L3: Maths Workshop - Integral Transform

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This lecture note is part of Level 3 Mathematics Workshop module for physics students. The whole module consists of:

1. Complex Analysis
2. Infinite Dimensional Vector Spaces
3. Calculus of Variations and Infinite Series
4. Integral Transforms (this lecture note only covers this part!!)

This lecture will cover Integral Transforms, *i.e.* Fourier and Laplace transform. There might be some typos or mistakes in this note (please let me know if you spot any typos/mistakes). This lecture note assumes knowledge of Complex Analysis (in particular, contour integral).

Contents

Chapter 1

Fourier transform

1.1 Review of complex analysis

Before proceeding, we will review the Jordan's lemma and the residue theorem again (covered in complex analysis).

Theorem (Residue theorem). Consider some complex function $f(z) : \mathbb{C} \to \mathbb{C}$. Suppose that $f(z)$ has a finite number of singularities: $z_1, z_2, \ldots, z_N$ inside some closed anti-clockwise contour γ. Then,

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{i=1}^{N} \operatorname*{res}_{z=z_i} f(z) \tag{1.1}$$

(see Fig. 1.1).

Theorem (Cauchy's theorem). If there is no singularity inside the closed contour γ, then:

$$\oint_\gamma f(z)\, dz = 0. \tag{1.2}$$

Lemma (Jordan's lemma). Consider some complex function $f(z) : \mathbb{C} \to \mathbb{C}$ which decays to zero as z goes to infinity:

$$f(z) \to 0 \quad \text{, as } |z| \to \infty. \tag{1.3}$$

Moreover, $f(z)$ is analytic inside the domain $\mathbb{C}$, except for a finite number of singularities. Now let us define γ_R^+ to be a semi-circular contour in the upper half complex plane and define γ_R^- to be a semi-circular contour in the lower half comples plane (see Fig. 1.2). Then the follwing integrals vanish in the limit $R \to \infty$:

$$\int_{\gamma_R^+} f(z) e^{i\lambda z}\, dz \to 0 \quad \text{, if } \lambda > 0 \tag{1.4}$$

$$\int_{\gamma_R^-} f(z) e^{i\lambda z}\, dz \to 0 \quad \text{, if } \lambda < 0. \tag{1.5}$$

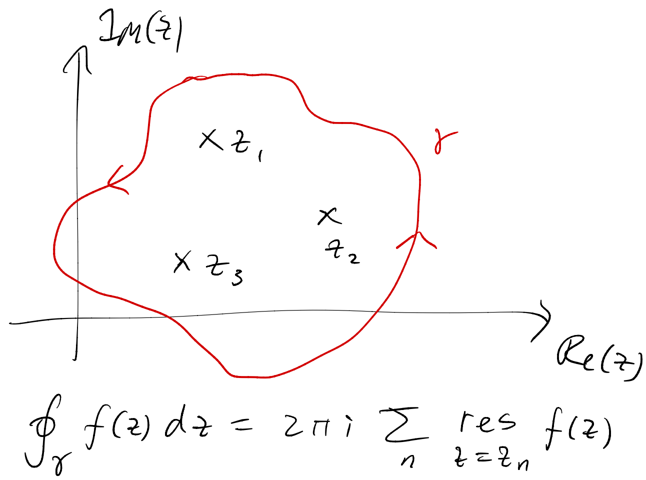

Figure 1.1: Residue theorem: the integral $\oint_\gamma f(z)\,dz$ over some closed contour is equal to the sum of all residues of the singularities $z_1, z_2, \ldots, z_N$ inside γ.

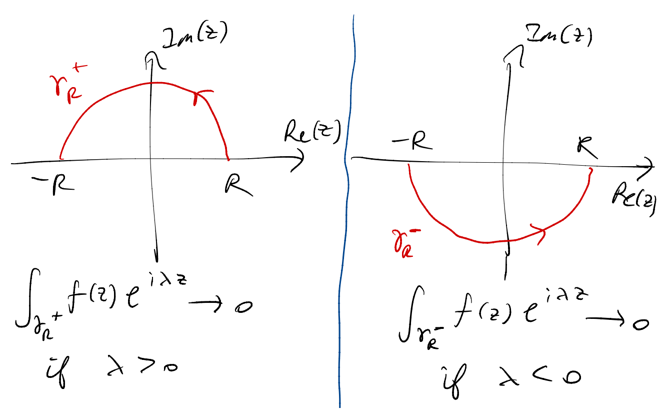

Figure 1.2: Jordan's lemma: the integral $\int_{\gamma_R^\pm} f(z)e^{i\lambda z}\,dz$ vanishes in the limit of $R \to \infty$, depending on the sign of λ.

1.2 Fourier transform

Definition (Square integrable). A function $f(x) : \mathbb{R} \to \mathbb{C}$ or $\mathbb{R}$ is called square-integrable if the following integral is finite:

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, dx = \text{finite}. \tag{1.6}$$

Definition (Fourier transform). Consider a function $f(x) : \mathbb{R} \to \mathbb{C}$ or $\mathbb{R}$, which is square-integrable and:

$$f(x) \to 0 \quad , \text{as } x \to \pm\infty. \tag{1.7}$$

The Fourier transform of $f(x)$ is another function $\tilde{f}(k) : \mathbb{R} \to \mathbb{C}$ or $\mathbb{R}$, which is defined to be:

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} \, dx. \tag{1.8}$$

Conversely, the inverse Fourier transform is defined to be:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{f}(k) e^{ikx} \, dk. \tag{1.9}$$

Note that $f(x)$ is a function of x and $\tilde{f}(k)$ is a function of k. In physics, the variable x and k have different physical meanings as we will see in the example below.

Example. Wavefunction in QM: $\psi(x) : \mathbb{R} \to \mathbb{C}$. ψ is complex and x is real. x indicates coordinate in space and the modulus square of the wavefunction $|\psi(x)|^2$ gives the probability. Consequently, $\psi(x)$ is square-integrable since the integral over the probability distribution should be equal to 1:

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 \, dx = 1. \tag{1.10}$$

The Fourier transform of $\psi(x)$ is $\tilde{\psi}(k)$, where k is the wavevector. We can then write $\psi(x)$ as (using the formula in (1.9)):

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\psi}(k) e^{ikx} \, dk. \tag{1.11}$$

In the above e^{ikx} is a plane wave with wavelength $2\pi/k$. So in the equation above, we are basically decomposing the wavefunction $\psi(x)$ into a sum of all plane waves e^{ikx} (with different k's) and $\tilde{\psi}(k)$ is the amplitude for each plane wave.

Example. The sound wave consists of compression and decompression of air. The sound wave detected on the microphone at time t is given by: $P(t) : \mathbb{R} \to \mathbb{R}$. Here P is the pressure, relative to the atmospheric pressure. The Fourier transform of $P(t)$ is $\tilde{P}(\omega)$, where ω is the angular frequency:

$$P(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{P}(\omega) e^{i\omega t} \, dt. \tag{1.12}$$

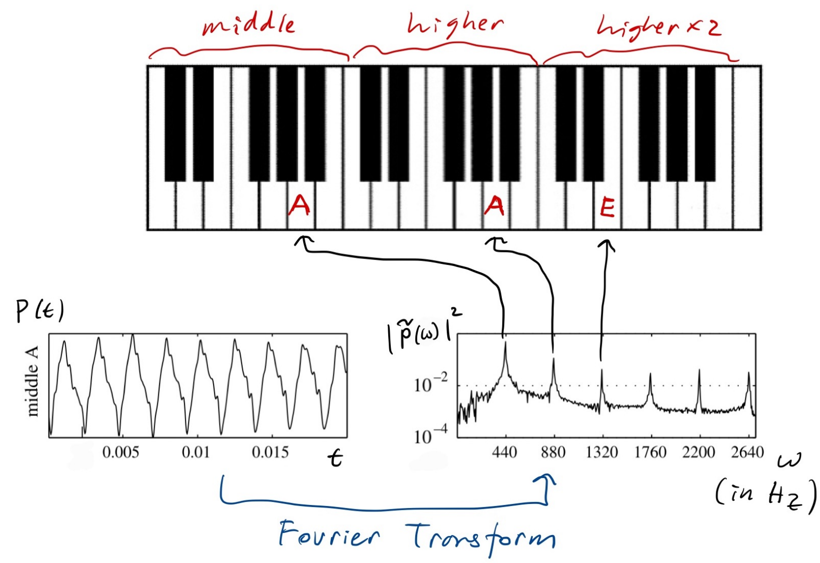

Figure 1.3: Sound wave from the middle A from the piano (bottom left). Bottom right panel is the Fourier transform of the sound wave (modulus squared), which gives the freqeuncy spectrum of the sound wave. [Plots adapted from M. R. Petersen, *The Math. Assoc. of America* (1995)]

