

---

# Implementation and Performance Analyses of a Highly Efficient Algorithm for Pressure-Velocity Coupling

---

Masterarbeit | Fabian Gabel, Studienbereich CE

Betreuer: Dipl.-Ing. U. Falk | Prof. Dr. rer. nat. M. Schäfer



TECHNISCHE  
UNIVERSITÄT  
DARMSTADT



**fnb** Fachgebiet  
Numerische Berechnungsverfahren  
im Maschinenbau





---

## Contents

---

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	State of Research and Knowledge . . . . .	1
1.2	Structure of the Thesis . . . . .	2
<b>2</b>	<b>Fundamentals of Continuum Physics for Thermo-Hydrodynamical Problems</b>	<b>4</b>
2.1	Conservation of Mass – Continuity Equation . . . . .	4
2.2	Conservation of Momentum – Cauchy-Equations . . . . .	4
2.3	Closing the System of Equations – Newtonian Fluids . . . . .	5
2.4	Conservation of Scalar Quantities . . . . .	5
2.5	Necessary Simplification of Equations . . . . .	5
2.5.1	Incompressible Flows and Hydrostatic Pressure . . . . .	5
2.5.2	Variation of Fluid Properties – The Boussinesq Approximation . . . . .	6
2.6	Final Form of the Set of Equations . . . . .	7
<b>3</b>	<b>Finite-Volume Methods for Incompressible Flows – Theoretical Basics</b>	<b>8</b>
3.1	Numerical Grid . . . . .	8
3.2	Approximation of Integrals and Derivatives . . . . .	9
3.3	Treatment of Non-Orthogonality of Grid Cells . . . . .	10
3.3.1	Minimum Correction Approach . . . . .	10
3.3.2	Orthogonal Correction Approach . . . . .	10
3.3.3	Over-Relaxed Approach . . . . .	10
3.3.4	Deferred Non-Orthogonal Correction . . . . .	10
3.4	Numerical Solution of Non-Linear Systems – Picard Iteration . . . . .	11
<b>4</b>	<b>Implicit Finite-Volume Method for Incompressible Flows – Segregated Approach</b>	<b>12</b>
4.1	Discretization of the Mass Balance . . . . .	12
4.2	A Pressure-Weighted Interpolation Method for Velocities . . . . .	12
4.3	Implicit Pressure Correction and the SIMPLE-Algorithm . . . . .	15
4.4	Discretization of the Mass Fluxes and the Pressure Correction Equation . . . . .	17
4.5	Discretization of the Momentum Balance . . . . .	18
4.5.1	Linearization and Discretization of the Convective Term . . . . .	18
4.5.2	Discretization of the Diffusive Term . . . . .	19
4.5.3	Discretization of the Source Terms . . . . .	20
4.6	Discretization of the Temperature Equation . . . . .	21
4.7	Boundary Conditions . . . . .	21
4.7.1	Dirichlet Boundary Conditions . . . . .	21
4.7.2	Treatment of Wall Boundaries . . . . .	22
4.7.3	Treatment of Block Boundaries . . . . .	23
4.8	Treatment of the Singularity of the Pressure Correction Equation with Neumann Boundaries . . . . .	24
4.9	Structure of the Assembled Linear Systems . . . . .	25
<b>5</b>	<b>Implicit Finite-Volume Method for Incompressible Flows – Fully Coupled Approach</b>	<b>28</b>
5.1	The Fully Coupled Algorithm – Pressure-Velocity Coupling Reconsidered . . . . .	28
5.2	Coupling to the Temperature Equation . . . . .	29
5.2.1	Decoupled Approach – Explicit Velocity-to-Temperature Coupling . . . . .	29
5.2.2	Implicit Velocity-to-Temperature Coupling . . . . .	30
5.2.3	Temperature-to-Velocity/Pressure Coupling – Newton-Raphson Linearization . . . . .	30
5.3	Boundary Conditions on Domain and Block Boundaries . . . . .	31
5.4	Assembly of Linear Systems – Final Form of Equations . . . . .	32
<b>6</b>	<b>CAFFA Framework</b>	<b>36</b>
6.1	PETSc Framework . . . . .	36
6.2	Grid Generation and Preprocessing . . . . .	36
6.3	Implementation of CAFFA . . . . .	37
6.3.1	The Message-Passing Model . . . . .	37
6.3.2	Convergence Control . . . . .	37

