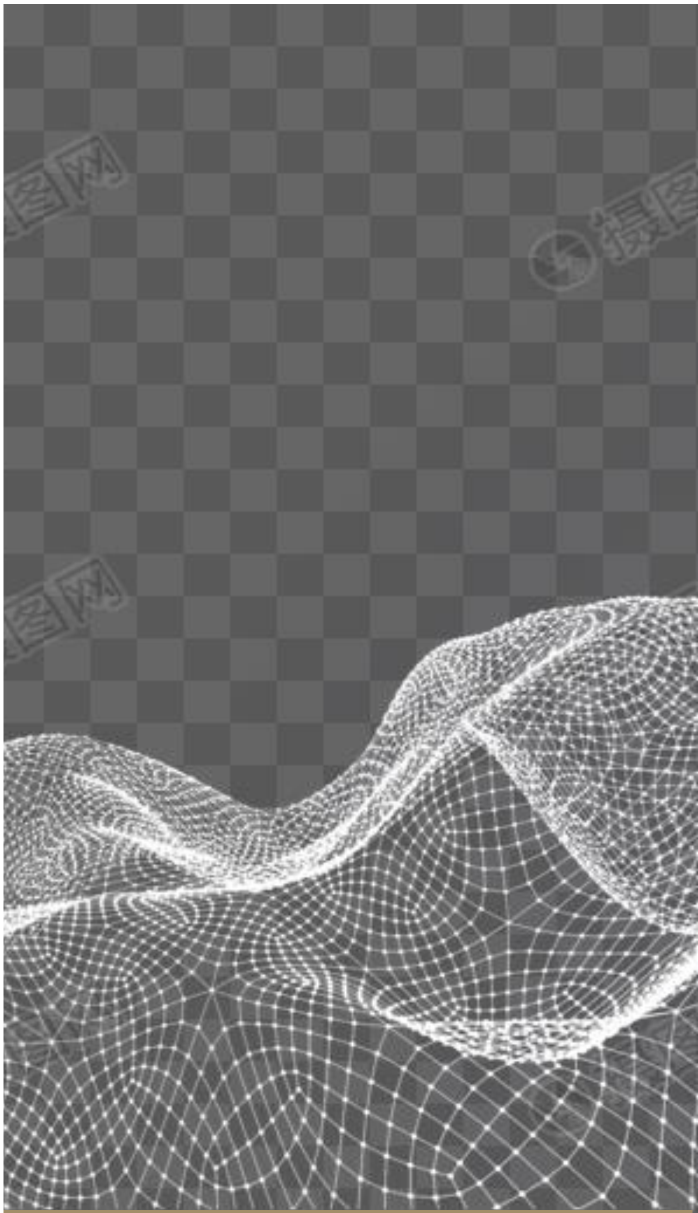
ΕΡΓΑΣΙΑ 2η
2020



ΑΝΑΓΝΩΡΙΣΗ ΠΡΟΤΥΠΩΝ

ΚΑΡΑΠΕΠΕΡΑ ΕΛΠΙΔΑ
57423

ΠΙΘΑΝΟΤΗΤΕΣ – ΤΥΧΑΙΕΣ ΜΕΤΑΒΛΗΤΕΣ

Ερώτημα 1:

$$P[A] - P[A \cap B] = \frac{N_1}{N_1+N_2+N_3+N_4} = \frac{N_1}{N_{total}}$$

$$P[B] - P[A \cap B] = \frac{N_2}{N_1+N_2+N_3+N_4} = \frac{N_2}{N_{total}}$$

$$P[A \cap B] = \frac{N_3}{N_1+N_2+N_3+N_4} = \frac{N_3}{N_{total}}$$

$$P[A|B] = \frac{P[A\cap B]}{P[B]} = \frac{P[A\cap B]}{P[B]-P[A\cap B]+P[A\cap B]} = \frac{\frac{N_3}{N_{total}}}{\frac{N_2}{N_{total}}+\frac{N_3}{N_{total}}} = \frac{N_3}{N_2+N_3}$$

BAYESIAN ΘΕΩΡΙΑ ΑΠΟΦΑΣΕΩΝ

Ερώτημα 2:

$p(\omega_1) = 0.3\ ,\ \ p(\omega_2) = 0.3\ \ \ \acute{\alpha}\rho\alpha \rightarrow\ \ \ p(\omega_3) = 0.4$

