Biological invasion in a predator-prey model with a free boundary

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January 23, 2019

AMS Subject Classification. 35K57, 35R35, 92C50, 92B99.

Keywords. Free Boundary, Biological invasions, Predator-Prey, Spreading-Vanishing Dichotomy.

Abstract

In this paper we study a predator-prey system with free boundary in a one-dimensional environment. The predator v is the invader which exists initially in a sub-interval $[0, s_0]$ of $[0, L]$ and has the Leslie-Gower terms that measure the loss in the predator population due to rarity of the prey. The prey u (the native species) is initially distributed over the whole region $[0, L]$. Our primary goal is to understand how the success or failure of the predator's invasion is affected by the initial datum v_0. We derive a spreading-vanishing dichotomy and give sharp criteria for spreading and vanishing in this model.

1 Introduction and statement of the main results

This paper is concerned with the existence and qualitative properties of solutions to a predator-prey system of semilinear parabolic type over a bounded spatial domain subject to free-boundary conditions. Inspired by former works (Chen and Shi [5] for instance) that study the nonlinear evolution of two species on an *unbounded spatial domain*, we focus on the case where indigenous population undergoes diffusion and growth in a *bounded* domain $[0, L]$ to be more realistic. We discuss some of the prior works in Subsection 1.1 below.

In this work, we consider system (1) over a bounded domain $[0, L]$ with Leslie-Gower type nonlinearity. The nonlinear evolution equations that u and v satisfy are as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = u_{xx} + u(1-u) - v\left(\dfrac{u}{u+m}\right) & \text{for } t>0 \text{ and } 0<x<L, \\ \dfrac{\partial v}{\partial t} = Dv_{xx} + kv\left(1 - \dfrac{bv}{u+a}\right) & \text{for } t>0 \text{ and } 0<x<s(t). \end{cases} \tag{1}$$

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The above equations are subject to the following initial, boundary and free-boundary conditions, for some $\mu > 0$

$$\begin{cases} s'(t) = -\mu v_x(t, s(t)) \text{ for all } t > 0, \\ s(0) = s_0 \text{ for all } x \in [0, s_0], \\ v(t, s(t)) = 0 \text{ for all } t > 0, \\ v(0, x) = v_0(x) \text{ for } x \in [0, s_0], \quad u(0, x) = u_0(x) \text{ for all } x \in [0, L], \\ u_x(t, 0) = v_x(t, 0) = 0, \quad u(t, L) = 0, \end{cases} \tag{2}$$

where all the parameters a, b, k, D, m and μ are positive.

Model (1), coupled with the conditions in (2), governs the dynamics of two species (u and v) over a bounded spatial domain $[0, L]$, where the function u (resp. v) stands for the population of the prey (resp. predator). The condition $v(t = 0, \cdot) \equiv v_0(\cdot)$ on $[0, s_0]$ conveys that v initially occupies only a subregion $[0, s_0] \subset [0, L]$ of the whole domain.

The nonlinear term $\frac{u}{u+m}$ in (1) is the Holling type-II functional response. This type of nonlinearity is commonly used in the ecological literature. We refer the reader to [4] for more details.

For species u who inhabit a finite region with a lethal exterior boundary point L (see the condition in (2), which are of Dirichlet boundary type). The evolution equation satisfied by v, namely the second equation in (1), holds over an evolving domain $(0, s(t))$ however. This brings a free-boundary nature to our problem. The first condition in (2), which is well-known as the Stefan condition, states that the speed at which the free-boundary expands is proportional to the population-gradient at this location.

Now we comment on some parameters in model (1)-(2) before we briefly discuss some prior works in Subsection 1.1. The domain size L is such that

$$L > \max\left\{\frac{\pi}{2}\sqrt{D/k}, \frac{\pi}{2}\right\}.$$

This choice of L is familiar: it appears as the critical domain size for the survival of a single species obeying a reaction-diffusion equation on the domain $[0, L]$, see Section 3.2 in [4]. We will see that this condition on L, together with additional conditions we derive later, play an essential role in the long-time asymptotic behaviours of the population densities u and v.

Initial data. The initial data u_0 and v_0 are assumed to satisfy

$$\begin{cases} u_0(x) \in C^2([0, L]), \quad v_0(x) \in C^2([0, s_0]), \\ u_0(0) = v_0(0) = v_0(s_0) = 0 \text{ for some } 0 < s_0 < L, \quad u_0(L) = 0, \\ u_0(x) > 0 \text{ for } x \in [0, L] \quad \text{and} \quad v_0(x) \geq 0 \text{ for all } x \in [0, s_0). \end{cases} \tag{3}$$

The parameters a, b and m In the evolution equation satisfied by v in system (1), the parameter a represents the extent to which prey resources provide protection to predator v. In all what follows, we assume that a, b and m satisfy the following hypothesis:

$$bm > 1 \text{ and } a < bm - 1. \tag{\textbf{H}}$$

Hypothesis (**H**) will be essential in proving our results about long-time asymptotic behaviours in Theorems 1.4 and 1.6.

1.1 Prior works

Many recent works [1, 2, 3, 11, 12, 19, 20, 21] studied predator-prey systems with the Leslie-Gower scheme. We will discuss the most relevant to our present work.

Chen and Shi [5] studied the following Holling-Tanner predator-prey model

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \triangle u + \alpha u - u^2 - \dfrac{uv}{u+\iota}, & t>0, x\in\Omega, \\ \dfrac{\partial v}{\partial t} = d_2 \triangle v + \gamma v(1-\dfrac{\varepsilon v}{u}), & t>0, x\in\Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & t>0, x\in\partial\Omega, \\ u(0,x)=u_0(x)>0,\ v(0,x)=v_0(x)\geq 0, & x\in\Omega, \end{cases} \tag{C.S.}$$

where u (resp. v) is the population of the prey (resp. predator). The term $\varepsilon v/u$, known as the Leslie-Gower term, measures the loss in predator v due to rarity of its favorite food u. The parameter ε is the number of prey required to support one predator at equilibrium when v equals $\frac{u}{\varepsilon}$. The nonlinearity $\frac{u}{u+\iota}$ in (C.S.) is the Holling type-II functional response. This type of nonlinearity is commonly used in the ecological literature (see [4] for details.) The parameter ι is a positive constant measuring the extent to which the environment provides protection to prey u. Chen and Shi [5] proved that the unique constant equilibrium of system (C.S.) is globally asymptotically stable.

The problem which describes the dynamical process of a new competitor invading the habitat of a native species originates from Du and Lin [9] who introduced the following free boundary problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u(m - b_1 u - c_1 v), & 0<r<h(t), \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + v(a_2 - b_2 u - c_2 v), & 0<r<+\infty, \\ h'(t) = -\mu u_r(t,h(t)), & t>0, \\ u_r(t,0)=v_r(t,0)=0, u(t,r)=0, & t>0, h(t)\leq r<+\infty, \\ u(0,r)=u_0(r), r\in[g(t),h(t)], & t=0, \\ u(0,r)=u_0(r), h(0)=h_0, & 0\leq r\leq h_0, \\ v(0,r)=v_0(r), & 0\leq r<+\infty. \end{cases} \tag{DL}$$

Du and Lin [9] considered two cases: (1) u is the superior competitor and v is the inferior competitor or (2) v is the superior competitor and u is the inferior competitor. When u is the superior competitor, [9] proves that a spreading-vanishing dichotomy holds. Namely, as $t\to+\infty$, either $h(t)\to\infty$ and $(u,v)\to(u^*,0)$, or $h_\infty<\infty$ and $(u,v)\to(0,v^*)$.
For more similar nonlinear free boundary problems, we refer the reader to [14, 15, 16, 17, 22, 23, 24, 25] and the references cited therein.

In the rest of this section, we state our main results. Subsection 1.2 shows existence and uniqueness of solutions to the model (1) subject to conditions (2). Subsection 1.4 shows the criteria on the parameters of the system in order to have specific asymptotic behaviours as $t\to\infty$.

The results of Subsection 1.4 address specifically the questions on whether the species vanish or spread throughout the domain $[0,L]$ after a large enough time.

1.2 Global existence of smooth solutions

Theorem 1.1. *Assume that* (u_0, v_0) *satisfying (3). Then for any* $\theta \in (0,1)$ *and there is* $T > 0$ *such that the problem* (1)-(2) *admits a unique solution* $(u(t,x), v(t,x), s(t))$ *for* $t \in [0,T]$. *Moreover,*

$$(u,v,s) \in C^{\frac{(1+\theta)}{2},1+\theta}(Q_u) \times C^{\frac{(1+\theta)}{2},1+\theta}(Q_v) \times C^{1+\frac{\theta}{2}}([0,T]),$$

where

$$Q_u = \left\{(t,x) \in \mathbb{R}^2 : \ t \in [0,T] \ \textit{and} \ x \in [0,L]\right\},$$

and

$$Q_v = \left\{(t,x) \in \mathbb{R}^2 : \ t \in [0,T] \ \textit{and} \ x \in [0,s(t)]\right\}.$$

The following lemma is essential in proving the existence of a global-in-time solution to the free-boundary problem (1)-(2).