6.3.3	Indexing of Variables and Treatment of Boundary Values . . . . .	38
6.3.4	Domain Decomposition and the Exchange of Ghost Values . . . . .	38
<b>7</b>	<b>Verification of the developed CAFFA Framework . . . . .</b>	<b>40</b>
7.1	The Method of Manufactured Solutions for Navier-Stokes Equations . . . . .	40
7.2	Manufactured Solution for the Navier-Stokes Equations and the Temperature Equation . . . . .	40
7.3	Measurement of Error and Calculation of Order . . . . .	41
7.4	Influence of the Under-Relaxation Factor for the Velocities . . . . .	42
<b>8</b>	<b>Comparison of Solver Concepts . . . . .</b>	<b>45</b>
8.1	Parallel Performance . . . . .	45
8.1.1	Employed Hardware and Software – The Lichtenberg-High Performance Computer . . . . .	45
8.1.2	Measures of Performance . . . . .	45
8.1.3	Preliminary Upper Bounds on Performance – The STREAM Benchmark . . . . .	46
8.1.4	Speed-Up Measurement and Impact of Coupling Algorithm for Analytic Test Case . . . . .	49
8.2	Weak Scaling Comparison of Coupled and Segregated Solution Algorithm . . . . .	50
8.3	Effect of Different Non-Orthogonal Correctore . . . . .	51
8.4	Realistic Testing Scenario – Complex Geometry . . . . .	51
8.5	Classical Benchmarking Case – Temperature-Driven Cavity Flow . . . . .	54
<b>9</b>	<b>Conclusion and Outlook . . . . .</b>	<b>57</b>
<b>10</b>	<b>TESTING ONLY . . . . .</b>	<b>58</b>
10.1	Effect of Different Non-Orthogonal Correctors . . . . .	58
	<b>References . . . . .</b>	<b>60</b>

## List of Figures

1	Vertex centered, cell centered and staggered variable arrangement. For the staggered arrangement, the arrows denote the velocity component and their location on the grid. The centered dot denotes the location of the pressure and further scalar quantities . . . . .	8
2	Block-structured grid consisting of two blocks . . . . .	9
3	Minimum correction, orthogonal correction and over-relaxed approach . . . . .	11
4	Possible interpretation of a virtual control volume (grey) located between nodes $P$ and $Q$ . . . . .	13
5	Non-matching grid cells with hanging nodes at a two-dimensional block boundary. Indexing is based on the face segments $S_l$ . . . . .	24
6	Non-zero structure of the linear systems used in the SIMPLE-algorithm for a block-structured grid consisting of one $2 \times 2 \times 2$ cell and one $3 \times 3 \times 3$ cell block . . . . .	27
7	Non-zero structure of block submatrices of the linear systems used in the coupled solution algorithm for a block-structured grid consisting of one $2 \times 2 \times 2$ cell block and one $3 \times 3 \times 3$ cell block. The blue coefficients represent the pressure-velocity coupling, the red coefficients correspond to the velocity-to-temperature coupling and the green coefficients result from the Newton-Raphson linearization technique. . . . .	33
8	Non-zero structure of the linear system used in the coupled solution algorithm for a block-structured grid consisting of one $2 \times 2 \times 2$ cell block and one $3 \times 3 \times 3$ cell block. The variables have been interlaced and the matrix consists of blocks as shown in figure 7. . . . .	34
9	Non-zero structure the linear system used in the coupled solution algorithm for a block-structured grid consisting of one $2 \times 2 \times 2$ cell block and one $3 \times 3 \times 3$ cell block. The variables have been ordered such that each major matrix block refers to only one variable or the coupling between exactly two variables. . . . .	35
10	Storage and update of ghost values in vectors related to variables on multi block domains. The blocks have been assigned to two different processes, <i>Proc 1</i> and <i>Proc 2</i> . The control volumes of the two-dimensional problem domain are indexed with respect to the process local indexing. . . . .	39
11	Comparison of calculated error for different under-relaxation factors $\alpha_u$ on a grid with different grid resolutions unknowns . . . . .	44
12	Sustainable memory bandwidth as determined by the STREAM benchmark (Triad) for different process binding options on one node of the MPI1 section . . . . .	47
13	Sustainable memory bandwidth as determined by the STREAM benchmark (Triad) for different process binding options on one node of the MPI2 section . . . . .	48

