

ON CERTAIN PROPERTIES OF PERTURBED FREUD-TYPE WEIGHT: A REVISIT

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ABSTRACT. In this paper, monic polynomials orthogonal with deformation of the Freud-type weight function are considered. These polynomials fulfill linear differential equation with some polynomial coefficients in their holonomic form. The aim of this work is explore certain characterizing properties of perturbed Freud type polynomials such as nonlinear recursion relations, finite moments, differential-recurrence and differential relations satisfied by the recurrence coefficients as well as the corresponding semiclassical orthogonal polynomials. We note that the obtained differential equation fulfilled by the considered semiclassical polynomials are used to study an electrostatic interpretation for the distribution of zeros based on the original ideas of Stieltjes.

1. INTRODUCTION

Suppose we have a family of polynomials $\{\psi_m(x)\}_{m=1}^{\infty}$ which are monic of degree m and that are orthogonal with respect to the positive weight $w(x)$ on the interval $[c, d]$, i.e.,

$$\langle \psi_m, \psi_k \rangle_w = \int_c^d \psi_m(x) \psi_k(x) w(x) dx = \Gamma_m \delta_{m,k}, \quad m, k = 0, 1, 2, \dots,$$

where $\Gamma_m > 0$ denotes the normalization constant [7, 29]. This value can be obtained from the square of the weighted L^2 -norm of $\psi_m(x)$ over $[c, d]$. Monic polynomial representation takes the form

$$\psi_n(x) = x^n + p(n)x^{n-1} + \dots$$

It is known that $\det(x_j^{i-1})_{i,j=1}^N = \prod_{1 \leq i < j \leq N} (x_i - x_j) = \det(\psi_{i-1}(x_j))_{i,j=1}^N$. The polynomials $\psi_n(x)$ can be generated by the Gram-Schmidt orthogonalization process [7, 18].

As it is known in [7, 18, 29], classical orthogonal polynomials obey Pearson's differential equation

$$\frac{d(\lambda(x)w(x))}{dx} = \tau(x)w(x), \quad (1.1)$$

where the polynomials $\lambda(x)$ and $\tau(x)$ are of degrees two and one respectively. Whereas polynomials for which the weight fulfills Eq. (1.1) with $\deg(\lambda) \geq 2$ or $\deg(\tau) \neq 1$ are said to be Semi-classical orthogonal polynomials [17].

For deformed orthogonality weight, if the moments exist and the corresponding monic orthogonal polynomials $\psi_n(z)$ for $n = 0, 1, 2, \dots$ obey linear recursive relation

$$\begin{cases} z\psi_n(z) = \psi_{n+1}(z) + \gamma_n \psi_{n-1}(z) + \alpha_n \psi_n(z), \\ \psi_0(z) = 1, \quad \gamma_0 \psi_{-1}(z) = 0. \end{cases}$$

The following relations in [3] are valid for a semiclassical weight w with $w(a) = w(b) = 0$.

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Lemma 1. [3]. Suppose that $v(x) = -\ln w(x)$ has a derivative in some Lipschitz order with a positive exponent [27]. The differential-difference coefficients obey the following formulas:

$$\psi'_n(z) = \gamma_n \mathcal{A}_n(z) \psi_{n-1}(z) - \mathcal{B}_n(z) \psi_n(z), \quad (1.2)$$

$$\psi'_{n-1}(z) = -\mathcal{A}_{n-1}(z) \psi_n(z) + [\mathcal{B}_n(z) + v'(z)] \psi_{n-1}(z), \quad (1.3)$$

where

$$\mathcal{A}_n(z) := \frac{1}{\Gamma_n} \int_a^b \frac{v'(z) - v'(\tau)}{z - \tau} \psi_n^2(y) w(\tau) d\tau, \quad (1.4)$$

$$\mathcal{B}_n(z) := \frac{1}{\Gamma_{n-1}} \int_a^b \frac{v'(z) - v'(\tau)}{z - \tau} \psi_n(\tau) \psi_{n-1}(\tau) w(\tau) d\tau. \quad (1.5)$$

Lemma 2. [3]. The coefficients $\mathcal{A}_n(z)$ and $\mathcal{B}_n(z)$ defined by Eq. (1.4) and Eq. (1.5) obeys

$$\begin{cases} \mathcal{B}_{n+1}(z) + \mathcal{B}_n(z) = -v'(z) + (z - \alpha_n) \mathcal{A}_n(z), & (M_1) \\ 1 + (z - \alpha_n)[\mathcal{B}_{n+1}(z) - \mathcal{B}_n(z)] = -\gamma_n \mathcal{A}_{n-1}(z) + \gamma_{n+1} \mathcal{A}_{n+1}(z). & (M_2) \end{cases}$$

We also mention another supplementary condition, that involves $\sum_{j=0}^{n-1} \mathcal{A}_j(z)$ and we will denote it by (M'_2) as this relation helps to obtain recurrence coefficients α_n and γ_n , as

$$v'(z) \mathcal{B}_n(z) + \sum_{j=0}^{n-1} \mathcal{A}_j(z) + \mathcal{B}_n^2(z) = \gamma_n \mathcal{A}_n(z) \mathcal{A}_{n-1}(z). \quad (M'_2)$$

Eq. (M'_2) can be perceived as an equation for $\sum_{j=0}^{n-1} \mathcal{A}_j(z)$. See, for instance, [2, 4].

The differential equation fulfilled by $\psi_n(z)$ is generated by eliminating $\psi_{n-1}(z)$ from ladder operators, and it is given as

$$\psi''_n(z) - \left(v'(z) + \frac{\mathcal{A}'_n(z)}{\mathcal{A}_n(z)} \right) \psi'_n(z) + \left(\mathcal{B}'_n(z) - \mathcal{B}_n(z) \frac{\mathcal{A}'_n(z)}{\mathcal{A}_n(z)} + \sum_{j=0}^{n-1} \mathcal{A}_j(z) \right) \psi_n(z) = 0, \quad (1.7)$$

where $\sum_{j=0}^{n-1} \mathcal{A}_j(z)$ is obtained from (M'_2) .

Lemma 3. Suppose we have a symmetric semi-classical weight $W_\sigma(x; t) = \exp(tx^2)w_0(x)$, with $t \in \mathbb{R}$ such that the moments of w_0 is finite. The recursive coefficient $\gamma_n(t)$ obeys the Volterra, or the Langmuir lattice, equation [31]

$$\frac{d\gamma_n(t)}{dt} = \gamma_n(t)(\gamma_{n+1}(t) - \gamma_{n-1}(t)). \quad (1.8)$$

Proof. See, for example, [31, Theorem 2.4]. □

In this paper, we consider to study semiclassical perturbed Freud-type measure

$$\begin{cases} d\mu_\sigma(x) = W_\sigma(x; t) dx = |x|^{2\sigma+1} \exp(-[cx^6 + t(x^4 - x^2)]) dx, \\ \sigma > 0, c > 0, t \in \mathbb{R}, \end{cases} \quad (1.9)$$

involving parameters t, σ , which will be used to represent the polynomials and in the L^2 norm. For simplicity, we may not sometimes display the parameters in the polynomials.

The motives for the choice of the perturbed orthogonality measure in (1.9) is as follows:- First, from some of the classical orthogonal polynomials, a new class of semiclassical (non-classical) orthogonal polynomials can be obtained by means of slight modifications on their orthogonality measure [25, 26]. Such measure deformation usually results in some difficulties, most of which have not been handled yet as noted in [25, 26]. Motivated by the works of P. Nevai et al. [26], a slight modification of a new orthogonality measure on non-compact support presents a new class of orthogonal polynomials if certain characterizing properties associated with the

considered polynomials are successfully obtained. Secondly, the choice of modified Freud-type measure is reasonable in the sense that this orthogonality measure emanates from quadratic transformation and Chihara's symmetrization of the modified Airy-type measure (cf. [7] for symmetrization process). This also leads to an investigation of certain fresh properties such as nonlinear differential-recurrence and differential equations satisfied by the recurrence coefficients as well as the perturbed polynomials themselves. The results obtained also motivate considerable applications; for instance, in modeling nonlinear phenomena, Soliton Theory and Random matrix theory [4] and in the crystal structure in solid-state physics, to mention a few.

2. SEMICLASSICAL PERTURBED FREUD-TYPE POLYNOMIALS

Semiclassical perturbed Freud polynomials $\{\mathcal{S}_n(x; t)\}_{n=0}^{\infty}$ on \mathbb{R} are real polynomials with their orthogonality weight given by

$$\begin{cases} d\mu_{\sigma}(x) = W_{\sigma}(x; t) dx = |x|^{2\sigma+1} \exp(-[cx^6 + t(x^4 - x^2)]) dx, \\ \sigma > 0, c > 0, t \in \mathbb{R}, \end{cases}$$

and the orthogonality condition is given by

$$\langle \mathcal{S}_n, \mathcal{S}_m \rangle_{W_{\sigma}} = \int_{-\infty}^{\infty} \mathcal{S}_n(x; t) \mathcal{S}_m(x; t) W_{\sigma}(x; t) dx = \hat{\Gamma}_n \delta_{mn}, \quad (2.1)$$

where δ_{mn} denotes the Kronecker delta function. It follows from Eq. (2.1) that the recursion relation takes the form

$$\begin{cases} \mathcal{S}_{n+1}(x; t) = -\gamma_n(t; \sigma) \mathcal{S}_{n-1}(x; t) + x \mathcal{S}_n(x; t), n \in \mathbb{N}, \\ \mathcal{S}_0 := 1 \text{ and } \gamma_0 \mathcal{S}_{-1} := 0. \end{cases} \quad (2.2)$$

If we multiply Eq. (2.2) with $\mathcal{S}_{n-1}(x; t) W_{\sigma}(x; t)$ and then integrate with respect to x and using orthogonality given in Eq. (2.1), we obtain

$$\gamma_n(t; \sigma) = \frac{1}{\hat{\Gamma}_{n-1}(t)} \langle x \mathcal{S}_n, \mathcal{S}_{n-1} \rangle_{W_{\sigma}} = \frac{\hat{\Gamma}_n(t)}{\hat{\Gamma}_{n-1}} > 0. \quad (2.3)$$

Observe that $\mathcal{S}_n(x; t)$ comprises the terms x^{n-r} , $r \leq n$ and is symmetric so that

$$\begin{cases} \mathcal{S}_n(-x; t) = (-1)^n \mathcal{S}_n(x; t), \\ \mathcal{S}_n(0; t) \mathcal{S}_{n-1}(0; t) = 0, \end{cases}$$

as the weight $W_{\sigma}(x; t)$ is even on \mathbb{R} . Using monic representation of considered polynomials $\mathcal{S}_n(x; t)$, associated with $W_{\sigma}(x; t)$, we have that

$$\mathcal{S}_n(x; t) = x^n + \chi(n; t) x^{n-2} + \dots + \mathcal{S}_n(0; t), \quad (2.4)$$

which can be expressed equivalently as [7],

$$\begin{cases} \mathcal{S}_{2i}(x; t) = x^{2i} + \chi(2i; t) x^{2i-2} + \dots + \mathcal{S}_{2i}(0), \\ \mathcal{S}_{2i+1}(x; t) = x^{2i+1} + \chi(2i+1; t) x^{2i-1} + \dots + s.x = x(x^{2i} + \chi(2i+1; t) x^{2i-2} + \dots + s), \end{cases}$$

where $s \in \mathbb{R}$. By substituting Eq. (2.4) into Eq. (2.2), we obtain

$$\begin{cases} \gamma_n(t) = \chi(n; t) - \chi(n+1; t), \\ \chi(0) := 0. \end{cases} \quad (2.5)$$

Imposing a telescoping iteration of terms of Eq. (2.5) gives

$$\sum_{k=0}^{n-1} \gamma_k(t, \sigma) = -\chi(n; t).$$

3. CERTAIN PROPERTIES OF THE CONSIDERED SEMICLASSICAL POLYNOMIALS

In this section, we explore certain characterizing properties for perturbed semi-classical Freud-type polynomials.

