1. What are the extrema (maxima, minima) of

$f(x) = \frac{1}{3}x^3 - \frac{7}{2}x^2 + 10x + 13$?

$\boxed{\begin{array}{l} \max_f = \{2\} \\ \min_f = \{5\} \end{array}}$

Recipe to find extremas of $f$,
1. compute $f'(x)$
2. Find its roots

$\Rightarrow f'(x) = \cancel{\frac{1}{3}} \cdot \cancel{3}x^2 - \frac{7}{\cancel{2}} \cdot \cancel{2}x + 10$

$\Rightarrow \boxed{f'(x) = x^2 - 7x + 10}$

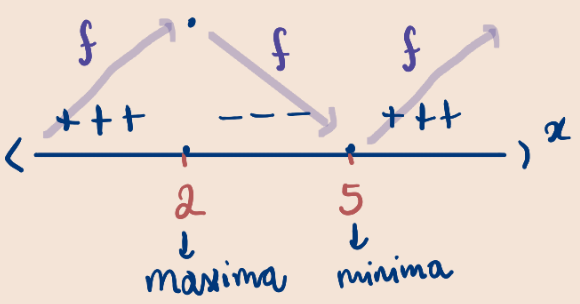
$f'(x) = 0 \Rightarrow x^2 - 7x + 10 = 0 = (x-5)(x-2) \Rightarrow \overbrace{\boxed{x_1 = 5 \ , \ x_2 = 2}}^{\text{extremas}}$

Maxima or minima?

⟨Second Derivative Test⟩

Let $x$ be an extrema of $f$. Then,
- $f''(x) > 0 \Rightarrow x$ is a minima
- $f''(x) < 0 \Rightarrow x$ is a maxima

⟨Analysis⟩




$f'(x) = x^2 - 7x + 10 \Rightarrow f''(x) = 2x - 7$

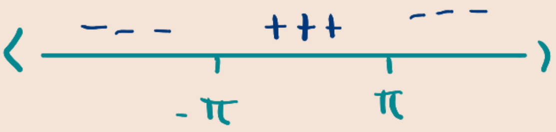
- $f''(2) = 4 - 7 = -3 < 0 \Rightarrow 2$ maxima
- $f''(5) = 10 - 7 = 3 > 0 \Rightarrow 5$ minima

---

2. For which interval is $f(x) = \dfrac{x^2}{\pi^2 - x^2}$ positive?

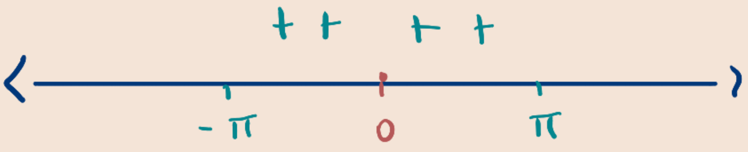
$f(x) = \dfrac{\overbrace{x^2}^{\geq 0}}{\pi^2 - x^2} > 0 \Rightarrow \pi^2 - x^2 > 0$

$\Rightarrow \underbrace{(\pi - x)\cdot(\pi + x)}_{= g(x)} > 0 \Rightarrow x < \pi, \ x > -\pi$




~~$x \in (-\pi, \pi)$~~

$\boxed{x \in (-\pi, 0) \cup (0, \pi)}$



3. Find maxima, minima of $f(x) = -\ln(x) + \sqrt{x}$

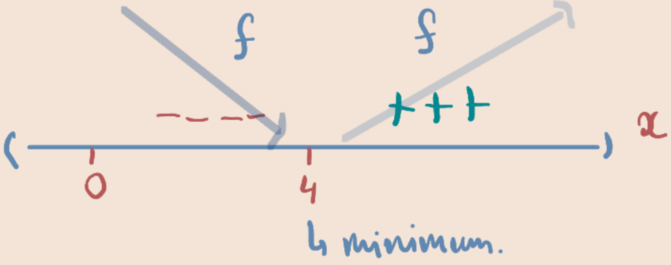
Recipe to find extremas of f,
1. compute $f'(x)$
2. Find its roots

