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Masterthesis

On Resolvent Estimates in L^p for the Stokes Operator in Lipschitz Domains

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Introduction

Chapter 1

Fundamentals

The purpose of this chapter is to collect basic definitions that will be used throughout the subsequent chapters. Furthermore, we want to formulate the main problem regarding the resolvent estimates of the Stokes operator. Throughout this chapter we let d always denote a natural number greater or equal to 2.

1.1 Lipschitz-Domains

In this first section we will establish the fundamental notions regarding bounded Lipschitz domains.

Definition 1.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded open connected set. We call Ω a *bounded Lipschitz domain* if there exist $r_0, M > 0$ such that for all $x \in \partial\Omega$ there exists a function $\eta_x \colon \mathbb{R}^{d-1} \to \mathbb{R}$ which is Lipschitz continuous and fulfills $\eta_x(0) = 0$ and $\|\nabla \eta_x\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} \leq M$, and a rotation $R_x \colon \mathbb{R}^d \to \mathbb{R}^d$ such that for all $0 < r \leq r_0$

$$R_x[\Omega - \{x\}] \cap D(r) = D_{\eta_x}(r)$$
$$R_x[\partial\Omega - \{x\}] \cap D(r) = I_{\eta_x}(r),$$

where

$$D(r) \coloneqq \{(x', x_d) \colon |x'| < r, |x_d| < 10d(M+1)r\}$$
$$D_{\eta_x}(r) \coloneqq \{(x', x_d) \colon |x'| < r, \eta_x(x') < x_d < 10d(M+1)r\}$$
$$I_{\eta_x}(r) \coloneqq \{(x', x_d) \colon |x'| < r, \eta_x(x') = x_d\}.$$

It is common to refer to sets of the form D_{η_x} as *Lipschitz cylinders*. If the point x in the definition of Lipschitz cylinders is not of particular importance we will denote the Lipschitz cylinder by $D_\eta(r)$.

If Ω is a bounded Lipschitz domain, $x \in \partial\Omega$ and $0 < r \leq r_0$, then we may define $U_{x,r} := \{x\} + R_x^{-1} D(r)$, where R_x is the rotation corresponding to x from Definition 1.1. This is all we need to define the Lipschitz character of a bounded Lipschitz domain Ω as suggested by Pipher and Verchota in [24, Sec. 5].

Definition 1.2. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and $x_1, \ldots, x_n \in \partial\Omega$ be such that $\{U_{x_i,r_0}\}_{i=1}^n$ covers $\partial\Omega$. Furthermore, let M be the constant from Definition 1.1. Then a constant $C > 0$ is said to depend on the *Lipschitz character of* Ω if it depends on M and n.

That the Lipschitz character is indeed a fruitful concept will be emphasized by the following theorem. This result is a crucial ingredient in the proof of the Rellich estimates in Chapter 4 as it provides a useful approximating property of Lipschitz domains. In short it enables us to approximate a bounded Lipschitz domain Ω by a sequence (Ω_j) of C^∞ domains in such a way that estimates on Ω_j with bounding constants that only depend on the Lipschitz characters may be extended to Ω when taking the limit. The original proof of this Theorem goes back to Nečas [23] and Verchota [34] . The presented version of this theorem appeared in Brown [2].

Theorem 1.3 (Nečas, Verchota)**.** *Let Ω be a Lipschitz domain. Then there exists a sequence of C^∞-domains (Ω_k) with uniform Lipschitz characters, corresponding homeomorphisms $\Lambda_k : \partial\Omega \to \partial\Omega_k$, functions $\vartheta_k : \partial\Omega \to \mathbb{R}^+$ and a smooth compactly supported vector field $h : \mathbb{R}^d \to \mathbb{R}^d$ which satisfy the following properties:*

i) There exists a covering of $\partial\Omega$ by coordinate cylinders which also serve as coordinate cylinders for $\partial\Omega_k$.

ii) The homeomorhpisms $\Lambda_k : \partial\Omega \to \partial\Omega_k$ satisfy

$$\sup_{Q\in\partial\Omega} |Q - \Lambda_k(Q)| \to 0 \quad \text{as } k \to \infty$$

and $\Lambda_k(P)$ approaches P nontangentially meaning that for all $k \in \mathbb{N}$

$$|P - \Lambda_k(P)| < (1 + \beta)\,\mathrm{dist}(\Lambda_k(P), \partial\Omega)$$

for some constant β depending only on d and the Lipschitz character of Ω.

iii) The normals ν_k of $\partial\Omega_k$ satisfy $\lim_{k\to\infty} \nu_k(\Lambda_k(P)) = \nu(P)$ a.e. for all $P \in \partial\Omega$

iv) The functions ϑ_k *satisfy* $\delta \le \vartheta_k \le \delta^{-1}$ *for some* $\delta > 0$, $\vartheta^k \to 1$ *pointwise a.e. and*

$$\int_E \vartheta_k(Q)\,\mathrm{d}\sigma(Q) = \int_{\Lambda_k(E)} \mathrm{d}\sigma_k(Q),$$

where $E \subset \partial\Omega$ *is measurable and* σ_k *denotes the surface measure on* Ω_k.

v) The vector field h *satisfies* $\langle h, \nu_k\rangle \ge c > 0$ *a.e. on each* $\partial\Omega_k$ *where* ν_k *denotes the unit inner normal to* $\partial\Omega_k$.

The next concept we introduce will allow us to talk about boundary values of functions which are defined on Ω by considering their nontangential behavior. The first step will be to define nontangential approach regions. Unfortunately, in the literature there exist at least two different concepts which will be introduced in the next definitions. In the following, by a cone we mean an open, circular, truncated cone with only one convex component.

Definition 1.4 (Regular family of cones, Verchota)**.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. If $q \in \partial\Omega$, then $\Gamma(q)$ will denote a cone with vertex q and one component in Ω. Assigning to each $q \in \partial\Omega$ one cone $\Gamma(q)$ the family $\{\Gamma(q)\colon q \in \partial\Omega\}$ will be called *regular* if there exist $x_1, \dots, x_{n_0} \in \partial\Omega$, $\tilde{r} > 0$ and rotations $\tilde{R}_{x_1}, \dots, \tilde{R}_{x_{n_0}}$ such that

$$\partial\Omega \subset \bigcup_{i=1}^{n_0} \{x_i\} + \tilde{R}_{x_i}^{-1} D(4\tilde{r}/5),$$

and such that there exist Lipschitz continuous functions $\tilde{\eta}_{x_i}\colon \mathbb{R}^{d-1} \to \mathbb{R}$ such that for all $\tilde{r} \le r \le \nu\tilde{r}$ with

$$\nu := 1 + [1 + [10d(M+1)]^2]^{1/2}$$

we have

$$\tilde{R}_{x_i}[\Omega - \{x_i\}] \cap D(r) = D_{\tilde{\eta}_{x_i}}(r)$$
$$\tilde{R}_{x_i}[\partial\Omega - \{x_i\}] \cap D(r) = I_{\tilde{\eta}_{x_i}}(r).$$

In addition for all i there exist cones α_i, β_i and γ_i with vertex at the origin and axis along the x_d-axis such that

$$\alpha_i \subset \overline{\beta_i} \setminus \{0\} \subset \gamma_i$$

and such that for all $q \in [\{x_i\} + \tilde{R}_{x_i}^{-1} D(4\tilde{r}/5)] \cap \partial\Omega$, we have

$$\tilde{R}_{x_i}^{-1}\alpha_i + \{q\} \subset \Gamma(q) \subset \overline{\Gamma(q)} \setminus \{q\} \subset \tilde{R}_{x_i}^{-1}\beta_i + \{q\},$$
$$\tilde{R}_{x_i}^{-1}\gamma_i + \{q\} \subset [\{x_i\} + \tilde{R}_{x_i}^{-1} D(\tilde{r})] \cap \Omega.$$

We will sometimes denote a regular cone as above by $\Gamma_{\mathrm{V}}(q)$.

