Liebeck's 136 Notes, 21st Century Edition

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1 Aerodynamic Variables

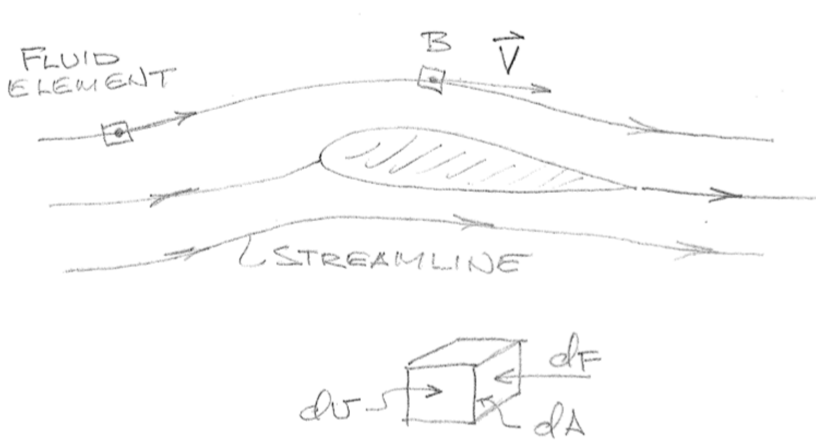


Figure 1: Fluid Element and Streamline

Table 1: Aerodynamic Variable Definitions

Pressure	p	=	$\lim_{d\Delta \to 0} \frac{dF}{dA}$, a scalar quantity
Density	ρ	=	$\lim_{dv \to 0} \frac{dw}{dv}$, a scalar quantity
Temperature (Kinetic Energy)	KE	=	$\frac{3}{2}kT$, scalar (k = Boltzmann Const.)
Perfect Gas Law	p	=	ρRT (R = Gas Const.)
Velocity	$\vec{V}$	=	Velocity of element B, a vector

1.1 Aerodynamic Forces and Moments

Figure 2 and 3: The force on a unit area of the airfoil is P, the normal force and τ, the tangential force. Integrating P and τ over the entire surface yields a resultant force $\vec{R}$ and a moment M.

1.1.1 Coefficients for 2D and 3D bodies

$$\begin{array}{lcl} \text{Lift Coefficient} & C_L & = & L/(q_\infty S) \\ \text{Drag Coefficient} & C_D & = & D/(q_\infty S) \\ \text{Moment Coefficient} & C_M & = & M/(q_\infty Sc) \end{array} \tag{1}$$

For 2D airfoils the area $S = C \cdot 1$. Define:

$$C_l \; L'/(q_\infty c), \quad C_d = D'/(q_\infty c), \quad C_m \; M'/(q_\infty c^2) \tag{2}$$

L', D', M' are defined per unit span. Define "Force" coefficients as:

$$C_p = \frac{p - p_\infty}{q_\infty} = \text{Pressure Coefficient} \tag{3}$$

$$C_f = \frac{\tau}{q_\infty} = \text{Skin Friction Coefficient} \tag{4}$$

where p_∞ is the free stream static pressure and $q_\infty = \frac{1}{2}\rho_\infty V_\infty^2$.

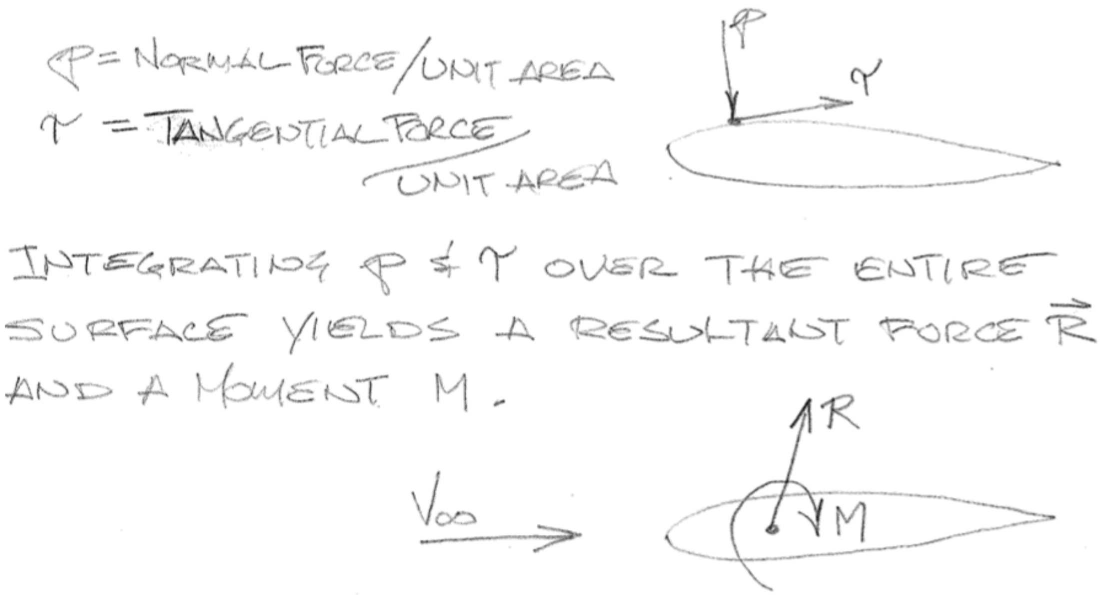


Figure 2: Force on Airfoil

Table 2: Variable Definitions for Airfoil

Lift	=	force perpendicular to $\vec{V}_\infty$.
Drag	=	Force parallel to $\vec{V}_\infty$
Chord	=	Max length line from trailing edge to leading edge = c
Normal Force	=	Force perpendicular to C=N
Chord Force	=	Force parallel to C=A
L	=	$N\cos\alpha - A\sin\alpha$
D	=	$N\sin\alpha + A\cos\alpha$
Dynamic Pressure	=	$q = \frac{1}{2}\rho_\infty V_\infty^2$
Reference Area	=	S (typically wing area)
Reference Length	=	C (typically wing chord)

1.2 Integration of P and T on a 2D Body

Along the upper surface of the airfoil (subscript u):

$$\begin{aligned} dN_u &= -P_u ds_u \cos\theta - T_u ds_u \sin\theta \\ dA_u &= -P_u ds_u \sin\theta + T_u ds_u \cos\theta \end{aligned}$$

Along the lower surface of the airfoil (subscript l):

$$\begin{aligned} dN_l &= P_l ds_l \cos\theta - T_l ds_l \sin\theta \\ dA_l &= P_l ds_l \sin\theta + T_l ds_l \cos\theta \end{aligned}$$

For the moment about the leading edge (LE), clockwise is **positive**.

$$\begin{aligned} dM_u &= (P_u\cos\theta + T_u\sin\theta)x ds_u + (-P_u\sin\theta + T_u\cos\theta)y ds_u \\ dM_l &= (-P_l\cos\theta + T_l\sin\theta)x ds_l + (P_l\sin\theta + T_l\cos\theta)y ds_l \end{aligned}$$

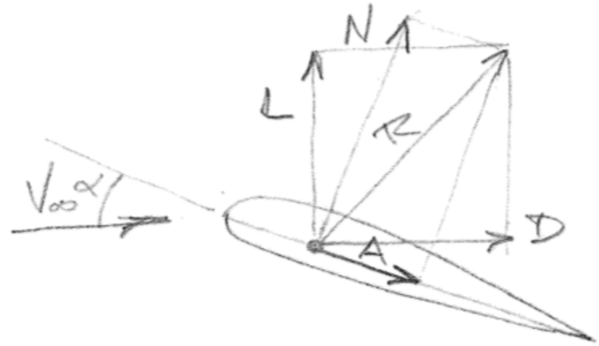


Figure 3: Force Definitions

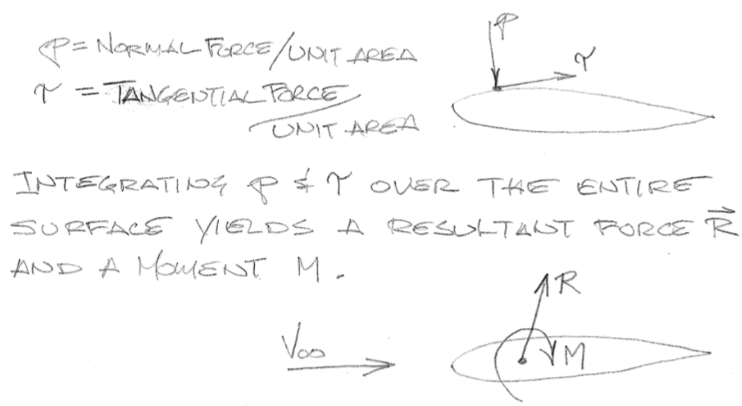
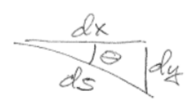


Figure 4: Force on Airfoil Surface and Resulting Force R and Moment M

For the surface element ds, $dx = ds\cos\theta$ and $dy = -ds\sin\theta$.



θ is measured clockwise positive from dx. ds is the surface of the airfoil.

	dN_u	$=$	$-P_u dx_u + T_u dy_u$
Convert to dx because we DO NOT KNOW dy	dN_u	$=$	$(-P_u + T_u \frac{dy_u}{dx})dx$
Same for lower side	dN_l	$=$	$p_l dx_l + T_l dy_l$
	dN_l	$=$	$(P_l + T_l \frac{dy_l}{dx})dx$
Integrate to get N, the normal force	N	$=$	$\int_{LE}^{TE}(dN_u + dN_l)$
Separate T and P terms	N	$=$	$\int_{LE}^{TE}(P_l - P_u)dx + \int_{LE}^{TE}(T_u \frac{dy_u}{dx} + T_l \frac{dy_l}{dx})dx$
Use C_n $\frac{N}{q_\infty c}$, C_p, C_f (convert to dimensionless)	C_n	$=$	$\int_0^1 (C_{p_l} - C_{p_u})d(\frac{x}{c}) + \int_0^1 (C_{f_u} \frac{dy_u}{dx} + C_{f_l} \frac{dy_l}{dx})d(\frac{x}{c})$
Similarly for A and M	C_a	$=$	$\int_0^1 (C_{p_u} \frac{dy_u}{dx} - C_{p_l} \frac{dy_l}{dx})d(\frac{x}{c}) + \int_0^1 (C_{f_u} + C_{f_l})d(\frac{x}{c})$
For the moment about the leading edge:	$C_{M_{LE}}$	$=$	$\int_0^1 (C_{p_u} - C_{p_l})\frac{x}{c} d(\frac{x}{c}) - \int_0^1 (C_{f_u} \frac{dy+u}{dx} + C_{f_l} \frac{dy_l}{dx})\frac{x}{c} d(\frac{x}{c})$
		$+$	$\int_0^1 (C_{p_u} \frac{dy_u}{dx} + C_{f_u})(\frac{y+u}{x})d(\frac{x}{c})$
		$+$	$\int_0^1 (-C_{p_l} \frac{dy_l}{dx} + C_{f_l})(\frac{y_l}{x})d(\frac{x}{c})$

1.3 Center of Pressure

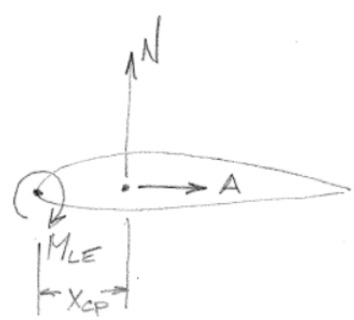


Figure 5: Center of Pressure and Moment at Leading Edge

Based on the definition of normal force N and chord force A, in Figure 5:

$$X_{CP} = -\frac{M_{LE}}{N} \tag{5}$$

For small α, N is close to L. So

$$X_{CP} = -\frac{M_{LE}}{L} \tag{6}$$

In Figure 6 the net moment is the same. (The moment is taken about different points but must always sum to the same value.)

$$M_{LE} = -\frac{c}{4}L + M_{c/4} = -X_{CP}L \tag{7}$$

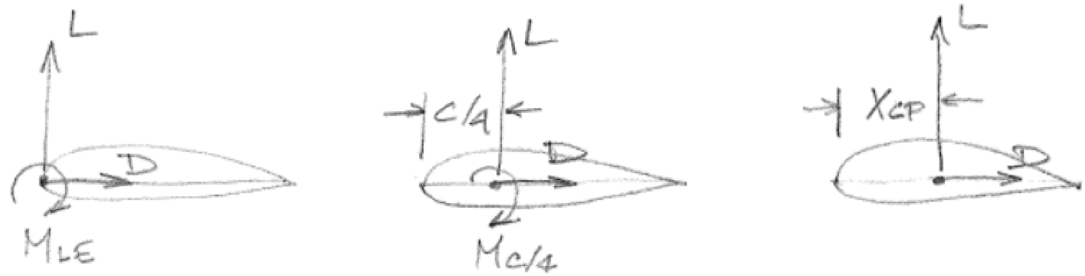


Figure 6: Center of Pressure and Moment in 3 Configurations

2 Drag, Fluid Statics, Dimensional Analysis

2.1 Comments on Drag

Recall the relation for C_a (Equation 8). The first term (containing C_p) is the pressure drag and the second (C_f) is the skin friction drag. The amount of each varies with the object's geometry. In Figure 7 the drag of the streamlined body is the SAME as the drag on a cylinder $1/10^{th}$ its diameter.

$$C_a = \int_0^1 (C_{p_u}\frac{dy_u}{dx} - C_{p_l}\frac{dy_l}{dx})d(\frac{x}{c}) + \int_0^1 (C_{f_u} + C_{f_l})d(\frac{x}{c}) \tag{8}$$

2.2 Fluid Statics: Buoyancy

Consider a static fluid (Figure 8).

$$\begin{aligned}
\text{Net pressure force} &= pdxdz - (p + \frac{dp}{dy}dy)\ dx\ dz \\
&= -\frac{dp}{dy}\ dx\ dy\ dz \\
\text{Gravity force} &= -\rho g\ dx\ dy\ dz \\
\text{static} &= \sum F_y = -\frac{dp}{dy}\ dx\ dy\ dz - \rho g\ dx\ dy\ dz = 0 \\
\text{Hydrostatic equation} &= dp = -\rho g dy \\
\text{For } \rho = const &= \int_{p1}^{p2} dp = -\rho g \int_{h1}^{h2} dy \\
&= p_2 - p_1 = -\rho g(h_2 - h_1 = \rho g \Delta h \\
\text{or} &= p + \rho g h = const
\end{aligned}$$

Consider a solid body immersed in a fluid (Figure 9). The vertical force on the body is $F = (p_2 - p_1)ab$. Integrating the hydrostatic equation:

$$p_2 - p_1 = \int_{p1}^{p2} dp = -\int_{h1}^{h2} \rho g dy = \int_{h2}^{h1} \rho g dy$$

$$F = ab\int_{h2}^{h1} \rho g dy$$

The **weight** of the column of fluid displaced by the solid body is

$$W = ab\int_{h2}^{h1} \rho g dy$$

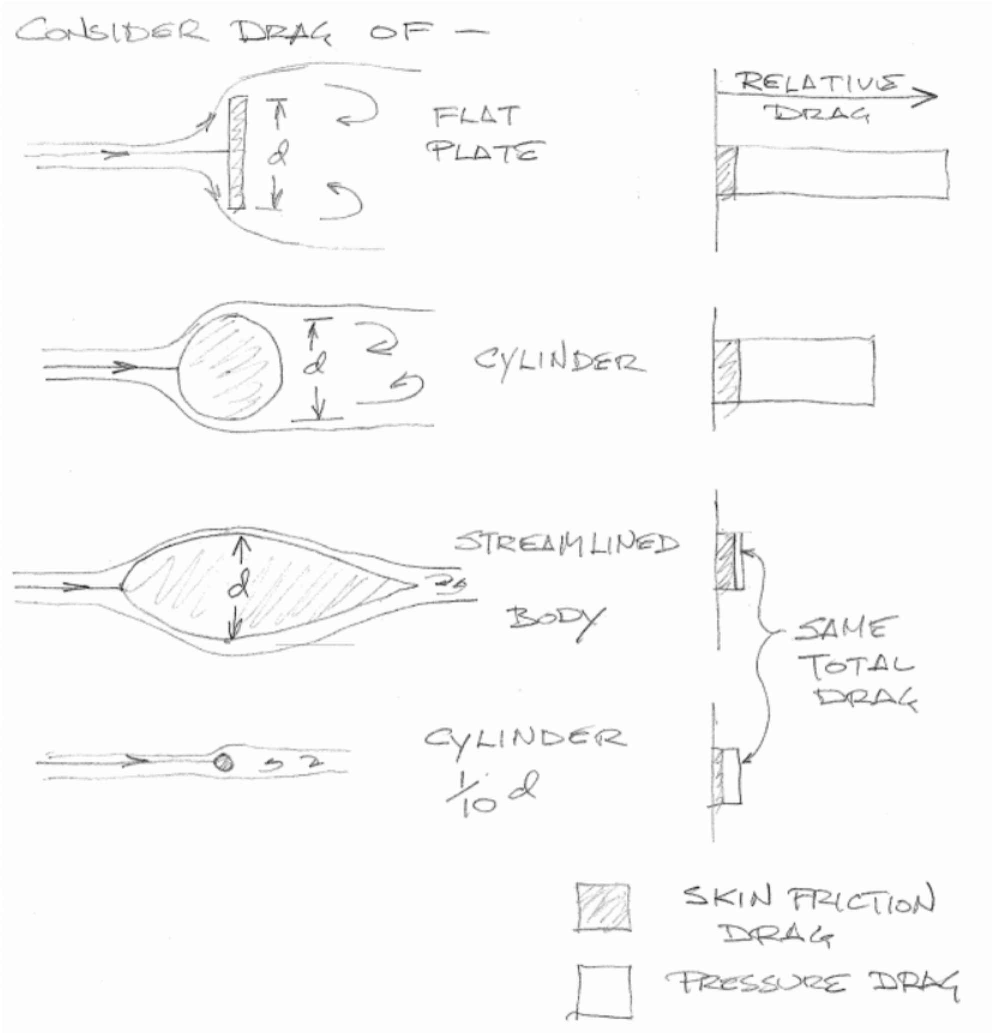


Figure 7: Drag Comparison for Plate, Cylinder, Streamlined Body

The buoyancy force on the body = the weight of the fluid displaced by the body (Archimedes Principle).
Note: ρ has not been assumed constant.

2.3 Dimensional Analysis

The aero force is $R = f(\rho_\infty,\ V_\infty,\ e,\ \mu_\infty,\ a_\infty)$. All variables can be expressed in terms of **fundamental dimensions** m, l, and t. Then the number of fundamental dimensions $K = 3$.
Given a physical relation $f_1(P_1,\ P_2, ..., P_N) = 0$ this can be presented as a relation with $n - K$ dimensionless products $\Pi' s$, $f_2(\Pi_1,\ \Pi_2, ..., \Pi_{N-K}) = 0$ where each Π is a product of the K fundamental dimensions.

$$\Pi_1 = f_3(P_1,\ P_2, ...,\ P_K,\ P_{K+1})$$
$$\Pi_1 = f_4(P_1,\ P_2, ...,\ P_K,\ P_{K+2})$$
$$...$$
$$\Pi_{N-K} = f_3(P_1,\ P_2, ...,\ P_K,\ P_N)$$

The dependent variable R should appear in only one of the Π terms.

Rewrite $R = f(...)$ as

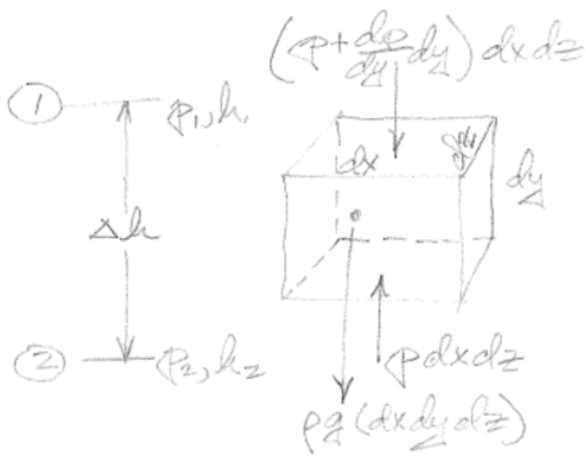


Figure 8: An element in a stagnant fluid

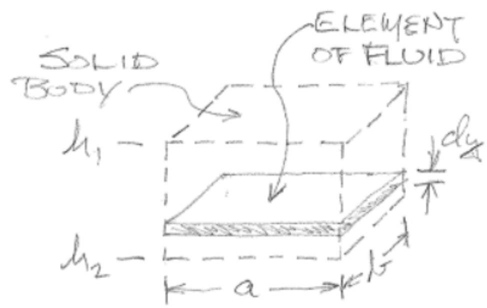


Figure 9: A body immersed in a static fluid

$$
\begin{gathered}
g(R,\ \rho_\infty,\ V_\infty,\ c,\ \mu_\infty,\ a_\infty)\\
[R] = mlt^{-2}\\
[\rho_\infty] = ml^{-3}\\
[V_\infty] = lt^{-1}\\
[c] = l\\
[\mu_\infty] = ml^{-1}t^{1}\\
[a_\infty] = lt^{-1}
\end{gathered}
$$

Here, $N - K = G - 3 = 3$.

$$
\begin{gathered}
f_2(\Pi_1,\ \Pi_2,\ \Pi_3) = 0\\
\Pi_1 = f_2(\rho_\infty,\ V_\infty,\ c,\ R)\\
\Pi_2 = f_3(\rho_\infty,\ V_\infty,\ c,\ \mu_\infty)\\
\Pi_3 = f_2(\rho_\infty,\ V_\infty,\ c,\ a_\infty)\\
\Pi_1 = \rho\infty^d V_\infty^b c^e R\\
[\Pi_1] = (ml^{-3})^d (lt^{-1})^b (l)^e (mlt^{-2})
\end{gathered}
$$

For Π to be dimensionless the exponents of m, l, and t must each sum to zero.

$$\begin{aligned} m &: d+1=0 \\ l &: -3d+b+e+1=0 \\ t &: -b-2=0 \\ & d=-1, \quad b=-2, \quad e=-2 \\ & \Pi_1 = R\rho_\infty^{-1}V_\infty^{-2}C^{-2} \\ & \frac{R}{\rho_\infty V_\infty^2 c^2} \rightarrow C_l = \frac{W/s}{\frac{1}{2}\rho_\infty V_\infty^2} \\ \text{Similarly} \quad & \Pi_2 = \frac{\rho_\infty V_\infty c}{\mu_\infty} = RM \quad \text{and} \quad \Pi_3 = \frac{V_\infty}{a_\infty} = M \\ \text{or} \quad & f_2\Big(\frac{R}{\frac{1}{2}\rho_\infty V_\infty^2 S}, \frac{\rho_\infty V_\infty c)}{\mu_\infty}, \frac{V_\infty}{a_\infty}\Big) = 0 \\ & C_L = f_6(RN, M_\infty) \end{aligned}$$

2.4 Vector Relations

Need to know addition: $\vec{A}+\vec{B}=\vec{C}$ and subtraction.