Επίσης ισχύουν:

$p(x = 1) = \sum_i p(x = 1|\omega_i) * p(\omega_i) = 0.3 * 0.3 + 0.2 * 0.3 + 0.1 * 0.4 = 0.19$

$p(x = 2) = \sum_i p(x = 2|\omega_i) * p(\omega_i) = 0.2 * 0.3 + 0.2 * 0.3 + 0.3 * 0.4 = 0.24$

$p(x = 3) = \sum_i p(x = 3|\omega_i) * p(\omega_i) = 0.1 * 0.3 + 0.4 * 0.3 + 0.15 * 0.4 = 0.21$

$p(x = 4) = \sum_i p(x = 4|\omega_i) * p(\omega_i) = 0.1 * 0.3 + 0.05 * 0.3 + 0.05 * 0.4 = 0.065$

$p(x = 5) = \sum_i p(x = 5|\omega_i) * p(\omega_i) = 0.2 * 0.3 + 0.1 * 0.3 + 0.3 * 0.4 = 0.21$

$p(x = 6) = \sum_i p(x = 6|\omega_i) * p(\omega_i) = 0.1 * 0.3 + 0.05 * 0.3 + 0.1 * 0.4 = 0.085$

Πλέον, μπορεί εύκολα να υπολογιστεί ο καινούριος πίνακας $P(\omega_i|x_i)$:

$$P(\omega_1|x = 1) = \frac{p(x = 1|\omega_1)*p(\omega_1)}{p(x=1)} = \frac{0.3*0.3}{0.19} = 0.4737$$

$$P(\omega_1|x = 2) = \frac{p(x = 2|\omega_1)*p(\omega_1)}{p(x=2)} = \frac{0.2*0.3}{0.24} = 0.25$$

$$P(\omega_1|x=3) = \frac{p(x=3|\omega_1)*p(\omega_1)}{p(x=3)} = \frac{0.1*0.3}{0.21} = 0.1429$$

$$P(\omega_1|x=4) = \frac{p(x=4|\omega_1)*p(\omega_1)}{p(x=4)} = \frac{0.1*0.3}{0.065} = 0.4615$$

$$P(\omega_1|x=5) = \frac{p(x=5|\omega_1)*p(\omega_1)}{p(x=5)} = \frac{0.2*0.3}{0.21} = 0.2857$$

$$P(\omega_1|x=6) = \frac{p(x=6|\omega_1)*p(\omega_1)}{p(x=6)} = \frac{0.1*0.3}{0.085} = 0.3529$$

$$P(\omega_2|x=1) = \frac{p(x=1|\omega_2)*p(\omega_2)}{p(x=1)} = \frac{0.2*0.3}{0.19} = 0.3158$$

$$P(\omega_2|x=2) = \frac{p(x=2|\omega_2)*p(\omega_2)}{p(x=2)} = \frac{0.2*0.3}{0.24} = 0.25$$

$$P(\omega_2|x=3) = \frac{p(x=3|\omega_2)*p(\omega_2)}{p(x=3)} = \frac{0.4*0.3}{0.21} = 0.5714$$

$$P(\omega_2|x=4) = \frac{p(x=4|\omega_2)*p(\omega_2)}{p(x=4)} = \frac{0.05*0.3}{0.065} = 0.2308$$

$$P(\omega_2|x=5) = \frac{p(x=5|\omega_2)*p(\omega_2)}{p(x=5)} = \frac{0.1*0.3}{0.21} = 0.1429$$

$$P(\omega_2|x=6) = \frac{p(x=6|\omega_2)*p(\omega_2)}{p(x=6)} = \frac{0.05*0.3}{0.085} = 0.1765$$

$$P(\omega_3|x=1) = \frac{p(x=1|\omega_3)*p(\omega_3)}{p(x=1)} = \frac{0.1*0.4}{0.19} = 0.2105$$

$$P(\omega_3|x=2) = \frac{p(x=2|\omega_3)*p(\omega_3)}{p(x=2)} = \frac{0.3*0.4}{0.24} = 0.5$$

$$P(\omega_3|x=3) = \frac{p(x=3|\omega_3)*p(\omega_3)}{p(x=3)} = \frac{0.15*0.4}{0.21} = 0.2857$$

$$P(\omega_3|x=4) = \frac{p(x=4|\omega_3)*p(\omega_3)}{p(x=4)} = \frac{0.05*0.4}{0.065} = 0.3077$$

$$P(\omega_3|x=5) = \frac{p(x=5|\omega_3)*p(\omega_3)}{p(x=5)} = \frac{0.3*0.4}{0.21} = 0.5714$$

$$P(\omega_3|x=6) = \frac{p(x=6|\omega_3)*p(\omega_3)}{p(x=6)} = \frac{0.1*0.4}{0.085} = 0.4706$$

