Agenda:

1. Extreme Value Theorem
2. Bisection Method
3. Squeeze Law in Context
4. The Exponential Function
5. Derivative as a Linear Approximation

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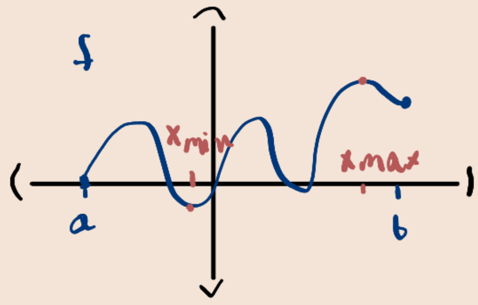
1. Extreme Value Theorem



if

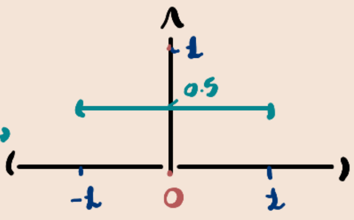
I) f continuous (drawn without lifting pen)

II) on a closed interval $[a,b]$

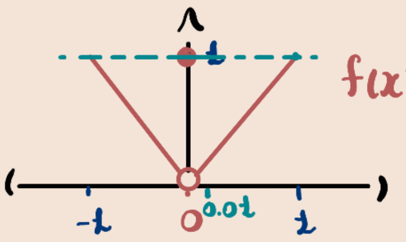
then there is a highest and lowest point.

$f(x_{max}) \geq f(x)$

$f(x_{min}) \leq f(x)$



Non-example I):



$f(x) = \begin{cases} |x|, & x \neq 0 \\ 1, & x = 0 \end{cases}$. Then $x_{max} = \{-1, 0, 1\}$

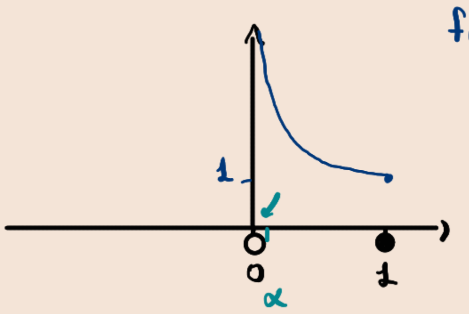
let $x_{min} = \alpha$.

then $f(x_{min}) = |\alpha|$.

but $f\left(\frac{x_{min}}{2}\right) = \frac{|\alpha|}{2} < |\alpha| = f(x_{min})$

$\Rightarrow$ no x_{min}!

Non-example II):



$f(x) = \frac{1}{x}$ on $(0,1]$. Then $\begin{cases} x_{min} = 1 \\ \text{If } x_{max} = \alpha \end{cases}$

then $f(\alpha) = \frac{1}{\alpha} \geq f(x)$

Choose $x_0 = \frac{\alpha}{2} \Rightarrow f(x_0) = \frac{1}{\left(\frac{\alpha}{2}\right)} = 2 \cdot \frac{1}{\alpha} = 2 \cdot f(\alpha)$

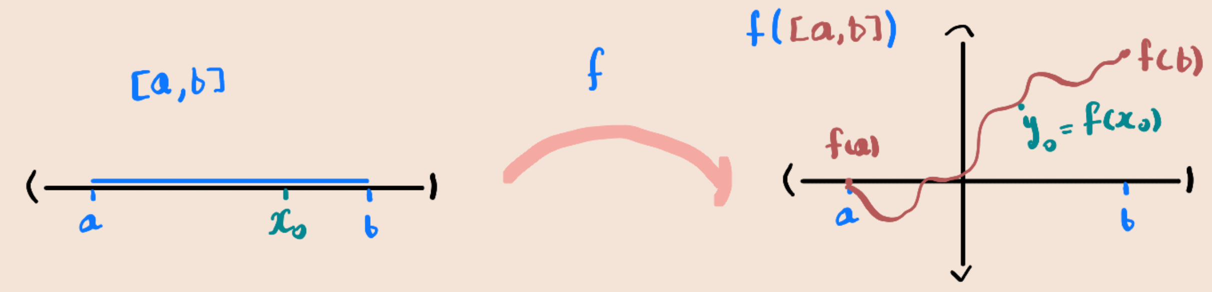
BUT $f(x_0) > f(\alpha)$

$\Rightarrow$ Contradiction $\Rightarrow$ no x_{max}!