2.4.1 Dot Product

$\vec{A} \bullet \vec{B} = |A||B|\cos\theta$ is a SCALAR. We can think of it as $|A|$(component of $\vec{B}$ || to $\vec{A}$).

2.4.2 Cross Product

$\vec{A} \times \vec{B} = (|A||B|\sin\theta)\vec{e} = \vec{G}$. The unit vector $\vec{e}$ is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$ using the **right-hand-rule**.

2.4.3 Orthogonal Coordinate System

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \text{and} \quad |r| = (x^2+y^2+z^2)^{1/2} \tag{9}$$

2.4.4 Scalar and Vector Fields

See Table 3.

Table 3: Scalar and Vector Fields

P	=	P(x, y, z, t)
ρ	=	$\rho(x,y,z,t)$
T	=	T(x, y, z, t)
$\vec{V}$	=	$V_x\vec{i}+V_y\vec{j}+V_z\vec{k}$
V_x	=	$V_x(x,y,z,t)$
V_y	=	$V_y(x,y,z,t)$
V_z	=	$V_z(x,y,z,t)$

2.4.5 Gradient of a Scalar Field

Consider a 2D pressure field $P_1 > P_2 > P_3$ where $P = P(x,y)$. (Figure 10) Then the gradient is:

$$\vec{\nabla}p = \frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} \tag{10}$$

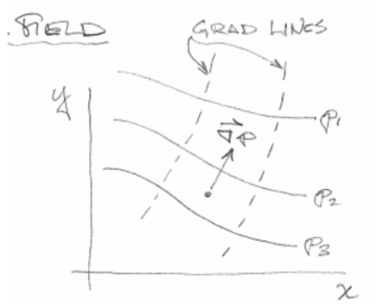


Figure 10: Gradient field with constant-pressure lines

$|\vec{\nabla}|$ represents the magnitude of the maximum rate of change of P per unit length.
The direction of $\vec{\nabla}p$ is the direction of the max rate of change of P. Generally:

$$\vec{\nabla}P = \frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k} \tag{11}$$

For this rate of change of P in an arbitrary direction $\vec{S}$, where $\vec{n}$ is the unit vector in that direction

$$\frac{dp}{ds} = \vec{\nabla}p \cdot \vec{h} \quad \text{and} \quad \frac{dp}{ds} = |\vec{\nabla}p|\cos\theta \tag{12}$$

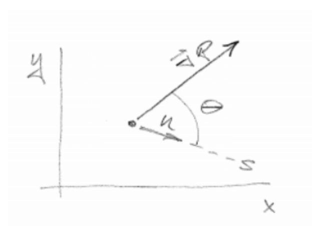


Figure 11: Vectors P and n

2.4.6 Divergence of a Vector Field

Consider a vector field where $\vec{V}(x, y, z)$ is the flow velocity.
Next consider a small fluid element of **fixed mass** moving along a streamline with velocity $\vec{V}$. The rate of change of the volume of the fluid element is given by the **divergence of** $\vec{V}$ (Equation 13).

$$\vec{\nabla} \bullet \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \tag{13}$$

2.4.7 Curl of a Vector Field

Again, consider a velocity field and a fluid element with velocity $\vec{V}$. Said fluid element may be rotating with angular velocity $\vec{\omega}$. It can be shown that $\vec{\omega} = 1/2\vec{\nabla} \times \vec{V}$ where curl(V) is given by Equation 14.

$$\text{Curl } \vec{V} = \vec{\nabla} \times \vec{V} \ \Big(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\Big)\vec{i} + \Big(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\Big)\vec{j} + \Big(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\Big)\vec{k} \tag{14}$$

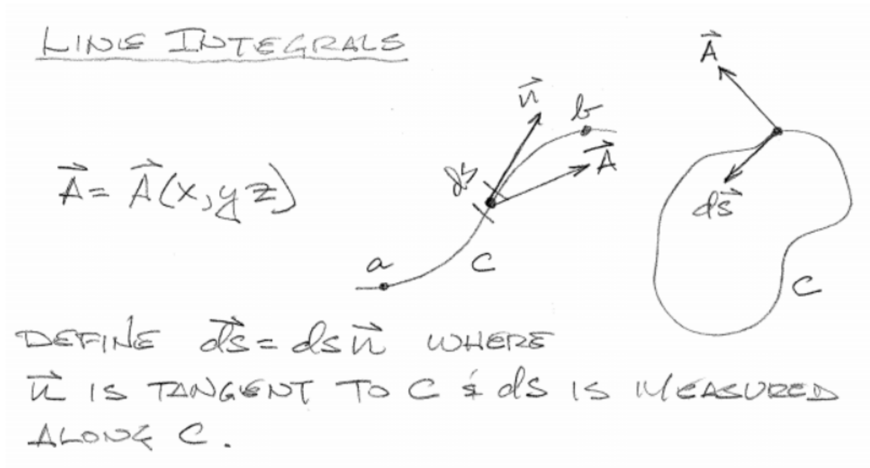


Figure 12: Line Integral on Open and Closed Curve

3 Integrals

3.1 Line Integrals

A line integral integrates a vector function along an open or closed curve.
Define $\vec{ds} = ds\vec{n}$ where $\vec{n}$ is tangent to the curve C and ds is measured along C. (Figure 12). The line integral of a vector $\vec{A}$ along C is Equation 15 where **counterclockwise along C is positive.**

$$\int_a^b \vec{A} \bullet \vec{ds} \quad \text{If C is closed:} \quad \oint_C \vec{A} \bullet \vec{ds} \tag{15}$$

3.2 Surface Integrals

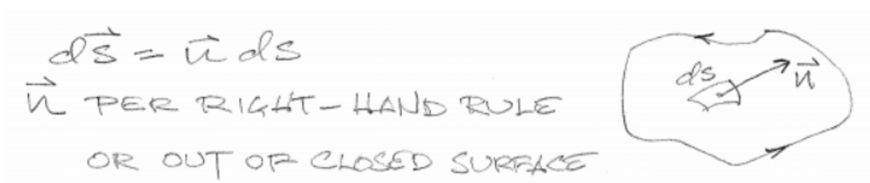


Figure 13: Surface Integral on Closed Surface

Again, $d\vec{s}$ $\vec{u}ds$. $\vec{u}$ is per right-hand-rule or out of the closed surface. There are three possibilities for surface integrals (Equation 16).

$$\begin{array}{ll} \text{Scalar function, vector result:} & \oiint_S p d\vec{s} \\ \text{Vector function, scalar result:} & \oiint_S \vec{A} \bullet \vec{ds} \\ \text{Vector function, vector result:} & \oiint_S \vec{A} \times \vec{ds} \end{array} \tag{16}$$

3.3 Volume Integrals

$$\iiint_V \rho dv \text{ Scalar result} \quad \iiint_V \vec{A} dv \text{ Vector result} \tag{17}$$

Note: These should be closed integrals but due to a collision between packages it became difficult to bring in the closed triple integral symbol.

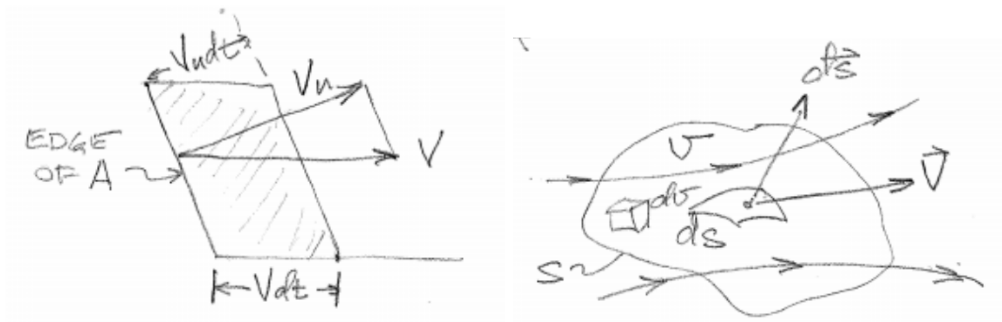


Figure 14: Continuity for an area A and a fixed control volume

3.3.1 Stokes Theorem

Converts between a line integral on a closed curve and a surface integral for the surface the curve encloses.

$$\oint_C \vec{A} \bullet \vec{ds} = \oiint_S (\vec{\nabla} \times \vec{A})\vec{ds} \tag{18}$$

3.3.2 Divergence Theorem

Converts between a surface integral and a volume integral for a vector integrand.

$$\oiint_S \vec{A} \bullet \vec{ds} = \iiint_V (\vec{\nabla} \bullet \vec{A}) dV \tag{19}$$

3.3.3 Gradient Theorem

Converts between a surface integral and a volume integral for a scalar integrand.

$$\oiint_S p\vec{ds} = \iiint_V \vec{\nabla} p dV \tag{20}$$

4 Continuity Equation

Consider an area A in a fluid flow field (Figure 14). The **volume** is $(V_u dt)A$. The **mass** is $\rho V_u dtA$ and the mass flow rate $\dot{m} = \rho V_u A$.
Consider a fixed control volume (Figure 14). The mass flow across ds is

$$\rho V_u ds = \rho \vec{V} \bullet \vec{ds}$$

$$\text{The net mass flow out of the volume V is} \quad B = \oiint_S \rho \vec{V} \bullet \vec{ds}$$

$$\text{The total mass in V is} \quad V = \iiint_V \rho dV$$

The rate of decrease (negative) of mass is the differential:

$$-\frac{\partial}{\partial t} \iiint_V \rho dV = c$$

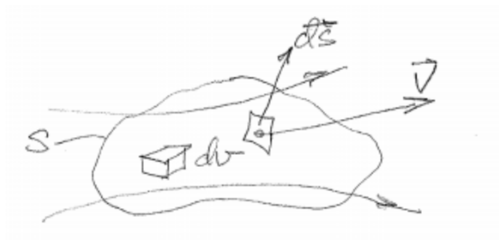


Figure 15: Momentum through an Arbitrary Control Volume

For continuity, $B = c$ (the net mass flow OUT equals the rate of decrease of mass in the control volume). Then write $B - c = 0$:

$$\boxed{\oiint_S \rho\vec{V} \bullet \vec{ds} + \frac{\partial}{\partial t}\iiint_V \rho dV = 0} \tag{21}$$

Since V is fixed in space (fixed control volume)	$\frac{\partial}{\partial t}\iiint_V \rho dV$	$=$	$\iiint_V \frac{\partial\rho}{\partial t} dV$
From the Divergence Theorem	$\oiint_S (\rho\vec{V}) \bullet \vec{ds}$	$=$	$\iiint_V \vec{\nabla} \bullet (\rho\vec{V}) dV$
Rewrite line 1 as	$\iiint_V \frac{\partial\rho}{\partial t} dV + \iiint_V \vec{\nabla} \bullet (\rho\vec{V}) dV$		
Combine integrals	$\iint_V \left[\frac{\partial\rho}{\partial t} + \vec{\nabla} \bullet (\rho\vec{V})\right] dV$	$=$	0
V is defined as arbitrary (switch to differential form)	$\frac{\partial\rho}{\partial t} + \vec{\nabla} \bullet (\rho\vec{V})$	$=$	0
In general, apply	ρ	$=$	$\rho(x, y, z, t)$
For steady flow	ρ	$=$	$\rho(x, y, z)$
This yields	$\oiint_C \rho\vec{V} \bullet \vec{ds}$	$=$	0
and	$\nabla \bullet (\rho\vec{V})$	$=$	0

5 Momentum Equation

Again, consider a control volume fixed in space (Figure 15).

Body Force	$=$	$\iiint_V \rho\vec{f} dV$ $\vec{f} = \frac{\text{Body Force}}{\text{Unit Mass}}$
Pressure Force	$=$	$-\oiint_S p\vec{ds}$ Due to force opposite $\vec{ds}$
Viscous Force	$=$	$\vec{F}_{vis}$ Total on surface S
Momentum Flux out of V	$=$	$\oiint_S (\rho\vec{V} \bullet \vec{ds})\vec{V}$
Momentum in V	$=$	$\iiint_V \rho\vec{V} dV$
Rate of change of momentum	$=$	$\frac{\partial}{\partial t}\iiint_V \rho\vec{V} dV$ is 0 if steady
Newton's 2nd Law		$\frac{d}{dt}(m\vec{V}) = \vec{F}$

$$\boxed{\frac{\partial}{\partial t}\iiint_V \rho\vec{V} dV + \oiint_S (\rho\vec{V} \bullet \vec{ds})\vec{V} = -\oiint_S p\vec{ds} + \iiint_V \rho\vec{f} dV + \vec{F_v}} \tag{22}$$

$$\text{Since V is fixed: } \frac{\partial}{\partial t}\iiint_V \rho\vec{V} dV = \iiint_V \frac{\partial}{\partial t}(\rho\vec{V}) dV$$

$$\text{Gradient Theorem: } -\oiint_{ds} p\vec{ds} = -\iiint_V \vec{\nabla} p dV$$

The integral form of the momentum equation (22) is a vector equation which can be written as 3 scalar equations. Define: $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$. Then the x-component of momentum is Equation 23) where $\rho\vec{V} \bullet \vec{ds}$ is a scalar.

$$\iiint_V \frac{\partial}{\partial t}(\rho u)dV + \iint_S (\rho\vec{V} \bullet \vec{ds})u = -\iiint_V \frac{\partial p}{\partial x}dV + \iiint_V \rho f_x dV + F_v|_x \tag{23}$$

We apply the divergence theorem, converting the surface integral to a volume integral.

$$\oiint_S (\rho\vec{V} \bullet \vec{ds})u = \oiint_S (\rho u\vec{V}) \bullet \vec{ds} = \iiint_V \vec{\nabla} \bullet (\rho u\vec{V})dV \tag{24}$$

Now all terms can be combined in a single volume integral.

$$\iiint_V \Big[\frac{\partial}{\partial t}(\rho u) + \vec{\nabla} \bullet (\rho u\vec{V}) + \frac{\partial p}{\partial x} - \rho f_x - F_{v_x}\Big]dV = 0 \tag{25}$$

Since V is arbitrary, switch to differential form

$$\frac{\partial}{\partial t}(\rho u) + \vec{\nabla} \bullet (\rho u\vec{V}) = -\frac{\partial p}{\partial x} + \rho f_x + F_{v_x} \tag{26}$$

Similar equations replacing x with y and z apply to the other dimensions.

5.1 Steady Inviscid Flow

For steady inviscid flow with no body force:

$$\oiint_S (\rho\vec{V} \bullet \vec{ds})\vec{V} = -\oiint_S p\vec{ds} \tag{27}$$

And the 3 components are Equation 28. These are the **Euler Equations**.

$$\begin{array}{rcl} \vec{\nabla} \bullet (\rho u\vec{V}) &=& -\frac{\partial p}{\partial x} \\ \vec{\nabla} \bullet (\rho v\vec{V}) &=& -\frac{\partial p}{\partial y} \\ \vec{\nabla} \bullet (\rho w\vec{V}) &=& -\frac{\partial p}{\partial z} \end{array} \tag{28}$$

6 Example: Airfoil Drag

Consider a 2D wing in a uniform onset flow. Draw a control volume such that:

1. abhi is far from the wing, then $P = P_\infty$ on abhi
2. u_1 and u_2 are parallel to x

List the surface forces on v:

1. On the boundary abhi, shear forces are zero, and the pressure force is

$$-\iint_{abhi} p\vec{ds}$$

2. The surface forces on cd and fg cancel
3. Let $\vec{R}$ be the net aerodynamic force on the wing. Then $-\vec{R}$ is the surface force on def

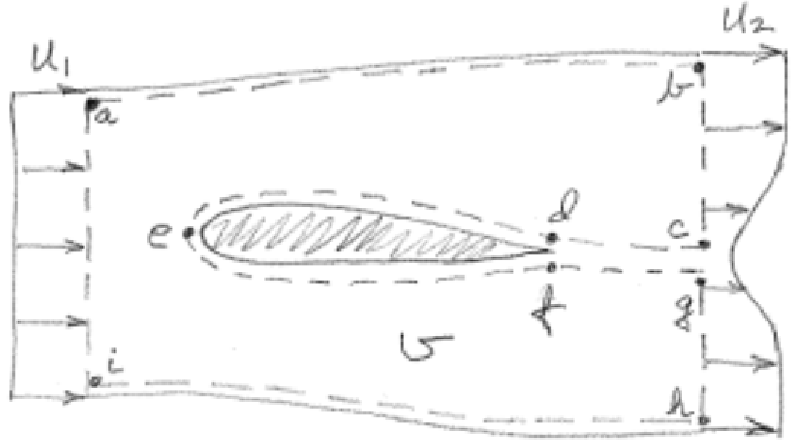


Figure 16: Example: Drag on an Airfoil in a Control Volume

The total surface force on v is

$$-\oiint_{abhi} p\vec{ds} - \vec{R}$$

Writing the integral form of the momentum equation:

$$\frac{\partial}{\partial t}\iiint_V \rho\vec{V}dV + \oiint_S (\rho\vec{V}\bullet\vec{ds})\vec{V} = -\oiint_{abhi} p\vec{ds} - \vec{R}$$

Since the surface force above is the total force (there are no body forces) on the fluid passing through V, and V is chosen to be large enough that $P = P_\infty$ everywhere on abhi:

$$\iint_{abhi} p\vec{ds} = 0$$

Assuming steady flow, $\frac{\partial}{\partial t}(\ ) = 0$ and the momentum equation reduces to

$$\oiint_S (\rho\vec{V}\bullet\vec{ds})\vec{V} = -\vec{R}$$

Or in terms of the aerodynamic force $\vec{R}$ **on** the wing

$$\vec{R} = \oiint_S (\rho\vec{V}\bullet\vec{ds})\vec{V}$$

The x-component gives the drag

$$D = \oiint_S (\rho\vec{V}\bullet\vec{ds})u$$

1. Since ab, hi, and def are streamlines, $\vec{V}$ is perpendicular to $\vec{ds}$ and $\vec{V}\bullet\vec{ds} = 0$
2. cd and fg are adjacent, hence the mass flux $\rho\vec{V}$ cancels
3. The only contribution to D comes from sections ai and bh

Figure 17: Fluid flowing through a fixed control volume V

$$D = -\int_i^a \rho_1 u_1^2 dy + \int_h^b \rho_2 u_2^2 dy$$

(The minus on ai is due to $\vec{V}$ being inward on $\vec{ds}$ along ai.)
Applying the continuity equation, which is

$$-\int_i^a \rho_u u_1 dy + \int_h^b \rho_2 u_2 dy = 0$$

Multiply the continuity equation by u_1, a constant. Then

$$\int_c^a \rho_1 u_1^2 dy = \int_h^b \rho_2 u_2 u_1 dy$$

Substitute in the drag equation. The final result is Equation 29.

$$D = \int_h^b \rho_2 u_2 (u_1 - u_2) dy \tag{29}$$

In a similar fashion, lift on a wing in a wind tunnel can be obtained by measuring the pressure distribution on the floor and ceiling of the test section. (In this case, ab and hi are not far enough away such that $p = p_\infty$ and they are not natural streamlines.)

6.1 Energy Equation

Consider a fixed amount of water within a closed boundary. This water is the **system**. The region outside of the system is the **surroundings**.
δq = heat added to the system from the surroundings
δw = work done on the system by the surroundings
δe = change in internal energy of the system due to $\delta q, \delta w$.
For conservation of energy:

$$\delta q + \delta w = \delta e$$

This is the **First Law of Thermodynamics.**

Apply the 1st Law to the fluid flowing through the control volume V, which is fixed in space. Let
B_1 = rate of heat added
B_2 = rate of work done
B_3 = rate of change of energy

$$B_1 + B_2 = B_3 \qquad \text{1st Law}$$

Define $\dot{q}$ as the rate of heat addition for unit mass. Then for the control volume V:

$$\text{Rate of heat addition} = \iiint_V \dot{q} \rho dV$$

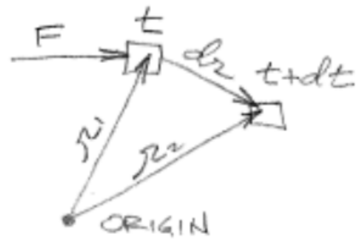


Figure 18: Work done on object by forces

Define $\dot{Q}_{visc}$ as the rate of heat addition due to viscosity. Then B_1 is the rate of heat addition.

Consider the work done on an object by the force $\vec{F}$, where the position of the object is defined by the radius vector $\vec{r}$ with a fixed origin.
$Work = \vec{F} \bullet \vec{dr}$
Rate of work is

$$\vec{F} \bullet \frac{dr}{dt} \vec{F} \bullet \vec{V}$$

On the surface S of the control volume, this pressure force is $-p\vec{ds}$

$$\text{Rate of work due to pressure} = -\oiint_S (p\vec{ds}) \bullet \vec{V}$$

Similarly, recalling $\vec{f}$ = body force per unit mass, the rate of work due to body force is

$$\iiint_V (\rho \vec{f} dv) \bullet \vec{V}$$

Define the rate of work due to shear as $\vec{\dot{w}}_{visc}$. Then B_2 (the rate of work done) is

$$B_2 = -\iint_S p\vec{V} \bullet \vec{ds} + \iiint_V \rho(\vec{f} \bullet \vec{V}) dv + \dot{W}_{visc}$$

Internal energy is typically the energy due to random molecular motion for a stationary system. The energy within the control volume must also include the kinetic energy due to $\vec{V}$ Said kinetic energy/unit mass is $1/2V^2$. The mass flow across ds is $\rho\vec{V} \bullet \vec{ds}$.

$$\text{Rate of flow of energy} = \oiint_S (\rho\vec{V} \bullet \vec{ds})(e + \frac{V^2}{2})$$

$$\text{Time rate of change of energy in v} = \frac{\partial}{\partial t} \iiint_V \rho(e + \frac{V^2}{2}) dv$$

Write $B_1 + B_2 = B_3$ (heat added + work done = rate of change of energy).

$$\iiint_V \dot{q}\rho dv + \dot{Q}_{visc} - \iint_S p\vec{V} \bullet \vec{ds} + \iiint_V \rho(\vec{f} \bullet \vec{V}) dv + \dot{W}_{visc} = \frac{\partial}{\partial t} \iiint_V \rho(e + \frac{V^2}{2}) dv + \oiint_S (\rho\vec{V} \bullet \vec{ds})(e + \frac{V^2}{2})$$

Applying the divergence theorem to convert the surface integrals to volume integrals and setting the integrand = 0, we get Equation 30.

$$\frac{\partial}{\partial t}\Big[\rho(e+\frac{V^2}{2})\Big]+\vec{\nabla}\bullet\Big[\rho(e+\frac{V^2}{2})\vec{V}\Big]=\rho\dot{q}-\vec{\nabla}\bullet(\rho\vec{V})+\rho(\vec{f}\bullet\vec{V})+\dot{Q}_{visc}+\dot{W}_{visc} \tag{30}$$

Where $\dot{Q}_{visc}$ and $\dot{W}_{visc}$ represent viscous terms to be defined explicitly later. Equation 30 is a PDE that defines the relationship of the flow field variables at any port in the control volume.