3.1. Finite moments. For certain semiclassical weights, it is known in [8, 9, 21] that the moments make link between the weight function and the theory of integrable equations, in particular, Painlevé-type equations [31].

Theorem 4. Suppose $x, t \in \mathbb{R}$ and $c, \sigma > 0$. The first moment $\eta_0(t; \sigma)$ associated with the weight (2.1) is finite.

Proof. For the weight given in Eq. (1.9), the moment $\eta_0(t; \sigma)$ takes the form

$$\eta_0(t; \sigma) = \int_{-\infty}^{\infty} W_{\sigma}(x; t) dx = 2 \int_0^{\infty} W_{\sigma}(x; t) dx. \quad (3.1)$$

For $\sigma > 0$ and $c > 0$, the function $W_{\sigma}(x; t) = x^{2\sigma+1} \exp(-[cx^6 + t(x^4 - x^2)])$ is continuous on $[0, \infty)$, and hence is integrable on $[0, \mathcal{K}]$ for any $\mathcal{K} > 0$. In order to show $\int_{\mathcal{K}}^{\infty} W_{\sigma}(x; t) dx$ is finite, we first note that $\lim_{x \rightarrow \infty} x^2 W_{\sigma}(x; t) = 0$; that is, there exists an $N > 0$ such that $x^2 W_{\sigma}(x; t) < 1$ whenever $x > N$ by definition. As $\int_N^{\infty} \frac{dx}{x^2} < \infty$, it follows, for $N > 0$, that $\int_N^{\infty} W_{\sigma}(x; t) dx < \infty$, particularly when $N = \mathcal{K}$. Hence, $\int_0^{\infty} W_{\sigma}(x; t) dx < \infty$. \square

The following result presents some conditions for differentiation and integration order for functions of two variables [20].

Lemma 5. [20, Theorem 16.11].

Let $J = (a, b) \subset \mathbb{R}$ be an open interval and $g : \mathbb{R} \times J \rightarrow \mathbb{R}$. Assume that

- (i) $g(x, t)$ has a derivative on \mathbb{R} with respect to t for almost all $x \in \mathbb{R}$,
- (ii) for every fixed $t \in J$, $\int_{-\infty}^{\infty} g(x, t) dx < \infty$,
- (iii) \exists an integrable function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $\forall t \in J$, $\left| \frac{\partial g(x, t)}{\partial t} \right| \leq h(x)$, which is true for almost all $x \in \mathbb{R}$.

It then follows that

$$\frac{d}{dt} \int_{-\infty}^{\infty} g(x, t) dx = \int_{-\infty}^{\infty} \frac{\partial g(x, t)}{\partial t} dx.$$

The following result shows how moments of high order behave for the weight function in Eq. (1.9).

Theorem 6. For $n \in \mathbb{N}_0$, the moments associated with the perturbed Freud weight given in (1.9) obey the following formulations

$$\begin{cases} \eta_{2n}(t; \sigma) &= \frac{d^n}{dt^n} \int_{-\infty}^{\infty} |x|^{2\sigma+1} \exp(-[cx^6 + t(x^4 - x^2)]) dx \\ &= \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \eta_{4n-2k}(t; \sigma) = \frac{d^n}{dt^n} \eta_0(t; \sigma), \\ \eta_{2n+1}(t; \sigma) &= 0. \end{cases} \quad (3.2)$$

Proof. Taking into account the weight in Eq. (1.9) is even on \mathbb{R} , let's take Freud-type weight defined on the positive x-axis; that is,

$$W_{\sigma}(x; t) := x^{2\sigma+1} \exp(-[cx^6 + t(x^4 - x^2)]), \quad x \in (0, \infty), \quad \sigma > 0, \quad t \in J \subset \mathbb{R}.$$

One can see that W_{σ} is a rapidly decreasing function [20].

Using Theorem 4, we can easily see that

$$\frac{\partial W_{\sigma}(x; t)}{\partial t} = (x^4 - x^2) x^{2\sigma+1} \exp(-[cx^6 + t(x^4 - x^2)]), \quad (3.3)$$

is continuous on \mathbb{R}^+ . For $t \leq 0$ and $x \in (1, \infty)$, we have $\exp(t(x^4 - x^2)) \leq 1$, since $ty^2 \leq 0$ for $y \in \mathbb{R}$. Thus,

$$\left| \frac{\partial W_\sigma(x; t)}{\partial t} \right| = \left| x^{2\sigma+1} (x^4 - x^2) \exp(-[cx^6 + t(x^4 - x^2)]) \right| \leq x^{2\sigma+k} \exp(-cx^6) := G(x),$$

for some bounding $k \in \mathbb{R}^+$ and $\sigma > 0$, with

$$\int_0^\infty G(x) dx = \int_0^\infty x^{2\sigma+k} \exp(-cx^6) dx = \frac{1}{6} \left(\frac{1}{c} \right)^{\frac{\sigma+4}{k}} \Gamma\left(\frac{2\sigma+8}{6}\right) < \infty,$$

where $\Gamma(z)$ denotes the Gamma function.

It then follows from Eq. (3.3) that

$$\left| \frac{\partial W_\sigma(x; t)}{\partial t} \right| = \left| x^{2\sigma+3} \exp(-[cx^6 + t(x^4 - x^2)]) \right| \leq x^{2\sigma+3} \exp(-cx^6 + Ax^2) := K(x),$$

for $t \in [0, A]$, $A \in \mathbb{R}^+$ and $K(x)$ is integrable for $x \in \mathbb{R}^+$. We see that all the conditions of Lemma 5 are fulfilled so that Eq. (3.2) can be proved using the principles of mathematical induction. For $n = 1$, we have

$$\begin{aligned} \frac{d}{dt} \eta_0(t, \sigma) &= \frac{d}{dt} \int_{-\infty}^\infty |x|^{2\sigma+1} \exp(-[cx^6 + t(x^4 - x^2)]) dx \\ &= (-1) \int_{-\infty}^\infty (x^4 - x^2) W_\sigma(x; t) dx = (-1)(\eta_4(t, \sigma) - \eta_2(t, \sigma)). \end{aligned}$$

We suppose, for inductive assumption, that

$$\frac{d^n}{dt^n} \eta_0(t, \sigma) = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \eta_{4n-2k}(t, \sigma) := \eta_{2n}(t, \sigma).$$

We need to show that

$$\eta_{2n+2}(t, \sigma) = \frac{d^{n+1}}{dt^{n+1}} \eta_0(t, \sigma).$$

We note that $x^{2n+2\sigma+1} \exp(-[cx^6 + t(x^4 - x^2)])$, $x \in \mathbb{R}^+$, $t \in J$, also obeys the conditions of Lemma 5. Then, by applying binomial expansion, we have

$$\begin{aligned} \frac{d^{n+1}}{dt^{n+1}} \eta_0(t, \sigma) &= \frac{d}{dt} \left(\frac{d^n}{dt^n} \eta_0(t, \sigma) \right) \\ &= \frac{d}{dt} \int_{\mathbb{R}} (-1)^n (x^4 - x^2)^n W_\sigma(x; t) dx = \int_{\mathbb{R}} (-1)^n (-1) (x^4 - x^2)^{n+1} W_\sigma(x; t) dx \\ &= \sum_{k=0}^n (-1)^{n+1} \binom{n+1}{k} \int_{-\infty}^\infty (x^4)^{n+1-k} (-x^2)^k W_\sigma(x; t) dx \\ &= \sum_{k=0}^n (-1)^{n+k+1} \binom{n+1}{k} \eta_{4n+4-2k}(t, \sigma) = \eta_{2n+2}(t, \sigma) \equiv \eta_0(t; n + \sigma + 1). \end{aligned}$$

Besides, moments of odd order vanish; i.e.,

$$\eta_{2n+1}(t, \sigma) = \int_{-\infty}^\infty x^{2n+1} W_\sigma(x; t) dx = 0, \quad n \in \mathbb{N}, \tag{3.2}$$

as the expression in the above integral is an odd function. \square

3.2. Concise formulation. The following result gives concise formulation for perturbed Freud-type polynomials $\mathcal{S}_n(x; t)$. For a similar result, [19, Lemma 3.2] .