Lemma 1.1 (Towards global solutions in time)**.** *Let* $(u, v, s(t))$ *be a solution of* (1)-(2) *for* $t \in [0,T]$ *for some* $T > 0$. *Then,*

(i) $0 < u(t,x) \le \max\{1, \|u_0\|_\infty\} := M_1$ *for all* $t \in [0,T]$ *and* $x \in [0,L)$;

(ii) $0 < v(t,x) \le \max\{M_1 + a, \|v_0\|_\infty\} := M_2$ *for all* $t \in [0,T]$ *and* $x \in [0,s(t))$;

(iii) $0 < s'(t) \le \Lambda$ *for all* $t \in (0,T]$,

where $\Lambda > 0$ *is a constant depending on* μ, D, k, $\|u_0\|_\infty$, $\|v_0\|_\infty$, $\|u_0'\|_{C[0,s_0]}$ *and* $\|v_0'\|_{C[0,s_0]}$.

From Theorem 1.1 and Lemma 1.1, we get the following global existence result.

Theorem 1.2. *Assume that* (u_0, v_0) *satisfies the condition (3), then for any* $\theta \in (0,1)$, *the free boundary problem* (1)-(2) *admits a unique solution*

$$(u(t,x), v(t,x), s(t))$$

which satisfies

$$(u,v,s) \in C^{\frac{(1+\theta)}{2},1+\theta}(Q_u) \times C^{\frac{(1+\theta)}{2},1+\theta}(Q_v) \times C^{1+\frac{\theta}{2}}([0,+\infty)),$$

where

$$Q_u = \{(t,x) \in \mathbb{R}^2 : t \in [0,+\infty) \ \textit{and} \ x \in [0,L]\},$$

and

$$Q_v = \{(t,x) \in \mathbb{R}^2 : t \in [0,+\infty) \ \textit{and} \ x \in [0,s(t)]\}.$$

1.3 Preliminaries

We start with a remark regarding the asymptotics of the free boundary $s(t)$:

Remark 1.1. *As we will see, Lemma 1.1 and Theorem 1.2 yield that* $s'(t) > 0$ *for all* $t > 0$. *This allows us to define the limit* s_∞ *as follows*

$$s_\infty := \lim_{t \to +\infty} s(t) \ \textit{in} \ [0,+\infty) \cup \{+\infty\}. \tag{4}$$

Then, we may have three different cases according to the relation between s_∞ *and L:*

$$\text{(i) } s_\infty < L, \quad \text{(ii) } s_\infty = L \quad \textit{or} \quad \text{(iii) } s_\infty > L.$$

Since L is finite, then if $s_\infty > L$, there exists $0 < T_* < \infty$ such that $s(T_*) = L$. In such case, the prey v exists in the whole region $[0, L]$ and the free-boundary problem (1) changes to the following fixed-boundary problem which holds over the whole interval $(0, L)$ (when $t > T_*$)

$$\begin{cases} \dfrac{\partial u}{\partial t} = u_{xx} + u(1-u) - v\dfrac{u}{u+m} & \text{for } t > T_* \text{ and } 0 < x < L, \\ \dfrac{\partial v}{\partial t} = Dv_{xx} + kv\left(1 - \dfrac{bv}{u+a}\right) & \text{for } t > T_* \text{ and } 0 < x < L, \end{cases} \tag{5}$$

with the conditions

$$\begin{cases} u_x(t,0) = v_x(t,0) = 0, & t > T_*, \\ u(t,L) = v(t,L) = 0, & t > T_*, \\ u(T_*, x) = u_{T_*}(x), & x \in [0, L], \\ v(T_*, x) = v_{T_*}(x), & x \in [0, L]. \end{cases} \tag{6}$$

The following theorem demonstrates rather strikingly that $s_\infty \neq L$. This rules out the possibility (ii), above:

Theorem 1.3. *Let s_∞ be as defined in (4) and $L > \max\left\{\frac{\pi}{2}\sqrt{D/k}, \frac{\pi}{2}\right\}$. Then we have a dichotomy for the relation between s_∞ and L. Namely, either $s_\infty < L$ or $s_\infty > L$.*

1.4 Spreading and Vanishing

The following statement is a comparison principle related to the free boundary problem (1)-(2). This comparison principle will help in deriving criteria for the spread or extinction/vanishing (as $t \to +\infty$) of the solutions to our system (1)-(2).

Lemma 1.2 (Comparison principle)**.** *Let (u, v, s) be a classical solution to the free-boundary problem (1)-(2) with initial data (u_0, v_0) and denote by*

$$M_1 := \max\{1, \|u_0\|_\infty\}.$$

(a) *Assume that $(\omega, \delta(t))$ satisfies*

$$\begin{cases} \omega_t - D\omega_{xx} \geq k\omega\left(1 - \dfrac{b\omega}{M_1 + a}\right) & \text{for } t > 0 \text{ and } 0 < x < \delta(t), \\ \omega_x(t,0) \leq 0, \quad \omega(t, \delta(t)) = 0, \\ \delta'(t) \geq -\mu\omega_x(t, \delta(t)). \end{cases} \tag{7}$$

If $\omega(0,x) \geq v_0(x)$ in $[0, L]$ and $\delta(0) \geq s(0)$, then

- *$\delta(t) \geq s(t)$ for all $t \geq 0$.*
- *$\omega(t,x) \geq v(t,x)$ for all $x \in [0, s(t)]$.*

(b) *Assume that* $(\nu, \sigma(t))$ *satisfies*

$$\begin{cases} \nu_t - D\nu_{xx} \leq k\nu\left(1 - \dfrac{b\nu}{a}\right) & \text{for } \; t > 0 \text{ and } 0 < x < \sigma(t), \\ \nu_x(t,0) \geq 0, \quad \nu(t,\sigma(t)) = 0, \\ \sigma'(t) \leq -\mu\nu_x(t,\sigma(t)). \end{cases} \tag{8}$$

If $\nu(0,x) \leq v_0(x)$ *in* $[0,L]$ *and* $\sigma(0) \leq s(0)$*, then*

- $\sigma(t) \leq s(t)$ *for all* $t \geq 0$*,*
- $\nu(t,x) \leq v(t,x)$ *for all* $x \in [0,\sigma(t)]$*.*

Theorem 1.4 (Spreading)**.** *Suppose that* (u,v,s) *is the solution of* (1) *subject to the conditions in* (2)*. If* $s_\infty > L$*, then we have*

$$\limsup_{t\to+\infty} u(t,x) \leq \bar{u}(x); \;\; \liminf_{t\to+\infty} u(t,x) \geq \underline{u}(x);$$

and

$$\limsup_{t\to+\infty} v(t,x) \leq \bar{v}(x); \;\; \liminf_{t\to+\infty} v(t,x) \geq \underline{v}(x).$$

Where $\bar{u}(x)$*,* $\underline{u}(x)$*,* $\bar{v}(x)$*,* $\underline{v}(x)$ *are determined in the Lemma 2.1.*

Theorem 1.5 (Vanishing)**.** *Suppose that* $(u,v,s(t))$ *is a solution of* (1)*-*(2)*. If* $s_\infty < L$*, then we have*

$$\liminf_{t\to+\infty} u(t,\cdot) \geq \bar{u}(x) \text{ for } x \in [0,L] \text{ and } \lim_{t\to+\infty} \|v(t,\cdot)\|_{C[0,s(t)]} = 0.$$

Here $\bar{u}(x)$ *be determined in the Lemma 2.1.*

Definition 1.1 (The notion of 'Vanishing' and 'Spreading')**.** *Based on the results of Theorems 1.4 and 1.5, we say that the species* v spreads successfully *if* $s_\infty > L$*. In such case, we have*

$$\limsup_{t\to+\infty} u(t,x) \leq \bar{u}(x); \;\; \liminf_{t\to+\infty} u(t,x) \geq \underline{u}(x);$$

and

$$\limsup_{t\to+\infty} v(t,x) \leq \bar{v}(x); \;\; \liminf_{t\to+\infty} v(t,x) \geq \underline{v}(x).$$

We say that the species v vanishes eventually *if* $s_\infty < L$*. In such case we have*

$$\liminf_{t\to+\infty} u(t,\cdot) \geq \bar{u}(x) \text{ for } x \in [0,L] \text{ and } \lim_{t\to+\infty} \|v(t,\cdot)\|_{C[0,s(t)]} = 0.$$

1.5 Criteria for spreading and vanishing

In this subsection, we find conditions on the parameters D, k, L, μ and $s(0) := s_0$ which determine whether the components of a solution (u,v,s) to the free-boundary problem (1), subject to the conditions (2), will spread or vanish as $t \to +\infty$.