For the case of steady, inviscid, adiabatic flow with no body forces, the equations become Equation 31.

$$\begin{aligned} \oiint_S \rho(e+\tfrac{V^2}{2})\vec{V}\bullet\vec{ds} &= \oiint_S p\vec{V}\bullet\vec{ds} \\ \vec{\nabla}\bullet\Big[\rho(e+\tfrac{V^2}{2})\vec{V}\Big] &= -\vec{\nabla}(\rho\vec{V}) \end{aligned} \tag{31}$$

For a calorically perfect gas: $e = C_vT$
For a perfect gas: $p = \rho RT$

6.2 Substantial Derivative

Consider a fluid element in a flow field from position 1 to 2. In general, flow properties are a function of both position and time, i.e. $\rho = \rho(x, y, z, t)$.
Using a Taylor series to expand this function about 1 (H.O.T. stands for Higher Order Terms):

$$\rho_2 = \rho_1 + \frac{\partial\rho}{\partial x}\Big)_1(x_2 = x_1) + \frac{\partial\rho}{\partial y}\Big)_1(y_2 - y_1) + H.O.T.$$

$$\frac{\rho_2 = \rho_1}{t_2 - t_1} = \frac{\partial\rho}{\partial x}\Big)_1\Big(\frac{x_2 - x_1}{t_2 - t_1}\Big) + \frac{\partial\rho}{\partial y}\Big)_1\Big(\frac{y_2 - y_1}{t_2 - t_1}\Big) + \frac{\partial\rho}{\partial z}\Big)_1\Big(\frac{z_2 - z_1}{t_2 - t_1}\Big)$$

Define the substantial derivative as Equation 32.

$$\frac{Dp}{Dt} = \lim_{t_2\to t_1}\frac{\rho_2-\rho_1}{t_2-t_1} \text{ and } \lim_{t_2\to t_1}\frac{x_2-x_1}{t_2-t_1} = u_1$$

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} \tag{32}$$

Physically, $D\rho/Dt$ is the rate of change of ρ as the fluid element moves through space. $\partial\rho/\partial t$ is the rate of change of ρ at a given point. For steady flow, $\partial\rho/\partial t = 0$ and $D\rho/Dt \neq 0$.

More generally

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Recall the definition of the gradient:

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \tag{33}$$

Thus

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{V}\bullet\vec{\nabla}) \tag{34}$$

A useful vector identity is

$$\vec{\nabla}\bullet(\rho\vec{V}) = \rho\vec{\nabla}\bullet\vec{V} + \vec{V}\bullet\vec{\nabla}\rho$$

Recall the continuity equation:

$$\begin{aligned} \tfrac{\partial\rho}{\partial t} + \vec{\nabla}\bullet(\rho\vec{V}) &= 0 \\ \text{Or } \tfrac{\partial\rho}{\partial t} + \vec{V}\bullet\vec{\nabla}\rho + \rho\vec{\nabla}\bullet\vec{V} &= 0 \\ \tfrac{D\rho}{Dt} + \rho\vec{\nabla}\bullet\vec{V} &= 0 \end{aligned} \tag{35}$$

Equation 35 is the substantial derivative form of the continuity equation. Similarly the momentum equation becomes Equation 36.

$$\rho\frac{Du}{Dt} = -\frac{\partial\rho}{\partial x} + \rho f_x + F_{x,visc} + \text{etc} \tag{36}$$

And the energy equation becomes Equation 37.

$$\rho\frac{D}{Dt}(e + \frac{V^2}{2}) = \rho\dot{q} - \vec{\nabla}\bullet(\rho\vec{V}) + \rho(\vec{f}\bullet\vec{V}) + \dot{Q}_{visc} + \dot{W}_{visc} \tag{37}$$

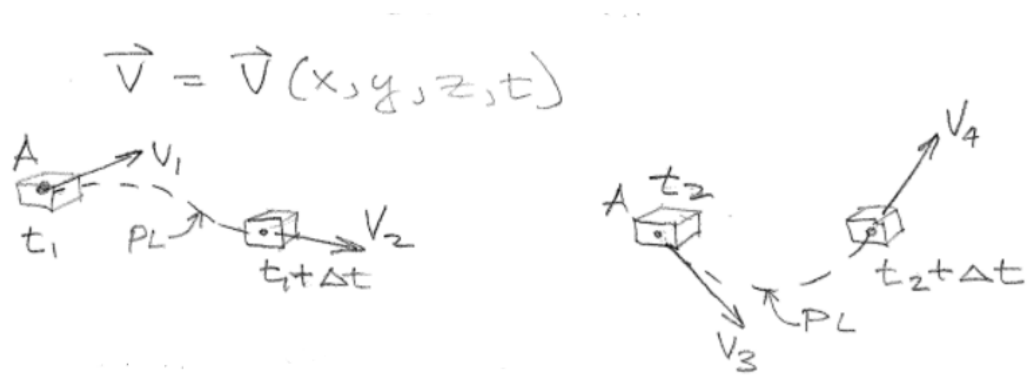


Figure 19: Pathline and Streamline for a Fluid Element

7 Pathlines and Streamlines

A **pathline** is the path a particular fluid element traces in space. At some time t_1, a fluid element passes through point A and traces the pathline shown in Figure 19. At some later time t_2, another fluid element passes through Point A and traces the pathline shown. If the flow is unsteady, the pathlines will be different.

A streamline is a curve whose tangent is in the same direction as the velocity vector at that point.
Pathline → Time Exposure
Streamline → Snapshot
For steady flows the streamlines and pathlines are the same.

Given a velocity field $\vec{V}(x, y, z)$, obtain the equation for the streamlines $f(x, y, z) = 0$. Let $\vec{ds}$ be an element of the streamline. For $\vec{ds}$ to be parallel to $\vec{V}$, it must be true that $\vec{ds} \times \vec{V} = 0$.

$$\vec{ds} = dx\vec{i} + dy\vec{j} + dz\vec{k},\ \vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\vec{ds} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ dx & dy & dz \\ u & v & w \end{vmatrix} = \begin{matrix} \vec{i}(wdy - vdz) \\ +\vec{j}(udz - wdx) \\ +\vec{k}(vdx - udy) \end{matrix} = 0$$

If a vector equals 0, all components are 0.

$$\begin{aligned} wdy - vdz &= 0 \\ udz - wdx &= 0 \\ vdx - udy &= 0 \end{aligned}$$

In principle, if $u,\ v,\ w$ are known as functions of x, y, and z, these equations can be integrated to obtain the equation for the corresponding streamlines $f(x, y, z) = 0$. In 2D: The slope of the streamline is dy/dx (Figure 20) and the slope of $\vec{V}$ is v/u. Since $dy/dx = v/u,\ vdx - udy = 0$.

7.1 Vorticity, Strain

In Figure 21, consider the y-distance moved in Δt. At Point A, this distance is $v\Delta t$.
At Point C, this distance is $(v + \frac{\partial v}{\partial x}dx)\Delta t$. At Point C relative to Point A, $C - A = \frac{\partial v}{\partial x}dx\Delta t$.

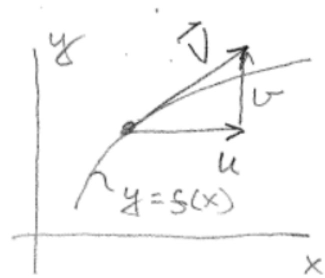


Figure 20: A Streamline in 2D Space

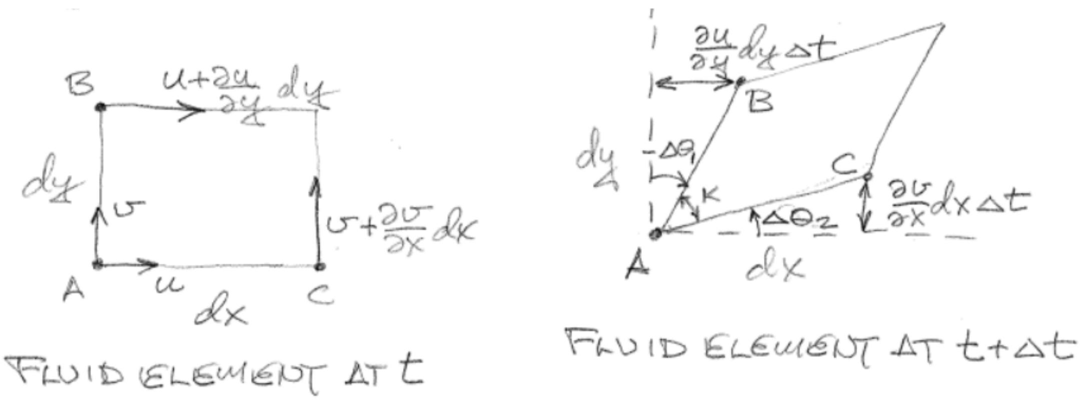


Figure 21: Strain on a Fluid Element

The rotation of $\bar{A}C$ is

$$\tan \Delta\theta_2 = \frac{\frac{\partial v}{\partial x} dx \Delta t}{dx} = \frac{\partial v}{\partial x}\Delta t$$

Or for small angles: $\Delta\theta_2 = \frac{\partial v}{\partial x}\Delta t$.
Similarly, for $\bar{A}B$, $\Delta\theta_1 = -\frac{\partial u}{\partial y}\Delta t$.

$$\text{Taking the limit: } \lim_{\Delta t \to 0} \frac{\Delta\theta_1}{\Delta t} = \frac{d\theta_1}{dt} - \frac{\partial u}{\partial y}, \quad \frac{d\theta_2}{dt} = \frac{\partial v}{\partial x}$$

The angular velocity of the fluid element is the average of the angular velocities of $\bar{A}B$, $\bar{A}C$. (Equation 38.)

$$\omega_2 = \frac{1}{2}\Big(\frac{d\theta_1}{dt} + \frac{d\theta_2}{dt}\Big) = \frac{1}{2}\Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big) \tag{38}$$

Similar arguments apply in the yz and xz planes. The angular velocity in xyz-space is then Equation 39.

$$\omega = \frac{1}{2}\Bigg[\Big(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\Big)\vec{i} + \Big(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\Big)\vec{j} + \Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big)\vec{k}\Bigg] \tag{39}$$

For convenience, **vorticity** is defined as Equation 40.

$$\vec{\zeta} = 2\vec{\omega} \tag{40}$$

Recalling the cross product of the operator $\vec{\nabla}$ with $\vec{V}$:

$$\vec{\nabla} \times \vec{V} = \vec{\zeta} \tag{41}$$

The curl of velocity equals vorticity (Equation 41).
If $\vec{\nabla} \times \vec{V} = 0$ everywhere, the flow is said to be **irrotational**. In the xy-plane, then:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \text{ (Irrotational Flow)} \tag{42}$$

7.2 Strain

The strain is defined as the change in the angle K of the fluid element. Strain is **positive** when K **decreases**.

$$\begin{aligned} \Delta K &= -\Delta\theta_2 = (-\Delta\theta_1) \\ \text{Strain } = -\Delta K &= \Delta\theta_2 - \Delta\theta_1 \end{aligned}$$

Recall $\Delta\theta_2 = \frac{\partial v}{\partial x}\Delta t$ and $\Delta\theta_1 = \frac{\partial u}{\partial y}\Delta t$. Define ϵ_{xy} as the rate of strain.

$$\epsilon_{xy} = -\frac{dK}{dt} = \frac{d\theta_2}{dt} - \frac{d\theta_1}{dt}$$

$$\epsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{43}$$

$$\begin{aligned} \text{Similarly: } \epsilon_{yz} &= \tfrac{\partial w}{\partial y} + \tfrac{\partial v}{\partial z} \\ \epsilon_{zx} &= \tfrac{\partial u}{\partial z} + \tfrac{\partial w}{\partial x} \end{aligned}$$

$$\text{Write the matrix } \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{44}$$

The diagonal of Equation 44 is $\vec{\nabla} \bullet \vec{V}$ which is the dilatation of the element. The off-diagonal is the rotation and strain.

7.3 Circulation

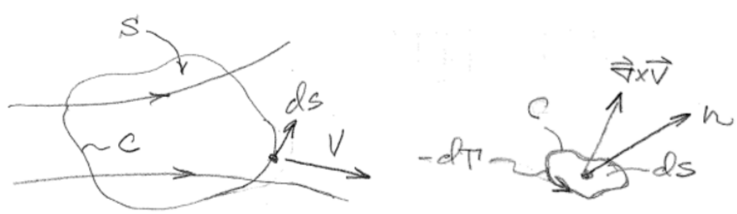


Figure 22: Circulation around a curve C in regular and differential form

Define the circulation Γ as Equation 45. This is minus due to clockwise Γ being positive.

$$\Gamma = -\oint_C \vec{V} \bullet \vec{ds} \tag{45}$$

Consider S as any open surface bounded by C (Figure 22). From Stokes Theorem:

$$\Gamma = -\oint_C \vec{V} \bullet \vec{ds} = -\iint_S (\vec{\nabla} \times \vec{V}) \bullet \vec{ds}$$

$$\text{Letting C shrink to a point} \quad d\Gamma = -(\vec{\nabla} \times \vec{V}) \bullet \vec{ds} = -(\vec{\nabla} \times \vec{V}) \bullet \vec{n} ds$$

$$\text{Divide by } ds \quad \frac{d\Gamma}{ds} = -(\vec{\nabla} \times \vec{V}) \bullet \vec{n}$$

$$\text{Recalling the definition of vorticity } \vec{\zeta} = \vec{\nabla} \times \vec{V} \quad \frac{d\Gamma}{ds} = -\vec{\zeta} \bullet \vec{n}$$

The vorticity normal to ds is equivalent to the circulation per unit area (Figure 22).

7.4 Stream Function

Consider a 2D steady flow. Recall the differential equation for a streamline is $dy/dx = v/u$. If $u(x, y)$ and $v(x, y)$ are known, the differential equation can be integrated to obtain $f(x, y) = c$, an algebraic equation for the streamline. Call this function $\bar{\Psi}(x, y) = c$, the **stream function** (Figure 23).

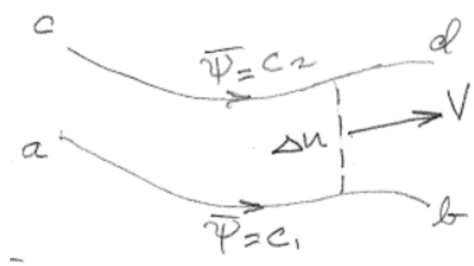


Figure 23: Stream function and stream lines

Different values of C will define different streamlines. Now define $\bar{\Psi}$ such that the difference $\Delta\bar{\Psi}$ between streamlines ab and cd is equal to the mass flow per unit depth.

$$\Delta\bar{\Psi} = c_2 - c_1 = \text{Mass flow}$$

Continuity requires that this applies everywhere along the streamtube: $\Delta\bar{\Psi} = \rho V \Delta n$.

$$\rho V = \lim_{\Delta n \to 0} \frac{\Delta\bar{\Psi}}{\Delta n} = \frac{\partial\bar{\Psi}}{\partial u}$$

$$\text{Also from continuity:} \quad \Delta\bar{\Psi} = \rho V \Delta n = \rho u \Delta y + \rho v(-\Delta x)$$

$$\text{Taking the limit:} \quad d\bar{\Psi} = \rho u dy - \rho v dx$$

$$\text{Also for } \bar{\Psi} = \bar{\Psi}(x, y) \quad d\bar{\Psi} = \frac{\partial\bar{\Psi}}{\partial x} dx + \frac{\partial\bar{\Psi}}{\partial y} dy$$

$$\text{Then:} \quad \rho u = \frac{\partial\bar{\Psi}}{\partial y}, \quad \rho v = -\frac{\partial\bar{\Psi}}{\partial x}$$

For **incompressible flow**, define $\Psi = \bar{\Psi}/\rho$. Equation 46 results. Given $\bar{\Psi}(x, y)$ or $\Psi(x, y)$, the velocities $u(x, y)$, $v(x, y)$ can be obtained by differentiation.

$$u = \frac{\partial\Psi}{\partial y}, \quad v = -\frac{\partial\Psi}{\partial x} \tag{46}$$

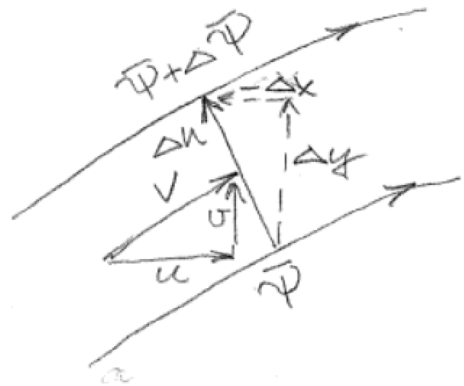


Figure 24: Stream function in Cartesian coordinates

7.4.1 Stream Function in Polar Coordinates

$$\begin{aligned}
\text{Start with the definition} \quad & \Delta\bar{\Psi} = \rho V \Delta n \\
\text{Then} \quad & \bar{\Psi} = (-\rho V_\theta)\Delta r + \rho V_r (r\Delta\theta) \\
\text{Take the limit} \quad & d\Psi - = \rho V_\theta dr + \rho r V_r d\theta \\
& d\Psi = \frac{\partial \Psi}{\partial r} dr + \frac{\partial \Psi}{\partial \theta} d\theta
\end{aligned}$$

$$\rho V_r = \frac{1}{r}\frac{\partial \bar{\Psi}}{\partial \theta}, \quad \rho V_\theta = -\frac{\partial \bar{\Psi}}{\partial r} \tag{47}$$

7.5 Velocity Potential

Recall that if a flow is **irrotational**, Equation 40 is 0. $\vec{\zeta} = \vec{\nabla} \times \vec{V} = 0$. Write the vector identity for a scalar function $\Phi(x, y)$ as $\vec{\nabla} \times (\vec{\nabla}\Phi) = 0$. If a flow is irrotational, there exists a scalar function Φ such that the velocity field is given by the gradient of Φ.

$$\begin{aligned}
\text{Then} \quad & \vec{V} = \vec{\nabla}\Phi = \frac{\partial \Phi}{\partial x}\vec{i} + \frac{\partial \Phi}{\partial y}\vec{j} + \frac{\partial \Phi}{\partial z}\vec{k} \\
\text{or} \quad & u = \frac{\partial \Phi}{\partial x}, \ v = \frac{\partial \Phi}{\partial y}, \ w = \frac{\partial \Phi}{\partial z}
\end{aligned}$$

7.5.1 Relationship between Stream Function and Potential

Φ = constant is an **equipotential line**.
A line everywhere tangent to $\vec{\nabla}\Phi$ is a **gradient line**, and since $\vec{\nabla}\Phi = \vec{V}$ this is also a streamline.
Gradient lines and equipotential lines are by definition **orthogonal**. Ψ = constant and Φ = constant are perpendicular.

For a 2D incompressible irrotational flow, $\Psi(x, y)$ = constant. Then, along a streamline:

$$d\Psi = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy = 0$$

$$\begin{aligned} \text{Also} \quad & d\Psi = -vdx + udy = 0 \\ \text{or} \quad & \left.\frac{dy}{dx}\right)_{\Phi=c} = \frac{v}{u} \end{aligned}$$

Similarly, $\Psi(x, y)$ = constant along an equipotential line, and:

$$d\Phi = \frac{\partial\Psi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy = 0$$

$$\begin{aligned} \text{Also} \quad & d\Phi = udx + vdy \\ \text{or} \quad & \left.\frac{dy}{dx}\right)_{\Phi=c} = -\frac{u}{v} \\ \text{Then} \quad & \left.\frac{dy}{dx}\right)_{\Phi=c} = -\frac{1}{\left.dy/dx\right)_{\Psi=c}} \end{aligned}$$

1. u and v are obtained by differentiating Φ in the same direction as $\vec{V}$ and Ψ in a direction normal to $\vec{V}$.
2. Φ is defined for irrotational flow only. Ψ does not require irrotational flow.
3. Φ applies to 3D and 2D flow. Ψ applies to 2D flow only.

Velocity Potential in Polar Coordinates

$$\begin{aligned} \text{By definition} \quad & \vec{V} = \vec{\nabla}\Phi \\ \text{In polar coordinates} \quad & \vec{\nabla} = \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\vec{e}_\theta \\ \text{Then} \quad & \vec{V} = \frac{\partial\Phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\vec{e}_\theta \end{aligned}$$

$$V_r = \frac{\partial\Phi}{\partial r}, \; V_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} \tag{48}$$

8 Bernoulli's Equation, Flow Measurement

8.1 Bernoulli's Equation

Consider the case of a steady, incompressible, inviscid flow. Write the x-component of the momentual equation as:

$$
\begin{aligned}
\text{Momentum} \quad & \rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \quad \frac{\partial u}{\partial t} = 0 \\
\text{Expand and multiply by } dx \quad & u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx + w\frac{\partial u}{\partial z}dx = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \\
\text{Consider 3D flow along a streamline} \quad & udz - wdx \; 0 \\
& vdx - udy = 0 \\
\text{Substituting} \quad & u\frac{\partial u}{\partial x}dx + u\frac{\partial u}{\partial y}dy + u\frac{\partial u}{\partial z}dz = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \\
\text{Recall} \quad & du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \\
& udu - -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \\
\text{or similarly} \quad & \frac{1}{2}d(u^2) = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \\
& \frac{1}{2}d(v^2) = -\frac{1}{\rho}\frac{\partial p}{\partial y}dy \\
& \frac{1}{2}d(w^2) = -\frac{1}{\rho}\frac{\partial p}{\partial z}dz \\
\text{Also} \quad & dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz \\
\text{Combine the 3 component equations} \quad & \frac{1}{2}d(V^2) = -\frac{1}{\rho}dp
\end{aligned}
$$

$$dp = -\rho V dV \tag{49}$$

Integrate Equation 49 between points 1 and 2 along a streamline:

$$\int_1^2 dp = -\rho\int_1^2 V dV, \;\; \rho \text{ is constant}$$

$$p_2 - p_1 = -\frac{1}{2}\rho(V_2^2 - V_1^2) \tag{50}$$

Equation 50 shows that $p + 1/2\rho V^2$ is constant along a streamline. For the general case where the flow may be rotational, the constant will vary from streamline to streamline. For **irrotational flow**, $p + 1/2\rho V^2$ everywhere. Thus, once the velocity field is known, the corresponding pressure field is known for a steady, incompressible, irrotational, inviscid flow.