$P(t)$ is real but $\tilde{P}(\omega)$ is complex in general. Fig. 1.3 bottom left shows the sound wave from the middle A from the piano. Fig. 1.3 bottom right shows its Fourier transform (modulus squared) $|\tilde{P}(\omega)|^2$ as a function of angular frequency ω (in units of Hz). The Fourier transform $|\tilde{P}(\omega)|^2$ tells us the frequency spectrum of this particular piano note. $|\tilde{P}(\omega)|^2$ shows a peak at $\omega \simeq 440\,\mathrm{Hz}$. This corresponds to the fundamental frequency of the middle A indeed. However, $|\tilde{P}(\omega)|^2$ also shows several other peaks at $\omega \simeq 880\,\mathrm{Hz}$, $1320\,\mathrm{Hz}$, $1760\,\mathrm{Hz}$ and so on. The peak at $\omega \simeq 880\,\mathrm{Hz}$ corresponds to the note of high A. The peak at $\omega \simeq 1320\,\mathrm{Hz}$ corresponds to the note of high E. So the middle A note contains a small percentage of E note. Actually Bach *et al.* realized that when note A and note E are played together, they sound nice. This is called harmony in music theory.

Notation and convention. The Fourier transform can also be denoted as:

$$\mathcal{F}[f](k) = \tilde{f}(k) \tag{1.13}$$

$$\mathcal{F}^{-1}[\tilde{f}](x) = f(x). \tag{1.14}$$

Sometimes, Fourier and inverse Fourier is defined with normalization factor $1/2\pi$ such as:

$$f(x) = \int_{-\infty}^{+\infty} \tilde{f}(k) e^{ikx} dk \quad \text{and} \quad \tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx. \tag{1.15}$$

Example. Useful formula relating to Gaussian integral:

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}. \tag{1.16}$$

Proof. We consider the 2-dimensional integral instead (where $\mathbf{r} = (x, y)^T$):

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, e^{-\frac{1}{2}(x^2+y^2)} = \int_{\mathbb{R}^2} d^2r \, e^{-\frac{1}{2}r^2}. \tag{1.17}$$

Next we use polar coordinates (where the area element is $d^2r = r\,dr\,d\theta$):

$$\begin{aligned} \int_{\mathbb{R}^2} d^2r \, e^{-\frac{1}{2}r^2} &= \int_0^{2\pi} d\theta \int_0^{\infty} dr \, re^{-\frac{1}{2}r^2} && (1.18)\\ &= 2\pi \int_0^{\infty} ds \, e^{-s} && (1.19)\\ &= 2\pi \left[-e^{-s}\right]_0^{\infty} && (1.20)\\ &= 2\pi. && (1.21) \end{aligned}$$

Therefore we prove (1.16).

1.2.1 Fourier transform of a Gaussian function

What is the Fourier transform $\tilde{f}(k)$ of the Gaussian function:

$$f(x) = e^{-\frac{1}{2}x^2}. \tag{1.22}$$

Using the formula in (1.8), the Fourier transform is:

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2} e^{-ikx} \, dx. \tag{1.23}$$

First, we need to complete the square of the exponent:

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2+2ikx)} \, dx && (1.24)\\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x+ik)^2} e^{-\frac{1}{2}k^2} \, dx && (1.25)\\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2} \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x+ik)^2} \, dx}_{I}. && (1.26) \end{aligned}$$

So what remains is to compute the integral:

$$I = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x+ik)^2} \, dx. \tag{1.27}$$

We cannot use the formula (1.16) directly since there is a complex number ik involved. Instead we have to promote x (real) into z (complex) and consider the following complex integration instead:

$$I_c = \oint_\gamma e^{-\frac{1}{2}(z+ik)^2}\, dz. \tag{1.28}$$

The integration is taken over the closed rectangular contour γ, which can be divided into 4 segments $\gamma = \gamma_u + \gamma_d + \gamma_l + \gamma_r$ (see Fig. 1.4). Later we are going to take the limit $R \to \infty$ in the contour γ. So the integral I_c can be written as a sum of 4 integrals over these line segments:

$$0 = I_c = \underbrace{\int_{\gamma_u} e^{-\frac{1}{2}(z+ik)^2}\, dz}_{-I} + \int_{\gamma_d} e^{-\frac{1}{2}(z+ik)^2}\, dz + \int_{\gamma_l} e^{-\frac{1}{2}(z+ik)^2}\, dz + \int_{\gamma_r} e^{-\frac{1}{2}(z+ik)^2}\, dz. \tag{1.29}$$

Also note that $I_c = 0$, since the closed contour γ does not contain any singularity. First, the integral over γ_u is just $-I$ (in the limit of $R \to \infty$), which is what we are looking for. Next the integral over γ_d is (note that $z = x - ik$ along γ_d):

$$\int_{\gamma_d} e^{-\frac{1}{2}(z+ik)^2} dz = \int_{-R}^{+R} e^{-\frac{1}{2}(x-ik+ik)^2} dx \tag{1.30}$$

$$= \sqrt{2\pi} \quad , \text{as } R \to \infty, \tag{1.31}$$

where we have used the formula in (1.16). Then the integral over γ_l is (note that $z = -R + iy$ along γ_l):

$$\int_{\gamma_l} e^{-\frac{1}{2}(z+ik)^2} dz = \int_{-k}^{0} e^{-\frac{1}{2}(-R+iy-ik)^2} dy \tag{1.32}$$

$$= \int_{-k}^{0} e^{-\frac{1}{2}[R^2+2Ri(k-y)-(k-y)^2]} dy \tag{1.33}$$

$$\to 0 \quad , \text{as } R \to \infty. \tag{1.34}$$

Therefore, (1.29) becomes:

$$0 = -I + \sqrt{2\pi} + 0 + 0, \tag{1.35}$$

so $I = \sqrt{2\pi}$. Substituting this into (1.29), we finally get:

$$\tilde{f}(k) = e^{-\frac{1}{2}k^2}. \tag{1.36}$$

Therefore, the Fourier transform of a Gaussian is also a Gaussian.

Definition (Fourier transform in higher dimension). We can easily extend the definitions in (1.8-1.9) to higher dimension:

$$\tilde{f}(\mathbf{k}) = \frac{1}{(2\pi)^{d/2}} \int d^d r\, f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{1.37}$$

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{d/2}} \int d^d k\, \tilde{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{1.38}$$

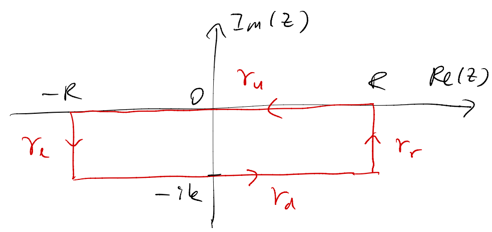

Figure 1.4: Rectangular contour for the integral $\oint_\gamma e^{-\frac{1}{2}(z+ik)^2}\,dz$.

1.3 Dirac delta function

Definition (Dirac delta function). The Dirac delta function $\delta(x)$ is defined such that:

$$\delta(x) = \begin{cases} \infty & \text{, if } x = 0 \\ 0 & \text{, if } x \neq 0 \end{cases}. \tag{1.39}$$

$$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1 \tag{1.40}$$

$$\int_{-\infty}^{+\infty} \delta(x-a) f(x)\,dx = f(a), \tag{1.41}$$

for any function $f(x)$, continuous at $x = a$. The last property of the Delta function above is also called the 'sieving' property. In fact, it is sufficient to have:

$$\int_{a-\epsilon}^{a+\epsilon} \delta(x-a) f(x)\,dx = f(a), \tag{1.42}$$

for any finite $\epsilon > 0$.