For the existence of such families of cones see the Appendix of Verchota [34].

In Verchota cones $\Gamma_{\mathrm{V}}(q)$ we have the properties that for all Ω there exists a constant $C > 0$ depending only on the Lipschitz character such that for all q, $p \in \partial\Omega$ and any $x \in \Gamma_V(p)$ we have that

$$|x - q| \geq C|x - p| \tag{1.1}$$

$$|x - q| \geq C|p - q|. \tag{1.2}$$

For a proof see Verchota [34, p. 9f.]

Definition 1.5 (Nontangential approach region, Shen)**.** For $\alpha > 1$ and $q \in \partial\Omega$ we define

$$\Gamma_\alpha(q) := \left\{x \in \Omega \setminus \partial\Omega \colon |x - q| < \alpha \operatorname{dist}(x, \partial\Omega)\right\}$$

If α is chosen sufficiently large (see Shen [27]) we call $\{\Gamma_\alpha(q) \colon q \in \partial\Omega\}$ a *family of nontangential approach regions*.

Note that in Shen cones $\Gamma_\alpha(q)$, we have that for $q, y \in \partial\Omega$ and $x \in \Gamma_\alpha(q)$

$$\begin{aligned} |q - y| &\leq |q - x| + |x - y| \leq \alpha \operatorname{dist}(x, \partial\Omega) + |x - y| \\ &\leq (\alpha + 1)|x - y| \end{aligned} \tag{1.3}$$

where α is the constant from Definition 1.5.

Depending on the type of cones used one may introduce similar concepts of nontangential convergence and nontangential maximal functions.

Definition 1.6. For a function u in Ω and a fixed family of nontangential approach regions $\{\Gamma_\alpha\}$, we define the nontangential maximal function $(u)^*_\alpha$ by

$$(u)^*_\alpha(q) = \sup\left\{|u(x)| \colon x \in \Gamma_\alpha(q)\right\} \tag{1.4}$$

for $q \in \partial\Omega$. For a fixed regular family of cones $\{\Gamma_{\mathrm{V}}(q)\}$ we define the nontangential maximal function $N(u)(q)$ via

$$N(u)(q) = \sup\left\{|u(x)| \colon x \in \Gamma_{\mathrm{V}}(q)\right\}.$$

Note that Tolksdorf [32] and Shen [27] show that the choice of α for the nontangential maximal function as in 1.4 does not affect their p-norms in an unpredictable way. In fact their p-norms for different α_1 and α_2 stay comparable with a constant only depending on d, α_1, α_2 and the Lipschitz character. We will therefore for a given bounded Lipschitz domain always assume that $\alpha > 1$ has been choosen big enough such that on the one hand condition (ii) from Theorem 1.3 is fulfilled and that on the other hand α is large enough such that $\{\Gamma_\alpha(q)\colon q \in \partial\Omega\}$ is a family of nontangential aproach regions. In the following we will thus ignore the parameter α in cones and nontangential maximal functions and tacitly assume that it was chosen appropriately. We further note that the functions $(u)^*$ and $N(u)$ will not be comparable in general, see the discussion in Tolksdorf.

The above mentioned constructions of cones are not limited to cones that lay in the interior of the domain Ω. In fact the same construction can be carried out for the exterior domain $\mathbb{R}^d \setminus \overline{\Omega}$ yielding cones that lay outside of Ω. While Verchota's cones from Definition 1.4 can be mirrored along the $x_d = 0$ plane in a suitable local coordinate system, Shen's cones from Definition 1.5 have to be modified in a natural way to give cones lying inside of $\mathbb{R}^d \setminus \overline{\Omega}$, namely

$$\Gamma_\alpha^{\text{ext}}(q) \coloneqq \left\{x \in \mathbb{R}^d \setminus \overline{\Omega}\colon |x - q| < \alpha \operatorname{dist}(x, \partial\Omega)\right\}.$$

As the name *nontangential approach region* suggests, for functions u living on Ω or $\mathbb{R}^d \setminus \overline{\Omega}$ there will be a notion of convergence of function values $u(x)$ as x goes to a point on $p \in \partial\Omega$. The idea is to restrict the set of directions from which one can approach p by only allowing sequences of points lying in cones $\Gamma(q)$.

Definition 1.7 (Nontangential convergence)**.** Let Ω be a bounded Lipschitz domain and $\{\Gamma(q)\colon q \in \partial\Omega\}$ be a family of nontangential approach regions with its exterior counterpart $\{\Gamma^{\text{ext}}(q)\colon q \in \partial\Omega\}$. Let furthermore u be a function on $\mathbb{R}^d \setminus \partial\Omega$ and f a function on $\partial\Omega$. We say that $u = f$ *in the sense of nontangential convergence from the inside* if

$$\lim_{\substack{x \to q \\ x \in \Gamma(q)}} u(x) = f(q), \quad \text{for a.e. } q \in \partial\Omega$$

and we say that $u = f$ *in the sense of nontangential convergence from the outside* if

$$\lim_{\substack{x \to q \\ x \in \Gamma^{\text{ext}}(q)}} u(x) = f(q), \quad \text{for a.e. } q \in \partial\Omega.$$

If both of the above limits exist and coincide we say that $u = f$ *in the sense of nontangential convergence*.

Usually the nontangential limits taken from inside and outside the domain will differ. For functions u on $\mathbb{R}^d \setminus \partial\Omega$ we will therefore often use the notation u_+ to denote the *inner* nontangential limit and u_- for the respective *outer* nontangential limit.

To put our new vocabulary to use, we will formulate an prove the divergence theorem for functions on bounded Lipschitz domains that do not have a trace but nontangential limits and integrable nontangential maximal functions. A similar statement was proven by Shen in [27, Thm. 7.1.6].

Proposition 1.8. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, a bounded Lipschitz domain and $f\colon \Omega \to \mathbb{C}^d$ smooth and $g\colon \partial\Omega \to \mathbb{C}$ measurable. Suppose that the nontangential limit f_+ exists almost everywhere and that the nontangential maximal function $(g)^*$ is integrable on $\partial\Omega$ and $|f_+| \leq (g)^*$ a.e.. Then Green's formula*

$$\int_{\partial\Omega} f_k(s)n_k(s)\,\mathrm{d}\sigma(s) = \int_{\Omega} \operatorname{div}(f)(x)\,\mathrm{d}x \tag{1.5}$$

holds, where n denotes the outer unit normal vector of $\partial\Omega$.

Proof. The proof rests heavily on the powerful Theorem 1.3 and uses its full capacity to uncover a very useful approximation argument.

