15.1 Oscillatory Motion

When a spring is neither stretched or compressed, the spring is at rest at a position called equilibrium. This equilibrium position is noted as $x = 0$. When a spring is displaced, there occurs a **Restoring Force** F_s,

Restoring Force of a Spring

$$F_s = -kx \tag{1}$$

Note the force is negative, which makes it act opposite to the displacement. This is due to the force acting in the opposite direction towards equilibrium.

If the restoring force of a spring was the only force acting on a system in the x direction,

$$\sum F_x = ma_x = -kx$$

Solving for acceleration we get the following,

$$a_x = \ddot{x} = -\frac{k}{m}x$$

This equation is correct for modeling the force of a spring, it is incomplete as it does not show how x is a function of other variables. For this we introduce the concept of differential equations. To formulate a differential equation, it is important to know how the system acts at various point.

- At the endpoints $x = \pm A$, the acceleration is $\ddot{x} = -(k/m)(\pm A)$
- At the origin, we notice there should be no acceleration, $x = 0$ so $\ddot{x} = 0$ but the speed is at a maximum
- The block would continue the motion past equilibrium to the opposite end point.
- This is all assuming the surface has no retarding force, so no friction.

I will be using the dot notation to indicate derivatives as it's simpler to show. The number of dots shows the differential order.

$$\ddot{x} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = a_x = -\frac{k}{m}x$$

15.2 Particle in Simple Harmonic Motion

If there was an object oscillating in the x direction due to a spring,

$$\ddot{x} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{k}{m}x$$

We should simplify this in order to solve this differential equation,

Angular Frequency of a Spring (1)

$$\omega = \sqrt{\frac{k}{m}} \tag{2}$$

This is an important simplification for reasons beyond the scope of this material. The symbol is called **omega** and has units $[\omega] = \frac{[rad]}{[s]}$

we can now rewrite our initial differential equation,

$$\ddot{x} = -\omega^2 x$$

If we decided to solve this differential equations, also known as finding out what is $x(t)$,

Position of an Object Under Simple Harmonic Motion

$$x(t) = A\cos(\omega t + \phi) \tag{3}$$

Where A is the amplitude of the motion, ϕ is the initial phase angle, and ω is the angular frequency of our motion. It is to be noted that A, ω, and ϕ are constants.

We will verify if this differential equation is a solution by taking the second derivative,

$$\dot{x} = -\omega A\sin(\omega t + \phi)$$
$$\ddot{x} = -\omega^2 A\cos(\omega t + \phi)$$

Since we know $x(t) = A\cos(\omega t + \phi)$,

$$\ddot{x} = -\omega^2 x$$

Additionally note that $(\omega t + \phi)$ is called the phase of the motion, whereas ϕ is called the initial phase angle.

We understand that the cosine function is periodic, therefore every 2π radians it will have the same value. The time it take for the motion to undergo one full cycle is T, which represents the **period**.

We can solve for the period, let there be two phases 2π radians apart, therefore one phase is T seconds ahead,

$$\begin{aligned} [\omega(t+T)+\phi] - (\omega t + \phi) &= 2\pi \\ \omega t + \omega T + \phi - \omega t - \phi &= 2\pi \\ \omega T &= 2\pi \end{aligned}$$

Rearranging we find,

$$\omega = \frac{2\pi}{T} \qquad T = \frac{2\pi}{\omega} \qquad f = \frac{\omega}{2\pi}$$

Angular Frequency of a Spring (2)

$$\omega = \sqrt{\frac{k}{m}} \tag{4}$$

$$\omega = \frac{2\pi}{T} \qquad T = \frac{2\pi}{\omega} \qquad \omega = 2\pi f \tag{5}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{6}$$

These are all the usable ways to rewrite our derivation, notice the dependencies of the variables. Notice how changing variables can increase or decrease the period or frequency of your motion.

We briefly went over the velocity of the wave which is typically given when solving for the phase angle. Generally, you divide the velocity by the position eliminating A, in attempt to solve for ϕ.