The Dirac delta function can also be thought of as the limit of a Gaussian function (normalized to 1) becoming narrower and narrower (see Fig. 1.5):

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}. \tag{1.43}$$

Another popular limiting definition of the delta function is:

$$\delta(x) = \lim_{a \to \infty} \frac{a}{\pi} \frac{1}{1 + a^2 x^2}. \tag{1.44}$$

1.3.1 Basic properties of delta function

Some basic properties of the delta function:

$$\delta(-x) = \delta(x) \tag{1.45}$$

$$\delta(ax) = \frac{1}{|a|}\delta(x). \tag{1.46}$$

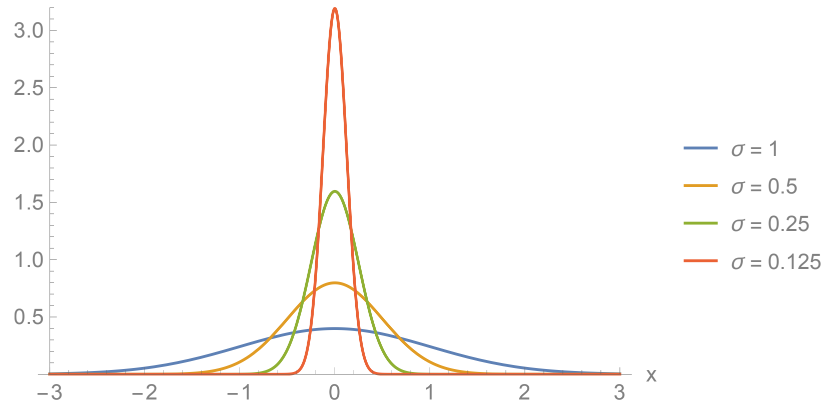

Figure 1.5: Dirac delta function as the limit $\lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$.

To prove (1.45), we can just look at equation (1.43). To prove (1.46), we first consider the case of $a > 0$, and we can write:

$$\int_{-\infty}^{+\infty} \delta(ax) f(x)\, dx = \int_{-\infty}^{+\infty} \delta(y) f\left(\frac{y}{a}\right) \frac{dy}{a} \tag{1.47}$$

$$= \frac{1}{a} f(0). \tag{1.48}$$

Next we consider the case of $a < 0$, and then following the same argument as above, we get:

$$\int_{-\infty}^{+\infty} \delta(ax) f(x)\, dx = \int_{+\infty}^{-\infty} \delta(y) f\left(\frac{y}{a}\right) \frac{dy}{a} \tag{1.49}$$

$$= \int_{-\infty}^{+\infty} \delta(x) f\left(\frac{x}{a}\right) \frac{dx}{-a} \tag{1.50}$$

$$= \frac{1}{-a} f(0). \tag{1.51}$$

Therefore, $\delta(ax) = \delta(x)/|a|$.

1.3.2 Derivatives of delta function

The delta function is differentiable and we call its derivative $\delta'(x)$. Some important property relating to $\delta'(x)$:

$$\int_{-\infty}^{+\infty} \delta'(x-a) f(x)\, dx = -f'(a). \tag{1.52}$$

Proof. We use integration by parts:

$$\int_{-\infty}^{+\infty} \delta'(x-a) f(x)\, dx = [\delta(x-a)f(x)]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \delta(x-a) f'(x)\, dx \tag{1.53}$$

$$= -f'(a). \tag{1.54}$$

We can also generalize equation (1.52) to the n^{th}-derivative of the delta function, $\delta^{(n)}(x)$:

$$\int_{-\infty}^{+\infty} \delta^{(n)}(x-a) f(x)\, dx = (-1)^n f^{(n)}(a). \tag{1.55}$$

1.3.3 Integrals of delta function

Definition (Heaviside step function). The Heaviside step function is defined to be:

$$\Theta(x) = \begin{cases} 1 & \text{, if } x \geq 0 \\ 0 & \text{, if } x < 0 \end{cases}. \tag{1.56}$$

The derivative of the Heaviside function is the delta function:

$$\Theta'(x) = \delta(x). \tag{1.57}$$

Proof. Consider:

$$\int_{-\infty}^{+\infty} \Theta'(x-a) f(x)\, dx = [\Theta(x-a)f(x)]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \Theta(x-a) f'(x)\, dx \tag{1.58}$$

$$= f(+\infty) - \int_{a}^{+\infty} 1 \times f'(x)\, dx - \int_{-\infty}^{a} 0 \times f'(x)\, dx \tag{1.59}$$

$$= f(+\infty) - [f(x)]_a^{+\infty} \tag{1.60}$$

$$= f(+\infty) - f(+\infty) + f(a) \tag{1.61}$$

$$= f(a). \tag{1.62}$$

Therefore, $\Theta'(x) = \delta(x)$.

1.3.4 Delta of some function $\delta(g(x))$

Let's consider the delta function of some polynomial, for example, $\delta(x^2 - a^2)$. It can be shown that this delta function can be broken up into simpler forms as follows:

$$\delta(x^2 - a^2) = \frac{1}{2|a|} \left[\delta(x+a) + \delta(x-a)\right]. \tag{1.63}$$

Proof. First consider the case of $a > 0$. Let's consider $\Theta(x^2 - a^2)$:

$$\Theta(x^2 - a^2) = \begin{cases} 1 & \text{, if } x > a \\ 0 & \text{, if } -a < x < a \\ 1 & \text{, if } x < -a \end{cases}. \tag{1.64}$$

This can also be written as:

$$\Theta(x^2 - a^2) = 1 - \Theta(x + a) + \Theta(x - a). \tag{1.65}$$

Now we differentiate:

$$\delta(x^2 - a^2)2x = -\delta(x + a) + \delta(x - a). \tag{1.66}$$

$$\delta(x^2 - a^2) = -\frac{\delta(x + a)}{2x} + \frac{\delta(x - a)}{2x} \tag{1.67}$$

Now let's consider:

$$\int_{-\infty}^{+\infty} \delta(x^2 - a^2) f(x)\, dx = -\int_{-\infty}^{+\infty} \delta(x + a)\frac{f(x)}{2x}\, dx + \int_{-\infty}^{+\infty} \delta(x - a)\frac{f(x)}{2x}\, dx \tag{1.68}$$

$$= \frac{1}{2a}\left[f(-a) + f(a)\right]. \tag{1.69}$$

Therefore,

$$\delta(x^2 - a^2) = \frac{1}{2a}\left[\delta(x - a) + \delta(x + a)\right]. \tag{1.70}$$

Next we need to consider the case for $a < 0$ to finally prove (1.63).

We can also generalize the result in (1.63). Suppose that $g(x)$ is some polynomial with roots $x_1, x_2, \ldots, x_N$. Then,

$$\delta(g(x)) = \sum_{i=1}^{N} \frac{\delta(x - x_i)}{|g'(x_i)|}. \tag{1.71}$$

Proof. Can you try to prove this? (Challenging.)

