Linear Regression

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Normal Error Model

Normal Error Model

Simple regression model $+$ Normality assumption:

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, \qquad i = 1, \ldots, n,$$

where the error terms ϵ_is are *independently and identically distributed (i.i.d.)* $N(0, \sigma^2)$ random variables.

MLE

Under the Normal error model:

- LS estimators $\hat{\beta}_0, \hat{\beta}_1$ are the *maximum likelihood estimator (MLE)* of β_0, β_1, respectively.
- The MLE of σ^2 is SSE/n.

not MSE $\left(= \frac{SSE}{n-2}\right)$

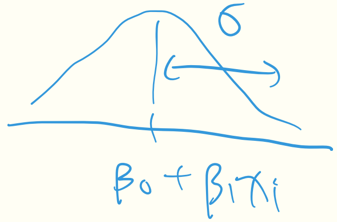
Sampling Distributions

$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x}) y_i}{S_{xx}}$, $S_{xx} = \sum (x_i - \bar{x})^2$

⇓ linear combination of y_i's

Under the Normal error model:

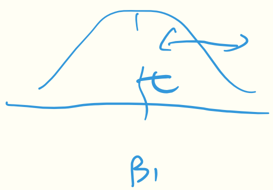
⇒ y_i indep't.



- $\hat{\beta}_0, \hat{\beta}_1$ are normally distributed:

$$\hat{\beta}_0 \sim N(\beta_0, \sigma^2\{\hat{\beta}_0\}), \quad \hat{\beta}_1 \sim N(\beta_1, \sigma^2\{\hat{\beta}_1\}).$$

$\hat{\beta}_1 \sim$



"bell shaped"

- SSE/σ^2 follows a χ^2 distribution with $n-2$ degrees of freedom, denoted by $\chi^2_{(n-2)}$.

$(n-2)$ = df of SSE

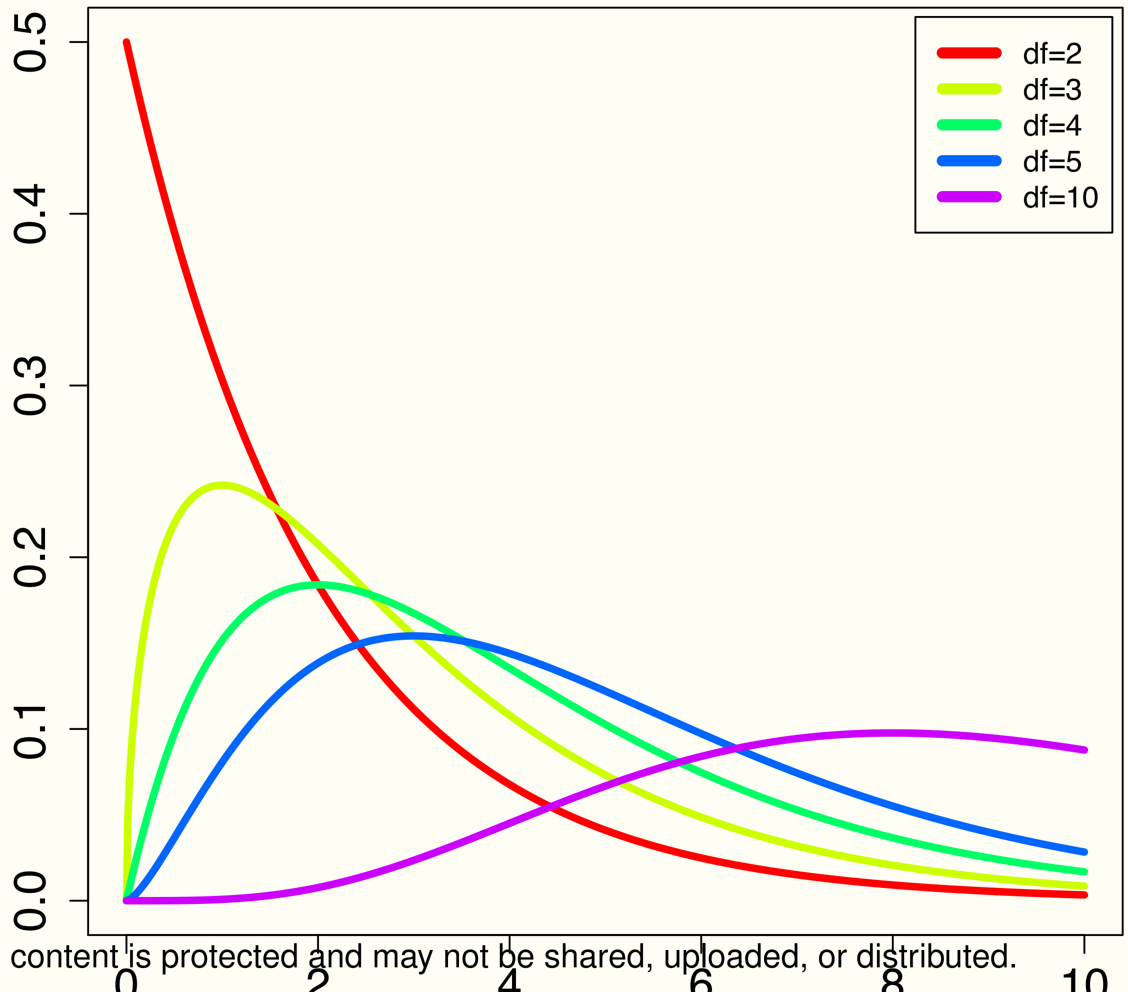
- SSE is independent with both $\hat{\beta}_0$ and $\hat{\beta}_1$.

follow chi-square (χ^2) distribution [not normal]

χ^2 Distributions

Figure: χ^2 distributions: probability density function

defined on $[0, +\infty)$: the positive real line



right-skewed
[longer right skill]

Interval estimator

Confidence Intervals of Regression Coefficients

Pivotal Quantity

SE

$$s\{\hat\beta_1\} = \sqrt{\frac{MSE}{\sum(x_i-\bar{x})^2}}$$

random　　mean

standardization

$$\frac{\hat{\beta}_1 - \beta_1}{s\{\hat{\beta}_1\}} \sim t_{(n-2)}$$

"estimated sd"

(not calculable) not a statistic ∵ involves unknown parameter β_1 (but know distribution)

- The numerator is the difference between the LS estimator $\hat{\beta}_1$ and its mean β_1.

  if divide by sd ↳ gives normal distribution?

- The denominator is the standard error of $\hat{\beta}_1$.
- This quantity follows a **known distribution**, $t_{(n-2)}$, t-distribution with $n-2$ degrees of freedom.

  → viewed as condition of $N(0,1)$ by using MSE to estimate σ^2 in standardization

  df (SSE) / df (MSE) ??

