# DRAFT: A Family of Sparse, Vertex-Transitive Graphs

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## 1 Motivation

In networks modeling interactions between agents (e.g. people, computers), the agents often have a limited capacity to form interactions. If the vertex degrees grow with the size of the network, the number of interactions will exceed the capacity of the agents, limiting the scale of the network. Sparse networks, such as trees, are not subject to this limitation on scaling. However, trees impose a hierarchical relationship between agents. In some applications, an egalitarian relationship between agents is desirable, formally described as vertex transitivity. In this paper we construct an infinite family $G_n$ of sparse ($|E|= O(|V|\log(\log(|V|)))$ (need to confirm this)), vertex-transitive graphs.

## 2 Construction

We recursively construct a family of vertex-transitive graphs $G_n = \langle V_n, E_n \rangle$. The vertices $v \in V_n$ are labeled by an $n$-sequence of integers. We define $G_0$ and $G_1$ as:

$$V_0 = \{\langle\rangle\}, \tag{1}$$

$$E_0 = \{\}, \tag{2}$$

$$V_1 = \{\langle 0\rangle, \langle 1\rangle\}, \tag{3}$$

$$E_1 = \{\{\langle 0\rangle, \langle 1\rangle\}\}. \tag{4}$$

We construct subsequent vertex sets from copies of the previous set, with each copy having a different integer appended to its vertex labels:

$$V_{n+1} = \bigcup_{k=0}^{C_n} \{v | w \in V_n \wedge v = w \nleftarrow k\}, \tag{5}$$

$$C_n \equiv |V_n|, \tag{6}$$

where $w \nleftarrow k$ denotes appending element $k$ to the end of sequence $w$. We note that a one-to-one mapping $z_n$ exists between the vertices of $G_n$ and the integers from 0 to $C_n - 1$:

$$z_n(v) = \sum_{i=0}^{n-1} C_i v_i. \tag{7}$$

We define even and odd subsets of $V_n$:

$$A_n = \{v | v \in V_n \wedge v_0 = 1\}, \tag{8}$$
$$B_n = \{v | v \in V_n \wedge v_0 = 0\}. \tag{9}$$

We also define the following mappings on the vertices of $G_n$:

$$(\phi_n(v))_i = \begin{cases} v_i + 1 \bmod (C_i + 1) & \text{if } i = n, \\ v_i & \text{otherwise,} \end{cases} \tag{10}$$

$$(\psi_n(v))_i = \begin{cases} v_i + 1 \bmod (C_i + 1) & \text{if } i = 0 \vee \forall j < i : v_j = C_j, \\ v_i & \text{otherwise,} \end{cases} \tag{11}$$

$$\theta_n(v) = \begin{cases} \psi_n(v) & \text{if } v \in A_n, \\ \psi_n^{-1}(v) & \text{if } v \in B_n, \end{cases} \tag{12}$$

with $i < n$.

To construct the edges of $G_{n+1}$ we define a "shortcut" function $s_n(j,k)$ with $j \in \{0, 1, C_n - 1\}$ and $k \in \{0, 1, 2, \ldots, C_n\}$ which determines the interconnections between copies of $G_n$.

$$s_n(j,k) = \begin{cases} k + 1 \bmod (C_n + 1) & \text{if } j = C_n - 1, \\ k - 1 \bmod (C_n + 1) & \text{if } j = 0, \\ k + j + 1 \bmod (C_n + 1) & \text{if } j \in \{1, 3, \ldots, C_n - 3\}, \\ k - j \bmod (C_n + 1) & \text{if } j \in \{2, 4, \ldots, C_n - 2\}. \end{cases} \tag{13}$$