Let's start by approximating Ω by a sequence $(\Omega)_l$ of C^∞ domains with uniform Lipschitz characters as described in Theorem 1.3. Remember that by Theorem 1.3 iv), the homeomorphisms $\Lambda_l\colon \partial\Omega \to \partial\Omega_l$ give rise to a tranformation rule of the form

$$\int_{\partial\Omega_l} f_k(s)n_k^{(l)}(s)\,\mathrm{d}\sigma_l(s) = \int_{\partial\Omega} \vartheta_l(x)\, f_k(\Lambda_l(x))\, n_k^{(l)}(\Lambda_l(x))\,\mathrm{d}\sigma(x). \tag{1.6}$$

The idea of the proof is based on the approximation argument performed in Brown [2, Prop. 2.4]. Additionally we have $\lim_{l\to\infty} \vartheta_l(x) = 1$ and $\lim_{l\to\infty} \Lambda_l(x) = x$ almost everywhere, where $\Lambda_l(x) \in \Gamma(x)$ for all $l \in \mathbb{N}$ thanks to Theorem 1.3 ii). Furthermore, we know that $\lim_{l\to\infty} n_k^{(l)}(\Lambda_l(x)) = n_k(x)$ almost everywhere by Theorem 1.3 and that f has a nontangential limit almost everywhere. This gives us that

$$\lim_{l\to\infty} \vartheta_l(x)\, f_k(\Lambda_l(x))\, n_k^{(l)}(\Lambda_l(x)) = f_k(x)n_k(x), \quad \text{a.e. } x \in \partial\Omega.$$

As this sequence of integrands is dominated by $\delta(g)^*$ with $(g)^* \in L^1(\partial\Omega)$ by assumption and δ the uniform bound to ϑ_l due to Theorem 1.3 iv), the dominated convergence theorem is applicable and yields

$$\lim_{l\to\infty} \int_{\partial\Omega} \vartheta_l(s)\, f_k(\Lambda_l(s))\, n_k^{(l)}(\Lambda_l(s))\,\mathrm{d}\sigma(s) = \int_{\partial\Omega} f_k(s)n_k(s)\,\mathrm{d}\sigma(s). \tag{1.7}$$

Now consider the left hand side of identity (1.6). By Green's formula [6, p. 711f.] we know that

$$\int_{\partial\Omega_l} f_k(s)n_k^{(l)}(s)\,\mathrm{d}\sigma_l(s) = \int_{\Omega_l} \operatorname{div}(f(x))\,\mathrm{d}x, \quad \text{for all } l \in \mathbb{N}.$$

As $\Omega_l \subseteq \Omega$ for all $l \in \mathbb{N}$, the monotone convergence Theorem leaves us with

$$\lim_{l\to\infty} \int_{\Omega_l} \operatorname{div}(f(x))\,\mathrm{d}x = \int_{\Omega} \operatorname{div}(f(x)\,\mathrm{d}x. \tag{1.8}$$

Gluing together equations (1.7) and (1.8) gives the claim. ☐

1.2 The Stokes Operator

In this section, we will introduce the Stokes operator on $\mathrm{L}^2(\Omega; \mathbb{C}^d)$ and $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ for general p and establish a relation to the *Dirichlet problem for the Stokes resolvent system*

$$\begin{aligned} -\Delta u + \nabla \phi + \lambda u &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.9}$$

where $\lambda \in \Sigma_\theta := \{z \in \mathbb{C} \colon \lambda \neq 0 \text{ and } |\arg(z)| < \pi - \theta\}$ and $\theta \in (0, \pi/2)$.

We beginn by defining the relevant function spaces. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $1 < p < \infty$. We define

$$\mathrm{C}^\infty_{c,\sigma}(\Omega) := \{\varphi \in \mathrm{C}^\infty_c(\Omega; \mathbb{C}^d) \colon \operatorname{div}(\varphi) = 0\},$$

which can serve as a suitable space of test functions. We can now close this space in $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ and the Sobolev space $\mathrm{W}^{1,p}(\Omega; \mathbb{C}^d)$ which gives

$$\mathrm{L}^p_\sigma(\Omega) := \overline{\mathrm{C}^\infty_{c,\sigma}(\Omega)}^{\mathrm{L}^p}$$

and

$$\mathrm{W}^{1,p}_{0,\sigma}(\Omega) := \overline{\mathrm{C}^\infty_{c,\sigma}(\Omega)}^{\mathrm{W}^{1,p}},$$

respectively. If $p = 2$, we will use the symbol $\mathrm{H}^1_{0,\sigma}(\Omega)$ to denote $\mathrm{W}^{1,2}_{0,\sigma}(\Omega)$ in order to emphasize that this space is a Hilbert space.

In order to define the Stokes operator, we introduce the following sesquilinear form

$$\mathfrak{a} \colon \mathrm{H}^1_{0,\sigma}(\Omega) \times \mathrm{H}^1_{0,\sigma}(\Omega) \to \mathbb{C}, \quad (u, v) \mapsto \int_\Omega \nabla u \cdot \overline{\nabla v} \, \mathrm{d}x.$$

Note that for $u \in \mathrm{H}^1_{0,\sigma}(\Omega)$ the gradient ∇u is a matrix and an element of the space $\mathrm{L}^2(\Omega; \mathbb{C}^{d\times d})$.

Definition 1.9. The *Stokes operator* A_2 on $\mathrm{L}^2_\sigma(\Omega)$ is given via

$$\begin{aligned} \mathcal{D}(A_2) &:= \Big\{ u \in \mathrm{H}^1_{0,\sigma}(\Omega) \colon \exists! f \in \mathrm{L}^2_\sigma(\Omega) \text{ s.t. } \forall v \in \mathrm{H}^1_{0,\sigma}(\Omega) \colon \mathfrak{a}(u, v) = \int_\Omega f \cdot \overline{v} \, \mathrm{d}x \Big\} \\ A_2 u &:= f, \end{aligned}$$

where $u \in \mathcal{D}(A_2)$ and f comes from the definition of the domain.

The following theorem from Mitrea and Monniaux [21, Thm 4.7] shows that our definition of the Stokes operator and the one used in Shen's paper [26] coincide. Another advantage of this characterization is the immediate link of the Stokes operator to the Stokes system.

Theorem 1.10. *If $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is a bounded Lipschitz domain and A_2 is the Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$ then*

$$\mathcal{D}(A_2) = \big\{u \in H^1_{0,\sigma}(\Omega) \colon \exists \pi \in \mathrm{L}^2(\Omega) \text{ s.t. } -\Delta u + \nabla\pi \in \mathrm{L}^2_\sigma(\Omega)\big\},$$

where the expression $\Delta u + \nabla\pi \in \mathrm{L}^2_\sigma(\Omega)$ needs to be understood in the distributional sense. For $u \in \mathcal{D}(A_2)$ and the corresponding pressure π we have

$$A_2 u = -\Delta u + \nabla\pi.$$

The following proposition summarizes some facts about the Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$. A proof can be found in Tolksdorf [32, Prop. 5.2.5].

Proposition 1.11. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and A_2 the Stokes operator as in Definition 1.9. Then we have*

a) A_2 is closed with dense domain. Furthermore, $0 \in \rho(A_2)$.

b) $\sigma(A) \subset [0,\infty)$ and for all $\theta \in (0,\pi]$ there exists $C > 0$ such that

$$\|\lambda(\lambda + A)^{-1}\|_{\mathfrak{L}(\mathrm{L}^2_\sigma(\Omega))} \leq C, \quad \text{for all } \lambda \in \mathbb{C} \setminus \overline{\Sigma}_\theta. \tag{1.10}$$

In particular $-A_2$ generates a bounded analytic semigroup on $\mathrm{L}^2_\sigma(\Omega)$.