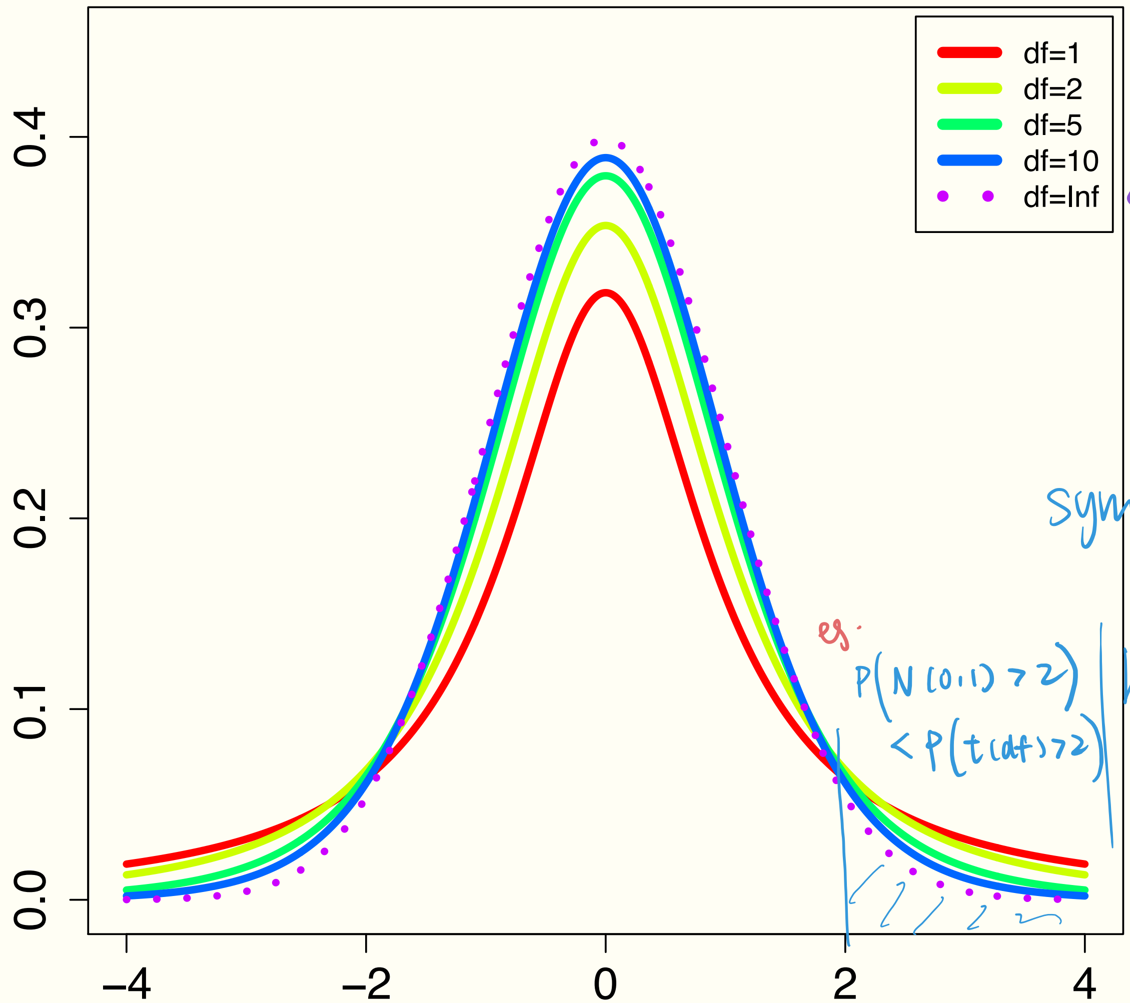
*

Fact

$$\left.\begin{array}{l} Z \sim N(0,1) \\ W \sim \chi^2(df) \end{array}\right\} \text{ independent}$$

then $$\frac{Z}{\sqrt{\frac{W}{df}}} \sim t(df)$$

Figure: t distributions: probability density function*



↓ more like

← $N(0,1)$ standard normal

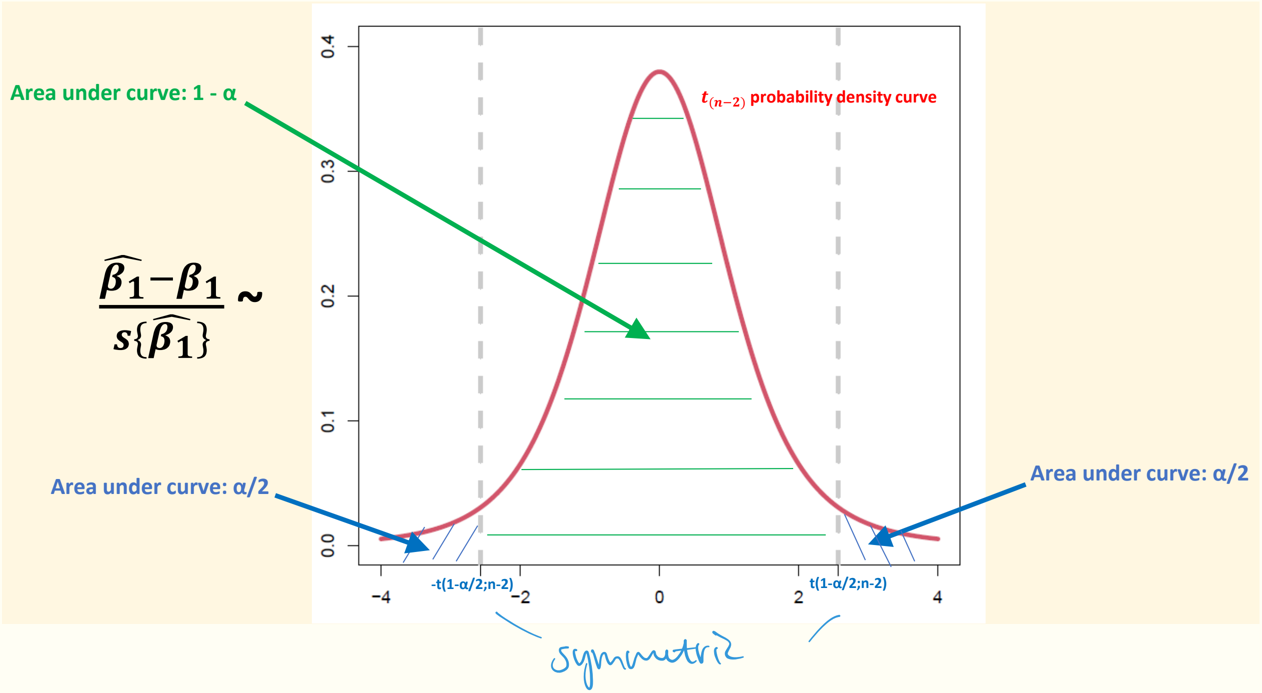
symmetric around 0,

or

heavy tailed (heavier than standard normal (more probabilities at tail))

eg. $P(N(0,1) > 2) < P(t(df) > 2)$

*t distribution with ∞ degrees of freedom is the standard normal $N(0, 1)$ distribution.



area inbetween

$$P\left(\left|\frac{\hat{\beta}_1-\beta_1}{s\{\hat{\beta}_1\}}\right| \leq t(1-\alpha/2; n-2)\right) = 1-\alpha \Rightarrow$$

$$P\left(\hat{\beta}_1 - t(1-\alpha/2; n-2)s\{\hat{\beta}_1\} \leq \beta_1 \leq \hat{\beta}_1 + t(1-\alpha/2; n-2)s\{\hat{\beta}_1\}\right) = 1-\alpha$$

rv : left end

falls between interval

random variable, sits right end

Confidence Interval

The $(1-\alpha)100\%$-confidence interval of β_1:

multiplier

$$\hat{\beta}_1 \pm t(1-\alpha/2; n-2)s\{\hat{\beta}_1\},$$

point estimator in middle

SE of the estimation

where $t(1-\alpha/2; n-2)$ is the $(1-\alpha/2)$100th percentile of $t_{(n-2)}$.

common recipe for CI : Estimator ± multiplier (α) × SE (estimator)

Confidence Coefficient: Accuracy

eg: $\alpha = 0.05 \Rightarrow 95\%$

- $(1 - \alpha)100\%$ is called the *confidence coefficient* or the *confidence level*.

