DRAFT: A Family of Sparse, Vertex-Transitive Graphs

Edward L. Platt

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1 Motivation

In networks modeling interactions between agents (e.g. people, computers), the agents often have a limited capacity to form interactions. If the vertex degrees grow with the size of the network, the number of interactions will exceed the capacity of the agents, limiting the scale of the network. Sparse networks, such as trees, are not subject to this limitation on scaling. However, trees impose a hierarchical relationship between agents. In some applications, an egalitarian relationship between agents is desirable, formally described as vertex transitivity. In this paper we construct an infinite family G_n of sparse ($|E| = O(|V| \log(\log|V|))$, small-diameter ($d = O(\log|V|)$), vertex-transitive graphs.

2 Construction

We recursively construct a family of vertex-transitive graphs $G_n = \langle V_n, E_n \rangle$. The vertices $v \in V_n$ are labeled by an n-sequence of integers. We define G_0 and G_1 as:

$$\begin{aligned} V_0 &= \{\langle\rangle\}, & (1) \\ E_0 &= \{\}, & (2) \\ V_1 &= \{\langle 0\rangle, \langle 1\rangle\}, & (3) \\ E_1 &= \{\{\langle 0\rangle, \langle 1\rangle\}\}. & (4) \end{aligned}$$

We construct subsequent vertex sets from copies of the previous set, with each copy having a different integer appended to its vertex labels:

$$\begin{aligned} V_{n+1} &= \bigcup_{k=0}^{C_n} \{v | w \in V_n \wedge v = w \nleftrightarrow k\}, & (5) \\ C_n &\equiv |V_n|, & (6) \end{aligned}$$

where $w \nleftrightarrow k$ denotes appending element k to the end of sequence w. We note that a one-to-one mapping z_n exists between the vertices of G_n and the integers from 0 to $C_n - 1$:

$$z_n(v) = \sum_{i=0}^{n-1} C_i v_i. \tag{7}$$

We define even and odd subsets of V_n:

$$\begin{aligned} A_n &= \{v \mid v \in V_n \wedge v_0 = 1\}, &(8)\\ B_n &= \{v \mid v \in V_n \wedge v_0 = 0\}. &(9) \end{aligned}$$

We also define the following mappings on the vertices of G_n:

$$(\phi_k(v))_i = \begin{cases} v_i + 1 \bmod (C_i + 1) & \text{if } i = k, \\ v_i & \text{otherwise,} \end{cases} \tag{10}$$

$$(\psi_n(v))_i = \begin{cases} v_i + 1 \bmod (C_i + 1) & \text{if } i = 0 \vee \forall j < i : v_j = C_j, \\ v_i & \text{otherwise,} \end{cases} \tag{11}$$

$$\theta_n(v) = \begin{cases} \psi_n(v) & \text{if } v \in A_n, \\ \psi_n^{-1}(v) & \text{if } v \in B_n, \end{cases} \tag{12}$$

with $i, k < n$.

To construct the edges of G_{n+1} we define a "shortcut" function $s_n(j, k)$ with $j \in \{0, 1, C_n{-}1\}$ and $k \in \{0, 1, 2, \ldots, C_n\}$ which determines the interconnections between copies of G_n.

$$s_n(j, k) = \begin{cases} k + 1 \bmod (C_n + 1) & \text{if } j = C_n - 1, \\ k - 1 \bmod (C_n + 1) & \text{if } j = 0, \\ k + j + 1 \bmod (C_n + 1) & \text{if } j \in \{1, 3, \ldots, C_n - 3\}, \\ k - j \bmod (C_n + 1) & \text{if } j \in \{2, 4, \ldots, C_n - 2\}. \end{cases} \tag{13}$$

We also define a parity selector function on the edges $e \in E_n$:

$$p_x(e) = v \quad \text{s.t. } v \in e \wedge v_0 = x. \tag{14}$$

The edges of G_{n+1} are then given by:

$$\begin{aligned} R_{n+1} &= \bigcup_{k=0}^{C_n} \bigcup_{e \in E_n} \{p_0(e) \nleftarrow k, p_1(e) \nleftarrow k\}, &(15)\\ S_{n+1} &= \bigcup_{k=0}^{C_n} \bigcup_{v \in A_n} \{v \nleftarrow k, \theta_n(v) \nleftarrow s_n(z_n(v), k)\}, &(16)\\ &= \bigcup_{k=0}^{C_n} \bigcup_{v \in B_n} \{v \nleftarrow k, \theta_n(v) \nleftarrow s_n(z_n(v), k)\}, &(17)\\ E_{n+1} &= R_{n+1} \cup S_{n+1}, &(18) \end{aligned}$$

noting that S_{n+1} can be written in terms of either the odd vertices A_n or the even vertices B_n. Eq. (15) replicates the edges of G_n among subsets of the vertices of G_{n+1}, while Eq. (16) creates one edge between each pair of the subsets.