2. Bisection Method.

Recall:

2. Intermediate value Theorem

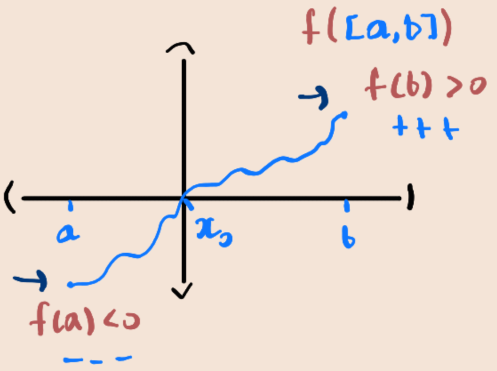


If f continuous then it sends closed intervals to closed intervals.

Why is this useful?

x_0 is a root if

If f(a) ±, f(b) ∓ then $f(x_0) = 0$ gives us existence of roots!



We know: f(a) < 0, f(b) > 0, f cont. on [a, b] ⇒ ∃ a root!

Algorithm: consider f(c), $c = \frac{a+b}{2}$.

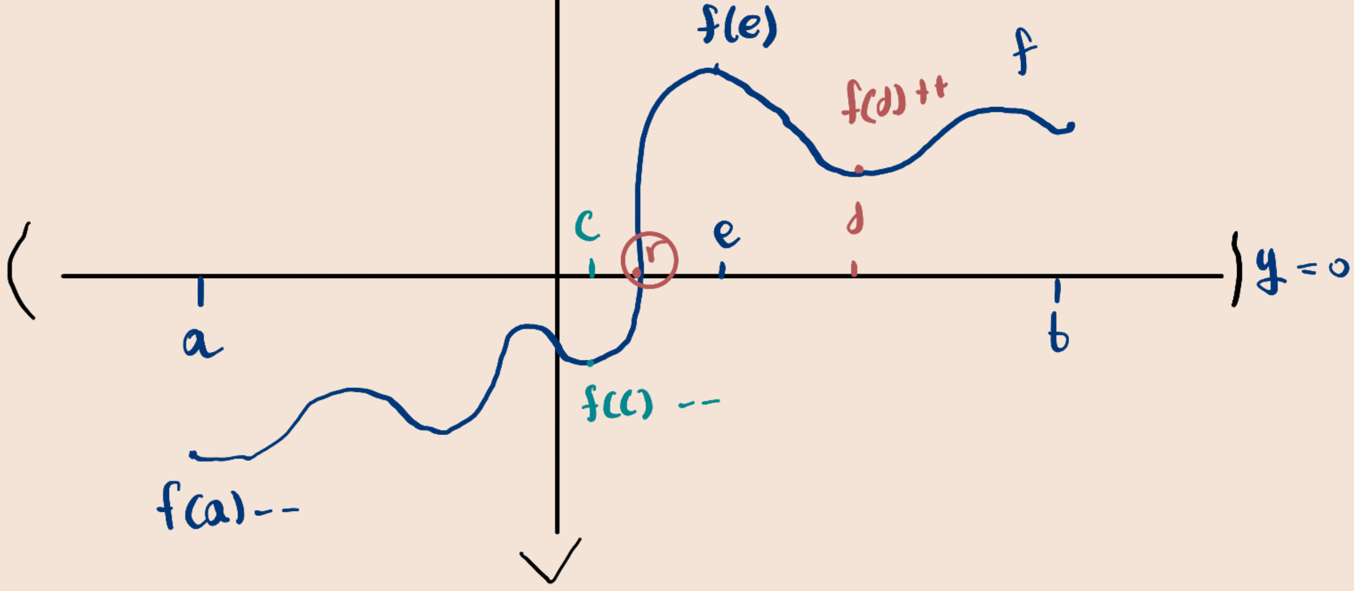
If f(c) > 0, replace f(b)

