#4 Systems of Two Linear Equations in Two Variables.

We start with two linear equations,

$$j: a_1x + b_1y = c_1$$
$$K: a_2x + b_2y = c_2$$

These are two lines in 2D!

Proof. If $\underset{\smile}{a}x+\underset{\smile}{b}y=\underset{\smile}{c}$ then we can solve for $\underline{\underline{y}}$!

then $by = c - ax$

then $y = \frac{c}{b} - \frac{a}{b}x$

$= -\frac{a}{b}x + \frac{c}{b}$. Choose $m = -\frac{a}{b}$ and $d = \frac{c}{b}$.

then $y = mx + d$ is an equation of a line with

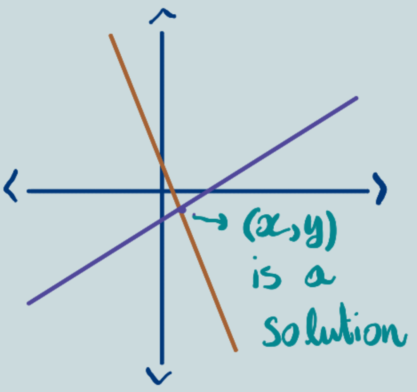
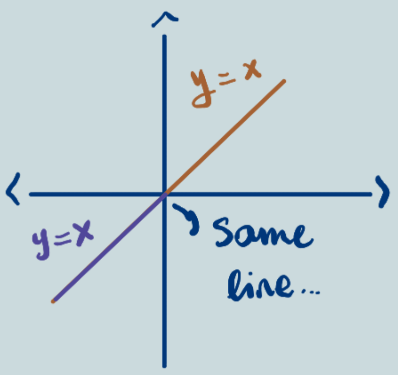
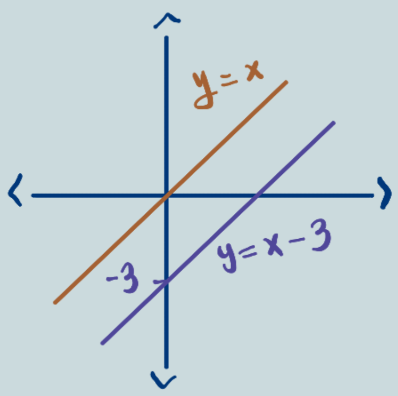
Slope $m = -\frac{a}{b}$ and $d = \frac{c}{b}$.

can be any number!

Done. Now, two lines in space. What are the odds?

$j: a_1x + b_1y = c_1$ becomes $j: y_1 = m_1x + d_1$

$K: a_2x + b_2y = c_2$ becomes $K: y_2 = m_2x + d_2$

Intersect once!	Intersect always...	Never Intersect.
		
$m_1 \neq m_2$	$m_1 = m_2$	$m_1 = m_2$
	$c_1 = c_2$	$c_1 \neq c_2$
One solution (x,y)	infinitely many Solutions	no Solutions (x,y)

These would also be different cases of solutions to the system of equations j, K. A pair (x,y) on both lines would satisfy both respective equations.

Question. How do we solve this system of equations?

$$-x + y = -3.5$$
$$x + 3y = 9.5$$

Two standard methods.

1. Elimination.

Notice that the coefficients of x are -1 and 1. This is great,

$$\begin{aligned} \boxed{-x} + y &= -3.5 \\ \boxed{x} + 3y &= 9.5 \end{aligned}$$

as adding both equations would give us an equation in one variable.

$$\begin{array}{r} \cancel{-x} + y = -3.5 \\ + \quad \cancel{x} + 3y = 9.5 \\ \hline 0 + \boxed{4y = 6} \end{array}$$

↳ nice!

We know how to solve the above equation.

$$y = \frac{3}{2}.$$

Finally, if $y = \frac{3}{2}$, and $x + 3y = 9.5$, then we can

- plug-in $y = \frac{3}{2}$: $x + 3y = x + 3\left(\frac{3}{2}\right) = 9.5$
- Solve for x : $x = 9.5 - 3 \cdot \frac{3}{2} = 5$

Then $(x, y) = (5, \frac{3}{2})$ satisfy both equations. check this!

2. Substitution.

We start by solving one equation in two variables.

Choose an equation
$$\begin{aligned} \boxed{-x + y = -3.5} \quad &(1) \\ x + 3y = 9.5 \quad &(2) \end{aligned}$$

Choose your favourite variable
$$-x + y = -3.5 \Rightarrow y = -3.5 + x \qquad (3)$$

Plug-in $-3.5 + x$ for y in (2)
$$x + 3(-3.5 + x) = 9.5 \overset{\text{group, solve}}{\Rightarrow} x = 5$$

Plug-in 5 for x in (3)
$$y = -3.5 + x \Rightarrow y = -3.5 + 5 \Rightarrow y = 1.5 = \frac{3}{2}$$

A bit more tedious. This is why elimination is nicer to work with: it is much easier to set up your equations to cancel a variable out.

Setting up your equations. Let us make this more precise.

$$\begin{aligned} 5x + 14y &= 45 \qquad (1) \\ 10x + 7y &= 27 \qquad (2) \end{aligned}$$

How would you eliminate x (or y)?

Observation. Notice that multiplying $5x$ by -2 gives us $-10x$.

$$\times(-2)\quad \begin{cases} \boxed{5x} + 14y = 45 & (1) \\ \boxed{10x} + 7y = 27 & (2) \end{cases}$$

Key. Multiply equation (1) by -2! Then,

$$5x + 14y = 45 \quad \xRightarrow{[\times -2]} \quad -\cancel{10x} - 28y = -90$$

$$\oplus \quad \cancel{10x} + 7y = 27$$

$$\boxed{-21y = -63}$$

$$\Rightarrow \quad \boxed{y = \frac{-63}{21} = 3}$$

Plugging in (1),

$$5x + 14y = 45 \Rightarrow 5x + 14 \cdot 3 = 45$$

$$\Rightarrow \quad \boxed{x = \frac{45 - 14 \cdot 3}{5} = \frac{3}{5}}$$

This should be good preparation for the worksheet.