1. $f'(x) = -\frac{1}{x} + \frac{1}{2\sqrt{x}}$

2. $-\frac{1}{x} + \frac{1}{2\sqrt{x}} = 0 \quad \times[x \cdot 2\sqrt{x}]$
$x \neq 0.$

extrema

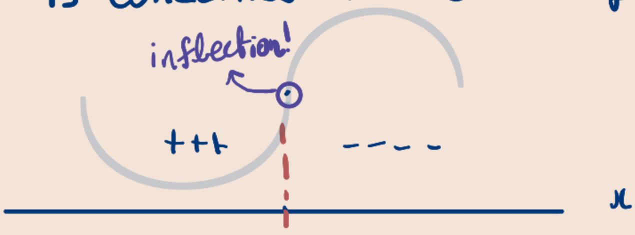
$$\Rightarrow -2\sqrt{x} + x = 0 \Rightarrow x = 2\sqrt{x} \Rightarrow \frac{x}{\sqrt{x}} = \frac{x^1}{x^{\frac{1}{2}}} = x^{\frac{1}{2}} = \sqrt{x} = 2 \Rightarrow \boxed{x = 4.}$$




$x = 4$
min

4. What value is a point of inflection of $f(x) = 2e^{-\frac{4}{x}}$?

$f''$ is concerned with concavity.




$\Rightarrow$

Inflection Point Recipe,
1. Compute $f''(x)$
2. Find its roots.

$$f(x) = 2e^{-\frac{4}{x}} \Rightarrow f'(x) = 2 \cdot e^{-\frac{4}{x}} \cdot \frac{4}{x^2}$$

$$\Rightarrow f''(x) = \underbrace{2 \cdot e^{-\frac{4}{x}}}_{>0} \cdot \underbrace{\frac{8}{x^3}}_{\neq 0} \Big[ \underbrace{\frac{2}{x} + -1}_{=0} \Big]$$

$$f''(x) = 0 \Rightarrow \frac{2}{x} + -1 = 0 \Rightarrow \boxed{x = 2}$$

5. Evaluate $\int \frac{\cos(\frac{\pi}{x})}{x^2} \cdot dx = \int \cos(\frac{\pi}{x}) \cdot \frac{1}{x^2} \cdot dx$

$$u = \frac{\pi}{x} \Rightarrow du = -\frac{\pi}{x^2} \cdot dx \Rightarrow \boxed{dx = \frac{du}{-\pi} \cdot x^2}$$

$$\int \cos(u) \cdot \frac{1}{\cancel{x^2}} \cdot \frac{\cancel{x^2}}{\pi} \cdot du$$

$$\frac{1}{\pi} \int \cos(u)\, du = \frac{1}{\pi} \sin(u) + C$$

$$\boxed{= \frac{1}{\pi} \sin\left(\frac{\pi}{x}\right) + C}$$

6. Compute $I = \int \frac{1}{\sqrt{9-x^2}} dx$.  $\boxed{\sin^2\theta + \cos^2\theta = 1}$

$x = 3\sin u \Rightarrow dx = 3\cos(u)\cdot du$

$I = \int \frac{1}{3\cos u} dx = \int \frac{1}{\cancel{3\cos u}} \cancel{3\cos(u)}\cdot du = \int du = u + C$

$\boxed{= \sin^{-1}\frac{x}{3} + C}$

$x = 3\sin u \Rightarrow \frac{x}{3} = \sin u \Rightarrow \boxed{u = \sin^{-1}\frac{x}{3}}$

---

7. $\int \sin(x)\cdot \ln(\underline{\cos x})\, dx = I$

$u = \cos x \Rightarrow du = -\sin x \cdot dx \Rightarrow dx = \frac{du}{-\sin x}$

$I = \int \cancel{\sin(x)}\cdot \ln(u)\cdot \frac{du}{-\cancel{\sin x}}$

$= -\int \ln(u)\, du$,  $\boxed{\int w\, dv = wv - \int v\, dw}$  $w = \ln(u) \Rightarrow \frac{1}{u} du = dw$  $dv = du \Rightarrow v = u$

$= -\left(u\cdot\ln(u) - \int \frac{1}{\cancel{u}}\cdot\cancel{u}\, du\right) = u\cdot(1 - \ln(u))$

$\boxed{= \cos x\cdot(1 - \ln(\cos x)) + C}$

---

8. Compute $I_n = \int \underbrace{x^n}_{=u} \underbrace{e^x dx}_{=dv}$ for $n \in \mathbb{N}$.  $\int u\, dv = uv - \int v\, du$

