A Family of Vertex Transitive Graphs

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1 Sylvester Number System

Let s_n be the nth element of Sylvester's sequence [1], defined as:

$$s_0 = 2 \tag{1}$$

$$s_{n+1} = 1 + \prod_{i=0}^{n} s_n. \tag{2}$$

A mixed-base number system can be constructed from Sylvester's sequence as follows:

Definition 1.1. A *Sylvester-radix* number a is a sequence of digits a_n such that: $a_n \in \mathbb{Z} : 0 \leq a_n < s_n$.

Lemma 1. There are $(s_n - 1)$ Sylvester-radix numbers of length n.

Proof. The Sylvester-radix numbers of length 1 are (0) and (1). $(s_1 - 1) = 2$ so the lemma holds for $n = 1$.

For $n > 1$, there are s_i possible values for each digit, with $0 \leq i < n$. The number of valid digit combinations is thus given by:

$$\prod_{i=0}^{n-1} s_i = s_n - 1 \qquad \text{(by (2))}.$$

□

Corollary 1. The place value of index i in a Sylvester-radix number is $(s_i - 1)$.

The integer value of a length-n Sylvester-radix number a is thus:

$$z(a) = \sum_{i=0}^{n-1} a_i (s_i - 1). \tag{3}$$

2 Nested Clique

We now define a family of graphs, which we call *nested cliques*. For $n > 0$, the nested clique G_n is defined recursively in terms of G_{n-1}. We will construct G_n from a union of subgraphs, each of which is isomorphic to G_{n-1}. Lower-order graphs are thus recursively *nested* within higher-order graphs. In addition to the edges internal to each subgraph, we will add one *external edge* to each vertex of G_n, such that each pair of subgraphs is connected by exactly one directed edge in each direction, making G_n homomorphic to a clique.

Definition 2.1. A nested clique sequence is a sequence of graphs $G_n = (V_n, E_n)$ such that G_n is a union of disjoint subgraphs isomorphic to G_{n-1} and of a set of edges connecting each vertex in V_n to another vertex belonging to a different subgraph.

The above definition implies the following useful lemma.

Lemma 2. The number of vertices $N_n = |V_n|$ in a nested clique sequence is given by the recurrence relation:

$$N_n = N_{n-1}(N_{n-1} + 1). \tag{4}$$

Proof. TODO □

2.1 Construction

We now construct a particular family of graphs satisfying the requirements of Definition 2.1.

2.1.1 Base Case

We choose a base case containing a single vertex, which we label $\varnothing$:

$$\begin{aligned} G_0 &= (V_0, E_0) && (5)\\ V_0 &= \{\varnothing\} && (6)\\ E_0 &= \{\}. && (7) \end{aligned}$$

Lemma 3. Whith the above base case G_0, the number of vertices in G_n is given by:

$$N_n = 1, 2, 6, 42, 1806, ... \tag{8}$$

2.1.2 Vertices

For the recursive case, it is necessary to distinguish between vertices in different nested subgraphs. Each vertex is labeled by a sequence of integers. For G_n each subgraph is assigned an integer in $[0, N_{n-1}]$. For all vertices in a particular subgraph, labels are constructed by taking the corresponding label in G_{n-1} and appending the subgraph's integer label. Formally:

$$V_n = \bigcup_{i=0}^{N_{n-1}} \{v \ll i | v \in V_{n-1}\}, \tag{9}$$

where $a \ll b$ denotes appending element b to the end of sequence a. The vertices of G_n are thus integer sequences of length n, with the ith digit in $[0, N_{n-1}]$. This set of labels is exactly the set of length-n Sylvester-radix numbers.

Theorem 1. The vertices of a nested clique G_n having $V_0 = \{\varnothing\}$ are isomorphic to length-n Sylester-radix numbers.

Proof. TODO □

2.1.3 Edges

It is not obvious whether any configuration of edges can satisfy Definition 2.1. We now construct a set of operators which map source vertices to target vertices in a manner satisfying 2.1.

References

[1] James J Sylvester. On a point in the theory of vulgar fractions. *American Journal of Mathematics*, 3(4):332–335, 1880.