Vector Calculus

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This lecture notes is a revision of undergraduate Vector Calculus.

I. CYLINDRICAL COORDINATE SYSTEM

If the point P has Cartesian coordinates (x, y, z), the cylindrical coordinates (ρ, ϕ, z) are then:

$$x = \rho\cos\phi \tag{1}$$
$$y = \rho\sin\phi \tag{2}$$
$$z = z, \tag{3}$$

and conversely,

$$\rho = \sqrt{x^2 + y^2} \tag{4}$$
$$\phi = \text{atan2}\left(\frac{y}{x}\right) \tag{5}$$
$$z = z, \tag{6}$$

where the function atan2 is defined to be:

$$\text{atan2}\left(\frac{y}{x}\right) = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x > 0 \text{ (I and IV)} \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \text{ (II)} \\ \tan^{-1}\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \text{ (III)} \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases}. \tag{7}$$

The range of (ρ, ϕ, z) are:

$$\rho - \pi \geq 0, \quad -\pi < \phi \leq \pi, \quad -\infty < z < \infty. \tag{8}$$

At point P, we define the unit vector $\hat{\boldsymbol{e}}_\rho, \hat{\boldsymbol{e}}_\phi$, and $\hat{\boldsymbol{e}}_z$ as follows. Consider the surface $\rho =$ constant, which represents a cylinder. The vector $\hat{\boldsymbol{e}}_\rho$ is defined to be perpendicular to this surface and in the direction of increasing ρ. Similarly consider the surface $\phi =$ constant, which is a half-plane. The vector $\hat{\boldsymbol{e}}_\phi$ is defined to be perpendicular to this surface and in the direction of increasing ϕ, and ditto $\hat{\boldsymbol{e}}_z$. If the point P has cylindrical coordinates (ρ, ϕ, z), then:

$$\hat{\boldsymbol{e}}_\rho = \cos\phi\hat{\boldsymbol{i}} + \sin\phi\hat{\boldsymbol{j}} \tag{9}$$
$$\hat{\boldsymbol{e}}_\phi = -\sin\phi\hat{\boldsymbol{i}} + \cos\phi\hat{\boldsymbol{j}} \tag{10}$$
$$\hat{\boldsymbol{e}}_z = \hat{\boldsymbol{k}}, \tag{11}$$

and conversely,

$$\hat{\boldsymbol{i}} = \cos\phi\hat{\boldsymbol{e}}_\rho - \sin\phi\hat{\boldsymbol{e}}_\phi \tag{12}$$
$$\hat{\boldsymbol{j}} = \sin\phi\hat{\boldsymbol{e}}_\rho + \cos\phi\hat{\boldsymbol{e}}_\phi \tag{13}$$
$$\hat{\boldsymbol{k}} = \hat{\boldsymbol{e}}_z. \tag{14}$$

Suppose that we have two vectors $\boldsymbol{a} = a_\rho\hat{\boldsymbol{e}}_\rho + a_\phi\hat{\boldsymbol{e}}_\phi + a_z\hat{\boldsymbol{e}}_z$ and $\boldsymbol{b} = b_\rho\hat{\boldsymbol{e}}_\rho + b_\phi\hat{\boldsymbol{e}}_\phi + b_z\hat{\boldsymbol{e}}_z$. The dot and cross product between $\boldsymbol{a}$ and $\boldsymbol{b}$ are then:

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_\rho b_\rho + a_\phi b_\phi + a_z b_z \tag{15}$$
$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{e}}_\rho & \hat{\boldsymbol{e}}_\phi & \hat{\boldsymbol{e}}_z \\ a_\rho & a_\phi & a_z \\ b_\rho & b_\phi & b_z \end{vmatrix}. \tag{16}$$

II. SPHERICAL COORDINATE SYSTEM

If the point P has Cartesian coordinates (x, y, z), its spherical coordinates (r, θ, ϕ) are then:

$$x = r\sin\theta\cos\phi \tag{17}$$
$$y = r\sin\theta\sin\phi \tag{18}$$
$$z = r\cos\theta, \tag{19}$$

and conversely,

$$r = \sqrt{x^2 + y^2 + z^2} \tag{20}$$
$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \tag{21}$$
$$\phi = \text{arctan2}\left(\frac{y}{x}\right). \tag{22}$$

The range of (r, θ, ϕ) is defined on:

$$r \geq 0, \quad 0 \leq \theta \leq \phi, \quad -\pi < \phi \leq \pi. \tag{23}$$