We also define a parity selector function on the edges $e \in E_n$:

$$p_x(e) = v \quad \text{s.t. } v \in e \wedge v_0 = x. \tag{14}$$

The edges of $G_{n+1}$ are then given by:

$$R_{n+1} = \bigcup_{k=0}^{C_n} \bigcup_{e \in E_n} \{p_0(e) \nleftrightarrow k, p_1(e) \nleftrightarrow k\}, \tag{15}$$

$$S_{n+1} = \bigcup_{k=0}^{C_n} \bigcup_{v \in A_n} \{v \nleftrightarrow k, \theta_n(v) \nleftrightarrow s_n(z_n(v), k)\}, \tag{16}$$

$$= \bigcup_{k=0}^{C_n} \bigcup_{v \in B_n} \{v \nleftrightarrow k, \theta_n(v) \nleftrightarrow s_n(z_n(v), k)\}, \tag{17}$$

$$E_{n+1} = R_{n+1} \cup S_{n+1}, \tag{18}$$

noting that $S_{n+1}$ can be written in terms of either the odd vertices $A_n$ or the even vertices $B_n$. Eq. (15) replicates the edges of $G_n$ among subsets of the vertices of $G_{n+1}$, while Eq. (16) creates one edge between each pair of the subsets.

# 3 Properties of $G_n$

**Lemma 1.** *The graph $G_n$ is $n$-regular for all $n \geq 0$.*

*Proof.* We proceed using induction on $n$. The base case $G_1$ is 1-regular by inspection. In the inductive case $G_{n+1}$, Eq. (15) reproduces the edges of $G_n$, which is $n$-regular by induction, contributing $n$ to the degree of each vertex. Eq. (16) adds one to the degree of each odd vertex. As $\theta_n$ is a bijective map between odd and even vertices, Eq. (16) also adds one to the degree of each even vertex, giving a total degree of $n+1$. □

**Lemma 2.** *The number of vertices and edges of the graph $G_n$ are given by the recurrence relations:*

$$\begin{aligned} C_0 = |V_0| &= 1, && (19)\\ C_n = |V_n| &= C_{n-1}(C_{n-1}+1), && (20)\\ |E_n| &= \frac{n}{2}C_n. && (21) \end{aligned}$$

*Proof.* The vertex set of $G_n$, given by Eq. (5), is a union of $C_{n-1}+1$ sets. The vertex labels within each set all end with the same element, and this element is unique to each set. The sets are thus disjoint. Each set contains $C_{n-1}$ elements, giving $C_{n-1}(C_{n-1}+1)$ elements. By Lemma 1, $G_n$ is $n$-regular and has $\frac{n}{2}|V_n| = \frac{n}{2}C_n$ edges. □

**Lemma 3.** *The mapping $\phi_{n-1}$ is an automorphism of $G_n$.*

*Proof.* Let $v$ be a vertex in $V_n$. $\phi_{n-1}$ has no effect on $v_0^{n-2}$ so it preserves the edges $R_n$, defined in Eq. (15). The vertex $v$ has exactly one other edge, from $S_n$ defined in Eq. (16):

$$e = \{v, \theta_n(v_0^{n-2} \nleftrightarrow s_n(z_n(v_0^{n-2}), v_{n-1})\}. \quad (22)$$

Applying $\phi_{n-1}$ to both vertices gives:

$$\begin{aligned} \tilde{e} &= \{v_0^{n-2} \nleftrightarrow (v_{n-1}+1) \bmod (C_{n-2}+1), && (23)\\ &\qquad \theta_n(v_0^{n-2} \nleftrightarrow s_n(z_n(v_0^{n-2}), (v_{n-1}+1) \bmod (C_{n-2}+1))\}, && (24)\\ &= \{v_0^{n-2} \nleftrightarrow \tilde{k}, \theta_n(v_0^{n-2} \nleftrightarrow s_n(z_n(v_0^{n-2}), \tilde{k})\}, && (25) \end{aligned}$$

where $\tilde{k} = (v_{n-1}+1) \bmod (C_{n-2}+1)$. The edge $\tilde{e}$ is also a member of $S_n$, showing that $\phi_{n-1}$ permutes the elements of $S_n$, and preserves all edges in $E_n$. □