Velocity and Acceleration of a Wave

$$v(t) = \dot{x} = -A\omega\sin(\omega t + \phi) \tag{7}$$

$$a(t) = \ddot{x} = -A\omega^2\cos(\omega t + \phi) \tag{8}$$

Since sine and cosine oscillate between ± 1 we can use this property,

$$v_{max} = -A\omega = -A\sqrt{\frac{k}{m}} \tag{9}$$

$$a_{max} = -A\omega^2 = -A\,\frac{k}{m} \tag{10}$$

15.3 Energy of the Simple Harmonic Oscillator

The kinetic energy for a system under simple harmonic motion would be,

$$E_K = \frac{1}{2}\,m\dot{x}^2 = \frac{1}{2}\,m(-A\omega\sin(\omega t + \phi))^2 = \frac{1}{2}\,m\omega^2 A^2\sin^2(\omega t + \phi)$$

Likewise, we know the potential energy would be,

$$U = -\int_x^\infty F_K \,\mathrm{d}x = \int_\infty^x kx \,\mathrm{d}x = \frac{1}{2}\,kx^2$$

$$U = \frac{1}{2}k(A\cos(\omega t + \phi))^2 = \frac{1}{2}\,kA^2\cos^2(\omega t + \phi)$$

We know that the total mechanical energy in a system is,

$$E_M = E_K + U$$

Therefore, we can find the solve for the total mechanical energy as we know the kinetic and and potential energy, (keep in mind that $\omega^2 = k/m$ and $\sin^2 x + \cos^2 x = 1$)

$$\begin{aligned}
E_M &= \frac{1}{2}\, mA^2\omega^2 \sin^2(\omega t + \phi) + \frac{1}{2}\, kA^2 \cos^2(\omega t + \phi) \\
&= \frac{1}{2}\, A^2 k \sin^2(\omega t + \phi) + \frac{1}{2}\, kA^2 \cos^2(\omega t + \phi) \\
&= \frac{1}{2}\, A^2 k(\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)) \\
&= \frac{1}{2}\, kA^2
\end{aligned}$$

Therefore we can use this result,

$$\frac{1}{2}\, mv^2 + \frac{1}{2}\, kx^2 = \frac{1}{2}\, kA^2$$

Solving for our speed, we find,

$$\begin{aligned}
v &= \pm\sqrt{\frac{k}{m}(A^2 - x^2)} \\
v &= \pm\omega\sqrt{(A^2 - x^2)}
\end{aligned}$$

Total Energy of Simple Harmonic Motion
Assuming only conservative forces are acting on the system,

$$E_M = E_K + U = \frac{1}{2}kA^2 \tag{11}$$

The speed at any point on the particles displacement is,

$$v = \pm\omega\sqrt{(A^2 - x^2)} \tag{12}$$

15.4 Comparing Simple Harmonic Motion with Uniform Circular Motion

In Uniform Circular Motion, we are aware that the angular acceleration $\alpha = 0$. From this we know then there is a constant angular velocity which we shall name ω. Let θ be the angular position,

$$\dot{\theta} = \omega$$

We will integrate to find the angular position with respect to time,

$$\int_{\theta_i}^{\theta_f} \mathrm{d}\theta = \int_0^t \omega \, \mathrm{d}t$$
$$\theta_f - \theta_i = \omega t$$

Note that since θ_i is the initial phase angle, which is ϕ. So rearranging,

$$\theta = \omega t + \phi$$

Since we know the x position anywhere along a circular motion of radius A is,

$$x = A\cos(\theta)$$

Plugging in theta, we find that,

$$x(t) = A\cos(\omega t + \phi)$$

And we can therefore take it's time derivative to find the acceleration and velocity. Key idea is that Simple Harmonic Motion and Uniform Circular Motion are the same.

