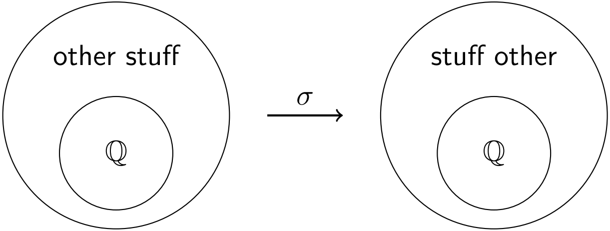
Something Else to Study

Question

What is the action of the Galois group on CRMs?

What is the Galois Group?

The Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ is the group of automorphisms $\mathbb{Q}(\zeta_{n-1}) \to \mathbb{Q}(\zeta_{n-1})$ fixing the elements of $\mathbb{Q}$. Visually, it can be represented like so:



What are dessins?

Definition

A K_n-**dessin** is a topological map whose underlying graph is bipartite with n vertices on each side.

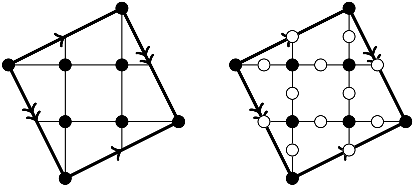


Figure: Bipartification of a CRM to obtain a dessin

K_n-dessins D give surfaces defined by polynomials over $\mathbb{Q}(\zeta_{n-1})$, which yields an action of $\mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ on K_n-dessins that we denote by D^σ for $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$.

Our results

Theorem

Given $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ *and a prime* $\mathfrak{p} \subseteq \mathbb{Z}[\zeta_{n-1}]$ *containing* p*, there is an isomorphism*

$$D_{\mathfrak{p}}^{\sigma} \simeq D_{\sigma\mathfrak{p}}$$

of K_n*-dessins.*

So this tells us the two actions of $\mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ on K_n-dessins are "equivalent".

Action I of the Galois group

Recall:

- Prime ideals $\mathfrak{p} \subseteq \mathbb{Z}[\zeta_{n-1}]$ give CRMs $M_\mathfrak{p}$.
- These $M_\mathfrak{p}$ induce K_n-dessins $D_\mathfrak{p}$.
- K_n-dessins $D_\mathfrak{p}$ give rise to surfaces that can be described by algebraic equations over $\mathbb{Q}(\zeta_{n-1})$.

$\mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ acts on K_n-dessins by acting on the coefficients of these equations.

Denote the action of $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ on $D_\mathfrak{p}$ by $D_\mathfrak{p}^\sigma$).

Action II of the Galois group

- $\mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ also permutes prime ideals of $\mathbb{Z}[\zeta_{n-1}]$.
- We saw earlier that prime ideals are in bijection with CRMs on n vertices.

Thus, we obtain a second action of $\mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ on K_n-dessins: $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ takes $D_{\mathfrak{p}}$ to $D_{\sigma\mathfrak{p}}$.

Statement & Proof Sketch

Theorem

Given $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ and a prime $\mathfrak{p} \subseteq \mathbb{Z}[\zeta_{n-1}]$ containing p, there is an isomorphism

$$D_{\mathfrak{p}}^{\sigma} \simeq D_{\sigma\mathfrak{p}}$$

of K_n-dessins.

Each $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{n-1})/\mathbb{Q})$ gives rise to an operation H_j on dessins (called a Wilson operator).

Proof.

1. Jones, Streit & Wolfart (2009) proved $D_{\mathfrak{p}}^{\sigma} \simeq H_j D_{\mathfrak{p}}$.
2. We proved $H_j D_{\mathfrak{p}} \simeq D_{\sigma\mathfrak{p}}$ using additonal results from our construction.

Thus $D_{\mathfrak{p}}^{\sigma} \simeq D_{\sigma\mathfrak{p}}$. □