Stat 343: Motivation for Numerical Optimization of the Likelihood

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Rainfall in Illinois, all storms from 1960 - 1964

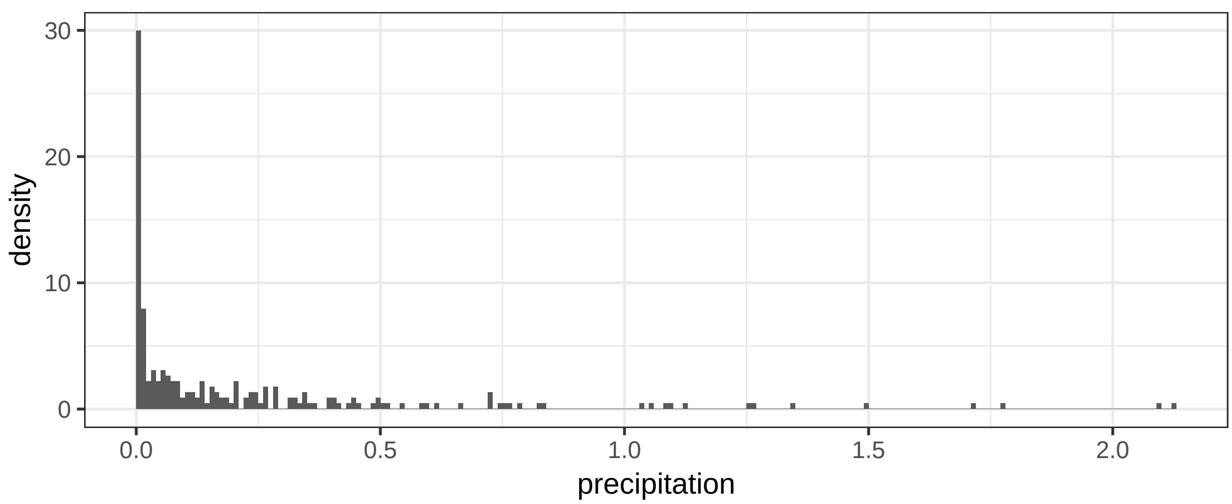
The code below reads in the data and makes an initial plot:

```
library(tidyverse)
```

```
## Warning: package 'tibble' was built under R version 3.5.2
## Warning: package 'purrr' was built under R version 3.5.2
```

```
# Precipitation in Illinois
# Amount of rainfall in 272 storms from 1960 to 1962
il_storms <- bind_rows(
  read_csv("http://www.evanlray.com/data/rice/Chapter%2010/illinois60.txt",
    col_names = FALSE),
  read_csv("http://www.evanlray.com/data/rice/Chapter%2010/illinois61.txt",
    col_names = FALSE),
  read_csv("http://www.evanlray.com/data/rice/Chapter%2010/illinois62.txt",
    col_names = FALSE),
  read_csv("http://www.evanlray.com/data/rice/Chapter%2010/illinois63.txt",
    col_names = FALSE),
  read_csv("http://www.evanlray.com/data/rice/Chapter%2010/illinois64.txt",
    col_names = FALSE)
)
names(il_storms) <- "precipitation"

ggplot(data = il_storms, mapping = aes(x = precipitation)) +
  geom_histogram(center = 0.005, binwidth = 0.01, mapping = aes(y = ..density..)) +
  theme_bw()
```



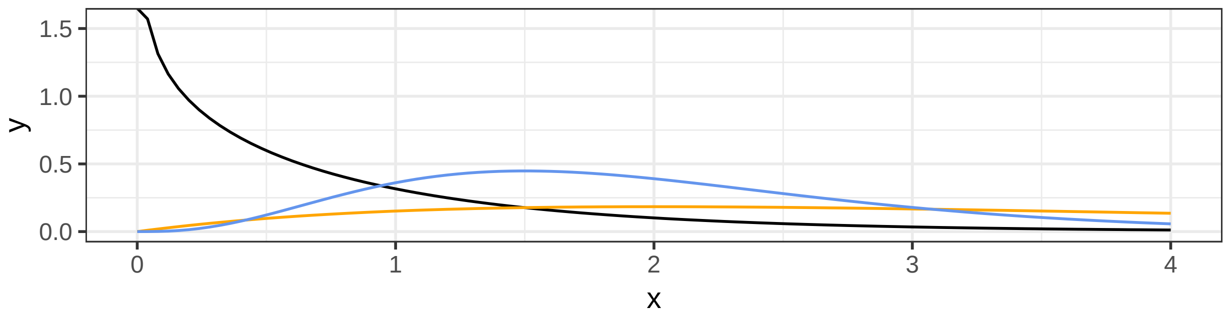
Let's model these data with a Gamma distribution and estimate the parameters by maximum likelihood

Gamma(α, λ) Distribution (copied from the "Common Distributions" handout)

A non-negative real number. Be careful – there are multiple other parameterizations used in other sources.

parameters	$\alpha \geq 0$: shape parameter, $\lambda > 0$: rate parameter
p.f.	$f(x\|\alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$
Mean	$\frac{\alpha}{\lambda}$
Variance	$\alpha\lambda^2$
R functions	`dgamma(..., shape =` α`, rate =` λ`), pgamma, qgamma, rgamma`