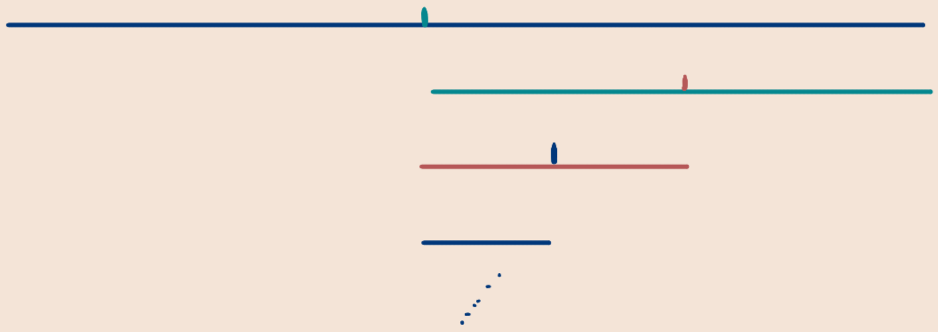
If f(c) < 0, replace f(a)

f(a)

root: f(root) = 0

+++
f(e)

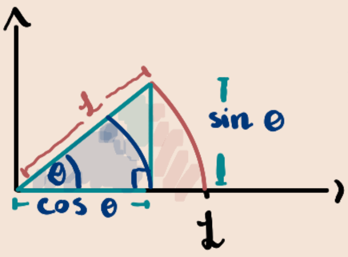




r is our root.

3. Squeeze Law in Context.

All this time we used $\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1.}$ Now, consider



$* \sin^2\theta + \cos^2\theta = 1$

Then $\begin{cases} A_b = \frac{\theta}{2} \cos^2\theta \\ A_{b+g} = \frac{1}{2} \cdot \cos\theta \cdot \sin\theta \\ A_{r+g+b} = \frac{\theta}{\cancel{360°}} \cdot \cancel{\pi} \cdot \frac{\cancel{180°}}{\cancel{\pi}} = \frac{\theta}{2}. \end{cases}$. Clearly $A_b \leq A_{b+g} \leq A_{b+g+o}$.

$\boxed{1 \text{ rad} = \frac{180°}{\pi}}$

Finally ...

$$A_b \leq A_{b+g} \leq A_{b+g+o}$$

$$\frac{\theta}{2} \cos^2\theta \leq \frac{1}{2} \cdot \cos\theta \cdot \sin\theta \leq \frac{\theta}{2}$$

$$\frac{\theta}{2} \cos^2\theta \leq \frac{\sin 2\theta}{4} \leq \frac{\theta}{2}$$

$\cos\theta \cdot \sin\theta = \frac{\sin 2\theta}{4}$

$$\frac{1}{2} \cos^2\theta \leq \frac{\sin 2\theta}{2\theta \cdot 2} \leq \frac{1}{2}$$

$$\lim_{\theta \to 0} \left(\cos^2\theta \leq \frac{\sin 2\theta}{2\theta} \leq 1 \right)$$

$$\Rightarrow 1 \leq \lim_{\theta \to 0} \frac{\sin 2\theta}{2\theta} \leq 1$$

$$\boxed{\Rightarrow \lim_{\theta \to 0} \frac{\sin 2\theta}{2\theta} = \lim_{x \to 0} \frac{\sin x}{x} = 1}$$

4. The Exponential Function.

We may define $e^x = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n$. There is a story behind this.