$I_n = x^n\cdot e^x - n \underbrace{\int e^x\cdot x^{n-1} dx}_{I_{n-1}} = x^n\cdot e^x - n\cdot e^x\cdot x^{n-1} + n\cdot(n-1)\, I_{n-2}$

$= x^n\cdot e^x - n\cdot e^x\cdot x^{n-1} + n\cdot(n-1)\, x^{n-2} e^x - n\cdot(n-1)(n-2)\, I_3$

$= e^x\left(x^n - n\, x^{n-1} + n\cdot(n-1)\, x^{n-2} - n\cdot(n-1)\cdot(n-2)\, x^{n-3} + \dots - n\dots 4\cdot 3\, x^2 + n!\, x^1 - n!\, x^0\right)$

$= \boxed{e^x \sum_{k=0}^{n} (-1)^k \cdot x^{n-k} \cdot \frac{n!}{n-k!} + C}$  $\begin{cases} k=0 & \frac{n!}{n!} = 1 \\ k=1 & \frac{n\cdot\cancel{n-1!}}{\cancel{n-1!}} = n \\ k=2 & \frac{n\cdot n-1\cdot \cancel{n-2!}}{\cancel{n-2!}} = n\cdot n-1 \end{cases}$

9. $\int \sec^2 x \tan x \, dx$

$$\frac{d}{dx}\left(\sec^2 x = \frac{1}{\cos^2 x}\right) = \frac{\overbrace{\cancel{(\cos^2 x)\cdot(1)'}}^{0} - 1\cdot(\cos^2 x)'}{(\cos^2 x)^2}$$

$$= \frac{2\cancel{\cos x}\cdot \sin x}{\cos^{\cancel{4}\,3} x} = 2\cdot \overbrace{\frac{\sin x}{\cos x}}^{\tan(x)} \cdot \overbrace{\frac{1}{\cos^2 x}}^{\sec^2(x)}$$

$$\frac{1}{2}\cdot\frac{d}{dx}\sec^2 x = \tan(x)\sec^2(x)$$

$$\cancel{\int}\frac{1}{2}\cdot\frac{d}{dx}\sec^2 x \,\cancel{dx} = \boxed{\frac{1}{2}\sec^2 x + C}$$

---

10. If $I_n = \int_0^1 (a-bx^3)^n \, dx$, find a relationship between $I_n, I_{n-1}$.

$$I_n = \int_0^1 (a-bx^3)^{n-1}(a-bx^3)\,dx = a\int_0^1 (a-bx^3)^{n-1} - \underbrace{\int_0^1 (a-bx^3)^{n-1}\cdot bx^3\,dx}_{=K}$$

$$= a I_{n-1} + \frac{(a-b)^n}{3n} - \frac{1}{3n}\cdot I_n \Rightarrow \boxed{\left(1+\frac{1}{3n}\right) I_n = a I_{n-1} + \frac{(a-b)^n}{3n}}$$

Computing K: $\int v\,du = uv - \int v\,du$

$$K = \int_0^1 \underbrace{(a-bx^3)^{n-1}\cdot bx^2}_{=dv}\cdot \underset{u}{x}\,dx = -\frac{x}{3}\cdot\frac{(a-bx^3)^n}{n}\Big|_0^1 + \frac{1}{3n}\underbrace{\int_0^1 (a-bx^3)^n\,dx}_{=I_n}$$

$$= \left(-\frac{x}{3}\cdot\frac{(a-bx^3)^n}{n}\Big|_0^1\right) + \frac{1}{3n}\cdot I_n = \left(-\frac{1}{3}\cdot\frac{(a-b)^n}{n}\right) + \frac{1}{3n}\cdot I_n$$

Mistake with exercise. set $a=b=1$. Then,

$$\int_0^1 (a-bx^3)\,dx = \int_0^1 (1-x^3)^n\,dx,$$ which evaluates to

$$\left(1+\frac{1}{3n}\right) I_n = a I_{n-1} + \frac{(a-b)^n}{3n} \;\underset{b=1}{\overset{a=1}{\Longleftrightarrow}}\; \boxed{\left(1+\frac{1}{3n}\right) I_n = I_{n-1}} + \overbrace{\cancel{\frac{(1-1)^n}{3n}}}^{=0}.$$