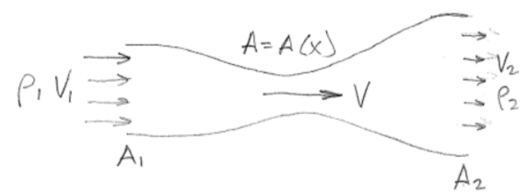


Figure 25: Incompressible flow in an axisymmetric duct

8.2 Incompressible Flow in a Duct

Consider an axisymmetric duct (Figure 25) where the area change is gradual enough to assume one-dimensional flow. For steady flow, continuity requires

$$\oiint_S \rho\vec{V} \bullet \vec{ds} = 0$$

$$\iint_{A_1} \rho\vec{V} \bullet \vec{ds} + \iint_{A+2} \rho\vec{V} \bullet \vec{ds} = 0$$

$$-\rho_1 A_1 V_1 + \rho_2 A_2 V_2 = 0 \text{ ds positive outwards}$$

$$\rho_1 A_1 V_1 \;\; \rho_2 A_2 V_2 \text{ or for incompressible flow } A_1 V_1 \;\; A_2 V_2$$

8.2.1 Velocity in a Wind Tunnel

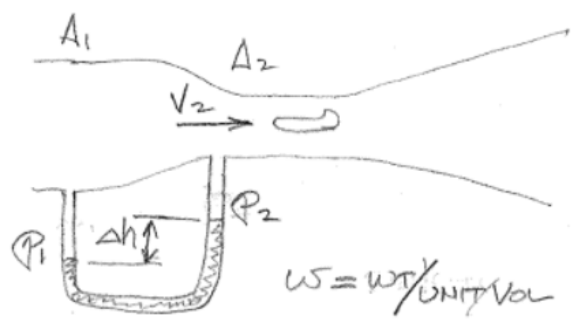


Figure 26: Incompressible flow in a wind tunnel with a manometer

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$$

$$V_2^2 = \frac{2}{\rho}(p_1 - p_2) + \left(\frac{A_2}{A_1}\right)^2 V_2^2 \quad \text{then} \quad V_2 = \sqrt{\frac{2(p_1 - p_2)}{\rho[1 - (A_2/A_1)^2]}} = \sqrt{\frac{2w\Delta h}{\rho[1 - (A_2/A_1)^2]}}$$

8.3 Measurement of Airspeed

Consider a pressure probe (Figure 27) pointing directly upstream. Then $p + 1/2\rho V^2 = p_0$. p_0 is the stagnation pressure. For this probe:

$$V = \sqrt{2\frac{p_0 - p}{\rho}}$$

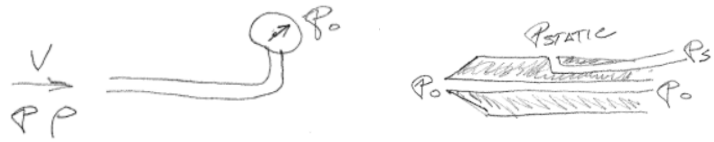


Figure 27: Pressure probe (left) and pitot-static probe (right)

The other probe in Figure 27 is a **pitot probe**. Adding a static port and measuring the pressure difference $p_0 - p$ gives the airspeed, assuming the density is known.

$$V\Big)_{IND} = \sqrt{2\frac{p_0 - p}{\rho_{SL}}}, \quad V\Big)_{TRUE} = \sqrt{2\frac{p_0 - p}{\rho}} \quad \text{and } V_{TRUE} = \sqrt{\frac{\rho_{SL}}{\rho}} V_{IND}$$

8.3.1 Pressure Coefficient

$$C_P = \frac{p - p_\infty}{1/2\rho V_\infty^2} = \frac{1/2\rho(V_\infty^2 - V^2)}{1/2\rho V_\infty^2} = 1 - \left(\frac{V}{V_\infty}\right)^2 \tag{51}$$

9 Incompressible, Inviscid Flow

9.1 Continuity Equation

$$\text{Recall the continuity equation} \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \bullet \rho\vec{V} = 0$$

$$\text{For steady, incompressible flow} \quad \vec{\nabla} \bullet \vec{V} = 0$$

If in addition the flow is irrotational, there exists a velocity potential Φ where $\vec{V} = \vec{\nabla}\Phi$. Then combine $\vec{\nabla} \bullet (\vec{\nabla}\Phi) = 0$ to obtain Equation 52: Laplace's Equation.

$$\boxed{\nabla^2\Phi = 0} \tag{52}$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

Recall the stream function for **2D flow** (Equation 46).

$$u = \frac{\partial\Psi}{\partial y}, \quad v = -\frac{\partial\Psi}{\partial x}$$

$$\text{The 2D continuity equation is} \quad \vec{\nabla} \bullet \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\text{In terms of } \Psi \text{ this becomes} \quad \frac{\partial^2\Psi}{\partial x \partial y} - \frac{\partial^2\Psi}{\partial y \partial x} = 0 \text{ (trivial)}$$

Recall Equation 42: $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ (condition for irrotational flow). In terms of Ψ this becomes:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = 0 \quad \text{Laplace's Equation}$$

1. Any irrotational, incompressible flow has a velocity potential that satisfies Laplace's Equation. If the flow is 2D, it has a stream function that also satisfies Laplace's Equation.

2. Conversely, any solution to Laplace's Equation represents Φ or Ψ for a flow.

3. Since Laplace's Equation is linear, separate solutions may be summed to obtain a combined solution $\Phi = \Phi_1 + \Phi_2 + ...$

9.2 Boundary Conditions

$$\begin{aligned}
\text{At infinity} \quad & u = \frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y} = V_\infty \\
& v = \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x} = 0 \\
\text{At the body} \quad & \vec{V} \bullet \vec{n} = \vec{\nabla} \Psi \bullet \vec{n} = 0 \\
& \frac{\partial \Phi}{\partial n} = 0 \\
& \Psi_{surface} = \text{const} \\
& \frac{\partial \Psi}{\partial s} = 0 \\
\text{For 2D flows} \quad & \frac{dy_{body}}{dx} = \left.\frac{v}{u}\right)_{surface}
\end{aligned}$$

Note: for viscous flows, the velocity is zero at the wall.

9.3 Uniform Flow

Consider uniform flow in the x-direction: $\vec{\nabla} \bullet \vec{V} = 0$ for incompressible flow, and $\vec{\nabla} \times \vec{V} = 0$ for irrotational flow. **This is physically possible!** (Equation 53.)

$$u = \frac{\partial \Phi}{\partial x} = V_\infty \quad v = \frac{\partial \Phi}{\partial y} = 0 \tag{53}$$

$$\begin{aligned}
\text{Integrating with respect to x} \quad & \Phi = V_\infty x + f(y) \\
\text{Integrating with respect to y} \quad & \Phi = const + q(x)
\end{aligned}$$

Since $\Phi(x, y)$ is the same function, $q(x) = V_\infty x + const$. Φ is always differentiated to obtain velocity, so the constant can be omitted.

$$\boxed{\Phi = v_\infty x} \quad \text{Uniform flow velocity potential}$$

Similarly, from $u = \frac{\partial \Psi}{\partial y} = V_\infty$ and $v = -\frac{\partial \Psi}{\partial x} = 0$, the stream function becomes

$$\boxed{\Psi = v_\infty y} \quad \text{Uniform flow stream function}$$

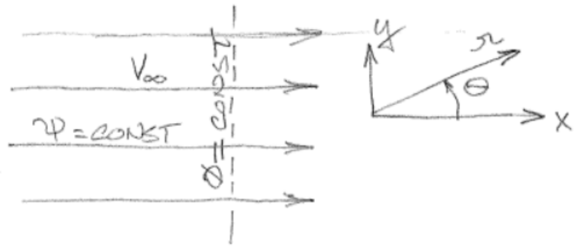


Figure 28: Uniform flow in polar coordinates

9.3.1 Uniform Flow in Polar Coordinates

$$\text{In polar coordinates} \quad \Phi = V_\infty r \cos(\theta)$$
$$\Psi = V_\infty r \sin(\theta)$$
$$V_r = \frac{\partial \Phi}{\partial r} = V_\infty \cos(\theta), \quad V_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -V_\infty \sin(\theta)$$
$$V_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = V_\infty \cos(\theta), \quad V_\theta = -\frac{\partial \Psi}{\partial r} = -V_\infty \sin(\theta)$$

9.4 Source (Sink) Flow

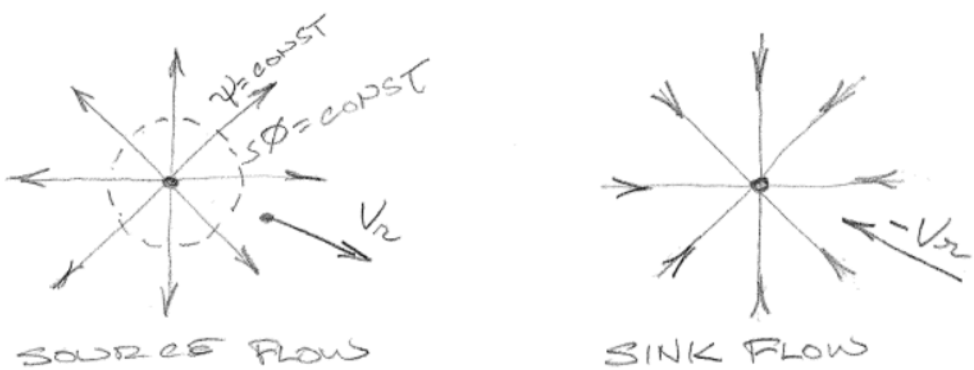


Figure 29: Source and Sink Flow

$$\text{Source or sink flow is defined by} \quad V_r = c/r, \quad V_\theta = 0$$
$$\text{Check continuity} \quad \vec{\nabla} \bullet \vec{V} = 0$$
$$\vec{\nabla} \bullet \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}$$
$$\vec{\nabla} \bullet \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(r \times \frac{c}{r}) = 0$$

Similarly, $\vec{\nabla} \times \vec{V} = 0$ so the flow is irrotational.

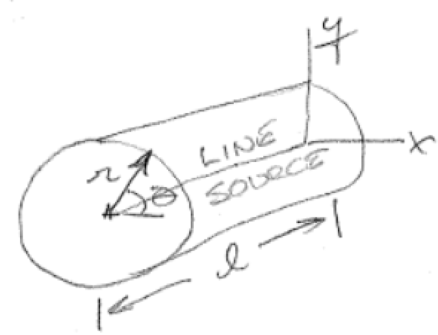


Figure 30: Line Source Flow

Consider a line source (Figure 30) of length l normal to the xy-plane. At a radius r, define:

$$\vec{ds} = l\vec{r}d\theta$$
$$d\dot{m} = \rho\vec{V} \bullet \vec{ds} = \rho V_r l r d\theta$$
$$\dot{m} = \int_0^{2\pi} \rho V_r l r d\theta = 2\pi\rho l r V_r \quad \text{mass/unit time}$$

Define $\dot{v} = \dot{m}/\rho$ as the volume flow / unit time. Then $\dot{v} = 2\pi l r V_r$.
Next, define $\Lambda = \dot{v}/l$ as the volume flow rate / unit length. Then $\Lambda = 2\pi r V_r$.

$$V_r = \frac{\Lambda}{d\pi r} \quad \Lambda = \text{Source strength, negative for sink}$$

9.4.1 Velocity Potential for a Source

$$V_r = \frac{\partial\Phi}{\partial r} = \frac{\Lambda}{2\pi r}$$
$$V_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = 0$$
$$\text{Integrating } \Phi = \frac{\Lambda}{2\pi}\log(r) + f(\theta)$$
$$\Phi = const + f(r)$$

$$\boxed{\Phi = \frac{\Lambda}{2\pi}\log(r)} \quad \text{Velocity potential for a source}$$

9.4.2 Stream Function for a Source

$$V_r = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = \frac{\Lambda}{2pir}, \quad V_\theta = -\frac{\partial\Psi}{\partial r} = 0$$
$$\text{Integrating } \Psi = \frac{\Lambda}{2\pi}\theta + f(r)$$

Notice that for $\Psi = const$ (streamline), $\theta = const$ which is a line from the origin.

$$\boxed{\Psi = \frac{\Lambda}{2\pi}\theta} \quad \text{Stream function for a source}$$

9.5 Uniform Flow + Source and Sink

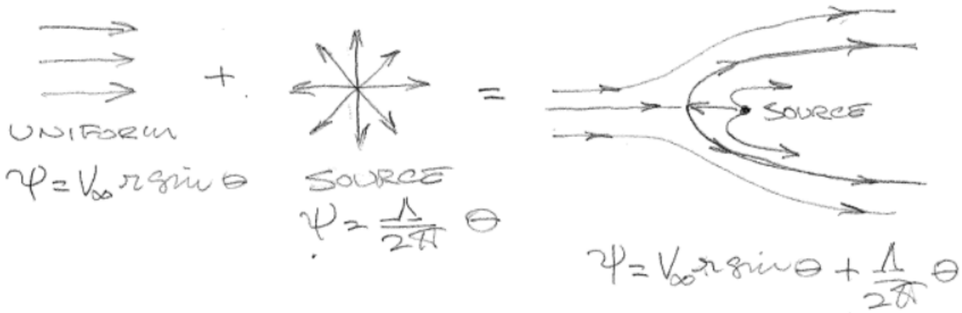


Figure 31: Uniform Flow + Source

In Figure 31 2 flows are superimposed. Both satisfy Laplace's equation, so the combination satisfies Laplace's equation. The stagnation point is defined by:

$$
\begin{aligned}
V_r &= \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = V_\infty \cos(\theta) + \frac{\Lambda}{2\pi r} = 0 \\
V_\theta &= -\frac{\partial \Psi}{\partial r} = -V_\infty \sin(\theta) = 0 \\
r &= \frac{\Lambda}{2\pi V_\infty}, \quad \theta = \pi
\end{aligned}
$$

Note the effect of V_∞ and Λ:
Increasing Λ moves the stagnation point forward.
Increasing V_∞ moves the stagnation point aft.

$$\text{Substitute } \frac{\Lambda}{2\pi V_\infty},\ \pi \text{ into } \Psi \quad \Psi = V_\infty \frac{\Lambda}{2\pi V_\infty}\sin(\pi) + \frac{\Lambda}{2\pi}\pi = \frac{\Lambda}{2} = \text{ Body streamline}$$

Because it contains the stagnation point, this streamline divides the source flow from the freestream.

9.5.1 Rankine Oval

Consider a source sink pair equidistant (distance b) from the origin in a uniform flow (Figure 32).

$$
\begin{aligned}
\Psi &= V_\infty r \sin(\theta) + \frac{\Lambda}{2\pi}\theta_1 - \frac{\Lambda}{2\pi}\theta_2 \\
\text{or } \Psi &= V_\infty r \sin(\theta) + \frac{\Lambda}{2\pi}(\theta_1 - \theta_2)
\end{aligned}
$$

Note: $\theta_1,\ \theta_2$ are functions of (r, θ) and b. To solve for the stagnation points A and B, first note that by symmetry they must lie on $\theta = 0,\ \pi$ and $V_\theta = 0$.

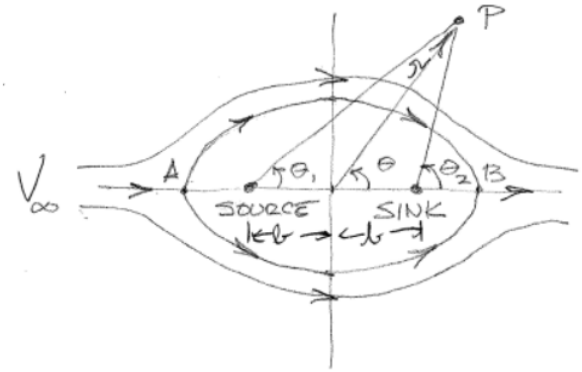


Figure 32: Rankine Oval formed by uniform flow, source, and sink

At point A: Freestream $= +V_\infty$

$$V_{SOURCE} = -V_r = -\frac{\Lambda}{2\pi(r-b)}$$

$$V_{SINK} = -V_r = -\frac{-\Lambda}{2\pi(\pi+b)}$$

$$V_a = V_\infty + \frac{\Lambda}{2\pi}\left(\frac{1}{\pi+b} - \frac{1}{\pi-b}\right) = 0 \text{ for stagnation}$$

$$V_a = V_\infty + \frac{\Lambda}{2\pi}\left[\frac{(r-b)-(+b)}{(\pi-b)(\pi+b)}\right] = V_\infty + \frac{\Lambda}{2\pi}\frac{-2b}{\pi^2-b^2} = 0$$

$$r = \sqrt{\frac{\Lambda b}{\pi V_\infty} + b^2}$$

The equation for a streamline is again:

$$\Phi = V_\infty r\sin(\theta) + \frac{\Lambda}{2\pi}(\theta_1 - \theta_2) = const$$

The streamline for the body contains the stagnation point A where $\theta = \theta_1 = \theta_2 = \pi$ and point B where $\theta = \theta_1 = \theta_2 = 0$. Substituting in the stream function gives:

$$V_\infty r\sin(\theta) + \frac{\Lambda}{2\pi}(\theta_1 - \theta_2) = 0$$

This can be shown to be the equation of an oval called the **Rankine oval**.

9.6 Doublet Flow

Consider a source sink pair at a distance l from each other (Figure 33).

$$\Psi = \frac{\Lambda}{2\pi}(\theta_1 - \theta_2) = -\frac{\Lambda}{2\pi}\Delta\theta \quad \Delta\theta = \theta_2 - \theta_1$$

Let $l \to 0$ while $l\Lambda = K$ remains constant.

$$\Psi = \lim_{l\to 0,\ K\to l\Lambda}\left(-\frac{\Lambda}{2\pi}d\theta\right) \quad \Lambda \to \infty \text{ as } l \to 0$$

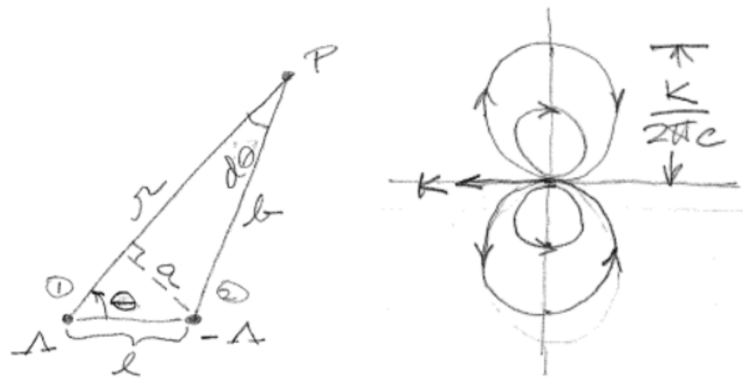


Figure 33: Doublet flow at a point P and on the xy-plane

For small $\Delta\theta \to d\theta \to 0$:

$$a = l\sin\theta, \quad b = r - l\cos\theta, \quad d\theta = a/b = \frac{l\sin\theta}{r - l\cos\theta}$$

$$\Psi = \lim_{l\to 0,\ K=\Lambda l}\Big(-\frac{\Lambda l\sin\theta}{2\pi(r - l\cos\theta)}\Big) = \lim_{l\to 0,\ K=\Lambda l}\Big(-\frac{K\sin\theta}{2\pi(r-\cos\theta)}\Big)$$

$$\Psi = -\frac{K\sin\theta}{2\pi r} \quad K = \text{ doublet strength}$$

To obtain the velocity potential, recall

$$V_r = \frac{\partial\Phi}{\partial r} = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = -\frac{K\cos(\theta)}{2\pi r^2}$$

$$\begin{aligned}
\text{Integrating with respect to } r \quad \Phi &= -\frac{K}{2\pi}\big(-\frac{1}{r}\big) \\
\text{or} \quad \Phi &= \frac{K}{2\pi}\cos(\theta) \\
\text{For the streamlines} \quad \Psi &= -\frac{K\sin(\theta)}{2\pi r} = const = c \\
\text{or} \quad r &= -\frac{K}{2\pi c}\sin(\theta)
\end{aligned}$$

The family of circles corresponds to the streamlines in Figure 33.

9.7 Flow over a Circular Cylinder

Consider the combination of a uniform flow and doublet (Figure 34).

$$\begin{aligned}
\Psi &= V_\infty r\sin(\theta) - \frac{K\sin(\theta)}{2\pi r} \\
\text{Let } R^2 &= \frac{K}{2\pi V_\infty} \\
\Psi &= V_\infty r\sin(\theta)\big(1 - \frac{R^2}{r^2}\big) \\
V_r &= \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = \Big(1 = \frac{R^2}{r^2}\Big)V_\infty\cos(\theta) \\
V_\theta &= -\frac{\partial\Psi}{\partial r} = -\Big[\big(V_\infty r\sin(\theta)\big)\frac{2R^2}{r^3} + \big(1 - \frac{R^2}{r^2}\big)V_\infty\sin(\theta)\Big] = -\Big(1 + \frac{R^2}{r^2}\Big)V_\infty\sin(\theta)
\end{aligned}$$

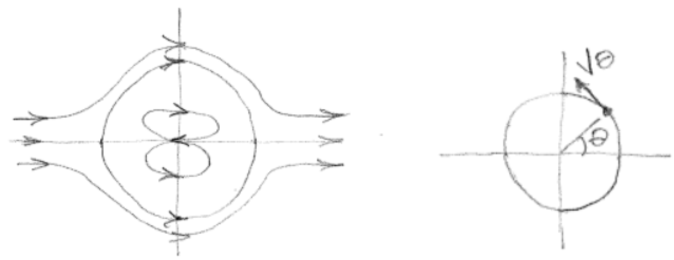


Figure 34: Flow over a circular cylinder constructed using a uniform flow and doublet

Solve for the stagnation point by setting $V_r = V_\theta = 0$.

$$\Big(1 - \frac{R^2}{r^2}\Big) V_\infty \cos\theta = 0, \quad \Big(1 + \frac{R^2}{r^2}\Big) V_\infty \sin\theta = 0$$

Stagnation occurs at 2 points: $(R, 0),\ (R, \pi)$. Also, the corresponding streamline containing the stagnation points is:

$$\Psi = V_\infty r \sin(\theta)\big(1 - \frac{R^2}{r^2}\big) = 0$$

For $r = R$, this is satisfied for all values of θ. Thus $\Psi = 0$ is a circle of radius:

$$R = \sqrt{\frac{\kappa}{2\pi V_\infty}}$$

on the cylinder $r\ R$. $V_\theta = -2V_\infty \sin(\theta),\ V_r\ 0$.

