}^{\,0} = 0$ → σε μόνιμη κατάσταση

$$\vec{\nabla}\vec{J} \Leftrightarrow \frac{1}{r_T}\frac{\partial (r_T J_{r_T})}{\partial r_T} + \frac{1}{r_T}\frac{\partial J_\varphi}{\partial \varphi} = 0$$

$$\Rightarrow \frac{\partial (r_T J_{r_T})}{\partial r_T} + J_0\left(\frac{a}{r_T}\right)^2 \cos\varphi = 0 \Rightarrow \frac{\partial (r_T J_{r_T})}{\partial r_T} = -J_0\left(\frac{a}{r_T}\right)^2\cos\varphi$$

Ολοκληρώνω: $\displaystyle\int_{r_T}^{a} \partial(r_T' J_{r_T'}) = -\int_{r_T' = r_T}^{a} J_0\left(\frac{a}{r_T'}\right)^2 \cos\varphi\, dr_T' + C$

$$\Rightarrow a J_{r_T}(a) - r_T J_{r_T}(r_T) = J_0 a^2 \cos\varphi\left[\frac{1}{a} - \frac{1}{r_T}\right] + C$$

$$\Rightarrow J_{r_T}(r_T, \varphi) = \frac{J_0 a^2}{r_T}\cos\varphi\left[\frac{1}{a} - \frac{1}{r_T}\right] + C$$

Εφαρμόζω συνοριακές συνθήκες: (1) $r_T = a$ (παίρνω το K_4 που είναι γνωστό)

ΝΔΦ σε επιφάνεια:

$$\hat{n}(\cancel{\vec{J}^+}^{\,0} - \vec{J}^-) = -\nabla_2 K - \cancel{\frac{\partial \rho}{\partial t}}^{\,0} \Rightarrow i_{r_T}(-C\, i_{r_T}) = -K_0\cos\varphi \Rightarrow \boxed{C = K_0 \cos\varphi}$$

$$\Rightarrow J_{r_T}(r_T, \varphi) = \frac{J_0 a^2}{r_T}\cos\varphi\left[\frac{1}{a} - \frac{1}{r_T}\right] + K_0\cos\varphi$$

$\varphi = \pi$: Για k_2:

ΝΔΦ για επιφάνεια

$$\hat{i}_n(\cancel{J^+}^{\,0} - J^-) = -\nabla_2 k - \cancel{\frac{\partial \rho}{\partial t}}^{\,0}$$

$$\Rightarrow \hat{i}_\varphi(-\cancel{J_{r_T}\hat{i}_{r_T}}^{\,0} - J_0 \frac{a^2}{r_T^2}\sin\varphi\,\hat{i}_\varphi) = \frac{1}{r_T}\frac{d(r_T k_2(r_T))}{dr_T} \Rightarrow$$

$$\Rightarrow J_0 \frac{a^2}{r_T^2}\cancel{\sin\pi}^{\,0} = \frac{1}{r_T}\cdot\frac{d(r_T k_2(r_T))}{dr_T}$$

$$\Rightarrow \boxed{k_2(r_T) = \frac{c}{r_T}}$$

$$k_2(a) + \cancel{k_4(\pi^-)}^{\,0} = 0$$

$$\Rightarrow \boxed{k_2(r_T) = 0}$$

$r_T = b$: Για $k_3(\varphi)$:

$$\hat{i}_n(J^+ - \cancel{J^-}^{\,0}) = -\nabla_2 k_3 - \cancel{\frac{d\rho}{\partial t}}^{\,0} \Rightarrow \cos\varphi\left[J_0\left(\frac{a^2}{b^2}\right) + \frac{k_0 - J_0 a}{b}\right] = \frac{1}{b}\frac{d(k_3(\varphi))}{d\varphi}$$

$$\Rightarrow dk_3(\varphi) = -\left[J_0\left(\frac{a^2}{b}\right) + k_0 - J_0 a\right]\cos\varphi\, d\varphi \Leftrightarrow \int_{\varphi=\varphi}^{\pi} d(k_3(\varphi)) = -J\int_{\varphi}^{\pi}\cos\varphi' d\varphi'$$

$$\Rightarrow k_3(\pi^-) - k_3(\varphi) = -J_0 a\left(\frac{a}{b} - 1\right) + k_0\Big]\sin\varphi$$

$$k_3(\pi^-) = k_2(b^+) = 0$$

$k_1(y)$ Συνοριακή στο $\varphi = -\pi/2$:

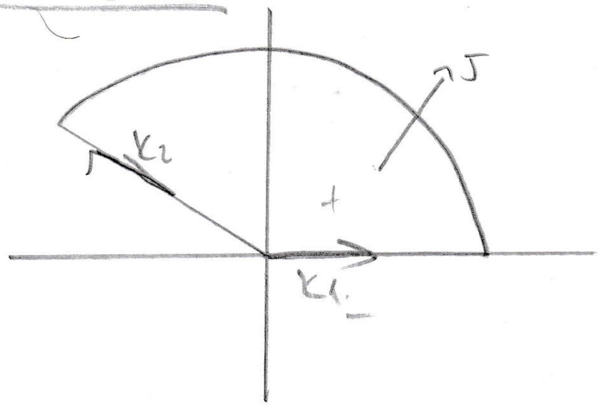
$$\hat{i}_n^{\,-\hat{i}_\varphi}(J^+ - \cancel{J^-}^{\,0}) = -\vec{\nabla}_2 k_1 - \cancel{\frac{d\rho}{\partial t}}^{\,0} \Rightarrow J_0\left(\frac{a}{y}\right)^2 = \frac{dk_1}{dy} \Rightarrow J_0 a^2 \int_y^a d\left(\frac{1}{y'}\right) = \int_y^a dk_1(y')$$

$$\Rightarrow J_0 a^2\left(\frac{1}{a} - \frac{1}{y}\right) = k_1(a) - k_1(y) \Rightarrow k_1(y) = J_0\frac{a^2}{y} - J_0 a + k_1(a)$$

Αφού $k_1(a) = k_4(-\pi/2^+)$ $\Rightarrow k_1(a) = +k_0\sin(-\pi/2) - k_0)$

$$\Rightarrow k_1(y) = J_0\frac{a^2}{y} + k_0 - J_0 a.$$

Άσκηση 7



$r_T < a,\ \varphi \in (0, 2\pi/3) \rightarrow \vec{J}\,(J_{r_T},\ J_0\cos\varphi)$

$r_T < a,\ \varphi = 0 \rightarrow \vec{k_1}\,(k_1, 0)$

$r_T < a,\ \varphi = 2\pi/3 \rightarrow \vec{k_2}\,(k_2, 0)$

Για το $J_T\,(r_T, \varphi)$ εφαρμόζω ΝΔΦ σε σημεία $(r_T, \varphi) \in (0, a) \times (0, 2\pi/3)$

→ κυλινδρικές

$$\vec{\nabla} J + \cancel{\frac{d\rho}{dt}}^{\,0} = 0 \Rightarrow \frac{1}{r_T}\frac{\partial (r_T J_{r_T})}{\partial r_T} + \frac{1}{r_T}\frac{\partial (J_0 \cos\varphi)}{\partial \varphi} = 0$$

$$\Rightarrow \frac{\partial (r_T J_{r_T})}{\partial r_T} + \frac{\partial (J_0 \cos\varphi)}{\partial \varphi} = 0$$

$$\Rightarrow \frac{\partial (r_T J_{r_T})}{\partial r_T} - J_0 \sin\varphi = 0$$

$$\Rightarrow \int_{r_T' = r_T}^{a} \partial (r_T' J_{r_T'}) = \int_{r_T' = r_T}^{r_T' = a} J_0 \sin\varphi \, dr_T' + c$$

$$\Rightarrow a \cancel{J_{r_T}(a)}^{\,0} - r_T J_{r_T} = J_0 \sin\varphi\,[a - r_T] + c(\varphi)$$

$$\Rightarrow \hat{i}_{r_T} J_{r_T} = J_0 \sin\varphi \left[\frac{r_T - a}{r_T}\right]$$