1.4 Fourier transform from Fourier series

In this Section, we will derive the Fourier transform from the Fourier series (heurestically). Let us consider some function $f(x)$, where x is real and defined on some domain $x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$. Here, f can be real or complex. Then $f(x)$ can be expanded in a complex Fourier series as:

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i\frac{2\pi n}{L}x}, \tag{1.72}$$

where:

$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-i\frac{2\pi n}{L}x}\, dx. \tag{1.73}$$

Note that if we expand $e^{i\frac{2\pi n}{L}x} = \cos\left(\frac{2\pi n}{L}x\right) + i\sin\left(\frac{2\pi n}{L}x\right)$ in (1.72), and assuming f to be real, we get the usual Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n}{L}x\right) + b_n \sin\left(\frac{2\pi n}{L}x\right)\right]. \tag{1.74}$$

The Fourier transform can be thought as the limit of $L \to \infty$ in equation (1.72). Defining $k = \frac{2\pi n}{L}$, equation (1.72) can be written as:

$$f(x) = \sum_n \Delta n\, c_n e^{ikx}. \tag{1.75}$$

Note that $\Delta n = 1$. Now from $k = \frac{2\pi n}{L}$, we have $\Delta k = \frac{2\pi}{L}\Delta n$. Substituting this to the above equation, we get:

$$f(x) = \frac{L}{2\pi}\sum_k \Delta k\, c_n e^{ikx}. \tag{1.76}$$

Now we subsitute (1.73) into the above equation to get:

$$f(x) = \frac{L}{2\pi}\sum_k \Delta k \left(\frac{1}{L}\int_{-L/2}^{L/2} f(x')e^{-ikx'}\,dx'\right) e^{ikx} \tag{1.77}$$

$$= \frac{1}{\sqrt{2\pi}}\sum_k \Delta k \left(\frac{1}{\sqrt{2\pi}}\int_{-L/2}^{L/2} f(x')e^{-ikx'}\,dx'\right) e^{ikx}. \tag{1.78}$$

Finally, taking the limit $L \to \infty$, the summation over k becomes a Riemann integral so we get:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dk\, \underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')e^{-ikx'}\,dx'\right)}_{\tilde{f}(k)} e^{ikx}, \tag{1.79}$$

where we have identified the term inside the bracket to be the Fourier transform $\tilde{f}(k)$. Now let us rearrange equation (1.79) slightly to get:

$$f(x) = \int_{-\infty}^{+\infty} dx' \left[f(x')\underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{ik(x-x')}\right)}_{\delta(x-x')}\right]. \tag{1.80}$$

Now the right hand side of the above equation has to be equal to $f(x)$ and therefore the term inside the parentheses $(\dots)$ must be a delta function! So we derive the following important formula:

Orhogonality and completeness relation.

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{ik(x-x')} = \delta(x-x') \tag{1.81}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\, e^{ix(k-k')} = \delta(k-k'). \tag{1.82}$$

1.5 Properties of Fourier transform

Now we will go through some basic properties of Fourier transform.

1.5.1 Linearities

For any functions $f(x)$ and $g(x)$ and numbers a and b, we have:

$$\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)] \tag{1.83}$$

$$\mathcal{F}^{-1}[\mathcal{F}[f(x)]] = \mathcal{F}[\mathcal{F}^{-1}[f(x)]] = f(x). \tag{1.84}$$

1.5.2 Derivatives

The Fourier transform of the derivative $\frac{df}{dx}$ is:

$$\mathcal{F}\left[\frac{df}{dx}\right] = ik\mathcal{F}[f]. \tag{1.85}$$

Proof. Using the definition of Fourier transform (1.8):

$$\begin{aligned}
\mathcal{F}\left[\frac{df}{dx}\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{df}{dx}e^{-ikx}\,dx && (1.86)\\
&= \frac{1}{\sqrt{2\pi}}\cancel{\left[f(x)e^{-ikx}\right]_{-\infty}^{+\infty}} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\frac{d}{dx}\left[e^{-ikx}\right]\,dx && (1.87)\\
&= \frac{ik}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{ikx}\,dx && (1.88)\\
&= ik\mathcal{F}[f]. && (1.89)
\end{aligned}$$

Note that $f(\pm\infty) = 0$. Similarly, we can generalize the above to n^{th} derivative:

$$\mathcal{F}\left[\frac{d^n f}{dx^n}\right] = (ik)^n\mathcal{F}[f]. \tag{1.90}$$

1.5.3 Fourier transform of $x^n f(x)$

Let's consider the Fourier transform of $xf(x)$:

$$\begin{aligned}
\mathcal{F}[xf] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)xe^{-ikx}\,dx && (1.91)\\
&= \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\frac{d}{dk}\left[e^{-ikx}\right]\,dx && (1.92)\\
&= i\frac{d}{dk}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}\,dx && (1.93)\\
&= i\frac{d}{dk}\mathcal{F}[f]. && (1.94)
\end{aligned}$$

More generally,

$$\mathcal{F}[x^n f] = \left(i\frac{d}{dk}\right)^n \mathcal{F}[f]. \tag{1.95}$$

1.5.4 Translation

Suppose we shift $f(x)$ by a. The Fourier transform of $f(x+a)$ is:

$$\mathcal{F}[f(x+a)] = e^{ika}\mathcal{F}[f(x)]. \tag{1.96}$$

Proof. Using definition of Fourier transform:

$$\begin{aligned}
\mathcal{F}[f(x+a)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x+a)e^{-ikx}\,dx &(1.97)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(y)e^{-ik(y-a)}\,dy &(1.98)\\
&= e^{ika}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(y)e^{-iky}\,dy &(1.99)\\
&= e^{ika}\mathcal{F}[f(x)]. &(1.100)
\end{aligned}$$

Similarly,

$$\mathcal{F}[e^{ik'x}f(x)](k) = \mathcal{F}[f(x)](k-k'). \tag{1.101}$$

Proof. Exercise for the readers.

1.5.5 Scaling

Let's say we scale x by a. The Fourier transform of $f(ax)$ is:

$$\mathcal{F}[f(ax)](k) = \frac{1}{a}\mathcal{F}[f(x)]\left(\frac{k}{a}\right). \tag{1.102}$$

Proof. Using definition of Fourier transform:

$$\begin{aligned}
\mathcal{F}[f(ax)](k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(ax)e^{-ikx}\,dx &(1.103)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(y)e^{-i\frac{k}{a}y}\,\frac{dy}{a} &(1.104)\\
&= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(y)e^{-i\frac{k}{a}y}\,dy &(1.105)\\
&= \frac{1}{a}\mathcal{F}[f(x)]\left(\frac{k}{a}\right). &(1.106)
\end{aligned}$$

1.5.6 Fourier transform of a real function

Suppose that $f(x)$ is real. The Fourier transform of $f(x)$ is:

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx}\, dx \tag{1.107}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \cos(kx)\, dx - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \sin(kx)\, dx \tag{1.108}$$

Here $\tilde{f}(k)$ is not necessarily real. Consider however the complex conjugate of $\tilde{f}(k)$:

$$\tilde{f}(k)^* = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \cos(kx)\, dx + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \sin(kx)\, dx \tag{1.109}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \cos(-kx)\, dx - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \sin(-kx)\, dx \tag{1.110}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i(-k)x}\, dx. \tag{1.111}$$

Therefore,

$$\tilde{f}(k)^* = \tilde{f}(-k). \tag{1.112}$$

Note that $\cos(x)$ is even and $\sin(x)$ is odd function.

Suppose that $f(x)$ is real and symmetric, then its Fourier transform is real:

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \cos(kx)\, dx. \tag{1.113}$$

Conversely, if $f(x)$ is real and antisymmetric, then its Fourier transform is imaginary:

$$\tilde{f}(k) = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \sin(kx)\, dx. \tag{1.114}$$

1.5.7 Fourier transform of delta function

The Fourier transform of delta function is a constant:

$$\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta(x) e^{-ikx}\, dx \tag{1.115}$$

$$= \frac{1}{\sqrt{2\pi}} \tag{1.116}$$

Alternatively, from the orthogonal relation (1.81), we have (putting $x' = 0$):

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x-0)}\, dk \tag{1.117}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi}}}_{\tilde{\delta}(k)} e^{ikx}\, dk. \tag{1.118}$$

1.5.8 Convolution

Definition (Convolution). The convolution between two functions $f(x)$ and $g(x)$ is defined to be:

$$(f * g)(x) = \int_{-\infty}^{+\infty} dy\, g(x-y) f(y). \tag{1.119}$$

Now let's insert the definition of inverse Fourier transforms:

$$g(x-y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, \tilde{g}(k) e^{ik(x-y)} \tag{1.120}$$

into equation (1.119) to get:

$$\begin{aligned}
(f * g)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dk\, f(y)\tilde{g}(k) e^{ik(x-y)} && (1.121)\\
&= \int_{-\infty}^{+\infty} dk\, \underbrace{\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dy\, f(y) e^{-iky} \right)}_{\tilde{f}(k)} \tilde{g}(k) e^{ikx} && (1.122)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, \underbrace{\sqrt{2\pi}\tilde{f}(k)\tilde{g}(k)}_{\widetilde{(f*g)}(k)} e^{ikx}. && (1.123)
\end{aligned}$$