Lemma 1.3. *If* $s_\infty < L$*, then* $s_\infty \leq \frac{\pi}{2}\sqrt{\frac{D}{k}}$*. Furthermore, if* $s_0 \geq \frac{\pi}{2}\sqrt{\frac{D}{k}}$ *then* $s_\infty > L$*.*

Theorem 1.6. *Suppose that* $s(0) := s_0 < \frac{\pi}{2}\sqrt{\frac{D}{k}}$ *in the free boundary problem* (1)*-*(2)*. Then,*

1. *if* $\displaystyle\int_0^{s_0} v_0(x)\,\mathrm{d}x \geq \max\left\{1, \frac{b\|v_0\|_\infty}{a}\right\} \times \frac{D}{\mu} \times \left(\frac{\pi}{2}\sqrt{\frac{D}{k}} - s_0\right)$, *the species v spreads successfully.*

2. *Let* $\delta = \dfrac{1}{2}\left(\dfrac{\frac{\pi}{2}\sqrt{D/k}}{s_0} - 1\right) > 0$ *and* $\beta = \dfrac{\pi^2}{8}\dfrac{D}{(1+\delta)^2 s_0^2} - \dfrac{k}{2} > 0$. *If*

$$\|v_0\|_\infty \leq \cos\left(\frac{\pi}{2+\delta}\right)\frac{\delta s_0^2 \beta(2+\delta)}{2\pi\mu},$$

then the species v vanishes eventually.

2 Proofs

2.1 Proofs of the results on global existence and the comparison principle

In this section, we first prove the local and global existence results of solution for the free boundary problem (1). We also derive a comparison principle which will be used several times in our proofs.

Proof of Theorem 1.1. We will use the contraction mapping principle on some functional spaces arranged after rewriting the problem in a domain without a free boundary. We follow the same steps, leading to local existence, followed in [10]. But we have to pay attention to the facts that our model is different from the one in [10] (especially the nonlinearities) and the spatial domain in our work is bounded.

We first straighten the free boundary and transform it to a "fixed" boundary through a common change of variables (appeared first in [6] in the case where the spatial domain is the whole real line): let $\eta \in C^3([0,\infty))$ such that

$$\begin{aligned} &\text{for all } y,\ |\eta'(y)| \leq \frac{2}{\sigma},\\ &\eta(y) = 1 \ \text{ if } \ |y - s_0| \leq \frac{\sigma}{4},\\ &\eta(y) = 0 \text{ if } \ |y - s_0| > \sigma, \end{aligned}$$

where we have chosen $\sigma = \frac{1}{2}\min\{L - s_0, s_0\}$. We then define

$$x = y + \eta(y)(s(t) - s_0), \quad 0 \leq y \leq L. \tag{9}$$

If $|s(t) - s_0| \leq \sigma/4$, the transformation $(x,t) \longrightarrow (y,t)$ is a diffeomorphism from $[0,L]$ to $[0,L]$: indeed fixing t so that $|s(t) - s_0| \leq \sigma/4$, we see that the transformation $x \to y$ is bijective since

$$\frac{\partial x}{\partial y} = 1 + (s(t) - s_0)\eta'(y) \geq 1 - |s(t) - s_0||\eta'(y)| \geq 1/2 > 0.$$

Moreover,

$$(0 \leq x \leq s(t)) \Longleftrightarrow (0 \leq y \leq s_0) \text{ and } (x = s(t)) \Longleftrightarrow (y = s_0). \tag{10}$$

Now we compute

$$\frac{\partial y}{\partial x} = \frac{1}{1+\eta'(y)(s(t)-s_0)} := Q_1(s(t), y(t)),$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{-\eta''(y)(s(t)-s_0)}{[1+\eta'(y)(s(t)-s_0)]^3} := Q_2(s(t), y(t)),$$

$$\text{and} \quad \frac{\partial y}{\partial t} = \frac{-s'(t)\eta(y)}{1+\eta'(y)(s(t)-s_0)} := Q_3(s(t), y(t)).$$

To simplify the presentation in the following steps, we denote by

$$U(t,y) = u(t,x), \quad V(t,y) = v(t,x),$$

$$F(U,V) = U(1-U) - V\frac{U}{U+m} \text{ and } G(U,V) = kV\left(1-\frac{bV}{U+a}\right).$$

Then problem (1) is transformed to the following '*fixed-boundary*' problem

$$\begin{cases} \dfrac{\partial U}{\partial t} = Q_1^2 U_{yy} + (Q_2 - Q_3)U_y + F(U,V), & t>0 \text{ and } 0<y<L, \\ \dfrac{\partial V}{\partial t} = DQ_1^2 V_{yy} + (DQ_2 - Q_3)V_y + G(U,V), & t>0 \text{ and } 0<y<s_0, \\ U_y(t,0) = U(t,L) = 0, & t>0, \\ V_y(t,0) = V(t,s_0) = 0, & t>0, \\ s'(t) = -\mu U_y(t,s_0), & t>0, \\ V(0,y) = V_0(y), & y\in[0,s_0], \\ U(0,y) = U_0(y), & y \in [0,L]. \end{cases} \tag{11}$$

As mentioned above, we will use the contraction mapping principle in order to prove the local existence of a solution. We let $\tilde{s} = -\mu U_0'(s_0)$ and choose T such that $0 < T \le \sigma/4(1+\tilde{s})$. We define the following functional spaces in terms of T.

$$\begin{aligned} X_{1T} &:= \{U \in C(Q_u) : U(0,y) = U_0(y) \text{ and } \|U-U_0\|_{C(Q_u)} \le 1\}, \\ X_{2T} &:= \{V \in C(Q_v) : V(0,y) = V_0(y) \text{ and } \|V-V_0\|_{C(Q_v)} \le 1\}, \\ X_{3T} &:= \{s \in C^1[0,T],\ \|s'-\tilde{s}\|_{C[0,T]} \le 1\}, \end{aligned} \tag{12}$$

where $Q_v = \{(t,y) : 0 \le t \le T \text{ and } 0 < y < s_0\}$. Then the space $X_T = X_{1T} \times X_{2T} \times X_{3T}$ is a complete metric with the metric

$$d((U_1,V_1,s_1),(U_2,V_2,s_2)) = \|U_1-U_2\|_{C(Q_u)} + \|V_1-V_2\|_{C(Q_v)} + \|s_1'-s_2'\|_{C[0,T]}.$$

We then have

$$|s(t)-s_0| \le \int_0^T |s'(\tau)|d\tau \le T(1+\tilde{s}) \le \frac{\sigma}{4},$$

and this guarantees that the mapping $(t,x) \to (t,y)$ is diffeomorphism.

By standard L^p theory and the Sobolev imbedding theorem, for any $(U,V,s(t)) \in X_T$ and for any $\theta \in (0,1)$, the following initial boundary value problem

$$\begin{cases} \dfrac{\partial \hat{U}}{\partial t} = Q_1^2 \hat{U}_{yy} + (Q_2 - Q_3)\hat{U}_y + F(U,V), & t>0 \text{ and } 0<y<L, \\ \dfrac{\partial \hat{V}}{\partial t} = DQ_1^2 \hat{V}_{yy} + (DQ_2 - Q_3)\hat{V}_y + G(U,V), & t>0 \text{ and } 0<y<s_0, \\ \hat{U}_y(t,0) = \hat{U}(t,L) = 0, & \\ \hat{V}_y(t,0) = \hat{V}(t,s_0) = 0, & \\ \hat{V}(0,y) = V_0(y), & y \in [0,s_0], \\ \hat{U}(0,y) = U_0(y), & y \in [0,L], \end{cases} \tag{13}$$

admits a unique bounded solution $(\hat{U},\hat{V}) \in C^{\frac{(1+\theta)}{2},1+\theta}(Q_u) \times C^{\frac{(1+\theta)}{2},1+\theta}(Q_v)$ such that

$$\|\hat{U}\|_{C^{\frac{(1+\theta)}{2},1+\theta}(Q_u)} \le C_1 \text{ and } \|\hat{V}\|_{C^{\frac{(1+\theta)}{2},1+\theta}(Q_v)} \le C_2,$$

where C_1 and C_2 depend on s_0, θ, $\|U_0\|_{C^2[0,s_0]}$ and $\|V_0\|_{C^2[0,s_0]}$.

Next, we define

$$\hat{s}(t) = s_0 - \mu \int_0^t \hat{U}_y(\tau, s_0) d\tau.$$

Then, $\hat{s}'(t) = -\mu \hat{U}_y(t,s_0) \in C^{\frac{\theta}{2}}[0,T]$ and $\|\hat{s}'\|_{C^{\frac{\theta}{2}}} \le C_3$, where C_3 depends on μ, h_0, θ, $\|U_0\|_{C^2[0,s_0]}$ and $\|V_0\|_{C^2[0,s_0]}$.