Θέτω οριακή συνθήκη για $\varphi = 0$

$$\hat{n} \cdot (\vec{J}(y = 0^+) - \cancel{J(y = 0^-)}^{\,0}) + \vec{\nabla} \vec{k_1} = 0 \Rightarrow \hat{\varphi}\,(\vec{J}(\varphi = 0, r)) + \vec{\nabla} \vec{k_1} = 0$$

$$\Rightarrow J_0 + \frac{\partial k_1}{\partial r} = 0 \Rightarrow \frac{\partial k_1}{\partial r_2} = -J_0$$

$$\Rightarrow k_1(r_2) = c_0 - J_0 r$$

για $r_T \Rightarrow k_1(a) = 0 \Rightarrow c_0 = J_0 a \Rightarrow k_1(r) = J_0(a - r)$

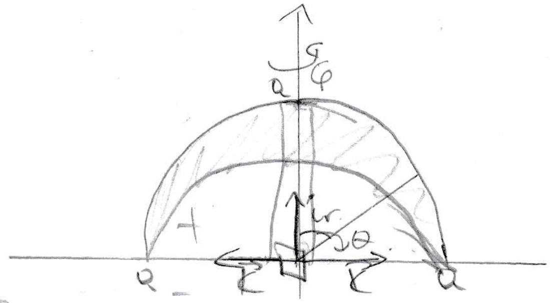
Οριακή συνθήκη για $\varphi = \frac{2\pi}{3}$

$$\hat{n}\,(\vec{0} - J(r, \varphi = 2\pi/3)) + \vec{\nabla} k_2 = 0 \Rightarrow -J_\varphi\,(r, \varphi = 2\pi/3) = -\frac{\partial k_2}{\partial r}$$

$$\Rightarrow k_2(r) = c_1 - J_0 r$$

Ασκηση 8

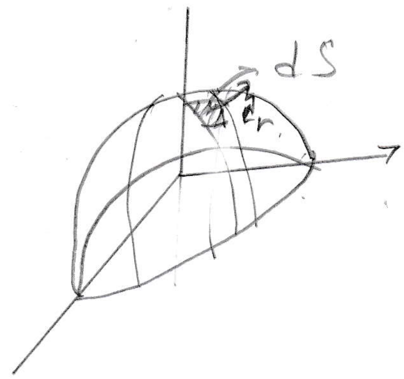
$\vec{J} = \hat{i}_\theta \frac{k_0}{r\sin\theta}$



$\vec{J}\left(0, \frac{k_0}{r\sin\theta}, 0\right)$? $r<a$, $0<\theta<\pi/2$

$\vec{K}_1(K_1, 0, 0)$? $\to \theta = \pi/2$

$i(r)$ νηματοειδής ροή για $\theta=0$, $r<a$.



ΝΔΦ $\Rightarrow \oint_S \vec{J}\, d\vec{S} + \int_V \frac{\partial\rho}{\partial t}\, dV = 0$

$i(r) = \oint_S \vec{J}\, d\vec{S}$ $\quad \vec{dS} = r\sin\theta\, d\theta\, dr\, \hat{i}_\theta$

άρα $i(r) = \int_{r'=r}^{a}\int_0^{2\pi} \frac{k_0}{\cancel{r\sin\theta}}\, \cancel{r\sin\theta}\, d\varphi\, dr' =$

$= k_0(a-r)\cdot 2\pi$ με $\boxed{\theta=0}$

Θα υπολογίσουμε την επιφανειακή συνιστώσα $\vec{K}$

ΝΔΦ σε 2 διαστάσεις

$\hat{i}_n(\vec{J}^+ - \cancel{\vec{J}^-}^{\,0}) + \vec{\nabla}K + \cancel{\frac{\partial\rho}{\partial t}}^{\,0} = 0$

$\Rightarrow \hat{i}_z(\vec{J}^+) + \frac{1}{r}\frac{\partial}{\partial r}(rK) = 0$

$\Rightarrow -\hat{i}_z\hat{i}_z\frac{k_0}{r} + \frac{1}{r}\frac{\partial}{\partial r}(rK) = 0 \Rightarrow rK = k_0 r + c \ (c \in \mathbb{R})$

Ισχύει ότι $K(a) = 0 \Rightarrow K(r) = k_0 + \frac{c}{r}$

$\Rightarrow k_0 = -\frac{c}{a}$

$\vec{K}(r) = \left(k_0 - \frac{k_0 a}{r}\right)\hat{i}_r \Rightarrow \vec{K}(r) = k_0\left(1 - \frac{a}{r}\right)\hat{i}_r$