$P(\omega_i / x_i)$						
	X=1	X=2	X=3	X=4	X=5	X=6
ω1	0.4737	0.25	0.1429	0.4615	0.2857	0.3529
ω_2	0.3158	0.25	0.5714	0.2308	0.1429	0.1765
ω_3	0.2105	0.5	0.2857	0.3077	0.5714	0.4706

$$P_{e_i} = 1 - \max_k\big(P(\omega_k, x_i)\big)$$

Άρα:

$P_{e_1} = 1 - \max_k\big(P(\omega_k, x_{\mathbf{1}})\big) = 1 - \max_k(0.4737, 0.3158, 0.2105) = 1 - 0.4737 =$
$= 0.5263$

$P_{e_2} = 1 - \max_k\big(P(\omega_k, x_{\mathbf{2}})\big) = 1 - \max_k(0.25, 0.25, 0.5) = 0.5$

$P_{e_3} = 1 - \max_k\big(P(\omega_k, x_{\mathbf{3}})\big) = 1 - \max_k(0.1429, 0.5714, 0.2857) = 0.4286$

$P_{e_4} = 1 - \max_k\big(P(\omega_k, x_4)\big) = 1 - \max_k(0.4615, 0.2308, 0.3077) = 0.5385$

$P_{e_5} = 1 - \max_k\big(P(\omega_k, x_5)\big) = 1 - \max_k(0.2857, 0.1429, 0.5714) = 0.4286$

$P_{e_6} = 1 - \max_k\big(P(\omega_k, x_{\mathbf{6}})\big) = 1 - \max_k(0.3529, 0.1765, 0.4706) = 0.5294$

$P_{e_{total}} = \sum_i P_{e_i} * P(x_i) =$

$= P_{e_1} * p(x_1) + P_{e_2} * p(x_2) + P_{e_3} * p(x_3) + P_{e_4} * p(x_4) + P_{e_5} * p(x_5) + P_{e_6} * p(x_6) =$
$= 0.5263 * 0.19 + 0.5 * 0.24 + 0.4286 * 0.21 + 0.5385 * 0.065 + 0.4286 * 0.21 +$
$0.5294 * 0.085 = 0.48$

Ερώτημα 3:

$\boldsymbol{p(\overline{x}|\omega_1) = \mathrm{N}(\overline{\mu_1}, \overline{\Sigma_1})}, \quad \boldsymbol{p(\overline{x}|\omega_2) = \mathrm{N}(\overline{\mu_2}, \overline{\Sigma_2})}$

$\boldsymbol{\overline{\mu_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}, \ \boldsymbol{\overline{\Sigma_1} = \begin{pmatrix} 4 & 1 \\ 1 & 9 \end{pmatrix}}, \quad \boldsymbol{\overline{\mu_2} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}}, \ \boldsymbol{\overline{\Sigma_2} = \begin{pmatrix} 4 & 1 \\ 1 & 9 \end{pmatrix}}$

$\mathbf{p(\omega_1) = 0,25}$

$p(\omega_1) = 0.25 \rightarrow p(\omega_2) = 0.75$

Εξίσωση επιφάνειας απόφασης:

$g_i(x) = W_i^T x + W_{i0}\,, \quad W_i = \Sigma^{-1}\mu_i\,, \quad W_{i0} = -\frac{1}{2}\mu_i^T \Sigma^{-1}\mu_i + \ln\big(P(\omega_i)\big)$

Οι εξισώσεις υπολογίστηκαν με τη βοήθεια του Matlab και το αποτέλεσμα είναι:

$W_1 = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, W_{10} = -1.6863, \qquad W_2 = \begin{bmatrix} -0.4857 \\ -0.0571 \end{bmatrix}, W_{20} = -0.802$

Επομένως, οι εξισώσεις θα είναι:
$g_1(x) = 0.2 * x_1 + 0.2 * x_2 - 1.6863$
$g_2(x) = -0.4857 * x_1 - 0.0571 * x_2 - 0.802$

Τελικά:
$d = g_1(x) - g_2(x) = 0.6857 * x_1 + 0.2571 * x_2 - 0.8843$

Και, επειδή $d = W^T x + W_0$*, αντιστοιχίζοντας με την πάνω εξίσωση θα είναι* $W = \begin{bmatrix} 0.6857 \\ 0.2571 \end{bmatrix}$ *και* $W_0 = -0.8843$.