$\uparrow$ larger more accurate, $\downarrow \alpha$

- Commonly used confidence coefficients are 95% ($\alpha = 0.05$), 90% ($\alpha = 0.1$), 99% ($\alpha = 0.01$).
- Confidence coefficient reflects **accuracy of the C.I.**: the larger (i.e., the smaller the α), the more accurate.

Confidence Interval Width: Precision

$$= \sqrt{\frac{MSE}{\sum_i (x_i - \bar{x})^2}} = \sqrt{\frac{MSE}{s_x^2 \cdot (n-1)}}$$

- The half-width: $t(1-\alpha/2; n-2)\underbrace{s\{\hat{\beta}_1\}}$
- The width reflects **precision of the C.I.**: the narrower, the more precise
- Factors influencing the precision:

  bigger $(1-\alpha)$ $\uparrow$ larger $(t(1-\frac{\alpha}{2}, n-2))$

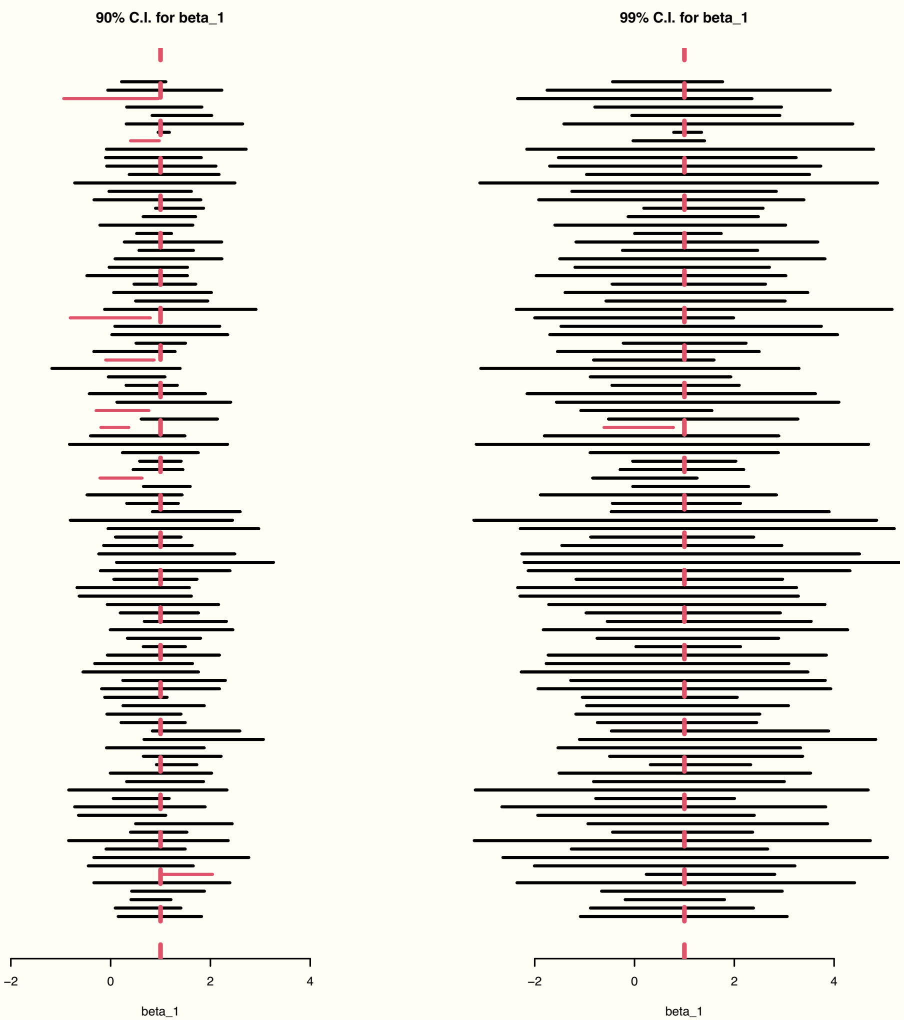
  - The larger the $\underbrace{\text{confidence coefficient}}_{(1-\alpha)}$ (more accurate), the wider the C.I. (less precise)

    smaller $s\{\hat{\beta}_1\}$

  - The larger the sample size n (more data), the narrower the C.I. (more precise)
  - The larger the SE (more uncertainty), the wider the C.I. (less precise)

Simulation Experiment

Figure: C.I.s of β_1: Left: 90% C.I.; Right: 99% C.I.



Heights

- $n = 928$, $\overline{X} = 68.316$, $\sum_{i=1}^{n}(X_i - \overline{X})^2 = 3038.761$, and

$$\hat{\beta}_0 = 24.54, \ \hat{\beta}_1 = 0.637, \ MSE = 5.031.$$

- $s\{\hat{\beta}_1\} = \sqrt{\frac{5.031}{3038.761}} = 0.0407.$
- 95%-confidence interval of β_1:

$$\begin{aligned} 0.637 \pm t(0.975; 926) \times 0.0407 &= 0.637 \pm 1.963 \times 0.0407 \\ &= [0.557, 0.717]. \end{aligned}$$

- We are 95% confident that the regression slope is between 0.557 and 0.717.

T-test for β_1

Normal error model: $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \quad i = 1, 2, \ldots n$

$\varepsilon_i \overset{i.i.d.}{\sim} N(0, \sigma^2)$

$\hat{\beta}_1 = \quad \sim N\left(\beta_1, \frac{\sigma^2}{\sum(x_i - \bar{x})^2}\right)$

- Null hypothesis: $H_0 : \beta_1 = \beta_1^{(0)}$, where $\beta_1^{(0)}$ is a given constant.
- **T-statistic**: Standardization of $\hat{\beta}_1$ under $H_0 : \beta_0 = \beta_1^{(0)}$

LSE — mean of $\hat{\beta}_1$ under H_0

$$T^* = \frac{\hat{\beta}_1 - \beta_1^{(0)}}{s\{\hat{\beta}_1\}}.$$

SE of $\hat{\beta}_1$

reference distribution

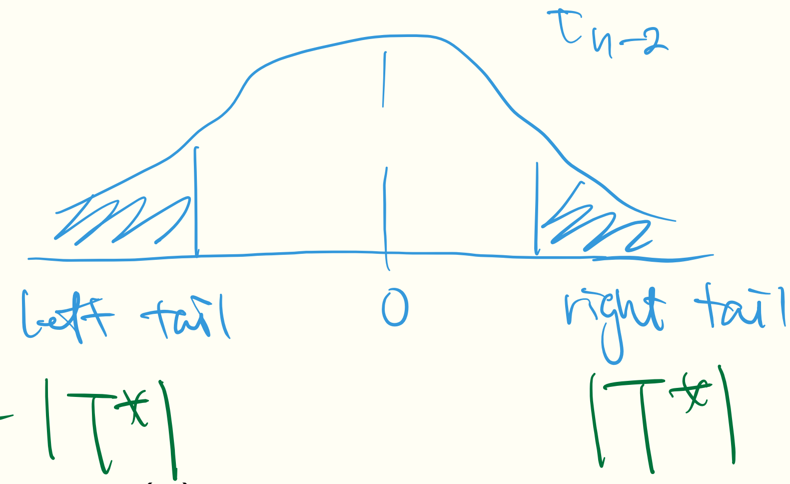
- **Null distribution**:

$$\text{Under } H_0 : \beta_1 = \beta_1^{(0)}, \quad T^* \sim t_{(n-2)}.$$

t distr. w/ $n-2$ df.

