DRAFT: A Family of Sparse, Vertex-Transitive Graphs

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1 Construction

We recursively construct a family of vertex-transitive graphs $G_n = \langle V_n, E_n \rangle$. The vertices $v \in V_n$ are labeled by an n-sequence of integers. We define G_0 and G_1 as:

$$V_0 = \{\langle\rangle\}, \quad (1)$$
$$E_0 = \{\}, \quad (2)$$
$$V_1 = \{\langle 0 \rangle, \langle 1 \rangle\}, \quad (3)$$
$$E_1 = \{\{\langle 0 \rangle, \langle 1 \rangle\}\}. \quad (4)$$

We construct subsequent vertex sets from copies of the previous set, with each copy having a different integer appended to its vertex labels:

$$V_{n+1} = \bigcup_{k=0}^{C_n} \{v | w \in V_n \wedge v = w \nleftarrow k\}, \quad (5)$$
$$C_n \equiv |V_n|, \quad (6)$$

where $w \nleftarrow k$ denotes appending element k to the end of sequence w. We note that a one-to-one mapping z_n exists between the vertices of G_n and the integers from 0 to $C_n - 1$:

$$z_n(v) = \sum_{i=0}^{n-1} C_i v_i. \quad (7)$$

We define even and odd subsets of V_n:

$$A_n = \{v | v \in V_n \wedge v_0 = 1\}, \quad (8)$$
$$B_n = \{v | v \in V_n \wedge v_0 = 0\}. \quad (9)$$

We also define the following mappings on the vertices of G_n:

$$(\phi_n(v))_i = \begin{cases} v_i + 1 \bmod (C_i+1) & \text{if } i = n, \\ v_i & \text{otherwise,} \end{cases} \tag{10}$$

$$(\psi_n(v))_i = \begin{cases} v_i + 1 \bmod (C_i+1) & \text{if } i = 0 \vee \forall j < i : v_j = C_j, \\ v_i & \text{otherwise,} \end{cases} \tag{11}$$

$$\theta_n(v) = \begin{cases} \psi_n(v) & \text{if } v \in A_n, \\ \psi_n^{-1}(v) & \text{if } v \in B_n, \end{cases} \tag{12}$$

with $i < n$.

To construct the edges of G_{n+1} we define a "shortcut" function $s_n(j,k)$ with $j \in \{0, 1, C_n - 1\}$ and $k \in \{0, 1, 2, \ldots, C_n\}$ which determines the interconnections between copies of G_n.

$$s_n(j,k) = \begin{cases} k+1 \bmod (C_n+1) & \text{if } j = C_n - 1, \\ k-1 \bmod (C_n+1) & \text{if } j = 0, \\ k+j+1 \bmod (C_n+1) & \text{if } j \in \{1, 3, \ldots, C_n - 3\}, \\ k-j \bmod (C_n+1) & \text{if } j \in \{2, 4, \ldots, C_n - 2\}. \end{cases} \tag{13}$$

We also define a parity selector function on the edges $e \in E_n$:

$$p_x(e) = v \quad \text{s.t. } v \in e \wedge v_0 = x. \tag{14}$$

The edges of G_{n+1} are then given by:

$$R_{n+1} = \bigcup_{k=0}^{C_n} \bigcup_{e \in E_n} \{p_0(e) \leftarrow\!\!\!\!\!+ k, p_1(e) \leftarrow\!\!\!\!\!+ k\}, \tag{15}$$

$$S_{n+1} = \bigcup_{k=0}^{C_n} \bigcup_{v \in A_n} \{v \leftarrow\!\!\!\!\!+ k, \theta_n(v) \leftarrow\!\!\!\!\!+ s_n(z_n(v), k)\}, \tag{16}$$

$$= \bigcup_{k=0}^{C_n} \bigcup_{v \in B_n} \{v \leftarrow\!\!\!\!\!+ k, \theta_n(v) \leftarrow\!\!\!\!\!+ s_n(z_n(v), k)\}, \tag{17}$$

$$E_{n+1} = R_{n+1} \cup S_{n+1}, \tag{18}$$

noting that S_{n+1} can be written in terms of either the odd vertices A_n or the even vertices B_n. Eq. (15) replicates the edges of G_n among subsets of the vertices of G_{n+1}, while Eq. (16) creates one edge between each pair of the subsets.

2 Properties of G_n

Lemma 1. *The graph G_n is n-regular for all $n \geq 0$.*

Proof. We proceed using induction on n. The base case G_1 is 1-regular by inspection. In the inductive case G_{n+1}, Eq. (15) reproduces the edges of G_n, which is n-regular by induction, contributing n to the degree of each vertex. Eq. (16) adds one to the degree of each odd vertex. As θ_n is a bijective map between odd and even vertices, Eq. (16) also adds one to the degree of each even vertex, giving a total degree of $n+1$. $\square$

Lemma 2. *The number of vertices and edges of the graph G_n are given by the recurrence relations:*

$$C_0 = |V_0| = 1, \tag{19}$$

$$C_n = |V_n| = C_{n-1}(C_{n-1}+1), \tag{20}$$

$$|E_n| = \frac{n}{2}C_n. \tag{21}$$

Proof. The vertex set of G_n, given by Eq. (5), is a union of $C_{n-1}+1$ sets. The vertex labels within each set all end with the same element, and this element is unique to each set. The sets are thus disjoint. Each set contains C_{n-1} elements, giving $C_{n-1}(C_{n-1}+1)$ elements. By Lemma 1, G_n is n-regular and has $\frac{n}{2}|V_n|= \frac{n}{2}C_n$ edges. $\square$