**Theorem 1.** *The graphs $G_n$ are vertex-transitive for all $n \geq 0$.*

*Proof.* The graph $G_0$ is trivially vertex-transitive. For higher order graphs, we recursively construct a mapping $\Phi_n$ that maps an arbitrary vertex $v \in V_n$ to an arbitrary vertex $\tilde{v} \in V_n$ and preserves the edges in $E_n$. Let $\delta_i = \tilde{v}_i - v_i \mod (C_i + 1)$. For the subsequence $v_0^0$, the map $\Phi_1 = \phi_0^{\delta_0}$ achieves the desired mapping and preserves the edges in $E_1$ by Lemma 3. Given an edge-preserving mapping $\Phi_n$ from $v_0^{n-1}$ to $\tilde{v}_0^{n-1}$, we can construct $\tau_n$. We define mappings $\pi_n$ and $\tau_n$:

$$
\begin{aligned}
\pi_n(0) &= 0 && (26)\\
\pi_n(s_n(v_0^{n-1}, 0)) &= s_n(\Phi_n(v_0^{n-1}), 0) && (27)\\
\tau_n(v) &= \Phi_n(v_0^{n-1}) \nleftrightarrow \pi_n(v_n). && (28)
\end{aligned}
$$

For the recursive edges defined in Eq. (15) $\{v_0^n, w_0^n\} \in R_{n+1}$:

$$
\begin{aligned}
\{\tau_n(v_0^n), \tau_n(v_0^n) &= \{\Phi_n(v_0^n) \nleftrightarrow \pi_n(k), \Phi_n(w_0^n) \nleftrightarrow \pi_n(k)\} && (29)\\
&= \{\tilde{v}_0^n \nleftrightarrow \tilde{k}, \Phi_n(\tilde{w}_0^n) \nleftrightarrow \tilde{k}\}, && (30)
\end{aligned}
$$

which is simply another member of $R_{n+1}$. As $\tau_n$ is bijective, it permutes the edges of $R_{n+1}$. For the shortcut edges defined in Eq. (16) $\{v_0^n, w_0^n\} \in S_{n+1}$, when $v_n = 0$:

$$
\begin{aligned}
\{\tau_n(v_0^n), \tau_n(w_0^n)\} &= \{\Phi_n(v_0^n) \nleftrightarrow \pi_n(0), && (31)\\
&\quad \Phi_n(w_0^n)) \nleftrightarrow \pi_n(s_n(z_n(v_0^n), 0))\} && (32)\\
&= \{\tilde{v}_0^n \nleftrightarrow 0, \tilde{w}_0^n \nleftrightarrow s_n(z_n(\Phi_n(v_0^n)), 0)\} && (33)\\
&= \{\tilde{v}_0^n \nleftrightarrow 0, \theta_n(\tilde{v}_0^n) \nleftrightarrow s_n(z_n(\tilde{v_0^n}), 0)\}, && (34)
\end{aligned}
$$

showing that $\tau_n$ permutes the subset of edges in $S_{n+1}$ for which $v_n = 0$. Consequently, because $\phi_n$ is an automorphism of $G_{n+1}$ by Lemma 3, $\tau_n$ also permutes the edges of the subsets for which $(\phi_n^{-k}(v_0^n))_n = 0$. Those edges are the subset of $S_{n+1}$ for which $v_n = k$, showing that $\Phi_n n + 1$ permutes all edges within $S_{n+1}$ and $E_{n+1}$. We finish by applying the automorphism $\phi_n^{\delta_n}$:

$$
\Phi_{n+1}(v) = \phi_n^{\delta_n}(\tau_n(v)), \qquad (35)
$$

completing the mapping from $\tilde{v}$ to $\tilde{w}$. □

# References


[1] I. Z. Bower. *The Foster Census.* Charles Babbage Research Centre, Winnipeg, 1988.