# DRAFT: A Family of Sparse, Vertex-Transitive Graphs

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## 1 Construction

We recursively construct a family of vertex-transitive graphs $G_n = \langle V_n, E_n \rangle$. The vertices $v \in V_n$ are labeled by an $n$-sequence of integers. We define the base case, $G_1$ as:

$$V_1 = \{\langle 0 \rangle, \langle 1 \rangle\}, \tag{1}$$

$$E_1 = \{\{\langle 0 \rangle, \langle 1 \rangle\}\}. \tag{2}$$

We construct subsequent vertex sets from copies of the previous set, with each copy having a different integer appended to its vertex labels:

$$V_{n+1} = \bigcup_{k=0}^{C_n} \{v | w \in V_n \wedge v = w \leftarrow\!\!+ k\}, \tag{3}$$

$$C_n \equiv |V_n|, \tag{4}$$

where $w \leftarrow\!\!+ k$ denotes appending element $k$ to the end of sequence $w$. We note that a one-to-one mapping $z_n$ exists between the vertices of $G_n$ and the integers from 0 to $C_n - 1$:

$$z_n(v) = \sum_{k=0}^{n-1} C_k v_k. \tag{5}$$

We also define the following mappings on the vertices of $G_n$:

$$(\phi_n(v))_k = \begin{cases} v_k + 1 \bmod (C_k + 1) & \text{if } k = n, \\ v_k & \text{otherwise,} \end{cases} \tag{6}$$

$$(\psi_n(v))_k = \begin{cases} v_k + 1 \bmod (C_k + 1) & \text{if } k = 0 \vee \forall j < k : v_j = C_j, \\ v_k & \text{otherwise.} \end{cases} \tag{7}$$

To construct the edges of $G_{n+1}$ we define a "shortcut" function $s_{n,j}(k)$ with $j \in \{1, 3, \ldots, C_n - 1\}$ and $k \in \{0, 1, 2, \ldots, C_n\}$ which determines the interconnections between copies of $G_n$.

$$s_n(j,k) \quad = \quad \begin{cases} k+1 \bmod (C_n+1) & \text{if } j = C_n - 1 \\ j+k+1 \bmod (C_n+1) & \text{otherwise.} \end{cases} \tag{8}$$

We also define a parity selector function on edges $e \in E_n$:

$$p_i(e) \quad = \quad v \quad \text{s.t. } v \in e \wedge v_0 = i. \tag{9}$$

The edges of $G_{n+1}$ are then given by:

$$E_{n+1} \quad = \quad \bigcup_{k=0}^{C_n} \Bigg[ \bigcup_{e \in E_n} \{p_0(e) \nleftrightarrow k, p_1(e) \nleftrightarrow k\} \tag{10}$$

$$\cup \bigcup_{v \in O_n} \{v \nleftrightarrow k, \psi_n(v) \nleftrightarrow s_n(z_n(v), k)\} \Bigg], \tag{11}$$

$$O_n \quad \equiv \quad \{v | v \in V_n \wedge v_0 = 1\}. \tag{12}$$

## 2 Properties of $G_n$

**Lemma 1.** *The graph $G_n$ is $n$-regular for all $n \geq 1$.*

*Proof.* We proceed using induction on $n$. The base case $G_1$ is 1-regular by inspection. In the inductive case $G_{n+1}$ the first union term of Eq. 10 reproduces the edges of $G_n$, which is $n$-regular by induction, contributing $n$ to the degree of each vertex. The second union term adds one to the degree of each odd vertex. As $\psi_n$ is a bijective map between odd and even vertices, the second union term also adds one to the degree of each even vertex, giving a total degree of $n+1$. □

**Lemma 2.** *The number of vertices and edges of the graph $G_n$ are given by the recurrence relations:*

$$C_0 = |V_0| \quad = \quad 1, \tag{13}$$

$$C_n = |V_n| \quad = \quad C_{n-1}(C_{n-1}+1), \tag{14}$$

$$|E_n| \quad = \quad \frac{n}{2} C_n. \tag{15}$$

*Proof.* The vertex set of $G_n$ (Eq. 3) is a union of $C_{n-1}+1$ sets. The vertex labels within each set all end with the same element, and this element is unique to each set. The sets are thus disjoint. Each set contains $C_{n-1}$ elements, giving $C_{n-1}(C_{n-1}+1)$ elements. By Lemma 1, $G_n$ is $n$-regular and has $\frac{n}{2}|V_n| = \frac{n}{2}C_n$ edges. □

**Theorem 1.** *The graphs $G_n$ are vertex-transitive for all $n \geq 0$.*

*Proof.* □

## References

[1] I. Z. Bower. *The Foster Census*. Charles Babbage Research Centre, Winnipeg, 1988.