Recalling Equation 51 we can write:

$$C_p = 1 - \Big(\frac{V}{V_\infty}\Big)^2 = 1 - 4\sin^2\theta$$

It is observed that the flow, and hence the pressure distribution, is **symmetric top and bottom and face and aft**. Thus there is no lift force and **no drag force**. This is called **D'Alembert's Paradox**. The pressure is symmetrically distributed (Figure 35).

This holds for all potential flow solutions without circulation, since there are no viscous forces and no separation.

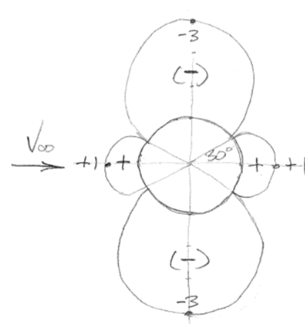


Figure 35: Polar plot of pressure coefficient C_p on a circular cylinder

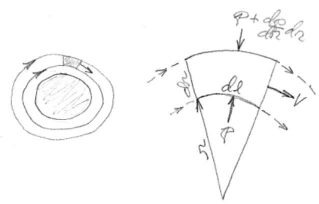


Figure 36: Vortex flow streamlines and differential element

9.8 Vortex Flow

Consider the flow of a fluid in concentric circles. For equilibrium, the centrifugal force must be balanced by the pressures on the surface of the element (Figure 36).

Assuming unit thickness, the volume of the element is $dldr$, and the centrifugal force is $(\rho dldr)V^2/r$. The net pressure force acting toward the center is:

$$\left(p + \frac{dp}{dr}dr\right)dl - pdl = \frac{dp}{dr}drdl$$

$$\begin{aligned}
\text{Isolating the forces} \quad & \frac{dp}{dr} == \rho\frac{V^2}{r} \\
\text{Euler's Equation} \quad & dp = -\rho V dV \\
\text{Then} \quad & -\rho V\frac{dV}{dr} = \rho\frac{V^2}{r} \\
\text{or} \quad & \frac{dV}{V} = -\frac{dr}{r} \\
\text{Integrating} \quad & \log(V) = -\log(r) + const \\
\text{then} \quad & \log(Vr) = const
\end{aligned}$$

$$\text{Vortex flow} \quad V = \frac{\text{const}}{r} \quad V_r = 0 \quad V_\theta = \frac{c}{r} \tag{54}$$

Calculate the circulation Γ around a circular streamline of radius r:

$$\Gamma = -\oint_C \vec{V} \bullet ds = -V_\theta 2\pi r = \frac{c}{r}2\pi r = 2\pi c \quad \text{then} \quad v = -\frac{\Gamma}{2\pi}$$

$$\boxed{V_\theta = -\frac{\Gamma}{2\pi r}} \quad \text{positive } \Gamma \text{ is clockwise}$$

To check on irrotationality, recall Equation 45 (definition of circulation).

$$\Gamma = -\iint_S (\vec{\nabla} \times \vec{V}) \bullet \vec{ds} = -2\pi c$$

Fo a 2D flow, $\vec{\nabla} \times \vec{V}$ and $\vec{ds}$ are both normal to the plane of the flow; thus

$$2\pi c = \iint_S |\vec{\nabla} \times \vec{V}| ds$$

Next, since Γ is invariant with r, let $r \to 0$ while $\Gamma = -2\pi c$.

$$\begin{aligned}
\text{Then} \quad & \iint |\vec{\nabla} \times \vec{V}| ds \to |\vec{\nabla} \times \vec{V}| ds \\
\text{or} \quad & 2\pi c = |\vec{\nabla} \times \vec{V}| ds \\
\text{then} \quad & |\vec{\nabla} \times \vec{V}| \to \infty \quad \text{as } r \to 0
\end{aligned}$$

Vortex flow is irrotational everywhere except at $r = 0$ where the vorticity is infinite. For the velocity potential:

$$\begin{aligned}
\frac{1}{r}\frac{\partial \Phi}{\partial \theta} &= V_\theta = -\frac{\Gamma}{2\pi r}, \quad \frac{\partial \Phi}{\partial r} = V_r = 0 \\
\frac{\partial \Phi}{\partial \theta} &= -\frac{\Gamma}{2\pi} \\
\text{Integrating} \quad \Phi &= -\frac{\Gamma}{2\pi}\theta + f(r) \\
\Phi &= f(\theta) \\
\text{then} \quad \Phi &= -\frac{\Gamma}{2\pi}\theta
\end{aligned}$$

For the stream function:

$$\begin{aligned}
\frac{1}{r}\frac{\partial \Psi}{\partial \theta} &= V_r = 0, \quad -\frac{\partial \Psi}{\partial \theta} = V_\theta = -\frac{\Gamma}{2\pi r} \\
\text{Integrating} \quad \Psi &= f(r) \\
\Psi &= \frac{\Gamma}{2\pi}\log(r) + f(\theta) \\
\text{then} \quad \Psi &= \frac{\Gamma}{2\pi}\log(r)
\end{aligned}$$

9.9 Circular Cylinder with Lift

The stream function for a vortex can be written with an arbitrary constant of integration:

$$\begin{aligned}
\Psi &= \frac{\Gamma}{2\pi}\log(r) + const \\
\text{Therefore, set} \quad const &= -\frac{\Gamma}{2\pi}\log(R) \\
\text{and } \Psi \text{ becomes} \quad \Psi &= \frac{\Gamma}{2\pi}\log\left(\frac{r}{R}\right)
\end{aligned}$$

Adding this to Ψ for the flow about a cylinder, we get the following equation where again $\Psi = 0$ for $r = R$.

$$\Psi = V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\log\left(\frac{r}{R}\right)$$

9.10 Stagnation Points

Set the velocity equal to 0 (condition for a stagnation point).

$$\begin{aligned}
V_r &= \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \left(1 - \frac{R^2}{r^2}\right) V_\infty \cos\theta = 0 \\
V_\theta &= -\frac{\partial \Psi}{\partial r} = -\left(1 + \frac{R^2}{r^2}\right) V_\infty \sin\theta - \frac{\Gamma}{2\pi r} = 0
\end{aligned}$$

On the cylinder, $r = R$.

$$\theta\Big|_{STAG} = \sin^{-1}\left(-\frac{\Gamma}{4\pi V_\infty R}\right)$$

Since $\Gamma > 0$, therefore $\theta > \pi$. There are three cases (Figure 37):

Figure 37: Stagnation points on a cylinder for varying values of Γ

1. $\Gamma < 4\pi V_\infty R$: 2 stagnation points symmetric about $3\pi/2$
2. $\Gamma = 4\pi V_\infty R$: 1 stagnation point at $3\pi/2$
3. $\Gamma > 4\pi V_\infty R$: Return to equation for $V_r = 0$ also satisfied by $\theta = \pm\pi/2$. Then

$$\theta = \mp\Big(1 + \frac{R^2}{r^2}\Big)V_\infty - \frac{\Gamma}{2\pi r} = 0$$

Solve for R

$$r = \frac{\Gamma}{4\pi V_\infty} \pm \sqrt{\Big(\frac{\Gamma}{4\pi V_\infty}\Big)^2 - R^2}$$

There are 2 stagnation points, one inside and one outside. (There exists a theoretical flow inside the cylinder, just as with the simple doublet.) The flow outside $r = R$ is the only one of interest, and it has the single **off-body** stagnation point.

9.11 Lift and Drag

The velocity on the surface of the cylinder is $V = V_\theta = -2V_\infty \sin(\theta) = \Gamma/(2\pi r)$. Then

$$C_p = 1 - \big(\frac{V}{V_\infty}\big)^2 = 1 - \Big[4\sin^2(\theta) + \frac{2\Gamma\sin(\theta)}{\pi R V_\infty} + \big(\frac{\Gamma}{2\pi R V_\infty}\big)^2\Big]$$

Recall the equation for Cd:

$$C_d = \frac{1}{c}\int_{LE}^{TE} C_{p_u}dy - \int_{LE}^{TE} C_{P_l}dy$$

Transform to polar coordinates: $y = R\sin(\theta),\ dy = R\cos(\theta),\ c = 2R$

$$C_d = \frac{1}{2}\int_{\pi}^{0} C_{p_u}\cos(\theta)d\theta - \frac{1}{2}\int_{\pi}^{2\pi} C_{p_l}\cos(\theta)d\theta$$

$$\text{Note:}\quad LE \to TE\Big)_u \Longrightarrow \pi \to 0, \quad LE \to TE\Big)_l \Longrightarrow \pi \to 2\pi$$

$$C_d = -\frac{1}{2}\int_{0}^{2\pi} C_D\cos(\theta)d\theta$$

$$\text{Substitute}\quad C_p(\theta) \text{ and note}\quad \int_0^{2\pi}\cos(\theta)d\theta = \int_0^{2\pi}\sin^2(\theta)\cos(\theta)d\theta = \int_0^{2\pi}\sin(\theta)\cos(\theta)d\theta = 0$$

$$\boxed{C_d = 0}\quad \text{D'Alembert's paradox again!}$$

Recall the equation for C_l

$$C_l = \frac{1}{c}\int_0^c C_{p_l}dx - \frac{1}{c}\int_0^c C_{p_u}dx$$

Transform $x = R\cos(\theta),\ dx = -R\sin(\theta)d\theta,\ c = 2R$

$$C_l = -\frac{1}{2}\int_{\pi}^{2\pi} C_{p_l}\sin(\theta)d\theta + \frac{1}{2}\int_{\pi}^{0} C_{p_u}\sin(\theta)d\theta$$

$$C_l = -\frac{1}{2}\int_{0}^{2\pi} C_p\sin(\theta)d\theta$$

Substitute $C_p(\theta)$ and note

$$\int_0^{2\pi}\sin(\theta)d\theta = \int_0^{2\pi}\sin^3(\theta)d\theta = 0, \qquad \int_0^{2\pi}\sin^2(\theta)d\theta = \pi$$

$$C_l = \frac{\Gamma}{RV_\infty}\text{By definition } L = \frac{1}{2}\rho_\infty V_\infty^2\ c\ C_l, \quad c = 2R$$

$$\text{or } L = \frac{1}{2}\rho_\infty V_\infty^2 2R\frac{\Gamma}{RV_\infty}$$

$$\boxed{L = \rho_\infty V_\infty \Gamma} \quad \text{Kutta-Joukowski Theorem}$$

9.11.1 Example: Flow over a cylinder

Consider the flow over a cylinder where $C_l = 5$. The velocity on the cylinder is given by

$$V = V_\theta = -2V_\infty\sin(\theta) - \frac{\Gamma}{2\pi R}$$

$$C_l = \frac{\Gamma}{RV_\infty} = 5$$

$$\text{Then}\quad V = -2V_\infty - \frac{5}{2\pi}V_\infty = -2.8V_\infty$$

$$\text{Versus } V_{max} = 2V_\infty \text{ for } C_l = 0$$

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2 = -6.8$$

$$\text{Versus } C_p = -4 \text{ for } C_l = 0$$

Note: $C_p(\theta),\ V/V_\infty(\theta),\ C_l$ are uniquely related, and do not depend on distinct values of $R,\ V_\infty,\ \rho_\infty$, or p_∞.

9.12 Real flow - cylinder with circulation

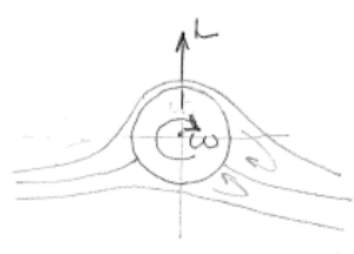


Figure 38: Lifting flow over a cylinder with circulation

Figure 38 shows a cylinder rotating at speed ω. This produces a curved flight path like a baseball's.

9.13 Theory of Lift

Recall circulation (Equation 45). Then in Figure 40:

$$\Gamma = \oint_A \vec{V} \bullet \vec{ds} \quad \text{and} \quad \oint_B \vec{V} \bullet \vec{ds} = 0$$

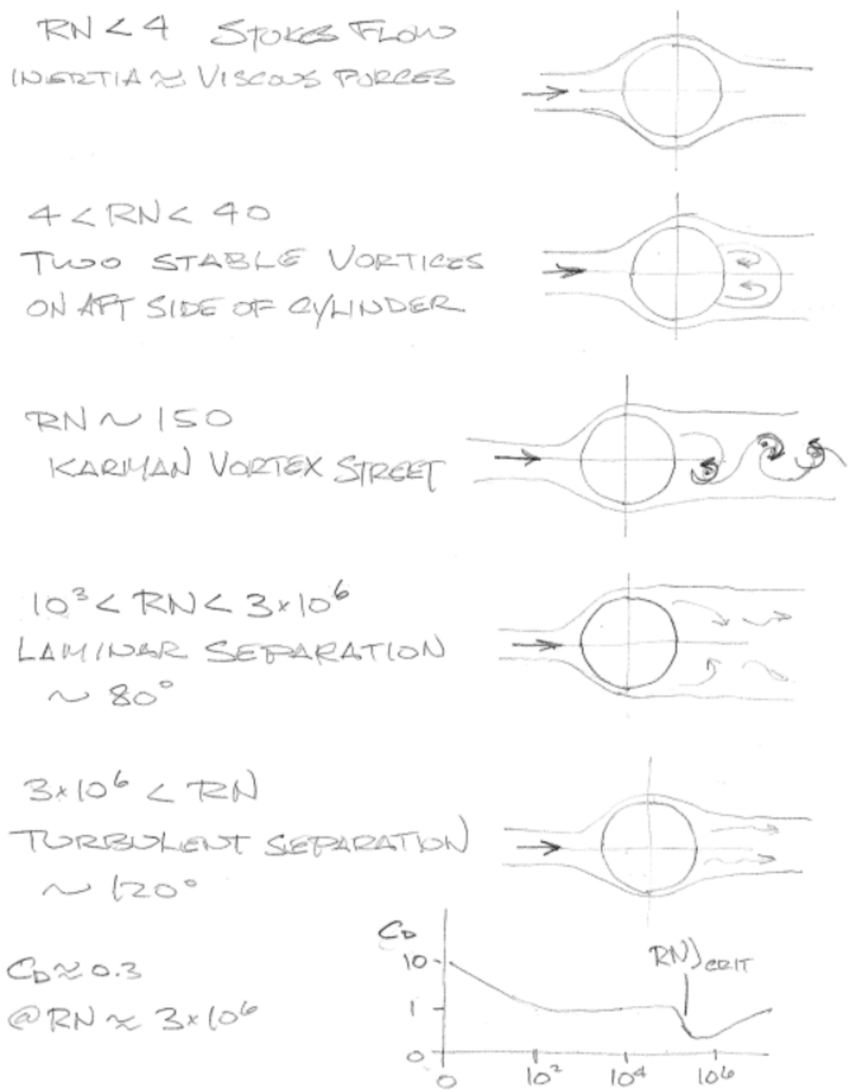


Figure 39: Real flow over cylinders at varying Reynolds numbers

Figure 40 shows a potential flow about an airfoil with $\Gamma = 0$. Viscosity will not permit this flow about the trailing edge. Adding a vortex and setting its strength Γ such that the flow passes smoothly off the trailing edge defines a streamline geometry that simulates the real flow. This value of Γ also correctly predicts the lift from

$$L = \rho V \Gamma$$

10 Source Panel Method

10.1 Point and Sheet Source

For a source sheet of varying strength $\lambda(s)$, apply Equation 55 (Figure 41).

$$\Phi(x, y) = \int_a^b \frac{\lambda(s)}{2\pi} ds \log(r) \tag{55}$$

Source-panel method represents an arbitrary body by a series of flat panels, each with a constant source strength λ_j. Thus, the velocity potential induced at $P(xy)$ due to the j^{th} panel is Equation 56 where the

Figure 40: Circulation around an airfoil and potential flow around an airfoil with $\Gamma = 0$

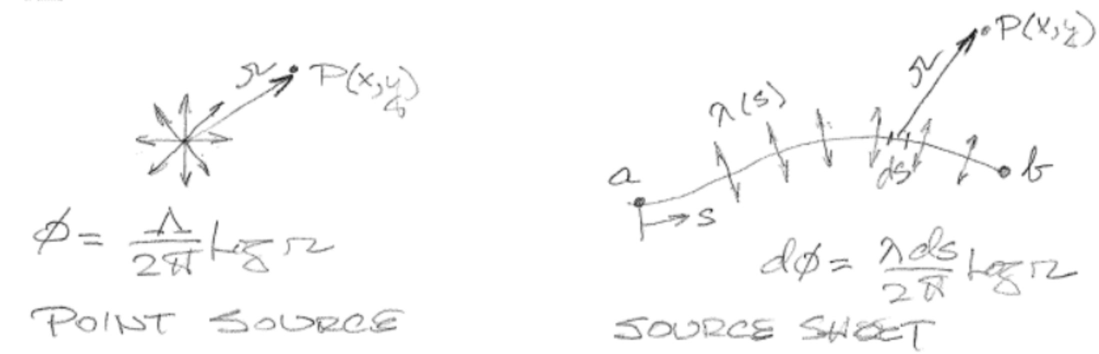


Figure 41: Point source and source sheet

integral is over the j^{th} panel (Figure 42).

$$\Delta\Phi_j = \frac{\lambda_j}{2\pi}\int_j \log(r_{pj})ds_j \tag{56}$$

In turn, the velocity potential at P due to all of the panels is Equation 57

$$\Phi(P) = \sum_{j=1}^{n}\Delta\Phi_j = \sum_{j=1}^{n}\frac{\lambda_j}{2\pi}\int_j \log(r_{pj})ds_j \tag{57}$$

where r_{p1} is given by

$$r_{p1} = \sqrt{(s-s_j)^2 + (y-y_j)^2}$$

Next, select $P(x,y)$ to be $P(x_i,y_i)$, the **control point** of the i^{th} panel. The boundary condition for the body to be a streamline is that the normal component of the velocity is zero. The free stream component normal to the i^{th} panel is

$$V_{\infty,n} = V_\infty \bullet n_i = V_\infty \cos\beta_i$$

V_∞ is positive away from the body.

The normal component of the velocity induced by the source panel is

$$V_n = \frac{\partial}{\partial n_i}\big[\Phi(x_i,y_i)\big]$$

(Again, V_n is positive away from the body.)

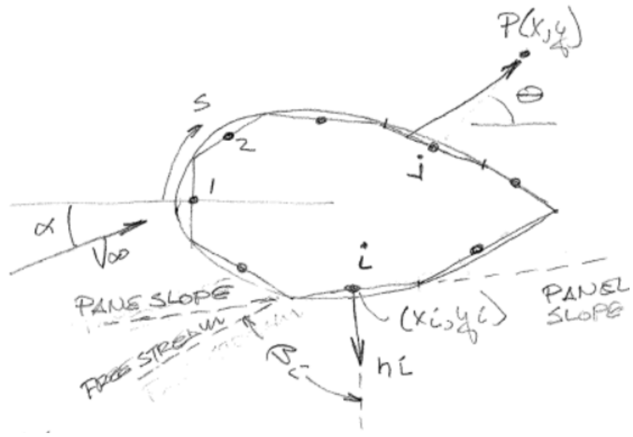


Figure 42: A body composed of source panels

Note: When $j = i$ in the sum for $\Phi(P)$, $r_{pj=i} = 0$ which is a singularity. It can be shown that the contribution to V_n is $\lambda_i/2$. Therefore, the expression for V_n becomes

$$V_n = \frac{\lambda_i}{2} + \sum_{j=1,\ j\neq i}^{n} \frac{\lambda_i}{2\pi} \int_j \frac{\partial}{\partial n_i} \log(r_{ij}) ds_j$$

The boundary condition for flow tangency is

$$V_{\infty n} + V_n = 0$$

$$\text{or} \quad \frac{\lambda_i}{2} + \sum_{j=1,\ j\neq i}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i} \log(r_{ij}) ds_j + V_\infty \cos\beta_i = 0$$

$$\text{Writing} \quad I_{ij} = \int_j \frac{\partial}{\partial n_i} \log(r_{ij} ds_j$$

It is noted that I_{ij} is a function of **panel geometry** alone, and it is not a function of the flow properties.

$$\text{Thus} \quad \frac{\lambda_i}{2} + \sum_{j=1,\ j\neq i}^{n} \frac{\lambda_j}{2\pi} I_{ij} + V_\infty \cos\beta_i = 0$$

is a set of n **linear algebraic equations** in the n unknowns λ_j. The solution for the source panel strengths λ_j provides a flow where the body surface is a streamline.

The velocity **tangent** to the surface at the i^{th} panel is given by

$$V_i = V_{\infty s} + V_s = V_\infty \sin\beta_i + \frac{\partial\Phi}{\partial s}$$

$$V_i = V_\infty \sin\beta_i + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial s} \log(r_{ij}) ds_j$$

(Here, the contribution of the i^{th} source panel to the tangential velocity there is by definition zero.)

Once V_i on the body surface is known, the corresponding pressure coefficient is simply

$$C_{pi} = 1 - \left(\frac{V_i}{V_\infty}\right)^2$$

And thus the pressure distribution over a non-lifting body of arbitrary geometry can be calculated. As a check, for a closed body, the sum of the source(sink) strengths must be zero (Equation 58).

$$\sum_{j=1}^{n} \lambda_j s_j = 0 \tag{58}$$

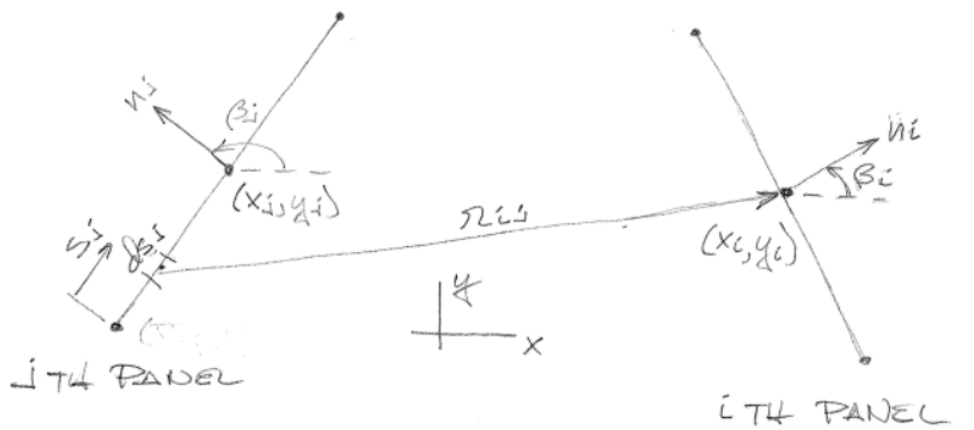


Figure 43: Corresponding i^{th} and j^{th} panels

$$\begin{aligned}
I_{ij} &= \int_j \frac{\partial}{\partial n_i}\big(\log r_{ij}\big) ds_j \\
\frac{\partial}{\partial n_i}\big(\log r_{ij}\big) &= \frac{1}{r_{ij}} \frac{\partial r_{ij}}{\partial n_i} \\
&= \frac{1}{r_{ij}} \frac{1}{2}\Big[(x_i - x_j)^2 + (y_i - y_j)^2\Big]^{-1/2} \cdot \Big[2(x_i - x_j)\frac{dx_i}{dn_i} + 2(y_i - y_j)\frac{dy_i}{du_i}\Big] \\
\frac{dx_i}{du_i} &= \cos\beta_i, \quad \frac{dy_i}{du_i} = \sin\beta_i \\
\frac{\partial}{\partial u_i}\big(\log r_{ij}\big) &= \frac{(x_i - x_j)\cos\beta_i + (y_i - y_j)\sin\beta_i}{(x_i - x_j)^2 + (y_i - y_j)^2}
\end{aligned}$$

This must be integrated over s_j to obtain a general expression for I_{ij} for two arbitrary panels. See pages 225-227 in text.