15.5 The Pendulum

The simple pendulum is another mechanical device that exhibits periodic motion. It generally consists of a particle like blob of mass m, at the end of a string of length l, and is fixed at one end. The forces causing the periodic motion is the tension in the string and gravity.

$$F_t = -mg\sin\theta \approx -mg\theta = m\ddot{s} \iff \ddot{s} = -g\theta$$

The distance traveled along the arc is s, we also use the small angle approximation such that $\sin\theta \approx \theta$. Recall,

$$s = l\theta \implies \ddot{s} = l\ddot{\theta} \iff \ddot{\theta} = \frac{\ddot{s}}{l}$$

Therefore, plugging them both together, we get

$$\ddot{\theta} = -\frac{g}{l}\theta$$

We know how to solve such differential equations, let

$$\omega = \sqrt{\frac{g}{l}}$$

We then get,

$$\ddot{\theta} = \omega^2\theta$$

Which we know has the solution,

$$\theta = \theta_{max}\cos(\omega t + \phi)$$

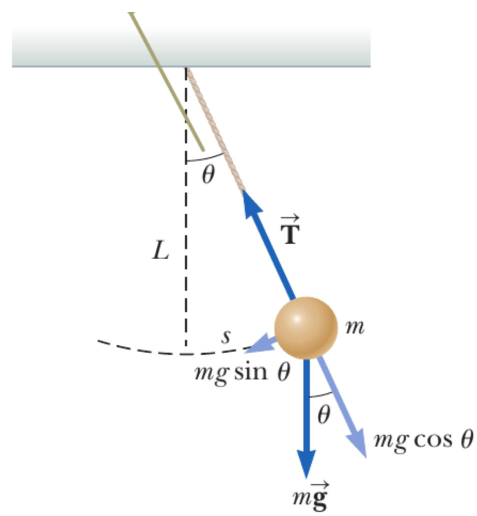

Figure 15.16 A simple pendulum.

A **simple pendulum** considers a point mass at the end of a string, whereas a **physical pendulum** considers the inertial effects of the mass.

Consider an object pivoted at point O, which is at a distance d from the center of mass. The magnitude of torque is $mgd\sin\theta$. Understanding that $\sum \tau_{ext} = I\alpha$.

$$-mgd\sin\theta = I\ddot{\theta}$$

By the small angle approximation,

$$\ddot{\theta} = -\left(\frac{mgd}{I}\right)\theta$$

We can simplify this down to a solvable differential equation,

$$\omega = \sqrt{\frac{mgd}{I}}$$

Therefore,

$$\ddot{\theta} = \omega^2\theta$$

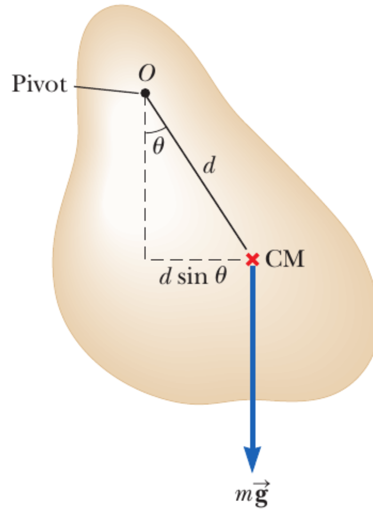

Figure 15.17 A physical pendulum pivoted at O.

Simple and Physical Pendulums

Simple and Physical

$$\theta = \theta_{max}\cos(\omega t + \phi) \tag{13}$$

For simple,

$$\omega = \sqrt{\frac{g}{l}} \tag{14}$$

For Physical,

$$\omega = \sqrt{\frac{mgd}{I}} \tag{15}$$

I is the moment of inertia and specific to the object and its shape.

15.6 Damped Oscillator

When there is a retarding force, we consider a damped oscillator which can be expressed as $\vec{\boldsymbol{R}} = -b\vec{\boldsymbol{v}}$, where b would be the dampening coefficient.

Damped Oscillator

$$x = Ae^{-(b/2m)t}\cos(\omega t + \phi) \tag{16}$$

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} \tag{17}$$

ω_0 is known as the natural frequency, $\omega_0 = \sqrt{k/m}$

15.7 Forced Oscillations

There is not too much to know in this chapter, just understand how to use one equation. When there is a force on an object acting in a sinusoidal way, we can find the amplitude of such a force by using the following equation,

Amplitude of Forced Oscillation

$$A = \frac{F_O/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + \frac{b\,\omega^2}{m}}} \tag{18}$$

where m is mass and ω_0 is,

$$\omega_0 = \sqrt{k/m} \tag{19}$$

Usually you can find ω of the equation of the force given, like $F = Kcos(\omega t + \phi)$, where K is some arbitrary value.