Decision Rules

half of p value



crit. value: $t(1-\frac{\alpha}{2}, n-2)$

At significance level α:

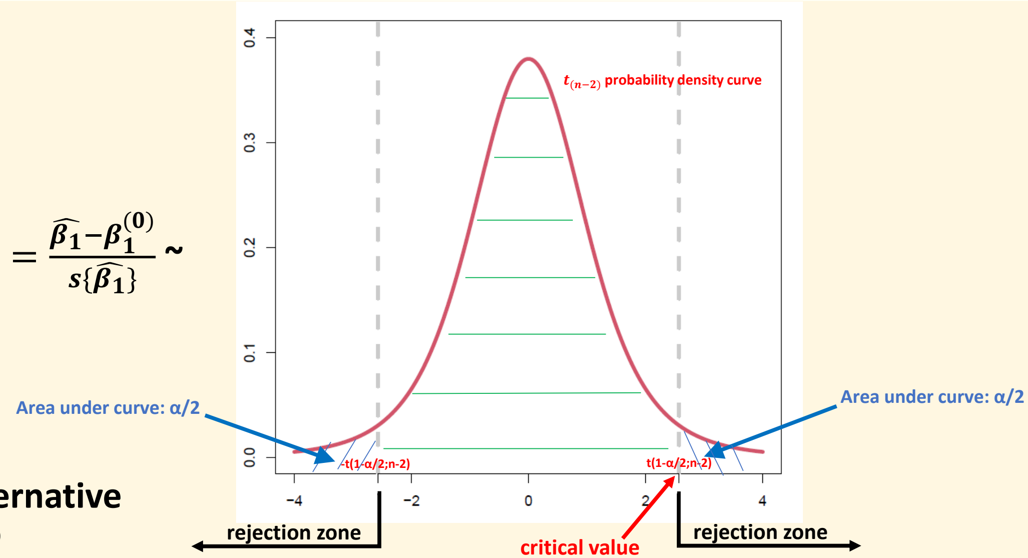
- *Two-sided alternative* $H_a : \beta_1 \neq \beta_1^{(0)}$: Reject H_0 if and only if $|T^*| > t(1 - \alpha/\mathbf{2}; n - 2)$; Or equivalently, reject H_0 if and only if pvalue:$= P(|t_{(n-2)}| > |T^*|) < \alpha$.

magnitude of T^*

- *Left-sided alternative* $H_a : \beta_1 < \beta_1^{(0)}$: Reject H_0 if and only if $T^* < t(\alpha; n - 2)$; Or equivalently, reject H_0 if and only if pvalue:$= P(t_{(n-2)} < T^*) < \alpha$.

- right-sided alt: $H_a: \beta_1 > \beta_1^{(0)}$

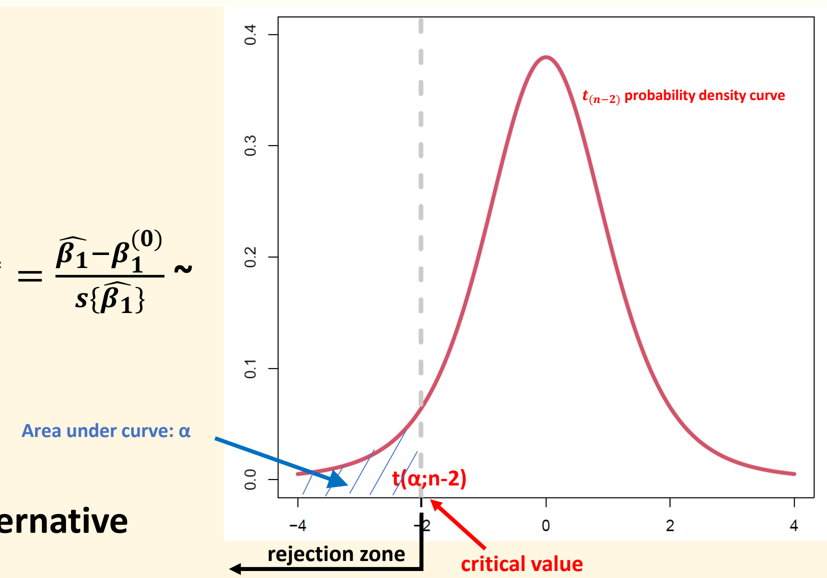
$$under\ H_0 : T^* = \frac{\widehat{\beta_1} - \beta_1^{(0)}}{s\{\widehat{\beta_1}\}} \sim$$



two-sided alternative

H_a: $\beta_1 \neq \beta_1^{(0)}$

$$under\ H_0 : T^* = \frac{\widehat{\beta_1} - \beta_1^{(0)}}{s\{\widehat{\beta_1}\}} \sim$$



Left-sided alternative

H_a: $\beta_1 < \beta_1^{(0)}$

Heights

Test whether there is a linear association between parent's height and child's height at significance level $\alpha = 0.01$.

- $H_0 : \beta_1 = 0$ *vs*. $H_a : \beta_1 \neq 0$.
- $T^* = \frac{\hat{\beta}_1 - 0}{s\{\hat{\beta}_1\}} = \frac{0.637}{0.0407} = 15.7$.
- **Critical value**: $t(1 - 0.01/2; 928 - 2) = 2.58$. Since the observed $|T^*| = |15.7| > 2.58$, reject the null hypothesis at level 0.01.
- **Pvalue**: $P(|t_{(926)}| > |15.7|) \approx 0$. Since *pvalue* $< \alpha = 0.01$, reject the null hypothesis at level 0.01.
- Conclusion: There is a significant association between parent's height and child's height at level 0.01.

Mean Response

Estimation of Mean Response

reg. line: $y = \underbrace{\beta_0 + \beta_1 x}_{\text{average } Y \text{ for } X = x}$

x_i: observation
x_h: any, can be hypothetical

The mean response at $X = X_h$ is $E(Y_h) = \beta_0 + \beta_1 X_h$.