14	Wall-clock time comparison for the SIMPLE-algorithm solving for an analytical solution on a grid with $64^3$ cells using different under-relaxation factors . . . . .	49
15	Wall-clock time and Speed-Up comparison for segregated and fully coupled solution algorithm solving for an analytical solution on a grid with $128^3$ cells . . . . .	50
16	Comparison of the number of outer iterations and the corresponding wall-clock time, needed to achieve a reduction of $1E-8$ of the initial residual, for different methods to resolve pressure-velocity coupling . . . . .	51
17	Average number of inner iterations for different process counts solving for an analytical solution with the fully coupled solution algorithm . . . . .	51
18	Sketch of the channel flow problem domain . . . . .	52
19	West and east boundary of the numerical grid for the channel flow problem . . . . .	53
20	Blocking for the two different obstacles within the problem domain of the channel flow . . . . .	53
21	Temperature field and velocity field in the coordinate direction of the gravitational force of the temperature-driven cavity flow problem for a cross section in the middle of the problem domain . . . . .	54
22	Skewing of an initially equidistant structured orthogonal grid via random movement of inner grid points within a small neighborhood of their original location. The neighborhood with maximal diameter is indicated by a circle around an inner grid point . . . . .	58
23	Number of needed outer iterations for different relative diameters of the neighborhood in which the movement of grid points takes place parametrized by the orthogonal corrector . . . . .	59

---

## List of Tables

---

1	Comparison of the errors of the velocity calculated by the segregated and the coupled solver for different grid resolutions and the resulting order of accuracy . . . . .	43
2	Comparison of the errors of the pressure calculated by the segregated and the coupled solver for different grid resolutions and the resulting order of accuracy . . . . .	43
3	Comparison of the errors of the temperature calculated by the segregated and the coupled solver for different grid resolutions and the resulting order of accuracy . . . . .	43
4	Characteristic problem properties used in the performance measurements solving for an analytic solution .	50
5	Characteristic problem properties used in the channel flow test case . . . . .	52
6	Performance analysis results of the channel flow problem for different numbers of unknowns comparing the segregated (SEG) to the fully coupled (CPLD) solution algorithm using one process on the MPI2 section of the HHLR supercomputer. . . . .	54
7	Characteristic problem properties used in the heated cavity flow test case . . . . .	55
8	Performance analysis results of the temperature-driven cavity flow problem comparing the SIMPLE-algorithm with segregated temperature solve (SEG), the fully coupled solution algorithm with segregated temperature solve (CPLD), the fully coupled solution algorithm with an implicit Boussinesq approximation (TCPLD) and the fully coupled solution algorithm using an implicit Boussinesq approximation and a semi-implicit Newton-Raphson linearization of the convective part of the temperature equation (NRCPLD). .	55