11 Airfoils

11.1 Airfoil Characteristics

1. In Figure 44: Airfoil thickness relates to structure and the range of α
2. Camber relates to the designed C_l, α_{ol} and designed α
3. Aerodynamic center: C_m is invariant with respect to C_l near $C/4$.
4. Reynolds Number: Increasing $RN = \frac{\rho V c}{\mu}$ increases $C_{l,max}$ and decreases C_d

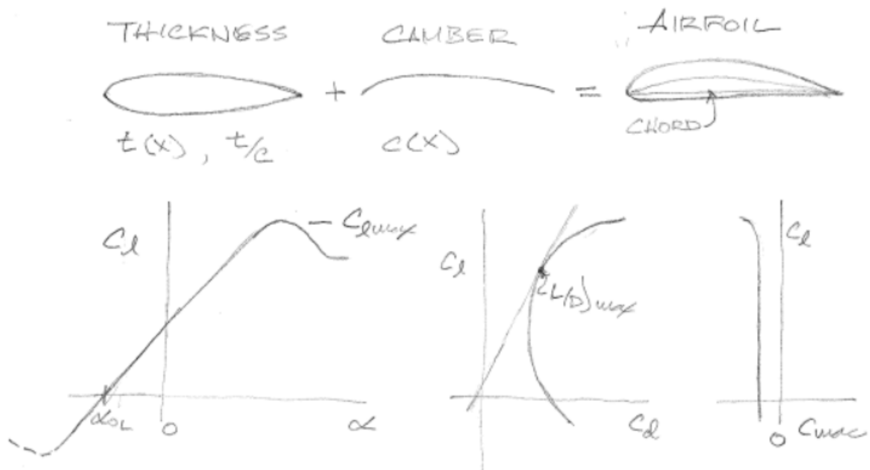


Figure 44: Characteristics of an airfoil and plots of $C_l,\ C_d,\ C_{M,ac}$

Figure 45: Attached flow and stall conditions on an airfoil

11.2 Vortex Sheet

Define $\gamma(s)$ as the strength of the vortex sheet / unit length. Then the vortex strength is γds. Recall that for a point vortex, $V = -\Gamma/(2\pi r)$. Then the velocity induced at P by the vortex sheet element ds is

$$dV = -\frac{\gamma ds}{2\pi r}$$

In this equation dV is $\perp$ to r. If the dV's in Figure 46 are to be summed from a to b, they must be added vectorially. Therefore, use the velocity potential

$$\Phi = -\frac{\Gamma\theta}{2\pi} \rightarrow d\Phi = -\frac{\gamma ds}{2\pi}\theta$$

$$\text{or} \quad \Phi(x,y) = -\frac{\Gamma\theta}{2\pi} = -\frac{1}{2\pi}\int_a^b \theta\gamma ds$$

The total circulation is Equation 59.

$$\Gamma = \int_a^b \gamma ds \tag{59}$$

Recall for a **source sheet**, tangential V is continuous across the sheet. Normal V changes direction 180$\circ$. For a **vortex sheet**: tangential V is discontinuous across the sheet and normal V is continuous.

In Figure 47, consider a section of the vortex sheet ds and the circulation Γ about ds.

$$\Gamma = -v_2 du + u_1 ds + v_1 du - u_2 ds = (u_1 - u_2)ds + (v_1 - v_2)du$$

Let the top and bottom approach the sheet, i.e. $dn \rightarrow 0$

$$\lim_{du\rightarrow 0} \Gamma = (u_1 - u_2)ds$$

$$\text{But also, } \Gamma = \gamma ds, \text{ therefore}$$

$$\gamma = u_1 - u_2$$

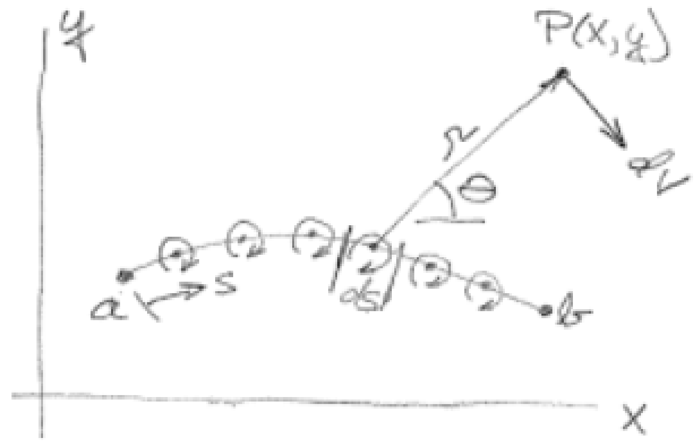


Figure 46: Airfoil surface as vortex sheet

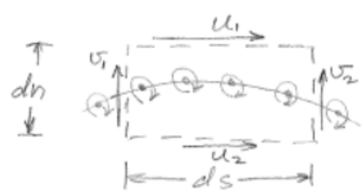


Figure 47: A section of the vortex sheet with circulation

The local velocity jump across the sheet is equal to the strength.
Analogous to the source panel method, represent an airfoil by a vortex sheet over its surface. Set the variation of $\gamma(s)$ such that the airfoil shape becomes a streamline. Then

$$L = \rho_\infty V_\infty \Gamma = \rho_\infty V_\infty \int \gamma ds$$

11.2.1 Thin Airfoil Approximation

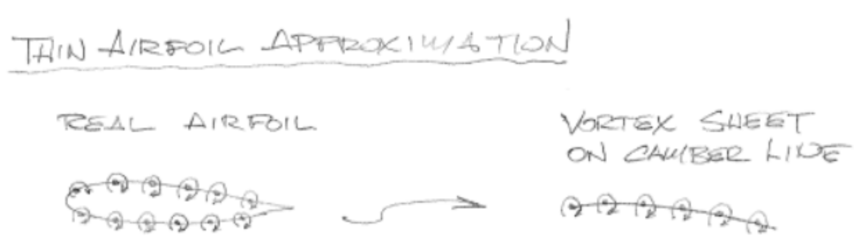


Figure 48: Approximation of a real airfoil as a vortex sheet

See Figure 48. **Camber** is the primary input to aerodynamic performance.

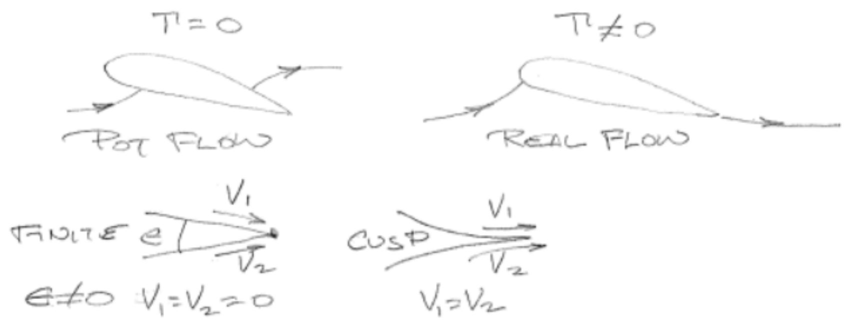


Figure 49: Flow at the trailing edge of the airfoil (Kutta Condition)

11.2.2 Trailing Edge / Kutta Condition

See Figure 49. The Kutta condition is Equation 60.

$$\gamma(TE) = V_1 - V_2 = 0 \tag{60}$$

11.3 Vortex Theorem of Kelvin

"The circulation around any path which is always made up of the same fluid particles is independent of the time."
"In a perfect fluid, no tangential forces can act, and hence the angular velocity of a fluid particle can never change since no couple can be exerted on it." -Clark Millikan

$$\frac{D\Gamma}{dt} = 0 \tag{61}$$

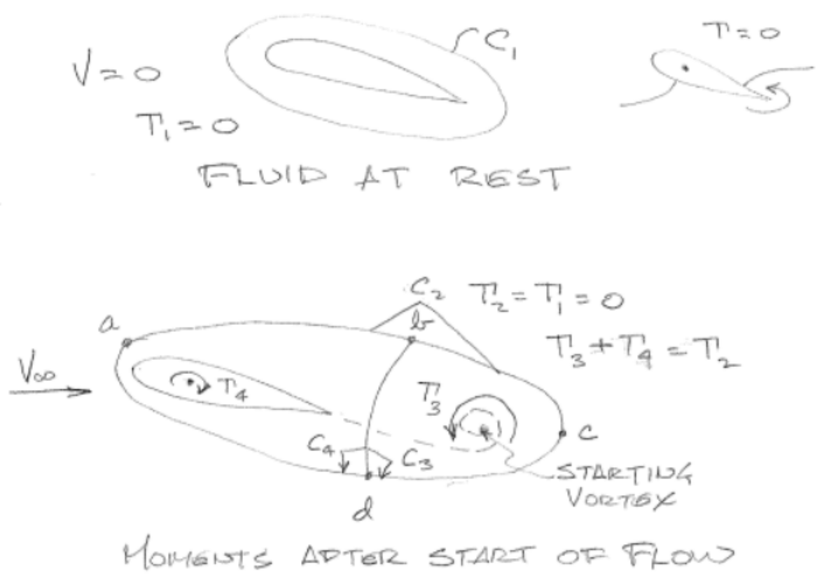


Figure 50: The starting vortex forms moments after the flow begins moving over the airfoil

Figure 50 shows that just as the airfoil starts to move, the high velocity gradient about the trailing edge forms a ”starting vortex” that is swept downstream.

$$\frac{D\Gamma}{Dt} = 0 \rightarrow \Gamma_1 = \Gamma_2 = 0$$
$$\Gamma_2 + \Gamma_4 = \Gamma_2 \; 0$$
$$\rightarrow \Gamma_4 = -\Gamma_3$$

Once a steady state is reached, the starting vortex Γ_3 is swept far downstream, and steady flow with circulation Γ_4 about the airfoil is established.

12 Thin Airfoil Theory

12.1 Vortex Sheet

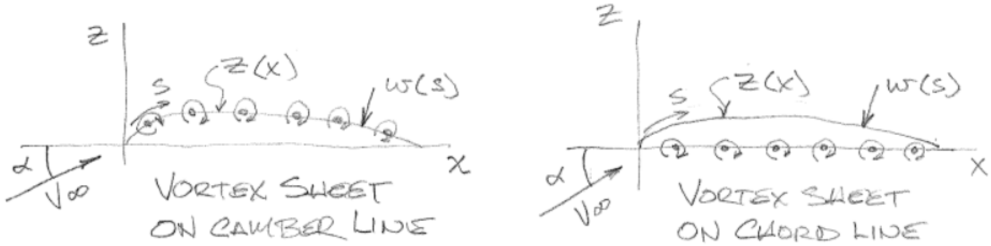


Figure 51: A vortex sheet on the camber line and a vortex sheet on the chord line

Define the airfoil as a vortex sheet on the camber line (Figure 51). Move the vortex sheet to the chord line $\gamma(s) \rightarrow \gamma(x)$ and $\gamma(c) = 0$, but set the strength of the vortex sheet such that the camber line (not the chord line) is a streamline.

$$V_{\infty n} + w(s) = 0$$

$V_\infty n$ is the component of V_∞ normal to the camber line. $w(s)$ is the velocity induced by the vortex sheet normal to the camber line (Figure 52).

$$V_{\infty n} = V_\infty \sin\left[\alpha + \arctan(-\frac{dz}{dx})\right]$$
$$\text{For small } \alpha \text{ and } dz/dx \quad V_{\infty n} = V_\infty\left[\alpha - \frac{dz}{dx}\right]$$

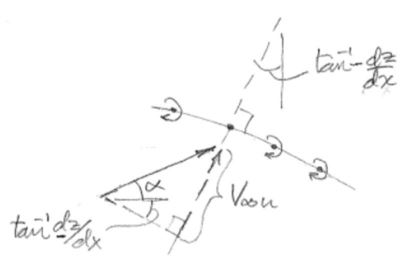


Figure 52: $V_{\infty n}$ on a vortex sheet

A consistent approximation is $w(s) = w(x)$ where $w(x)$ is the component of velocity induced by the vortex sheet normal to the chord line (Figure 53).

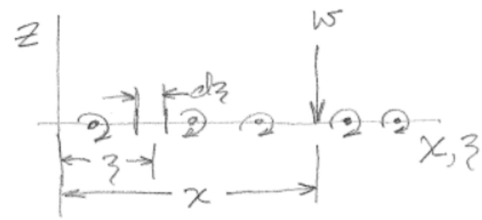


Figure 53: Approximating $w(s) = w(x)$ for a vortex sheet

$$\gamma = \gamma(\zeta)$$
$$dw(x) = -\frac{\gamma(\zeta)d\zeta}{2\pi(x-\zeta)}$$
$$w(x) = -\int_0^c \frac{\gamma(\zeta)d\zeta}{2\pi(x-\zeta)}$$

Recalling $V_{\infty n} + w(s) = 0, \quad V_{\infty n} = V_\infty(\alpha - dz/dx)$

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\zeta)d\zeta}{x-\zeta} = V_\infty(\alpha - dz/dx) \tag{62}$$

Equation 62 is called the "Fundamental Equation of Thin Airfoil Theory" - it states that the camber line is a streamline.

12.2 Symmetric Airfoil

For a symmetric airfoil, $dz/dx = 0$ so

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\zeta)d\zeta}{x-\zeta} = V_\infty(\alpha) \tag{63}$$

Since thickness is not included, Equation 62 becomes Equation 63. This is an exact expression for the inviscid, incompressible flow over a **flat plate** at angle of attack α. To solve for $\gamma(\zeta)$:

Apply the transformation $\quad \zeta = \frac{c}{2}(1-\cos\theta), \quad d\zeta = \frac{c}{2}\sin\theta d\theta$

Let the fixed point x correspond to a particular $\theta = \theta_0,\ x_0 = \frac{c}{2}(1-\cos\theta_0)$

The limits on the integral become $\theta = 0,\ \theta = \pi \quad \frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta - \cos\theta_0} = V_\infty\alpha$

The solution to this integral equation is given by $\quad \gamma(\theta) = 2\alpha V_\infty \frac{1+\cos\theta}{\sin\theta}$

This can be checked by substitution

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta - \cos\theta_0} = \frac{V_\infty\alpha}{\pi}\int_0^\pi \frac{(1+\cos\theta)d\theta}{\cos\theta-\cos\theta_0}$$

And using the integral result

$$\frac{1}{2\pi}\int_0^\pi \frac{\cos u\theta d\theta}{\cos\theta - \cos\theta_0)} = \frac{\pi\sin u\theta_0}{\sin\theta_0}$$
$$\frac{V_\infty\alpha}{\pi}\int_0^\pi \frac{(1+\cos\theta)d\theta}{\cos\theta-\cos\theta_0} = \frac{V_\infty\alpha}{\pi}\big(0+\pi\big) = V_\infty\alpha$$

The trailing edge condition $\gamma(\pi) = 0$ must be evaluated using L'Hopital's Rule:

$$\gamma(\pi) = 2\alpha V_\infty \frac{1-\cos\pi}{\sin\pi} = 2\alpha V_\infty \frac{-\sin\pi}{\cos\pi} = 0$$

The total circulation about the airfoil is

$$\Gamma = \int_0^c \gamma(\zeta)d\zeta = \frac{c}{2}\int_0^\pi \gamma(\theta)\sin\theta d\theta$$

$$\Gamma = \alpha c V_\infty \int_0^\pi \big(1+\cos\theta\big)d\theta = \pi\alpha c V_\infty$$

$$\text{and the lift is} \quad L = \rho_\infty V_\infty \Gamma = \pi\alpha c \rho_\infty V_\infty^2$$

$$C_l = \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 c} = \frac{\pi\alpha c\rho_\infty V_\infty^2}{\frac{1}{2}\rho_\infty V_\infty^2 c} = 2\pi\alpha, \quad \boxed{\frac{dc_l}{d\alpha} = 2\pi}$$

The moment about the leading edge (Figure 54) is

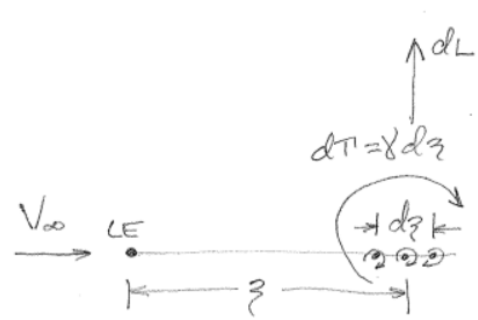


Figure 54: Moment about the leading edge of the wing

$$dM = -\zeta\ dL$$

$$dL = \rho_\infty V_\infty\ d\Gamma$$

$$dL = \rho_\infty V_\infty \gamma(\zeta)\ d\zeta$$

$$M_{LE} = -\int_0^c \rho_\infty V_\infty \gamma(\zeta)\zeta\ d\zeta = -\rho)\infty V_\infty \int_0^c \zeta\lambda(\zeta)\ d\zeta$$

$$\text{Transforming} \quad \zeta = \frac{c}{2}\big(1-\cos\theta\big), \quad d\zeta = \frac{c}{2}\sin\theta\ d\theta$$

$$M_{LE} = -\rho_\infty V_\infty \frac{c^2}{4}\int_0^\pi \gamma(\theta)(1-\cos\theta)\sin\theta\ d\theta$$

$$\gamma(\theta) = 2\alpha V_\infty \frac{1+\cos\theta}{\sin\theta}$$

$$M_{LE} = -\frac{1}{2}\rho_\infty V_\infty^2 c^2\alpha \int_0^\pi (1-\cos^2\theta)d\theta$$

$$M_{LE} = -q_\infty c^2 \alpha\pi/2$$

$$C_{M,LE} = \frac{M_{LE}}{q_\infty c^2} = -\frac{\pi}{2}\alpha$$

$$\text{Using} \quad c_l = 2\pi\alpha \rightarrow C_{M,LE} = -c_l/4$$

$$\text{Recalling} \quad C_{M,LE} = -c_l/4 + C_{M,ac/4}, \quad \text{then } \boxed{C_{M,c/4} = 0}$$

12.3 Cambered Airfoil

Recall

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\zeta)\ d\zeta}{x-\zeta} = V_\infty\Big(\alpha - \frac{dz}{dx}\Big)$$

where $dz/dx \neq 0$ and is a function of x. Again, transform to θ:

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\ d\theta}{\cos\theta-\cos\theta_0} = V_\infty\Big(\alpha - \frac{dz}{dx}\Big)$$

$\gamma(\theta)$ where $\gamma(\pi)=0$ will provide a solution where the camber line is a streamline. Here, the solution to the integral equation is:

$$\gamma(\theta) = 2V_\infty\left(A_0\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right)$$

The first term is similar to the symmetric case, and the rest is a Fourier series. A_0 and the A_n depend on $dz/dx,\ \alpha$.
Substituting this solution:

$$\frac{1}{\pi}\int_0^\pi \frac{A_0(1+\cos\theta)d\theta}{\cos\theta-\cos\theta_0} + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^\pi \frac{A_n\sin n\theta\sin\theta\ d\theta}{\cos\theta-\cos\theta_0} = \alpha - \frac{dz}{dx}$$

This first term is evaluated as in the symmetric case, and the A_n terms use

$$\int_0^\pi \frac{\sin n\theta\sin\theta d\theta}{\cos\theta-\cos\theta_0} = -\pi\cos n\theta_0$$

This yields

$$A_0 - \sum_{n=1}^{\infty} A_n\cos n\theta_o = \alpha - \frac{dz}{dx}$$

$$\text{or,}\quad \frac{dz}{dx} = (\alpha - A_0) + \sum_{n=1}^{\infty} A_n\cos n\theta_0$$

Note: dz/dx is the slope of the camber line at $x=\frac{c}{2}(1-\cos\theta_0)$. The coefficients of the Fourier cosine series are obtained from

$$\alpha - A_0 = \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}\ d\theta_0$$
$$A_n = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos n\theta_0\ d\theta_0$$

where dz/dx is a function of θ_0 from $0\to\pi$, or $0\to c$.

- A_0 depends on $\alpha, dz/dx$, while the A_n depend only on dz/dx.
- Note that $\gamma(\pi)=0$ is satisfied and that when $dz/dx=0,\ \gamma(\theta)$ reduces to the symmetric case.

Solutions for c_l and c_m:

$$\Gamma = \int_0^c \gamma(\zeta)\ d\zeta = \frac{c}{2}\int_0^\pi \gamma(\theta)\sin\theta\ d\theta$$
$$\Gamma = cV_\infty\Big[A_0\int_0^\pi(1+\cos\theta)d\theta + \sum_{n=1}^{\infty}A_n\int_0^\pi \sin n\theta\sin\theta\ d\theta\Big]$$
$$\int_0^\pi(1+\cos\theta)d\theta = \pi,\quad \int_0^\pi \sin^2\theta = \pi/2,\quad \int_0^\pi \sin n\theta\sin\theta\ d\theta = 0\ n\neq 1$$
$$\Gamma = cV_\infty\big(\pi A_0 + \frac{\pi}{2}A_1\big)$$
$$L = \rho_\infty V_\infty\Gamma = \frac{1}{2}\rho_\infty V_\infty^2 c\pi(2A_0 + A_1)$$
$$c_l = \pi(2A_0 + A_1)$$

Substituting for $A_0,\ A_1$

$$c_l = 2\pi\Big[\alpha + \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta_0 - 1)d\theta_0\Big]$$

$$\text{Again} \quad \frac{dc_l}{d\alpha} = 2\pi$$

$$\text{Write} \quad c_l = \frac{dc_l}{d\alpha}(\alpha - \alpha_{OL}) = 2\pi(\alpha - \alpha_{OL})$$

$$\text{Where} \quad \alpha_{OL} = -\frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta_0 - 1)d\theta_0$$

For $C_{M,LE}$ recall

$$M_{LE} = -\rho_\infty V_\infty \int_0^c \zeta x(\zeta)\ d\zeta$$

$$M_{LE} = -\frac{\rho_\infty V_\infty c^2}{4}\int_0^c (1-\cos\theta)\sin\theta\gamma(\theta)\ d\theta$$

$$\text{or} \quad C_{M,LE} = -\frac{1}{2V_\infty}\int_0^c \gamma(\theta)(1-\cos\theta)\sin\theta\ d\theta$$

$$\text{Substituting} \quad \gamma(\theta) = 2V_\infty\Big[A_0\frac{1+\cos\theta}{\sin\theta} + \sum_{u=1}^{\infty} A_n \sin n\theta\Big]$$

$$\text{and evaluating} \quad \int_0^\pi \sin n\theta \sin\theta\ d\theta \text{ etc}$$

$$\text{yields} \quad C_{M,LE} = -\frac{\pi}{2}\Big(A_0 + A_1 - \frac{A_2}{2}\Big)$$

$$\text{Subsituting} \quad c_l = \pi(2A_0 - A_2) \quad \text{yields}$$

$$C_{M,LE} = -\Big[\frac{c_l}{4} + \frac{\pi}{4}(A_1 - A_2)\Big]$$

Again, recalling $C_{M,LE} = -c_l/4 + C_{M,c/4}$ yields

$$C_{M,c/4} = \frac{\pi}{4}(A_2 - A_1)$$

Since $A_1,\ A_2$ depend on camber but not $\alpha,\ C_{M,c/4}$ is invariant with α.

$$\frac{X_{CP}}{c} = -\frac{C_{M,LE}}{c_l} = \frac{c}{4}\Big[1 + \frac{\pi}{c_l}(A_1 - A_2)\Big]$$

12.4 Vortex Panel Method

is analogous to the source panel method (Figure 55).

