Significance level: Reject H_0 if p-value $< \alpha$

Size: $\underbrace{P(\text{Type I Error} \mid H_0 \text{ true})}_{\text{also sometimes denoted by } \alpha} \leq \alpha$

Power: ~~Reject H_0 correctly~~

$$\text{Power} = P(\text{Reject } H_0 \mid \cancel{H_A \text{ true}}\ H_0 \text{ false})$$

Neyman-Pearson Lemma

Suppose that H_0 and H_A are simple hypotheses and that the test that rejects whenever the likelihood ratio is less than c has ~~significance level~~ size α. Then any other test with significance level less than or equal to α has power less than or equal to the power of the likelihood ratio test.

Pf: Let $f_0(\underset{\sim}{x})$ denote the pdf or pmf of the data if H_0 true,
$f_A(\underset{\sim}{x})$ " " " if H_A true,

~~Let R denote the rejection region of the test (we reject H_0 if $\underset{\sim}{x} \in R$)~~ } drop this,

Define the "decision functions"

$$d(x) = \begin{cases} 1 \text{ if we reject } H_0 \text{ based on the LRT} : \dfrac{f_0(\underset{\sim}{x})}{f_A(\underset{\sim}{x})} < c, \text{ or } \cancel{f_0(\underset{\sim}{x}) - cf_A(\underset{\sim}{x}) < 0} \;\; cf_A(\underset{\sim}{x}) - f_0(\underset{\sim}{x}) > 0 \\ 0 \text{ otherwise (fail to reject } H_0) \end{cases}$$

$$d^*(x) = \begin{cases} 1 \text{ if we reject } H_0 \text{ based on some other test} \\ 0 \text{ otherwise (fail to reject } H_0) \end{cases}$$

~~Power other~~
Want to show: Power other test $\le$ Power LRT

$$\int_{R^*} f_A(x)\,dx \le \int_R f_A(x)\,dx$$

$$\int d^*(x) f_A(x)\,dx \le \int d(x) f_A(x)\,dx$$

$$0 \le \int d(\underset{\sim}{x}) f_A(\underset{\sim}{x})\,dx - \int d^*(\underset{\sim}{x}) f_A(\underset{\sim}{x})\,dx \quad (1)$$

Note: $d^*(\underset{\sim}{x})[c f_A(\underset{\sim}{x}) - f_0(\underset{\sim}{x})] \le d(\underset{\sim}{x})[c f_A(\underset{\sim}{x}) - f_0(\underset{\sim}{x})]$

- If $d(\underset{\sim}{x}) = 1$ then $cf_A(\underset{\sim}{x}) - f_0(\underset{\sim}{x}) > 0$, so dividing both sides we get $d^*(\underset{\sim}{x}) \le 1$ which is true
- If $d(\underset{\sim}{x}) = 0$ then $cf_A(\underset{\sim}{x}) - f_0(\underset{\sim}{x}) \le 0$, so $d^*(\underset{\sim}{x}) \cdot [cf_A(\underset{\sim}{x}) - f_0(\underset{\sim}{x})] \le 0$ is true.

Integrating both sides wrt $\underset{\sim}{x}$ we get

$$\int d^*(\underset{\sim}{x})\{cf_A(\underset{\sim}{x})-f_0(\underset{\sim}{x})\}d\underset{\sim}{x} \leq \int d(\underset{\sim}{x})\{cf_A(\underset{\sim}{x})-f_0(\underset{\sim}{x})\}d\underset{\sim}{x}$$

$$\Rightarrow c\int d^*(\underset{\sim}{x})f_A(\underset{\sim}{x})d\underset{\sim}{x} - \int d^*(\underset{\sim}{x})f_0(\underset{\sim}{x})d\underset{\sim}{x} \leq c\int d(\underset{\sim}{x})f_A(\underset{\sim}{x}) - \int d(\underset{\sim}{x})f_0(\underset{\sim}{x})d\underset{\sim}{x}$$

$$\Rightarrow 0 \leq \underbrace{\underbrace{\int d(\underset{\sim}{x})f_0(\underset{\sim}{x})d\underset{\sim}{x}}_{\substack{P(\text{Type I Error; LRT})\\ \alpha}} - \underbrace{\int d^*(\underset{\sim}{x})f_0(\underset{\sim}{x})d\underset{\sim}{x}}_{\substack{P(\text{Type I Error})\\ \text{other test}\\ \leq\alpha}}}_{\text{non-negative}} \leq c\left\{\int d(\underset{\sim}{x})f_A(\underset{\sim}{x})d\underset{\sim}{x} - \int d^*(\underset{\sim}{x})f_A(\underset{\sim}{x})d\underset{\sim}{x}\right\}$$

$$\therefore\quad 0 \leq \int d(\underset{\sim}{x})f_A(\underset{\sim}{x})d\underset{\sim}{x} - \int d^*(\underset{\sim}{x})f_A(\underset{\sim}{x})d\underset{\sim}{x},$$

which is what we wanted to show above (eq 1)

$0 \leq$ Power of LRT $-$ Power of other test