Με σκοπό να κάνουμε το σύστημα 1 διάστασης αντί για 2 ορίζουμε μια νέα μεταβλητή $y = W^T x$*. Τότε, οι υπό συνθήκη πιθανότητες θα είναι:*
$P(\boldsymbol{x}|\omega_{\boldsymbol{i}}) = N(\mu_i, \Sigma_i) \rightarrow \ P(W^T x|\omega_{\boldsymbol{i}}) = N(W^T \mu_i, W^T \Sigma_i W^T)$

άρα
$P(W^T x|\omega_{\boldsymbol{1}}) = N(1.1999,\ 2.8282)$
$P(W^T x|\omega_{\boldsymbol{2}}) = N(-1.6285,\ 2.8282)$

Ισχύει ότι η πιθανότητα ορθής ταξινόμησης είναι:
$P_c = P(\omega_1) * \int_{R_1} P(W^T x|\omega_1) * d(W^T x) + P(\omega_2) * \int_{R_2} P(W^T x|\omega_2) * d(W^T x) =$
$= 0.25 * \int_{W_0}^{\infty} N(1.1999,\ 2.8282) dy + 0.75 * \int_{-\infty}^{W_0} N(-1.6285,\ 2.8282) dy =$
$= 0.25 * \int_{0.8843}^{\infty} N(1.1999,\ 2.8282) dy + 0.75 * \int_{-\infty}^{0.8843} N(-1.6285,$
$2.8282) dy =$
$= 0.25 - 0.25 * \Phi(\frac{0.8843 - 1.1999}{\sqrt{2.8282}}) + 0.75 * \Phi(\frac{0.8843 + 1.6285}{\sqrt{2.8282}}) \cong$ ***0.84***

Άρα το ολικό σφάλμα θα είναι $P_e = 1 - P_c = 1 - 0.84 = 0.16$

Ερώτημα 4:

A)

$\boldsymbol{P(x|\omega_1) = N(2, 0.5)}$, $\boldsymbol{P(x|\omega_2) = N(1.5, 0.2)}$

άρα $\mu_1 = 2, \sigma_1^2 = 0.5, \mu_2 = 1.5, \ \sigma_2^2 = 0.2$

$\boldsymbol{P(\omega_1) = \frac{1}{3}}, \quad \boldsymbol{P(\omega_2) = \frac{2}{3}}, \quad \boldsymbol{\lambda = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}}$

Είναι: $P(x|\omega_i) = \frac{1}{\sigma_i * \sqrt{2\pi}} e^{-\frac{(x-\mu_i)^2}{2\sigma_i^2}}$

Άρα:

$$P(x|\omega_1) = \frac{1}{\sqrt{0.5*2\pi}} e^{-\frac{(x-2)^2}{2*0.5}} = \frac{1}{\sqrt{\pi}} e^{-(x-2)^2}$$

$$P(x|\omega_2) = \frac{1}{\sqrt{0.2*2\pi}} e^{-\frac{(x-1.5)^2}{2*0.2}} = \frac{1}{\sqrt{0.4\pi}} e^{-\frac{(x-1.5)^2}{0.4}}$$