- An unbiased estimator of $E(Y_h)$:

$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \cdot \bar{x}$

$E(\hat{Y}_h) = E(\hat{\beta}_0) + E(\hat{\beta}_1) X_h$
$= \beta_0 + \beta_1 x_2$

$$\widehat{Y}_h \overset{\text{def.}}{=} \hat{\beta}_0 + \hat{\beta}_1 X_h = \underline{\bar{Y} + \hat{\beta}_1 (X_h - \bar{X})}.$$

$\operatorname{var}(\hat{Y}_h) = \operatorname{var}(\bar{y}) +$
$\operatorname{var}[\hat{\beta}_1 \cdot (x_h - \bar{x})] +$
$2\operatorname{cov}(\bar{Y}, \hat{\beta}_1 (x_h - x))$

- $\sigma^2\{\widehat{Y}_h\} = \sigma^2 \left[\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum_{i=1}^n (X_i - \bar{X})^2} \right]$. HW2

- Standard error of $\widehat{Y}_h$:

$$s\{\widehat{Y}_h\} = \sqrt{MSE \left[\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\underline{\sum_{i=1}^n (X_i - \bar{X})^2}} \right]}.$$

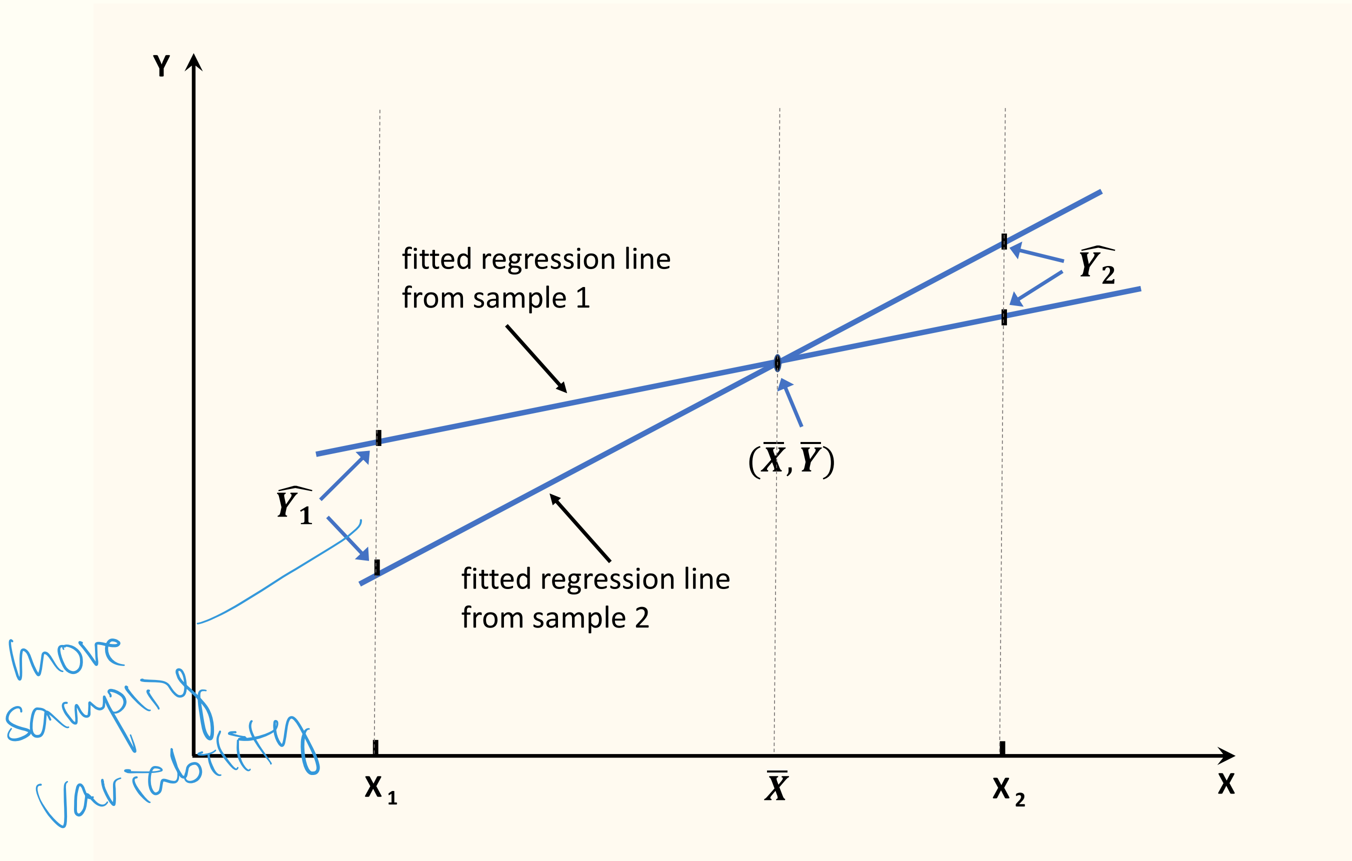
estimated

$\sqrt{\sigma^2\{\hat{Y}_h\}} = \sigma\{\hat{Y}_h\}$

$(n-1) \cdot s^2_x$
sample variance of x

$\operatorname{var}(z_1 + z_2)$
$= \operatorname{var}(z_1) + \operatorname{var}(z_2)$
$+ 2\operatorname{cov}(z_1, z_2)$

- The larger the sample size, or the larger the dispersion of X values, the smaller the SE of $\widehat{Y}_h$.
- The further X_h from $\overline{X}$, the larger the SE of $\widehat{Y}_h$.



Sampling Distribution of $\widehat{Y}_h$

Under the Normal error model:

- $\widehat{Y}_h$ is normally distributed:

$$\widehat{Y}_h \sim \text{Normal}(E(Y_h), \sigma^2\{\widehat{Y}_h\})$$

- Pivotal quantity:

target "parameter"

$$\frac{\widehat{Y}_h - E(Y_h)}{s(\widehat{Y}_h)} \sim t_{(n-2)}$$

Confidence Intervals of $E(Y_h)$

The $(1-\alpha)100\%$ confidence interval of $E(Y_h)$:

$$\widehat{Y}_h \pm t(1-\alpha/2; n-2)s(\widehat{Y}_h)$$

Heights

What is the average height of children of 70*in* parents?

- $n = 928$, $\overline{X} = 68.316$, $\sum_{i=1}^{n}(X_i - \overline{X})^2 = 3038.761$ and $\hat{\beta}_0 = 24.54$, $\hat{\beta}_1 = 0.637$, $MSE = 5.031$
- $\widehat{Y}_h = 24.54 + 0.637 \times 70 = 69.2$
- $s\{\widehat{Y}_h\} = \sqrt{5.031 \times \left\{\frac{1}{928} + \frac{(70-68.316)^2}{3038.761}\right\}} = 0.1$
- 95%-confidence interval: $69.2 \pm 1.963 \times 0.1 = [69, 69.40]$
- We are 95% confident that the average height of children of 70*in* parents is between $[69in, 69.40in]$.

Prediction of New Outcome

estimation: consider fixed

↳ consider both (fixed + random)

↳ harder

Predict a **future outcome** at $X = X_h$:

assume ε_h is independent with $\varepsilon_1, \varepsilon_2 \cdots \varepsilon_n$

Assumption:

$$Y_{h(new)} = \underbrace{\beta_0 + \beta_1 X_h}_{\text{fixed}} + \overset{\text{random}}{\epsilon_h}$$

- Predict $Y_{h(new)}$ by the estimated mean response at $X = X_h$:

$$\widehat{E(Y_h)} = \widehat{Y}_h = \hat{\beta}_0 + \hat{\beta}_1 X_h = \overline{Y} + \hat{\beta}_1 (X_h - \overline{X})$$

- ϵ_h is assumed to be uncorrelated with $\overset{\varepsilon_1, \varepsilon_2, \cdots \varepsilon_n}{\epsilon_i}$s $\rightarrow$ $Y_{h(new)}$ is uncorrelated with the observed Y_is.