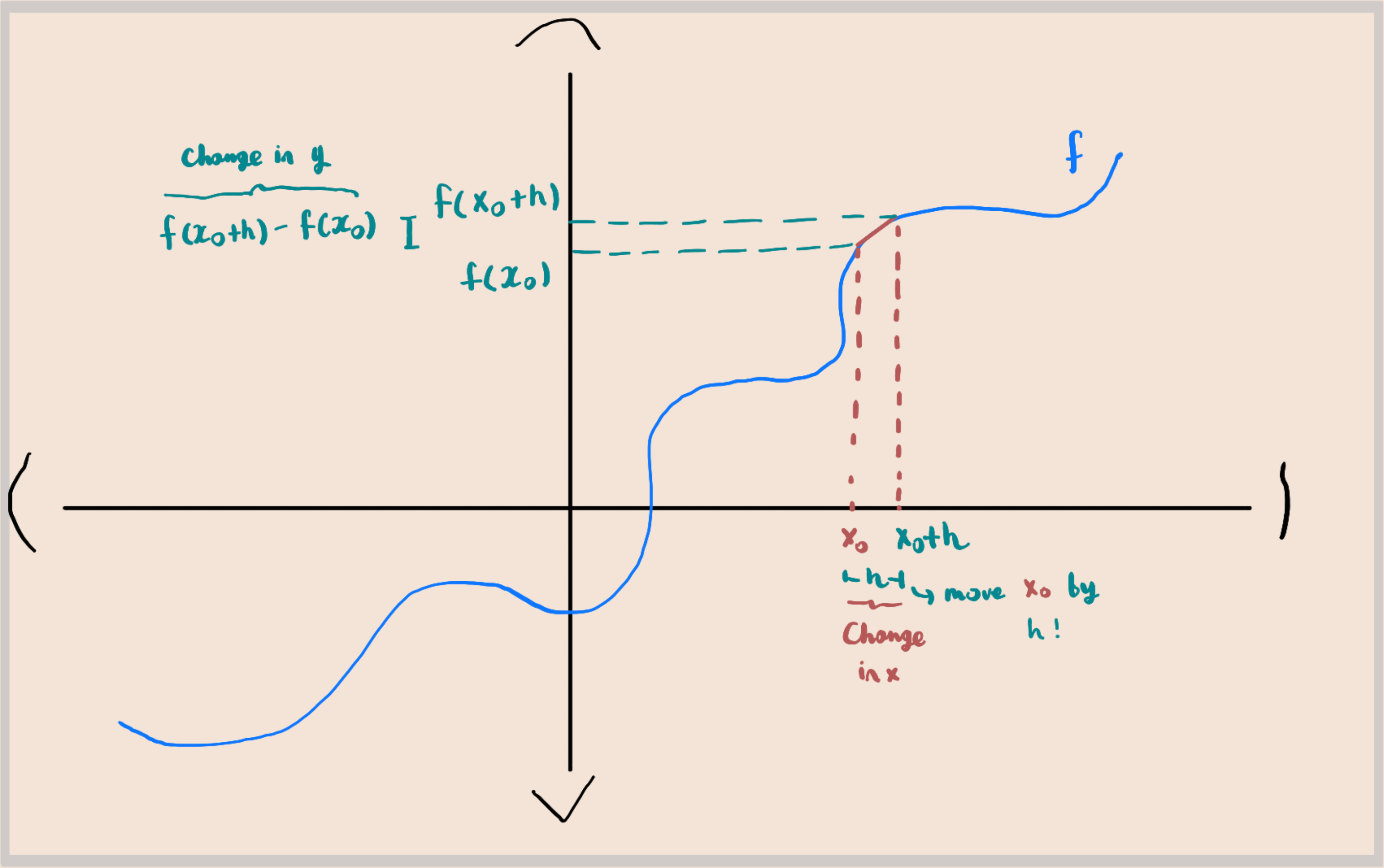
$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n := e^1 \simeq 2.71\ldots$$