We are now ready to consider the mapping, defined on X_T by

$$\Phi : (U,V,s(t)) \mapsto (\hat{U},\hat{V},\hat{s}(t)),$$

in order to seek a fixed point. We first confirm that, for T small enough, Φ maps X_T into itself: indeed, if we take T such that

$$0 < T \le \min\{C_1^{\frac{-2}{1+\theta}}, C_2^{\frac{-2}{1+\theta}}, C_3^{\frac{-2}{\theta}}\},$$

we then have

$$\|\hat{U} - U_0\|_{C(Q_u)} \le \|\hat{U}\|_{C^{0,\frac{1+\theta}{2}}(Q_u)} T^{\frac{1+\theta}{2}} \le C_1 T^{\frac{1+\theta}{2}} \le 1,$$

$$\|\hat{V} - V_0\|_{C(Q_v)} \le \|\hat{V}\|_{C^{0,\frac{1+\theta}{2}}(Q_v)} T^{\frac{1+\theta}{2}} \le C_2 T^{\frac{1+\theta}{2}} \le 1,$$

and

$$\|\hat{s}' - \tilde{s}\|_{C[0,T]} \le \|\hat{s}'\|_{C^{\frac{\theta}{2}}[0,T]} T^{\frac{\theta}{2}} \le C_3 T^{\frac{\theta}{2}} \le 1.$$

In other words, Φ maps X_T into X_T. Let us now verify that Φ is a contraction for sufficiently small T. Let $(\hat{U}_i, \hat{V}_i, \hat{s}_i) \in X_T$ for $i = 1,2$. Setting $\bar{U} = \hat{U}_1 - \hat{U}_2$, and $\bar{V} = \hat{V}_1 - \hat{V}_2$, then we have

$$\frac{\partial \bar{U}}{\partial t} = Q_1^2(s_2(t),y(t))\bar{U}_{yy} + [Q_2(s_2(t),y(t)) - Q_3(s_2(t),y(t))]\bar{U}_y + \mathbf{F}_1,$$

for $t > 0$ and $0 < y < L$,

and $$\frac{\partial \bar{V}}{\partial t} = DQ_1^2(s_2(t),y(t))\bar{V}_{yy} + (DQ_2(s_2(t),y(t)) - Q_3(s_2(t),y(t)))\bar{V}_y + \mathbf{F}_2,$$

for $t > 0$ and $0 < y < s_0$,

together with the initial-boundary conditions

$$\begin{aligned}
&\bar U(t,0)=\bar U(t,L)=0, && t>0,\\
&\bar V(t,0)=\bar V(t,s_0)=0, && t>0,\\
&\bar U(0,y)=0, && 0\le y\le L,\\
&\bar V(0,y)=0, && 0\le y\le s_0,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{F}_1 = \ & [Q_1^2(s_1(t),y(t))-Q_1^2(s_2(t),y(t))]\partial_{yy}\hat U_1\\
&+[Q_2(s_1(t),y(t))-Q_2(s_2(t),y(t))]\partial_y\hat U_1\\
&-[Q_3(s_1(t),y(t))-Q_3(s_2(t),y(t)]\partial_y\hat U_1\\
&+F(U_1,V_1)-F(U_2,V_2),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{F}_2 = \ & [DQ_1^2(s_1(t),y(t))-DQ_1^2(s_2(t),y(t))]\partial_{yy}\hat V_1\\
&+[DQ_2(s_1(t),y(t))-DQ_2(s_2(t),y(t))]\partial_y\hat V_1\\
&-[Q_3(s_1(t),y(t))-Q_3(s_2(t),y(t))]\partial_y\hat V_1\\
&+G(U_1,V_1)-G(U_2,V_2).
\end{aligned}$$

Using standard L^p estimates and the Sobolev embedding theorem we then get

$$\|\bar U\|_{C^{\frac{1+\theta}{2},1+\theta}(Q_u)} \le C_4(\|U_1-U_2\|_{C(Q_u)}+\|V_1-V_2\|_{C(Q_v)}+\|s_1-s_2\|_{C^1[0,T]}),$$

$$\|\bar V\|_{C^{\frac{1+\theta}{2},1+\theta}}(Q_v) \le C_5(\|U_1-U_2\|_{C(Q_u)}+\|V_1-V_2\|_{C(Q_v)}+\|s_1-s_2\|_{C^1[0,T]}),$$

$$\text{and } \|\bar s_1'-\bar s_2'\|_{C^{\frac{1+\theta}{2},1+\theta}([0,T])}$$

$$\le C_6(\|U_1-U_2\|_{C(Q_u)}+\|V_1-V_2\|_{C(Q_v)}+\|s_1-s_2\|_{C^1[0,T]}),$$

where $C_4,\ C_5,\ C_6>0$ depend on Q_i and C_i for $i=1,2,3$. Furthermore,

$$\begin{aligned}
&\|\bar U\|_{C(Q_u)}+\|\bar V\|_{C(Q_v)}+\|\bar s_1'-\bar s_2'\|_{C[0,T]}\\
&\le T^{\frac{1+\theta}{2}}\|\bar U\|_{C^{\frac{1+\theta}{2},1+\theta}(Q_u)} + \ T^{\frac{1+\theta}{2}}\|\bar V\|_{C^{\frac{1+\theta}{2},1+\theta}(Q_v)}\\
&+\ T^{\frac{\theta}{2}}\|\bar s_1'-\bar s_2'\|_{C^{\frac{1+\theta}{2},1+\theta}([0,T])}.
\end{aligned}$$

From the above estimates we can conclude that, if $T\in(0,1]$, then

$$\begin{aligned}
&\|\bar U\|_{C(Q_u)}+\|\bar V\|_{C(Q_v)}+\|\bar s_1'-\bar s_2'\|_{C([0,T])}\\
&\le C_7T^{\frac{\theta}{2}}(\|U\|_{C(Q_u)}+\|V\|_{C(Q_v)}+\|s_1'-s_2'\|_{C([0,T])}),
\end{aligned}$$

where $C_7:=\max\{C_4,C_5,C_6\}$. Thus, choosing

$$T=\frac{1}{2}\min\left\{1,\frac{L-s_0}{8(1+\tilde h)},C_1^{\frac{-2}{1+\theta}},C_2^{\frac{-2}{1+\theta}},C_3^{\frac{-2}{\theta}},C_7^{\frac{-2}{\theta}}\right\},$$

we get that Φ is a contraction mapping on the set X_T. Therefore, Φ admits a unique fixed point in X_T and this completes the proof of short-time existence of a solution to (1). □

As mentioned in Section 1 above, Lemma 1.1 is the main key leading to global existence in time. We will prove this lemma and then turn to the proof of the global existence theorem.

Proof of Lemma 1.1. Consider the following initial value problem

$$\bar{u}'(t) = \bar{u}(1-\bar{u}) \text{ for } t>0, \quad \bar{u}(0) = \|u_0\|_\infty := \sup_{x\in[0,L]} u_0(x). \tag{14}$$

The comparison principle applied to the function $u-\bar{u}$, yields that

$$u(t,x) \le \bar{u}(t) \le \max\{1, \|u_0\|_\infty\} := M_1,$$

for all $t\in[0,T]$ and $x\in[0,L]$. Similarly, considering the initial value problem

$$\bar{v}'(t) = k\bar{v}(1-\frac{\bar{v}}{M_1+a}) \text{ for } t>0, \quad \bar{v}(0) = \|v_0\|_\infty := \sup_{x\in[0,L]} v_0(x), \tag{15}$$

the comparison principle again yields that

$$v(t,x) \le \bar{v}(t) \le \max\{M_1+a, \|v_0\|_\infty\}$$

for all $t\in[0,T]$ and $x\in[0,s(t)]$. Moreover, the strong maximum principle yields that $u(t,x)>0$ for all $(t,x)\in[0,T]\times[0,L)$ and $v>0$ for $t\in[0,T]$ and $x\in[0,s(t))$. Since $v(t,s(t))=0$, Hopf Lemma then implies that $v_x(t,s(t))<0$ for all $t\in(0,T]$. It then follows from the free-boundary condition in (2) that $s'(t)>0$ for $t\in(0,T]$.