- Represents a body with panels of constant vortex strength γ_i/unit length.
- Seeks the set $\gamma_i,\ i = 1 \to n$ such that the body surface is a streamline.
- Can be applied directly to cases with **lift**.

The velocity potential $\Delta\Phi(x,y)$ induced by the vortex panel j is given by

$$\Delta\Phi_k(x,y) = -\frac{1}{2\pi}\int_j \theta_{Pj} X_j ds_j$$

Where the integral is over the j^{th} panel. The total velocity potential at P from all panels is Equation 64.

$$\Phi(x,y) = \sum_{j-1}^{n}\Phi_j = -\sum_{j-1}^{n}\frac{\gamma_i}{2\pi}\int_j \theta_{Pj} ds_j \tag{64}$$

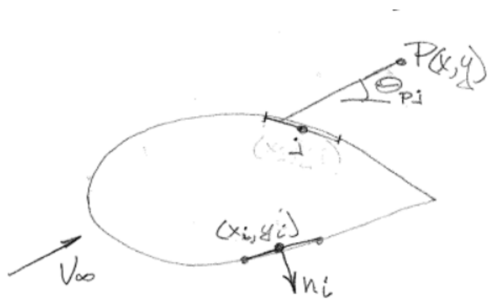


Figure 55: Airfoil with vortex panels

where

$$\theta_{pj} = \arctan\left(\frac{y - y_j}{x - x_j}\right)$$

Next, P is located at the control point (x_i, y_i) of the i^{th} panel.

The velocity potential at (x_i, y_i) is

$$\Phi(x_i, y+i) = -\sum_{j=1}^{n} \frac{\gamma_j}{2\pi} \int_j \theta_{ij} ds_j$$

$$\text{Where} \quad \theta_{ij} = \arctan\left(\frac{y_i - y_j}{x_i - x_j}\right)$$

For the body to be a streamline, the normal velocity on each panel must = 0. The normal velocity induced by the vortex panels is:

$$V_n(x_i, y_i) = \frac{\partial}{\partial n_i}\Big[\Phi(x_i, y_i)\Big]$$
$$= -\sum_{j=1}^{n} \frac{\gamma_j}{2\pi} \int_j \frac{\partial \theta_{ij}}{\partial n_i} ds_j$$

And again the normal velocity of the freestream is

$$V_{\infty n} \ V_{infty} \cos\beta_i$$

The boundary condition becomes

$$V_{\infty n} + V_n = 0$$

$$\text{or} \quad V_\infty \cos\beta_i - \sum_{j=1}^{n} \frac{\gamma_j}{2\pi} \int_j \frac{\partial}{\partial n_i} \theta_{ij} ds_j$$

$$\text{Defining} \quad I_{ij} = \int_j \frac{\partial \theta_{ij}}{\partial n_j} ds_j$$

These integrals are functions of the body (panel) geometry alone. Thus:

$$V_\infty \cos\beta_i - \sum_{j=1}^{u} \frac{\gamma_i}{2\pi} I_{ij} = 0$$

is a set of u linear algebraic equations that can be solved for the u unknown vortex strengths γ_j.

Since the vortex panels by definition provide circulation, lift can result. This implies that the **Kutta Condition** must be satisfied at the trailing edge, as in Figure 56: $\gamma(TE) = 0$.

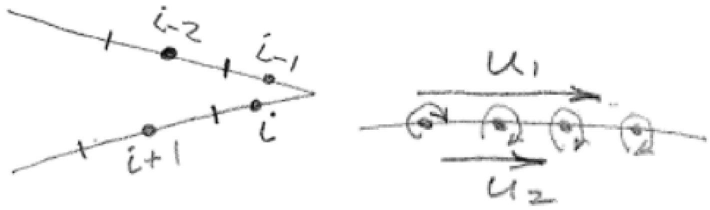


Figure 56: Trailing edge of airfoil with Kutta Condition

To approximate numerically, set $\gamma_i + \gamma_{i-1} = 0$. Applying this gives $n+1$ equations for n unknowns, an overdetermined system. This is accommodated by not applying the flow tangency condition at one of the control points, and thus removing one of the u tangency equations. The remaining $n-1$ equations plus the Kutta condition equation now make up the set of n equations for the n unknowns γ_i.

To calculate the resulting tangential velocity, recall the condition for the velocity next to a vortex sheet: $\gamma = u_1 - u_2$. Inside the body, set the velocity to zero then $\gamma = u_1 - 0 = u_1$. Thus the velocity tangent to the body is equal to the local vortex strength γ_i. The total circulation is Equation 65.

$$\Gamma = \sum_{j=1}^{n} \gamma_j s_j \tag{65}$$

And the lift is Equation 66

$$L = \rho_\infty V_\infty \sum_{j-1}^{n} \gamma_j s_j \tag{66}$$

12.5 Real Airfoils

See Figure 57.

12.6 Viscous Flow - Airfoil Drag

Laminar boundary layer:

$$\delta = \frac{5x}{\sqrt{RN_x}}, \quad RN_x = \frac{\rho_\infty V_\infty x}{\mu}$$
$$\rightarrow \delta N \sqrt{x} \quad c_f = \frac{1.328}{\sqrt{RN_x}}$$

Turbulent boundary layer:

$$\delta = \frac{0.37x}{(RN_x)^{1/5}} \quad c_f = \frac{0.074}{(RN_c)^{1/5}}$$
$$\text{where} \quad c_f = \frac{\tau}{\frac{1}{2}\rho_\infty V_\infty^2}$$

For **one side** of a flat plate: $c_f = c_d$

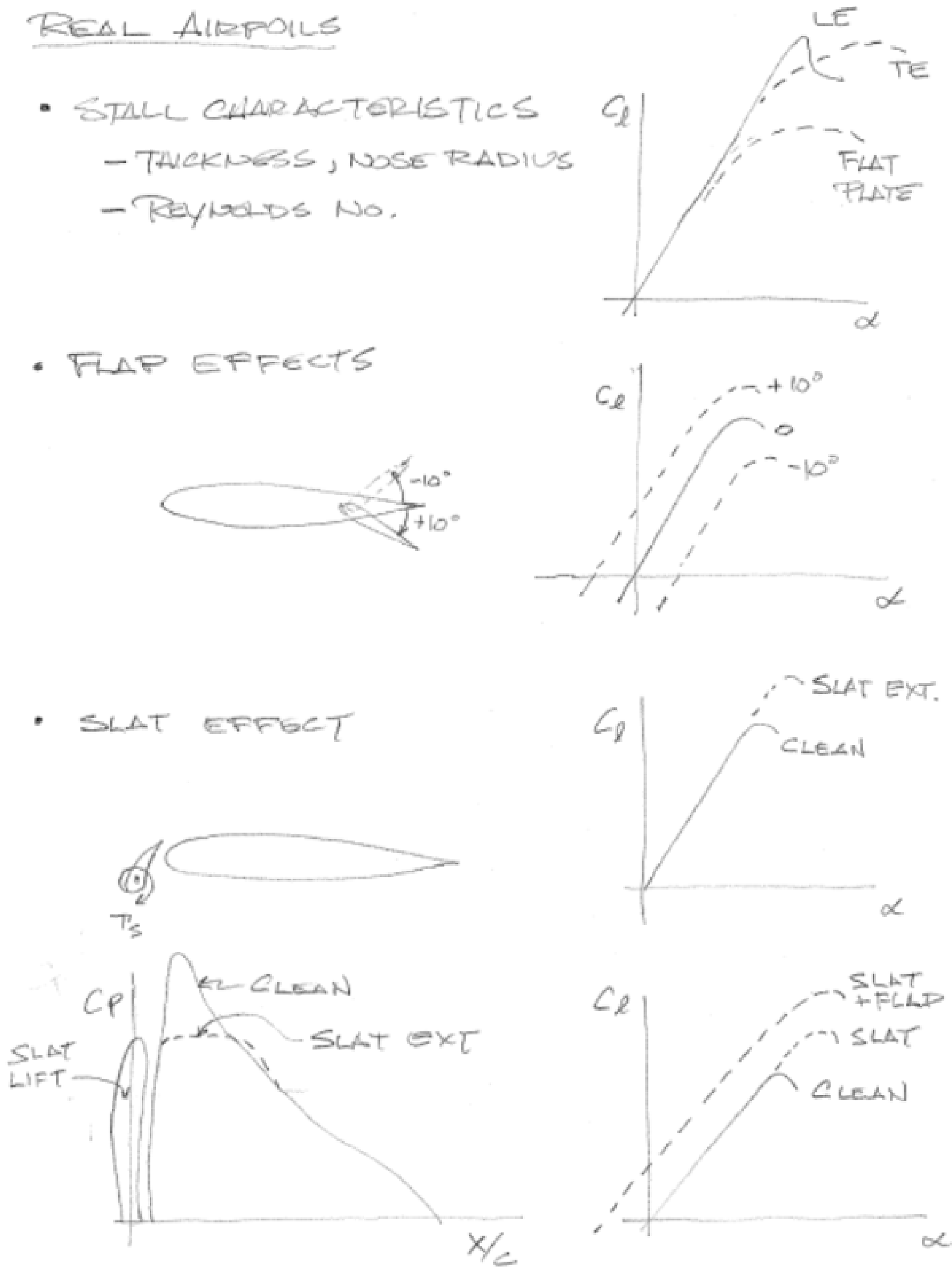


Figure 57: c_l for real airfoils with flaps and slats

13 Wings

13.1 Finite Wings

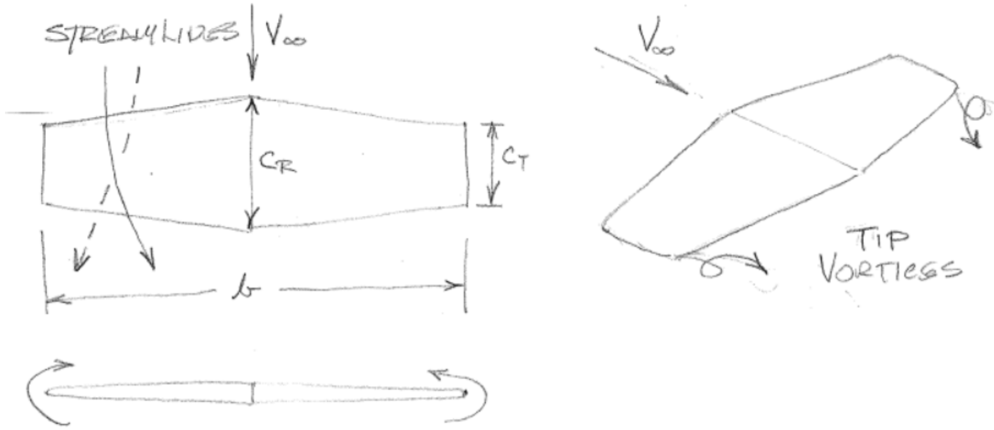


Figure 58: Streamlines and tip vortices over a finite wing

In Figure 58, the trailing vortices create a **downwash velocity** w at the wing. This results in the local velocity at the wing being canted downward relative to the freestream velocity.

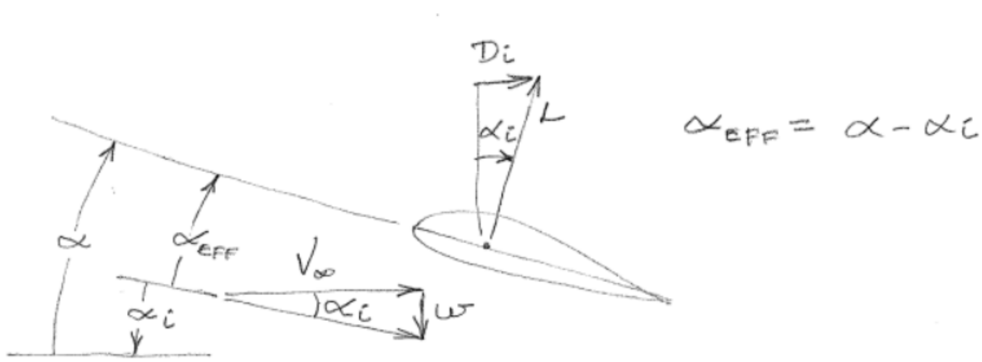


Figure 59: Effective angle of attack vs real angle

Since the local lift is $\perp$ to the local velocity, the resulting cant of the lift vector creates **induced drag** D_i (Figure 59).

13.2 Biot-Savart Law

Consider a vortex filament in space (Figure 60) of strength Γ - The segment $\vec{dl}$ induces a velocity at P given by Equation 67.

$$d\vec{V} = \frac{\Gamma}{4\pi}\frac{\vec{dl} \times \vec{r}}{|\vec{r}|^2} \tag{67}$$

For a straight filament of infinite length, Equation 67 becomes:

$$\vec{V} = \int_{-\infty}^{\infty} \frac{\Gamma}{4\pi}\frac{\vec{dl} \times \vec{r}}{|\vec{r}|^3}$$

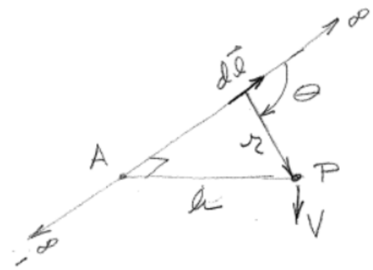


Figure 60: Biot-Savart Law applied to a vortex filament line

Here, $\vec{dl} \times \vec{r} = dl \cdot r \sin\theta$ and $\vec{V}$ is down.

$$V = \frac{\Gamma}{4\pi} \int_{-\infty}^{\infty} \frac{\sin\theta}{r^2} dl$$

Defining h as the normal distance from P to the filament:

$$r = \frac{h}{\sin\theta}, \quad l = \frac{h}{\tan\theta}, \quad dl = -\frac{h}{\sin^2\theta} d\theta$$

$$V = -\frac{\Gamma}{2\pi h} \int_{\pi}^{0} \sin\theta \; d\theta = \frac{\Gamma}{2\pi h}$$

For a **semi-infinite** vortex filament, where the filament begins at the point A of the normal to P:

$$V = \frac{1}{2}\left(\frac{\Gamma}{2\pi h}\right) = \frac{\Gamma}{4\pi h}$$

Heimholtz' Vortex Theorems:

1. The strength of a vortex filament is constant along its length.
2. A vortex filament cannot end in a fluid; it must extend to the boundaries of the fluid (possibly $\pm\infty$) or form a closed path.

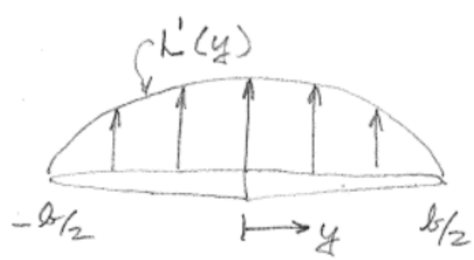


Figure 61: Lift distribution along span of real wing

13.2.1 General thoughts on a finite wing

- Lift will vary with spanwise location (Figure 61)
- Lift = 0 at wingtips
- $L'(y) = \rho_\infty V_\infty \Gamma(y)$
- $\Gamma(y)$ will vary with y.

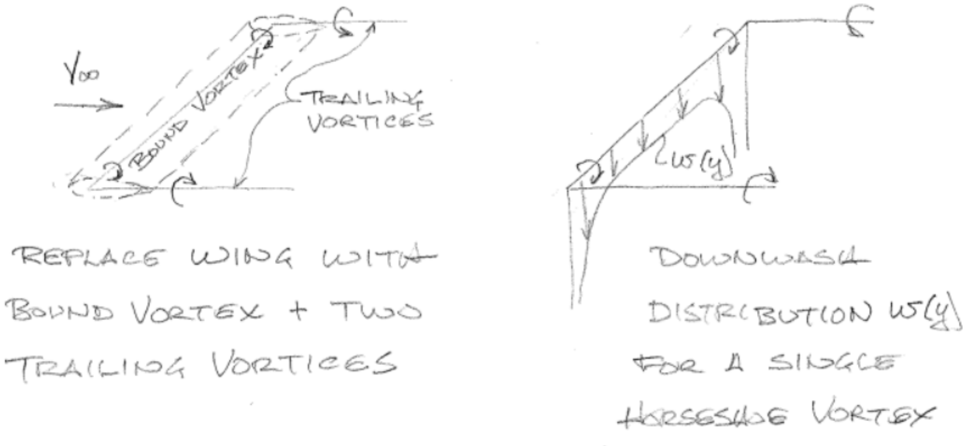


Figure 62: Prandtl's Lifting Line Theory for a wing

13.3 Prandtl's Lifting Line Theory

From the Biot-Savart Law:

$$w(y) = -\frac{\Gamma}{4\pi(b/2 + y)} - \frac{\Gamma}{4\pi(b/2 - y)} = -\frac{\Gamma}{4\pi}\frac{b}{(b/2)^2 - y^2}$$

which shows that $w(y) \to \infty$ at $y = \pm b/2$ (Figure 62). Next, consider a finite number of horseshoe vortices superimposed along the lifting line (Figure 63).

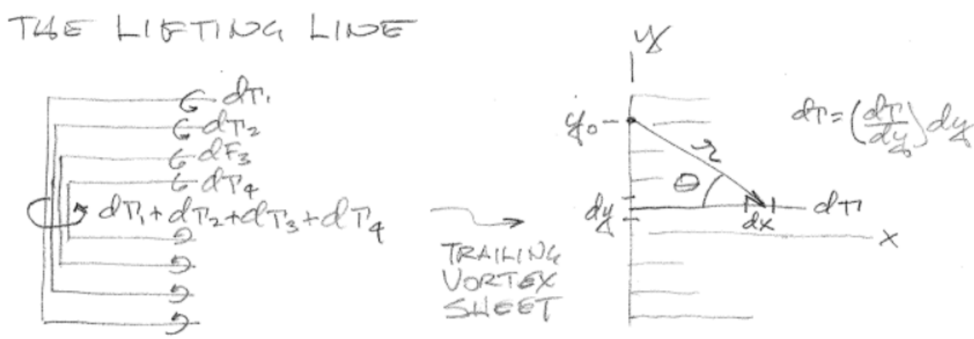


Figure 63: Horseshoe vortices superimposed along Prandtl's lifting line

$$dw(y_0) = -\frac{(d\Gamma/dy)dy}{4\pi(y - y_0)} \tag{68}$$

Note: $d\Gamma/dy$ is negative since Γ is decreasing as y increases toward the wingtip. However w is positive up for the sketch. Therefore Equation 68 for dw requires the minus sign.

The total velocity $w(y_0)$ is obtained by integrating over the entire trailing vortex sheet:

$$w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{y - y_0} dy$$

The induced angle of attack at the spanwise station y_0 is Equation 70.

$$\alpha_i(y_0) = \arctan\Big(-\frac{w(y_0)}{V_\infty}\Big) = -\frac{w(y_0)}{V_\infty} \quad \text{small angle} \tag{69}$$

$$\text{or} \quad \alpha_i(y_0) = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2}\frac{(d\Gamma/dy)}{y-y_0}dy \tag{70}$$

The lift coefficient of the airfoil at y_0 is:

$$c_l = a_0\big[\alpha_{EFF}(y_0) - \alpha_{OL}\big] = 2\pi\big[\alpha_{EFF}(y_0) - \alpha_{OL}\big]$$
$$\text{Also} \quad L = \frac{1}{2}\rho_\infty V_\infty^2 c(y_0)c_l = \rho_\infty V_\infty \Gamma(y_0)$$
$$\text{or} \quad c_l = \frac{2\Gamma(y_0)}{V_\infty c(y_0)}$$

Combining and solving for α_{EFF}, we get Equation 71.

$$\alpha_{EFF}(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{OL} \tag{71}$$

By definition, $\alpha_{EFF} = \alpha - \alpha_i$. Combining the expressions for $alpha_{EFF}$ (Equation 71) and α_i (Equation 70), we get Equation 72, the **fundamental equation of Prandtl's Lifting Line Theory**.

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{OL}(y_0) + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2}\frac{(d\Gamma/dy)}{y-y_0}dy \tag{72}$$

Equation 72 is an integral-differential equation with the unknown $\Gamma(y)$. For a given wing design, α, c, α_{OL} and V_∞ are known at each spanwise station. Solution of the fundamental equation yields $\Gamma(y)$ from $-b/2$ to $b/2$.