---

## List of Algorithms

---

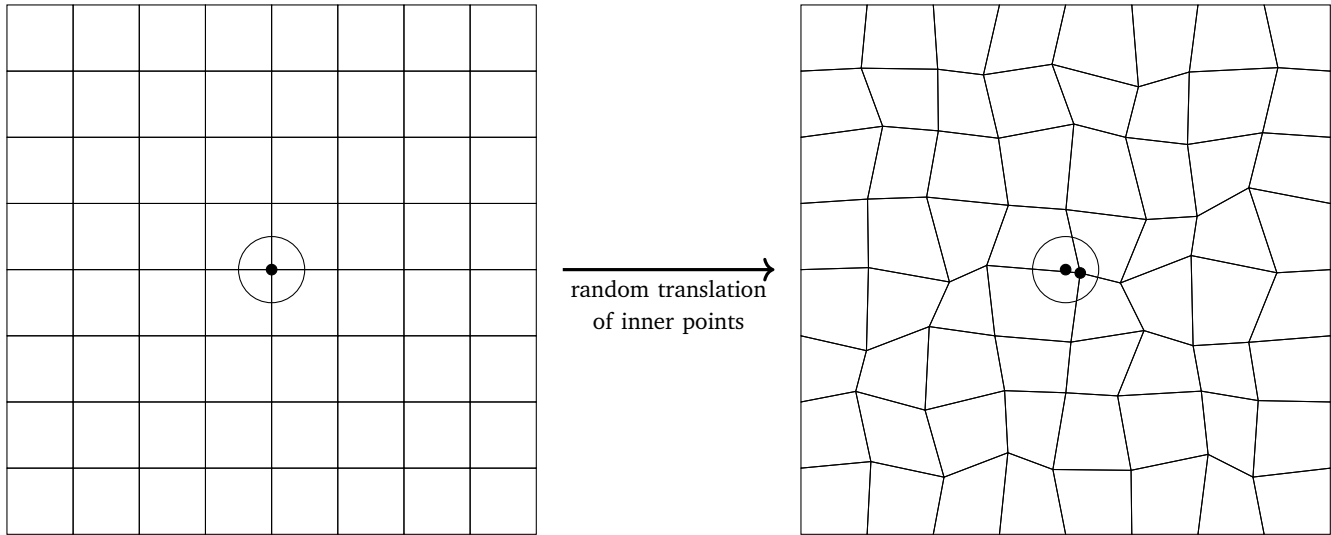
1	SIMPLE-Algorithm . . . . .	17
2	Fully Coupled Solution Algorithm . . . . .	29



## 10 TESTING ONLY

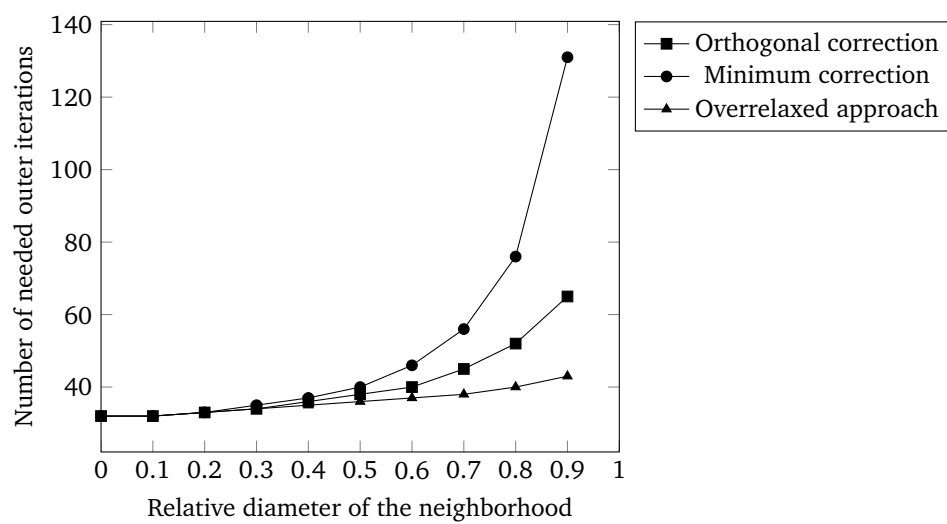
### 10.1 Effect of Different Non-Orthogonal Correctors

Section 3.3 introduced different ways to address the degradation of the grid quality due to non-orthogonality of the grid. This section compares the three different correctors, the orthogonal correction, the minimum correction and the overrelaxed approach, which are used in the discretization process of the gradients at cell boundary faces. For this purpose the grid generator program of section ?? was extended to generate skewed grids. For this purpose a random number generator was used to randomly move each inner grid point within a specified neighborhood of the original location. Figure 22 illustrates the concept for a two-dimensional 8 grid. To maintain the grid's integrity after the movement of the grid points the neighborhoods were limited by half the distance  $\Delta x$  of two neighboring grid points.



