## Agenda

1. Sketching Functions
2. Integration ...

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## 1. Sketching Functions.

Consider some non-simple function...

$$f(x) = \frac{x^2}{4-x^2}$$

We prepare a checklist of 6 things,

☐ Domain of f
☐ y-intercept, x-intercept
☐ Horizontal Asymptotes
☐ Vertical Asymptotes
☐ First Derivative Analysis ( $f\nearrow$, $f\searrow$, local extremas)
☐ Second Derivative Analysis ( inflection points, concavity $\nearrow\swarrow$ )

Tedious, but let's start.

☑ Domain of $f(x) = \dfrac{x^2}{4-x^2}$, $2, -2 \notin D_f$

- $D_f = \mathbb{R} \setminus \{2, -2\}$

  - $f(-2) = \dfrac{(-2)^2}{4-(-2)^2} = \dfrac{(-2)^2}{0}$

  - $f(2) = \dfrac{(2)^2}{4-(2)^2} = \dfrac{(2)^2}{0}$

☑ y-intercept, x-intercept

$$f(x) = \frac{x^2}{4-x^2}$$

y-intercept ↓ set $x=0$

$f(x=0) = f(0) = 0$

x-intercept ↓ set $y=0$

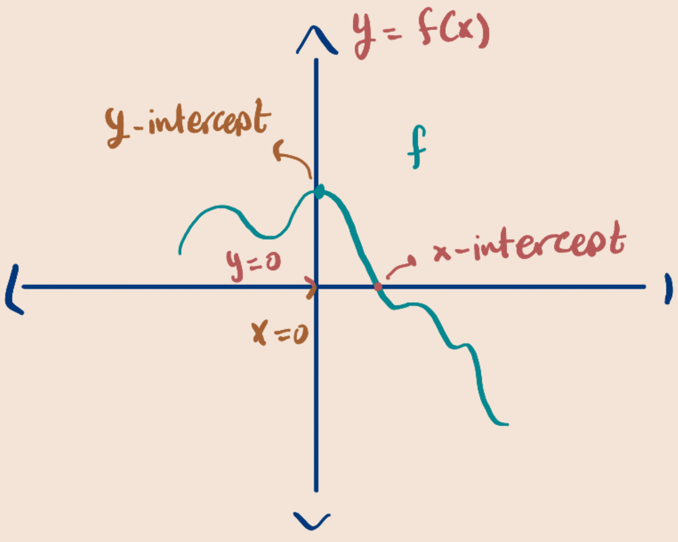
$f(x) = 0$

$\rightarrow \dfrac{x^2}{4-x^2} = 0$  $x(4-x^2)$

$\rightarrow x^2 = 0$

$\rightarrow x = 0$

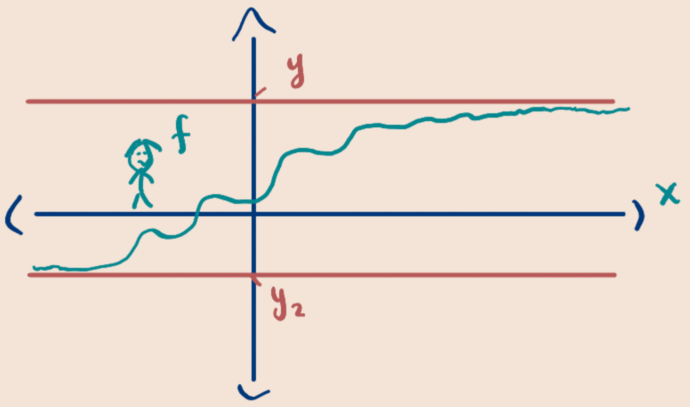
$(0,0)$ : x,y intercept.



## ☑ Horizontal Asymptotes.

$f(x) = \dfrac{x^2}{4-x^2}$

Recall: $\lim_{x \to \pm\infty} f(x)$




$\lim_{x \to \pm\infty} f(x) = \dfrac{x^2}{4-x^2}$

$\rightarrow \lim_{x \to \pm\infty} \left( \dfrac{x^2}{4-x^2} = \dfrac{\cancel{x^2}}{\cancel{x^2}\left(\frac{4}{x^2} - 1\right)} = \dfrac{1}{\underbrace{\frac{4}{x^2}}_{\to 0} - 1} \right) = \dfrac{1}{0-1} = \underbrace{-1}_{\text{horizontal asymptote!}}$