With these results at hand we can now give a quick recap of the solution theory to (1.9). Let $f \in \mathrm{L}^2_\sigma(\Omega)$ and $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi/2)$. By the previous theorem and proposition we know that there exists a unique $u \in \mathcal{D}(A_2) \subseteq \mathrm{H}^1_{0,\sigma}(\Omega)$ and some $\pi \in \mathrm{L}^2(\Omega)$ such that

$$-\Delta u + \nabla\pi + \lambda u = A_2 u + \lambda u = f.$$

For general $f = \mathrm{L}^2(\Omega; \mathbb{C}^d)$ we use the *Helmholtz projection* $\mathbb{P}_2$ to get

$$\Delta u + \nabla\pi + \lambda u + (I - \mathbb{P}_2)f = \mathbb{P}_2 f + (I - \mathbb{P}_2)f = f,$$

where u and π now correspond to $\mathbb{P}_2 f \in \mathrm{L}^2_\sigma(\Omega)$. On bounded Lipschitz domains the orthogonal complement to $\mathbb{P}_2[\mathrm{L}^2(\Omega; \mathbb{C}^d)] = \mathrm{L}^2_\sigma(\Omega)$ is characterized via

$$\mathrm{L}^2_\sigma(\Omega)^\perp = \big\{f \in \mathrm{L}^2(\Omega; \mathbb{C}^d) \colon f = \nabla\phi, \text{ for some } \phi \in \mathrm{L}^2(\Omega)\big\}.$$

A proof of this fact can be found in the book of Sohr [28, Lemma 2.5.3]. Using this result we find $g \in \mathrm{L}^2(\Omega)$ such that $\nabla g = (I - \mathbb{P}_2)f$ in the sense of distributions and we see that

$$-\Delta u + \nabla(\pi + g) + \lambda u = f.$$

Consequently, we see that solving the resolvent equation for the Stokes operator and solving the Stokes resolvent system (1.9) are two sides of the saime coin. Furthermore, we may deduce from the resolvent estimate (1.10) that the solution u which apparently is not affected by the additional part $(I - \mathbb{P}_2)f$ fulfills the inequality

$$|\lambda|^{-1}\|u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} = |\lambda|^{-1}\|(A_2 + \lambda)^{-1}\mathbb{P}_2 f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \leq C\|f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)},$$

where C depends only on θ. By the calculations above it is understandable why this estimate on u instead of (1.10) is sometimes called *resolvent estimate*.

In order to develop an L^p-theory for system (1.9), one way is to study the Stokes operator on subspaces of $\mathrm{L}^p(\Omega;\mathbb{C}^d)$. More precisely, we are interested in estimating solutions $u \in \mathrm{H}^1_0(\Omega;\mathbb{C}^d)$, in $\mathrm{L}^p(\Omega;\mathbb{C}^d)$ provided that the right hand side of the Stokes resolvent system (1.9) is an element of the space $\mathrm{L}^2(\Omega;\mathbb{C}^d) \cap \mathrm{L}^p(\Omega;\mathbb{C}^d)$. This is once again just one side of the aforementioned coin. The other side just asks for a resolvent estimate on the Stokes operator, hoping that in analogy to Proposition 1.11 this leads to an analytic semigroup.

Definition 1.12. Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ be a bounded Lipschit domain. If $p > 2$ we define the Stokes operator A_p via its *part of* A_2 *in* $\mathrm{L}^p_\sigma(\Omega)$.

$$\begin{aligned} \mathcal{D}(A_p) &:= \Big\{u \in \mathcal{D}(A_2) \cap \mathrm{L}^p_\sigma(\Omega)\colon A_2 u \in \mathrm{L}^p_\sigma(\Omega)\Big\} \\ A_p u &:= A_2 u, \quad u \in \mathcal{D}(A_p). \end{aligned}$$

For $p > 2$ there exists an analog to Theorem 1.10. The peculiar range of p for which this theorem holds is due to the fact that the boundedness of the Helmholtz projection on $\mathrm{L}^p(\Omega;\mathbb{C}^d)$ is a crucial ingredient to the proof and a fundamental pillar of the L^p-theory of the Stokes equations. More details about the mechanics of the Helmholtz projection can be found in Tolksdorf [32, Sec. 5.1].

Theorem 1.13 (Thm. 5.2.11, [32])**.** *Let* $\Omega \subseteq \mathbb{R}^d$*,* $d \geq 2$*, is a bounded Lipschitz domain. Then there exists* $\varepsilon > 0$ *such that for all*

$$2 < p < \frac{2d}{d-1} + \varepsilon$$

the domain of the Stokes operator A_p is characterized as

$$\mathcal{D}(A_p) = \big\{u \in \mathrm{W}^{2,p}_{0,\sigma}(\Omega) \colon \exists \pi \in \mathrm{L}^p(\Omega) \text{ s.t. } -\Delta u + \nabla \pi \in \mathrm{L}^p_\sigma(\Omega)\big\},$$

where the expression $\Delta u + \nabla \pi \in \mathrm{L}^p_\sigma(\Omega)$ needs to be understood in the distributional sense. For $u \in \mathcal{D}(A_p)$ and the corresponding pressure π we have

$$A_p u = -\Delta u + \nabla \pi.$$

For $p < 2$ there exist various ways to define the Stokes operator. One adequate way is to dualize the operator $A_{p'}$, where $p' = p/(1+p)$ is the Hölder conjugate exponent. In order to carry out this construction we need the spaces $\mathrm{L}^p_\sigma(\Omega)$ to exhibit the same behavior regarding dualization as the spaces $\mathrm{L}^p(\Omega; C^d)$.

Lemma 1.14 (Lem 5.2.13, [32])**.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

the spaces $\mathrm{L}^p_\sigma(\Omega)$ and $(\mathrm{L}^{p'}_\sigma(\Omega))^*$ are isomorphic, where $(\mathrm{L}^{p'}_\sigma(\Omega))^*$ denotes the antidual space and $p' = p/(p-1)$ is the Hölder conjugate exponent of p. The isomorphism Ψ is given by

$$[\Psi f](g) = \int_\Omega f \cdot \overline{g}\, \mathrm{d}x, \quad g \in \mathrm{L}^{p'}_\sigma(\Omega).$$

Now we define the Stokes operator for $p < 2$ as announced.

Definition 1.15. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and let $\varepsilon > 0$ be as in Lemma 1.14. Let furthermore

$$\frac{2d}{d+1} - \varepsilon < p < 2$$

and Ψ be the isomorphism from Lemma 1.14. Then the *Stokes operator on* $\mathrm{L}^p_\sigma(\Omega)$ is defined via

$$\begin{aligned} \mathcal{D}(A_p) &\coloneqq \big\{u \in \mathrm{L}^p_\sigma(\Omega) \colon \Psi u \in \mathcal{D}(A^*_{p'})\big\} \\ A_p &= \coloneqq \Psi^{-1} A^*_{p'} \Psi u, \end{aligned}$$

where $p' = p/(1-p)$ denotes the Hölder conjugate exponent of p and $A^*_{p'}$ the adjoint operator to $A_{p'}$.

Without investing too much additional work, it is now possible to prove the following Theorem.

Theorem 1.16 (Thm. 5.2.9 and Thm. 5.2.17, [32])**.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

the operator A_p is closed with dense domain. Furthermore $0 \in \rho(A_p)$.

The natural question arises when comparing Theorem 1.16 with the Hilbert space counterpart Theorem 1.11: Does the Stokes operator generate a bounded analytic semigroup in L^p_σ? An affirmative answer was given by Shen in 2012 with his seminal paper [26] by proving the necessary resolvent estimates for $d \geq 3$.

Theorem 1.17 (Shen)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$. There exists $\varepsilon > 0$, such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

there exists a constant $C > 0$ such that for every $f \in \mathrm{L}^p_\sigma(\Omega)$ and all $\lambda \in \Sigma_\theta$ the inequality

$$|\lambda| \|(\lambda + A_p)^{-1}\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \leq C \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)}$$

holds. In particular $-A_p$ is the generator of a bounded analytic semigroup on $\mathrm{L}^p_\sigma(\Omega)$.

This Theorem gave an affirmative answer to Taylor's conjecture in [30]. Curiously this positive result is limited to $d \geq 3$ even though Shen states that the approach he developed should also work in the case $d = 2$. This sets the starting point for the present thesis. In the subsequent chapters we will not only present Shen's approach to the problem of the resolvent estimates, we will furthermore extend his results whenever possible to the two dimensional case.