At the point P, the unit vectors $\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta$, and $\hat{\boldsymbol{e}}_\phi$ are defined as follows. Consider the surface $r =$ constant, which represents a sphere. The vector $\hat{\boldsymbol{e}}_r$ is defined to be perpendicular to this surface and in the direction of increasing r. Similarly, the surface $\theta =$ constant represents a cone and the vector $\hat{\boldsymbol{e}}_\theta$ is defined to be perpendicular to this surface and in the direction of increasing θ. Ditto ϕ. If the point P has spherical coordinates (r, θ, ϕ), then:

$$\hat{\boldsymbol{e}}_r = \sin\theta\cos\phi\hat{\boldsymbol{i}} + \sin\theta\sin\phi\hat{\boldsymbol{j}} + \cos\theta\hat{\boldsymbol{k}} \tag{24}$$
$$\hat{\boldsymbol{e}}_\theta = \cos\theta\cos\phi\hat{\boldsymbol{i}} + \cos\theta\sin\phi\hat{\boldsymbol{j}} - \sin\theta\hat{\boldsymbol{k}} \tag{25}$$
$$\hat{\boldsymbol{e}}_\phi = -\sin\phi\hat{\boldsymbol{i}} + \cos\phi\hat{\boldsymbol{j}}, \tag{26}$$

and conversely,

$$\hat{\boldsymbol{i}} = \sin\theta\cos\phi\hat{\boldsymbol{e}}_r + \cos\theta\cos\phi\hat{\boldsymbol{e}}_\theta - \sin\phi\hat{\boldsymbol{e}}_\phi \tag{27}$$
$$\hat{\boldsymbol{j}} = \sin\theta\sin\phi\hat{\boldsymbol{e}}_r + \cos\theta\sin\phi\hat{\boldsymbol{e}}_\theta + \cos\phi\hat{\boldsymbol{e}}_\phi \tag{28}$$
$$\hat{\boldsymbol{k}} = \cos\theta\hat{\boldsymbol{e}}_r - \sin\theta\hat{\boldsymbol{e}}_\theta. \tag{29}$$

III. GRADIENT OF A SCALAR FIELD

Lets consider a scalar field $f(\boldsymbol{r})$. In Cartesian coordinates, the gradient of the scalar field $f(\boldsymbol{r})$ is defined by:

$$\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}}. \tag{30}$$

Some properties:

1. The (spatial) rate of change of f in the direction of the unit vector $\hat{\boldsymbol{e}}$ is given by: $\hat{\boldsymbol{e}} \cdot \nabla f$.

2. The normal vector to the surface $f =$ constant and passing through P is given by ∇f, evaluated at P.

Now we shall derive the expression for ∇f in cylindrical coordinate system. Let us write:

$$\nabla f = g_\rho \hat{\boldsymbol{e}}_\rho + g_\phi \hat{\boldsymbol{e}}_\phi + g_z \hat{\boldsymbol{e}}_z, \tag{31}$$

where g_ρ, g_ϕ, and g_z are to be found. First $g_\rho = \hat{\boldsymbol{e}}_\rho \cdot \nabla f$ is the spatial rate of change of f in the direction of $\hat{\boldsymbol{e}}_\rho$:

$$g_\rho = \lim_{\delta\rho\to 0} \frac{f(\rho+\delta\rho,\phi,z) - f(\rho,\phi,z)}{\delta\rho} = \frac{\partial f}{\partial \rho}. \tag{32}$$

Next $g_\phi = \hat{\boldsymbol{e}}_\phi \cdot \nabla f$ is the spatial rate of change of f in the direction of $\hat{\boldsymbol{e}}_\phi$:

$$g_\phi = \lim_{\delta\phi\to 0} \frac{f(\rho,\phi+\delta\phi,z) - f(\rho,\phi,z)}{\rho\delta\phi} = \frac{1}{\rho}\frac{\partial f}{\partial \phi}. \tag{33}$$

The denominator is the distance between point $(\rho, \phi + \delta\phi, z)$ and (ρ, ϕ, z). Similarly for g_z. Therefore,

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{e}}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{e}}_\phi + \frac{\partial f}{\partial z}\hat{\boldsymbol{e}}_z. \tag{34}$$

Now the expression for ∇f in spherical coordinate system is:

$$\nabla f = g_r \hat{\boldsymbol{e}}_r + g_\theta \hat{\boldsymbol{e}}_\theta + g_\phi \hat{\boldsymbol{e}}_\phi. \tag{35}$$

For instance, $g_\phi = \hat{\boldsymbol{e}}_\phi \cdot \nabla f$ is the spatial rate of change of f in the direction of $\hat{\boldsymbol{e}}_\phi$:

$$g_\phi = \lim_{\delta\phi\to 0} \frac{f(r,\theta,\phi+\delta\phi) - f(r,\theta,\phi)}{r\sin\theta\delta\phi} = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}. \tag{36}$$

The denominator is the distance between point $(r, \theta, \phi + \delta\phi)$ and (r, θ, ϕ). Finally,

$$\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{e}}_\phi. \tag{37}$$

IV. DIFFERENTIAL DISPLACEMENTS

Let us consider the displacement vector from point (x, y, z) to $(x + dx, y + dy, z + dz)$ in the Cartesian coordinates. This displacement vector can be written as a sum of 3 displacement vectors: one going from (x, y, z) to $(x + dx, y, z)$ and then another one from $(x + dx, y, z)$ to $(x + dx, y + dy, z)$ and finally from $(x + dx, y + dy, z)$ to $(x + dx, y + dy, z + dz)$:

$$d\boldsymbol{r} = dx\hat{\boldsymbol{\imath}} + dy\hat{\boldsymbol{\jmath}} + dz\hat{\boldsymbol{k}}, \tag{38}$$

where the coefficients are the corresponding distances.