Lemma 7. Suppose we have the perturbed Freud-type weight given in (1.9). Concise formulation of the corresponding polynomials, in terms of recurrence coefficient $\gamma_j(t; \sigma)$, is given by

$$\begin{cases} \mathcal{S}_q(x; t) = \sum_{k=0}^{\lfloor \frac{q}{2} \rfloor} \Psi_k(q) x^{q-2k}, \\ \Psi_0(q) = 1, \quad \text{for } k \in \{1, 2, \dots, \lfloor \frac{q}{2} \rfloor\}, q \in \mathbb{N}, \end{cases} \quad (3.1a)$$

where

$$\Psi_k(q) = (-1)^k \sum_{j_1=1}^{q+1-2k} \gamma_{j_1}(t; \sigma) \sum_{j_2=j_1+2}^{q+3-2k} \gamma_{j_2}(t; \sigma) \sum_{j_3=j_2+2}^{q+5-2k} \gamma_{j_3}(t; \sigma) \cdots \sum_{j_k=j_{k-1}+2}^{q-1} \gamma_{j_k}(t; \sigma). \quad (3.1b)$$

Proof. Since the perturbed Freud-type polynomials $\mathcal{S}_q(x; t)$ are symmetric and monic of degree q , and for a fixed $t \in \mathbb{R}$, we have $\mathcal{S}_q(-x) = (-1)^q \mathcal{S}_q(x)$, so that

$$\mathcal{S}_{2q}(x; t) = \sum_{j=0}^q g_{2q-2j} x^{2q-2j}; \quad \mathcal{S}_{2q+1}(x; t) = \sum_{j=0}^q g_{2q-2j+1} x^{2q-2j+1}, \quad (3.0)$$

where $g_{q-2k} = \Psi_k(q)$ with $\Psi_0(q) = 1$ and $\Psi_k(q) = 0$ for $k > \lfloor \frac{q}{2} \rfloor$. If we substitute Eq. (3.1a) into Eq. (2.2) and if we compare the coefficients of x , we obtain

$$\begin{cases} \Psi_k(q+1) - \Psi_k(q) = -\gamma_q(t; \sigma) \Psi_{k-1}(q-1), \\ \Psi_0(q) = 1. \end{cases} \quad (3.1)$$

Eq. (3.1b) can be proved by employing induction on k . For $k = 1$, we see that

$$\Psi_1(q) - \Psi_1(q-1) = -\gamma_{q-1}, \quad (3.2)$$

By employing a telescoping sum of terms in Eq. (3.2), we obtain

$$\Psi_1(q) = - \sum_{j_1=0}^{q-1} \gamma_{j_1}(t; \sigma), \quad \forall q \geq 1.$$

Let's assume that, for every $q \in \mathbb{N}$, Eq. (3.1b) holds true for values up to $k-1$, i.e.,

$$\Psi_{k-1}(n) = (-1)^{k-1} \sum_{j_1=1}^{q+3-2k} \gamma_{j_1}(t; \sigma) \sum_{j_2=j_1+2}^{q+5-2k} \gamma_{j_2}(t; \sigma) \sum_{j_3=j_2+2}^{q+7-2k} \gamma_{j_3}(t; \sigma) \cdots \sum_{j_{k-1}=j_{k-2}+2}^{q-1} \gamma_{j_{k-1}}(t; \sigma). \quad (3.3)$$

Eq. (3.1) can be repeatedly used to obtain

$$\begin{aligned} \Psi_k(q) &= \Psi_k(q-1) - \gamma_{q-1} \Psi_{k-1}(q-2), \\ &= \Psi_k(q-2) - \gamma_{q-2} \Psi_{k-1}(q-3) - \gamma_{q-1} \Psi_{k-1}(q-2), \\ &= \Psi_k(q-3) - \gamma_{q-3} \Psi_{k-1}(q-4) - \gamma_{q-2} \Psi_{k-1}(q-3) - \gamma_{q-1} \Psi_{k-1}(q-2), \\ &\quad \vdots \\ &= -\gamma_{2k-1} \Psi_{k-1}(2k-2) - \gamma_{2k} \Psi_{k-1}(2k-1) - \cdots - \gamma_{q-2} \Psi_{k-1}(q-3) - \gamma_{q-1} \Psi_{k-1}(q-2). \end{aligned} \quad (3.4)$$

Substituting Eq. (3.3) into Eq. (3.4) yields Eq. (3.1b) and hence the required result. \square

Lemma 5 is alternately given as follows.

Proposition 8. The following formulation also holds for monic perturbed Freud-type polynomials $\mathcal{S}_q(x; t)$:

$$\mathcal{S}_q(x; t) = x^q + \sum_{r=1}^{\lfloor \frac{q}{2} \rfloor} (-1)^r \left(\sum_{k \in W(q, r)} \gamma_{k_1} \gamma_{k_2} \cdots \gamma_{k_{r-1}} \gamma_{k_r} \right) x^{q-2r},$$

where $W(q, r) = \{k \in \mathbb{N}^r \mid k_{j+1} \geq k_j + 2 \text{ for } 1 \leq j \leq r-1, 1 \leq k_1, k_r < q\}$, and $\lfloor \frac{q}{2} \rfloor = \begin{cases} \frac{q}{2}, & q \text{ is even,} \\ \frac{q-1}{2}, & q \text{ is odd.} \end{cases}$

3.3. Normalization constant. The normalization constant $\hat{\Gamma}_m$ in Eq. (2.1) for the weight in Eq. (1.9) takes the form

$$\hat{\Gamma}_m = \langle \mathcal{S}_m, \mathcal{S}_m \rangle_{W_\sigma} = \|\mathcal{S}_m\|_{W_\sigma}^2 = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \Psi_k(m) \eta_{2m-2k}(t; \sigma), \quad (3.5)$$

where $\Psi_k(m)$ is given in Eq. (3.1b). Eq. (3.5) is equivalently given by

$$\hat{\Gamma}_m(t) = \int_{-\infty}^{\infty} \mathcal{S}_m^2(x, t) W_\sigma(x; t) dx.$$

By using variable transformation $x^2 = \xi$, we have different normalization parties as follows:

$$\begin{aligned} \hat{\Gamma}_{2m}(t) &= \int_{-\infty}^{\infty} \mathcal{S}_{2m}^2(x, t) W_\sigma(x; t) dx \\ &= 2 \int_0^{\infty} \mathcal{S}_{2m}^2(\sqrt{\xi}, t) |\xi|^{\sigma+\frac{1}{2}} \exp(-[c\xi^3 + t(\xi^2 - \xi)]) \frac{1}{2\sqrt{\xi}} d\xi \\ &= \int_0^{\infty} \tilde{P}_m^2(\xi, t) \xi^{-\frac{1}{2}} |\xi|^{\sigma+\frac{1}{2}} \exp(-[c\xi^3 + t(\xi^2 - \xi)]) d\xi =: \tilde{h}_m(t), \end{aligned}$$

and

$$\begin{aligned} \hat{\Gamma}_{2m+1}(t) &= \int_{-\infty}^{\infty} \mathcal{S}_{2m+1}^2(x, t) W_\sigma(x; t) dx \\ &= 2 \int_0^{\infty} \mathcal{S}_{2m+1}^2(\sqrt{\xi}, t) |\xi|^{\sigma+\frac{1}{2}} \exp(-[cs^3 + t(s^2 - s)]) \frac{1}{2\sqrt{\xi}} d\xi \\ &= \int_0^{\infty} \tilde{P}_m^2(\xi, t) \xi^{\frac{1}{2}} |\xi|^{\sigma+\frac{1}{2}} \exp(-[c\xi^3 + t(\xi^2 - \xi)]) d\xi =: \widehat{h}_m(t), \end{aligned}$$

We now see that

$$\begin{aligned} \mathcal{S}_{2m}(\sqrt{\xi}, t) &= (\sqrt{\xi})^{2m} + \chi(2m, t)(\sqrt{\xi})^{2m-2} + \dots + \mathcal{S}_{2m}(0, t) \\ &= \xi^n + \tilde{p}(m, t)\xi^{m-1} + \dots + \tilde{P}_m(0, t) := \tilde{P}_m(\xi, t), \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_{2m+1}(\sqrt{\xi}, t) &= (\sqrt{\xi})^{2m+1} + \chi(2m, t)(\sqrt{\xi})^{2m-1} + \dots + k \cdot \sqrt{\xi}, \quad k \in \mathbb{R}, \\ &= \sqrt{\xi}(\xi^m + \widehat{p}(m, t)\xi^{m-1} + \dots + k) := \sqrt{\xi}\widehat{P}_m(\xi, t). \end{aligned}$$

The above polynomials $\tilde{P}_m(\xi, t)$ and $\widehat{P}_m(\xi, t)$ are recognized as monic semiclassical Airy-type polynomials with corresponding orthogonality weights

$$w_1(x; t) = \xi^{-\frac{1}{2}} |\xi|^{\sigma+\frac{1}{2}} \exp(-[c\xi^3 + t(\xi^2 - \xi)]), \quad (3.6a)$$

$$w_2(x; t) = \xi^{\frac{1}{2}} |\xi|^{\sigma+\frac{1}{2}} \exp(-[c\xi^3 + t(\xi^2 - \xi)]), \quad (3.6b)$$

both defined over $(0, \infty)$ respectively. (See [7] for symmetrization process and quadratic transformation). The corresponding Hankel determinants for the weights in (3.6) can be given by

$$\widetilde{D}_m(t) := \det \left(\int_0^{\infty} \xi^{i+j-\frac{1}{2}} \xi^{\sigma+\frac{1}{2}} \exp(-[c\xi^3 + t(\xi^2 - \xi)]) d\xi \right)_{i,j=0}^{n-1} = \prod_{l=0}^{m-1} \tilde{h}_l(\xi),$$

$$\widehat{D}_m(t) := \det \left(\int_0^\infty \xi^{i+j+\frac{1}{2}} \xi^{\sigma+\frac{1}{2}} \exp(-[c\xi^3 + t(\xi^2 - \xi)]) \, d\xi \right)_{i,j=0}^{n-1} = \prod_{l=0}^{m-1} \widehat{h}_l(\xi)$$

respectively. Hence,

$$\Delta_n(t) = \prod_{j=0}^{n-1} \Gamma_j(t) = \begin{cases} \widetilde{D}_{k+1} \widehat{D}_k & n = 2k+1, \\ \widehat{D}_k \widehat{D}_k & n = 2k. \end{cases}$$

It is good to mention here that investigation of asymptotics of the Hankel determinants when n is large has been an interesting subject for many years; for instance, for Gaussian weight is studied in Chen et al. in [23]. See also the monograph by Szegő [29] as we will not address this as it goes beyond the scope of the paper.

3.4. Nonlinear recursion relation. In this section, we explore certain nonlinear recurrence relations associated with the semi-classical weight given in (1.9).