Now we turn to prove our claim that $s'(t)\le\Lambda$ in $(0,T]$. To this end, we compare v to the auxiliary function ω defined by

$$\omega(t,x) = M_2[2M(s(t)-x) - M^2(s(t)-x)^2]$$

for $t\in[0,T]$ and $x\in[s(t)-M^{-1}, s(t)]$, where we have chosen (reasons for this choice will become clear in the next steps)

$$M = \max\left\{\frac{1}{s_0}, \frac{\sqrt{2}}{2}, \sqrt{\frac{\kappa}{2D}}, \frac{\|u'\|_{C[0,s_0]}}{M_1}, \frac{\|v'\|_{C[0,s_0]}}{M_2}\right\}. \tag{16}$$

We have

$$\begin{cases} \omega_t - D\omega_{xx} \ge 2DM_2M^2 \ge kv \ge kv\left(1-\dfrac{v}{u+a}\right) = v_t - Dv_{xx}, \\ \omega(t,s(t)) = 0 = v(t,s(t)), \\ \omega(t,s(t)-M^{-1}) = M_2 \ge v(t,s(t)-M^{-1}). \end{cases} \tag{17}$$

We note that the choice made for M in (16) leads to $\omega(0,x)\ge v_0(x)$: for a fixed $x\in[s_0-M^{-1}, s_0]$, we have

$$v_0(x) = -\int_x^{s_0} v_0'(s)ds \le (s_0-x)\|v_0'\|_{C[0,s_0]},$$

and

$$\omega(0,x) = M_2M(s_0-x)[2-M(s_0-x)] \ge M_2M(s_0-x).$$

Thus, if M satisfies (16), we get $\omega(0,x) \geq v(0,x)$ for all x in $[s_0 - M^{-1}, s_0]$. From (17), the comparison principle yields that $\omega(t,x) \geq v(t,x)$ for all $t \in [0,T]$ and $x \in [s(t) - M^{-1}, s(t)]$. Since $\omega(t,s(t)) = 0 = v(t,s(t))$, we then obtain that

$$\partial_x v(t,s(t)) \geq \partial_x \omega(t,s(t)) = -2MM_2.$$

This, together with the free-boundary condition in (2), implies that $s'(t) \leq \Lambda$ where $\Lambda := 2\mu MM_2$. The proof of Lemma 1.1 is now complete. $\square$

Having Lemma 1.1 in hand, we are now ready to prove the global existence result we stated in Theorem 1.2.

Proof of Theorem 1.2. In view of Theorem 1.1, we let $T_{\max}$ be the maximal existence time of the solution. Now we need to show $T_{\max} = +\infty$. Suppose to the contrary that $T_{\max} < +\infty$. By Lemma 1.1, there exists a positive constant M, independent of $T_{\max}$, such that for all $t \in [0, T_{\max})$ and $x \in [0,L]$, we have

$$0 \leq u(t,x),\ v(t,x),\ s'(t) \leq M,$$

and $v(t,x) = 0$ when $x \in [s(t), L]$.
Fix $\varepsilon \in (0, T_{\max})$ and $T' > T_{\max}$. By standard regularity theory, there exists M' which depends only on ε, T' and M, such that

$$\max\{\|u(t,\cdot)\|_{C^2[0,L]}, \|v(t,\cdot)\|_{C^2[0,L]}\} \leq M',$$

for all $t \in [\varepsilon, T_{\max})$. Following the same steps in the proof of Theorem 1.1, we can find $\delta > 0$, which depends only on M' and M, such that the solution of (1) with the initial time $T_{\max} - \frac{\delta}{2}$ can be uniquely extended to the time $T_{\max} + \frac{\delta}{2}$. This however contradicts the maximality of $T_{\max}$. Eventually, we have $T_{\max} = +\infty$ and the proof of Theorem 1.2 is complete. $\square$

In order to prove the Theorem 1.3. We need the following lemmas.

Lemma 2.1 ($s_\infty = L$)**.** *Suppose that (u,v,s) is the solution of (1) subject to the conditions in (2). If $s_\infty = L$, then we have*

$$\limsup_{t\to+\infty} u(t,x) \leq \bar{u}(x);\ \ \liminf_{t\to+\infty} u(t,x) \geq \underline{u}(x);$$

and

$$\limsup_{t\to+\infty} v(t,x) \leq \bar{v}(x);\ \ \liminf_{t\to+\infty} v(t,x) \geq \underline{v}(x).$$

Where $\bar{u}(x)$, $\underline{u}(x)$, $\bar{v}(x)$, $\underline{v}(x)$ are determined in the following proof.

Proof. The proof mainly uses the upper and lower solution method. Suppose $s_\infty = L$. We start by letting that $\bar{u}(t,x)$ satisfies

$$\begin{cases} \bar{u}_t - \bar{u}_{xx} = \bar{u}(1-\bar{u}), & \text{for all } t > 0 \text{ and } 0 < x < L, \\ \bar{u}_x(t,0) = 0,\ \bar{u}(t,L) = 0, & t > 0, \\ \bar{u}(0,x) = u_0(x), & 0 < x < L. \end{cases} \tag{18}$$

By the comparison principle, we know $u(t,x) \le \bar{u}(t,x)$ for $t \ge 0$ and $0 \le x \le L$. Since $L > \frac{\pi}{2}$, appealing to Proposition 3.2. and Proposition 3.3. of [10] we obtain that $\lim_{t\to+\infty} \bar{u}(t,x) = \bar{u}(x)$ uniformly in $x \in [0,L]$, where $\bar{u}(x) > 0$ satisfies

$$\begin{cases} -\bar{u}_{xx} = \bar{u}(1-\bar{u}), & 0 < x < L, \\ \bar{u}_x(0) = 0,\ \bar{u}(L) = 0, & t > 0, \end{cases} \tag{19}$$

Hence,

$$\limsup_{t\to+\infty} u(t,x) \le \bar{u}(x) \text{ uniformly in } \ x \in [0,L].$$

Since $s_\infty = L$, then for any $\epsilon > 0$, there exists $T_1 > 0$ such that $u(t,x) < \bar{u}(x) + \epsilon$ for all $0 < x < L$ when $t > T_1$. Thus we consider $\bar{v}_\epsilon$ which satisfies

$$\begin{cases} \bar{v}_{\epsilon t} - D\bar{v}_{\epsilon xx} = k\bar{v}_\epsilon \left(1 - \dfrac{b\bar{v}_\epsilon}{\bar{u}(x) + a + \epsilon}\right) & \text{for all } t > T_1 \text{ and } 0 < x < L, \\ \bar{v}_{\epsilon x}(t,0) = 0, \bar{v}_\epsilon(t,L) = 0, & t > T_1, \\ \bar{v}_\epsilon(T_1,x) = v(T_1,x), & x \in [0,L]. \end{cases} \tag{20}$$

From the comparison principle, we know that $v(t,x) \le \bar{v}_\epsilon(t,x)$ for $t \ge 0$ and $0 \le x \le L$.

As above, $L > \frac{\pi}{2}\sqrt{\frac{D}{k}}$ yields that $\lim_{t\to+\infty} \bar{v}_\epsilon(t,x) = \bar{v}_\epsilon(x)$, where $\bar{v}_\epsilon(x)$ satisfies

$$\begin{cases} -D\bar{v}_{\epsilon xx} = k\bar{v}_\epsilon \left(1 - \dfrac{b\bar{v}_\epsilon}{\bar{u}(x) + a + \epsilon}\right), & 0 < x < L, \\ \bar{v}_{\epsilon x}(0) = 0 = \bar{v}_\epsilon(L) = 0, \end{cases} \tag{21}$$

As ϵ is arbitrary, it then follows that

$$\limsup_{t\to+\infty} v(t,x) \le \bar{v}(x) \text{ uniformly in any compact subset of } [0,L).$$

Where $\bar{v}(x)$ satisfies

$$\begin{cases} -D\bar{v}_{xx} = k\bar{v}\left(1 - \dfrac{b\bar{v}}{\bar{u}(x) + a}\right), & 0 < x < L, \\ \bar{v}_x(0) = 0 = \bar{v}(L) = 0, \end{cases} \tag{22}$$

Now, we note that there exists $T_2 > T_1$ such that $v(t,x) < \bar{v}(x) + \varepsilon$ when $t > T_2$, $0 < x < L$.

Similarly, we consider the following problem about $\underline{u}(t,x)$

$$\begin{cases} \underline{u}_t - \underline{u}_{xx} = \underline{u}\left(1 - \underline{u} - \dfrac{\bar{v}(x) + \varepsilon}{m}\right) & \text{for all } t > T_2 \text{ and } 0 < x < L, \\ \underline{u}_x(t,0) = 0, \underline{u}(t,L) = 0, & t > T_2, \\ \underline{u}(T_2,x) = u(T_2,x), & x \in [0,L]. \end{cases} \tag{23}$$

Again, because of the comparison principle, we obtain $u(t,x) \ge \underline{u}(t,x)$ for $t \ge 0$ and $0 \le x \le L$.

Since $L > \frac{\pi}{2}$, then $\lim_{t\to+\infty} \underline{u}(t,x) = \underline{u}(x)$ *uniformly in* $[0,L]$. Here $\underline{u}(x)$ satisfies

$$\begin{cases} -\underline{u}_{xx} = \underline{u}\left(1 - \underline{u} - \dfrac{\bar{v}(x) + \varepsilon}{m}\right), & 0 < x < L, \\ \underline{u}_x(0) = 0, \underline{u}(L) = 0, & t > T_2, \end{cases} \tag{24}$$

Consequently, $\liminf_{t\to+\infty} u(t,x) \geq \underline{u}(x)$ uniformly in $[0,L]$. We mention that the positivity of $\underline{u}(x)$ follows from the assumption (**H**).

