ELECENG 3TQ3
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1 Lecture 6

1.1 Discrete Random Variables

We assign number to the outcomes in sample space. Once we observe a number, we refer to that observation by a random variable. Each random variable has a range we denote as S_X.

A probability model always begins with an experiment. Each random variable is directly related to this experiment. There are 3 types of relationships (how random variables can be related to your experiment):

- The random variable is the observation
- The random variable is a function of the observation
- The random variable is a function of another random variable

Random variable is the observation: The experiment is to attach a photo detector to an optical fiber and count the number of photons arriving in a one microsecond time interval. Each observation is a random variable X. The range of X is $S_X = \{0, 1, 2, \dots\}$. In this case S_X and the range X is identical.

A better example is if we wanted to determine the number of cars passing through a particular intersection at a particular time during a particular time interval range.

Random variable is a function of an observation: If we had a class of 30 students and wanted to determine the number of students who got an A- or better. Then the sample space would be the final grades, but the random variable is the number of students who achieved an A- or better.

Random variable is is a function of another random variable: number of new daily COVID cases is a random variable X, we introduce a new random variable Y as estimated cost to OHIP. Obviously Y depends on X hence $Y = f(X)$.

Definition: A random variable consists of an experiment with a probabilistic measure $P[\cdot]$ defined on a sample space S and a function that assigns a real number to each outcome in the sample space of the experiment.

1.2 Discrete vs. Continuous

Example of a discrete random variable are the number of cars in a parking lot, or the number of lights in a room. Since we have a number for each value in the range S_X as there cannot be $0.51323\dots$ of a car, we understand these variables as discrete.

Example of a continuous random variable is the amount of time it takes to travel from Square One to McMaster University. Since the range S_X is time, time has an infinite range and is continuous. So if the range is continuous, then the random variable is continuous.

Definition: X is a discrete random variable if the range of X is a countable set.

Definition: X is a finite random variable if the range is a finite set.

1.3 Probability Mass Function

The probability mass function (PMF) of the discrete random variable X

$$P_X(x) = P[X = x] \tag{1}$$

The above means, what is the probability the random variable X takes the value x.

1.4 Equivalent Properties to PMF

$\forall x \in S_X,\ P_X(x) \geq 0$, means that the probability of any outcome cannot be a negative number.

$\sum_{x \in S_X} P_X(x) = 1$, means the sum of all outcomes in the range of where the probability is define is equal to 1.

For any event $B \subset S_X$, the probability that X is in the set B is $P[B] = \sum_{x \in B} P_X(x)$, means that the probability that X is in B is the sum of all the probabilities in B.

1.5 Bernoulli Distribution

X is a Bernoulli distribution (p) random variable if the PMF of X has the following form

$$P_X(x) = \begin{cases} 1 - p, & x = 0 \\ p, & x = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The parameter $p \in (0, 1)$ or simply $0 < p < 1$.

1.6 Geometric Random Variables

X has a geometric (p) random variable if it has a distribution

$$P_X(x) = \begin{cases} p(1-p)^{(x-1)} & x = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The parameter p is in the range $0 < p < 1$.

1.7 Pascal Distribution

If you conduct an experiment with N trials including up to and including k successes. Then if we define the success probability of success as p. Then we have a pascal distribution if the PMF has the following form

$$P_X(k,p) = \binom{x-1}{k-1} p^k (1-p)^k \tag{4}$$

Example: find the pmf of random variables described as number of tests until we get k failed tests.

1.8 Discrete Uniform Random Variable

If all the outcomes have the same probability of outcomes then the pmf can be defined as follows.

The distribution is a function on the limits of it's sample space range.

Let X be in the range $K \leq X \leq L$. In other words, $S_X = K, \ldots, L$. The pmf $U(K, L)$ of a uniform random variable is

$$P[X = x] = \frac{1}{L - K + 1} \tag{5}$$

1.9 Poisson Random Variable

Models phenomenon occurring randomly in time. Each instance in time is random, but there is a known average number occurring in a given time.

$$f(x) = \begin{cases} \frac{\alpha^x \exp -\alpha}{x!}, & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

An example would be the number of customers arriving to the Ministry of Transportation per time interval.

1.10 Cumulative Distribution Function

A cumulative distributive function is denoted as follows: $F_X(x) = P[X \leq x]$. For any discrete random variable X with range $S_X = x_1, x_2, \ldots$ satisfying $x_1 \leq x_2 \leq \ldots$ the following properties hold.

- $F(-\infty) = 0$ and $F(\infty) = 1$
- For all $x' \geq x, F(x') \geq F(x)$
- For $x_i \in S_X$ and let ϵ be an arbitrarily small number $F(x_i) - F(x_i - \epsilon) = P_X(x_i)$
- $F_X(x) = F_X(x_i)$ for all x such that $x_i \leq x \leq x_{i+1}$