Theorem 9. *For the semiclassical weight in (2.1), the recurrence coefficient $\gamma_n(t; \sigma)$ fullfill the following difference relations*

$$6c [\gamma_n(\Xi_{n-1} + \Xi_n + \Xi_{n+1}) + \gamma_{n-1}\gamma_n\gamma_{n+1}] + 4t\Xi_n - 2t\gamma_n = n + (2\sigma + 1)\Omega_n, \quad (3.7)$$

with initial conditions given by

$$\begin{cases} \gamma_1(t; \sigma) = \frac{\|x^2\|_t^2}{\|1\|_t^2} = \frac{\eta_2(t; \sigma)}{\eta_0(t; \sigma)} = \frac{\int_{-\infty}^\infty x^2 W_\sigma(x; t) dx}{\int_{-\infty}^\infty W_\sigma(x; t) dx}, \\ \gamma_0 = 0, \end{cases} \quad (3.8)$$

where Ξ_n and Ω_n are, respectively, given by

$$\Xi_n = \gamma_n(t; \sigma) [\gamma_{n-1}(t; \sigma) + \gamma_n(t; \sigma) + \gamma_{n+1}(t; \sigma)], \quad (3.9)$$

and

$$\Omega_n = \frac{1 - (-1)^n}{2} = \begin{cases} 1, & \text{for } n \text{ is odd} \\ 0, & \text{for } n \text{ is even.} \end{cases} \quad (3.10)$$

Proof. (i) Applying similar procedure due to Freud as given in [30, Section 2] (see also [26]), let's consider the following integral

$$\mathbb{J}_n = \frac{1}{\hat{\Gamma}_n} \int_{-\infty}^\infty [\mathcal{S}_n(x; t) \mathcal{S}_{n-1}(x; t)]' W_\sigma(x; t) dx, \quad (3.11)$$

where $\hat{\Gamma}_n$ is given in (3.5). Eq. (3.11) is equivalently given by

$$\begin{aligned} \mathbb{J}_n &= \frac{1}{\hat{\Gamma}_n} [\langle \mathcal{S}'_n, \mathcal{S}_{n-1} \rangle_{W_\sigma} + \langle \mathcal{S}_n, \mathcal{S}'_{n-1} \rangle_{W_\sigma}] \\ &= \frac{1}{\hat{\Gamma}_n} \int_{-\infty}^\infty (nx^{n-1} + V_{n-2}) \mathcal{S}_{n-1}(x; t) W_\sigma(x; t) dx = \frac{\hat{\Gamma}_{n-1}}{\hat{\Gamma}_n} n, \end{aligned} \quad (3.12)$$

where $V_{n-2} \in \mathbb{P}_{n-2}$. We also see that by evaluating Eq. (3.11) using technique of integration, we arrive at

$$\begin{aligned} \mathbb{I}_n \hat{\Gamma}_n &= [\mathcal{S}_n(x; t) \mathcal{S}_{n-1}(x; t) W_\sigma(x; t)]_{-\infty}^\infty - \int_{-\infty}^\infty \mathcal{S}_n(x; t) \mathcal{S}_{n-1}(x; t) W'_\sigma(x; t) dx \\ &= -(2\sigma + 1) \int_{-\infty}^\infty \frac{\mathcal{S}_n(x; t) \mathcal{S}_{n-1}(x; t)}{x} W_\sigma(x; t) dx + 6c \int_{-\infty}^\infty x^5 \mathcal{S}_n(x; t) \mathcal{S}_{n-1}(x; t) W_\sigma(x; t) dx \\ &\quad + 4t \int_{-\infty}^\infty x^3 \mathcal{S}_n(x; t) \mathcal{S}_{n-1}(x; t) W_\sigma(x; t) dx - 2t \int_{-\infty}^\infty x \mathcal{S}_n(x; t) \mathcal{S}_{n-1}(x; t) W_\sigma(x; t) dx, \end{aligned} \quad (3.13)$$

in consideration of the fact that $\left[\mathcal{S}_n(x; t) \mathcal{S}_{n-1}(x; t) W_\sigma(x; t) \right]_{-\infty}^{\infty} = 0$ as the weight (2.1) vanishes at the boundary terms when $x \rightarrow \pm\infty$ due to symmetry property of the weight W_σ ; hence it follows that

$$\int_{-\infty}^{\infty} \mathcal{S}_n(x; t) \frac{1}{x} \mathcal{S}_{n-1}(x; t) W_\sigma(x; t) dx = 0, \quad (3.14a)$$

for n is even and, when n is odd, we have that

$$\int_{-\infty}^{\infty} \mathcal{S}_{n-1}(x; t) \frac{\mathcal{S}_n(x; t)}{x} W_\sigma(x; t) dx = \hat{\Gamma}_{n-1}, \quad (3.14b)$$

as $\frac{\mathcal{S}_n(x; t)}{x}$ is a polynomial of degree $n - 1$. Thus, we have

$$\int_{-\infty}^{\infty} \frac{\mathcal{S}_{n-1}(x; t) \mathcal{S}_n(x; t)}{x} W_\sigma(x; t) dx = \Omega_n \hat{\Gamma}_{n-1}, \quad (3.14c)$$

where Ω_n is given in (3.10). Let's us employ the following iterated recurrence relation from Eq. (2.2) to obtain

$$\begin{aligned} x^5 \mathcal{S}_n(x; t) &= \mathcal{S}_{n+5}(x; t) + (\gamma_n + \gamma_{n+1} + \gamma_{n+2} + \gamma_{n+3} + \gamma_{n+4}) \mathcal{S}_{n+3}(x; t) \\ &\quad + [\gamma_n (\Xi_{n-1} + \Xi_n + \Xi_{n+1}) + \gamma_{n-1} \gamma_n \gamma_{n+1}] \mathcal{S}_{n+1}(x; t) \\ &\quad + [\gamma_n \gamma_{n-2} \Xi_{n-1} + \gamma_{n-2} \gamma_{n-1} \gamma_n \gamma_{n+1} + \gamma_n \gamma_{n-1} \gamma_{n-2} \gamma_{n-3}] \mathcal{S}_{n-3}(x; t) \\ &\quad + (\gamma_n \gamma_{n-1} \gamma_{n-2} \gamma_{n-3} \gamma_{n-4}) \mathcal{S}_{n-5}(x; t), \end{aligned} \quad (3.15a)$$

$$\begin{aligned} x^4 \mathcal{S}_n(x; t) &= \mathcal{S}_{n+4}(x; t) + (\gamma_n + \gamma_{n+1} + \gamma_{n+2} + \gamma_{n+3}) \mathcal{S}_{n+2}(x; t) \\ &\quad + [\gamma_n (\gamma_{n-1} + \gamma_n + \gamma_{n+1}) + \gamma_{n+1} (\gamma_n + \gamma_{n+1} + \gamma_{n+2})] \mathcal{S}_n(x; t) \\ &\quad + \gamma_n \gamma_{n-1} (\gamma_{n-2} + \gamma_{n-1} + \gamma_n + \gamma_{n+1}) \mathcal{S}_{n-2}(x; t) + (\gamma_n \gamma_{n-1} \gamma_{n-2} \gamma_{n-3}) \mathcal{S}_{n-4}(x; t), \end{aligned} \quad (3.15b)$$

$$\begin{aligned} x^3 \mathcal{S}_n(x; t) &= (\gamma_n + \gamma_{n+1} + \gamma_{n+2}) \mathcal{S}_{n+1}(x; t) + \mathcal{S}_{n+3}(x; t) \\ &\quad + \gamma_n \gamma_{n-1} \gamma_{n-2} \mathcal{S}_{n-3}(x; t) + \gamma_n (\gamma_{n-1} + \gamma_n + \gamma_{n+1}) \mathcal{S}_{n-1}(x; t), \end{aligned} \quad (3.15c)$$

$$x^2 \mathcal{S}_n(x; t) = (\gamma_n + \gamma_{n+1}) \mathcal{S}_n(x; t) + \gamma_n \gamma_{n-1} \mathcal{S}_{n-2}(x; t) + \mathcal{S}_{n+2}(x; t). \quad (3.15d)$$

By using the identities (3.15) and Eq. (1.1) for the weight (1.9) together with Eqs. (3.14) into (3.13), we obtain

$$\begin{aligned} n \hat{\Gamma}_{n-1} &= \mathbb{I}_n \hat{\Gamma}_n = 6c [(\gamma_n + \gamma_{n-1}) \Xi_n + (\gamma_n \Xi_{n+1} + \gamma_n \gamma_{n-1} \gamma_{n-2})] \hat{\Gamma}_{n-1} \\ &\quad - 2t \gamma_n \hat{\Gamma}_{n-1} - (2\sigma + 1) \Omega_n \hat{\Gamma}_{n-1} + 4t [\gamma_n (\gamma_{n-1} + \gamma_n + \gamma_{n+1})] \hat{\Gamma}_{n-1}, \end{aligned} \quad (3.16)$$

which simplifies, using the fact that $\hat{\Gamma}_{n-1} \neq 0$, to

$$\begin{aligned} n + (2\sigma + 1) \Omega_n &= 6c [(\gamma_n + \gamma_{n-1}) \Xi_n + (\gamma_n \Xi_{n+1} + \gamma_n \gamma_{n-1} \gamma_{n-2})] \\ &\quad + 4t [\gamma_n (\gamma_{n-1} + \gamma_n + \gamma_{n+1})] - 2t \gamma_n, \end{aligned} \quad (3.17)$$

where Ω_n is given in (3.10). Note that Eq. (3.16) and Eq. (3.12) yield Eq. (3.7). \square

Remark 10. Quite similar non-linear discrete equations like Eq. (3.17) can be obtained in [13, Eq. (23), p. 5] and we also refer to [1, 9, 31].

The following result gives the differential-recurrence relation for the weight (1.9).

Theorem 11. For the semiclassical weight in (2.1), the coefficients $\gamma_n(t; \sigma)$ obey Toda-type formulation

$$\frac{d\gamma_n}{dt} = \gamma_n [(\gamma_{n+1} - \Xi_{n+1}) - (\gamma_{n-1} - \Xi_{n-1})], \quad (3.18)$$

where Ξ_n is given in Eq. (3.9).

Proof. In order to prove this result, we first differentiate the normalization constant $\hat{\Gamma}_n(t)$ with respect to t as

$$\begin{aligned} \frac{d\hat{\Gamma}_n}{dt} &= 2 \left\langle \frac{d\mathcal{S}_n}{dt}, \mathcal{S}_n \right\rangle_{W_\sigma} + \langle (x^2 - x^4) \mathcal{S}_n, \mathcal{S}_n \rangle_{W_\sigma}, \\ &= 2 \int_{-\infty}^{\infty} \frac{d\mathcal{S}_n(x; t)}{dt} \mathcal{S}_n(x; t) W_\sigma(x; t) dx + \int_{-\infty}^{\infty} x^2 \mathcal{S}_n^2(x; t) W_\sigma(x; t) dx \\ &\quad - \int_{-\infty}^{\infty} x^4 \mathcal{S}_n^2(x; t) W_\sigma(x; t) dx. \end{aligned} \quad (3.19)$$

We see from Eq. (3.19) that the first integral vanishes by orthogonality as $\frac{d\mathcal{S}_n}{dt} \in \mathcal{P}_{n-1}$. Using the recursive relation in Eq. (2.2) and orthogonality fact, we now have

$$\frac{d}{dt} \hat{\Gamma}_n = (\gamma_n + \gamma_{n+1}) \hat{\Gamma}_n - (\Xi_n + \Xi_{n+1}) \hat{\Gamma}_n = [(\gamma_n - \Xi_n) + (\gamma_{n+1} - \Xi_{n+1})] \hat{\Gamma}_n, \quad (3.20)$$

Besides, if we differentiate Eq. (2.3) with respect to t , we obtain

$$\frac{d}{dt} \gamma_n = \frac{d}{dt} \left(\frac{\hat{\Gamma}_n}{\hat{\Gamma}_{n-1}} \right) = \gamma_n \left[\frac{d}{dt} \ln \hat{\Gamma}_n - \frac{d}{dt} \ln \hat{\Gamma}_{n-1} \right] = \gamma_n [(\gamma_{n+1} - \gamma_{n-1}) - [\Xi_{n+1} - \Xi_{n-1}]], \quad (3.21)$$

and substituting Eq. (3.20) into (3.21) leads to the required result. \square

The following result presents nonlinear differential-recurrence relation of high order associated with the weight (2.1); we quote ideas of the proof from [22].