Furthermore, for any fixed $l>0$, there exists $T_3 > T_2$ such that $s(T_3) > \max\left\{l, \frac{\pi}{2}\sqrt{\frac{D}{k}}\right\}$ when $t > T_3$ and $u(t,x) > \underline{u}(x) - \varepsilon$ when $t > T_3$ and $0 < x < L$. Then we let $\underline{v}(t,x)$ satisfies

$$\begin{cases} \underline{v}_t - D\underline{v}_{xx} = k\underline{v}\left(1 - \dfrac{b\underline{v}}{\underline{u}(x)+a-\varepsilon}\right) & \text{for all } t > T_3 \text{ and } 0 < x < s(T_3), \\ \underline{v}_x(t,0) = 0, \underline{v}(t,s(T_3)) = 0, & t > T_3, \\ \underline{v}(T_3,x) = v(T_3,x), & x \in [0,s(T_3)]. \end{cases} \tag{25}$$

Thus, we obtain $v(t,x) \geq \underline{v}(t,x)$ for $t \geq 0$ and $0 \leq x \leq l$. $\lim_{t\to+\infty} \underline{v}(t,x) = \underline{v}(x)$ uniformly in $x \in [0,l]$.

$$\begin{cases} -D\underline{v}_{xx} = k\underline{v}\left(1 - \dfrac{b\underline{v}}{\underline{u}(x)+a-\varepsilon}\right), & 0 < x < s(T_3), \\ \underline{v}_x(0) = 0, \underline{v}(s(T_3)) = 0, & t > T_3, \end{cases} \tag{26}$$

Hence, the arbitrariness of l implies that

$$\liminf_{t\to+\infty} v(t,x) \geq \underline{v}(x) \text{ uniformly in any compact subset of } [0,L).$$

□

Proof of Theorem 1.3. By the proof of Theorem 1.1, we can easily get the following estimates

$$\|u\|_{C^{1+\theta,(1+\theta)/2}(G_u)} + \|v\|_{C^{1+\theta,(1+\theta)/2}(G_v)} + \|h(t)\|_{C^{1+\theta/2}([0,\infty))} \leq C, \tag{27}$$

where C depends on s_∞, on the initial data (u_0, v_0) and s_0 and on $\theta \in (0,1)$. We have denoted by

$$G_u = \{(t,x) \in [0,\infty) \times [0,L]\} \;\text{ and }\; G_v = \{(t,x) \in [0,\infty) \times [0,s(t)]\}.$$

From Lemma 2.1, we have

$$\liminf_{t\to+\infty} v(t,x) \geq \underline{v}(x) > 0.$$

Then, a sequence $(t_k, x_k) \in (0,\infty) \times [0,s(t)]$ with $t_k \to \infty$ as $k \to \infty$ exists such that $v(t_k,x_k) \geq \underline{v}(x)/2$ for all $k = 1,2,3,\cdots$. Hence $x_k < s(t_k)$ and so $0 \leq x_k < s(t_k) < s_\infty = L$. Thus, up to subsequence we have $x_k \to x_0 \in (0, s_\infty)$ as $k \to \infty$.
Define

$$u_k(t,x) := u(t+t_k,x) \text{ and } v_k(t,x) := v(t+t_k,x),$$

for $t \in (-t_k,\infty)$ and $x \in [0, s(t+t_k)]$. From (27) and standard parabolic regularity, it follows that $\{(u_k,v_k)\}$ has a subsequence $\{(u_{k_i}, v_{k_i})\}$ such that $(u_{k_i}, v_{k_i}) \to (\breve{u}, \breve{v})$ as $k_i \to \infty$, where $(\breve{u}, \breve{v})$ satisfies

$$\begin{cases} \dfrac{\partial \breve{u}}{\partial t} = \breve{u}_{xx} + \breve{u}(1-\breve{u}) - \breve{v}\dfrac{\breve{u}}{\breve{u}+m} & \text{for } (t,x) \in (-\infty,+\infty) \times (0,L), \\ \dfrac{\partial \breve{v}}{\partial t} = D\breve{v}_{xx} + \breve{v}(k - \dfrac{b\breve{v}}{\breve{u}+a}) & \text{for } (t,x) \in (-\infty,+\infty) \times (0,L), \end{cases} \tag{28}$$

together with $\check{v}(t, s_\infty) = 0$ for all $t \in (-\infty, +\infty)$.
We note that $\check{v}(0, x_0) = \lim_{k_i \to \infty} v(t_{k_i}, x_{t_{k_i}}) \geq \underline{v}(x)/2$. It follows from the maximum principle that $\check{v} > 0$ in $(-\infty, +\infty) \times (0, L)$. Thus, we can apply Hopf Lemma at the point $(0, L)$ and conclude that

$$\check{v}_x(0, L) < 0.$$

As a consequence, one can find a uniform constant $\kappa > 0$ such that

$$\partial_x v(t_{k_i}, s(t_{k_i})) = \partial_x v_{k_i}(0, s(t_{k_i})) < -\kappa < 0, \quad \text{for } i \text{ large enough.}$$

The latter, together with the Stefan condition $s'(\cdot) = -\mu v_x(\cdot, s(\cdot))$, implies that $s'(t_{k_i}) > \mu\kappa$, for i large enough. On the other hand, our assumption that $s_\infty = L$ leads to $s'(t) \to 0$ as $t \to \infty$ (see Lemma 3.3 of [10]) and this contradicts with $s'(t_{k_i}) > \mu\kappa$ (for large enough i). So this shows that $s_\infty \neq L$. □

2.2 Long time asymptotics: proofs of the vanishing and spreading criteria

The first result we prove in this subsection is that in Lemma 1.2, which is a comparison principle for system (1)-(2).

Proof of Lemma 1.2. We will prove **(a)** only, as the proof of **(b)** is similar.

Step 1: We consider the case $\delta(0) > s(0)$. In such case, we have $\delta(t) > s(t)$ for small t and we are left to prove $\delta(t) > s(t)$ for all $t \geq 0$. Suppose this is not true, then there exists $T > 0$ such that $\delta(T) = s(T)$ and, for such T, we have $\delta'(T) \leq s'(T)$. Since $\omega(0, x) \geq v_0(x)$, then by the maximum principle applied to $v - w$, with the second equation of (1) in hand, we get $\omega > v$ for all $(t, x) \in [0, T] \times (0, s(t))$. By Hopf Lemma, as $\omega(T, s(T)) = v(T, s(T))$, we get that $\omega_x(T, s(T)) < v_x(T, s(T))$. Appealing now to the free-boundary condition, ($s'(t) = -\mu v_x(t, s(t))$ for all $t > 0$) in (2), we obtain

$$\delta'(T) \geq -\mu\omega_x(T, \delta(T)) > -\mu v_x(T, s(T)) = s'(T),$$

which contradicts with $\delta'(T) \leq s'(T)$. Thus, $\delta(t) > s(t)$ for all $t \geq 0$. Using the comparison principle between $(0, \omega)$ and (u, v) where $x \in [0, s(t)] \subset [0, \delta(t)]$, we obtain that $\omega \geq v$ for all $x \in [0, s(t)]$ and $t \geq 0$.

Step 2: In the general case, we have $\delta(0) \geq s(0)$. We construct the parametric functions $(v_\varepsilon, s_\varepsilon)$, for $\varepsilon > 0$, such that

$$s'_\varepsilon(t) = -\mu(1 - \varepsilon)\partial_x v_\varepsilon(t, s_\varepsilon(t))$$

with suitable initial data $(v_\varepsilon(0, x), s_\varepsilon(0))$ such that $\delta(0) > s_\varepsilon(0)$. Using the result of Step 1, followed by passing to the limit $\varepsilon \to 0$, we obtain the desired inequalities. □

Now we turn to the

Proof of Theorem 1.4. Since $s_\infty > L$ and $L > \max\left\{\frac{\pi}{2}\sqrt{\frac{D}{k}}, \frac{\pi}{2}\right\}$, then there exists T_* such that $s(T_*) = L > \max\left\{\frac{\pi}{2}\sqrt{\frac{D}{k}}, \frac{\pi}{2}\right\}$ when $t = T_*$ and the system (1) becomes (5) with the conditions (6). In such case we are studying a fixed boundary problem. With minor modifications, the proof can be done by following the same lines in the proof of Lemma 2.1 above. This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5. Suppose to the contrary that

$$\limsup_{t\to+\infty} v(t,\cdot)_{C[0,s(t)]} = \delta > 0.$$

Then combining with (27), a sequence $(t_k, x_k) \in (0,\infty) \times [0, s(t)]$ with $t_k \to \infty$ as $k \to \infty$ exists such that $v(t_k, x_k) \geq \delta/2$ for all $k = 1, 2, 3, \cdots$. Hence $x_k < s(t_k)$ and so $0 \leq x_k < s(t_k) < s_\infty < L$. Thus, up to subsequence (if necessary) we have $x_k \to x_0 \in (0, s_\infty)$ as $k \to \infty$. From the Proof of Theorem 1.3, we know that $\check{v}_x(0, s_\infty) < 0$. As a consequence, one can find a uniform constant $\kappa_1 > 0$ such that

$$\partial_x v(t_{k_i}, s(t_{k_i})) = \partial_x v_{k_i}(0, s(t_{k_i})) < -\kappa_1 < 0, \text{ for } i \text{ large enough.}$$