Now in cylindrical coordinates, the displacement vector from (ρ, ϕ, z) to $(\rho + d\rho, \phi + d\phi, z + dz)$ is given by:

$$d\boldsymbol{r} = d\rho\hat{\boldsymbol{e}}_\rho + \rho d\phi\hat{\boldsymbol{e}}_\phi + dz\hat{\boldsymbol{e}}_z, \tag{39}$$

where $d\rho$ is the distance along the edge from (ρ, ϕ, z) to $(\rho + d\rho, \phi, z)$ and $\rho d\phi$ is the distance along the edge from $(\rho + d\rho, \phi, z)$ to $(\rho + d\rho, \phi + d\phi, z)$ and so on. Finally in spherical coordinates, the displacement vector from (r, θ, ϕ) to $(r + dr, \theta + d\theta, \phi + d\phi)$ is given by:

$$d\boldsymbol{r} = dr\hat{\boldsymbol{e}}_r + rd\theta\hat{\boldsymbol{e}}_\theta + r\sin\theta d\phi\hat{\boldsymbol{e}}_\phi. \tag{40}$$

V. VOLUME INTEGRALS

Let us consider the volume integral of some scalar field $f(\boldsymbol{r})$ over some region V: $\int_V f(\boldsymbol{r})\, dV$. To do this, we divide the region into N volume elements. Let us label δV_i to be the volume of the i^{th} volume element and let us select a point $\boldsymbol{r}_i$ inside this volume element. The volume integral of a scalar field f over this region V is defined to be:

$$\int_V f(\boldsymbol{r})\, dV = \lim_{N\to\infty} \sum_{i=1}^{N} f(\boldsymbol{r}_i)\delta V_i. \tag{41}$$

Similarly, the volume integral of a vector field $\boldsymbol{F}(\boldsymbol{r})$ over the region V is defined to be:

$$\int_V \boldsymbol{F}(\boldsymbol{r})\, dV = \lim_{N\to\infty} \sum_{i=1}^{N} \boldsymbol{F}(\boldsymbol{r}_i)\delta V_i. \tag{42}$$

When we take the limit $N \to \infty$, the diameter of each volume element δV_i should go to zero. It can be shown that the limit exists as along as the function f (or $\boldsymbol{F}$) is sufficiently smooth on V and on the boundary of V.

Cylindrical coordinates. To evaluate the volume integral in cylindrical polar coordinates, we first divide the region of integration into volume element whose surfaces are of the form $\rho =$ constant, $\phi =$ constant, and $z =$ constant. The volume of this volume element is:

$$\delta V = \rho\delta\rho\delta\phi\delta z. \tag{43}$$

In terms of integration this is:

$$\int_V f(\boldsymbol{r})\,dV = \int \rho\,d\rho \int d\phi \int dz\, f(\rho,\phi,z). \tag{44}$$

Spherical coordinates. Similarly to evaluate the volume integral in the spherical polar coordinates, the volume element is defined by $r =$ constant, $\theta =$ constant, and $\phi =$ constant surfaces. The volume of this volume element is:

$$\delta V = r^2 \sin\theta \delta r \delta\theta \delta\phi. \tag{45}$$

Thus,

$$\int_V f(\boldsymbol{r})\,dV = \int r^2 dr \int \sin\theta d\theta \int d\phi\, f(r,\theta,\phi). \tag{46}$$

Example. Consider the following integral:

$$\int_V (x^2+y^2)z\,dV, \tag{47}$$

where the region of integration is a cone with height h and angle α. The apex of the cone is at the origin and the axis of the cone coincides with the positive z-axis.

Using symmetry, we can evaluate this integral in the cylindrical coordinate system:

$$\int_V (x^2+y^2)z\,dV = \int_{-\pi}^{\pi} d\phi \int_0^h dz \int_0^{z\tan\alpha} \rho d\rho\ \left(\rho^2 z\right) \tag{48}$$

$$= 2\pi \int_0^h dz\ \left(\frac{1}{4} z^5 \tan^4\alpha\right) \tag{49}$$

$$= \frac{\pi}{12} h^6 \tan^4\alpha. \tag{50}$$

Note that we perform the integration over ρ first. For fixed z, the integration range of ρ is from 0 to $z\tan\alpha$.

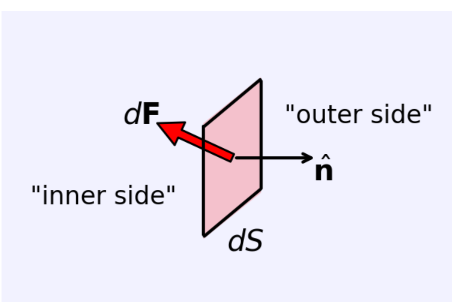

Figure 1. The normal unit vector $\hat{\mathbf{n}}$ of a surface element dS is a unit vector which is perpendicular to dS and point from the 'inner' to the 'outer' side of dS.