Pivotal Quantity

↑ variance in prediction

Under Normal error model:

- $\widehat{Y}_h - Y_{h(new)} \sim$ Normal$(0, \sigma^2(pred_h))$, where

$$\sigma^2(pred_h) := Var(\widehat{Y}_h - Y_{h(new)}) = \sigma^2(\widehat{Y}_h) + \sigma^2(Y_{h(new)})$$

$$= \sigma^2(\widehat{Y}_h) + \sigma^2 = \sigma^2\left[\mathbf{1} + \frac{1}{n} + \frac{(X_h - \overline{X})^2}{\sum_{i=1}^n (X_i - \overline{X})^2}\right]$$

also random [if non-random, var = 0]

- Pivotal quantity: $\frac{\widehat{Y}_h - Y_{h(new)}}{s(pred_h)} \sim t_{(n-2)}$, where

$$s(pred_h) = \sqrt{MSE\left[\mathbf{1} + \frac{1}{n} + \frac{(X_h - \overline{X})^2}{\sum_{i=1}^n (X_i - \overline{X})^2}\right]}$$

Prediction Intervals

The $(1-\alpha)100\%$ prediction interval of $Y_{h(new)}$:

$$\widehat{Y}_h \pm t(1-\alpha/2; n-2)s(pred_h)$$

Prediction vs. Estimation

- $Y_{h(new)}$ – a "moving target" (random variable) vs. $E(Y_h)$ – a fixed quantity (non-random).
- Two sources of variations in the prediction process: Variability from $\widehat{Y}_h$ and variability from the target $Y_{h(new)} \rightarrow s(pred_h) > s(\widehat{Y}_h)$.

  for E, only from $\widehat{Y}_h$
- At a given X value, the prediction interval of a new outcome is wider than the confidence interval of the mean response.

Heights

What would be the predicted height of the child of a 70*in* couple?

- $n = 928$, $\overline{X} = 68.316$, $\sum_{i=1}^{n}(X_i - \overline{X})^2 = 3038.761$, and $\hat{\beta}_0 = 24.54$, $\hat{\beta}_1 = 0.637$, $MSE = 5.031$
- Predicted height: $\widehat{Y}_h = 24.54 + 0.637 \times 70 = 69.2$
- Standard error:

$$s\{pred_h\} = \sqrt{5.031 \times \left\{\mathbf{1} + \frac{1}{928} + \frac{(70 - 68.316)^2}{3038.761}\right\}} = 2.25$$

- 95% prediction interval: $69.2 \pm 1.8831 \times 2.25 = [64.75, 73.56]$
- We are 95% confident that the child's height will be between $[64.75in, 73.56in]$.

Extrapolation

Extrapolation occurs when predicting the outcome at an X value that lies outside of the observed data range.

- Every model has a **range of validity**.
- A model may be inappropriate when it is extended outside of the range of the observations upon which it was built.
- Extrapolation is less reliable than interpolation and need to be handled with caution.

Analysis of Variance

Analysis of Variance

- Basic idea: attributing variation in the data to different sources through **decomposition of the total variation.**
- In regression, the variation in the observations comes from:
  - variation in the error term
  - variation in X

$$y = \underbrace{\beta_0 + \beta_1 x_i}_{} + \underbrace{\varepsilon_i}_{}$$

Partition of Total Deviation

$\hat{y}_i$: fitted value

$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$

- **Total deviation:** difference between Y_i and the sample mean $\overline{Y}$:

ref. point

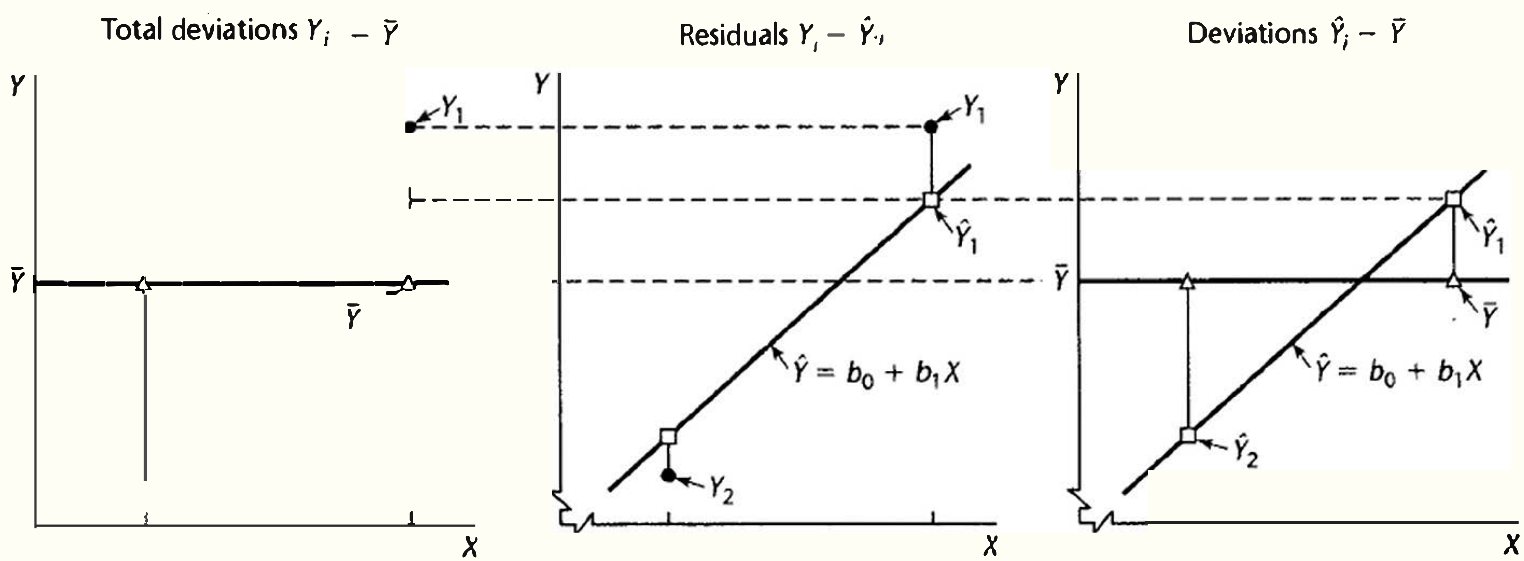
$$Y_i - \overline{Y}, \quad i = 1, \cdots, n.$$

- Total deviation can be decomposed into the sum of two terms:

$$Y_i - \overline{Y} = \underbrace{(Y_i - \widehat{Y}_i)}_{e_i} + (\widehat{Y}_i - \overline{Y}), \qquad i = 1, \ldots, n$$

- I.e., the *deviation of the observed value around the fitted regression line (residual)* and the *deviation of the fitted value from the sample mean*.