Chapter 2

Estimating Fundamental Solutions

The purpose of this section is to study fundamental solutions of the Stokes resolvent problem and to deduce related estimates which will be crucial for the next chapters. Before working on the Stokes resolvent problem we will take a look at the atoms of the fundamental solution of this problem: the Hankel functions.

As a basis for the subsequent sections and chapters let us fix recurring quantities regarding sectors in the complex plane $\mathbb{C}$.

Let $\theta \in (0, \pi/2)$ and $\lambda \in \Sigma_\theta$ as in Section 1.2. The polar form of λ is given as $\lambda = r\mathrm{e}^{\mathrm{i}\tau}$ with $0 < r < \infty$ and $-\pi + \theta < \tau < \pi - \theta$. Now set

$$k = \sqrt{r}\,\mathrm{e}^{\mathrm{i}(\pi+\tau)/2}.$$

Then we have

$$k^2 = -\lambda \quad \text{and} \quad \frac{\theta}{2} < \arg(k) < \pi - \frac{\theta}{2}$$

as it holds

$$\arg(k) = \frac{\pi + \tau}{2} > \frac{\pi}{2} + \frac{-\pi + \theta}{2} = \frac{\theta}{2}$$

on the one hand and

$$< \frac{\pi}{2} + \frac{\pi - \theta}{2} = \pi - \frac{\theta}{2}$$

on the other hand. The preceding calculation gives rise to the following estimate:

$$\operatorname{Im}(k) > \sqrt{|\lambda|}\sin(\theta/2) > 0. \tag{2.1}$$

Indeed, we have

$$\operatorname{Im}(k) = \sqrt{r}\sin\left(\frac{\pi+\tau}{2}\right) = \sqrt{|\lambda|}\sin\left(\frac{\pi+\tau}{2}\right) \quad \text{and} \quad \frac{\theta}{2} < \frac{\pi+\tau}{2} < \pi - \frac{\theta}{2}$$

which gives for τ with $\frac{\pi+\tau}{2} \leq \frac{\pi}{2}$ that $\sin(\frac{\pi+\tau}{2}) \geq \sin(\frac{\theta}{2})$ and for τ with $\frac{\pi+\tau}{2} > \frac{\pi}{2}$ that $\sin(\frac{\pi+\tau}{2}) > \sin(\pi - \frac{\theta}{2}) = \sin(\frac{\theta}{2})$.

2.1 Hankel Functions and the Helmholtz equation

Before diving into fundamental solutions of the Stokes resolvent problem, we will first consider a fundamental solution for the (scalar) Helmholtz equation in $\mathbb{R}^d$

$$-\Delta u + \lambda u = 0.$$

One fundamental solution with pole at the origin is given by

$$G(x;\lambda) = \frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}} \cdot \frac{1}{|x|^{d-2}} \cdot (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|), \tag{2.2}$$

see McLean [20, Eq. (9.14)], where $H^{(1)}_\nu(z)$ is the Hankel function of the first kind which according to Lebedev [19, Sec. 5.11] can be also be written as

$$H^{(1)}_\nu(z) = \frac{2^{\nu+1}\mathrm{e}^{\mathrm{i}(z-\nu\pi)}z^\nu}{\mathrm{i}\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \int_0^\infty \mathrm{e}^{2z\mathrm{i}s} s^{\nu-\frac{1}{2}}(1+s)^{\nu-\frac{1}{2}}\,\mathrm{d}s. \tag{2.3}$$

This formula holds for $\nu > -\frac{1}{2}$ and $0 < \arg(z) < \pi$. We will usually set

$$\nu = \nu_d = \frac{d}{2} - 1 \quad \text{and} \quad z = k|x|.$$

Note that by (2.1) we will always have $\operatorname{Im}(z) > 0$. Since $\nu_d < \nu_{d+1}$ for all $d \geq 2$ and $\nu_2 = 0$, formula (2.3) will hold for all dimensions $d \geq 2$ and all $x \in \mathbb{R}^d$.

In the case $d = 2$, formula (2.2) simplifies to

$$G(x;\lambda) = \frac{\mathrm{i}}{4} H^{(1)}_0(k|x|). \tag{2.4}$$

In the case $d = 3$, one has an even easier formula, namely

$$G(x;\lambda) = \frac{\mathrm{i}}{4(2\pi)^{1/2}} \cdot \frac{1}{|x|} \cdot (k|x|)^{1/2} H^{(1)}_{1/2}(k|x|) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\pi|x|}, \tag{2.5}$$

which is due to an easy formula for $H^{(1)}_{1/2}(z)$ in Lebedev, see [19, Eq. (5.8.4)] and [20, Eq. (9.15)].

Our first estimate is concerned with estimates on the fundamental solution and its derivatives for the (scalar) Helmholtz equation. The main concern of this lemma is with the asymptotic behavior of $G(\,\cdot\,,\lambda)$ for large values of $|x|$.

Lemma 2.1. *Let $\lambda \in \Sigma_\theta$. Then*

$$|\nabla^l_x G(x;\lambda)| \leq \frac{C_l \mathrm{e}^{-c\sqrt{|\lambda|}|x|}}{|x|^{d-2+l}} \tag{2.6}$$

for any integer $l \geq 0$ if $d \geq 3$ and for $l \geq 1$ if $d = 2$. Here, $c > 0$ depends only on θ and C_l depends only on d, l and θ.

Let $d = 2$. Then $|G(x;\lambda)| = o(1)$ as $|x| \to \infty$.

Proof. We start with the case $l = 0$ and $d \geq 3$. Let $\mathrm{Im}(z) > 0$ and $\nu - \frac{1}{2} \geq 0$. Then (2.3) gives

$$|H_\nu^{(1)}(z)| \leq C_d \, \mathrm{e}^{-\mathrm{Im}(z)} |z|^\nu \int_0^\infty \mathrm{e}^{-2s\,\mathrm{Im}(z)} s^{\nu-\frac{1}{2}} (1+s)^{\nu-\frac{1}{2}} \, \mathrm{d}s, \tag{2.7}$$

where $C_d > 0$ depends only on d. We apply the substitution rule with $t = s - (1/2)$ and calculate

$$\begin{aligned}
\mathrm{e}^{\frac{-\mathrm{Im}(z)}{2}} \int_0^\infty \mathrm{e}^{-s\,\mathrm{Im}(z)} s^{\nu-\frac{1}{2}} (1+s)^{\nu-\frac{1}{2}} \, \mathrm{d}s &= \int_0^\infty \mathrm{e}^{-(s+\frac{1}{2})\,\mathrm{Im}(z)} s^{\nu-\frac{1}{2}} (1+s)^{\nu-\frac{1}{2}} \, \mathrm{d}s \\
&= \int_{\frac{1}{2}}^\infty \mathrm{e}^{-t\,\mathrm{Im}(z)} \Big(t^2 - \frac{1}{4}\Big)^{\nu-\frac{1}{2}} \, \mathrm{d}t \\
&\leq \int_0^\infty e^{-t\,\mathrm{Im}(z)} t^{2\nu-1} \, \mathrm{d}t \\
&= \int_0^\infty e^{-u} u^{2\nu-1} \, \mathrm{Im}(z)^{1-2\nu} \, \mathrm{Im}(z)^{-1} \, \mathrm{d}u \\
&= \mathrm{Im}(z)^{-2\nu} \int_0^\infty \mathrm{e}^{-u} u^{2\nu-1} \, \mathrm{d}u \\
&= C_\nu \, \mathrm{Im}(z)^{-2\nu},
\end{aligned}$$

where we also used the substitution rule with $u = t\,\mathrm{Im}(z)$. Now we multiply (2.7) by $|z|^\nu$ and reuse the previous estimate to arrive at

$$|z|^\nu |H_\nu^{(1)}(z)| \leq C_d \, C_\nu \, |z|^{2\nu} \, |\,\mathrm{Im}(z)|^{-2\nu} \mathrm{e}^{-\frac{\mathrm{Im}(z)}{2}},$$

which for $z = k|x|$ gives

$$|kx|^\nu |H_\nu^{(1)}(k|x|)| \leq C \sin(\theta/2)^{-2\nu} \, \mathrm{e}^{-\frac{1}{2}\sin(\theta/2)\sqrt{|\lambda|}|x|}, \tag{2.8}$$

where $C > 0$ depends only on d and we used (2.1) to estimate

$$(|kx|)^{2\nu} \cdot |\,\mathrm{Im}(k|x|)|^{-2\nu} = |\lambda|^\nu \cdot |\,\mathrm{Im}(k)|^{-2\nu} \leq \sin(\theta/2)^{-2\nu}.$$