**Figure 22:** Skewing of an initially equidistant structured orthogonal grid via random movement of inner grid points within a small neighborhood of their original location. The neighborhood with maximal diameter is indicated by a circle around an inner grid point

To measure the effect of different non-orthogonal corrections on the solution process, tests for different skewed grids were performed, measuring the number of needed outer iterations. The grid skewness was parametrized with the relative diameter  $\alpha$  of the maximal neighborhood, which constrains the movement of grid points.  $\alpha \in [0, 1)$ , where  $\alpha = 0$  corresponds to no movement of grid points at all and  $\alpha = 1$  would move grid points up to  $\frac{\Delta x}{2}$  away from their original location. The choice of  $\alpha = 1$  is not permitted to maintain the grid's integrity. Such a choice would permit the grid generator to place two grid points at the same location.



**Figure 23:** Number of needed outer iterations for different relative diameters of the neighborhood in which the movement of grid points takes place parametrized by the orthogonal corrector



---

## References

---

- [1] ACHARYA, S., BALIGA, B. R., KARKI, K., MURTHY, J. Y., PRAKASH, C., AND VANKA, S. P. Pressure-based finite-volume methods in computational fluid dynamics. *Journal of Heat Transfer* 129, 4 (Jan 2007), 407–424.
- [2] ANDERSON, D. A., TANNEHILL, J. C., AND PLETCHER, R. H. *Computational Fluid Mechanics and Heat Transfer*. Hemisphere Publishing Corporation, Washington, 1984.
- [3] ARIS, R. *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. Prentice-Hall, Inc., 1962.
- [4] BALAY, S., ABHYANKAR, S., ADAMS, M. F., BROWN, J., BRUNE, P., BUSCHELMAN, K., EIJKHOUT, V., GROPP, W. D., KAUSHIK, D., KNEPLEY, M. G., MCINNES, L. C., RUPP, K., SMITH, B. F., AND ZHANG, H. PETSc users manual. Tech. Rep. ANL-95/11 - Revision 3.5, Argonne National Laboratory, 2014.
- [5] BALAY, S., ABHYANKAR, S., ADAMS, M. F., BROWN, J., BRUNE, P., BUSCHELMAN, K., EIJKHOUT, V., GROPP, W. D., KAUSHIK, D., KNEPLEY, M. G., MCINNES, L. C., RUPP, K., SMITH, B. F., AND ZHANG, H. PETSc Web page. <http://www.mcs.anl.gov/petsc>, 2014.
- [6] BALAY, S., GROPP, W. D., MCINNES, L. C., AND SMITH, B. F. Efficient management of parallelism in object oriented numerical software libraries. In *Modern Software Tools in Scientific Computing* (1997), E. Arge, A. M. Bruaset, and H. P. Langtangen, Eds., Birkhäuser Press, pp. 163–202.
- [7] BONFIGLIOLI, A., CAMPOBASSO, S., CARPENTIERI, B., AND BOLLHÖFER, M. A parallel 3d unstructured implicit rans solver for compressible and incompressible cfd simulations. In *Parallel Processing and Applied Mathematics*, R. Wyrzykowski, J. Dongarra, K. Karczewski, and J. Waśniewski, Eds., vol. 7204 of *Lecture Notes in Computer Science*. Springer Berlin Heidelberg, 2012, pp. 313–322.