VI. SURFACE INTEGRALS

Consider a smooth surface S, which can be open or closed. For an open surface S, we can arbitrarily define one side of the surface to be the 'inner' side and the other side to be the 'outer' side, see Fig. 1. For a closed surface S, the volume enclosed by the surface is always defined to be the 'inner' side. At any point P on S, we may define the outward normal unit vector $\hat{\boldsymbol{n}}$ to be perpendicular to S and point in the direction of the 'outer' side, see Fig. 1

Let us divide the surface S into N curved surface elements. The surface area of the i^{th} surface element is δS_i. We can then choose any point $\boldsymbol{r}_i$ on this surface element. The surface integral of a scalar field $f(\boldsymbol{r})$ over S is defined to be:

$$\int_S f(\boldsymbol{r})\,dS = \lim_{N\to\infty} \sum_{i=1}^{N} f(\boldsymbol{r}_i)\delta S_i, \tag{51}$$

where the limit is taken such that the diameter of each surface element goes to zero as $N \to \infty$. Similarly, the surface integral of a vector field $\boldsymbol{F}(\boldsymbol{r})$ over S is defined to be:

$$\int_S \boldsymbol{F}(\boldsymbol{r})\,dS = \lim_{N\to\infty} \sum_{i=1}^{N} \boldsymbol{F}(\boldsymbol{r}_i)\delta S_i. \tag{52}$$

If the surface is closed, the integral is indicated by a small circle: $\oint$.

Force by a static fluid. Consider a static fluid with pressure distribution $p(\boldsymbol{r})$. Let us consider some surface element δS_i. The force acting on the 'inner' fluid by the 'outer' fluid is then:

$$\delta \boldsymbol{F}_i = -p(\boldsymbol{r}_i)\hat{\boldsymbol{n}}(\boldsymbol{r}_i)\delta S_i, \tag{53}$$

where $\hat{\boldsymbol{n}}_i$ is a normal unit vector pointing to the 'outer' fluid, see Fig. 1 Therefore for an arbitrary curved surface S, the net force acting on the 'inner' fluid by the 'outer' fluid is given by:

$$\boldsymbol{F} = \lim_{N\to\infty} \sum_{i=1}^{N} -p(\boldsymbol{r}_i)\hat{\boldsymbol{n}}(\boldsymbol{r}_i)\delta S_i = -\int_S p(\boldsymbol{r})\hat{\boldsymbol{n}}(\boldsymbol{r})\,dS. \tag{54}$$

Volume flow rate. The volume flow rate is defined to be the volume of fluid that passes through a given surface per unit time. Let's consider a surface element δS_i. Let us define $\hat{\boldsymbol{n}}_i$ to be the unit normal perpendicular to this surface element. Suppose that $\boldsymbol{r}_i$ is any point on the surface element. The fluid velocity on this surface element is then $\boldsymbol{u}(\boldsymbol{r}_i)$. During a small time interval δt, the fluid flowing through δS_i will be contained inside an oblique cylinder with volume: $\boldsymbol{u}(\boldsymbol{r}_i)\cdot\hat{\boldsymbol{n}}_i\delta S_i\delta t$. Therefore the volume flow rate across this surface element is $\boldsymbol{u}(\boldsymbol{r}_i)\cdot\hat{\boldsymbol{n}}_i\delta S_i$. Now for an arbitrary surface S, the volume flow rate across this surface is:

$$\int_S \boldsymbol{u}(\boldsymbol{r})\cdot\hat{\boldsymbol{n}}(\boldsymbol{r})\,dS = \lim_{N\to\infty} \sum_{i=1}^{N} \boldsymbol{u}(\boldsymbol{r}_i)\cdot\hat{\boldsymbol{n}}_i\delta S_i. \tag{55}$$

This volume flow rate through the surface S is also called the flux of the fluid through S. In general, we call $\int_S \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,dS$ to be the flux of $\boldsymbol{F}$ through the surface S.

Cylindrical coordinates. Let us calculate the surface integral in the cylindrical coordinate system. Here, the surface element is usually defined by $\rho =$ constant (most common), $\phi =$ constant , or $z =$ constant surface. For example, suppose that we want to evaluate the surface integral of a scalar field $f(\boldsymbol{r})$ over a cylindrical surface $\rho = R =$ constant. The area of a surface element on this cylinder is:

$$\delta S = R\delta\phi\delta z. \tag{56}$$

The surface integral can then be written as:

$$\int_S f(\boldsymbol{r})\, dS = \int R d\phi \int dz\, f(R, \phi, z). \tag{57}$$

Spherical coordinates. In spherical coordinate system, the surface element is usually defined by the $r =$ constant (most common), $\theta =$ constant, or $\phi =$ constant surface. For example, consider the surface integral of a scalar field $f(\boldsymbol{r})$ over the sphere $r = R =$ constant. The area of the surface element is:

$$\delta S = R^2 \sin\theta\delta\theta\delta\phi. \tag{58}$$

Therefore, the surface integral can be written as:

$$\int_S f(\boldsymbol{r})\, dS = \int R^2 \sin\theta\, d\theta \int d\phi\, f(R, \theta, \phi). \tag{59}$$

Example. Consider the following surface integral:

$$\int_S x^2\, dS, \tag{60}$$

where the integration range is over the cylindrical surface $x^2 + y^2 = R^2$ between $z = 0$ and $z = h$.

