1 Square probabilities

Chutes and Ladders runs as a Markov chain, meaning that the probability of landing on one square depends only on where you were at the previous step. I am using the board configuration in basicboard.png.

I want the probabilities of landing on a square to add to 100 at each step, so I need to choose what counts as "landing" on a square. Here I make the arbitrary choice that the square you hit before going on a chute or ladder is the square you "landed" on. Then if you want to calculate the probabilities such that landing is where you end up at the end of your turn, you can add the probability of the "source" square to the probability of the "sink" square to get that formulation of the problem.

Then:

$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$ on turn 1, 0 otw

In general:

$$P(n_t) = \frac{1}{6}\sum_{i=1}^{6} p((n-i)_{t-1}) \tag{1}$$

when there are no chutes or ladders. When there are chutes or ladders, the sources and sinks of the chutes/ladders must be included when calculating the conditional probabilities. For example, the probability of landing on square 18 becomes:

$$P(18_t) = \frac{1}{6}P(17_{t-1})+\frac{1}{6}P(16_{t-1})+\frac{1}{6}P(6_{t-1})+\frac{1}{6}P(15_{t-1})+\frac{1}{6}P(14_{t-1})+\frac{1}{6}P(4_{t-1})+\frac{1}{6}P(13_{t-1})+\frac{1}{6}P(12_{t-1}) \tag{2}$$

Thus the probability of landing on squares just past sinks increases, and the probability of landing on squares just past sources decreases. A plot of the probability of landing on each square over 120 steps (each line is a square) is in ProbabilitiesBySquare.pdf. There is also a version without the accumulated probability in square 100 to make it more readable. The same data for 100 time steps can be found in ProbabilitiesBySquareByStep.txt.

2 Game Length

The probability of being at each square at a given time is a probability distribution conditional over the time t. Then the expected game length is an expectation over these conditional probability distributions:

$$E(t) = \sum_{t=1}^{\infty} tP(100|t) = 1P(100|t=1) + 2P(100|t=2) + \ldots \tag{3}$$

Each conditional probability is a sum over all combinations of squares that could have gotten the player to the finish in precisely the given number of steps. Since it is a Markov chain, though, it is possible to simply keep the probabilities of all squares from the previous time step. Then, in principle, it is possible to calculate the probability of winning at every time step; in practice, we can approximate the expectation well by calculating it for a very large number of time steps. Games are generally not very long, so this will not be a significant underestimate of the expectation. Based on my code over 100,000 time steps (at which point the probabilities are smaller than precision error), the expected game length is 20.5712. I believe this is incorrect, as simulations show the length to be closer to 37.23 turns.

3 Effect of multiple players

Players do not interact in chutes and ladders. This means that the length of a multi-player game can be thought of as multiple trials of a single-player game. Then the length of a 2-player game is equal to the minimum of two trials of a single-player game, and the problem is to find the expectation of the minimum of two iid random variables.

Considering the probability mass function of this case with two players X and Y, we have:

$$P(X \wedge Y winatt) = P(Xlose|t)P(Ywin|t)+P(Xwin|t)P(Ylose|t)+P(Xwin|t)P(Ywin|t) \tag{4}$$

$$= 2P(100|t)(1 - P(100|t)) + P(100|t)^2 \tag{5}$$

where $P(100|t)$ is the probability of landing on square 100 on turn t. Then the expectation is:

$$E(t) = \sum_{t=1}^{\infty} t(2P(100|t)(1 - P(100|t)) + P(100|t)^2) \tag{6}$$

Note that the pmf is related to the binomial distribution. This fact can be used to extend this problem to m players:

$$P(atleast1winatt) = \sum_{i=1}^{m} (mchoosei)P(100|t)^i(1 - P(100|t))^{m-i} \tag{7}$$

$$= 1 - (1 - P(100|t))^m \tag{8}$$

which can be used to calculate the expectation as usual. Using these equations, I calculate the expected game length to be 40.996 for 2 players and 61.277 for 3, but the lengths are longer, not shorter, so there is something wrong with my code, and I was not able to find it in time. I believe the problem is with the code and not the math behind it.

I don't have time to fix the code, but I ran simulations in the background to compare against my analytical results. For these data, I have the expected game length to be approximately 37.23 turns for one player (based on N=1,000,000), 25.14 for two players (based on N=100,000), and 20.91 for three (based on N=100,000).

$$ \tag{9} $$