1 Exponential Distribution

The Poisson distribution defined a random variable to be the number of flaws along a length of copper wire (we were able to find the probability of n flaws along the length of the cable).

Now we are interested in the **distance between flaws**.

Let X be the length from any starting point on the wire until the first flaw is detected. Thus we see that the distribution of X can be found using our knowledge from the distribution of the number of flaws.

In general, let the random variable N denote the number of flaws in x millimetres of wire. If the mean number of flaws is λ per millimetre, N has a Poisson distribution with mean λx. We assume that the wirer is longer than the value of x.

We need to find the probability of finding no flaws in x length of the cable.

$$P(N=0) = \frac{e^{-\lambda x}(\lambda x)^0}{0!} = e^{-\lambda x} \tag{1}$$

Now we need to find the c.d.f (which sums up the probability of finding no flaws before x)

$$F(x) = P(X \leq x) = \int_0^x e^{-\lambda u}\ 1du = 1 - e^{-\lambda x}, \quad x \geq 0 \tag{2}$$

From this we see that

$$P(X > x) = P(X \geq x) = 1 - F(x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}, \ \ x \geq 0 \tag{3}$$

This equation models the probability of not finding a flaw after length x, but we can see it is the exact same equation as (1)

Now we differentiate to find the p.d.f for finding no flaws in various lengths

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0 \tag{4}$$

The probability density function of X $f(x) = \lambda e^{-\lambda x}$ has a domain $0 \leq x \leq \infty$

If the random variable X has an exponential distribution with parameter λ, then

$$\mu = E(X) = \frac{1}{\lambda} \tag{5}$$
$$\mu^2 = V(X) = \frac{1}{\lambda^2} \tag{6}$$

Example 4.21 In a large corporate office, user logins to the system can be modelled by a poisson process with a mean of 25 logins per hour. What is the probability of no logins in an interval of six minutes?

Ans. $6min = 6/60 = 0.1hours$

$$P(X > 0.1) = \int_{0.1}^{\infty} 25e^{-25x}\,dx = e^{-2.5} = 0.082 \tag{7}$$

Example 4.22 Let X denote the time between detections of a particle with a Geiger counter and assume that X has an exponential distribution with $E(X) = 1.4$ minutes. Find the probability that we detect a particle within 30 seconds of starting the counter.

Ans.

$$P(X < 0.5) = F(0.5) = 1 - e^{0.5/1.4} = 0.30$$

Example 4.22 Contd. What is the probability that we will find a detection within the next 30 seconds after knowing there was no detection after 3 minutes?

Ans.

$$P(X < 3.5|X > 3) = P(3 < X < 3.5)/P(X > 3) \tag{8}$$

Since we know

$$P(3 < X < 3.5) = F(3) - F(3.5) = 0.035 \tag{9}$$

Thus we see that

$$P(X > 3.5|X > 3) = \frac{0.035}{0.117} = 0.30 \tag{10}$$

But we realize that its the same as though we simply waited for 30 seconds

$$P(X < 0.5) = F(0.5) = 1 - e^{-0.5/1.4} = 0.30 \tag{11}$$

From this we see that the exponential random variable follows the **Loss of Memory Property** which is simply put mathematically as

$$P(X < t_1 + t_2|X > t_1) = P(X < t_2) \tag{12}$$

This simply means that it does not matter how long you have waited previously, the probability of a future event happening in the next n minutes is the same as if you just started the experiment and waited n minutes.