Using cylindrical coordinates,

$$\int_S x^2\, dS = \int_{-\pi}^{\pi} R\, d\phi \int_0^h dz\, \left(R^2 \cos^2\phi\right) = R^3 h\pi. \tag{61}$$

VII. DIVERGENCE OF A VECTOR FIELD

Consider some volume of fluid V. Let S be the surface surrounding this fluid volume. The total volume flow rate of the fluid out of this region V is given by

$$\oint_S \boldsymbol{u}\cdot\hat{\boldsymbol{n}}\, dS. \tag{62}$$

If this is positive, that means there is a 'source' of fluid, or the fluid is expanding. If this is negative, that means there is a 'sink' of fluid, or the fluid is contracting. The rate of expansion (or contraction) of the fluid within this region V is defined to be:

$$\frac{1}{V}\oint_S \boldsymbol{u}\cdot\hat{\boldsymbol{n}}\, dS. \tag{63}$$

We can find the rate of expansion (or contraction) of the fluid at point P inside V by taking the limit:

$$\nabla\cdot\boldsymbol{u} = \lim_{V\to 0}\frac{1}{V}\oint_S \boldsymbol{u}\cdot\hat{\boldsymbol{n}}\, dS. \tag{64}$$

This rate of expansion (or contraction) of the fluid at point P is called the divergence of $\boldsymbol{u}$, evaluated at point P, which is denoted by $\nabla\cdot\boldsymbol{u}$. By analogy, we can also find the divergence of any vector field $\boldsymbol{F}$, $\nabla\cdot\boldsymbol{F}$. $\nabla\cdot\boldsymbol{F}$ is a scalar field.

Divergence in Cartesian coordinates. We shall calculate the divergence of a vector field $\boldsymbol{F}$ in the Cartesian coordinates. First we write:

$$\boldsymbol{F} = F_x(x, y, z)\hat{\boldsymbol{\imath}} + F_y(x, y, z)\hat{\boldsymbol{\jmath}} + F_z(x, y, z)\hat{\boldsymbol{k}}. \tag{65}$$

Let us consider a small cube with surface with side length 2ℓ, centred on the point P, whose coordinates are (x, y, z). The surface integral $\oint_{\partial V} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\, dS$ will have contributions from all the six sides. Let's first consider the top side:

$$\int_{\text{top}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\, dS = 4\ell^2 F_z(x, y, z+\ell) \tag{66}$$

$$= 4\ell^2 F_z(x, y, z+\ell) + 4\ell^3 \left.\frac{\partial F_z}{\partial z}\right|_P + \mathcal{O}(\ell^4). \tag{67}$$

Next we consider the bottom side:

$$\int_{\text{bottom}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\, dS = -4\ell^2 F_z(x, y, z-\ell) \tag{68}$$

$$= -4\ell^2 F_z(x, y, z) + 4\ell^3 \left.\frac{\partial F_z}{\partial z}\right|_P + \mathcal{O}(\ell^4). \tag{69}$$

Similarly for the left, right, front, and back sides. The net contribution form all the six sides is:

$$\oint_{\partial V} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\, dS = 8\ell^3\left(\left.\frac{\partial F_x}{\partial x}\right|_P + \left.\frac{\partial F_y}{\partial y}\right|_P + \left.\frac{\partial F_z}{\partial z}\right|_P\right) + \mathcal{O}(\ell^4). \tag{70}$$

Now using the definition of divergence of $\boldsymbol{F}$, evaluated at point P,

$$(\nabla\cdot\boldsymbol{F})_P \tag{71}$$

$$= \lim_{\ell\to 0}\frac{1}{8\ell^3}\left[8\ell^3\left(\left.\frac{\partial F_x}{\partial x}\right|_P + \left.\frac{\partial F_y}{\partial y}\right|_P + \left.\frac{\partial F_z}{\partial z}\right|_P\right) + \mathcal{O}(\ell^4)\right] \tag{72}$$

$$= \left.\frac{\partial F_x}{\partial x}\right|_P + \left.\frac{\partial F_y}{\partial y}\right|_P + \left.\frac{\partial F_z}{\partial z}\right|_P. \tag{73}$$