The latter, together with the Stefan condition $s'(\cdot) = -\mu v_x(\cdot, s(\cdot))$, implies that $s'(t_{k_i}) > \mu\kappa_1$ for i large enough. On the other hand, our assumption that $s_\infty < L$ leads to $s'(t) \to 0$ as $t \to \infty$ and thus a contradiction with $s'(t_{k_i}) > \mu\kappa_1$ (for large enough i). Therefore,

$$\lim_{t\to+\infty} \|v(t,\cdot)\|_{C[0,s(t)]} = 0.$$

Now we prove that $\liminf_{t\to+\infty} u(t,\cdot) \geq \omega(x)$ for $x \in [0, L]$. Since $\lim_{t\to+\infty} \|v(t,\cdot)\|_{C[0,s(t)]} = 0$, then for any $\varepsilon \in (0,1)$, there exists $T > 0$ such that $0 \leq v(t,x) \leq \varepsilon$ for $t > T$ and for all $x \in [0, L]$. We then get

$$\begin{cases} u_t - u_{xx} \geq u\left(1 - u - \dfrac{\varepsilon}{m}\right) & \text{for all } t \geq T \text{ and } x \in [0, L], \\ u_x(t,0) = u(t,L) = 0, & t \geq T, \\ u(T,x) > 0. \end{cases} \tag{29}$$

Thanks to the comparison principle, we have $u(t,x) \geq \omega(t,x)$ for $t \geq T$ and $0 \leq x \leq L$. Where $\omega(t,x)$ satisfies the following

$$\begin{cases} \omega_t - \omega_{xx} = \omega\left(1 - \omega - \dfrac{\varepsilon}{m}\right) & \text{for all } t \geq T \text{ and } x \in [0, L], \\ \omega_x(t,0) = \omega(t,L) = 0, & t \geq T, \\ \omega(T,x) = u(T,x). \end{cases} \tag{30}$$

Since $L > \dfrac{\pi}{2}$, the arbitrariness of ε follows that $\lim_{t\to+\infty} \omega(t,\cdot) = \bar{u}(x)$ uniformly in $x \in [0, L]$. Where $\bar{u}(x)$ be determined in the Lemma 2.1.

We then have $\liminf_{t\to+\infty} u(t,\cdot) \geq \bar{u}(x)$ for $x \in [0, L]$. This completes the proof of Theorem 1.5. $\square$

Proof of Lemma 1.3. By Theorem 1.5, we know that if $s_\infty < L$ then

$$\liminf_{t\to+\infty} u(t,\cdot) \geq \omega(x) > 0 \text{ for } x \in [0, L] \text{ and } \lim_{t\to+\infty} \|v(t,\cdot)\|_{C[0,s(t)]} = 0.$$

In the following, we assume on the contrary that $s_\infty > \dfrac{\pi}{2}\sqrt{D/k}$ while $s_\infty < L$. Then there exists $T > 0$ such that $s(T) > \dfrac{\pi}{2}\sqrt{D/k}$ and $u(t,x) > 0$ for all $t > T$ and $0 < x < s(T)$.

Let $\underline{v}(t,x)$ be the solution of the following equation

$$\begin{cases} \underline{v}_t - D\underline{v}_{xx} = k\underline{v}(1-\frac{b\underline{v}}{a}) & \text{for all } t > T \text{ and } 0 < x < s(T), \\ \underline{v}_x(t,0) = \underline{v}(t,s(T)) = 0, & t > T, \\ \underline{v}(T,x) = v(T,x), & 0 < x < s(T). \end{cases} \tag{31}$$

By the comparison principle, we have $\underline{v}(t,x) \le v(t,x)$ for all $t > T$ and $0 < x < s(T)$. Since $s(T) > \dfrac{\pi}{2}\sqrt{D/k}$, then by Proposition 3.3 in [4],

$$\underline{v}(t,x) \to W(x) > 0$$

as $t \to +\infty$ uniformly in any compact subset of $(0, s(T))$, where W is the unique positive solution of

$$\begin{cases} DW_{xx} + kW\left(1 - \dfrac{bW}{a}\right) = 0 & \text{for all } 0 < x < s(T), \\ W_x(0) = W(s(T)) = 0. \end{cases} \tag{32}$$

So, for each x we have

$$\liminf_{t\to+\infty} v(t,x) \ge \liminf_{t\to+\infty} \underline{v}(t,x) = W(x) > 0.$$

This is a contradiction to Theorem 1.5. Therefore, $s_\infty < L$ implies that

$$s_\infty < \frac{\pi}{2}\sqrt{D/k}.$$

Finally, since $s'(t) > 0$ for $t > 0$, then with the above result we can see that $s_\infty > L$ when $s_0 \ge \dfrac{\pi}{2}\sqrt{D/k}$. □

Proof of Part 1 of Theorem 1.6. We consider the following problem:

$$\begin{cases} \partial_t\omega_1 - D\partial_{xx}\omega_1 = k\omega_1\left(1 - \dfrac{b\omega_1}{a}\right), & \text{for all } t > 0 \text{ and } 0 < x < \underline{s}_1(t), \\ \partial_x\omega_1(t,0) = 0, & t > 0, \\ \omega_1(t,\underline{s}_1(t)) = 0, & t > 0, \\ \underline{s}_1'(t) = -\mu\omega_{1x}(t,\underline{s}_1(t)) \text{ and } \underline{s}_1(0) = s_0 & t > 0, \\ \omega_1(0,x) = v_0(x), & 0 \le x \le s_0. \end{cases} \tag{33}$$

By the comparison principle stated in Lemma 1.2, we have $\underline{s}_1(t) \le s(t)$ and $\omega_1(t,x) \le v(t,x)$ for all $t > 0$ such that $0 < x < \underline{s}_1(t)$. Now we focus on system (33):
Similar to the argument done in Lemma 3.7 of [7], we first consider the case $\|v_0\|_{C[0,s_0]} \le \dfrac{a}{b}$ and conclude that $\omega_1(t,x) < a/b$. Assuming that $\underline{s}_1(\infty) < L$, a straightforward computation leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\underline{s}_1(t)} \omega_1(t,x)\,\mathrm{d}x \;=\; \int_0^{\underline{s}_1(t)} \partial_t\omega_1(t,x)\,\mathrm{d}x + \underline{s}_1'(t)\omega_1(t,\underline{s}_1(t))$$

$$= \int_0^{\underline{s}_1(t)} D\partial_{xx}\omega_1 \,\mathrm{d}x + \int_0^{\underline{s}_1(t)} k\omega_1(1-\frac{b\omega_1}{a})\,\mathrm{d}x$$
$$= \frac{-D\underline{s}_1'(t)}{\mu} + \int_0^{\underline{s}_1(t)} k\omega_1(1-\frac{b\omega_1}{a})\,\mathrm{d}x.$$

Integration from 0 to t yields

$$\int_0^{\underline{s}_1(t)} \omega_1(t,x)\,\mathrm{d}x = \int_0^{s_0} v_0(x)\mathrm{d}x + \frac{D}{\mu}(s_0 - \underline{s}_1(t)) + \int_0^t \int_0^{\underline{s}_1(\tau)} k\omega_1(1-\frac{b\omega_1}{a})\,\mathrm{d}x\,\mathrm{d}\tau. \tag{34}$$

Since $0 < \omega_1(t,x) < a/b$, then for $t > 0$ and $x \in [0, \underline{s}_1(t)]$, we have

$$\int_0^t \int_0^{\underline{s}_1(\tau)} k\omega_1(1-\frac{b\omega_1}{a})\,\mathrm{d}x\,\mathrm{d}\tau \geq \int_0^1 \int_0^{\underline{s}_1(\tau)} k\omega_1(1-\frac{b\omega_1}{a})\,\mathrm{d}x\,\mathrm{d}\tau > 0. \tag{35}$$

If $\underline{s}_1(\infty) < L$, then we have $\underline{s}_1(\infty) \leq \frac{\pi}{2}\sqrt{D/k}$ and $\lim_{t\to+\infty} \|\omega_1(t,\cdot)\|_{C[0,\underline{s}_1(t)]} = 0$. Thus, passing to the limit as $t \to +\infty$, we get

$$\int_0^{s_0} v_0(x)\,\mathrm{d}x < \frac{D}{\mu}\left(\frac{\pi}{2}\sqrt{\frac{D}{k}} - s_0\right),$$

which is a contradiction to our assumption. Therefore, we must have $\underline{s}_1(\infty) > L$, and this in turn implies that $s_\infty > L$.

