DRAFT: A Family of Sparse, Vertex-Transitive Graphs

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1 Construction

We recursively construct a family of vertex-transitive graphs $G_n = \langle V_n, E_n \rangle$. The vertices $v \in V_n$ are labeled by an n-sequence of integers. We define G_0 and G_1 as:

$$
\begin{aligned}
V_0 &= \{\langle\rangle\}, & (1)\\
E_0 &= \{\}, & (2)\\
V_1 &= \{\langle 0 \rangle, \langle 1 \rangle\}, & (3)\\
E_1 &= \{\{\langle 0 \rangle, \langle 1 \rangle\}\}. & (4)
\end{aligned}
$$

We construct subsequent vertex sets from copies of the previous set, with each copy having a different integer appended to its vertex labels:

$$
\begin{aligned}
V_{n+1} &= \bigcup_{k=0}^{C_n} \{v | w \in V_n \wedge v = w \leftarrow\!\!\!+ k\}, & (5)\\
C_n &\equiv |V_n|, & (6)
\end{aligned}
$$

where $w \leftarrow\!\!\!+ k$ denotes appending element k to the end of sequence w. We note that a one-to-one mapping z_n exists between the vertices of G_n and the integers from 0 to $C_n - 1$:

$$
z_n(v) = \sum_{k=0}^{n-1} C_k v_k. \quad (7)
$$

We define even and odd subsets of V_n:

$$
\begin{aligned}
A_n &= \{v | v \in V_n \wedge v_0 = 1\}, & (8)\\
B_n &= \{v | v \in V_n \wedge v_0 = 0\}. & (9)
\end{aligned}
$$

We also define the following mappings on the vertices of G_n:

$$(\phi_n(v))_i = \begin{cases} v_i + 1 \bmod (C_i + 1) & \text{if } i = n, \\ v_i & \text{otherwise}, \end{cases} \tag{10}$$

$$(\psi_n(v))_i = \begin{cases} v_i + 1 \bmod (C_i + 1) & \text{if } i = 0 \vee \forall j < i : v_j = C_j, \\ v_i & \text{otherwise}, \end{cases} \tag{11}$$

$$\theta_n(v) = \begin{cases} \psi_n(v) & \text{if } v \in A_n, \\ \psi_n^{-1}(v) & \text{if } v \in B_n, \end{cases} \tag{12}$$

with $i \leq n$.

To construct the edges of G_{n+1} we define a "shortcut" function $s_n(j,k)$ with $j \in \{0, 1, C_n - 1\}$ and $k \in \{0, 1, 2, \ldots, C_n\}$ which determines the interconnections between copies of G_n.

$$s_n(j,k) = \begin{cases} k + 1 \bmod (C_n + 1) & \text{if } j = C_n - 1, \\ k - 1 \bmod (C_n + 1) & \text{if } j = 0, \\ k + j + 1 \bmod (C_n + 1) & \text{if } j \in \{1, 3, \ldots, C_n - 3\}, \\ k - j \bmod (C_n + 1) & \text{if } j \in \{2, 4, \ldots, C_n - 2\}. \end{cases} \tag{13}$$

We also define a parity selector function on edges $e \in E_n$:

$$p_i(e) = v \quad \text{s.t. } v \in e \wedge v_0 = i. \tag{14}$$

The edges of G_{n+1} are then given by:

$$R_{n+1} = \bigcup_{k=0}^{C_n} \bigcup_{e \in E_n} \{p_0(e) \nleftarrow k, p_1(e) \nleftarrow k\}, \tag{15}$$

$$S_{n+1} = \bigcup_{k=0}^{C_n} \bigcup_{v \in A_n} \{v \nleftarrow k, \theta_n(v) \nleftarrow s_n(z_n(v), k)\}, \tag{16}$$

$$= \bigcup_{k=0}^{C_n} \bigcup_{v \in B_n} \{v \nleftarrow k, \theta_n(v) \nleftarrow s_n(z_n(v), k)\}, \tag{17}$$

$$E_{n+1} = R_{n+1} \cup S_{n+1}, \tag{18}$$

noting that S_{n+1} can be written in terms of either the odd vertices A_n or the even vertices B_n. Eq. (15) replicates the edges of G_n among subsets of the vertices of G_{n+1}, while Eq. (16) creates one edge between each pair of the subsets.

2 Properties of G_n

Lemma 1. *The graph G_n is n-regular for all $n \geq 0$.*

Proof. We proceed using induction on n. The base case G_1 is 1-regular by inspection. In the inductive case G_{n+1} Eq. (15) reproduces the edges of G_n, which is n-regular by induction, contributing n to the degree of each vertex. Eq. (16) adds one to the degree of each odd vertex. As θ_n is a bijective map between odd and even vertices, Eq. (16) also adds one to the degree of each even vertex, giving a total degree of $n+1$. $\square$

Lemma 2. *The number of vertices and edges of the graph G_n are given by the recurrence relations:*

$$
\begin{aligned}
C_0 = |V_0| &= 1, && (19)\\
C_n = |V_n| &= C_{n-1}(C_{n-1}+1), && (20)\\
|E_n| &= \frac{n}{2}C_n. && (21)
\end{aligned}
$$

Proof. The vertex set of G_n, given by Eq. (5), is a union of $C_{n-1}+1$ sets. The vertex labels within each set all end with the same element, and this element is unique to each set. The sets are thus disjoint. Each set contains C_{n-1} elements, giving $C_{n-1}(C_{n-1}+1)$ elements. By Lemma 1, G_n is n-regular and has $\frac{n}{2}|V_n|= \frac{n}{2}C_n$ edges. $\square$

Theorem 1. *The graphs G_n are vertex-transitive for all $n \geq 0$.*

Proof. We begin by showing that ϕ_{n-1} is an automorphism of G_n. Let v be a vertex in V_n. ϕ_{n-1} has no effect on v_0^{n-2} so it preserves the edges defined in Eq. (15). The vertex v has exactly one other edge:

$$e = \{v, \theta_n(v_0^{n-2} \nleftrightarrow s_n(z_n(v_0^{n-2}), v_{n-1})\}. \tag{22}$$

Applying ϕ_{n-1} to both vertices gives:

$$
\begin{aligned}
\tilde{e} &= \{v_0^{n-2} \nleftrightarrow (v_{n-1}+1) \bmod (C_{n-2}+1), && (23)\\
&\qquad \theta_n(v_0^{n-2} \nleftrightarrow s_n(z_n(v_0^{n-2}), (v_{n-1}+1) \bmod (C_{n-2}+1))\}, && (24)\\
&= \{v_0^{n-2} \nleftrightarrow \tilde{k}, \theta_n(v_0^{n-2} \nleftrightarrow s_n(z_n(v_0^{n-2}), \tilde{k})\}, && (25)
\end{aligned}
$$

where $\tilde{k} = (v_{n-1}+1) \bmod (C_{n-2}+1)$. The edge $\tilde{e}$ is also a member of S_n in Eq. (16), showing that ϕ_{n-1} permutes the elements of S_n, and preserves all edges in E_n.

TODO: show that ϕ_{n-k} when combined with a permutation $\pi_{n,k}$ preserves edges. Use ϕ to construct an automorphism mapping any v to any w. $\square$

References

[1] I. Z. Bower. *The Foster Census.* Charles Babbage Research Centre, Winnipeg, 1988.