Therefore,

$$\widetilde{(f * g)}(k) = \sqrt{2\pi}\tilde{f}(k)\tilde{g}(k). \tag{1.124}$$

1.5.9 Parseval's theorem

Theorem (Parseval's theorem). For any function $f(x)$ and $g(x)$, we have:

$$\int_{-\infty}^{+\infty} f(x)^* g(x)\, dx = \int_{-\infty}^{+\infty} \tilde{f}(k)^* \tilde{g}(k)\, dk. \tag{1.125}$$

Proof. We substitute the inverse Fourier transform for $f(x)$ and $g(x)$ into the left hand side of the above equation to get:

$$\int_{-\infty}^{+\infty} dx\, f(x)^* g(x) = \int_{-\infty}^{+\infty} dx \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, \tilde{f}(k)^* e^{-ikx} \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk'\, \tilde{g}(k') e^{ik'x} \right) \tag{1.126}$$

$$= \frac{1}{2\pi} \underbrace{\int_{-\infty}^{+\infty} dx}_{} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk'\, \tilde{f}(k)^* \tilde{g}(k') \underbrace{e^{-i(k-k')x}}_{2\pi\delta(k-k')}. \tag{1.127}$$

Next we perform integral over x. The exponential becomes a delta function (using orthogonality relation (1.82)). Therefore,

$$\int_{-\infty}^{+\infty} dx\, f(x)^* g(x) = \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk'\, \tilde{f}(k)^* \tilde{g}(k')\delta(k-k') \tag{1.128}$$

$$= \int_{-\infty}^{+\infty} dk\, \tilde{f}(k)^* \tilde{g}(k). \tag{1.129}$$

1.6 Applications of Fourier transform

1.6.1 Wave equation

Consider an infinite one-dimensional string along x-axis. Let's denote the tranverse displacement to be $y(x,t)$. $y(x,t)$ satisfies the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}, \tag{1.130}$$

where $v > 0$ is the speed of wave. We will try to solve this partial differential equation using 2-dimensional Fourier transform (1-dimension for space and 1-dimension for time):

$$y(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega\, \tilde{y}(k,\omega) e^{i(kx-\omega t)} \tag{1.131}$$

$$\tilde{y}(k,\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dt\, y(x,t) e^{-i(kx-\omega t)}. \tag{1.132}$$

Thus from (1.131), the wave can be described as the superposition of plane waves $\cos(kx-\omega t)$ & $\sin(kx-\omega t)$, where k is the wavevector and ω is the angular frequency. k and ω are related to the wavelength λ and period T of the plane wave *via*:

$$k = \frac{2\pi}{\lambda} \tag{1.133}$$

$$\omega = \frac{2\pi}{T}. \tag{1.134}$$

To find the relationship between k and ω, we substitute (1.131) into (1.130) to get:

$$\frac{\partial^2}{\partial t^2}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega\, \tilde{y}(k,\omega) e^{i(kx-\omega t)}\right] = v^2 \frac{\partial^2}{\partial x^2}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega\, \tilde{y}(k,\omega) e^{i(kx-\omega t)}\right] \tag{1.135}$$

$$-\omega^2\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega\, \tilde{y}(k,\omega) e^{i(kx-\omega t)}\right] = -v^2 k^2 \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega\, \tilde{y}(k,\omega) e^{i(kx-\omega t)}\right]. \tag{1.136}$$

Therefore,

$$v = \frac{\omega}{k} = \frac{\lambda}{T}. \tag{1.137}$$

1.6.2 Fourier transform of some common functions

Example. What is the Fourier transform of e^{-x^2}? We know from Sec.1.2.1, the Fourier transform of $e^{-\frac{1}{2}x^2}$ is $e^{-\frac{1}{2}k^2}$. Therefore:

$$e^{-\frac{1}{2}x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, e^{-\frac{1}{2}k^2} e^{ikx}. \tag{1.138}$$

Next let's define $y = \frac{x}{\sqrt{2}}$ to get:

$$e^{-y^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, e^{-\frac{1}{2}k^2} e^{ik\sqrt{2}y}. \tag{1.139}$$

Next we define $k' = k\sqrt{2}$ to get:

$$e^{-y^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk'\, \frac{1}{\sqrt{2}} e^{-\frac{1}{4}k'^2} e^{ik'y}. \tag{1.140}$$

Therefore, the Fourier transform of e^{-x^2} is $\frac{1}{\sqrt{2}}e^{-\frac{1}{4}k^2}$.

Example. The Lorentzian function is defined to be:

$$f(x) = e^{-|x|}. \tag{1.141}$$

The Fourier transform of Lorentzian function is:

$$\tilde{f}(k) = \sqrt{\frac{2}{\pi}} \frac{1}{k^2+1}. \tag{1.142}$$

Proof. Using the definition of Fourier transform,

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-|x|} e^{-ikx}\, dx. && (1.143)\\
&= \frac{1}{\sqrt{2\pi}} \left(\int_0^{+\infty} e^{-x} e^{-ikx}\, dx + \int_{-\infty}^{0} e^{x} e^{-ikx}\, dx \right) && (1.144)\\
&\vdots && (1.145)\\
&= \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2}. && (1.146)
\end{aligned}$$

(Can you complete the derivation?) Similarly one can show that if $f(x) = \sqrt{\frac{\pi}{2}}e^{-|x|}$, then the Fourier transform is $\tilde{f}(k) = \frac{1}{k^2+1}$.

1.6.3 Application of Parseval's theorem

Example. Calculate the integral:

$$I = \int_0^{\infty} \frac{dk}{(a^2+k^2)^2}. \tag{1.147}$$

Rearranging, we get:

$$I = \frac{1}{a^4}\int_0^\infty \frac{dk}{\left(1+\frac{k^2}{a^2}\right)^2}. \tag{1.148}$$

Now let's define: $k' = k/a$ to get:

$$\begin{aligned}
I &= \frac{1}{a^3}\int_0^\infty \frac{dk'}{(1+k'^2)^2} && (1.149)\\
&= \frac{1}{2a^3}\int_{-\infty}^{\infty} dk\, \underbrace{\frac{1}{(1+k^2)}}_{\tilde{f}(k)^*}\, \underbrace{\frac{1}{(1+k^2)}}_{\tilde{f}(k)} && (1.150)\\
&= \frac{1}{2a^3}\int_{-\infty}^{\infty} dx\, f(x)^* f(x), && (1.151)
\end{aligned}$$

using Parseval's theorem. From previous Section, we know that if $\tilde{f}(k) = \frac{1}{1+k^2}$, then $f(x) = \sqrt{\frac{\pi}{2}}e^{-|x|}$. Therefore,

$$\begin{aligned}
I &= \frac{\pi}{4a^3}\int_{-\infty}^{+\infty} dx\, e^{-2|x|} && (1.152)\\
&= \frac{\pi}{2a^3}\int_0^\infty dx\, e^{-2x} && (1.153)\\
&= \frac{\pi}{4a^3}. && (1.154)
\end{aligned}$$

1.6.4 Differential equation

Example. Let's consider:

$$\frac{d^2 f}{dx^2} - f = \delta(x). \tag{1.155}$$

The solution to this differential equation can be written as:

$$f(x) = f_{\mathrm{h}}(x) + f_{\mathrm{p}}(x), \tag{1.156}$$

where f_h is the homogenous solution and f_p is the particular solution.

The homogenous solution f_h satisfies:

$$\frac{d^2 f_h}{dx^2} - f_h = 0. \tag{1.157}$$

Solving this, we get:

$$f_h(x) = Ae^x + Be^{-x}, \tag{1.158}$$

where A and B are constants.