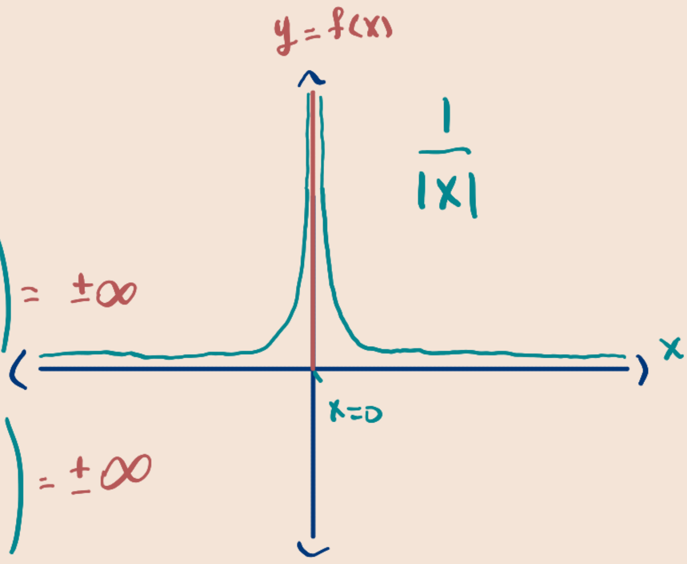
## ☑ Vertical Asymptotes

$f(x) = \dfrac{x^2}{4-x^2}$

Recall: $D_f = \mathbb{R} \setminus \underbrace{\{2,-2\}}_{\text{candidates for vertical asymptotes!}}$

1. $\lim_{x \to 2} \left( \dfrac{x^2}{4-x^2} = \dfrac{1}{\frac{4}{x^2} - 1} = \dfrac{1}{\lim_{x \to 2} \frac{4}{x^2} - 1} \right) = \pm\infty$

2. $\lim_{x \to -2} \left( \dfrac{x^2}{4-x^2} = \dfrac{1}{\frac{4}{x^2} - 1} = \dfrac{1}{\lim_{x \to -2} \frac{4}{x^2} - 1} \right) = \pm\infty$




$\Rightarrow$ 2, -2 are vertical asymptotes.

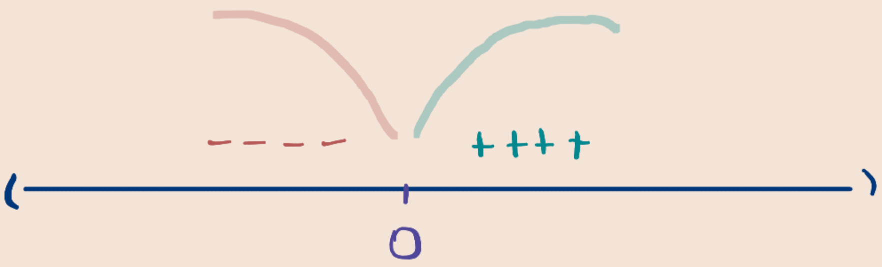
## ☑ First Derivative Analysis ( f ↗, f ↙, local extremas)

$f(x) = \dfrac{x^2}{4-x^2} \Rightarrow f'(x) = \dfrac{2x \cdot (4-x^2) - x^2 \cdot (-2x)}{(4-x^2)^2} =$

$\boxed{\left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}}$

$= \dfrac{8x}{(4-x^2)^2}$. Let $f'(x) = 0$, then

$8 \dfrac{x}{\underbrace{(4-x^2)^2}_{>0}} = 0 \Rightarrow 8x = 0 \Rightarrow x = 0$ is a minima.




further, f is

- increasing on $(0, \infty)$
- decreasing on $(-\infty, 0)$

(excluding discontinuities)

## ☑ Second Derivative Analysis ( $\underbrace{\text{inflection points}}_{f''(x)=0}$, concavity ↗↙ )

$$f(x) = \frac{x^2}{4-x^2} \Rightarrow f''(x) = \left[f'(x)\right]' = \left[\frac{8x}{(4-x^2)^2}\right]'$$

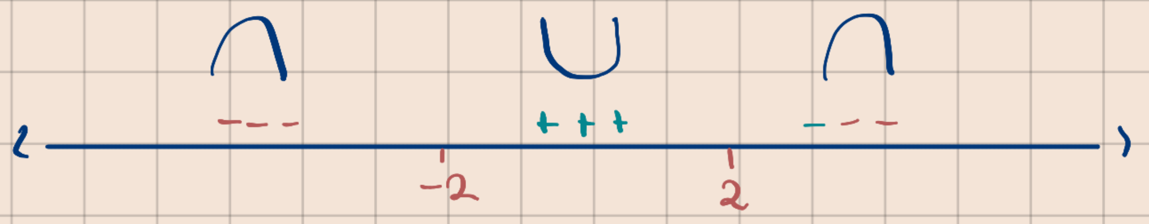
$$\boxed{\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}}$$

$$= \frac{8\cdot(4-x^2)^2 - (8x)\cdot\left(2\cdot(-2x)(4-x^2)\right)}{(4-x^2)^4}$$