Using the same procedure, the divergence in cylindrical and spherical coordinate system can also be found. For cylindrical coordinate system, we consider instead a volume, which is surrounded by $\rho =$ constant, $\phi =$

constant, and z = constant surfaces. The divergence of $\boldsymbol{F} = F_\rho \hat{\boldsymbol{e}}_\rho + F_\phi \hat{\boldsymbol{e}}_\phi + F_z \hat{\boldsymbol{e}}_z$ in cylindrical coordinates is:

$$\nabla \cdot \boldsymbol{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}. \tag{74}$$

and the divergence of $\boldsymbol{F} = F_r \hat{\boldsymbol{e}}_r + F_\theta \hat{\boldsymbol{e}}_\theta + F_\phi \hat{\boldsymbol{e}}_\phi$ in spherical coordinates is:

$$\nabla \cdot \boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}. \tag{75}$$

VIII. GAUSS' DIVERGENCE THEOREM

Consider a finite region V, which is bounded by the surface S. Let us divide this region into N volume elements. Let us label the volume of the i^{th} volume element is δV_i and the surface bounding the volume element is δS_i. Suppose we also choose a point $\boldsymbol{r}_i$ inside the i^{th} volume element. The divergence of this volume element is then:

$$\nabla \cdot \boldsymbol{F}(\boldsymbol{r}_i) = \lim_{\delta V_i \to 0} \frac{1}{\delta V_i} \oint_{\delta S_i} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dS \tag{76}$$

$$\simeq \frac{1}{\delta V_i} \oint_{\delta S_i} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dS. \tag{77}$$

Now we can sum this over all volume elements:

$$\sum_{i=1}^{N} \nabla \cdot \boldsymbol{F}(\boldsymbol{r}_i) \delta V_i \simeq \sum_{i=1}^{N} \oint_{\delta S_i} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dS. \tag{78}$$

The sum on the RHS contains contributions from both 'interior' and 'exterior' surfaces. However the contributions from the 'interior' surfaces cancel with each other since the outward unit normal point in opposite direction for an 'interior' surface adjacent to each other. Thus we have:

$$\sum_{i=1}^{N} \nabla \cdot \boldsymbol{F}(\boldsymbol{r}_i) \delta V_i \simeq \oint_{S} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dS. \tag{79}$$

As we take the limit $N \to \infty$, the approximation becomes equality and the LHS becomes a volume integral and we arrive at the Gauss' divergence theorem:

$$\int_V \nabla \cdot \boldsymbol{F}\, dV = \oint_S \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dS. \tag{80}$$

From the equation above we can insert $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\mathbf{e}$, where $\mathbf{e}$ is an arbitrary constant vector to get another form of Gauss' theorem for a scalar field $f(\mathbf{r})$:

$$\int_V \nabla f\, dV = \oint_S f \hat{\boldsymbol{n}}\, dS. \tag{81}$$

Continuity equation. Suppose that the fluid has mass density $\rho(\boldsymbol{r}, t)$ and velocity $\boldsymbol{u}(\boldsymbol{r}, t)$. Let us consider an arbitrary and fixed volume V in space. The mass of the fluid insider this volume at time t is given by:

$$m_V(t) = \int_V \rho(\boldsymbol{r}, t)\, dV. \tag{82}$$

Now the rate of increase of this fluid mass is:

$$\frac{dm_V}{dt} = \int_V \frac{\partial \rho}{\partial t}\, dV. \tag{83}$$

We also know the volume flow rate of the fluid passing a surface element δS is: $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} \delta S$. Hence the rate of the fluid mass passing this surface is $\rho \boldsymbol{u} \cdot \hat{\boldsymbol{n}} \delta S$. Therefore the rate of fluid mass flowing out of the volume V is:

$$\oint_{\partial V} \rho \boldsymbol{u} \cdot \hat{\boldsymbol{n}}\, dS = -\frac{dm_V}{dt}. \tag{84}$$

This rate must be equal to negative $\frac{dm_V}{dt}$. Using Gauss' divergence theorem we then have:

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u})\right] dV = 0. \tag{85}$$

Since the volume V is arbitrary, and assuming the integrand to be a continuous function, the integrand must then be the zero function and thus we obtain the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \tag{86}$$

IX. CURL OF A VECTOR FIELD

Let $\boldsymbol{F}(\boldsymbol{r})$ be some vector field defined over some region. Consider a small closed curve C inside this region. Let's define A to be the area enclosed by this curve. Let us also define P to be a point on A and define the unit vector $\hat{\boldsymbol{n}}$ to be perpendicular to A with direction determined by the right-hand screw rule. The curl of $\boldsymbol{F}$ at the point P is defined to be:

$$(\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}} = \lim_{A \to 0} \frac{1}{A} \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}. \tag{87}$$

Taking a rectangular shape for the curve C, we obtain $\nabla \times \boldsymbol{F}$ in the Cartesian coordinates:

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}. \tag{88}$$

We can also find $\nabla \times \boldsymbol{F}$ in cylindrical and spherical coordinates. For example, in cylindrical coordinates, to find the z-component of $\nabla \times \boldsymbol{F}$, we should consider a closed contour C, whose curve segments are defined by ρ = constant and ϕ = constant. The result is:

$$\nabla \times \boldsymbol{F} = \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{\boldsymbol{e}}_\rho + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\hat{\boldsymbol{e}}_\phi + \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\phi) - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi}\right)\hat{\boldsymbol{e}}_z. \tag{89}$$

In spherical coordinates, this is:

$$\nabla \times \boldsymbol{F} = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\phi) - \frac{1}{r\sin\theta}\frac{\partial F_\theta}{\partial\phi} \right) \hat{\boldsymbol{e}}_r + \left(\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rF_\phi) \right) \hat{\boldsymbol{e}}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) - \frac{1}{r}\frac{\partial F_r}{\partial\theta} \right) \hat{\boldsymbol{e}}_\phi. \tag{90}$$

X. STOKES' THEOREM

Consider a finite open surface S (which can be curved) with perimeter C. Let us divide this surface into N surface elements, each with area δA_i, perimeter δC_i, and unit normal $\hat{\boldsymbol{n}}_i$ in the right hand screw rule direction. Let us also define $\boldsymbol{r}_i$ to be a point on the i^{th} surface element. The curl of a vector field $\boldsymbol{F}(\boldsymbol{r})$ at $\boldsymbol{r}_i$ is then approximately given by:

$$[\nabla \times \boldsymbol{F}(\boldsymbol{r}_i)] \cdot \hat{\boldsymbol{n}}_i \simeq \frac{1}{\delta A_i} \oint_{\delta C_i} \boldsymbol{F} \cdot d\boldsymbol{r}. \tag{91}$$

Now summing over all surface elements, we then obtain:

$$\sum_{i=1}^{N} [\nabla \times \boldsymbol{F}(\boldsymbol{r}_i)] \cdot \hat{\boldsymbol{n}}_i \delta A_i \simeq \sum_{i=1}^{N} \oint_{\delta C_i} \boldsymbol{F} \cdot d\boldsymbol{r}. \tag{92}$$

The summation on the RHS consists of contribution from 'interior' and 'exterior' edges. However for two adjacent perimeters, the line integrals over the shared edge cancel since they traverse in opposite direction. Thus we get:

$$\sum_{i=1}^{N} [\nabla \times \boldsymbol{F}(\boldsymbol{r}_i)] \cdot \hat{\boldsymbol{n}}_i \delta A_i \simeq \oint_{C} \boldsymbol{F} \cdot d\boldsymbol{r}. \tag{93}$$

Finally, taking the limit $N \to \infty$, this approximation becomes exact and we get the Stokes' theorem:

$$\int_S (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\, dS = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}. \tag{94}$$

Irrotational flow. In fluid mechanics, the curl of the fluid velocity is called the vorticity and it is a measure how much the fluid is rotating at that point. The fluid flow is said to be irrotational if: $\nabla \times \boldsymbol{u} = 0$. The following three statements are equivalent:

1. A vector field $\boldsymbol{F}(\boldsymbol{r})$ is irrotational, *i.e.* $\nabla \times \boldsymbol{F} = 0$.
2. The line integral of $\boldsymbol{F}$ around any closed path C is zero.
3. The line integral of $\boldsymbol{F}$ between two points in space is independent of the path taken.

Theorem. If $\boldsymbol{F}(\boldsymbol{r})$ is irrotational, then there exists a scalar field $\phi(\boldsymbol{r})$ such that $\boldsymbol{F} = \nabla\phi$.

Proof. Let us choose a fixed reference point O. Since $\boldsymbol{F}$ is irrotational, the line integral of $\boldsymbol{F}$ from O to P is independent of the path taken. Let's denote the value this integral to be:

$$\phi(P) - \phi(O) = \int_{OP} \boldsymbol{F} \cdot d\boldsymbol{r}. \tag{95}$$

To evaluate this line integral, we usually parameterise this path in terms of a parameter s:

$$x(s), \quad y(s), \quad z(s). \tag{96}$$

The parameter s is usually chosen to be the distance along the path such that $s = 0$ at O and $s = s_1$ at P so that the line integral becomes:

$$\phi(s_1) - \phi(0) = \int_0^{s_1} \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{ds}\, ds. \tag{97}$$

Now the Fundamental Theorem of Calculus states that if:

$$G(s_1) - G(0) = \int_0^{s_1} g(s)\, ds, \tag{98}$$

then we must have

$$g(s) = \frac{dG}{ds}, \quad \text{for all } s. \tag{99}$$

Therefore in our example, we must have:

$$\boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{ds} = \frac{d\phi}{ds} \tag{100}$$

$$= \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} \tag{101}$$

$$= \nabla\phi \cdot \frac{d\boldsymbol{r}}{ds}. \tag{102}$$

Therefore

$$(\boldsymbol{F} - \nabla\phi) \cdot \underbrace{\frac{d\boldsymbol{r}}{ds}}_{\hat{\boldsymbol{t}}} = 0. \tag{103}$$

Note that $\frac{d\boldsymbol{r}}{ds} = \hat{\boldsymbol{t}}$ is a unit vector, tangent to the path at the point under consideration. Therefore the above statement implies $\boldsymbol{F} - \nabla\phi = 0$ or $\boldsymbol{F} - \nabla\phi$ is perpendicular to $\hat{\boldsymbol{t}}$. However since the path we choose is arbitrary, the first conclusion is the correct one.