Theorem 12. The coefficients $\gamma_n(t; \sigma)$ for the weight in Eq. (1.9) fulfills the following nonlinear differential-recurrence equation

$$\left\{ \begin{aligned} \frac{d^2 \gamma_n}{dt^2} &= \frac{1}{6c} [n + (2\sigma + 1)\Omega_n - \vartheta(t)] + (-\gamma_{n-1} - \gamma_{n+1}) \gamma_n^4 \\ &\quad + \left(-\gamma_{n-2}\gamma_{n-1} - \gamma_{n-1}^2 - 6\gamma_{n-1}\gamma_{n+1} - \gamma_{n+1}^2 - \gamma_{n+1}\gamma_{n+2} + 2\gamma_{n-1} + 2\gamma_{n+1} \right) \gamma_n^3 \\ &\quad + \left(\gamma_{n-3}\gamma_{n-2}\gamma_{n-1} + \gamma_{n-2}^2\gamma_{n-1} + 2\gamma_{n-2}\gamma_{n-1}^2 - 4\gamma_{n-2}\gamma_{n-1}\gamma_{n+1} + \gamma_{n-1}^3 - 5\gamma_{n-1}^2\gamma_{n+1} - 4\gamma_{n-1}\gamma_{n+1}\gamma_{n+2} \right. \\ &\quad \left. - 5\gamma_{n-1}\gamma_{n+1}^2 + \gamma_{n+1}^3 + 2\gamma_{n+1}^2\gamma_{n+2} + \gamma_{n+1}\gamma_{n+2}^2 + \gamma_{n+1}\gamma_{n+2}\gamma_{n+3} + 8\gamma_{n-1}\gamma_{n+1} - \gamma_{n-1} - \gamma_{n+1} \right) \gamma_n^2 \\ &\quad + \left(\gamma_{n-4}\gamma_{n-3}\gamma_{n-2}\gamma_{n-1} + \gamma_{n-3}^2\gamma_{n-2}\gamma_{n-1} + 2\gamma_{n-3}\gamma_{n-2}^2\gamma_{n-1} + 2\gamma_{n-3}\gamma_{n-2}\gamma_{n-1}^2 + \gamma_{n-2}^3\gamma_{n-1} \right. \\ &\quad \left. + 3\gamma_{n-2}^2\gamma_{n-1}^2 + 3\gamma_{n-2}\gamma_{n-1}^3 - 2\gamma_{n-2}\gamma_{n-1}\gamma_{n+1}\gamma_{n+2} + \gamma_{n-1}^4 - 2\gamma_{n-1}^2\gamma_{n+1}^2 \right. \\ &\quad \left. - 2\gamma_{n-1}^2\gamma_{n+1}\gamma_{n+2} + \gamma_{n+1}^4 + 3\gamma_{n+1}^3\gamma_{n+2} + 3\gamma_{n+1}^2\gamma_{n+2}^2 + 2\gamma_{n+1}^2\gamma_{n+2}\gamma_{n+3} + \gamma_{n+1}\gamma_{n+2}^3 \right. \\ &\quad \left. + 2\gamma_{n+1}\gamma_{n+2}^2\gamma_{n+3} + \gamma_{n+1}\gamma_{n+2}\gamma_{n+3}^2 - 2\gamma_{n-2}^2\gamma_{n-1} + \gamma_{n+1}\gamma_{n+2}\gamma_{n+3}\gamma_{n+4} - 2\gamma_{n-3}\gamma_{n-2}\gamma_{n-1} \right. \\ &\quad \left. - 4\gamma_{n-2}\gamma_{n-1}^2 + 2\gamma_{n-2}\gamma_{n-1}\gamma_{n+1} - 2\gamma_{n-1}^3 + 2\gamma_{n-1}^2\gamma_{n+1} + 2\gamma_{n-1}\gamma_{n+1}\gamma_{n+2} - 2\gamma_{n+1}^3 \right. \\ &\quad \left. - 4\gamma_{n+1}^2\gamma_{n+2} - \gamma_n^2 - 2\gamma_{n+1}\gamma_{n+2}^2 - 2\gamma_{n+1}\gamma_{n+2}\gamma_{n+3} - 2\gamma_{n-1}\gamma_{n+1} - 2\gamma_n\gamma_{n-1} - 2\gamma_n\gamma_{n+1} - \gamma_{n+1}\gamma_{n-1} \right) \gamma_n, \\ \vartheta(t) &= 4t\Xi_n - 2\gamma_n t = 2t\gamma_n [2(\gamma_{n-1} + \gamma_n + \gamma_{n+1}) - 1]. \end{aligned} \right.$$

where Ω_n and Ξ_n are given in (3.10) and (3.9) respectively.

Proof. For the proof, we refer similar ideas in [22]. \square

3.5. Differential-Recurrence relation. Chen and Feigin [6] obtained ladder operators for a semiclassical weight $\tilde{w}(x)|x-t|^\theta$, where $x, \theta, t \in \mathbb{R}$ and $\tilde{w}(x)$ is classical weight function. In Filipuk et al. [12], it is shown that the recurrence coefficients for the quartic Freud weight $|x|^{2\alpha+1}e^{-x^4+tx^2}$, $x, t \in \mathbb{R}$, $\alpha > -1$ are related to the solutions of the Painlevé IV and the first discrete Painlevé equation. Clarkson et al. [9] provided a systematic study on Freud weights and some generalized work for [6].

Lemma 13. [22] *The monic orthogonal polynomials $P_n(x; t)$ with respect to the semiclassical Freud-type weight (1.9)*

$$w_\alpha(x) = |x|^\alpha w_0(x),$$

where

$$w_0(x) := e^{-v_0(x)} \text{ with } v_0(x) := cx^6 + t(x^4 - x^2).$$

on \mathbb{R} satisfy the differential-difference recurrence relation

$$P'_n(x) = \gamma_n(t)\mathcal{A}_n(x)P_{n-1}(x) - \mathcal{B}_n(x)P_n(x),$$

where

$$\mathcal{A}_n(x) := \frac{1}{\Gamma_n} \int_{-\infty}^{\infty} \frac{v'_0(x) - v'_0(\tau)}{x - \tau} P_n^2(\tau) w(\tau) d\tau, \quad (3.22a)$$

$$\mathcal{B}_n(x) := \frac{1}{\Gamma_{n-1}} \int_{-\infty}^{\infty} \frac{v'_0(x) - v'_0(\tau)}{x - \tau} P_n(\tau) P_{n-1}(\tau) w(\tau) d\tau + \frac{\alpha[1 - (-1)^n]}{2x}. \quad (3.22b)$$

Proof. For the proof, we refer to [22]. See also similar works in [5]. \square

Lemma 14. $\mathcal{A}_n(z)$ and $\mathcal{B}_n(z)$ defined by Lemma 13 satisfy the relation:

$$\mathcal{A}_n(z) = \frac{v'_0(z)}{z} + \frac{\mathcal{B}_n(z) + \mathcal{B}_{n+1}(z)}{z} - \frac{\alpha}{z^2}. \quad (3.23)$$

Proof. Be the definition of $\mathcal{A}_n(z)$, we rewrite it as

$$\begin{aligned} \mathcal{A}_n(z) &= \frac{1}{z\Gamma_n} \left\{ \int_{-\infty}^{\infty} \frac{v'_0(z) - v'_0(\tau)}{z - \tau} y P_n^2(\tau) w(\tau) d\tau + \int_{-\infty}^{\infty} [v'_0(z) - v'_0(\tau)] P_n^2(\tau) w(\tau) d\tau \right\} \\ &= \frac{1}{z\Gamma_n} \left\{ \int_{-\infty}^{\infty} \frac{v'_0(z) - v'_0(\tau)}{z - \tau} [P_{n+1}(\tau) + \gamma_n P_{n-1}(\tau)] P_n(\tau) w(\tau) d\tau + v'_0(z) \Gamma_n \right. \\ &\quad \left. - \int_{-\infty}^{\infty} P_n^2(\tau) \left[\frac{\alpha}{\tau} w(\tau) - w'(\tau) \right] d\tau \right\} \\ &= \frac{1}{z} \left\{ \mathcal{B}_{n+1}(z) - \frac{\alpha}{2z} [1 - (-1)^{n+1}] + \mathcal{B}_n(z) - \frac{\alpha}{2z} [1 - (-1)^n] \right\} + \frac{v'_0(z)}{z}, \\ &= \frac{\mathcal{B}_n(z) + \mathcal{B}_{n+1}(z)}{z} - \frac{\alpha}{z^2} + \frac{v'_0(z)}{z}, \end{aligned}$$

which completes the proof. \square

Lemma 15. [18, Chapter 3]. *The functions $\mathcal{A}_n(z)$, $\mathcal{B}_n(z)$ and $\sum_{k=0}^{n-1} \mathcal{A}_k(z)$ satisfy the identity*

$$\mathcal{B}_n^2(z) + v'(z)\mathcal{B}_n(z) + \sum_{k=0}^{n-1} \mathcal{A}_k(z) = \gamma_n \mathcal{A}_n(z) \mathcal{A}_{n-1}(z). \quad (3.24)$$

We, next, apply the ladder coefficients to the case of perturbed Freud weight as follows.

3.5.1. *Ladder operator relations for the weight (1.9).* For the perturbed Freud-type weight (1.9),

$$v(x) = -\ln W_\sigma(x; t) = -(2\sigma + 1) \ln |x| + cx^6 + t(x^4 - x^2), \quad x \in \mathbb{R}, \quad (3.25)$$

we have

$$v'(x) = -\frac{(2\sigma + 1)}{x} + 6cx^5 + t(4x^3 - 2x),$$

and hence

$$\frac{v'(x) - v'(\tau)}{x - \tau} = \frac{2\sigma + 1}{x\tau} + 6c\{x^4 + x^3\tau + x^2\tau^2 + x\tau^3 + \tau^4\} + 4t(x^2 + x\tau + \tau^2) - 2t.$$

Theorem 16. *The monic orthogonal polynomials $\mathcal{S}_n(x; t)$ with respect to the weight in (1.9) defined on \mathbb{R} obey the relation*

$$\mathcal{S}'_n(x; t) = \gamma_n(t)\mathcal{A}_n(x; t)\mathcal{S}_{n-1}(x; t) - \mathcal{B}_n(x; t)\mathcal{S}_n(x; t)$$

where

$$\mathcal{A}_n(x; t) = 6cx^4 + 6c(\gamma_n + \gamma_{n+1})x^2 + 6c(\Xi_{n+1} + \Xi_n) + 4tx^2 + 4t(\gamma_n + \gamma_{n+1}) - 2t, \quad (3.26a)$$

$$\mathcal{B}_n(x; t) = \left(\frac{2\sigma + 1}{x}\right)\Omega_n + 6c\gamma_n x^3 + 6c\Xi_n x + 4tx\gamma_n, \quad (3.26b)$$

where the expressions Ξ_n and Ω_n are given in (3.9) and (3.10) respectively.