- [8] BROWN, J., KNEPLEY, M. G., MAY, D. A., MCINNES, L. C., AND SMITH, B. Composable linear solvers for multiphysics. In *Proceedings of the 2012 11th International Symposium on Parallel and Distributed Computing* (Washington, DC, USA, 2012), ISPDC '12, IEEE Computer Society, pp. 55–62.
- [9] CHEN, Z., AND PRZEKWAŚ, A. A coupled pressure-based computational method for incompressible/compressible flows. *Journal of Computational Physics* 229, 24 (2010), 9150 – 9165.
- [10] CHOI, S. K. Note on the use of momentum interpolation method for unsteady flows. *Numerical Heat Transfer, Part A: Applications* 36, 5 (1999), 545–550.
- [11] CHRISTON, M. A., GRESHO, P. M., AND SUTTON, S. B. Computational predictability of time-dependent natural convection flows in enclosures (including a benchmark solution). *International Journal for Numerical Methods in Fluids* 40, 8 (2002), 953–980.
- [12] DARWISH, M., SRAJ, I., AND MOUKALLED, F. A coupled finite volume solver for the solution of incompressible flows on unstructured grids. *Journal of Computational Physics* 228, 1 (2009), 180 – 201.
- [13] DARWISH, F. MOUKALLED, M. A unified formulation of the segregated class of algorithms for fluid flow at all speeds. *Numerical Heat Transfer, Part B: Fundamentals* 37, 1 (2000), 103–139.
- [14] DE VAHL DAVIS, G. Natural convection of air in a square cavity: A bench mark numerical solution. *International Journal for Numerical Methods in Fluids* 3, 3 (1983), 249–264.
- [15] ELMAN, H., HOWLE, V. E., SHADID, J., SHUTTLEWORTH, R., AND TUMINARO, R. A taxonomy and comparison of parallel block multi-level preconditioners for the incompressible navier-stokes equations. *J. Comput. Phys.* 227, 3 (Jan. 2008), 1790–1808.
- [16] ELMAN, H., HOWLE, V. E., SHADID, J., AND TUMINARO, R. A parallel block multi-level preconditioner for the 3d incompressible navier-stokes equations. *Journal of Computational Physics* 187 (2003), 504–523.
- [17] FALGOUT, R., AND YANG, U. hypre: A library of high performance preconditioners. In *Computational Science — ICCS 2002*, P. Sloot, A. Hoekstra, C. Tan, and J. Dongarra, Eds., vol. 2331 of *Lecture Notes in Computer Science*. Springer Berlin Heidelberg, 2002, pp. 632–641.
- [18] FALK, U., AND SCHÄFER, M. A fully coupled finite volume solver for the solution of incompressible flows on locally refined non-matching block-structured grids. In *Adaptive Modeling and Simulation 2013* (Barcelona, Spain, June 2013), J. P. M. de Almeida, P. Diez, C. Tiago, and N. Parež, Eds., pp. 235–246.

