Agenda

1. Separable Differential Equations

2. Application: Newton's Law of Cooling.

Bonus. A non-separable differential equation!

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1. Separable Differential Equations.

We start with a simple differential equation,

$$\frac{dy}{dx} = 2x \, , \quad y(0) = 0$$

with the goal of finding $y(x)$.

$\boxed{1}$ Separate. Take algebraic action!

$$\left(\frac{dy}{\cancel{dx}} = 2x \right) \times \cancel{dx} \Rightarrow dy = 2x \quad dx$$

$\boxed{2}$ Collect. (here, the terms are already collected.)

$$\Rightarrow \int 1 \cdot dy = \int 2x \; dx$$

$\boxed{3}$ Integrate!

$$\Rightarrow \quad y = x^2 + C$$

$\boxed{4}$ Find C. Here, utilize the initial condition $y(0)=0$.

$$y(0) = 0 = 0^2 + C \Rightarrow C = 0$$

$\boxed{5}$ Rewrite

$$y(x) = x^2 + \underbrace{C}_{=0} = x^2.$$

Question. what would $y(x)$ look like for $y(0) = -5$?

Exercise. Solve

$$\frac{dy}{dx} = y \quad , \quad y(0) = 1$$

More diff. eqns ....

The solution to

initial condition

$$\frac{dy}{dt} = -3yt \, , \quad \boxed{y(0) = 1.}$$

should involve a simple separation of variables.

$\boxed{1}$ Separate.

$$\left(\frac{dy}{\cancel{dt}} \times \cancel{dt} = -3yt \right) \times dt$$

$$\Rightarrow \qquad dy = 3t \; dt$$

$\boxed{2}$ collect.

$$\left(dy = -3\cancel{y}t \, dt \right) \cdot \frac{1}{\cancel{y}}$$

$$\frac{1}{y} \, dy = -3t \, dt$$

$\boxed{3}$ Integrate!

$$\boxed{(\ln(x))' = \frac{1}{x}}$$

$$\int \frac{1}{y} \, dy = -3 \int t \, dt$$

$$\Rightarrow \qquad \ln(y) = -3\left(\frac{t^2}{2}\right) + c$$

$$\Rightarrow \quad y(t) = e^{\ln(y)} = e^{c} \cdot e^{-\frac{3}{2} \cdot t^2} \, , \qquad c \in \mathbb{R}$$

$\boxed{4}$ Find c. with $y(0) = 1$, we get

$$1 = y(0) = e^{c} \cdot \underbrace{e^{-\frac{3}{2} \cdot 0^2}}_{=1} = e^{c}$$

$$\Rightarrow \quad e^{c} = 1$$

$\boxed{5}$ Rewrite

$$y(t) = \overbrace{e^{c}}^{=1} \cdot e^{-\frac{3}{2} \cdot t^2} = e^{-\frac{3}{2} \cdot t^2}$$

2. Application: Newton's Law of Cooling.

The goal is to find temperature T as a function of time t.

T(t) = 

nice figure.

$$\frac{dT}{dt} = \overset{>0,\ \text{rate coefficient}}{K}(\underset{\hookrightarrow\ \text{Ambient Temp.}}{A} - T(t))$$

1 Separate.

$$\left(\frac{dT}{dt} = K(A-T(t))\right) \times dt$$

$$dT = K(A-T(t))\ dt$$

2 collect.

$$\boxed{dT} = K(A-\boxed{T(t)})\ dt \quad \div (A-T(t))$$

$$\frac{1}{(A-T(t))}\cdot dT = K\ dt$$

3 Integrate!

$$\int \frac{1}{(A-T(t))}\cdot dT = K\int dt$$

$$\Rightarrow -\ln(A-T(t)) = Kt + C$$

$\cancel{\frac{K^2}{2}+C}$, K is constant!

Integration check...

$$(-\ln(A-T(t)))' = \frac{-1}{A-T(t)} \cdot \underbrace{\overbrace{(A-T)'}^{-\cancel{1}}}_{=\frac{d}{dT}(\overset{=0}{A}-T)} = \frac{1}{A-T(t)}$$

↳ A constant

4 ~~Find C~~ Set boundaries...

$$\int_{\underset{\text{initial temp.}}{T(0)}}^{\overset{\text{temp at time } t\ \leftarrow}{T(t)}} \frac{1}{(A-T(t))}\cdot dT = K\int_{\underset{\rightarrow\ \text{initial } t}{t=0}}^{\overset{\text{time } t}{t=t}} dt = Kt\Big|_{t=0}^{t=t}$$

$$\Rightarrow -\ln(A-T)\Big|_{T(0)}^{T(t)} = \ln\left(\frac{A-T(0)}{A-T(t)}\right) = Kt$$

5 ~~R~~ewrite T(t).

$$\Rightarrow \left(\frac{A-T(0)}{A-T(t)} = e^{Kt}\right) \quad [\times e^{-Kt} \times A-T(t)]$$

$$\Rightarrow A-T(t) = (A-T(0))e^{-Kt}$$

$$\Rightarrow \boxed{T(t) = A + (\overset{T(0)}{T_0}-A)e^{-Kt}}$$

Bonus. A non-separable differential equation!

The equation is non-separable, meaning previous methods will not be successful.

$$y' = y + x - 1$$

This exercise focuses on insight. Let's try to make progress together.

$$\overset{(-y)}{\Rightarrow} \quad \underbrace{y' - y}_{\#} = x - 1$$

#: difference of a function and its derivative.

→ quotient rule? → product rule?

Question. Can i multiply by some function f to make

$$\underbrace{y' \cdot f + f' \cdot y}_{\text{product rule } (y \cdot f)'} = f \cdot (\underbrace{y' - y}_{\text{original expression}}) \; ? \qquad (*)$$

It would be so nice, because $\int y' \cdot f + f' \cdot y \; dx = y \cdot f$!

We need to set a constraint $f' = -f$ for (*) to work, because then

$$y' \cdot f + \underset{-f}{\underbrace{f'}} \cdot y = y' \cdot f - f \cdot y = f \cdot (y' - y)$$

Finally: what f solves $f' = -f$? None but the e^{-x} ! Then...

$$y' - y = x - 1$$

$$\overset{\times e^{-x}}{\Rightarrow} \quad y' e^{-x} + y \cdot \underbrace{(e^{-x})'}_{= -1} = e^{-x} \cdot (x - 1)$$

$$\int dx \Rightarrow \int y' e^{-x} + y \cdot (e^{-x})' \, dx = \int e^{-x} \cdot (x - 1) \, dx$$

$$\Rightarrow \quad y \cdot e^{-x} = \int e^{-x} \cdot (x - 1) \, dx \qquad [\times e^{x}]$$

$$\Rightarrow \quad y = e^{x} \cdot \int e^{-x} \cdot (x - 1) \, dx$$

$= $  $+ c$.

some $f^{\underline{n}}$ of x

Why are you still here? Solve $\int e^{-x} \cdot (x - 1) \, dx$!