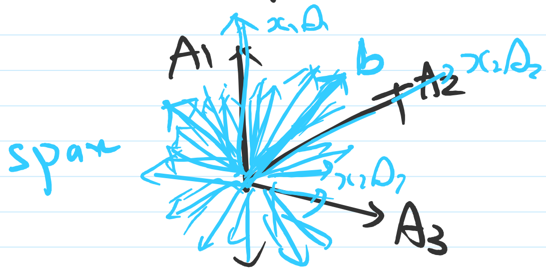
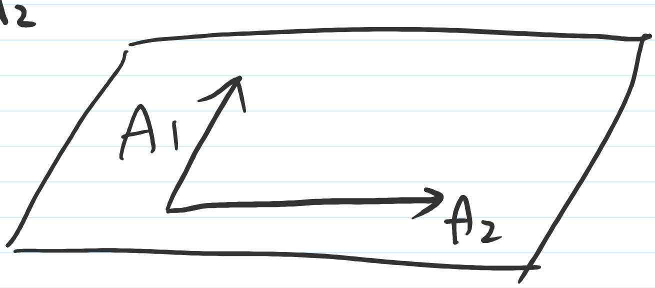
Span

The span of vectors $A_1, A_2, \dots, A_k$ is the collection of all linear combinations of $A_1, A_2, \dots, A_k$



Example: The span of two vectors $A_1 \in \mathbb{R}^2$ and $A_2 \in \mathbb{R}^2$ is the entire $\mathbb{R}^2$ if $A_1 \neq cA_2$



on a plane

Theorem: Linear system $Ax=b$, $A \in \mathbb{R}^{n \times k}$

$$\begin{bmatrix} A_1 & A_2 & \cdots & A_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

has a solution if and only if the vector b is within the span, that is, the space spanned by column vectors $A_1, A_2, \dots, A_k$.

Example:
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ b \\ \\ \end{bmatrix}$$

The linear system has a solution if and only if

The linear system has a solution if and only if b is in the space spanned by $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$.

This system has a solution when $b=\begin{bmatrix}2\\5\\1\end{bmatrix}$ but has <u>no</u> solution when $b=\begin{bmatrix}2\\5\\6\end{bmatrix}$.

Linearly independence

The vectors A_i, $i=1\ldots k$ are <u>linearly independent</u> if the only linear combination of all vectors which results in a zero vector is the trivial combination.

Example: $x_1\begin{bmatrix}1\\2\\3\end{bmatrix}+x_2\begin{bmatrix}1\\0\\1\end{bmatrix}+x_3\begin{bmatrix}1\\3\\4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

trivial combination: $0\begin{bmatrix}1\\2\\3\end{bmatrix}+0\begin{bmatrix}1\\0\\1\end{bmatrix}+0\begin{bmatrix}1\\3\\4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

another combination: $3\begin{bmatrix}1\\2\\3\end{bmatrix}+(-1)\begin{bmatrix}1\\0\\1\end{bmatrix}+(-2)\begin{bmatrix}1\\3\\4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

We thus know $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\3\\4\end{bmatrix}$ are <u>not</u> linearly independent.

2.1 Important subspaces in $\mathbb{R}^n$

Subspace

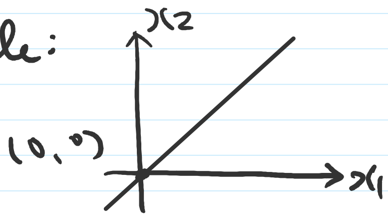
A subspace of $\mathbb{R}^n$ is a set of vectors in $\mathbb{R}^n$ that has three properties.

1. The 0-vector is in the set.
2. For each vector u in the set and each vector v in the set, the vector $u+v$ is in the set.
3. For each vector u in the set and each scalar $c \in \mathbb{R}$, the vector cu is in the set.

$$\left[\begin{array}{l} S \text{ is a subspace if} \\ 1.\ 0 \in S \\ 2.\ \forall u \in S,\ \forall v \in S,\ u+v \in S \\ 3.\ \forall u \in S,\ \forall c \in \mathbb{R},\ cu \in S \end{array}\right]$$

$\in$: belongs in
$\forall$: for all

Example:



A line passing through (0,0) is a subspace of $\mathbb{R}^2$

A line not passing through the origin is not a subspace in $\mathbb{R}^2$.

Four foundamental subspaces of $\mathbb{R}^n$

Column space
null space
row space
left null space

Column space of a matrix

The column space of a matrix is a set of all linear combinations of column vectors of matrix A. It is denoted as $C(A)$ or $Range(A)$

span. column space. range.

Example: Determine if the vector $\begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$ is in $C(A)$,

where $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \end{bmatrix}$.

$\begin{bmatrix} \\ \\ -4 \end{bmatrix}$

where $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$.

Ans: The system $x_1 \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$

has a solution if and only if $\begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$ is in the collection

of linear combinations of $\left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix} \right\}$.

We do Gaussian elimination on the augmented

matrix $\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ 3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. There are

∞ many solutions. And $\begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$ is in $C(A)$. $\blacksquare$

Properties:

a. $C(A)$ is the <u>span</u> of column vectors of matrix A.

b. The smallest possible column space is for zero matrix $A = 0$.
$C(A) = \{0\}$. $\{\ \}$ represents a collection.

c. Any invertible matrix (nonsingular matrix) A, $A \in \mathbb{R}^{n \times n}$, has $C(A) = \mathbb{R}^n$.

Null space of matrix

The null space of a matrix A is the set of all solutions of $Ax = 0$, where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, 0 is a vector with m elements.

The null space is denoted as $N(A)$ or $\mathrm{Null}(A)$.

Theorem: For any matrix $A \in \mathbb{R}^{m \times n}$

$$N(A) \oplus C(A^T) = \mathbb{R}^n,$$

where $N(A) \oplus C(A^T)$ is the set of vectors $x+y$ for each $x \in N(A)$ and each $y \in C(A^T)$.

$\oplus$ is called the direct sum of two sets.

Properties:

a. When $A \in \mathbb{R}^{n\times n}$ is a square matrix and A is nonsingular, $N(A)=\{0\}$, which is a set with only zero vector.

According to the theorem, $C(A^T)=\mathbb{R}^n$. It is because A is a square matrix and A is invertible, A^T is also invertible and $A^T \in \mathbb{R}^{n\times n}$. By property (c) of the column space, we know that $C(A^T)=\mathbb{R}^n$. Therefore $N(A)=\{0\}$ to let $N(A)\oplus C(A^T)=\mathbb{R}^n$ hold.

b. If A is not square, $A \in \mathbb{R}^{m\times n}$, $m \neq n$, $N(A)$ can be anything between $\{0\}$ and $\mathbb{R}^n$.

Basis of subspace

A basis for a subspace H of $\mathbb{R}^n$ is a set of vectors such that (i) the vectors in the basis are linearly independent

(ii) the vectors in the basis span the subspace H.

a verb

Example: $e_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $e_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$,