

DRAFT: A Family of Sparse, Vertex-Transitive Graphs

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1 Construction

We recursively construct a family of vertex-transitive graphs $G_n = \langle V_n, E_n \rangle$. The vertices $v \in V_n$ are labeled by an n -sequence of integers. We define G_0 and G_1 as:

$$V_0 = \{\langle \rangle\}, \quad (1)$$

$$E_0 = \{\}, \quad (2)$$

$$V_1 = \{\langle 0 \rangle, \langle 1 \rangle\}, \quad (3)$$

$$E_1 = \{\{\langle 0 \rangle, \langle 1 \rangle\}\}. \quad (4)$$

We construct subsequent vertex sets from copies of the previous set, with each copy having a different integer appended to its vertex labels:

$$V_{n+1} = \bigcup_{k=0}^{C_n} \{v | w \in V_n \wedge v = w \leftarrow k\}, \quad (5)$$

$$C_n \equiv |V_n|, \quad (6)$$

where $w \leftarrow k$ denotes appending element k to the end of sequence w . We note that a one-to-one mapping z_n exists between the vertices of G_n and the integers from 0 to $C_n - 1$:

$$z_n(v) = \sum_{k=0}^{n-1} C_k v_k. \quad (7)$$

We define even and odd subsets of V_n :

$$A_n = \{v | v \in V_n \wedge v_0 = 1\}, \quad (8)$$

$$B_n = \{v | v \in V_n \wedge v_0 = 0\}. \quad (9)$$

We also define the following mappings on the vertices of G_n :

$$(\phi_n(v))_i = \begin{cases} v_i + 1 \bmod (C_i + 1) & \text{if } i = n, \\ v_i & \text{otherwise,} \end{cases} \quad (10)$$

$$(\psi_n(v))_i = \begin{cases} v_i + 1 \bmod (C_i + 1) & \text{if } i = 0 \vee \forall j < i : v_j = C_j, \\ v_i & \text{otherwise,} \end{cases} \quad (11)$$

$$\theta_n(v) = \begin{cases} \psi_n(v) & \text{if } v \in A_n, \\ \psi_n^{-1}(v) & \text{if } v \in B_n, \end{cases} \quad (12)$$

with $i \leq n$.

To construct the edges of G_{n+1} we define a “shortcut” function $s_n(j, k)$ with $j \in \{0, 1, C_n - 1\}$ and $k \in \{0, 1, 2, \dots, C_n\}$ which determines the interconnections between copies of G_n .

$$s_n(j, k) = \begin{cases} k + 1 \bmod (C_n + 1) & \text{if } j = C_n - 1, \\ k - 1 \bmod (C_n + 1) & \text{if } j = 0, \\ k + j + 1 \bmod (C_n + 1) & \text{if } j \in \{1, 3, \dots, C_n - 3\}, \\ k - j \bmod (C_n + 1) & \text{if } j \in \{2, 4, \dots, C_n - 2\}. \end{cases} \quad (13)$$

We also define a parity selector function on edges $e \in E_n$:

$$p_i(e) = v \quad \text{s.t. } v \in e \wedge v_0 = i. \quad (14)$$

The edges of G_{n+1} are then given by:

$$R_{n+1} = \bigcup_{k=0}^{C_n} \bigcup_{e \in E_n} \{p_0(e) \leftarrow k, p_1(e) \leftarrow k\}, \quad (15)$$

$$S_{n+1} = \bigcup_{k=0}^{C_n} \bigcup_{v \in A_n} \{v \leftarrow k, \theta_n(v) \leftarrow s_n(z_n(v), k)\}, \quad (16)$$

$$= \bigcup_{k=0}^{C_n} \bigcup_{v \in B_n} \{v \leftarrow k, \theta_n(v) \leftarrow s_n(z_n(v), k)\}, \quad (17)$$

$$E_{n+1} = R_{n+1} \cup S_{n+1}, \quad (18)$$

noting that S_{n+1} can be written in terms of either the odd vertices A_n or the even vertices B_n . Eq. (15) replicates the edges of G_n among subsets of the vertices of G_{n+1} , while Eq. (16) creates one edge between each pair of the subsets.

2 Properties of G_n

Lemma 1. *The graph G_n is n -regular for all $n \geq 0$.*

Proof. We proceed using induction on n . The base case G_1 is 1-regular by inspection. In the inductive case G_{n+1} Eq. (15) reproduces the edges of G_n , which is n -regular by induction, contributing n to the degree of each vertex. Eq. (16) adds one to the degree of each odd vertex. As θ_n is a bijective map between odd and even vertices, Eq. (16) also adds one to the degree of each even vertex, giving a total degree of $n + 1$. \square

Lemma 2. *The number of vertices and edges of the graph G_n are given by the recurrence relations:*

$$C_0 = |V_0| = 1, \quad (19)$$

$$C_n = |V_n| = C_{n-1}(C_{n-1} + 1), \quad (20)$$

$$|E_n| = \frac{n}{2} C_n. \quad (21)$$

Proof. The vertex set of G_n , given by Eq. (5), is a union of $C_{n-1} + 1$ sets. The vertex labels within each set all end with the same element, and this element is unique to each set. The sets are thus disjoint. Each set contains C_{n-1} elements, giving $C_{n-1}(C_{n-1} + 1)$ elements. By Lemma 1, G_n is n -regular and has $\frac{n}{2}|V_n| = \frac{n}{2}C_n$ edges. \square

Theorem 1. *The graphs G_n are vertex-transitive for all $n \geq 0$.*

Proof. We begin by showing that ϕ_{n-1} is an automorphism of G_n . Let v be a vertex in V_n . ϕ_{n-1} has no effect on v_0^{n-2} so it preserves the edges defined in Eq. (15). The vertex v has exactly one other edge:

$$e = \{v, \theta_n(v_0^{n-2} \leftarrow s_n(z_n(v_0^{n-2}), v_{n-1}))\}. \quad (22)$$

Applying ϕ_{n-1} to both vertices gives:

$$\tilde{e} = \{v_0^{n-2} \leftarrow (v_{n-1} + 1) \bmod (C_{n-2} + 1), \quad (23)$$

$$\theta_n(v_0^{n-2} \leftarrow s_n(z_n(v_0^{n-2}), (v_{n-1} + 1) \bmod (C_{n-2} + 1)))\}, \quad (24)$$

$$= \{v_0^{n-2} \leftarrow \tilde{k} \bmod (C_{n-2} + 1), \theta_n(v_0^{n-2} \leftarrow s_n(z_n(v_0^{n-2}), \tilde{k}))\}, \quad (25)$$

where $\tilde{k} = (v_{n-1} + 1) \bmod (C_{n-2} + 1)$. The edge \tilde{e} is also a member of S_n in Eq. (16), showing that ϕ_{n-1} permutes the elements of S_n , and preserves all edges in E_n .

TODO: show that ϕ_{n-k} when combined with a permutation $\pi_{n,k}$ preserves edges. Use ϕ to construct an automorphism mapping any v to any w . \square

References

- [1] I. Z. Bower. *The Foster Census*. Charles Babbage Research Centre, Winnipeg, 1988.