9 Cumulative Distributive Functions

A **cumulative distributive function** is defined as follows

$$F(x) = \int_{-\infty}^{x} f(t)dt \tag{1}$$

for $-\infty < x < \infty$

Hence $P(a < X \leq b) = F(b) - F(a)$

Example 4.3 The current in a thin copper wire measured in milliamperes as a continuous random variable X. Assume the range of X is $[4.9, 5.1]$ mA, and assume the probability density function $f(x) = 5$ for $4.9 \leq x \leq 5.1$. Find the cumulative distributive function.

Ans. if $x < 4.9, f(x) = 0$ therefore

$$F(x) = 0, \text{ for } x < 4.9$$

if $4.9 \leq x < 5.1$ then

$$F(x) = \int_{4.9}^{x} f(t)dt = \int_{4.9}^{x} 5dt = 5x - 5(4.9) = 5x - 24.5 \text{ for } 4.9 \leq x \leq 5.1$$

if $x \geq 5.1$ then

$$F(x) = 1 \text{ for } x > 5.1$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 4.9 \\ 5x - 24.5 & 4.9 \leq x < 5.1 \\ 1 & 5.1 \leq x \end{cases}$$

Recall the relation

$$f(x) = \frac{d}{dx}F(x) \tag{2}$$

Example 4.45 Reaction time X has a cumulative distributive function

$$F(x) \begin{cases} 0 & x < 0 \\ 1 - e^{-0.10x} & 0 \leq x \end{cases}$$

Find the p.d.f

Ans. Differentiate F

$$f(x) = \begin{cases} 0 & x < 0 \\ 0.10e^{-0.10x} & 0 \leq x \end{cases}$$

Example 4.17 Given

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.20x & 0 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

Find $P(X < 2.8)$, $P(X > 6)$, and $P(X < -2)$

Ans.

$$\begin{aligned} P(X < 2.8) &= F(2.8) = 0.20(2.8) = 0.56 \\ P(X > 6) &= 1 - F(6) = 1 - 1 = 0 \\ P(X < -2) &= F(-2) = 0 \end{aligned}$$

Example 4.46 X is the number of minutes after 8:00am that you arrive at work. The p.d.f is

$$f(x) = \begin{cases} 0 & x < 0 \\ 0.10e^{-0.10x} & 0 \le x \end{cases}$$

Find the probability we arrive before 8:40am on two or more days out of a week(5 days).

Ans. First we need to determine the c.d.f

$$F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 0.10e^{-0.10x} = 1 - e^{-0.10x} & 0 \ge x \end{cases}$$

Now we need to find the probability we arrive before 40 minutes

$$p = F(40) = 1 - e^{-4} = 0.9817$$

Now we need to use binomial distributions. Let Y be the discrete random variable for the amount of times we arrived before 8:40

$$\begin{aligned} P(Y = 0) &= \binom{5}{0}(0.9817)^0(1 - 0.9817)^5 \approx 0 \\ P(Y = 1) &= \binom{5}{1}(0.9817)^0(1 - 0.9817)^4 \approx 0 \end{aligned}$$

So we know $P(Y \ge y) = 1 - P(Y < y)$

$$P(Y \ge 2) = 1 - 0 - 0 = 1$$

Suppose that X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X, denoted by μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx \tag{3}$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\mu^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 \tag{4}$$

The **standard deviation** of X is $\sigma = +\sqrt{\sigma^2}$

Example 4.6 Suppose

$$f(x) = \begin{cases} 5 & 4.9 < x < 5.1 \\ 0 & \text{else} \end{cases}$$

Find the mean and variance of X.
Ans.

$$E[X] = \int_{4.9}^{5.1} xf(x) \ dx = \int_{4.9}^{5.1} 5x \ dx = \frac{5}{2}(5.1^2 - 4.9^2) = 5$$

$$V[X] = \int_{4.9}^{5.1} (x-\mu)^2 f(x) \ dx = \int_{4.9}^{5.1} 5(x-5)^2 \ dx = 0.0033$$

It is important to note that just like the discrete case

$$E[h(X)] = \int_{-\infty}^{\infty} h(x)f(x) \ dx \tag{5}$$

Example 4.7 Referring to Example 4.6. Suppose X represents the current in milliamperes and suppose there exists a resistance of $R = 100$ ohms. The power function P in watts is given by the function $P = 10^{-6}RI^2$. To recall the current has the following p.d.f

$$f(x) = \begin{cases} 5 & 4.9 \le x < 5.1 \\ 0 & \text{else} \end{cases}$$

Find the expected power.

Ans. We know that $P(I) = 10^{-6}(100)I^2 = 10^{-4}I^2$, and $I = X$. So we are trying to find the mean of

$$E[P(X)] = (10^{-4}) \int_{4.9}^{5.1} x^2 \ dx = 0.00050$$

Example 4.45 X has a p.d.f

$$f(x) = \begin{cases} \dfrac{2}{x^3} & x > 1 \\ 0 & \text{else} \end{cases}$$

Find the mean and variance
Ans.

$$E[X] = \int_1^{\infty} \frac{2}{x^3}x \ dx = 2 \quad E[X^2] = \int_1^{\infty} \frac{2}{x^3}x^2 \ dx = \infty$$

Since $V[X] = E[X^2] - (E[X])^2 = \infty$

Example.(Inverse Problem) Distribution of gravel sold (tons) in a week is a continuous X with p.d.f

$$f(x) = \begin{cases} \frac{3}{2}(1 - x^2) & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

Find the median, also called the 50^{th} percentile.

Ans. We need to find an $X = m$ such that $F(m) = 0.50$.

$$F(m) = \int_0^m \frac{3}{2}(1 - x^2)\ dt = \frac{3}{2}(m - \frac{m^3}{3})$$

Now we solve for m as we set the condition that $F(m) = 0.50$

$$0.5 = \frac{3}{2}(m - \frac{m^3}{3})$$

$$m^3 - 3m + 1 = 0$$

By trial and error, we find $m = 0.347$