Proof. From (3.22a), we obtain

$$\begin{aligned} \mathcal{A}_n(x; t) &= \frac{1}{\hat{\Gamma}_n} \int_{\mathbb{R}} \mathcal{S}_n^2(\tau) \left(\frac{v'(x) - v'(\tau)}{x - \tau} \right) W_\sigma(\tau; t) d\tau \\ &= \frac{1}{\hat{\Gamma}_n} \int_{\mathbb{R}} \mathcal{S}_n^2(\tau) \left(\frac{2\sigma + 1}{x\tau} + 6c\{x^4 + x^3\tau + x^2\tau^2 + x\tau^3 + \tau^4\} + 4t(x^2 + x\tau + \tau^2) - 2t \right) W_\sigma(\tau; t) d\tau \\ &= 6cx^4 + 6c(\gamma_n + \gamma_{n+1})x^2 + 6c(\Xi_{n+1} + \Xi_n) + 4tx^2 + 4t(\gamma_n + \gamma_{n+1}) - 2t, \end{aligned} \quad (3.27)$$

and the integral in (3.27) vanishes due to symmetry of W_σ .

Besides, by using Eq. (3.22b), orthogonality and Eq. (2.2), we have that

$$\begin{aligned} \mathcal{B}_n(x; t) &= \frac{1}{\hat{\Gamma}_{n-1}} \int_{\mathbb{R}} \mathcal{S}_n(\tau) \mathcal{S}_{n-1}(\tau) \left(\frac{2\sigma + 1}{xy} + 6c\{x^4 + x^3y + x^2y^2 + xy^3 + y^4\} \right. \\ &\quad \left. + 4t(x^2 + xy + y^2) - 2t \right) W_\sigma(y; t) dy \\ &= 6c\gamma_n x^3 + 6c\Xi_n x + 4tx\gamma_n + \left(\frac{2\sigma + 1}{x}\right)\Omega_n, \end{aligned} \quad (3.28)$$

where Ξ_n and Ω_n are given respectively in (3.9) and (3.10). \square

Remark 17. *It is good to mention that there is a similar result in [10] for differential-recurrence relation for sextic Freud-type weight; whereas our considered weight in Eq. (1.9) can be perceived as generalized measure deformation using $d\mu(x; t) = e^{t(x^4 - x^2)}d\mu(x; 0)$. For a similar procedure, one can see [16] where the authors used classical measure deformation via $d\mu(x; t) = e^{tx^2}d\mu(x; 0)$ for Laguerre-type weight.*

3.6. Shohat's quasi-orthogonality method. Shohat [28] studied a strategy using quasi-orthogonality, to find differential-difference relation for a general semiclassical weight function. Bonan, Freud, Mhaskar and Nevai are renowned experts who used this method in their work [26]. The idea of quasi-orthogonality is well articulated in [24, 11, 28]). Our goal in this section is to apply this method to the case of perturbed Freud-type weight in (1.9) [9, Section 4.5]. Following the ideas in [26], we notice that monic perturbed Freud-type polynomials obey quasi-orthogonality of order $m = 7$ and therefore

$$x \frac{d\mathcal{S}_n(x; \tau)}{dx} = \sum_{k=n-6}^n u_{n,k} \mathcal{S}_k(x; \tau), \quad (3.29)$$

where the expression $u_{n,k}$ is obtained by

$$u_{n,k} = \frac{1}{\Gamma_k} \int_{-\infty}^{\infty} x \frac{d\mathcal{S}_n}{dx}(x; \tau) \mathcal{S}_k(x; t) W_{\sigma}(x; \tau) dx, \quad (3.30)$$

with $n - 6 \leq k \leq n$ and $\Gamma_k \neq 0$. By employing integration techniques, for $n - 6 \leq j \leq n - 1$, we have

$$\begin{aligned} \Gamma_k u_{n,k} &= \left[x \mathcal{S}_k(x; t) \mathcal{S}_n(x; t) W_{\sigma}(x; t) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dx} (x \mathcal{S}_k(x; t) W_{\sigma}(x; t)) \mathcal{S}_n(x; t) dx \\ &= - \int_{-\infty}^{\infty} \left[\mathcal{S}_n(x; t) \mathcal{S}_k(x; t) + x \mathcal{S}_n(x; t) \frac{\mathcal{S}_k}{x}(x; t) \right] W_{\sigma}(x; t) dx \\ &\quad - \int_{-\infty}^{\infty} x \mathcal{S}_n(x; t) \mathcal{S}_j(x; t) \frac{dW_{\sigma}(x, t)}{dx}(x; t) dx, \end{aligned} \quad (3.31)$$

$$\begin{aligned} &= - \int_{-\infty}^{\infty} \mathcal{S}_n(x; t) \mathcal{S}_j(x; t) (-6cx^6 - 4tx^4 + 2tx^2 + 2\sigma + 1) W_{\sigma}(x; t) dx \\ &= \int_{-\infty}^{\infty} (6cx^6 + 4tx^4 - 2tx^2 - (2\sigma + 1)) \mathcal{S}_n(x; t) \mathcal{S}_j(x; t) W_{\sigma}(x; t) dx, \end{aligned} \quad (3.32)$$

since

$$x \frac{dW_{\sigma}(x, t)}{dx} = [-6cx^6 - 4tx^4 + 2tx^2 + 2\sigma + 1] W_{\sigma}(x; t).$$

The following relations follow from iterating the recurrence given in Eq. (2.2):

$$\begin{aligned} x^6 \mathcal{S}_n(x; t) &= \mathcal{S}_{n+6}(x; t) + (\gamma_n + \gamma_{n+1} + \gamma_{n+2} + \gamma_{n+3} + \gamma_{n+4} + \gamma_{n+5}) \mathcal{S}_{n+4}(x; t) \\ &\quad + \left[\gamma_{n+3} (\gamma_n + \gamma_{n+1} + \gamma_{n+2} + \gamma_{n+3} + \gamma_{n+4}) + \gamma_{n+2} (\gamma_n + \gamma_{n+1} + \gamma_{n+2} + \gamma_{n+3}) + \Xi_n + \Xi_{n+1} \right] \mathcal{S}_{n+2}(x; t) \\ &\quad + \left[\gamma_{n+1} \gamma_{n+2} [\gamma_n + \gamma_{n+1} + \gamma_{n+2} + \gamma_{n+3}] + [(\gamma_n + \gamma_{n+1})(\Xi_n + \Xi_{n+1})] \right. \\ &\quad \left. + \gamma_n \gamma_{n-1} (\gamma_{n-2} + \gamma_{n-1} + \gamma_n + \gamma_{n+1}) \right] \mathcal{S}_n(x; t) \\ &\quad + \gamma_n \gamma_{n-1} \left[\Xi_{n-1} + \Xi_n + \Xi_{n+1} + \gamma_{n-1} \gamma_{n+1} + \gamma_{n-2} (\gamma_{n-3} + \gamma_{n-2} + \gamma_{n-1} + \gamma_n + \gamma_{n+1}) \right] \mathcal{S}_{n-2}(x; t) \\ &\quad + \gamma_n \gamma_{n-1} \gamma_{n-2} \gamma_{n-3} \left[\gamma_{n-4} + \gamma_{n-3} + \gamma_{n-2} + \gamma_{n-1} + \gamma_n + \gamma_{n+1} \right] \mathcal{S}_{n-4}(x; t) \\ &\quad + (\gamma_n \gamma_{n-1} \gamma_{n-2} \gamma_{n-3} \gamma_{n-4} \gamma_{n-5}) \mathcal{S}_{n-6}(x; t), \end{aligned} \quad (3.33a)$$

$$\begin{aligned} x^4 \mathcal{S}_n(x; t) &= \mathcal{S}_{n+4}(x; t) + (\gamma_n + \gamma_{n+1} + \gamma_{n+2} + \gamma_{n+3}) \mathcal{S}_{n+2}(x; t) \\ &\quad + [\gamma_n (\gamma_{n-1} + \gamma_n + \gamma_{n+1}) + \gamma_{n+1} (\gamma_n + \gamma_{n+1} + \gamma_{n+2})] \mathcal{S}_n(x; t) \\ &\quad + \gamma_n \gamma_{n-1} (\gamma_{n-2} + \gamma_{n-1} + \gamma_n + \gamma_{n+1}) \mathcal{S}_{n-2}(x; t) + (\gamma_n \gamma_{n-1} \gamma_{n-2} \gamma_{n-3}) \mathcal{S}_{n-4}(x; t), \end{aligned} \quad (3.33b)$$

$$x^2 \mathcal{S}_n(x; t) = \mathcal{S}_{n+2}(x; t) + (\gamma_n + \gamma_{n+1}) \mathcal{S}_n(x; t) + \gamma_n \gamma_{n-1} \mathcal{S}_{n-2}(x; t), \quad (3.33c)$$

By substituting Eq. (3.33) into Eq. (3.32), we obtain the coefficients $\{\mathfrak{f}_{n,j}\}_{j=n-4}^{n-1}$ in Eq. (3.29) as:

$$\mathfrak{u}_{n,n-6} = 6c \left(\prod_{j=0}^5 \gamma_{n-j} \right) = 6c [\gamma_n \gamma_{n-1} \gamma_{n-2} \gamma_{n-3} \gamma_{n-4} \gamma_{n-5}], \quad \mathfrak{u}_{n,n-5} = 0, \quad (3.34a)$$

$$\mathfrak{u}_{n,n-4} = 6c \left(\prod_{j=0}^3 \gamma_{n-j} \right) [\gamma_{n-4} + \gamma_{n-3} + \gamma_{n-2} + \gamma_{n-1} + \gamma_n + \gamma_{n+1}], \quad \mathfrak{u}_{n,n-3} = 0, \quad (3.34b)$$

$$\begin{aligned} \mathfrak{u}_{n,n-2} &= \gamma_n \gamma_{n-1} \left[6c \{ \Xi_{n-2} + \Xi_{n-1} + \Xi_n + \Xi_{n+1} + \gamma_{n-1} \gamma_{n-2} + \gamma_{n+1} (\gamma_{n-2} + \gamma_{n-1}) \} \right. \\ &\quad \left. + 4t (\gamma_{n-2} + \gamma_{n-1} + \gamma_n + \gamma_{n+1}) - 2t \right], \end{aligned} \quad (3.34c)$$

$$\mathfrak{u}_{n,n-1} = 0. \quad (3.34d)$$

For the case when $k = n$, we use integration technique in Eq. (3.30) to obtain

$$\begin{aligned} \Gamma_n \mathfrak{f}_{n,n} &= \int_{-\infty}^{\infty} x \frac{d\mathcal{S}_n(x; t)}{dx} \mathcal{S}_n(x; t) W_{\sigma}(x; t) dx = -\frac{1}{2} \int_{-\infty}^{\infty} \mathcal{S}_n^2(x; t) \left[W_{\sigma}(x; t) + x \frac{dW_{\sigma}(x; t)}{dx} \right] dx \\ &= -\frac{1}{2} \Gamma_n + \int_{-\infty}^{\infty} \mathcal{S}_n^2(x; t) (3cx^6 - 2tx^4 + tx^2 - \sigma - \frac{1}{2}) W_{\sigma}(x; t) dx \\ &= 3c \int_{-\infty}^{\infty} x^6 \mathcal{S}_n^2(x; t) W_{\sigma}(x; t) dx - 2t \int_{-\infty}^{\infty} x^4 \mathcal{S}_n^2(x; t) W_{\sigma}(x; t) dx \\ &\quad + t \int_{-\infty}^{\infty} x^2 \mathcal{S}_n^2(x; t) W_{\sigma}(x; t) dx - (\sigma + 1) \Gamma_n. \end{aligned} \quad (3.35)$$