Figure: Partition of total deviation



quantity variation due to x_i

Decomposition of Total Variation

$$Y_i - \bar{Y} = (Y_i - \hat{Y}_i) + (\hat{Y}_i - \bar{Y})$$

$$\sum (Y_i - \bar{Y})^2 = \sum \big((Y_i - \hat{Y}_i) + (\hat{Y}_i - \bar{Y})\big)^2 = \sum (Y_i - \hat{Y}_i)^2 + 2\sum (Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) + \sum (\hat{Y}_i - \bar{Y})^2$$

0

- Taking sum of squares of the total deviations and noting that the sum of the cross product terms vanishes:

$$SSTO = \sum_{i=1}^{n}(Y_i - \overline{Y})^2 = \underbrace{\sum_{i=1}^{n}(Y_i - \widehat{Y}_i)^2}_{SSE} + \underbrace{\sum_{i=1}^{n}(\widehat{Y}_i - \overline{Y})^2}_{SSR}.$$

$e_i = Y_i - \hat{Y}_i$

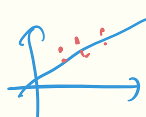
residual

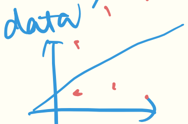
SSE (how well fit data)

SSR slope

variation fitted value around sample mean

- Decomposition of total variation:





$$SSTO = SSE + SSR$$

$$\sum (Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) = \sum e_i(\hat{Y}_i - \bar{Y}) = \sum e_i \hat{Y}_i - \left(\sum e_i\right)\bar{Y}$$

$$= 0 - 0 \times \bar{Y}$$

$$= 0$$

ANOVA: Sums of Squares

Total Sum of Squares (SSTO)

Quantify variation of the observations around the sample mean:

$$SSTO := \sum_{i=1}^{n} (Y_i - \overline{Y})^2, \quad d.f.(SSTO) = n - 1.$$

$\Sigma (Y_i - \bar{Y}) = 0$

1 linear constraint on total deviation

Error Sum of Squares (SSE)

Quantify variation of the observations around the fitted regression line:

$$SSE = \sum_{i=1}^{n}(Y_i - \widehat{Y}_i)^2, \quad d.f.(SSE) = n - 2.$$

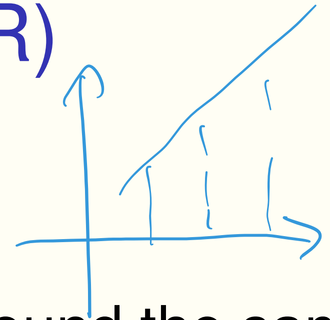
$= \sum_i e_i^2$

$\sum_i e_i = 0$, $\sum e_i x_i = 0$: 2 linear constraints in residuals

Regression Sum of Squares (SSR)

SSTO = SSE + SSR

df (n-1) = (n-2) + 1



?

Quantify variation of the fitted values around the sample mean:

HW 2

dispersion of x-value

y variation explained by model

$$SSR = \sum_{i=1}^{n}(\widehat{Y}_i - \overline{Y})^2 = \hat{\beta}_1^2 \sum_{i=1}^{n}(X_i - \overline{X})^2, \quad d.f.(SSR) = 1.$$

→ slope of fitted regression line

- $SSR = SSTO - SSE$: reduction of uncertainty in Y by utilizing the predictor X through a linear regression model
- The larger the fitted regression slope or the more the dispersion of X values, the larger SSR

Mean Squares

Sum of Squares divided by its degree of freedom:

$$MS \overset{\text{def}}{=} SS/d.f.(SS).$$

- Mean squared error:

$$MSE = \frac{SSE}{\text{d.f.}(SSE)} = \frac{SSE}{n-2}$$

- Regression mean square:

$$MSR = \frac{SSR}{\text{d.f.}(SSR)} = \frac{SSR}{1}$$

for simple regression with only 1 x var.

ANOVA: F Tests

Expected Values of SS and MS

Under simple regression model:

- Expected values of SS:

$$E(SSE) = (n-2)\sigma^2, \quad E(SSR) = \sigma^2 + \beta_1^2 \sum_{i=1}^{n}(X_i - \overline{X})^2.$$

HW2

$\div (n-2)$

copy $\because df = 1$

- Expected values of MS:

$$E(MSE) = \sigma^2, \qquad E(MSR) = \sigma^2 + \beta_1^2 \sum_{i=1}^{n}(X_i - \overline{X})^2.$$

- $E(MSR) \geq E(MSE)$ and "$=$" holds iff $\beta_1 = 0$.

no linear association

Sampling Distributions of SS

Under Normal error model:

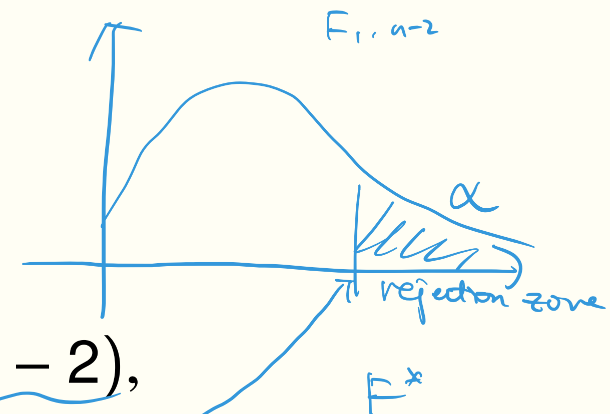
- $SSE \sim \sigma^2 \chi^2_{(n-2)}$
- SSE and SSR are independent.

F Test

- $H_0 : \beta_1 = 0$ vs. $H_a : \beta_1 \neq 0$
- F ratio: $F^* = \frac{MSR}{MSE} = \frac{SSR/1}{SSE/(n-2)}$

F distribution

- Null distribution: $F^* \underset{H_0:\beta_1=0}{\sim} F_{1,n-2}$.
- Decision rule at the significance level α:

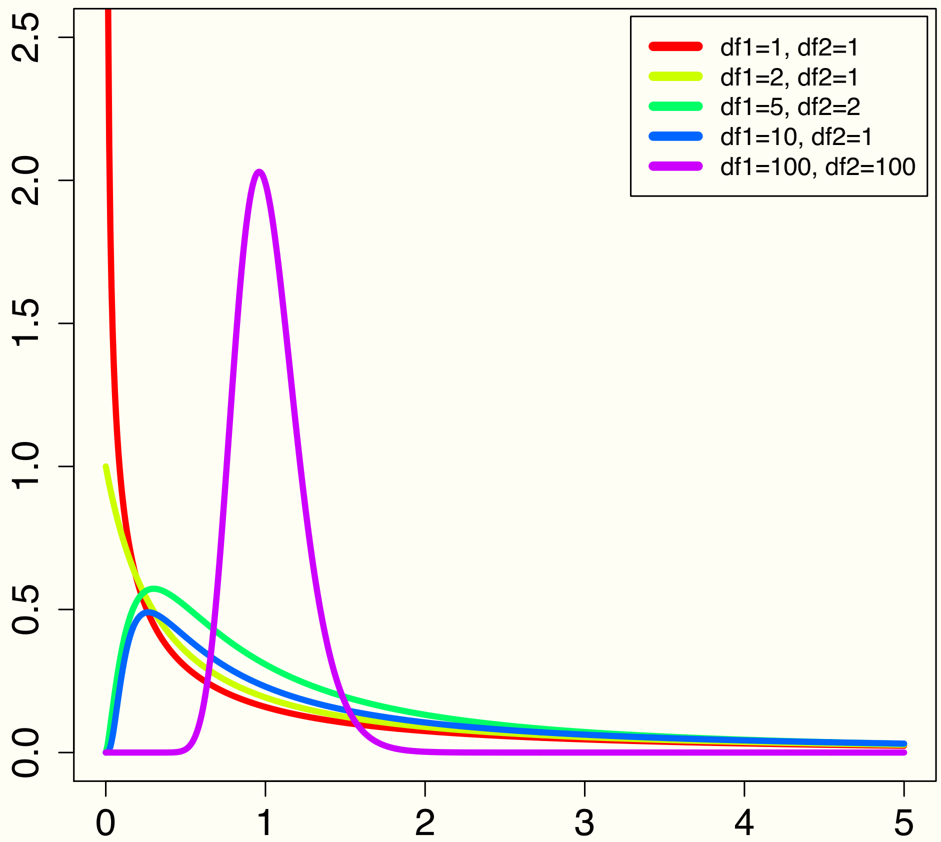


$$\text{reject } H_0 \text{ if } \quad F^* > \underbrace{F(1-\alpha; 1, n-2)}_{\text{critical value}},$$

where $F(1-\alpha; 1, n-2)$ is the $(1-\alpha)100$th percentile of the $F_{1,n-2}$ distribution.