The lift distribution is obtained from:

$$L(y_0) = \rho_\infty V_\infty \Gamma(y_0)$$

The total lift is Equation 74

$$L = \rho_\infty V_\infty \int_{-b/2}^{b/2}\Gamma(y)dy \tag{73}$$

$$c_L = \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 S} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2}\Gamma(y)dy \tag{74}$$

Induced drag is given by Equation 75.

$$D_i = L\sin\alpha_i = L\alpha_i \quad \text{for small angles} \tag{75}$$

Total induced drag is then:

$$D_i = \int_{-b/2}^{b/2} L(y)\alpha_i(y)dy = \rho_\infty V_\infty \int_{-b/2}^{b/2}\Gamma(y)\alpha_i(y)dy$$
$$\text{and} \quad c_{Di} = \frac{Di}{\frac{1}{2}\rho_\infty V_\infty^2 S} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2}\Gamma(y)\alpha_i(y)dy$$

13.3.1 Notes on Fundamental Equation of Lifting Line Theory

$$
\begin{aligned}
c_l &= 2\pi(\alpha_{EFF} - \alpha_{OL}) \\
&= 2\pi(\alpha - \alpha_i - \alpha_{OL}) \\
\alpha &= \frac{c_l}{2\pi} + \alpha_{OL} + \alpha_i \\
L &= \frac{1}{2}\rho_\infty V_\infty^2 c(y_0) c_l(y_0) = \rho_\infty V_\infty \Gamma(y_0) \\
&\rightarrow c_l(y_0) = \frac{2\Gamma(y_0)}{V_\infty c(y_0)} \\
\alpha_i(y_0) &= \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{y_0 - y} dy
\end{aligned}
$$

$$
\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{OL}(y_0) + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{y_0 - y} dy
$$

13.4 Elliptic Lift Distribution

Assume an elliptic circulation distribution given by Equation 76

$$
\Gamma(y) = \Gamma_0 \sqrt{1 - 4\left(\frac{y}{b}\right)^2} \tag{76}
$$

1. Γ_0 is the circulation at midspan.
2. $L(y) = \rho_\infty V_\infty \Gamma(y)$ is the elliptic lift distribution.
3. $\Gamma(b/2) = \Gamma(-b/2) = 0$

Note: This $\Gamma(y)$ has not been obtained as a direct solution of the fundamental equation. At this point, it is simply specified.

Calculation of Downwash:

$$
\frac{d\Gamma}{dy} = -\frac{4\Gamma_0 y}{b^2\left[1 - 4(y/b)^2\right]^{1/2}}
$$

$$
w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{y - y_0} dy = \frac{\Gamma_0}{\pi b^2}\int_{-b/2}^{b/2} \frac{y}{\left[1 - 4(y/b)^2\right]^{1/2}(y_0 - y)} dy
$$

$$
\text{Transform via} \quad y = \frac{b}{2}\cos\theta, \quad dy = -\frac{b}{2}\sin\theta \, d\theta
$$

$$
\rightarrow w(\theta_0) = -\frac{\Gamma_0}{2\pi b}\int_0^\pi \frac{\cos\theta \, d\theta}{\cos\theta - \cos\theta_0} = -\frac{\Gamma_0}{2b}
$$

The downwash is constant for an elliptic lift distribution. Also:

$$
\alpha_i(y_0) = -\frac{w(y_0)}{V_\infty} = -\frac{\Gamma_0}{2bV_\infty} \quad \text{constant}
$$

Note: w and $\alpha_i \to 0$ as $b \to \infty$ (2D flow). For the total lift:

$$L = \rho_\infty V_\infty \Gamma_0 \int_{-b/2}^{b/2} \sqrt{1 - 4\left(\frac{y}{b}\right)^2} dy = \rho_\infty V_\infty \Gamma_0 \frac{b}{2} \int_0^\pi \sin^2\theta \, d\theta$$

$$\to L = \rho_\infty V_\infty \Gamma_0 \frac{b}{4}\pi$$

$$\text{or} \quad \Gamma_0 = \frac{4L}{\rho_\infty V_\infty b\pi}$$

$$\text{Using} \quad L = \frac{1}{2}\rho_\infty V_\infty^2 S c_L \quad \text{yields}$$

$$\Gamma_0 = \frac{2V_\infty S c_L}{b\pi}$$

$$\text{and} \quad \alpha_i \quad \text{becomes} \quad \alpha_i = \frac{S c_L}{\pi b^2}$$

$$\text{Define aspect ratio as} \quad AR = \frac{b^2}{S} \quad \text{then} \quad \alpha_i = \frac{c_L}{\pi AR}$$

$$\text{Recall} \quad C_{Di} = \frac{2}{V_\infty S} \int_{-b/2}^{b/2} \Gamma(y)\alpha_i(y) dy$$

$$C_{Di} = \frac{2\alpha c \Gamma_o b}{2V_\infty S} \int_0^\pi \sin^2\theta \, d\theta = \frac{\pi \alpha_i \Gamma_o b}{2V_\infty S}$$

$$\text{or} \quad C_{D1} = \frac{\pi b}{2V_\infty S}\left(\frac{C_L}{\pi R}\right)\frac{2V_\infty S C_L}{b\pi}$$

$$\boxed{C_{Di} = \frac{C_L^2}{\pi AR}}$$

Consider a wing composed of the same airfoils with no geometric twist. It is desired to have an elliptic lift distribution.

$$\text{No twist} \to \alpha = const$$

$$\text{Elliptic} \to \alpha_i = const$$

$$\alpha_{EFF} = \alpha - \alpha_i = const$$

$$\text{Common Airfoil} \to \alpha_o, \ \alpha_{0L} = const$$

$$C_l = \alpha_o(\alpha_{EFF} - \alpha_{0L}) = const$$

$$L(y) = q_\infty c \cdot C_l$$

$$\to c(y) = \frac{L(y)}{q_\infty C_l} \quad \text{must vary elliptically}$$

Few wings have elliptic planforms, however, the lift distribution on most wings is similar to elliptic.

$$C_{Di} = \frac{C_L^2}{\pi ARe}, \quad 0.9 < e \leqslant 1 \quad e = 1 \text{ Ellipse}$$

13.5 Notes on Elliptic Lift Distribution

The fundamental equation applies for an arbitrary wing geometry where α, α_{0L} and c are specified functions of y. In this case, $\Gamma(y)$ is the solution that is sought. However if $\Gamma(y)$ is specified, then the fundamental equation is reduced to a simple algebraic equation where the solution is $c(y)$.

$$\text{For} \quad \Gamma(y) = \Gamma_o\sqrt{1 - 4(y/b)^2}$$

$$L(y) = \rho_\infty V_\infty \Gamma_o \sqrt{1 - 4(y/b)^2} = C_l \frac{1}{2}\rho_\infty V_\infty^2 c$$

$$\text{Then} \quad C_l c = \frac{2\Gamma_o}{V_\infty}\sqrt{1 - 4(y/b)^2}$$

This the product $C_l c$ must vary elliptically. For common airfoils and zero twist, this yields an elliptic planform.

However, C_l intself could in principle provide the elliptic variation while $c = const$. Recall:

$$C_l = \alpha_0(\alpha - \alpha_{0L} - \alpha_i)$$
$$\alpha \rightarrow \text{Twist}$$
$$\alpha_{0L} \rightarrow \text{Camber}$$
$$\alpha_i = \text{Const for elliptic } \Gamma(y)$$

Thus $\alpha(y)$ and $\alpha_{0L}(y)$ could be set to provide (or at least contribute to) an elliptic $\Gamma(y)$ where $c(y)$ itself is not elliptic. In this case, if the angle of attack of the wing is changed, a constant $\Delta\alpha$ is added at **all** spanwise stations:

$$C_l = \alpha_0(\alpha + \Delta\alpha - \alpha_{0L} - \alpha_i)$$

and the product $C_l c$ will no longer vary elliptically.

However, for the special case of common airfoils with zero twist, the resulting $C_l c$ will remain elliptic at all angles of attack since $C_l = const$ and $c(y)$ is elliptic.

13.6 Shrenk Approximation

Based on a comparison of experimental results, Shrenk showed the lift distribution on a non-elliptic planform with zero twist lies approximately halfway between that of an elliptic planform of the same area (Figure 64).

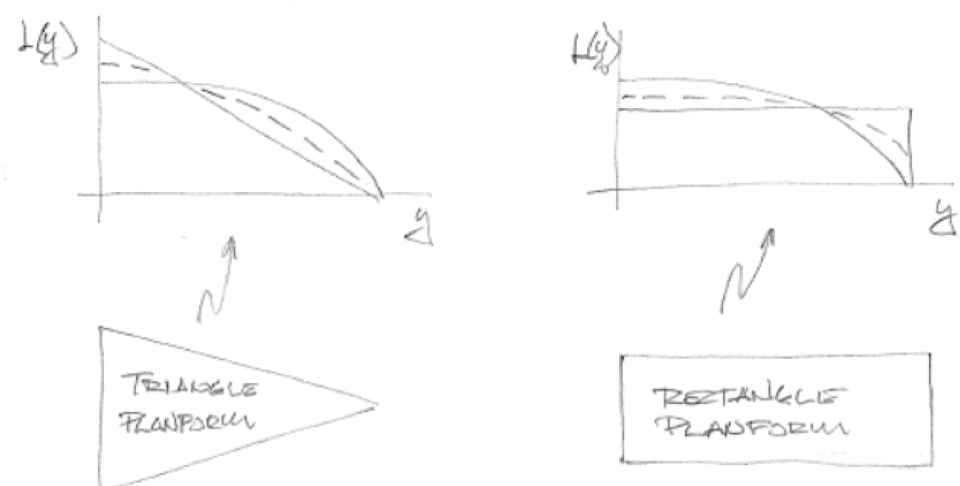


Figure 64: Shrenk approximation for a non-elliptic planform

13.7 General Lift Distribution

Consider the spanwise transformation:

$$y = -\frac{b}{2}\cos\theta, \quad \theta = 0,\ y = -\frac{b}{2}, \quad \theta = \pi,\ y = +\frac{b}{2}$$

The elliptic circulation distribution becomes

$$\Gamma(y) = \Gamma_o\Big[1 - 4\big(\frac{y}{b}\big)^2\Big]^{1/2} = \Gamma_o\Big[1 - 4\frac{b^2}{4}\frac{\cos^2\theta}{b^2}\Big]^{1/2}$$
$$\text{or} \quad \Gamma(\theta) = \Gamma_o \sin\theta$$

For the general case, assume

$$\Gamma(\theta) = 2bV_\infty \sum_1^N A_n \sin n\theta$$

$$\frac{d\Gamma}{dy} = \frac{d\Gamma}{d\theta}\frac{d\theta}{dy} = 2bV_\infty \sum_1^N nA_n \cos n\theta \frac{d\theta}{dy}$$

Substitute in the fundamental equation:

$$\alpha(\theta_0) = \frac{2b}{\pi c(\theta_0} \sum_1^N A_n \sin n\theta + \alpha_{0L}(\theta_0) + \frac{1}{\pi}\int_0^\pi \frac{\sum_1^N nA_n \cos n\theta}{\cos\theta - \cos\theta_0} d\theta$$

$$\text{Recall} \quad \int_0^\pi \frac{\cos n\theta \; d\theta}{\cos\theta - \cos\theta_0} = \frac{\pi \sin n\theta}{\sin\theta_0}$$

The Fundamental Equation becomes Equation 77.

$$\alpha(\theta_0) = \frac{2b}{\pi c(\theta_0)} \sum_1^N A_n \sin n\theta_0 + \alpha_{0L}(\theta_0) + \sum_1^N nA_n \frac{\sin n\theta_0}{\sin\theta_0} \tag{77}$$

b, $c(\theta_0)$ and $\alpha_{0L}(\theta_0)$ and the airfoil section properties are given by the geometry of the wing. Next select N representative spanwise stations θ_0's and write the fundamental equation for each. This yields N linear algebraic equations that can be solved for the N unknowns A_1, $A_2 ... A_n$.
This yields the circulation distribution $\Gamma(\theta_0)$.

For the lift coefficient:

$$C_L = \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y) dy = \frac{2b^2}{S}\sum_1^N A_n \int_0^\pi \sin n\theta \; sin\theta \; d\theta$$

$$\int_0^\pi \sin^2\theta \; d\theta = \frac{\pi}{2}, \quad \int_0^\pi \sin n\theta \; \sin\theta \; d\theta = 0, \quad n > 1$$

$$\rightarrow C_L = A_1 \pi \frac{b^2}{S} = A_1 \pi AR$$

(Although C_L depends on A_1 alone, allthe A_n's must be solved for to obtain A_1.)

For the induced drag coefficient:

$$C_{Di} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y)\alpha_i(y) dy$$

$$= \frac{2b^2}{S}\int_0^\pi \big(\sum_1^N A_n \sin n\theta\big)\alpha_i(\theta) \sin n\theta \; d\theta$$

$$\alpha_i(y_0) = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{y - y_0} dy = \frac{1}{\pi}\sum_1^N nA_n \int_0^\pi \frac{\cos u\theta}{\cos\theta - \cos\theta_0} d\theta$$

$$\text{Then} \quad \alpha_i(\theta) = \sum_1^N nA_n \frac{\sin n\theta}{\sin\theta}$$

Substituting into the equation for C_{DI}:

$$C_{Di} = \frac{2b^2}{S}\int_0^\pi \big(\sum_1^N A_n \sin n\theta\big)\big(\sum_1^N nA_n \sin n\theta\big) d\theta$$

The integrand is a product of two summations whose terms have the form $\sin m\theta \ \sin k\theta$. However:

$$\int_0^{\pi} \sin m\theta \ \sin k\theta \ d\theta = \begin{cases} 0, & m \neq k \\ \pi/2, & m = k \end{cases}$$

Thus C_{Di} becomes

$$C_{Di} = \frac{2b^2}{S}\Big(\sum_1^N nA_n^2\Big)\frac{\pi}{2} = \pi AR\sum_1^N nA_n^2$$

$$C_{Di} = \pi AR\Big(A_1^2 + \sum_2^N nA_n^2\Big) - \pi ARA_1^2\Big[1 + \sum_2^N \big(\frac{A_n}{A_1}\big)^2\Big]$$

Recalling that $C_l \ A_1\pi AR$:

$$C_{Di}\frac{C_L^2}{\pi AR}(1+S), \quad S = \sum_2^N n\big(\ A_n/A_1\big)^2$$

Note: $A_1 > A_2 > ...A_N \ \rightarrow \ S$ is small. Also, $S \geqslant 0$.

$$\boxed{C_{Di} \geqslant \frac{C_L^2}{\pi AR} \rightarrow \text{Elliptic Distance provides minimum } C_{Di}}$$

The expression for C_{Di} is typically Equation 78

$$\boxed{C_{Di} = \frac{C_L^2}{\pi ARe}, \quad e \leqslant 1 \text{ is the span efficiency factor}} \tag{78}$$

13.8 Lift Curve Slope for a Finite Wing

For a 2D Airfoil, $\alpha_0 = dC_l/d\alpha$. Define the lift curve slope for a finite wing as Equation 79.

$$\alpha = \frac{dC_L}{d\alpha} \tag{79}$$

When a finite wing is at the geometric angle α, it experiences a reduced angle α_{EFF} given by Equation 80.(Figure 65).

$$\alpha_{EFF} = \alpha - \alpha_i, \quad \text{where } \alpha_i = \frac{C_L}{\pi AR} \text{ (Elliptic)} \tag{80}$$

Consider an elliptic wing with zero twist. Since $C_l = const$ across the span:

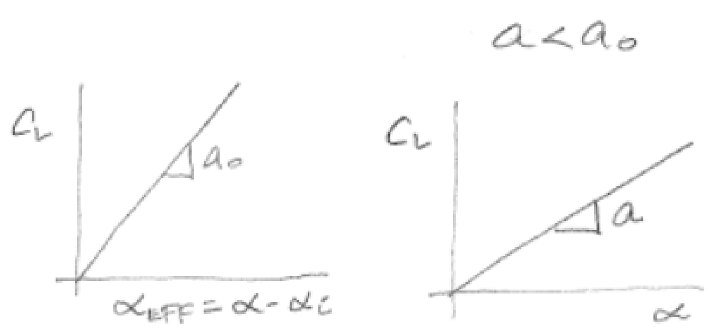


Figure 65: C_L slope for α and α_{EFF}. The slope α is less than α_0

$$C_L = C_l$$

Since the wing experiences α_{EFF}:

$$\frac{dC_L}{d\alpha_{EFF}} = \frac{dC_L}{d(\alpha - \alpha_i)} = \alpha_0$$

Integrating:

$$C_L = a_0(\alpha - \alpha_i) + const$$
$$C_L = a_0(\alpha - \frac{C_L}{\pi AR}) + const$$
$$C_L(1 + \frac{a_o}{\pi AR}) = a_0\alpha$$
$$\frac{dC_L}{d\alpha} = a = \frac{a_0}{1 + a_0/(\pi AR)(1+\tau)}$$

13.9 Summary of Wing Drag

Wing drag originates from 3 sources:

1. D_f due to skin friction
2. D_p due to separation (Pressure drag)
3. D_i due to lift

Define $C_{DP} = \frac{D_f + D_p}{q_\infty S}$ as the profile drag coefficient. C_{DP} is regarded as a function of the airfoil geometry and Reynolds Number, and is a weak function of C_l.

$$C_{Di} = \frac{D_i}{q\infty S} = \frac{C_L^2}{\pi ARe}$$

$$\begin{aligned}
\text{Therefore total wing drag is} \quad & C_D = C_{DP} + \frac{D_L^2}{\pi ARe} \\
\text{For two of different aspect ratio:} \quad & C_{D_1} = C_{DP_1} + \frac{C_L^2}{\pi AR_1 e}, \quad C_{D_2} = C_{DP_2} + \frac{D_L^2}{\pi AR_2 e} \\
\text{Assuming the same airfoil and twist:} \quad & C_{DP_1} = C_{DP_2} \quad \text{and} \quad e_1 = e_2 = e \\
\text{Taking the difference:} \quad & C_{D_1} - C_{D_2} = \frac{C_L^2}{\pi e}(\frac{1}{AR_1} - \frac{1}{AR_2})
\end{aligned}$$

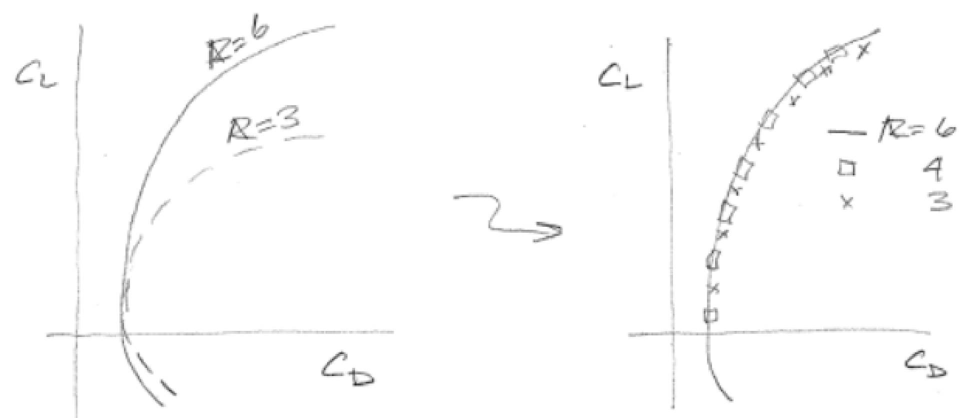


Figure 66: Drag polars for varying AR

Using the relation:

$$C_{D_1} = C_{D_2} + \frac{C_L^2}{\pi e}(\frac{1}{6} + \frac{1}{AR_2})$$

The data for $AR = 4,\ 3$, etc. collapses onto the polar for $AR = 6$ (Figure 66). Similarly, using the 3D lift curve slope definition (Equation 81), the lift curve slopes collapse to a common curve in Figure 67.

$$a = \frac{a_0}{1 + a_0/\pi AR)(1+\tau)} \tag{81}$$

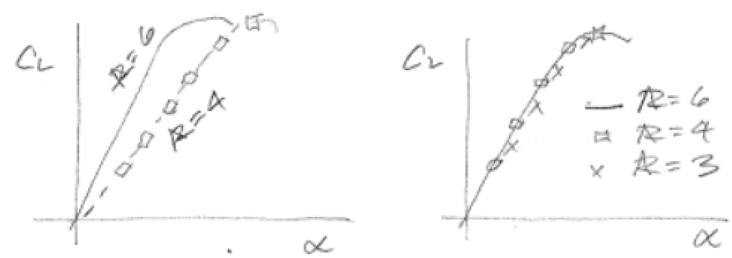


Figure 67: The lift curve slopes collapse to a common curve for varying AR

$C_f = \tau/(\frac{1}{2}\rho V^2) = \tau/q$ and $D = \tau \times S_{wet}$ = drag force due to skin friction for a **flat plate** (Figure 68).

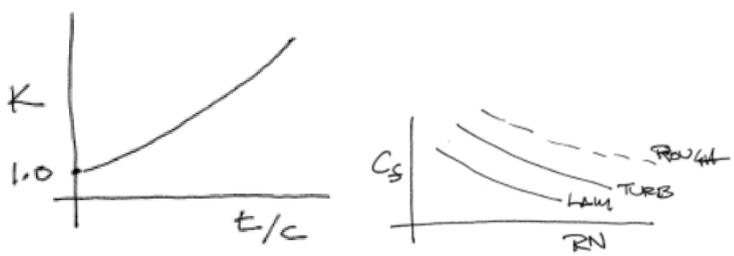


Figure 68: Drag and C_f on a flate plate

$$\text{Define } f = KC_fS_{wet} \text{ as drag area}$$
$$f = D/q$$
$$C_D = \frac{D}{qS_{ref}} = \frac{f}{S_{ref}}$$
$$\text{Then } f_{total} = \sum f_i = \sum K_i C_{fi} S_{wet,\ i}$$
$$C_D = \frac{f_{total}}{S_{ref}}$$

$$C_D = C_{DP} + C_{Di} = C_{DP} + \frac{C_L^2}{\pi ARe}$$
$$D/L = C_D/C_L = \frac{C_{DP}}{C_L} + \frac{C_L}{\pi ARe}$$
$$\frac{d(C_d/C_L)}{dC_L} = -\frac{C_{DP}}{C_L} + \frac{1}{\pi ARe} = 0$$
$$C_{DP} = \frac{C_L^2}{\pi ARe} \text{ for } C_D/C_L\Big)_{min} \text{ or } C_L/C_D\Big)_{max}$$
$$C_L\Big)_{L/D\ max} = \big(\pi AReC_{DP}\big)^{1/2}$$
$$\frac{C_L}{C_D}\Big)_{max} = \frac{C_L}{C_{DP} + \frac{C_L^2}{\pi ARe}} = \frac{\big(\pi AReC_{DP}\big)^{1/2}}{2C_{DP}} = \Big(\frac{\pi ARe}{4C_{DP}}\Big)^{1/2}$$
$$C_{DP} = \sum f_i/S_{ref} = \sum \frac{K_i C_{fi} S_{wet,\ i}}{S_{ref}} = \bar{K}\bar{C_f}\frac{S_{wet}}{S_{ref}}$$
$$\frac{C_L}{C_D}\Big)_{max} = \Big(\frac{\pi b^2 e}{4\bar{K}\bar{C_f}S_{wet}}\Big)^{1/2} = Big(\frac{\pi}{4}e\frac{b^2}{\bar{K}\bar{C_f}S_{wet}}\Big)^{1/2}$$
$$\text{Optimum (max) lift:} \quad e = 1,\ \bar{K} = 1,\ C_f(RN)$$
$$\text{Then} \quad L/D\Big)_{max} = \text{const}(AR_{wet})^{1/2}$$