By using the recursive relation given in Eq. (2.2) for Eq. (3.35), we have that

$$x^2 \mathcal{S}_n^2 = (\mathcal{S}_{n+1} + \gamma_n \mathcal{S}_{n-1})^2 = \mathcal{S}_{n+1}^2 + 2\gamma_n \mathcal{S}_{n+1} \mathcal{S}_{n-1} + \gamma_n^2 \mathcal{S}_{n-1}^2, \quad (3.36a)$$

$$\begin{aligned} x^4 \mathcal{S}_n^2 &= x^2 (\mathcal{S}_{n+1}^2 + 2\gamma_n \mathcal{S}_{n+1} \mathcal{S}_{n-1} + \gamma_n^2 \mathcal{S}_{n-1}^2) = x^2 \mathcal{S}_{n+1}^2 + 2\gamma_n (x \mathcal{S}_{n+1})(x \mathcal{S}_{n-1}) + \gamma_n^2 x^2 \mathcal{S}_{n-1}^2 \\ &= (\mathcal{S}_{n+2} + \gamma_{n+1} \mathcal{S}_n)^2 + 2\gamma_n (\mathcal{S}_{n+2} + \gamma_{n+1} \mathcal{S}_n)(\mathcal{S}_n + \gamma_{n-1} \mathcal{S}_{n-2}) + \gamma_n^2 (\mathcal{S}_n + \gamma_{n-1} \mathcal{S}_{n-2})^2 \\ &= \mathcal{S}_{n+2}^2 + 2(\gamma_{n+1} + \gamma_n) \mathcal{S}_{n+2} \mathcal{S}_n + (\gamma_{n+1} + \gamma_n)^2 \mathcal{S}_n^2 + 2\gamma_n \gamma_{n-1} \mathcal{S}_{n+2} \mathcal{S}_{n-2} \\ &\quad + 2\gamma_n \gamma_{n-1} (\gamma_n + \gamma_{n+1}) \mathcal{S}_n \mathcal{S}_{n-2} + \gamma_n^2 \gamma_{n-1}^2 \mathcal{S}_{n-2}^2, \end{aligned} \quad (3.36b)$$

and so by orthogonality, we have that

$$\begin{aligned} \int_{-\infty}^{\infty} x^6 \mathcal{S}_n^2(x; t) W_{\sigma}(x; t) dx &= (\Gamma_{n+3} + \gamma_{n+2}^2 \Gamma_{n+1}) + 2(\gamma_{n+1} + \gamma_n) \gamma_{n+1} \gamma_{n+2} \Gamma_n + (\gamma_{n+1} + \gamma_n)^2 (\Gamma_{n+1} + \gamma_n^2 \Gamma_{n-1}) \\ &= (\gamma_n \gamma_{n+1} \gamma_{n+2}) \Gamma_n + \Xi_{n+2} \gamma_{n+1} \Gamma_n + \gamma_{n+1} (\Xi_n + \Xi_{n+1}) \Gamma_n + \gamma_n (\Xi_n + \Xi_{n+1}) \Gamma_n \\ &\quad + \gamma_{n-1} \gamma_n \gamma_{n+1} \Gamma_n + \gamma_n \Xi_{n-1} \Gamma_n, \end{aligned} \quad (3.37a)$$

$$\int_{-\infty}^{\infty} x^2 \mathcal{S}_n^2(x; t) W_{\sigma}(x; t) dx = \Gamma_{n+1} + \gamma_n^2 \Gamma_{n-1} = (\gamma_{n+1} + \gamma_n) \Gamma_n, \quad (3.37b)$$

$$\begin{aligned} \int_{-\infty}^{\infty} x^4 \mathcal{S}_n^2(x; t) W_{\sigma}(x; t) dx &= \Gamma_{n+2} + (\gamma_{n+1} + \gamma_n)^2 \Gamma_n + \gamma_n^2 \gamma_{n-1}^2 \Gamma_{n-2} \\ &= [(\gamma_{n+1} + \gamma_n + \gamma_{n-1}) \gamma_n + (\gamma_{n+2} + \gamma_{n+1} + \gamma_n) \gamma_{n+1}] \Gamma_n \\ &= (\Xi_n + \Xi_{n+1}) \Gamma_n, \end{aligned} \quad (3.37c)$$

using $\Gamma_{n+1} = \gamma_{n+1} \Gamma_n$, the difference equation Eq. (3.7) and Ξ_n is given by Eq. (3.9).

By rearranging Eq. (3.7) and taking $n \rightarrow n - 1$ in Eq. (3.7), we have

$$2t \Xi_n - t \gamma_n = \frac{n + (2\sigma + 1)\Omega_n}{2} - 3c [\gamma_n (\Xi_{n-1} + \Xi_n + \Xi_{n+1}) + \gamma_{n-1} \gamma_n \gamma_{n+1}], \quad (3.38a)$$

$$2t\Xi_{n+1} - t\gamma_{n+1} = \frac{n+1+(2\sigma+1)\Omega_{n+1}}{2} - 3c[\gamma_{n+1}(\Xi_n + \Xi_{n+1} + \Xi_{n+2}) + \gamma_n\gamma_{n+1}\gamma_{n+2}]. \quad (3.38b)$$

By combining Eqs. (3.38a) and Eq. (3.38b), we obtain

$$\begin{aligned} & -2t \int_{-\infty}^{\infty} x^4 S_n^2(x; t) W_{\sigma}(x; t) dx + t \int_{-\infty}^{\infty} x^2 S_n^2(x; t) W_{\sigma}(x; t) dx \\ &= -(t\gamma_n - 2t\Xi_n) - (t\gamma_{n+1} - 2t\Xi_{n+1}) \\ &= -3c \left[\gamma_n(\Xi_{n-1} + \Xi_n + \Xi_{n+1}) + \gamma_{n-1}\gamma_n\gamma_{n+1} + \gamma_{n+1}(\Xi_n + \Xi_{n+1} + \Xi_{n+2}) + \gamma_n\gamma_{n+1}\gamma_{n+2} \right] \\ &\quad + \frac{2n+1+(2\sigma+1)(\Omega_n+\Omega_{n+1})}{2} \\ &= -3c \left[\gamma_n(\Xi_{n-1} + \Xi_n + \Xi_{n+1}) + \gamma_{n-1}\gamma_n\gamma_{n+1} + \gamma_{n+1}(\Xi_n + \Xi_{n+1} + \Xi_{n+2}) + \gamma_n\gamma_{n+1}\gamma_{n+2} \right] \\ &\quad + n + (\sigma+1), \end{aligned} \quad (3.39)$$

Hence from Eq. (3.37a) and Eq. (3.39), Eq. (3.35) becomes

$$\begin{aligned} u_{n,n} &= \frac{1}{\Gamma_n} \left\{ 3c \int_{-\infty}^{\infty} x^6 S_n^2(x; t) W_{\sigma}(x; t) dx - (\sigma+1)\Gamma_n - 2t \int_{-\infty}^{\infty} x^4 S_n^2(x; t) W_{\sigma}(x; t) dx \right. \\ &\quad \left. + t \int_{-\infty}^{\infty} x^2 S_n^2(x; t) W_{\sigma}(x; t) dx \right\} \\ &= 3c \left[(\gamma_n\gamma_{n+1}\gamma_{n+2}) + \Xi_{n+2}\gamma_{n+1} + \gamma_{n+1}(\Xi_n + \Xi_{n+1})\Gamma_n + \gamma_n(\Xi_n + \Xi_{n+1}) + \gamma_{n-1}\gamma_n\gamma_{n+1}\Gamma_n + \gamma_n\Xi_{n-1} \right] \\ &\quad - (\sigma+1) - 3c \left[\gamma_n(\Xi_{n-1} + \Xi_n + \Xi_{n+1}) + \gamma_{n-1}\gamma_n\gamma_{n+1} + \gamma_{n+1}(\Xi_n + \Xi_{n+1} + \Xi_{n+2}) + \gamma_n\gamma_{n+1}\gamma_{n+2} \right] \\ &\quad + n + (\sigma+1) \\ &= n. \end{aligned} \quad (3.40)$$

Combining Eq. (3.34) with Eq. (3.29) gives

$$x \frac{dS_n}{dx} = u_{n,n-6} S_{n-6}(x; t) + u_{n,n-4} S_{n-4}(x; t) + u_{n,n-2} S_{n-2}(x; t) + u_{n,n} S_n(x; t). \quad (3.41a)$$

Rewriting S_{n-4} and S_{n-2} into Eq. (3.41a) in terms of S_n and S_{n-1} using Eq. (2.2), we obtain

$$S_{n-2}(x; t) = \frac{xS_{n-1}(x; t) - S_n(x; t)}{\gamma_{n-1}}, \quad (3.41b)$$

$$S_{n-3}(x; t) = \frac{xS_{n-2}(x; t) - S_{n-1}(x; t)}{\gamma_{n-2}} = \frac{x^2 - \gamma_{n-1}}{\gamma_{n-1}\gamma_{n-2}} S_{n-1}(x; t) - \frac{x}{\gamma_{n-1}\gamma_{n-2}} S_n(x; t), \quad (3.41c)$$

$$S_{n-4}(x; t) = \frac{xS_{n-3}(x; t) - S_{n-2}(x; t)}{\gamma_{n-3}} = \frac{x^3 - (\gamma_{n-1} + \gamma_{n-2})x}{\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}} S_{n-1}(x; t) - \frac{x^2 - \gamma_{n-2}}{\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}} S_n(x; t), \quad (3.41d)$$

$$\begin{aligned} S_{n-6}(x; t) &= \left\{ \frac{x^5 - (\gamma_{n-1} + \gamma_{n-2} + \gamma_{n-3} + \gamma_{n-4})x + (\gamma_{n-1}\gamma_{n-3} + \gamma_{n-1}\gamma_{n-4} + \gamma_{n-2}\gamma_{n-4})}{\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}\gamma_{n-4}\gamma_{n-5}} \right\} S_{n-1}(x; t) \\ &\quad - \left\{ \frac{x^4 - (\gamma_{n-2} + \gamma_{n-3} + \gamma_{n-4})x + \gamma_{n-2}\gamma_{n-4}}{\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}\gamma_{n-4}\gamma_{n-5}} \right\} S_n(x; t). \end{aligned} \quad (3.41e)$$

Substituting Eq. (3.34), Eq. (3.40), Eq. (3.41b), Eq. (3.41d) and Eq. (3.41e) into Eq. (3.41a) yields the required result.

