A Family of Vertex Transitive Graphs

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1 Sylvester Number System

Let s_n be the nth element of Sylvester's sequence [1], defined as:

$$s_0 = 2 \tag{1}$$

$$s_{n+1} = 1 + \prod_{i=0}^{n} s_n. \tag{2}$$

A mixed-base number system can be constructed from Sylvester's sequence as follows:

Definition 1.1. A *Sylvester-radix* number a is a sequence of digits such that: $a_i \in \mathbb{Z} : 0 \leq a_i < s_i$.

We denote the set of n-digit Sylvester-radix numbers as $\mathcal{S}_n$ and the set of all Sylvester-radix numbers as $\mathcal{S}$.

Lemma 1. There are $(s_n - 1)$ Sylvester-radix numbers of length n.

Proof. The Sylvester-radix numbers of length 1 are $\langle 0 \rangle$ and $\langle 1 \rangle$. $(s_1 - 1) = 2$ so the lemma holds for $n = 1$.

For $n > 1$, there are s_i possible values for each digit, with $0 \leq i < n$. The number of valid digit combinations is thus given by:

$$\prod_{i=0}^{n-1} s_i = s_n - 1 \qquad \text{(by (2))}.$$

$\square$

Corollary 1. The place value of index i in a Sylvester-radix number is $(s_i - 1)$.

The integer value of a length-n Sylvester-radix number a is thus:

$$z(a) = \sum_{i=0}^{n-1} a_i(s_i - 1). \tag{3}$$

2 Nested Clique

We now define a family of graphs, which we call *nested cliques*. For $n > 0$, the nested clique G_n is defined recursively in terms of G_{n-1}. We will construct G_n from a union of subgraphs, each of which is isomorphic to G_{n-1}. Lower-order graphs are thus recursively *nested* within higher-order graphs. In addition to the edges internal to each subgraph, we will add one *external edge* to each vertex of G_n, such that each pair of subgraphs is connected by exactly one directed edge in each direction, making G_n homomorphic to a clique.

Definition 2.1. A nested clique sequence is a sequence of graphs $G_n = (V_n, E_n)$ such that G_n is a union of disjoint subgraphs isomorphic to G_{n-1} and of a set of edges connecting each vertex in V_n to another vertex belonging to a different subgraph.

The above definition implies the following useful lemma.

Lemma 2. The number of vertices $N_n = |V_n|$ in a nested clique sequence is given by the recurrence relation:

$$N_n = N_{n-1}(N_{n-1} + 1). \tag{4}$$

Proof. TODO □

2.1 Construction

We now construct a particular family of graphs satisfying the requirements of Definition 2.1.

2.1.1 Base Case

We choose a base case containing a single vertex, which we label $\varnothing$:

$$G_0 = (V_0, E_0) \tag{5}$$
$$V_0 = \{\varnothing\} \tag{6}$$
$$E_0 = \{\}. \tag{7}$$

Lemma 3. Whith the above base case G_0, the number of vertices in G_n is given by:

$$N_n = 1, 2, 6, 42, 1806, ... \tag{8}$$

2.1.2 Vertices

For the recursive case, it is necessary to distinguish between vertices in different nested subgraphs. Each vertex is labeled by a sequence of integers. For G_n each subgraph is assigned an integer in $[0, N_n)$. For all vertices in a particular subgraph, labels are constructed by taking the corresponding label in G_{n-1} and appending the subgraph's integer label. Formally:

$$V_n = \bigcup_{i=0}^{N_{n-1}} \{v \ll i \mid v \in V_{n-1}\}, \tag{9}$$

where $a \ll e$ denotes appending element e to the end of sequence a. The vertices of G_n are thus integer sequences of length n, with the digit at index i in $[0, N_i - 1]$. This set of labels is exactly the set of length-n Sylvester-radix numbers.

Theorem 1. The vertices of a nested clique G_n having $V_0 = \{\varnothing\}$ are isomorphic to length-n Sylester-radix numbers.

Proof. TODO $\square$

2.1.3 Edges

It is not obvious whether any configuration of edges can satisfy Definition 2.1. We will construct these edges below. First, we define notation that will be needed for the construction.

Let σ_i be the cyclic permutation that increments an integer, modulo s_i:

$$\sigma_i x = (x + 1) \bmod s_i. \tag{10}$$

We now define the kth order *skip operator* S_k. This operator on a Sylvester-radix number to permute digits 0 . . . k. We use $\|$ to denote the concatanetion operator for sequences.

Definition 2.2.

$$\begin{aligned} S_0 v &= \langle \sigma_0 v_0, v_1, v_2, \ldots \rangle & (11)\\ z'_i(v) &\equiv z(S_{i-1} v_{0:i}) & (12)\\ S_i v &= S_{i-1} v_{0:i} \,\|\, \langle v_i + 1 + z'_i(v) \rangle \,\|\, v_{i+1:n}. & (13) \end{aligned}$$

References

[1] James J Sylvester. On a point in the theory of vulgar fractions. *American Journal of Mathematics*, 3(4):332–335, 1880.