Inviscid fluid. A fluid with zero viscosity is said to be inviscid. If the flow of an inviscid fluid is found to be irrotational at a particular instant, it can be shown that the flow will remain irrotational for all subsequent times (Kelvin's theorem).

XI. SOME USEFUL VECTOR IDENTITIES

For any scalar fields f and g, and vector fields $\boldsymbol{F}$ and $\boldsymbol{G}$, we have the following identities:

$$\nabla(fg) = f\nabla g + g\nabla f \tag{104}$$
$$\nabla \cdot (f\boldsymbol{F}) = f\nabla \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \nabla f \tag{105}$$
$$\nabla \times (f\boldsymbol{F}) = f\nabla \times \boldsymbol{F} - \boldsymbol{F} \times \nabla f \tag{106}$$
$$\begin{aligned}\nabla(\boldsymbol{F} \cdot \boldsymbol{G}) &= (\boldsymbol{G} \cdot \nabla)\boldsymbol{F} + (\boldsymbol{F} \cdot \nabla)\boldsymbol{G} \\ &\quad + \boldsymbol{G} \times (\nabla \times \boldsymbol{F}) + \boldsymbol{F} \times (\nabla \times \boldsymbol{G})\end{aligned} \tag{107}$$
$$\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot (\nabla \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G}) \tag{108}$$
$$\begin{aligned}\nabla \times (\boldsymbol{F} \times \boldsymbol{G}) &= \boldsymbol{F}\nabla \cdot \boldsymbol{G} - \boldsymbol{G}\nabla \cdot \boldsymbol{F} \\ &\quad + (\boldsymbol{G} \cdot \nabla)\boldsymbol{F} - (\boldsymbol{F} \cdot \nabla)\boldsymbol{G}\end{aligned} \tag{109}$$
$$\nabla \cdot (\nabla f) = \nabla^2 f \tag{110}$$
$$\nabla \times (\nabla f) = \mathbf{0} \tag{111}$$
$$\nabla \cdot (\nabla \times \boldsymbol{F}) = 0 \tag{112}$$
$$\nabla \times (\nabla \times \boldsymbol{F}) = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F} \tag{113}$$

Laplacian in cylindrical and spherical coordinates. Recall the gradient in cylindrical coordinates:

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{e}}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{e}}_\phi + \frac{\partial f}{\partial z}\hat{\boldsymbol{e}}_z, \tag{114}$$

and the divergence in cylindrical coordinates:

$$\nabla \cdot \boldsymbol{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}. \tag{115}$$

Combining the two, we get Laplacian in cylindrical coordinates:

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{116}$$

Similarly, the Laplacian in the spherical coordinates:

$$\begin{aligned}\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) \\ &\quad + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}.\end{aligned} \tag{117}$$

Green's First Theorem. For a region V, bounded by the surface S, we have:

$$\int_V \left(f\nabla^2 g + \nabla f \cdot \nabla g\right) dV = \oint_S f\nabla g \cdot \hat{\boldsymbol{n}}\, dS. \tag{118}$$

Proof. We start from the Gauss' divergence theorem:

$$\int_V \nabla \cdot \boldsymbol{F}\, dV = \oint_S \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dS. \tag{119}$$

Putting $\boldsymbol{F} = f\nabla g$, we then get:

$$\oint_S f\nabla g \cdot \hat{\boldsymbol{n}}\, dS = \int_V \nabla \cdot (f\nabla g)\, dV \tag{120}$$
$$= \int_V \left[f\nabla \cdot (\nabla g) + \nabla g \cdot \nabla f\right] dV, \tag{121}$$

which is the one we want to prove.

Green's Second Theorem. For a region V, bounded by the surface S, we have:

$$\int_V \left(f\nabla^2 g - g\nabla^2 f\right) dV = \oint_S (f\nabla g - g\nabla f) \cdot \hat{\boldsymbol{n}}\, dS. \tag{122}$$

Green's Theorem in the Plane. For a surface S on an x-y plane, with perimeter C, we have:

$$\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_C \left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right) dt. \tag{123}$$

Proof. Let us define a vector field $\boldsymbol{F} = P(x, y)\hat{\boldsymbol{\imath}} + Q(x, y)\hat{\boldsymbol{\jmath}}$. Applying Stokes theorem on $\boldsymbol{F}$, we get:

$$\int_S \nabla \times \boldsymbol{F} \cdot \hat{\boldsymbol{k}}\, dS = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r} \tag{124}$$
$$\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dS = \oint_C (Pdx + Qdy). \tag{125}$$