DRAFT: A Family of Sparse, Vertex-Transitive Graphs

Edward L. Platt

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1 Construction

We recursively construct a family of vertex-transitive graphs $G_n = \langle V_n, E_n \rangle$. The vertices $v \in V_n$ are labeled by an n-sequence of integers. We define the base case, G_1 as:

$$V_1 = \{\langle 0 \rangle, \langle 1 \rangle\}, \tag{1}$$

$$E_1 = \{\{\langle 0 \rangle, \langle 1 \rangle\}\}. \tag{2}$$

We construct subsequent vertex sets from copies of the previous set, with each copy having a different integer appended to its vertex labels:

$$V_{n+1} = \bigcup_{k=0}^{C_n} \{v | w \in V_n \wedge v = w \nleftrightarrow k\}, \tag{3}$$

$$C_n \equiv |V_n|, \tag{4}$$

where $w \nleftrightarrow k$ denotes appending element k to the end of sequence w. We note that a one-to-one mapping z_n exists between the vertices of G_n and the integers from 0 to $C_n - 1$:

$$z_n(v) = \sum_{k=0}^{n-1} C_k v_k. \tag{5}$$

We also define the following mappings on the vertices of G_n:

$$(\phi_n(v))_k = \begin{cases} v_k + 1 \bmod (C_k + 1) & \text{if } k = n, \\ v_k & \text{otherwise,} \end{cases} \tag{6}$$

$$(\psi_n(v))_k = \begin{cases} v_k + 1 \bmod (C_k + 1) & \text{if } k = 0 \vee \forall j < k : v_j = C_j, \\ v_k & \text{otherwise.} \end{cases} \tag{7}$$

To construct the edges of G_{n+1} we define a "shortcut" function $s_{n,j}(k)$ with $j \in \{1, 3, \ldots, C_n - 1\}$ and $k \in \{0, 1, 2, \ldots, C_n\}$ which determines the interconnections between copies of G_n.

$$s_n(j,k) \quad = \quad \begin{cases} k - 1 \bmod (C_n + 1) & \text{if } j = C_n - 1 \\ j + k + 1 \bmod (C_n + 1) & \text{otherwise.} \end{cases} \tag{8}$$

We also define a parity selector function on edges $e \in E_n$:

$$p_i(e) \quad = \quad v \quad \text{s.t. } v \in e \wedge v_0 = i. \tag{9}$$

The edges of G_{n+1} are then given by:

$$E_{n+1} \quad = \quad \bigcup_{k=0}^{C_n} \Bigg[\bigcup_{e \in E_n} \{p_0(e) \leftarrow\!\!\!\!+ k, p_1(e) \leftarrow\!\!\!\!+ k\} \tag{10}$$

$$\cup \bigcup_{v \in O_n} \{v \leftarrow\!\!\!\!+ k, \psi_n(v) \leftarrow\!\!\!\!+ s_n(z_n(v), k)\} \Bigg], \tag{11}$$

$$O_n \quad \equiv \quad \{v | v \in V_n \wedge v_0 = 1\}. \tag{12}$$

2 Properties of G_n

Lemma 1. *The graph G_n is n-regular for all $n \geq 1$.*

Proof. We proceed using induction on n. The base case G_1 is 1-regular by inspection. In the inductive case G_{n+1} the first union term of Eq. 10 reproduces the edges of G_n, which is n-regular by induction, contributing n to the degree of each vertex. The second union term adds one to the degree of each odd vertex. As ψ_n is a bijective map between odd and even vertices, the second union term also adds one to the degree of each even vertex, giving a total degree of $n+1$. □

Lemma 2. *The number of vertices and edges of the graph G_n are given by the recurrence relations:*

$$C_0 = |V_0| \quad = \quad 1, \tag{13}$$

$$C_n = |V_n| \quad = \quad C_{n-1}(C_{n-1} + 1), \tag{14}$$

$$|E_n| \quad = \quad \frac{n}{2} C_n. \tag{15}$$

Proof. The vertex set of G_n (Eq. 3) is a union of $C_{n-1} + 1$ sets. The vertex labels within each set all end with the same element, and this element is unique to each set. The sets are thus disjoint. Each set contains C_{n-1} elements, giving $C_{n-1}(C_{n-1} + 1)$ elements. By Lemma 1, G_n is n-regular and has $\frac{n}{2}|V_n| = \frac{n}{2}C_n$ edges. □

Theorem 1. *The graphs G_n are vertex-transitive for all $n \geq 0$.*

Proof. Let $\phi_i(k)$ map the vertex v □

References

[1] I. Z. Bower. *The Foster Census.* Charles Babbage Research Centre, Winnipeg, 1988.