Using (2.2), we estimate for $d \geq 3$ setting $\nu = \frac{d}{2} - 1$

$$|G(x;\lambda)| \leq C \, |x|^{2-d} e^{-c\sqrt{|\lambda|}|x|}$$

and it is clear that the generic constants depends on d and θ. This gives the estimate for $l = 0$ and $d \geq 3$.

Using the relation for the derivatives of Hankel functions which one finds in the book of Lebedev [19, Eq. (5.6.3)],

$$\frac{\mathrm{d}}{\mathrm{d}z} \Big\{ z^{-\nu} H_\nu^{(1)}(z) \Big\} = -z^{-\nu} H_{\nu+1}^{(1)}(z),$$

we inductively establish the estimate (2.6) for $l \geq 1$ and $d \geq 2$: For $1 \leq j \leq d$, we calculate using the product and chain rule

$$\begin{aligned} |\nabla_x G(x;\lambda)| &\leq C \cdot \left\{ |x|^{1-d} \cdot (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|) - |x|^{2-d} (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}}(k|x|) \cdot k \right\} \\ &\leq C \cdot |x|^{1-d} \left\{ (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|) - (k|x|)^{\frac{d}{2}} H^{(1)}_{\frac{d}{2}}(k|x|) \right\}, \end{aligned} \tag{2.9}$$

where $C > 0$ is a generic constant that depends on d. Note that the first summand in (2.9) does not arise in the case $d = 2$ as is easily seen from equation (2.4). The terms in the bracket can now be estimated individually by (2.8). The extension of this proof to orders of differentiation $l \geq 2$ is straightforward using the Leibniz product rule for higher derivatives.

Now for the last part of the proof, let us verify the claim regarding the asymptotic behavior of $|G(x;\lambda)|$ if $d = 2$. Based on the integral representation (2.3), Lebedev derived an asymptotic expansion for the Hankel function [19, Sec. 5.11, Eq. (5.11.3)]. For $\nu = (d/2) - 1 = 0$ and $z = k|x|$ this expansion reads

$$H^{(1)}_0(z) = \left(\frac{2}{\pi z}\right)^{1/2} \mathrm{e}^{\mathrm{i}(z-(1/4)\pi)} \left[1 + O(|z|^{-1})\right].$$

As $\mathrm{Im}(z) > 0$ we see that $|H^{(1)}_0(k|x|)| = O(|x|^{-1/2})$. Due to the simple structure of $G(x;\lambda)$ for $d = 2$ as shown in equation (2.4), the claim follows easily. □

In the derivation of the next estimates we will use the following useful interior estimate for solutions of Poisson's equation which we write down for further use.

Lemma 2.2. *Let* $r > 0$ *and* $x \in \mathbb{R}^d$, $d \geq 2$. *If* $w \in \mathrm{C}^k(\mathrm{B}(x,r)) \cap \mathrm{C}^0(\overline{\mathrm{B}(x,r)})$ *is a solution to* $\Delta w = f$ *in* $\mathrm{B}(x,r)$, *then*

$$|\nabla^l w(x)| \leq C r^{-l} \sup_{\mathrm{B}(x,r)} |w| + C \max_{0 \leq j \leq l-1} \sup_{\mathrm{B}(x,r)} r^{j-l+2} |\nabla^j f|, \quad l \leq k,. \tag{2.10}$$

where $C > 0$ *only depends on* d *and* l.

Proof. If $l = 1$, then estimate (2.10) is a consequence of the *comparison principle* and a proof of this fact can be found in the book of Gilbarg and Trudinger [12, Sec. 3.4, Eq. (3.16)]. We can now use this estimate to inductively deduce the estimates for higher derivatives. Note that by translating from x to 0 and rescaling like

$$u_r(x) := u(rx) \quad \text{and} \quad f_r(x) := r^2 f(rx)$$

we may assume that $\Delta w = f$ in $\mathrm{B}(0,1)$ and that it suffices to prove

$$|\nabla^l w(0)| \leq C \sup_{\mathrm{B}(0,1)} |w| + C \max_{0\leq j\leq l-1} \sup_{\mathrm{B}(0,1)} |\nabla^j f|. \tag{2.11}$$

for $l > 1$. By the Schwartz theorem we have that if w solves Poisson's equation with right hand side f and w and f are sufficiently regular, then $\nabla^l w$ solves Poisson's equation with right hand side $\nabla^l f$. We can thus estimate inductively

$$\begin{aligned}
|\nabla^l w(0)| &\leq C_l \sup_{\mathrm{B}(0,1/2^{l-1})} |\nabla^{l-1} w| + C_l \sup_{\mathrm{B}(0,1/2^{l-1})} |\nabla^{l-1} f| \\
&\leq C_l \sup_{\mathrm{B}(0,1/2^{l-2})} |\nabla^{l-2} w| + C_l \left\{ \sup_{\mathrm{B}(0,1)} |\nabla^{l-2} f| + \sup_{\mathrm{B}(0,1)} |\nabla^{l-1} f| \right\} \\
&\leq \dots \\
&\leq C_l \sup_{\mathrm{B}(0,1)} |w| + C_l \sum_{j=0}^{l-1} \sup_{B(0,1)} |\nabla^j f|
\end{aligned}$$

which readily yields the desired estimate. $\square$

We will need the following asymptotic expansions for the function $z^\nu H_\nu^{(1)}(z)$ in $\mathbb{C} \setminus (-\infty, 0]$. The derivation of these asymptotic expansions is based on asymptotic expansions of the *Bessel functions of the first* and *the second kind* and can be found in Tolksdorf [32, Sec. 4.2]:

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{\mathrm{i}}{\pi} z^2 \log(z) + \omega z^2 + O(|z|^4 | \log(z)) \quad \text{if } d = 4, \tag{2.12}$$

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{2^\nu \Gamma(\nu - 1)}{4\pi\mathrm{i}} z^2 + \omega z^3 + O(|z|^4) \quad \text{if } d = 5, \tag{2.13}$$

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{2^\nu \Gamma(\nu - 1)}{4\pi\mathrm{i}} z^2 + O(|z|^4 | \log z|) \quad \text{if } d = 6, \tag{2.14}$$

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{2^\nu \Gamma(\nu - 1)}{4\pi\mathrm{i}} z^2 + O(|z|^4) \quad \text{if } d \geq 7. \tag{2.15}$$