Lemma 3. *The mapping ϕ_{n-1} is an automorphism of G_n.*

Proof. Let v be a vertex in V_n. ϕ_{n-1} has no effect on v_0^{n-2} so it preserves the edges R_n, defined in Eq. (15). The vertex v has exactly one other edge, from S_n defined in Eq. (16):

$$e = \{v, \theta_n(v_0^{n-2} \leftarrow\!\!\!+ s_n(z_n(v_0^{n-2}), v_{n-1})\}. \tag{22}$$

Applying ϕ_{n-1} to both vertices gives:

$$\begin{aligned} \tilde{e} &= \{v_0^{n-2} \leftarrow\!\!\!+ (v_{n-1}+1) \bmod (C_{n-2}+1), &(23)\\ &\qquad \theta_n(v_0^{n-2} \leftarrow\!\!\!+ s_n(z_n(v_0^{n-2}), (v_{n-1}+1) \bmod (C_{n-2}+1))\}, &(24)\\ &= \{v_0^{n-2} \leftarrow\!\!\!+ \tilde{k}, \theta_n(v_0^{n-2} \leftarrow\!\!\!+ s_n(z_n(v_0^{n-2}), \tilde{k})\}, &(25)\end{aligned}$$

where $\tilde{k} = (v_{n-1}+1) \bmod (C_{n-2}+1)$. The edge $\tilde{e}$ is also a member of S_n, showing that ϕ_{n-1} permutes the elements of S_n, and preserves all edges in E_n. $\square$

Theorem 1. *The graphs G_n are vertex-transitive for all $n \geq 0$.*

Proof. The graph G_0 is trivially vertex-transitive. For higher order graphs, we recursively construct a mapping Φ_n that maps an arbitrary vertex $v \in V_n$ to an arbitrary vertex $\tilde{v} \in V_n$ and preserves the edges in E_n. Let $\delta_i = \tilde{v}_i - v_i \bmod (C_i+1)$. For the subsequence v_0^0, the map $\Phi_1 = \phi_0^{\delta_0}$ achieves the desired mapping and preserves the edges in E_1 by Lemma 3. Given an edge-preserving mapping Φ_n from v_0^{n-1} to $\tilde{v}_0^{n-1}$, we can construct τ_n. We define mappings π_n and τ_n:

$$\pi_n(0) = 0 \tag{26}$$

$$\pi_n(s_n(v_0^{n-1}, 0)) = s_n(\Phi_n(v_0^{n-1}), 0) \tag{27}$$

$$\tau_n(v) = \Phi_n(v_0^{n-1}) \leftarrow\!\!\!+ \pi_n(v_n). \tag{28}$$

For the recursive edges defined in Eq. (15) $\{v_0^n, w_0^n\} \in R_{n+1}$:

$$\begin{aligned}
\{\tau_n(v_0^n), \tau_n(v_0^n) &= \{\Phi_n(v_0^n) \nleftarrow \pi_n(k), \Phi_n(w_0^n) \nleftarrow \pi_n(k)\} && (29)\\
&= \{\tilde{v}_0^n \nleftarrow \tilde{k}, \Phi_n(\tilde{w}_0^n) \nleftarrow \tilde{k}\}, && (30)
\end{aligned}$$

which is simply another member of R_{n+1}. As τ_n is bijective, it permutes the edges of R_{n+1}. For the shortcut edges defined in Eq. (16) $\{v_0^n, w_0^n\} \in S_{n+1}$, when $v_n = 0$:

$$\begin{aligned}
\{\tau_n(v_0^n), \tau_n(w_0^n)\} &= \{\Phi_n(v_0^n) \nleftarrow \pi_n(0), && (31)\\
&\qquad \Phi_n(w_0^n)) \nleftarrow \pi_n(s_n(z_n(v_0^n), 0))\} && (32)\\
&= \{\tilde{v}_0^n \nleftarrow 0, \tilde{w}_0^n \nleftarrow s_n(z_n(\Phi_n(v_0^n)), 0)\} && (33)\\
&= \{\tilde{v}_0^n \nleftarrow 0, \theta_n(\tilde{v}_0^n) \nleftarrow s_n(z_n(\tilde{v_0^n}), 0)\}, && (34)
\end{aligned}$$

showing that τ_n permutes the subset of edges in S_{n+1} for which $v_n = 0$. Consequently, because ϕ_n is an automorphism of G_{n+1} by Lemma 3, τ_n also permutes the edges of the subsets for which $(\phi_n^{-k}(v_0^n)_n = 0$. Those edges are the subset of S_{n+1} for which $v_n = k$, showing that $\Phi_n n + 1$ permutes all edges within S_{n+1} and E_{n+1}. We finish by applying the automorphism $\phi_n^{\delta_n}$:

$$\Phi_{n+1}(v) = \phi_n^{\delta_n}(\tau_n(v)), \qquad (35)$$

completing the mapping from $\tilde{v}$ to $\tilde{w}$. □

References

[1] I. Z. Bower. *The Foster Census.* Charles Babbage Research Centre, Winnipeg, 1988.