Thermo–Hyperelastic Tube Model: Governing Equations and Weak Forms

1 Problem Overview

This simulation models:

- transient heat conduction,

- finite-strain hyperelasticity,

- thermal expansion,

- and time-dependent mechanical excitation.

The coupling is *one-way*:

$$T \;\longrightarrow\; F_{\text{th}}(T) \;\longrightarrow\; F_e \;\longrightarrow\; u,$$

meaning temperature affects mechanics, but mechanics does not influence the heat equation. This reduced thermoelastic viewpoint follows the classical formulation in Bonnet [TE-1] and is consistent with finite-strain hyperelastic frameworks such as Ogden [ME-2].

The mechanics are solved *quasi-statically* (no inertial effects), matching the implementation.

2 Thermal Model

2.1 Strong Form

The transient heat equation is:

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T).$$

Boundary conditions used in the simulation are:

- Top surface: prescribed temperature (time-ramped Dirichlet),

- Bottom surface: convection to an ambient temperature $T_\infty(t)$,

- Lateral surface: insulated (zero flux).

This model and assumptions follow the standard treatment in Lewis et al. [HT-1].

2.2 Weak Form

Let w denote a test function from the same finite-element space. Find T^{n+1} such that:

$$\int_{\Omega} \rho c_p \frac{T^{n+1} - T^n}{\Delta t} \, w \, dV + \int_{\Omega} k \, \nabla T^{n+1} \cdot \nabla w \, dV \tag{1}$$

$$+ \int_{\Gamma_{\text{bottom}}} h \, T^{n+1} w \, dS = \int_{\Gamma_{\text{bottom}}} h \, T_{\infty} \, w \, dS. \tag{2}$$

This weak form is classical in heat-transfer FEM [HT-1].

3 Finite-Strain Hyperelasticity

3.1 Kinematics

Total deformation gradient:
$$F = I + \nabla u.$$

Thermal deformation gradient:
$$F_{\text{th}} = (1 + \alpha(T - T_{\text{ref}}))I.$$

Elastic part:
$$F_e = F \, F_{\text{th}}^{-1}.$$

Elastic measures:
$$B_e = F_e F_e^T, \qquad J_e = \det(F_e).$$

The thermal strain concept matches Ogden [ME-2].

3.2 Neo-Hookean Energy

$$\psi(F_e) = \frac{\mu}{2}(\text{tr}(B_e) - 3) - \mu \ln J_e + \frac{\lambda}{2}(\ln J_e)^2.$$

Lamé parameters:
$$\mu = \frac{E}{2(1 + \nu)}, \qquad \lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}.$$

This specific functional form follows the course template [ME-1] and fits into the general hyperelastic theory of Ogden [ME-2].

3.3 Cauchy Stress

$$\sigma = \frac{\mu}{J_e}(B_e - I) + \frac{\lambda \ln J_e}{J_e} I.$$

This is the standard compressible Neo-Hookean Cauchy stress from Ogden [ME-2].

3.4 Boundary Conditions in the Simulation

- Top face: fully fixed ($\mathbf{u} = 0$),
- Bottom face: prescribed time-dependent displacement in the z-direction (ramp + sinusoidal),
- Remaining surfaces: traction-free.

These match the implemented loading conditions.

3.5 Weak Equilibrium Form

Seek u such that:

$$\int_\Omega P : \nabla w \, dV = 0,$$

for all admissible test functions w, where

$$P = \frac{\partial \psi}{\partial F}.$$

With the above boundary conditions, natural tractions vanish on free surfaces. This is the canonical hyperelastic weak form [ME-2].

4 Coupling Strategy

A staggered (partitioned) scheme is used:

$$T^{n+1} \; \rightarrow \; F_{\text{th}}(T^{n+1}) \; \rightarrow \; u^{n+1}.$$

This reduced thermoelastic treatment is consistent with Bonnet's formulation [TE-1] and compatible with Ogden's hyperelastic framework [ME-2].

5 Post-Processing

Deviatoric stress:

$$s = \sigma - \frac{1}{3}(\text{tr}\,\sigma)I.$$

Von Mises equivalent stress:

$$\sigma_{\text{vm}} = \sqrt{\frac{3}{2}\, s : s}.$$

These stress measures are standard in finite-strain solid mechanics [ME-2].

References

[HT-1] Lewis, R.W., Morgan, K., Thomas, H.R., Seetharamu, K.N. *The Finite Element Method in Heat Transfer Analysis.* Wiley.

[ME-1] *Energy-functional_New_Template1.pdf.* Course material on Neo-Hookean strain energy.

[ME-2] Ogden, R.W. *Non-Linear Elastic Deformations.* Dover, 1997.

[TE-1] Bonnet, M. "Thermoélasticité", ENSTA Paris lecture notes.