The next Lemma will be concerned with estimating the difference $G(x;\lambda) - G(x;0)$ (and derivatives of this difference) of the fundamental solution to the scalar Helmholtz equation and the fundamental solution for $-\Delta = 0$ in $\mathbb{R}^d$ which is given by

$$G(x;0) \coloneqq \begin{cases} -\frac{1}{2\pi} \log(|x|), & \text{for } d = 2, \\ c_d \frac{1}{|x|^{d-2}}, & \text{for } d > 2, \end{cases} \tag{2.16}$$

where the coefficient c_d is given as as a multiple of the surface measure of the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$

$$c_d = \frac{1}{(d-2)\omega_d}, \quad \text{with} \quad \omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = |\mathbb{S}^{d-1}|. \tag{2.17}$$

Note that $c_2 = (2\pi)^{-1}$. By rearranging terms and using the functional equation of the Gamma function

$$\begin{aligned} \Gamma(z+1) &= z\,\Gamma(z), \quad z \in \mathbb{C}, \mathrm{Re}(z) > 0 \\ \Gamma(1) &= 1, \end{aligned} \tag{2.18}$$

we get

$$(d-2)\omega_d = 2\left(\frac{d}{2}-1\right)\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = 2\left(\frac{d}{2}-1\right)\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}-1)(\frac{d}{2}-1)} = \frac{4\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}-1)},$$

and thus we will also sometimes use the equivalent definition

$$c_d := \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}. \tag{2.19}$$

Furthermore, the leading coefficient of the asymptotic expansions of the Hankel functions (2.12)-(2.15) for $d \geq 3$ will be denoted as

$$a_d := \frac{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2}-1)}{\mathrm{i}\pi}. \tag{2.20}$$

The coefficients a_d and c_d are related in the following way:

$$c_d = \frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}}\, a_d.$$

This allows us to write for $d \geq 3$

$$G(x;\lambda) - G(x;0) = \frac{i}{4(2\pi)^{\frac{d}{2}-1}} \cdot \frac{1}{|x|^{d-2}}\left\{(k|x|)^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(k|x|) - a_d\right\}. \tag{2.21}$$

The following lemma will help us to estimate the expression (2.21) together with its derivatives and their 2-dimensional counterparts.

Lemma 2.3. *Let $\lambda \in \Sigma_\theta$. Then*

$$\left|\nabla_x^l\left\{G(x;\lambda) - G(x;0)\right\}\right| \leq C|\lambda||x|^{4-d-l} \tag{2.22}$$

if $d \geq 5$ and $l \geq 0$, where C depends only on d, l and θ. If $d = 3$ or 4, *estimate* (2.22) *holds for $l \geq 1$ and if $d = 2$, the estimate holds for $l \geq 3$.*

Proof. (a) In this part we will show that the desired estimate (2.22) holds if we assume that $|\lambda||x|^2 > (1/2)$. In this case, Lemma 2.1 gives

$$\left|\nabla_x^l\left\{G(x;\lambda) - G(x;0)\right\}\right| \leq C\left\{\frac{\mathrm{e}^{-c\sqrt{|\lambda|}|x|}}{|x|^{d-2+l}} + \frac{1}{|x|^{d-2+l}}\right\} \leq C\frac{|\lambda|}{|x|^{d-4+l}},$$

where C depends only on d, l and θ. Therefore, for the remaining proof we will suppose $|\lambda||x|^2 \leq (1/2)$.

(b) In this step we show that we can restrict ourselves to proving (2.22) in three cases: (1) $d \geq 5$ and $l = 0$; (2) $d = 3$ or 4 and $l = 1$; (3) $d = 2$ and $l = 3$.

Suppose (2.22) holds in case (1) and let $l > 1$. If we set $w(x) = G(x;\lambda) - G(x;0)$, we have $\Delta_x w = \lambda G(x;\lambda)$ in $\mathbb{R}^d \setminus \{0\}$. For $f = \lambda G(x;\lambda)$, estimate (2.10) now gives

$$\begin{aligned}
|\nabla^l w(x)| &\leq Cr^{-l} \sup_{\mathrm{B}(x,r)} |w| + C \max_{0\leq j\leq l-1} \sup_{\mathrm{B}(x,r)} r^{j-l+2} |\nabla^j f| \\
&\leq Cr^{-l} \sup_{y\in\mathrm{B}(x,r)} |\lambda||y|^{4-d} + C \sum_{j=0}^{l-1} \sup_{y\in\mathrm{B}(x,r)} r^{j-l+2}|\lambda||y|^{2-d-j} \\
&= Cr^{-l}|\lambda| \left| x - r\frac{x}{|x|} \right|^{4-d} + C \sum_{j=0}^{l-1} r^{j-l+2}|\lambda| \left| x - r\frac{x}{|x|} \right|^{2-d-j},
\end{aligned}$$

for all $0 < r < |x|$, where we used (2.22) with $l = 1$ for the first summand and (2.6) to estimate the second summand. We choose $r = \frac{|x|}{2}$ and receive

$$\begin{aligned}
|\nabla^l w(x)| &\leq C|\lambda||x|^{-l}|x|^{4-d} + C \sum_{j=0}^{l-1} |x|^{j-l+2}|\lambda||x|^{2-d-j} \\
&\leq C|\lambda||x|^{4-d-l}.
\end{aligned}$$

The proof for case (2) is completely analogous if one sets

$$w(x) = \nabla_x(G(x;\lambda) - G(x;0)) \quad \text{and} \quad f(x) = \lambda\nabla_x G(x;\lambda).$$

Also case (3) is proven in a similar fashion.

(c) In this step we prove (2.22) for $d \geq 5$ and $l = 0$. First, note that for the functions

$$\begin{aligned}
g(x) &:= (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|), \quad g(0) = a_d, \\
h(z) &:= z^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(z), \quad h(0) = a_d,
\end{aligned}$$

the mean value theorem yields the estimate

$$|g(x) - g(0)| \leq |x| \sup_{y\in\mathrm{B}(0,|x|)} |\nabla g(y)| \leq |x||k| \sup_{y\in\mathrm{B}(0,|x|)} \left| \Big(\frac{\mathrm{d}}{\mathrm{d}z} h\Big)(k|y|) \right|.$$

Using representation (2.21), we estimate

$$\begin{aligned}
|G(x;\lambda) - G(x;0)| &\leq C|x|^{2-d} \cdot |k||x| \max_{\substack{|z|\leq|k||x| \\ \mathrm{Im}(z)>0}} \left| \frac{\mathrm{d}}{\mathrm{d}z} \left\{ z^{\frac{d}{2}-1} H_{\frac{d}{2}-1}(z) \right\} \right| \\
&= C|x|^{2-d} \cdot |k||x| \max_{\substack{|z|\leq|k||x| \\ \mathrm{Im}(z)>0}} \left| z^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-2}(z) \right|, \qquad (2.23)
\end{aligned}$$

where for the last equality we used another useful relation that can be found in Lebedev [19, Eq. (5.6.3)],

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{z^{\nu}H_{\nu}^{(1)}(z)\right\} = z^{\nu}H_{\nu-1}^{(1)}(z). \tag{2.24}$$

Since the asymptotic expansions yield that $|z^{\nu}H_{\nu}^{(1)}(z)| \leq C_{\nu}$ for $\nu > 0$ and $|z| \leq 1$ with $\mathrm{Im}(z) > 0$, it follows from (2.23) that

$$|G(x;\lambda) - G(x;0)| \leq C|x|^{2-d} \cdot |k||x| \cdot |k||x| \max_{\substack{|z|\leq|k||x| \\ \mathrm{Im}(z)>0}} \left|z^{\frac{d}{2}-2}H_{\frac{d}{2}-2}^{(1)}(z)\right| \leq C|\lambda||x|^{4-d}.$$

(d) Now we consider the case $d = 4$ and $l = 1$. The asymptotic expansion (2.12) gives that

$$\left|\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{zH_1^{(1)}(z) - a_4}{z^2}\right\}\right| \leq C|z|^{-1} \tag{2.25}$$

for all $|z| \leq \frac{1}{2}$ with $\mathrm{Im}(z) > 0$. Since

$$\frac{G(x;\lambda) - G(x;0)}{\lambda} = -\frac{C(zH_1^{(1)}(z) - a_4)}{z^2},$$

where $z = k|x|$. With (2.25) we conclude that

$$\left|\frac{\nabla_x\{G(x;\lambda) - G(x;0)\}}{\lambda}\right| \leq C|k| \left|\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{zH_1^{(1)}(z) - a_4}{z^2}\right\}\Big|_{z=k|x|}\right| \leq C|k||k|^{-1}|x|^{-1},$$

which after rearrangement of the involved terms gives the claim.

