Advanced Econometrics (2ST123)

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1 Nonlinear Regression Model

1.1 Model Transformation and Estimation

Consider the nonlinear regression model

$$y_i = e^{\beta_0} e^{\beta_1 x_i} x_i^{\beta_2} e^{\varepsilon_i}, \tag{1}$$

where ε_i denotes an error term. Taking logarithms of both sides yields

$$\log y_i = \beta_0 + \beta_1 x_i + \beta_2 \log x_i + \varepsilon_i, \tag{2}$$

which is linear in the parameters. This transformation allows the use of ordinary least squares (OLS) for estimation under the usual assumptions on the error term.

Let $y = (\log y_1, \ldots, \log y_n)^\top$ and define the design matrix

$$X = (\,1\,)\, x_1 \log x_1 \vdots\vdots\vdots 1 x_n \log x_n. \tag{3}$$

The OLS estimator is then given by

$$\hat{\beta} = (X^\top X)^{-1} X^\top y. \tag{4}$$

1.2 Fitted Values and Analytical Maximization

Since estimation is performed in logarithms, fitted values in the original scale are obtained by

$$\hat{y}_i = e^{\hat{\beta}_0} e^{\hat{\beta}_1 x_i} x_i^{\hat{\beta}_2}. \tag{5}$$

To find the value of x that maximizes $\hat{y}(x)$, consider the logarithm of the fitted function,

$$\log \hat{y}(x) = \hat{\beta}_0 + \hat{\beta}_1 x + \hat{\beta}_2 \log x. \tag{6}$$

Differentiating with respect to x and setting the derivative equal to zero yields

$$\hat{\beta}_1 + \frac{\hat{\beta}_2}{x} = 0, \tag{7}$$

which implies that the maximizer is given by

$$x^* = -\frac{\hat{\beta}_2}{\hat{\beta}_1}. \tag{8}$$

2 Monte Carlo Study of Model Misspecification

2.1 Data Generating Process

The true data generating process (DGP) is specified as

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon, \tag{9}$$

where $\varepsilon \sim \mathcal{N}(0, \sigma^2 I)$ and $\beta = (20, 0.03, 0.7, 1.4)^\top$. In the Monte Carlo study, the explanatory variables are treated as fixed, and all randomness in y is induced by the error term ε, consistent with the probabilistic framework advocated by Haavelmo.

2.2 Omitted Variable Bias

The misspecified model excludes the variable x_2 and is given by

$$y = \beta_0 + \beta_1 x_1 + \beta_3 x_3 + u. \tag{10}$$

Since x_2 affects y and is generally correlated with the included regressors, the estimator $\hat{\beta}_1$ is biased. As a consequence, the estimator $\hat{\beta}_1$ is inconsistent and the bias does not vanish as the sample size increases. The Monte Carlo simulation illustrates this phenomenon by comparing the average simulated estimate of β_1 with its true value and by examining whether the confidence interval constructed from the simulated distribution covers the true parameter.

3 Nonlinear Least Squares Theory

3.1 Orthogonality Condition

Consider the nonlinear regression model

$$y_i = m_i(\beta) + e_i, \tag{11}$$

with $e_i \sim \mathcal{N}(0, \sigma^2)$. The objective function is

$$E[e^\top e] = E[(y - m(\beta))^\top (y - m(\beta))]. \tag{12}$$

Differentiating with respect to β and setting the gradient equal to zero yields the first-order condition

$$J(\beta)^\top (y - m(\beta)) = 0, \tag{13}$$

where $J(\beta)$ denotes the Jacobian matrix of partial derivatives of $m(\beta)$ with respect to the parameter vector β.

3.2 OLS as a Special Case of NLS

If the conditional mean function is linear, $m(\beta) = X\beta$, then the Jacobian reduces to $J(\beta) = X$. In this case, the orthogonality condition becomes

$$X^\top (y - X\beta) = 0, \tag{14}$$

which corresponds to the normal equations for the OLS estimator. Hence, OLS is a special case of nonlinear least squares.

3.3 Jacobian and Identifiability

Consider the nonlinear regression model

$$m_i(\beta) = \beta_0 \exp(\beta_1 x_{i1} + \beta_2 x_{i2}). \tag{15}$$

The elements of the Jacobian matrix are given by $\partial m_i(\beta) \frac{}{\partial \beta_0 = \exp(\beta_1 x_{i1} + \beta_2 x_{i2}), \frac{\partial m_i(\beta)}{\partial \beta_1} = \beta_0 x_{i1} \exp(\beta_1 x_{i1} + \beta_2 x_{i2}), \frac{\partial m_i(\beta)}{\partial \beta_2} =}$

To assess identifiability, suppose that two parameter vectors $\beta^{(1)}$ and $\beta^{(2)}$ produce the same conditional expectation function for all observations. Taking logarithms of the resulting equality yields a linear equation in $(1, x_{i1}, x_{i2})$. Since the regressor matrix with rows $(1, x_{i1}, x_{i2})$ has full column rank by assumption, equality of the conditional expectations implies $\beta^{(1)} = \beta^{(2)}$. Hence, the model is identifiable.