DRAFT: A Family of Sparse, Vertex-Transitive Graphs

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1 Construction

We recursively construct a family of vertex-transitive graphs $G_n = \langle V_n, E_n \rangle$. The vertices $v \in V_n$ are labeled by an n-sequence of integers. We define the base case, G_1 as:

$$\begin{aligned} V_0 &= \{\langle 0 \rangle, \langle 1 \rangle\}, && (1) \\ E_0 &= \{\{\langle 0 \rangle, \langle 1 \rangle\}\}. && (2) \end{aligned}$$

We construct subsequent vertex sets from copies of the previous set, with each copy having a different integer appended to its vertex labels:

$$\begin{aligned} V_{n+1} &= \bigcup_{k=0}^{C_n} \{v | w \in V_n \wedge v = w \nleftrightarrow k\}, && (3) \\ C_n &\equiv |V_n|, && (4) \end{aligned}$$

where $w \nleftrightarrow k$ denotes appending element k to the end of sequence w. We note that a one-to-one mapping z_n exists between the vertices of G_n and the integers from 0 to $C_n - 1$, and define a successor mapping ψ_n:

$$\begin{aligned} z_n(v) &= \sum_{k=0}^{n-1} C_k v_k, && (5) \\ (\psi_n(v))_k &= \begin{cases} v_k + 1 \bmod C_k & \text{if } \forall j < k \quad v_j = C_j \\ v_k & \text{otherwise.} \end{cases} && (6) \end{aligned}$$

To construct the edges of G_{n+1} we define a "shortcut" function $s_{n,j}(k)$ with $j \in \{1, 3, \ldots, C_n - 1\}$ and $k \in \{0, 1, 2, \ldots, C_n\}$ which determines the interconnections between copies of G_n.

$$s_n(j,k) = \begin{cases} k - 1 \bmod (C_n + 1) & \text{if } j = C_n - 1 \\ j + k + 1 \bmod (C_n + 1) & \text{otherwise.} \end{cases} \quad (7)$$

We also define a parity selector function on edges $e \in E_n$:

$$p_i(e) = v \quad \text{s.t. } v \in e \wedge v_0 = i. \tag{8}$$

The edges of G_{n+1} are then given by:

$$\begin{aligned} E_{n+1} &= \bigcup_{k=0}^{C_n} \left[\bigcup_{e \in E_n} \{p_0(e) \leftarrow\!\!\!+ k, p_1(e) \leftarrow\!\!\!+ k\} \right. && (9) \\ & \quad \left. \cup \bigcup_{v \in O_n} \{v \leftarrow\!\!\!+ k, \psi_n(v) \leftarrow\!\!\!+ s_n(z_n(v), k)\} \right], && (10) \\ O_n &\equiv \{v | v \in V_n \wedge v_0 = 1\}. && (11) \end{aligned}$$

2 Properties of G_n

Lemma 1. *The graph G_n is n-regular for all $n \geq 1$.*

Proof. We proceed using induction on n. The base case G_1 is 1-regular by inspection. In the inductive case G_{n+1} the first union term of Eq. 9 reproduces the edges of G_n, which is n-regular by induction, contributing n to the degree of each vertex. The second union term adds one to the degree of each odd vertex. As ψ_n is a bijective map between odd and even vertices, the second union term also adds one to the degree of each even vertex, giving a total degree of $n+1$. □

Lemma 2. *The number of vertices and edges of the graph G_n are given by the recurrence relations:*

$$\begin{aligned} C_0 = |V_0| &= 1, && (12) \\ C_n = |V_n| &= C_{n-1}(C_{n-1}+1), && (13) \\ |E_n| &= \frac{n}{2} C_n. && (14) \end{aligned}$$

Proof. The vertex set of G_n (Eq. 3) is a union of $C_{n-1}+1$ sets. The vertex labels within each set all end with the same element, and this element is unique to each set. The sets are thus disjoint. Each set contains C_{n-1} elements, giving $C_{n-1}(C_{n-1}+1)$ elements. By Lemma 1, G_n is n-regular and has $\frac{n}{2}|V_n| = \frac{n}{2}C_n$ edges. □

Theorem 1. *The graphs G_n are vertex-transitive for all $n \geq 0$.*

Proof. Define ϕ, show preserves edges. □

References

[1] I. Z. Bower. *The Foster Census.* Charles Babbage Research Centre, Winnipeg, 1988.