- 
- [19] FERZIGER, J. H., AND PERIĆ, M. *Numerische Strömungsmechanik*. Springer Verlag, Berlin, 2002.
- [20] GALPIN, P. F., AND RAITHY, G. D. Numerical solution of problems in incompressible fluid flow: Treatment of the temperature-velocity coupling. *Numerical Heat Transfer* 10, 2 (1986), 105–129.
- [21] GEE, M., SIEFERT, C., HU, J., TUMINARO, R., AND SALA, M. ML 5.0 smoothed aggregation user’s guide. Tech. Rep. SAND2006-2649, Sandia National Laboratories, 2006.
- [22] GRAY, D. D., AND GIORGINI, A. The validity of the boussinesq approximation for liquids and gases. *International Journal of Heat and Mass Transfer* 19, 5 (1976), 545 – 551.
- [23] GROPP, W., LUSK, E., AND SKJELLUM, A. *Using MPI: portable parallel programming with the message-passing interface*, 2. ed. The MIT Press, Cambridge, Massachusetts, 1999.
- [24] GROPP, W. D., KAUSHIK, D. K., KEYES, D. E., AND SMITH, B. F. High performance parallel implicit cfd. *Parallel Computing* 27 (2000), 337–362.
- [25] HACKBUSCH, W. *Theorie und Numerik elliptischer Differentialgleichungen mit Beispielen und Übungsaufgaben*. Teubner-Studienbücher Mathematik. Teubner, Stuttgart, 1996.
- [26] HAGER, G., AND WELLEIN, G. *Introduction to High Performance Computing for Scientists and Engineers*. CRC, Boca Raton, 2011.
- [27] HENDERSON, A. Paraview guide, a parallel visualization application, 2007.
- [28] JASAK, H. *Error Analysis and Estimation for the Finite Volume Method with Applications to Fluid Flows*. PhD thesis, Imperial College of Science, Technology and Medicine, Jun 1996.
- [29] KARIMIAN, S. M., AND STRAATMAN, A. G. Benchmarking of a 3d, unstructured, finite volume code of incompressible navier-stokes equation on a cluster of distributed-memory computers. *High Performance Computing Systems and Applications, Annual International Symposium on O* (2005), 11–16.
- [30] KLAIJ, C. M., AND VUIK, C. Simple-type preconditioners for cell-centered, colocated finite volume discretization of incompressible reynolds-averaged navier-stokes equations. *International Journal for Numerical Methods in Fluids* 71, 7 (2013), 830–849.
- [31] KUNDU, P. K., COHEN, I. M., AND DOWNLING, D. R. *Fluid Mechanics*, 5 ed. Elsevier, 2012.
- [32] LANGE, C. F., SCHÄFER, M., AND DURST, F. Local block refinement with a multigrid flow solver. *International Journal for Numerical Methods in Fluids* 38, 1 (2002), 21–41.
- [33] LILEK, U., MUZAFERIJA, S., PERIĆ, M., AND SEIDL, V. An implicit finite-volume method using nonmatching blocks of structured grid. *Numerical Heat Transfer, Part B: Fundamentals* 32, 4 (1997), 385–401.
- [34] LIU, X., TAO, W., AND HE, Y. A simple method for improving the simpler algorithm for numerical simulations of incompressible fluid flow and heat transfer problems. *Engineering Computations* 22, 8 (2005), 921–939.
- [35] MAJUMDAR, S. Role of underrelaxation in momentum interpolation for calculation of flow with nonstaggered grids. *Numerical Heat Transfer* 13, 1 (1988), 125–132.
- [36] MANGANI, L., BUCHMAYR, M., AND DARWISH, M. Development of a novel fully coupled solver in openfoam: Steady-state incompressible turbulent flows in rotational reference frames. *Numerical Heat Transfer, Part B: Fundamentals* 66, 6 (2014), 526–543.
- [37] MAPLE. *version 18*. Waterloo Maple Inc. (Maplesoft), Waterloo, Ontario, 2014.
- [38] MATLAB. *version 8.4.150421 (R2014b)*. The MathWorks Inc., Natick, Massachusetts, 2014.
- [39] MCCALPIN, J. D. Stream: Sustainable memory bandwidth in high performance computers. Tech. rep., University of Virginia, Charlottesville, Virginia, 1991-2007. A continually updated technical report. <http://www.cs.virginia.edu/stream/>.
- [40] MCCALPIN, J. D. Memory bandwidth and machine balance in current high performance computers. *IEEE Computer Society Technical Committee on Computer Architecture (TCCA) Newsletter* (Dec. 1995), 19–25.