F Distributions

Figure: F distributions: probability density function



F: intrinsically 2 sided, useful when more than 1 x variable

t: need for 1 sided,

In simple linear regression, the F-test is equivalent to the two-sided t-test for testing $H_0 : \beta_1 = 0$ versus $H_a : \beta_1 \neq 0$.

- $F^* = (T^*)^2$
- $F(1-\alpha; 1, n-2) = t^2(1-\alpha/2; n-2)$.

$\therefore \quad F^* > F(1-\alpha; 1, n-2) \iff |T^*| > t(1-\alpha/2; n-2)$

ANOVA Table for Simple Regression

Source of Variation	SS	d.f.	MS=SS/d.f.	F^*
Regression	$SSR = \sum_{i=1}^{n}(\widehat{Y}_i - \overline{Y})^2$	1	$MSR = SSR/1$	MSR/MSE
Error	$SSE = \sum_{i=1}^{n}(Y_i - \widehat{Y}_i)^2$	$n-2$	$MSE = SSE/(n-2)$	
Total	$SSTO = \sum_{i=1}^{n}(Y_i - \overline{Y})^2$	$n-1$		

Heights

Source of Variation	SS	d.f.	MS=SS/d.f.	F^*
Regression	$SSR = 1234$	1	$MSR = 1234$	245
Error	$SSE = 4659$	926	$MSE = 5.03$	
Total	$SSTO = 5893$	927		

- Test whether there is a linear association between parent's height and child's height at significance level $\alpha = 0.01$.
- $F(0.99; 1, 926) = 6.66 < F^* = 245$, so reject $H_0 : \beta_1 = 0$ and conclude that there is a significant linear association between parent's height and child's height.

Coefficient of Determination

Coefficient of Determination R^2

A descriptive measure for **linear association** between X and Y:

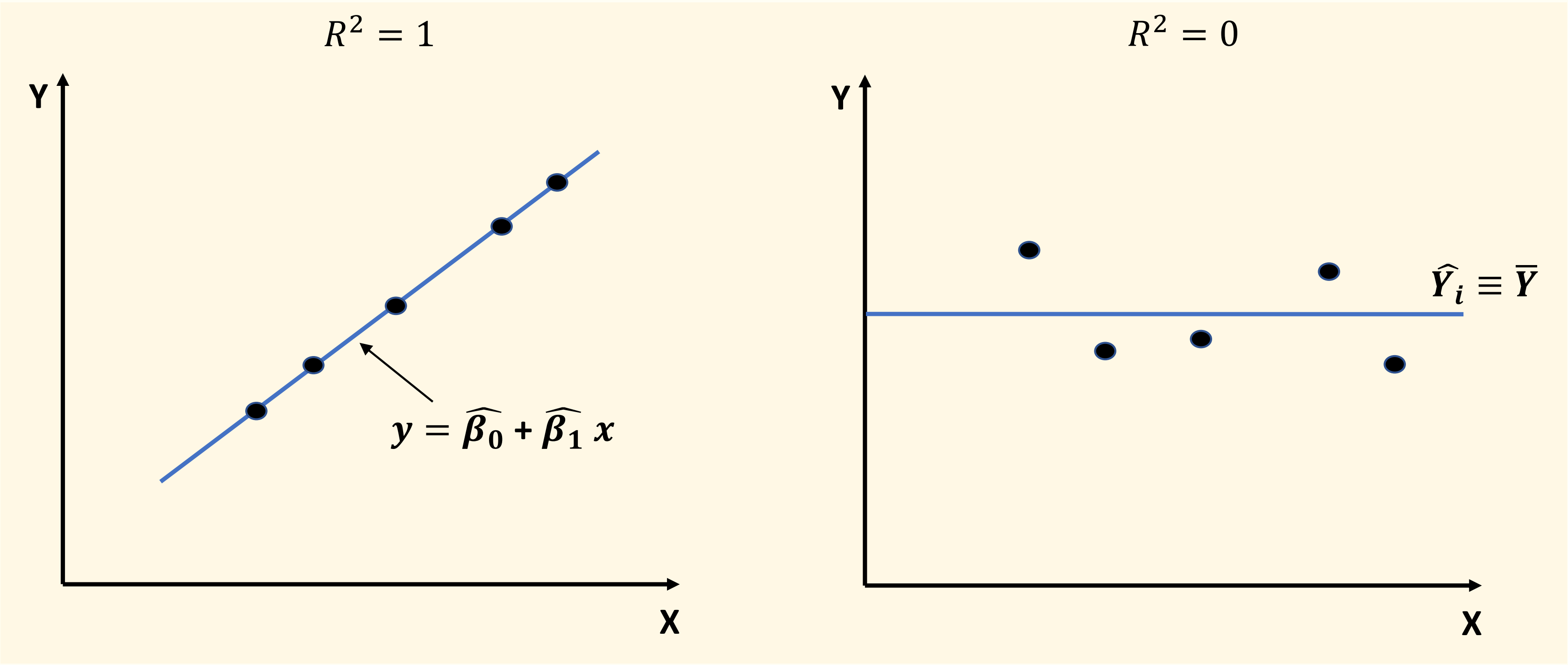
$$R^2 = \frac{SSR}{SSTO} = 1 - \frac{SSE}{SSTO}.$$

- Heights: $R^2 = \frac{1234}{5893} = 0.209$. 20% of variation in child's height may be explained by the variation in parent's height.

Properties of R^2

- $0 \leq R^2 \leq 1$.
- If all observations fall on one straight line, then $R^2 = 1$.
  - X accounts for all variation in the observations.
- If the fitted regression line is horizontal, i.e., $\hat{\beta}_1 = 0$, then $R^2 = 0$.
  - X is of no use in explaining variation in the observations.
  - There is no evidence of linear association between X and Y in the data.

Figure:



$R^2 = 1$
Y
X
$y = \widehat{\beta_0} + \widehat{\beta_1}\, x$
$R^2 = 0$
Y
X
$\widehat{Y}_i \equiv \overline{Y}$

Caution with Interpreting R^2

When the relationship between X and Y is nonlinear, R^2 is not a meaningful measure.

- *"A large R^2 means that the estimated regression line must be a good fit of the data". Not necessarily!*
- *"A near zero R^2 means that X and Y are not related". Not necessarily!*