3 Properties of G_n

Lemma 1. *The graph G_n is n-regular for all $n \geq 0$.*

Proof. We proceed using induction on n. The base case G_1 is 1-regular by inspection. In the inductive case G_{n+1}, Eq. (15) reproduces the edges of G_n, which is n-regular by induction, contributing n to the degree of each vertex. Eq. (16) adds one to the degree of each odd vertex. As θ_n is a bijective map between odd and even vertices, Eq. (16) also adds one to the degree of each even vertex, giving a total degree of $n+1$. $\square$

Lemma 2. *The number of vertices and edges of the graph G_n are given by the recurrence relations:*

$$\begin{aligned} C_0 = |V_0| &= 1, && (19)\\ C_n = |V_n| &= C_{n-1}(C_{n-1}+1), && (20)\\ |E_n| &= \frac{n}{2}C_n. && (21)\end{aligned}$$

Proof. The vertex set of G_n, given by Eq. (5), is a union of $C_{n-1}+1$ sets. The vertex labels within each set all end with the same element, and this element is unique to each set. The sets are thus disjoint. Each set contains C_{n-1} elements, giving $C_{n-1}(C_{n-1}+1)$ elements. By Lemma 1, G_n is n-regular and has $\frac{n}{2}|V_n|= \frac{n}{2}C_n$ edges. $\square$

Lemma 3. *The mapping ϕ_{n-1} is an automorphism of G_n.*

Proof. Let v be a vertex in V_n. ϕ_{n-1} has no effect on v_0^{n-2} so it preserves the edges R_n, defined in Eq. (15). The vertex v has exactly one other edge, from S_n defined in Eq. (16):

$$e \quad = \quad \{v, \theta_n(v_0^{n-2} \nleftarrow s_n(z_n(v_0^{n-2}), v_{n-1})\}. \qquad (22)$$

Applying ϕ_{n-1} to both vertices gives:

$$\begin{aligned} \tilde{e} &= \{v_0^{n-2} \nleftarrow (v_{n-1}+1) \bmod (C_{n-2}+1), && (23)\\ &\qquad \theta_n(v_0^{n-2} \nleftarrow s_n(z_n(v_0^{n-2}), (v_{n-1}+1) \bmod (C_{n-2}+1))\}, && (24)\\ &= \{v_0^{n-2} \nleftarrow \tilde{k}, \theta_n(v_0^{n-2} \nleftarrow s_n(z_n(v_0^{n-2}), \tilde{k})\}, && (25)\end{aligned}$$

where $\tilde{k} = (v_{n-1}+1) \bmod (C_{n-2}+1)$. The edge $\tilde{e}$ is also a member of S_n, showing that ϕ_{n-1} permutes the elements of S_n, and preserves all edges in E_n. $\square$

Theorem 1. *The graphs G_n are vertex-transitive for all $n \geq 0$.*

Proof. The graph G_0 is trivially vertex-transitive. For higher order graphs, we recursively construct a mapping Φ_n that maps an arbitrary vertex $v \in V_n$ to an arbitrary vertex $\tilde{v} \in V_n$ and preserves the edges in E_n. Let $\delta_i = \tilde{v}_i - v_i \mod (C_i + 1)$. For the subsequence v_0^0, the map $\Phi_1 = \phi_0^{\delta_0}$ achieves the desired mapping and preserves the edges in E_1 by Lemma 3. Given an edge-preserving mapping Φ_n from v_0^{n-1} to $\tilde{v}_0^{n-1}$, we can construct τ_n. We define mappings π_n and τ_n:

$$\begin{aligned}
\pi_n(0) &= 0 && (26)\\
\pi_n(s_n(v_0^{n-1}, 0)) &= s_n(\Phi_n(v_0^{n-1}), 0) && (27)\\
\tau_n(v) &= \Phi_n(v_0^{n-1}) \nleftrightarrow \pi_n(v_n). && (28)
\end{aligned}$$

For the recursive edges defined in Eq. (15) $\{v_0^n, w_0^n\} \in R_{n+1}$:

$$\begin{aligned}
\{\tau_n(v_0^n), \tau_n(v_0^n) &= \{\Phi_n(v_0^n) \nleftrightarrow \pi_n(k), \Phi_n(w_0^n) \nleftrightarrow \pi_n(k)\} && (29)\\
&= \{\tilde{v}_0^n \nleftrightarrow \tilde{k}, \Phi_n(\tilde{w}_0^n) \nleftrightarrow \tilde{k}\}, && (30)
\end{aligned}$$

which is simply another member of R_{n+1}. As τ_n is bijective, it permutes the edges of R_{n+1}. For the shortcut edges defined in Eq. (16) $\{v_0^n, w_0^n\} \in S_{n+1}$, when $v_n = 0$:

$$\begin{aligned}
\{\tau_n(v_0^n), \tau_n(w_0^n)\} &= \{\Phi_n(v_0^n) \nleftrightarrow \pi_n(0), && (31)\\
&\quad\ \Phi_n(w_0^n)) \nleftrightarrow \pi_n(s_n(z_n(v_0^n), 0))\} && (32)\\
&= \{\tilde{v}_0^n \nleftrightarrow 0, \tilde{w}_0^n \nleftrightarrow s_n(z_n(\Phi_n(v_0^n)), 0)\} && (33)\\
&= \{\tilde{v}_0^n \nleftrightarrow 0, \theta_n(\tilde{v}_0^n) \nleftrightarrow s_n(z_n(\tilde{v_0^n}), 0)\}, && (34)
\end{aligned}$$

showing that τ_n permutes the subset of edges in S_{n+1} for which $v_n = 0$. Consequently, because ϕ_n is an automorphism of G_{n+1} by Lemma 3, τ_n also permutes the edges of the subsets for which $(\phi_n^{-k}(v_0^n))_n = 0$. Those edges are the subset of S_{n+1} for which $v_n = k$, showing that $\Phi_n n + 1$ permutes all edges within S_{n+1} and E_{n+1}. We finish by applying the automorphism $\phi_n^{\delta_n}$:

$$\Phi_{n+1}(v) = \phi_n^{\delta_n}(\tau_n(v)), \qquad (35)$$

completing the mapping from $\tilde{v}$ to $\tilde{w}$. □

4 Acknowledgements

References

[1] I. Z. Bower. *The Foster Census*. Charles Babbage Research Centre, Winnipeg, 1988.