Για να επιλέξω το ω_1 *πρέπει να ισχύει η ανίσωση:*

$$\frac{P(x|\omega_1)}{P(x|\omega_2)} \geq \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} * \frac{P(\omega_2)}{P(\omega_1)} \Rightarrow \frac{\frac{1}{\sqrt{\pi}} e^{-(x-2)^2}}{\frac{1}{\sqrt{0.4\pi}} e^{-\frac{(x-1.5)^2}{0.4}}} \geq \frac{2-1}{3-1} * \frac{\frac{2}{3}}{\frac{1}{3}} \Rightarrow$$

$$\Rightarrow \sqrt{0.4} * e^{-(x-2)^2 + \frac{(x-1.5)^2}{0.4}} \geq \frac{1}{2} * 2 \Rightarrow e^{-(x-2)^2 + \frac{(x-1.5)^2}{0.4}} \geq 1.5811 \Rightarrow$$

$$\Rightarrow -(x-2)^2 + \frac{(x-1.5)^2}{0.4} \geq \ln(1.5811) \Rightarrow -(x-2)^2 + \frac{(x-1.5)^2}{0.4} \geq 0.4581 \Rightarrow$$

$$\Rightarrow -x^2 + 4*x - 4 + 2.5*(x^2 - 3x + 2.25) \geq 0.4581 \Rightarrow$$

$$\Rightarrow 1.5*x^2 - 3.5*x + 1.625 \geq 0.4581 \Rightarrow 1.5*x^2 - 3.5*x + 1.1669 \geq 0$$

Άρα αν $x < 0.403$ ή $x > 1.93$ η βέλτιστη λύση είναι ω_1 και αν $0.403 < x < 1.93$ ω_2.

Το ελάχιστο κόστος είναι:

$$C = p(\omega_1) * \left[\lambda_{11} \int_{R_1} f_{x_1}(x_1|H_1)dx + \lambda_{21} \int_{R_2} f_{x_1}(x_1|H_1)dx\right] + p(\omega_2) *$$
$$\left[\lambda_{22} \int_{R_2} f_{x_1}(x_1|H_2)dx + \lambda_{12} \int_{R_1} f_{x_1}(x_1|H_2)dx\right] =$$

$$= \frac{1}{3} * \left[\int_{-\infty}^{0.403} P(x|\omega_1)dx + \int_{1.93}^{\infty} P(x|\omega_1)dx + 3\int_{0.403}^{1.93} P(x|\omega_2)dx\right] + \frac{2}{3} * \left[\int_{0.403}^{1.93} P(x|\omega_1)dx + 2\int_{-\infty}^{0.403} P(x|\omega_1)dx + 2\int_{1.93}^{\infty} P(x|\omega_2)dx\right] =$$

$$= \frac{1}{3} * \left[\int_{-\infty}^{0.403} N(2, 0.5)dx + \int_{1.93}^{\infty} N(2, 0.5)dx + 3\int_{0.403}^{1.93} N(1.5, 0.2)dx\right] + \frac{2}{3} * \left[\int_{0.403}^{1.93} N(2, 0.5)dx + 2\int_{-\infty}^{0.403} N(2, 0.5)dx + 2\int_{1.93}^{\infty} N(1.5, 0.2)dx\right] =$$

$$= \frac{5}{3} * \int_{-\infty}^{0.403} N(2, 0.5)dx + \frac{5}{3}\int_{1.93}^{\infty} N(2, 0.5)dx + \frac{5}{3}\int_{0.403}^{1.93} N(1.5, 0.2)dx =$$

$$= \frac{5}{3} * \Phi\left(\frac{0.403-2}{\sqrt{0.5}}\right) + \frac{5}{3} * (1 - \Phi\left(\frac{1.93-2}{\sqrt{0.5}}\right)) + + \frac{5}{3}\int_{-\infty}^{1.93} N(1.5, 0.2)dx + \frac{5}{3}\int_{-\infty}^{0.403} N(1.5, 0.2)dx =$$

$$= \frac{5}{3} * \Phi(-2.2585) + \frac{5}{3} - \frac{5}{3} * \Phi(-0.099) + \frac{5}{3}\Phi\left(\frac{1.93-1.5}{\sqrt{0.2}}\right) - \frac{5}{3}\Phi\left(\frac{0.403-1.5}{\sqrt{0.2}}\right) =$$

$$= 1.4$$

B)

Παρατηρούμε ότι το αποτέλεσμα που προκύπτει από τον επισυναπτόμενο κώδικα (ergasia2.m) έχει απόκλιση 0.19048%, η οποία πιθανότατα ευθύνεται σε στρογγυλοποιήσεις. Δεν ευθύνεται σε μικρό μέγεθος των διανυσμάτων, καθώς δοκιμάστηκαν και μεγαλύτερα διανύσματα, χωρίς να αλλάξει το αποτέλεσμα.