- 
- [41] MCINNES, L. C., SMITH, B., ZHANG, H., AND MILLS, R. T. Hierarchical krylov and nested krylov methods for extreme-scale computing. *Parallel Comput.* 40, 1 (Jan. 2014), 17–31.
- [42] MILLER, T. F., AND SCHMIDT, F. W. Use of a pressure-weighted interpolation method for the solution of the incompressible navier-stokes equations on a nonstaggered grid system. *Numerical Heat Transfer* 14, 2 (1988), 213–233.
- [43] MUZAFERIJA, S. *Adaptive finite volume method for flow predictions using unstructured meshes and multigrid approach*. PhD thesis, University of London, 1994.
- [44] OBERKAMPE, W. L., AND TRUCANO, T. G. Verification and validation in computational fluid dynamics. *Progress in Aerospace Sciences* 38, 3 (2002), 209 – 272.
- [45] OLIVEIRA, P. J., AND ISSA, R. I. An improved piso algorithm for the computation of buoyancy-driven flows. *Numerical Heat Transfer, Part B: Fundamentals* 40, 6 (2001), 473–493.
- [46] PATANKAR, S., AND SPALDING, D. A calculation procedure for heat, mass and momentum transfer in three-dimensional parabolic flows. *International Journal of Heat and Mass Transfer* 15, 10 (1972), 1787 – 1806.
- [47] PERIĆ, M. Analysis of pressure-velocity coupling on nonorthogonal grids. *Numerical Heat Transfer* 17 (Jan. 1990), 63–82.
- [48] POPE, S. B. *Turbulent Flows*. Cambridge University Press, New York, 2000.
- [49] R., H. M. Methods of conjugate gradients for solving linear systems. *J. Res. Nat. Bur. Standards* 49 (1952), 409.
- [50] RHIE, C. M., AND CHOW, W. L. Numerical study of the turbulent flow past an airfoil with trailing edge separation. *AIAA Journal* 21 (Nov. 1983), 1525–1532.
- [51] SAAD, Y. *Iterative methods for sparse linear systems*. SIAM, 2003.
- [52] SAAD, Y., AND SCHULTZ, M. H. Gmres: A generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Stat. Comput.* 7, 3 (July 1986), 856–869.
- [53] SALARI, K., AND KNUPP, P. Code verification by the method of manufactured solutions. Tech. Rep. SAND2000-1444, Sandia National Labs., Albuquerque, NM (US); Sandia National Labs., Livermore, CA (US), Jun 2000.
- [54] SCHÄFER, M. *Numerik im Maschinenbau*. Springer Verlag, Berlin, 1999.
- [55] SHEU, T. W. H., AND LIN, R. K. Newton linearization of the incompressible navier–stokes equations. *International Journal for Numerical Methods in Fluids* 44, 3 (2004), 297–312.
- [56] SILVESTER, D., ELMAN, H., KAY, D., AND WATHEN, A. Efficient preconditioning of the linearized navier–stokes equations for incompressible flow. *Journal of Computational and Applied Mathematics* 128, 1–2 (2001), 261 – 279. Numerical Analysis 2000. Vol. VII: Partial Differential Equations.
- [57] SPURK, J. H., AND AKSEL, N. *Strömungslehre: Einführung in die Theorie der Strömungen*, 8. ed. Springer Verlag, Berlin, 2010.
- [58] TAYLOR, G. I., AND GREEN, A. E. Mechanism of the production of small eddies from large ones. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 158, 895 (1937), pp. 499–521.
- [59] TUREK, S., AND SCHMACHTEL, R. Fully coupled and operator-splitting approaches for natural convection flows in enclosures. *International Journal for Numerical Methods in Fluids* 40, 8 (2002), 1109–1119.
- [60] VAKILIPOUR, S., AND ORMISTON, S. J. A coupled pressure-based co-located finite-volume solution method for natural-convection flows. *Numerical Heat Transfer, Part B: Fundamentals* 61, 2 (2012), 91–115.
- [61] VAN DOORMAAL, J. P., AND RAITHEY, G. D. Enhancements of the simple method for predicting incompressible fluid flows. *Numerical Heat Transfer* 7, 2 (1984), 147–163.
- [62] ZHANG, S., ZHAO, X., AND BAYYUK, S. Generalized formulations for the rhie–chow interpolation. *Journal of Computational Physics* 258, 0 (2014), 880 – 914.