$$= 8\,\frac{3x^2+4}{(4-x^2)^3}, \text{ let } f''(x)=0.$$

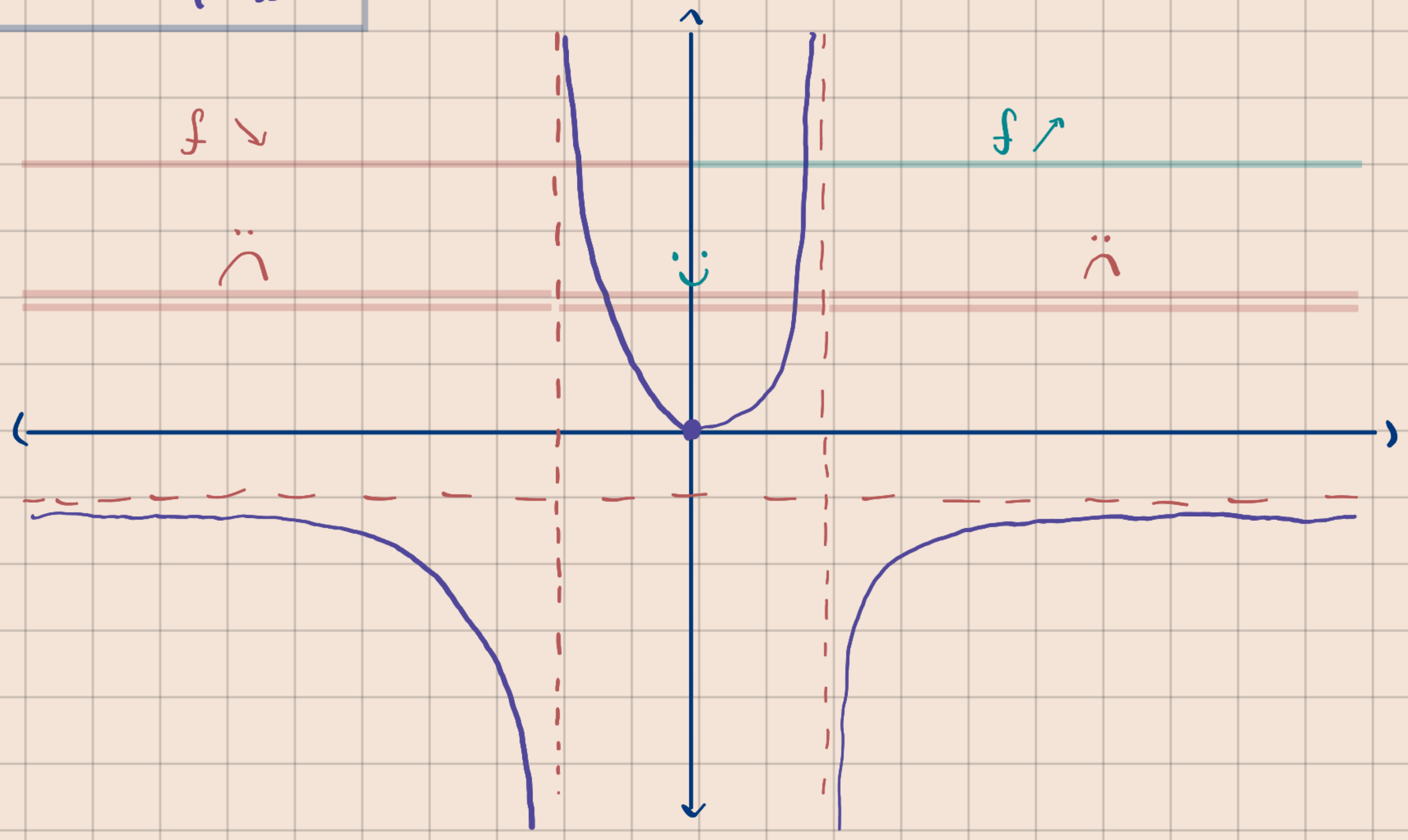
$$3x^2+4=0 \Rightarrow 3x^2=-4 \Rightarrow x^2 = \frac{-4}{3} \Rightarrow f''(x) \neq 0$$

Assuming $(4-x^2)^3 = 0 \Rightarrow x = \pm 2$




### Altogether...

$$\boxed{f(x) = \frac{x^2}{4-x^2}}$$




- $(0,0)$: x,y intercept.
- $-1$ is a horizontal asymptote.
- $2, -2$ are vertical asymptotes.
- $f$ is increasing on $(0, \infty)$ ; decreasing on $(-\infty, 0)$
- $f$ is concave up $\ddot{\cup}$ on $(-2, 2)$; concave down $\ddot{\cap}$ on $(-\infty, -2) \cup (2, \infty)$