Then define $e^x = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n$

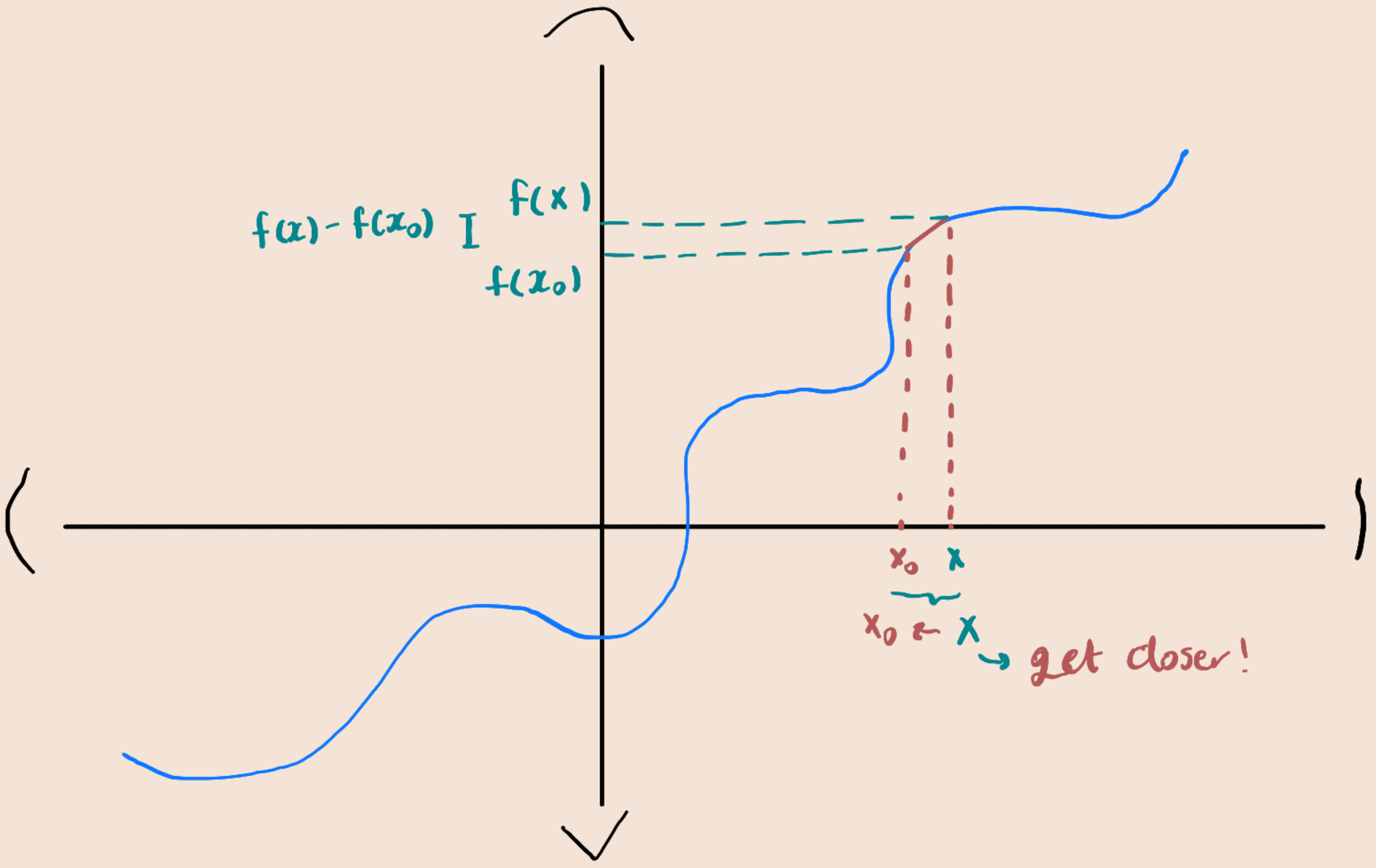
Exercise: $e^x \cdot e^y = e^{x+y}$ using limit defⁿ

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n \lim_{n\to\infty} \left(1+\frac{y}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{(x+y)}{n}\right)^n$$

5. Derivative as a Linear Approximation.



Graph: Equivalent formulation of $f'(x)$.



So $f'(x_0) := \dfrac{f(x)-f(x_0)}{x-x_0}$. Equivalently one may define

$$f(x+h) = f(x) + \text{something!}$$

$$+ \underbrace{m \cdot h}_{\text{linear term}} + \underbrace{E_x(h)}_{\text{error term.}}$$