If $\|v_0\|_{C[0,s_0]} > \dfrac{a}{b}$, we consider the following problem

$$\begin{cases} \partial_t\omega_2 - D\partial_{xx}\omega_2 = k\omega_2(1-\frac{b\omega_2}{a}) & \text{for all } t > 0 \text{ and } 0 < x < \underline{s}_2(t), \\ \partial_x\omega_2(t,0) = 0, & t > 0, \\ \omega_2(t,\underline{s}_2(t)) = 0, & t > 0, \\ \underline{s}_2'(t) = -\mu\omega_2(t,\underline{s}_2(t)), & t > 0, \\ \underline{s}_2(0) = s_0, & \\ \omega_2(0,x) = \dfrac{a}{b\|v_0\|_{C[0,s_0]}} v_0(x), & 0 \leq x \leq s_0. \end{cases} \tag{36}$$

From Lemma 1.2, we have $\underline{s}_2(t) \leq s(t)$ and $\omega_2(t,x) \leq v(t,x)$. Note that $\|\omega_2(0,\cdot)\|_\infty = a/b$ and this leads to $\omega_2(t,x) < a/b$. Then, as we did above, we can conclude that

$$\int_0^{s_0} \frac{a}{b\|v_0\|_\infty} v_0(x)\,\mathrm{d}x < \frac{D}{\mu} \times \left(\frac{\pi}{2}\sqrt{\frac{D}{k}} - s_0\right)$$

and get a contradiction. Eventually this leads to $s_\infty > L$. The proof of Part 1 of Theorem 1.6 is now complete. □

Proof of Part 2 of Theorem 1.6. Let

$$\bar{s}(t) = s_0(1+\delta - \frac{\delta}{2}e^{-\beta t}) \ \text{ for } \ t \geq 0, \quad V(y) = \cos\frac{\pi y}{2} \text{ for } 0 \leq y \leq 1,$$

$$\text{and } \bar{v}(t,x) = \bar{M}e^{-\beta t} V\left(\frac{x}{\bar{s}(t)}\right) \text{ for } 0 \leq x \leq \bar{s}(t),$$

where $\delta = \frac{1}{2}\left(\frac{\frac{\pi}{2}\sqrt{D/k}}{s_0} - 1\right) > 0$ (since $s_0 < \frac{\pi}{2}\sqrt{\frac{D}{k}}$) and

$$\beta = \frac{\pi^2}{8}\frac{D}{(1+\delta)^2 s_0^2} - \frac{k}{2} > 0 \text{ because } s_0(1+\delta) < \frac{\pi}{2}\sqrt{\frac{D}{k}}.$$

Let $\bar{M} = \frac{\|v_0\|_\infty}{\cos\left(\frac{\pi}{2+\delta}\right)}$. If $\|v_0\|_\infty \le \cos\left(\frac{\pi}{2+\delta}\right)\frac{\delta s_0^2 \beta(2+\delta)}{2\pi\mu}$, then a computation leads to

$$\begin{cases} \text{for all } t > 0 \text{ and } 0 < x < \bar{s}(t), \\ \bar{v}_t - D\bar{v}_{xx} - k\bar{v}(1 - \frac{\bar{v}}{M_1 + a}) \ge \bar{M}e^{-\beta t} V((\frac{\pi}{2})^2 \frac{D}{(1+\delta)^2 s_0^2} - k - \beta) \ge 0, \\ \bar{v}_x(t,0) = 0, \ t > 0, \\ \bar{v}(t, \bar{s}(t)) = 0, \ t > 0, \\ \bar{s}'(t) + \mu\bar{v}_x(t,\bar{s}(t)) \ge \frac{\delta s_0 \beta e^{-\beta t}}{2}\left[1 - \frac{2\mu\pi\bar{M}}{\delta s_0^2\beta(2+\delta)}\right] \ge 0, \ t > 0. \end{cases} \tag{37}$$

Since $s_0 \le \bar{s}(0)$ and $\bar{v}(0,x) \ge v_0(x)$ on $[0, s_0]$ we get $s(t) \le \bar{s}(t)$ on $[0, +\infty)$. Taking $t \to +\infty$ yields

$$s_\infty \le \bar{s}(\infty) = s_0(1+\delta) < \frac{\pi}{2}\sqrt{D/k}.$$

By the Lemma 1.3, we complete the proof. □

3 Summary and conclusions

In this paper, we have studied a Leslie-Gower and Holling-type II predator-prey model in one-dimensional environment. The predator v is the invader which exists initially in an interval $[0, s_0]$ and has the Leslie-Gower terms which measure the loss in the predator population due to rarity of the prey. The prey u is the native species lives in the whole region $[0, L]$. In this setting, we obtained several results:

1. Lemma 1.3 provides a sufficient condition for spreading success or spreading failure via a comparison between spreading front $x = s(t)$ and the threshold $\frac{\pi}{2}\sqrt{\frac{D}{k}}$.

2. Theorem 1.6 reveals that when $s_0 < \frac{\pi}{2}\sqrt{\frac{D}{k}}$, if the total initial population in the region $[0, s_0]$, $\int_0^{s_0} v_0(x)\ dx$, is greater than

$$\max\left\{1, \frac{b\|v_0\|_\infty}{a}\right\} \cdot \frac{D}{\mu} \cdot \left(\frac{\pi}{2}\sqrt{\frac{D}{k}} - s_0\right),$$

then spreading takes place. By contrast, the invasion by species v fails (and the species v vanishes eventually) if the maximal initial population density, $\|v_0\|_\infty$, in the region $[0, s_0]$ is less than a positive number given explicitly in part 2 of the theorem.

3. According to Theorem 1.4 and Theorem 1.5, we can say that the species v spreads successfully if $s_\infty > L$ and

$$\limsup_{t\to+\infty} u(t,x) \leq \bar{u}(x); \quad \liminf_{t\to+\infty} u(t,x) \geq \underline{u}(x);$$

and

$$\limsup_{t\to+\infty} v(t,x) \leq \bar{v}(x); \quad \liminf_{t\to+\infty} v(t,x) \geq \underline{v}(x).$$

And we can say that the specie v vanish eventually if $s_\infty < L$ and

$$\liminf_{t\to+\infty} u(t,\cdot) \geq \bar{u}(x) \text{ for } x \in [0,L] \text{ and } \lim_{t\to+\infty} \|v(t,\cdot)\|_{C[0,s(t)]} = 0.$$

We also have the following conclusions:

1. Suppose that $s_0 < \frac{\pi}{2}\sqrt{D/k}$ in the free boundary problem (1)-(2). Then,

   (a) there exists $\bar{\mu} > 0$ depending on v_0 such that $s_\infty > L$ whenever $\mu \geq \bar{\mu}$. The value of $\bar{\mu}$ is given by

$$\bar{\mu} = \max\left\{1, \frac{b\|v_0\|_\infty}{a}\right\} \cdot D \cdot \left(\frac{\pi}{2}\sqrt{\frac{D}{k}} - s_0\right) \cdot \left(\int_0^{s_0} v_0(x)\,\mathrm{d}x\right)^{-1}. \tag{38}$$

   (b) there exists $\underline{\mu} > 0$, depending on v_0, such that $s_\infty < L$ whenever $\mu \leq \underline{\mu}$. Hence, by Theorem 1.5 we have

$$\liminf_{t\to+\infty} u(t,\cdot) \geq \bar{u}(x) \text{ for } x \in [0,L] \text{ and } \lim_{t\to+\infty} \|v(t,\cdot)\|_{C[0,s(t)]} = 0.$$

2. Moreover, from Theorem 1.6 we can easily have other criteria for spreading in terms of the diffusion coefficient D, *for any* s_0. Let $D^* = \frac{4ks_0^2}{\pi^2}$ for any s_0. Then,

   (i) $0 < D \leq D^*$ implies that spreading occurs;

   (ii) if $D > D^*$, then the statement $\mu \geq \bar{\mu}$ is equivalent to spreading occurs, and $\mu \leq \underline{\mu}$ implies that vanishing occurs.

Acknowledgements

The authors would like to thank the referees and editors for their helpful suggestions which helped improve our work.

Funding

Y. Liu and Z. Guo were supported by National Science Foundation of China (No. 11771104), Program for Chang Jiang Scholars and Innovative Research Team in University (IRT-16R16). Y. Liu was supported by the Innovation Research for the Postgraduates of Guangzhou University under Grant No.2017GDJC-D05. M. El Smaily and L. Wang acknowledge support received through NSERC Discovery Grants from the Natural Sciences and Engineering Research Council of Canada (NSERC).

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have made equally important contributions to this article. All authors read and approved the final manuscript.

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