Now the particular solution f_p satisfies:

$$\frac{d^2 f_p}{dx^2} - f_p = \delta(x). \tag{1.159}$$

First notice that the Fourier transform of $\delta(x)$ is $1/\sqrt{2\pi}$ (see Sec 1.5.7). Therefore we can write the above equation as:

$$\frac{d^2 f_p}{dx^2} - f_p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{ikx}\, dk$$

Now we write f_p in terms of its Fourier transform:

$$f_p(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{f}_p(k) e^{ikx}\, dk. \tag{1.160}$$

Substituting (1.160) into (1.155), we get:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{f}_p(k)(-k^2 - 1) e^{ikx}\, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{ikx}\, dk \tag{1.161}$$

$$\tilde{f}_p(k) = -\frac{1}{\sqrt{2\pi}} \frac{1}{k^2 + 1}. \tag{1.162}$$

Therefore,

$$f_p(x) = -\frac{1}{2\pi} \underbrace{\int_{-\infty}^{+\infty} \frac{e^{ikx}}{k^2 + 1} dk}_{I}. \tag{1.163}$$

What remains is to calculate the integral:

$$I = \int_{-\infty}^{+\infty} \frac{e^{ikx}}{k^2 + 1} dk. \tag{1.164}$$

(Actually we know that the answer is $\sim e^{-|x|}$ from the previous Section.) However for this exercise, let us compute the above integral using aomplex analysis. We go the complex plane and consider:

$$I_c = \oint_\gamma \frac{e^{izx}}{z^2 + 1} dz. \tag{1.165}$$

Let's first consider the case of $x > 0$. Then we consider a semi-circular closed contour on the upper half of the complex plane as shown in Fig. 1.6 left. The integral can be writtean as:

$$\oint_\gamma \frac{e^{izx}}{z^2 + 1} dz = \int_{\gamma_R^+} \frac{e^{izx}}{z^2 + 1} dz + \int_{\gamma_0} \frac{e^{izx}}{z^2 + 1} dz. \tag{1.166}$$

The first term on the right hand side is zero by Jordan's lemma and the second term on the right hand side is I (in the limit $R \to \infty$).

Now we will evaluate the left hand side in equation (1.166). The function:

$$\frac{e^{izx}}{z^2 + 1} = \frac{e^{izx}}{(z - i)(z + i)}$$

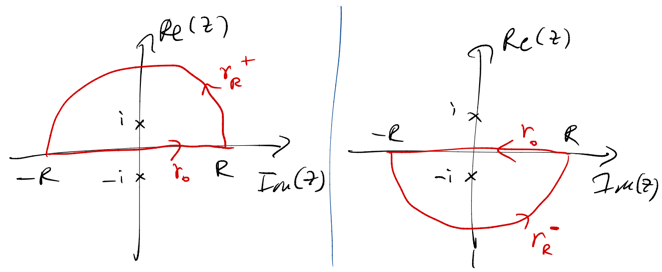

Figure 1.6: The contour considered to calculate $\oint_\gamma \frac{e^{izx}}{z^2+1}dz$.

has singularity at $z = i$ and $z = -i$. Both singularities are of order 1. The contour γ encloses the singularity at $z = i$. The residue at $z = i$ is:

$$\underset{z=i}{\text{res}} \frac{e^{izx}}{(z-i)(z+i)} = \lim_{z\to i} \frac{e^{izx}}{(z-i)(z+i)} \times (z-i) \tag{1.167}$$

$$= \lim_{z\to i} \frac{e^{izx}}{(z+i)} \tag{1.168}$$

$$= \frac{e^{-x}}{2i}. \tag{1.169}$$

Therefore, the left hand side in equation (1.166) is:

$$\oint_\gamma \frac{e^{izx}}{z^2+1}dz = 2\pi i \underset{z=i}{\text{res}} \frac{e^{izx}}{z^2+1} \tag{1.170}$$

$$= \pi e^{-x}. \tag{1.171}$$

Therefore, (1.166) becomes:

$$I = \pi e^{-x}. \tag{1.172}$$

Now we consider the case of $x < 0$. In this case, we have to consider the semi-circular closed contour in the lower half of the complex plane (see Fig. 1.6 right). Finally, the result is:

$$I = \pi e^{-|x|}. \tag{1.173}$$

Therefore, the particular solution to the differential equation (1.155) is given by:

$$f_p(x) = -\frac{1}{2}e^{-|x|}. \tag{1.174}$$

Example. Consider the following differential equation:

$$\frac{d^2 f}{dx^2} - f = e^{-x^2}. \tag{1.175}$$

The solution can be written as a sum of the homogenous and particular solution: $f(x) = f_h(x) + f_p(x)$. To find the particular solution, we write $f_p(x)$ and e^{-x}

in terms of its Fourier decomposition:

$$f_p(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, \tilde{f}_p(k) e^{ikx} \tag{1.176}$$

$$e^{-x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, \frac{1}{\sqrt{2}} e^{-\frac{1}{4}k^2} e^{ikx}. \tag{1.177}$$

Substituting (1.176-1.177) into (1.176), we get:

$$\tilde{f}_p(k) = \sqrt{2\pi}\, \underbrace{\frac{1}{\sqrt{2\pi}} \frac{-1}{k^2+1}}_{\tilde{f}_1(k)} \underbrace{\frac{1}{\sqrt{2}} e^{-\frac{1}{4}k^2}}_{\tilde{f}_2(k)}. \tag{1.178}$$

Using convolution theorem, we can see that $f_p(x)$ is the convolution of $f_1(x)$ and $f_2(x)$ where $f_1(x)$ and $f_2(x)$ are the inverse Fourier transform of $\tilde{f}_1(k)$ and $\tilde{f}_2(k)$. Thus:

$$\begin{aligned} f_p(x) &= (f_1 * f_2)(x) && (1.179)\\ &= \int_{-\infty}^{+\infty} dy\, f_2(x-y) f_1(y) && (1.180)\\ &= -\int_{-\infty}^{+\infty} dy\, \frac{1}{\sqrt{2\pi}} e^{-(x-y)^2} \sqrt{\frac{\pi}{2}} e^{-|y|} && (1.181)\\ &= -\frac{1}{2} \int_{-\infty}^{+\infty} dy\, e^{-(x-y)^2} e^{-|y|}. && (1.182) \end{aligned}$$

1.6.5 Integral equation

Fourier transform can also be used to solve integral equation, which involves convolution.

Example. Find $h(x)$, which satisfies the following integral equation:

$$h(x) = e^{i3x} + \int_{-\infty}^{+\infty} h(x-y) e^{-|y|}\, dy. \tag{1.183}$$

First, we notice that the second term on the left hand side is just a convolution:

$$h(x) = e^{i3x} + (h(x) * e^{-|x|})(x). \tag{1.184}$$

Taking Fourier transform, we get:

$$\begin{aligned} \tilde{h}(k) &= \mathcal{F}[e^{i3x}](k) + \sqrt{2\pi}\tilde{h}(k) \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2} && (1.185)\\ &= \mathcal{F}[e^{i3x}](k) + \frac{2}{1+k^2} \tilde{h}(k) && (1.186) \end{aligned}$$

Now we need to find the Fourier transform of e^{i3x}:

$$\mathcal{F}[e^{i3x}](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, e^{i3x} e^{-ikx} \tag{1.187}$$

$$= \sqrt{2\pi}\delta(k-3). \tag{1.188}$$

Therefore,

$$\tilde{h}(k) = \sqrt{2\pi}\frac{k^2+1}{k^2-1}\delta(k-3). \tag{1.189}$$

Now we invert this to find $h(x)$:

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, \sqrt{2\pi}\frac{k^2+1}{k^2-1}\delta(k-3)e^{ikx} \tag{1.190}$$

$$= \frac{3^2+1}{3^2-1}e^{i3x} \tag{1.191}$$

$$= \frac{5}{4}e^{i3x}. \tag{1.192}$$

1.7 Discrete Fourier transform

We consider two cases:

1.7.1 x is continuous and confined to a finite interval $x \in [0, L]$

Consider some function $f(x)$, where x is defined on the interval $x \in [0, L]$ (this includes periodic functions with period L). Then $f(x)$ can be expanded in terms of Fourier series as follow:

$$f(x) = \frac{1}{\sqrt{L}} \sum_k \tilde{f}(k) e^{ikx} \tag{1.193}$$

$$\tilde{f}(k) = \frac{1}{\sqrt{L}} \int_0^L dx\, f(x) e^{-ikx}. \tag{1.194}$$

Compared to the full Fourier transform above, here the wavector k is discetized into:

$$k = \tilde{n}\frac{2\pi}{L} \quad \text{, where} \tag{1.195}$$

$$\tilde{n} = 0, \pm 1, \pm 2, \ldots. \tag{1.196}$$

1.7.2 x (and hence f) is discrete and confined to a finite interval $x \in [0, L]$

Now let's suppose that the variable x is discretized into:

$$x = n\Delta x \quad \text{, where} \tag{1.197}$$

$$n = 0, 1, 2, \ldots, N-1, \tag{1.198}$$

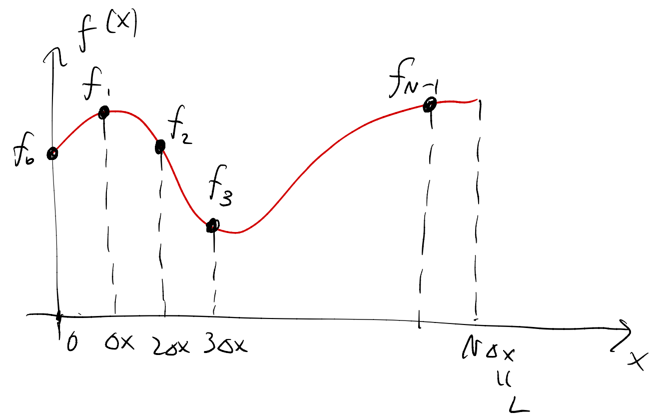

Figure 1.7: Some function $f(x)$, where x is confined to a finite interval $x \in [0, L]$. Furthermore, x (and hence f) is also discretized into $0, \Delta x, 2\Delta x, \ldots (N-1)\Delta x$.

where $N = \frac{L}{\Delta x}$. Consequently, $f(x)$ is also discretized into f_n (see Fig. 1.7). Now the discrete Fourier transform of f_n is:

$$f_n = \frac{1}{\sqrt{N\Delta x}} \sum_{\tilde{n}=0}^{N-1} \tilde{f}_{\tilde{n}} e^{i\frac{2\pi}{N}\tilde{n}n} \tag{1.199}$$

$$\tilde{f}_{\tilde{n}} = \frac{1}{\sqrt{N\Delta x}} \sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi}{N}\tilde{n}n}. \tag{1.200}$$

We can write the discrete Fourier transform above in the matrix form as follow:

$$\begin{pmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_{N-1} \end{pmatrix} = \frac{1}{\sqrt{N\Delta x}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-i\frac{2\pi}{N}} & e^{-i\frac{2\pi}{N}2} & & \\ 1 & e^{-i\frac{2\pi}{N}2} & e^{-i\frac{2\pi}{N}4} & & \\ \vdots & & & \ddots & \\ 1 & & & & e^{-i\frac{2\pi}{N}(N-1)^2} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix}. \tag{1.201}$$

Example. Consider some signal $h(x) = \cos(x)$ for $x \in [0, 2\pi]$. Suppose that we only know the measurements of $h(x)$ at 4 different intervals:

$$\begin{array}{rcl} x_0 = 0 & \leftrightarrow & h_0 = 1 \\ x_1 = \frac{\pi}{2} & \leftrightarrow & h_1 = 0 \\ x_2 = \pi & \leftrightarrow & h_2 = -1 \\ x_3 = \frac{3\pi}{2} & \leftrightarrow & h_3 = 0 \end{array}. \tag{1.202}$$

Now the Fourier transform of h_n is (note that $N = 4$ and $\Delta x = \frac{\pi}{2}$):

$$\tilde{h}_{\tilde{n}} = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{3} f_n e^{-i\frac{\pi}{2}\tilde{n}n}. \tag{1.203}$$

In matrix form, this is:

$$\begin{pmatrix} \tilde{h}_0 \\ \tilde{h}_1 \\ \tilde{h}_2 \\ \tilde{h}_3 \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-i\frac{\pi}{2}} & e^{-i\pi} & e^{-i\frac{3\pi}{2}} \\ 1 & e^{-i\pi} & e^{-i2\pi} & e^{-i3\pi} \\ 1 & e^{-i\frac{3\pi}{2}} & e^{-i3\pi} & e^{-i\frac{9\pi}{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \tag{1.204}$$

$$= \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{1.205}$$

Note that the computation of $\{\tilde{h}_{\tilde{n}}\}$ above scales with the matrix size $\sim N^2$. However, notice that there is also a lot of symmetries inside the matrix. In fact, there exists a fast algorithm called Fast Fourier Transform to compute this matrix multiplication, which scales as $\sim N \log N$ (see Wikipedia).

1.8 Fourier transform in quantum mechanics

1.8.1 Hilbert space

Definition (Hilbert space). The Hilbert space $\mathcal{H}$ is a space of all functions $f(x) : \mathbb{R} \to \mathbb{C}$, such that:

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, dx < \infty \quad \text{(square-integrable).}$$

We also define an inner product between two functions $f(x) \in \mathcal{H}$ and $g(x) \in \mathcal{H}$ to be:

$$\langle f | g \rangle = \int_{-\infty}^{+\infty} f(x)^* g(x) \, dx. \tag{1.206}$$

The norm of some function $f(x) \in \mathcal{H}$ is defined to be:

$$||f(x)|| = \langle f | f \rangle \, . \tag{1.207}$$

Two functions $f(x)$ and $g(x)$ are said to be orthogonal to each other if:

$$\langle f | g \rangle = 0. \tag{1.208}$$

If $f(x)$ and $g(x)$ have unit norm and orthogonal to each other, then they are said to be orthonormal.

Definition (Linear operator). A linear operator $\hat{A}$ operates on some function $f(x) \in \mathcal{H}$ to give another function, which also belongs to the Hilbert space:

$$\hat{A}f(x) \in \mathcal{H}. \tag{1.209}$$

Definition (Eigenvalue and eigenfunction). Let $\hat{A}$ be a linear operator. $\lambda_n \in \mathbb{C}$ is called the eigenvalue and $\phi_n(x) \in \mathcal{H}$ is called the eigenfunction if they both satisfy:

$$\hat{A}\phi_n(x) = \lambda_n \phi_n(x). \tag{1.210}$$

Here $n = 1, 2, \ldots$ is just a label to distinguish the different eigenvalues. We say that a set of orthogonal eigenfunctions $\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots\}$ spans the Hilbert space $\mathcal{H}$ if for any function $f(x) \in \mathcal{H}$, $f(x)$ can be written as a linear combination of $\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots\}$:

$$f(x) = \sum_n c_n \phi_n(x), \tag{1.211}$$

where $c_n \in \mathbb{C}$. Note that if n is continuous instead of discrete (*e.g.* $n \in \mathbb{R}$), the sum inside (1.211) becomes an integral:

$$f(x) = \int dn\, c(n)\phi(n, x). \tag{1.212}$$

1.8.2 Wavefunction in position and momentum representation

The wavefunction $\psi(x)$ belongs to the Hilbert space: $\psi(x) \in \mathcal{H}$. Usually, the wavefunction is normalized to 1:

$$\langle \psi | \psi \rangle = 1. \tag{1.213}$$

In quantum mechanics, physical observables A's are represented as linear operators $\hat{A}$'s with eigenvalues and eigenfunctions:

$$\hat{A}\phi_n(x) = \lambda_n \phi_n(x). \tag{1.214}$$

The eigenvalues $\{\lambda_1, \lambda_2, \ldots\}$ give all the possible values of measurement of A. The probability of measuring λ_n is given by the inner product squared:

$$\mathrm{Prob}(A = \lambda_n) = |\langle \phi_n | \psi \rangle|^2. \tag{1.215}$$

After measurement of A, say we get $A = \lambda_m$, the wavefunction collapses into the eigenfunction $\phi_m(x)$.