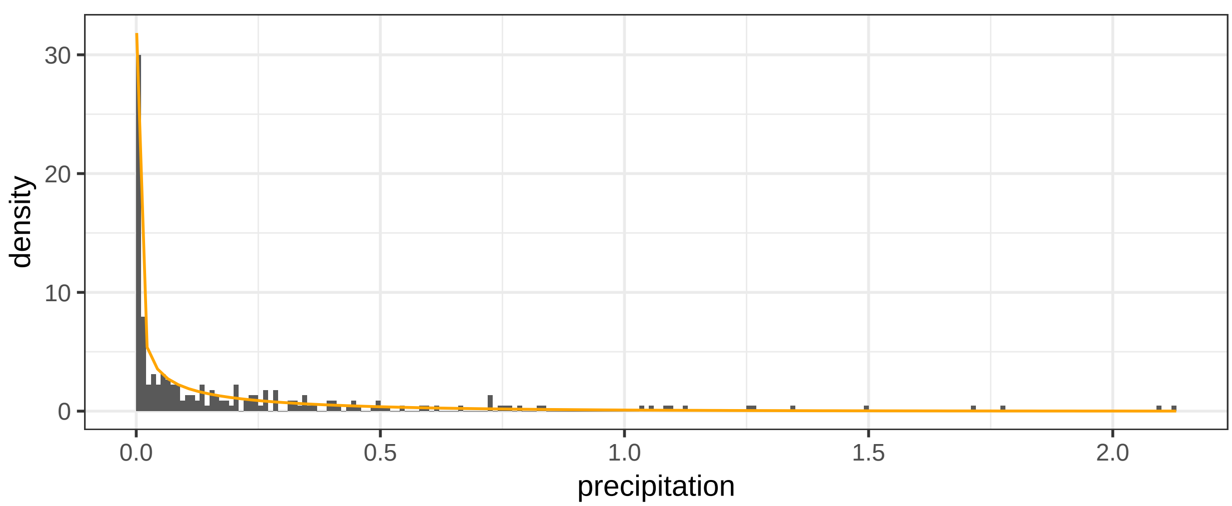
```
ggplot(data = data.frame(x = c(0, 4)), mapping = aes(x = x)) +
  stat_function(fun = dgamma, args = list(shape = 0.8, rate = 1), color = "black") +
  stat_function(fun = dgamma, args = list(shape = 2, rate = 0.5), color = "orange") +
  stat_function(fun = dgamma, args = list(shape = 4, rate = 2), color = "cornflowerblue") +
  theme_bw()
```



Preview

For our data set, the maximum likelihood estimates are approximately $\hat{\alpha}_{MLE} = 0.440813$ and $\hat{\beta}_{MLE} = 1.964378$

```
ggplot(data = il_storms, mapping = aes(x = precipitation)) +
  geom_histogram(center = 0.005, binwidth = 0.01, mapping = aes(y = ..density..)) +
  stat_function(fun = dgamma, args = list(shape = 0.440813, rate = 1.964378), color = "orange") +
  theme_bw()
```



Maximum Likelihood for a Gamma(α, β)

Suppose we model $X_1, \ldots, X_n$ as i.i.d. with each $X_i \sim \text{Gamma}(\alpha, \beta)$.

The likelihood function is

$$
\begin{aligned}
\mathcal{L}(\alpha, \beta | x_1, \ldots, x_n) &= \prod_{i=1}^{n} \frac{\lambda^\alpha}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\lambda x_i} \\
&= \left\{ \frac{\lambda^\alpha}{\Gamma(\alpha)} \right\}^n \left\{ \prod_{i=1}^{n} x_i^{\alpha-1} \right\} e^{-\lambda \sum_{i=1}^{n} x_i}
\end{aligned}
$$

The log-likelihood function is

$$
\ell(\alpha, \beta | x_1, \ldots, x_n) = n\alpha \log(\lambda) - n \log \{\Gamma(\alpha)\} + \sum_{i=1}^{n} (\alpha - 1) \log(x_i) - \lambda \sum_{i=1}^{n} x_i
$$

The partial derivatives are

$$
\frac{\partial}{\partial \alpha} \ell(\alpha, \beta | x_1, \ldots, x_n) = n \log(\lambda) - n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum_{i=1}^{n} \log(x_i) \tag{1}
$$

$$
\frac{\partial}{\partial \beta} \ell(\alpha, \beta | x_1, \ldots, x_n) = \frac{n\alpha}{\lambda} - \sum_{i=1}^{n} x_i \tag{2}
$$

Setting Equation (2) equal to 0 and solving for λ, we obtain

$$
\lambda = \frac{n\alpha}{\sum_{i=1}^{n} x_i} = \frac{\alpha}{\bar{x}} \tag{3}
$$

Substituting Equation (3) into Equation (1) and setting equal to 0, we obtain

$$
n \log(\alpha) - n \log(\bar{x}) + \sum_{i=1}^{n} \log(x_i) - n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = 0 \tag{4}
$$

However, there is no way to rearrange Equation (4) to solve for α! We are stuck!!!