3 Probability

Probability is a function assigned to the events of a sample space and represent our belief in what the outcomes of an experiment are likely to be.

Suppose our sample space S has finite points $\{s_1, s_2, .., s_N\}$. We assign non-negative numbers p_i (probability) to each point in the sample space s_i where $i = 1, 2, .., N$. The following must always be true

$$\sum_{i=1}^{N} p_i = 1 \tag{1}$$

In other words, the sum of the probability for each point in our sample space S has to equal to 1.

We can further idea as follows: define for any event $\mathcal{E}$

$$P(\mathcal{E}) = \sum_{i:s_i \in \mathcal{E}} p_i$$

In words, the equation states that the probability of an event $\mathcal{E}$ is the sum of the probabilities that belong to the points in the sample space which coincide with $\mathcal{E}$. This concept is very simple but the equation above looks very daunting, an example should help.

Example 2.16 There is a sample space S with the sample points $\{a, b, c, d\}$ with respected probabilities $\{0.1, 0.3, 0.5, 0.1\}$. Let there be an event $\mathcal{A} = \{a, b\}$, and let there be an event $\mathcal{B} = \{b, c, d\}$. Find the probability of $\mathcal{A}$ and the probability of $\mathcal{B}$.

Ans. $P(\mathcal{A}) = 0.1 + 0.3 = 0.4$; $P(\mathcal{B}) = 0.3 + 0.5 + 0.1 = 0.9$.
Notice the sum of the probability of all events in the sample space equals to 1.

There is a special case which considers equally likely outcomes corresponds to $p_i = \frac{1}{N}$, where N is the total number of outcomes. If we look at the event $\mathcal{E}$ then it's always the total number of outcomes which produce $\mathcal{E}$ over the total number of outcomes N.

$$P(\mathcal{E}) = \frac{|\mathcal{E}|}{N} \tag{2}$$

Example 2.18 There are 50 components, containing 47 working and 3 non-working. If we sample 6 components, what is the probability of getting exactly 2 defectives
Ans. The total number of outcomes

$$N = \binom{50}{6} = 15890700$$

The number of sample points in our sample space in $\mathcal{E}$

$$|\mathcal{E}| = \binom{3}{2}\binom{47}{4} = 535095$$

Thus the probability of $\mathcal{E}$

$$P(\mathcal{E}) = \frac{|\mathcal{E}|}{N} = \frac{535095}{15890700} = 0.034$$

If we decided to repeat the calculations of Example 2.18 with X number of defective parts in our sample of 6 components, we get the following distribution

X	$P[X]$
0	0.676
1	0.290
2	0.034
3	0.010

Exercise 2.66 Credit cards contain 16 digits. But only 10^6 digits are valid. If you were to pick a 16 digit number at random, what is the probability it is a working credit card number.
Ans. The total number of credit card number can be calculate by utilizing the rule of multiplicity

$$N = 10^6$$

The number of card numbers of 16 digits which actually work is

$$|\mathcal{E}| = 10^8$$

Thus the probability of choosing a valid credit card number is

$$P(\mathcal{E}) = \frac{|\mathcal{E}|}{N} = \frac{10^8}{10^{16}}$$

Exercise 2.68 A message can follow different server paths on a network to get to the same destination. The first server stack consists of 5 servers, the second server stack contains 5 servers, and the third server stack has 4 servers.

1. How many paths are available for a message to take?
   Ans.

$$N = 5 \times 5 \times 4 = 100$$

2. What is the probability the message will go through the first server on the third server stack?
   Ans.

$$P(\mathcal{E}) = \frac{|\mathcal{E}|}{N} = \frac{5 \times 5 \times 1}{100} = \frac{1}{4}$$

3.1 Axioms of Probability

A probability P is assigned to each subset of outcomes (event $\mathcal{E}$) in a collection of subset of outcomes (the sample space S) of a random experiment. P has to satisfy the following assumptions:

1. $P(S) = 1$: The probability for all events in the sample space must be equal to 1
2. $0 \leq P(\mathcal{E}) \leq 1$: The probability for any subset of events in the sample space must be greater than zero and less than or equal to 1
3. $E_1 \cap E_2 = \emptyset \implies P(E_1 \cup E_2) = P(E_1) + P(E_2)$: If two events are **mutually disjoint** (they happen at the same time, but do not influence the other), then the probability of one event or the other occurring is the sum of their individual probabilities

By using the result of axiom 3, we are able to extend it to n events.

If $E_i \cap E_j = \emptyset$ for all $i \neq j$ then

$$P(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) \tag{3}$$

If the number of potential events is infinite, then

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i) \tag{4}$$

If A and B are not mutually exclusive, then we can draw Venn diagrams to show

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{5}$$

The reason we subtract $P(A \cap B)$ is because we counted that specific area twice when adding the probability of A and B

Likewise, for events A, B, C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \tag{6}$$

You can prove this using visual proofs or using a direct proof by elements.

Exercise 2.82 $P(A) = 0.3$, $P(B) = 0.2$, $P(A \cap B) = 0.1$: what is $P(A \cup B)$?

Ans. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.2 + 0.3 - 0.1 = 0.4$

Exercise 2.95 A bar code must have black bars separated by white bars. There are 3 narrow black bars denoted by b, and two wide black bars denoted by B. Likewise, there are 3 narrow white spaces denoted by w, and only 1 wide white space denoted by W(A bar considers only the black, a space considers only the white).

1. What is the probability the first bar is wide or the second bar is wide?

   Ans. Let A be the event the first bar is wide, and let B be the event the second bar is wide.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{\frac{4!}{1!3!} + \frac{4!}{1!3!} - \frac{3!}{3!}}{\frac{5!}{2!3!}} = 0.7 \end{aligned}$$

2. What is the probability the first nor the second bar is wide?

   Ans.

$$P(A' \cap B') = \frac{\frac{3!}{1!2!}}{\frac{5!}{2!3!}} = \frac{3}{10} = 0.3$$

   The easier calculation is simply the complement

$$P(A' \cap B') = 1 - P((A' \cup B')') = 1 - P(A \cap B) = 1 - 0.7 = 0.3$$

3. What is the probability the first bar is wide or the second bar is narrow?

Ans. Let A be the event the first bar is wide, and let B be the event the second bar is narrow.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{\frac{4!}{1!3!} + \frac{4!}{2!2!} - \frac{3!}{1!2!}}{\frac{5!}{2!3!}} = 0.7$$

Example 2.97 Passwords are exactly 8 characters wrong with repetitions allowed from 26 lower case, 26 upper case, and 10 integers. If $A = \{\text{only letters}\}$ and $B = \{\text{ only integers}\}$ Find:

1. $P(A \cup B)$

   Ans.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{52^8}{62^8} + \frac{10^8}{62^8} - 0$$

2. $P(A' \cup B)$

   Ans.

$$P(A' \cup B) = \frac{10^8}{62^8}$$

3. $P(\text{exactly 1 or two integers})$

   Ans. Let A be the event of exactly 1 integer, and let B be the event of exactly 2 integers

$$P(A) = \frac{\binom{8}{1} \times 10 \times 52^7}{62^8}$$

$$P(B) = \frac{\binom{8}{2} \times 10^2 \times 52^6}{62^8}$$

$$P(A \cup B) = P(A) + P(B) = 0.6302$$