DRAFT: A Family of Sparse, Vertex-Transitive Graphs

Edward L. Platt

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1 Construction

We recursively construct a family of vertex-transitive graphs $G_n = \langle V_n, E_n \rangle$. The vertices $v \in V_n$ are labeled by an n-sequence of integers. We define G_0 and G_1 as:

$$
\begin{aligned}
V_0 &= \{\langle\rangle\}, & (1)\\
E_0 &= \{\}, & (2)\\
V_1 &= \{\langle 0 \rangle, \langle 1 \rangle\}, & (3)\\
E_1 &= \{\{\langle 0 \rangle, \langle 1 \rangle\}\}. & (4)
\end{aligned}
$$

We construct subsequent vertex sets from copies of the previous set, with each copy having a different integer appended to its vertex labels:

$$
V_{n+1} = \bigcup_{k=0}^{C_n} \{v | w \in V_n \wedge v = w \leftarrow\!\!+ k\}, \tag{5}
$$

$$
C_n \equiv |V_n|, \tag{6}
$$

where $w \leftarrow\!\!+ k$ denotes appending element k to the end of sequence w. We note that a one-to-one mapping z_n exists between the vertices of G_n and the integers from 0 to $C_n - 1$:

$$
z_n(v) = \sum_{k=0}^{n-1} C_k v_k. \tag{7}
$$

We also define the following mappings on the vertices of G_n:

$$
(\phi(v))_k = \begin{cases} v_k + 1 \bmod (C_k + 1) & \text{if } k = n, \\ v_k & \text{otherwise,} \end{cases} \tag{8}
$$

$$
(\psi_n(v))_k = \begin{cases} v_k + 1 \bmod (C_k + 1) & \text{if } k = 0 \vee \forall j < k : v_j = C_j, \\ v_k & \text{otherwise,} \end{cases} \tag{9}
$$

with $k \leq n$.

To construct the edges of G_{n+1} we define a "shortcut" function $s_n(j,k)$ with $j \in \{0, 1, C_n - 1\}$ and $k \in \{0, 1, 2, \ldots, C_n\}$ which determines the interconnections between copies of G_n.

$$s_n(j,k) \quad = \quad \begin{cases} k + 1 \bmod (C_n + 1) & \text{if } j = C_n - 1 \\ j + k + 1 \bmod (C_n + 1) & \text{if } j \in \{1, 3, \ldots, C_n - 3\} \\ k - 1 \bmod (C_n + 1) & \text{if } j = 0 \\ k - j \bmod (C_n + 1) & \text{if } j \in \{2, 4, \ldots, C_n - 2\} \end{cases} \qquad (10)$$

We also define a parity selector function on edges $e \in E_n$:

$$p_i(e) \quad = \quad v \quad \text{s.t. } v \in e \wedge v_0 = i. \qquad (11)$$

The edges of G_{n+1} are then given by:

$$R_{n+1} \quad = \quad \bigcup_{k=0}^{C_n} \bigcup_{e \in E_n} \{p_0(e) + k, p_1(e) + k\}, \qquad (12)$$

$$S_{n+1} \quad = \quad \bigcup_{k=0}^{C_n} \bigcup_{v \in A_n} \{v + k, \psi_n(v) + s_n(z_n(v), k)\}, \qquad (13)$$

$$= \quad \bigcup_{k=0}^{C_n} \bigcup_{v \in B_n} \{v + k, \psi_n(v) + s_n(z_n(v), k)\}, \qquad (14)$$

$$E_{n+1} \quad = \quad R_{n+1} \cup S_{n+1}, \qquad (15)$$

where A_n and B_n are the sets of odd and even vertices in V_n, respectively. Eq. 12 replecates the edges of G_n among subsets of the vertices, while Eq. 13 creates one edge between each pair of the subsets.

2 Properties of G_n

Lemma 1. *The graph G_n is n-regular for all $n \geq 0$.*

Proof. We proceed using induction on n. The base case G_1 is 1-regular by inspection. In the inductive case G_{n+1} the Eq. 12 reproduces the edges of G_n, which is n-regular by induction, contributing n to the degree of each vertex. Eq. 13 adds one to the degree of each odd vertex. As ψ_n is a bijective map between odd and even vertices, Eq. 13 also adds one to the degree of each even vertex, giving a total degree of $n + 1$. $\qquad \square$

Lemma 2. *The number of vertices and edges of the graph G_n are given by the recurrence relations:*

$$C_0 = |V_0| \quad = \quad 1, \qquad (16)$$

$$C_n = |V_n| \quad = \quad C_{n-1}(C_{n-1} + 1), \qquad (17)$$

$$|E_n| \quad = \quad \frac{n}{2} C_n. \qquad (18)$$

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Proof. The vertex set of G_n (Eq. 5) is a union of $C_{n-1} + 1$ sets. The vertex labels within each set all end with the same element, and this element is unique to each set. The sets are thus disjoint. Each set contains C_{n-1} elements, giving $C_{n-1}(C_{n-1} + 1)$ elements. By Lemma 1, G_n is n-regular and has $\frac{n}{2}|V_n| = \frac{n}{2}C_n$ edges. $\qquad\square$

Theorem 1. *The graphs G_n are vertex-transitive for all $n \geq 0$.*

Proof. We begin by showing that ϕ_{n-1} is an automorphism of G_n. Let v be a vertex in V_n. ϕ_{n-1} has no effect on v_0^{n-2} so it preserves the edges defined in Eq. 12. The vertex $\phi_{n-1}(v)$ has exactly one other edge, connecting it to:

$$\psi_{n-2}(v_0^{n-2}) \leftthreetimes s_n(z_n(v_0^{n-2}), v_{n-1} + 1), \tag{19}$$

$$\tag{20}$$

$$\square$$

References

[1] I. Z. Bower. *The Foster Census.* Charles Babbage Research Centre, Winnipeg, 1988.