For example, the position and momentum operators are:

$$\hat{x} = x \tag{1.216}$$

$$\hat{p} = -i\hbar \frac{d}{dx}. \tag{1.217}$$

Now the eigenvalues and eigenfunctions of the position operator is:

$$\hat{x}\ \underbrace{\delta(x-x_n)}_{\text{eigenfunction}} = \underbrace{x_n}_{\text{eigenvalue}}\ \underbrace{\delta(x-x_n)}_{\text{eigenfunction}}. \tag{1.218}$$

The eigenfunctions are orthogonal to each other. For example, consider ($x_n \neq x_m$):

$$\langle \delta(x-x_n)|\delta(x-x_m)\rangle = \int_{-\infty}^{+\infty} \delta(x-x_n)\delta(x-x_m)\, dx. \tag{1.219}$$

Now what is the probability of measuring the particle at location x_n?

$$\begin{aligned}\text{Prob}(x=x_n) &= |\,\langle \delta(x-x_n)|\psi\rangle\,|^2 & (1.220)\\ &= \left|\int_{-\infty}^{+\infty} \delta(x-x_n)\psi(x)\, dx\right|^2 & (1.221)\\ &= |\psi(x_n)|^2, & (1.222)\end{aligned}$$

which is unsurprising.

Now the eigenvalues and eigenfunctions of the momentum operator is:

$$\hat{p}\ \underbrace{\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p}{\hbar}x}}_{\text{eigenfunction}} = \underbrace{p}_{\text{eigenvalue}}\ \underbrace{\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p}{\hbar}x}}_{\text{eigenfunction}}. \tag{1.223}$$

We can easily check by substituting $\hat{p} = -i\hbar\frac{d}{dx}$ to the left hand side in the above equation. Furthermore, the eigenfunctions are also orthogonal to each other since:

$$\begin{aligned}\left\langle \frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p}{\hbar}x}\,\middle|\,\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p'}{\hbar}x}\right\rangle &= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{i\left(\frac{p}{\hbar}-\frac{p'}{\hbar}\right)x} & (1.224)\\ &= \frac{1}{\hbar}\delta\left(\frac{p-p'}{\hbar}\right) & (1.225)\\ &= \delta(p-p'). & (1.226)\end{aligned}$$

Furthermore, the set of all eigenfunctions $\left\{\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p}{\hbar}x}\right\}$ spans the whole Hilbert space. Therefore, for any wavefunction $\psi(x) \in \mathcal{H}$, $\psi(x)$ can be written as a linear combination of $\left\{\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p}{\hbar}x}\right\}$:

$$\psi(x) = \int_{-\infty}^{+\infty} dp\,\tilde{\psi}(p)\frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p}{\hbar}x}. \tag{1.227}$$

Very often we use the units where $\hbar = 1$ so the above equation becomes:

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dp\,\tilde{\psi}(p)e^{ipx}, \tag{1.228}$$

	Position representation	Momentum representation
Wavefunction	$\psi(x) \in \mathcal{H}$	$\tilde{\psi}(p) \in \mathcal{H}^*$
Position operator $\hat{x}$	x	$i\frac{d}{dp}$
Eigenfunctions of $\hat{x}$	$\{\delta(x - x_n)\}$	$\left\{\frac{1}{\sqrt{2\pi}}e^{-ipx}\right\}$
Momentum operator $\hat{p}$	$-i\frac{d}{dx}$	p
Eigenfunctions of $\hat{p}$	$\left\{\frac{1}{\sqrt{2\pi}}e^{ipx}\right\}$	$\{\delta(p - p_n)\}$

Table 1.1: Summary of wavefunctions and operators in position and momentum representation. (Note that we are using units where $\hbar = 1$.)

which is just the definition of Fourier transform. Now let's calculate the probability of measuring the momentum of the particle (in the units where $\hbar = 1$):

$$\text{Prob}(p) = \left|\left\langle \frac{1}{\sqrt{2\pi}}e^{ipx} \middle| \psi(x)\right\rangle\right|^2 \tag{1.229}$$

$$= \left|\int_{-\infty}^{+\infty} dx \frac{1}{\sqrt{2\pi}}e^{ipx}\psi(x)\right|^2 \tag{1.230}$$

Substituting (1.228) into the above equation, we get:

$$\text{Prob}(p) = \left|\frac{1}{2\pi}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dp'\, e^{-ipx}\tilde{\psi}(p')e^{ip'x}\right|^2 \tag{1.231}$$

$$= \left|\int_{-\infty}^{+\infty} dp'\, \tilde{\psi}(p')\delta(p - p')\right|^2 \tag{1.232}$$

$$= \left|\tilde{\psi}(p)\right|^2. \tag{1.233}$$

Now we can define the dual Hilbert space $\mathcal{H}^*$, which consists of all functions $\tilde{\psi}(p)$, where $\tilde{\psi}(p)$ is the Fourier transform of $\psi(x)$. We call $\psi(x) \in \mathcal{H}$ and $\tilde{\psi}(p) \in \mathcal{H}^*$ the wavefunction in the position and momentum representation respectively, which are related by Fourier Transform:

$$\psi(x) \underset{\text{transform}}{\overset{\text{Fourier}}{\longleftrightarrow}} \tilde{\psi}(p). \tag{1.234}$$

Table. 1.1 gives the summary of these two representations.

Example. Simple harmonic oscillator. Let's work in units of $\hbar = m = \omega = 1$. The Hamiltonian can be written as:

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{x}^2. \tag{1.235}$$

The eigenvalues and eigenfunctions satisfy (in position representation):

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right)\phi_n(x) = E_n\phi_n(x). \tag{1.236}$$

The solutions to this differential equation are known from quantum mechanics textbook:

$$E_n = n + \frac{1}{2} \tag{1.237}$$

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}} \frac{e^{-\frac{1}{2}x^2}}{\pi^{1/4}} H_n(x), \tag{1.238}$$

where $H_n(x)$ is the Hermite polynomial, defined to be:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{1.239}$$

The first few Hermite polynomials are:

$$H_0(x) = 1 \tag{1.240}$$

$$H_1(x) = 2x \tag{1.241}$$

$$H_2(x) = 4x^2 - 2 \tag{1.242}$$

$$\vdots$$

It can also be shown that the eigenfunctions are orthonormal to each other, *i.e.* $\langle \phi_m | \phi_n \rangle = \delta_{mn}$. Now let's find out the eigenfunctions $\{\phi_n(x)\}$ in the momentum representation: $\{\tilde{\phi}_n(p)\}$. $\{\tilde{\phi}_n(p)\}$ are related to $\{\phi_n(x)\}$ *via* Fourier transform:

$$\phi_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{\phi}_n(p) e^{ipx}\, dp. \tag{1.243}$$

Substituting (1.243) into (1.236), we get:

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{\phi}_n(p) e^{ipx}\, dp = E_n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{\phi}_n(p) e^{ipx}\, dp \tag{1.244}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{\phi}_n(p) \left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right) e^{ipx}\, dp = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} E_n \tilde{\phi}_n(p) e^{ipx}\, dp \tag{1.245}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{\phi}_n(p) \left(\frac{1}{2}p^2 - \frac{1}{2}\frac{d^2}{dp^2}\right) e^{ipx}\, dp = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} E_n \tilde{\phi}_n(p) e^{ipx}\, dp \tag{1.246}$$

$$\left(\frac{1}{2}p^2 - \frac{1}{2}\frac{d^2}{dp^2}\right) \tilde{\phi}_n(p) = E_n \tilde{\phi}_n(p), \tag{1.247}$$

which is exactly the same differential equation as (1.236). So the eigenfunctions $\{\tilde{\phi}_n(p)\}$ in the momentum representation have the same functional form as in (1.238).

Chapter 2

Laplace transform