4. THE DIFFERENTIAL EQUATION

Theorem 18. *For the semiclassical weight in (1.9), the corresponding monic orthogonal polynomials $S_n(x; t)$ obey a linear ODE (with rational coefficients) as*

$$\frac{d^2}{dx^2} S_n(x; t) + \tilde{U}_n(x; t) \frac{d}{dx} S_n(x; t) + \tilde{W}_n(x; t) S_n(x; t) = 0, \quad (4.1)$$

where

$$\begin{aligned}\tilde{U}_n(x; t) &= -6cx^5 - t(4x^3 - 2x) + \frac{(2\sigma + 1)}{x} \\ &\quad - \left[\frac{24cx^3 + 2[6c(\gamma_n + \gamma_{n+1}) + 4t]x}{6cx^4 + 6c(\gamma_n + \gamma_{n+1})x^2 + 6c(\Xi_{n+1} + \Xi_n) - 2t + 4t(x^2 + \gamma_n + \gamma_{n+1})} \right]\end{aligned}\quad (4.2a)$$

$$\begin{aligned}\tilde{W}_n(x; t) &= 18c\gamma_n x^2 + 6c\Xi_n - \frac{(2\sigma + 1)\Omega_n}{x^2} + 4t\gamma_n \\ &\quad + \gamma_n \left(6cx^4 + 6c(\gamma_n + \gamma_{n+1})x^2 + 6c(\Xi_{n+1} + \Xi_n) - 2t + 4t(x^2 + \gamma_n + \gamma_{n+1}) \right) \\ &\quad \times \left(6cx^4 + 6c(\gamma_n + \gamma_{n-1})x^2 + 6c(\Xi_{n-1} + \Xi_n) - 2t + 4t(x^2 + \gamma_n + \gamma_{n-1}) \right) \\ &\quad - \left[\left(6cx^5 + (6c\gamma_n + 4t)x^3 - \frac{2\sigma + 1}{x} + (6c\Xi_n + 4t\gamma_n - 2t)x + \frac{(2\sigma + 1)\Omega_n}{x} \right. \right. \\ &\quad \left. \left. + \frac{24cx^3 + 2[6c(\gamma_n + \gamma_{n+1}) + 4t]x}{6cx^4 + 6c(\gamma_n + \gamma_{n+1})x^2 + 6c(\Xi_{n+1} + \Xi_n) - 2t + 4t(x^2 + \gamma_n + \gamma_{n+1})} \right) \right. \\ &\quad \left. \times \left(6c\gamma_n x^3 + (6c\Xi_n + 4t\gamma_n)x + \frac{(2\sigma + 1)\Omega_n}{x} \right) \right] \\ &\equiv -\mathcal{B}_n(x; t) \left[v'(x) + \mathcal{B}_n(x; t) + \frac{\mathcal{A}'_n(x; t)}{\mathcal{A}_n(x; t)} \right] + \gamma_n \mathcal{A}_n(x; t) A_{n-1}(x; t) + \mathcal{B}'_n(x; t),\end{aligned}\quad (4.2b)$$

where Ω_n and Ξ_n are given in Eqs. (3.10) and (3.9) respectively.

Proof. For the proof, consult similar ideas in [21] and [22]. \square

Remark 19. One can expand Eq. (4.2) via symbolic packages such as Mathematica (Maple), however the resulting expression may look quite cumbersome.

5. APPLICATION OF EQ. (4.1) FOR ELECTROSTATIC ZERO DISTRIBUTION

The authors in [14] considered a perturbation of quartic Freud weight ($w(x) = \exp(-x^4)$) by the addition of a fixed charged point of mass δ at the origin; the corresponding polynomials are Freud-type polynomials (see the recent work in [15]). For semiclassical orthogonality measure, it was shown in [14] that these polynomials obey a second-order linear differential equation of the form (1.7), and the electrostatic model is in sight as in [18]. Application of Eq. (4.1) for electrostatic zero distribution is also mentioned. Following these ideas, a similar work for the perturbed Freud-type weight in (1.9) is given in a recent paper [22] using the obtained differential equation in Section 4.

6. CONCLUSIONS

By introducing a time variable to scaled sextic Freud-type measure upon deformation (perturbation), we have found certain fresh characterizing properties: some recursive relations, moments of finite order, concise formulation and orthogonality relation, nonlinear difference equation for recurrence coefficients as well as the corresponding polynomials and certain properties of the zeros of the corresponding polynomials. This work derived certain non-linear difference equations, Toda-like equations, and differential equations for the recurrence coefficients of the corresponding orthogonal polynomials under consideration. Special attention, using the method of Shohat's quasi-orthogonality and ladder operators, is given to characterize the Freud-type weight (1.9). Such semiclassical symmetric weight in (1.9) follows from quadratic transformation and symmetrization as in [7]. By combining the three-term recurrence relation with the difference-recurrence relation, a second-order differential equation fulfilled by polynomials associated with the semiclassical weight (1.9) is obtained. Application of the resulting differential equation in Eq. (4.1) for electrostatic zero distribution is also noted. Following this work, investigation of these recurrence coefficients in connection with certain (discrete) integrable systems will be a prominent continuation of this study.

REFERENCES

- [1] Aptekarev, A., Branquinho A., Marcellán F., Toda-type differential equations for the recurrence coefficients of orthogonal polynomials and Freud transformation, *J. Comput. Appl. Math.*, Elsevier, **78** (1997), 139–160.
- [2] Basor E. and Chen Y., Painlevé V and the distribution function of a discontinuous linear statistic in the Laguerre unitary ensembles. *J. Phys. A*, **42** (2009), 035203.
- [3] Chen Y. and Ismail M. E. H., Ladder operators and differential equations for orthogonal polynomials. *J. Phys. A* **30** (1997), 7817–7829.
- [4] Chen Y. and Its A., Painlevé III and a singular linear statistics in Hermitian random matrix ensembles. I. *J. Approx. Theory* **162** (2010), 270–297.
- [5] Chen Y. and Ismail, M.E.H., Ladder operators and differential equations for orthogonal polynomials, *J. Phys. A*, **30** (1997), 7817.
- [6] Chen Y. and Feigin M.V., Painlevé IV and degenerate Gaussian unitary ensembles, *J. Phys. A*, **39** (2006), 12381.
- [7] Chihara T.S., An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, (1978).
- [8] Clarkson P.A. and Jordaan K., The relationship between semiclassical Laguerre polynomials and the fourth Painlevé equation, *Constructive Approximation*, Springer, **39** (2014), 223–254.
- [9] Clarkson P.A., Jordaan K. and Kelil A., A generalized Freud weight, *Studies in Applied Mathematics*, Wiley Online Library, **136** (2016), 288–320.
- [10] Clarkson, P.A.; Jordaan, K., A Generalized Sextic Freud Weight, arXiv preprint arXiv:2004.00260, (2020).
- [11] Driver, K. and Jordaan, K., Zeros of quasi-orthogonal Jacobi polynomials, *SIGMA*, **12** (2016), p. 042.
- [12] Filipuk G., Van Assche W. and Zhang L., The recurrence coefficients of semi-classical Laguerre polynomials and the fourth Painlevé equation, *J. Phys. A*, **45** (2012), 205201.
- [13] Freud, G., On the coefficients in the recursion formulae of orthogonal polynomials, In *Math. Proc. R. Ir. Acad. Section A: Mathematical and Physical Sciences*, JSTOR, 1–6, (1976).
- [14] Garrido, Á., Arvesu Carballo, J., and Marcellán F., An electrostatic interpretation of the zeros of the Freud-type orthogonal polynomials, *ETNA*, (2005).
- [15] Garza, L.E., Huertas, E.J. and Marcellan, F., On Freud-Sobolev type orthogonal polynomials, *Afrika Matematika*, **30** (2019), 505–528.
- [16] Han, P. and Chen, Y., The recurrence coefficients of a semi-classical Laguerre polynomials and the large n asymptotics of the associated Hankel determinant, *Random Matrices: Theory and Applications*, **6** (2017), p.1740002.
- [17] Hendriksen E. and van Rossum H., Semi-classical orthogonal polynomials, *Polynômes Orthogonaux et Applications*, Springer, 354 – 361, (1985).
- [18] Ismail M.E.H., Classical and Quantum Orthogonal Polynomials in One Variable, *Encyclopedia of Mathematics and its Applications*, 98, Cambridge University Press, Cambridge, (2005).
- [19] Ismail, M.E.H.; Mansour, Z.S.I. q -analogues of Freud weights and non-linear difference equations, *Adv. Appl. Math.*, **45** (2010), 518–547.
- [20] Jost J., Postmodern analysis, Springer Science & Business Media, (2006).
- [21] Kelil A. S., Properties of a class of generalized Freud polynomials, University of Pretoria, PhD Thesis, (2018).
- [22] Kelil A.S., R Appadu A. On Semi-Classical Orthogonal Polynomials Associated with a Modified Sextic Freud-Type Weight. *Mathematics*, **8** (2020), 1250.
- [23] Lyu S. L., Chen Y. and Fan E. G., Asymptotic gap probability distributions of the Gaussian unitary ensembles and Jacobi unitary ensembles. *Nuclear Phys. B*, **926** (2018), 639–670.
- [24] Maroni P., Prolégomènes à l'étude des polynômes orthogonaux semi-classiques, *Ann. Mat. Pura Appl.* **149** (1987), 165–184.
- [25] Nevai P., Orthogonal polynomials associated with $\exp(-x^4)$, Second Edmonton Conference on Approximation Theory, Z. Ditzian, A. Meir, S.D. Riemenschneider, and A. Sharma (Eds.), *CMS Conf. Proc.*, Amer. Math. Soc., Providence, RI, **3** (1983), 263–285.
- [26] Nevai P., Géza Freud, orthogonal polynomials and Christoffel functions. A case study, *J. A. T. Elsevier* **48** (1986), 3–167.
- [27] M. O. Searcoid, Metric spaces, Springer (2006).
- [28] Shohat J., A differential equation for orthogonal polynomials, *Duke Math. J.* **5**, 401–417, (1939).
- [29] Szegő G., Orthogonal Polynomials, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., (1975).
- [30] Van Assche W., Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials, In *Difference equations, special functions and orthogonal polynomials*, World Sci., 687–725, (2007).
- [31] Van Assche W., Orthogonal polynomials and Painlevé equations, Cambridge University Press, 27 (2017).