(e) For the case $d = 3$ and $l = 1$, we get from equation (2.19) and a well known fact of the Gamma function, $\Gamma(1/2) = \sqrt{\pi}$, the following identity:

$$G(x;\lambda) - G(x;0) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\pi|x|} - \frac{c_3}{|x|} = \frac{\mathrm{e}^{\mathrm{i}k|x|} - 1}{4\pi|x|}.$$

Now we calculate

$$\begin{aligned}\frac{\partial}{\partial x_j}\left\{\frac{\mathrm{e}^{\mathrm{i}k|x|} - 1}{|x|}\right\} &= \frac{\partial}{\partial x_j}\left\{\frac{\mathrm{e}^{\mathrm{i}k|x|} - 1 - \mathrm{i}k|x|}{|x|}\right\} = \frac{\partial}{\partial x_j}\left\{\sum_{n=2}^{\infty}\frac{(\mathrm{i}k|x|)^n}{n!} \cdot \frac{1}{|x|}\right\} \\ &= \sum_{n=2}^{\infty}\frac{(\mathrm{i}k)^n}{n!}(n-1) \cdot \frac{x_j}{|x|}|x|^{n-2}\end{aligned}$$

which in turn implies

$$\left|\frac{\partial}{\partial x_j}\left\{\frac{\mathrm{e}^{\mathrm{i}k|x|} - 1}{|x|}\right\}\right| \leq |\lambda| \sum_{n=2}^{\infty}\frac{n-1}{n!}|k|^{n-2}|x|^{n-2} \leq C\,|\lambda|$$

since $|\lambda||x| \leq (1/2)$.

(f) For the last case $d = 2$ and $l = 3$, we will directly calculate the estimate using the asymptotic expansion of $H_0^{(1)}(z)$ with $z = k|x|$. The calculations are omitted from this chapter. Instead, they can be found in the appendix of this thesis, see A.1. $\square$

Remark 2.4. In the situation of Lemma 2.3, one can show for $|\lambda||x|^2 \leq (1/2)$ by considering the asymptotic expansions that

$$|G(x;\lambda) - G(x;0)| \leq \begin{cases} C\sqrt{|\lambda|} & \text{if } d = 3, \\ C|\lambda|\{|\log(|\lambda||x|^2)| + 1\} & \text{if } d = 4. \end{cases}$$

Also using the asymptotic expansions it can be shown that if $d = 2$, then

$$|\nabla_x^l\{G(x;\lambda) - G(x;0)\}| \leq C|\lambda||x|^{2-l}\{|\log(|\lambda||x|^2)| + 1\},$$

for $l \in \{1, 2\}$.

2.2 The Stokes Resolvent Problem

We will now analyze fundamental solutions to the *Stokes resolvent problem*

$$\begin{aligned} -\Delta u + \nabla\phi + \lambda u &= 0 \\ \operatorname{div} u &= 0 \end{aligned} \tag{2.26}$$

in $\mathbb{R}^d$ with $\lambda \in \Sigma_\theta$ with the goal to deduce helpful estimates for the following chapters. The fundamental solutions to the (scalar) Helmholtz equation and the Laplace equation will form the main ingredients for the following matrix of fundamental solutions to the Stokes resolvent problem with pole at the origin:

$$\Gamma_{\alpha\beta}(x;\lambda) = G(x;\lambda)\delta_{\alpha\beta} - \frac{1}{\lambda}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\Big\{G(x;\lambda) - G(x;0)\Big\}, \quad \alpha,\beta = 1,\ldots,d. \tag{2.27}$$

As the matrix of fundamental solutions $\Gamma(x;\lambda) = (\Gamma_{\alpha\beta}(x;\lambda))_{d\times d}$ carries two arguments it cannot be confused with the Gamma function. Having formula (2.27) at sight, the following observations are obvious:

$$\Gamma_{\alpha\beta}(x;\lambda) = \Gamma_{\beta\alpha}(x;\lambda), \quad \overline{\Gamma_{\alpha\beta}(x;\lambda)} = \Gamma_{\alpha\beta}(x;\bar{\lambda}) \quad \text{and} \quad \Gamma_{\alpha\beta}(x;\lambda) = \Gamma_{\alpha\beta}(-x;\lambda).$$

For the pressure, we define the vector of fundamental solutions

$$\Phi_\beta(x) = -\frac{\partial}{\partial x_\beta}\Big\{G(x;0)\Big\} = \frac{x_\beta}{\omega_d|x|^d}, \quad \beta = 1,\ldots,d. \tag{2.28}$$

We note that $\Phi_\beta(x) = \Phi_\beta(-x)$.

Using the fact that $\Delta_x G(x;\lambda) = \lambda G(x;\lambda)$ in $\mathbb{R}^d \setminus \{0\}$, one can see that on $\mathbb{R}^d \setminus \{0\}$ and for all $1 \le \beta \le d$

$$\begin{aligned} (-\Delta_x + \lambda)\Gamma_{\alpha\beta}(x;\lambda) + \frac{\partial}{\partial x_\alpha}\Big\{\Phi_\beta(x)\Big\} &= 0, \\ \frac{\partial}{\partial x_\alpha}\Big\{\Gamma_{\alpha\beta}(x;\lambda)\Big\} &= 0, \quad \text{for } 1 \le \alpha \le d. \end{aligned} \tag{2.29}$$

Note that in the last equation the summation convention was used.

We now keep up to the spirit of this exhausting chapter by proving further estimates, this time for the fundamental solutions to the Stokes resolvent problem (2.26).

Theorem 2.5. *Let $\lambda \in \Sigma_\theta$. Then for any $d \ge 3$ and $l \ge 0$*

$$\left|\nabla_x^l \Gamma(x;\lambda)\right| \le \frac{C}{(1+|\lambda||x|^2)|x|^{d-2+l}} \tag{2.30}$$

where C depends only on d, l and θ. For $d = 2$ and $l \ge 1$ the same estimate holds.

Proof. Let $|\lambda||x|^2 > (1/2)$. Then there exist constants C_a, C_b, C_c such that

$$\begin{aligned} e^{-c\sqrt{|\lambda|}|x|}(1+|\lambda||x|^2) &\le C_a, \\ 1 &\le \frac{C_b|\lambda||x|^2}{1+|\lambda||x|^2}, \\ e^{-c\sqrt{|\lambda|}|x|} &\le \frac{C_c|\lambda||x|^2}{1+|\lambda||x|^2}, \end{aligned}$$

where c is the constant from Lemma 2.1. Using these estimates and Lemma 2.1 gives

$$\begin{aligned} \left|\nabla_x^l \Gamma(x;\lambda)\right| &\le \left|\nabla_x^l G(x;\lambda)\right| + \frac{1}{|\lambda|}\left|\nabla_x^{l+2} G(x;\lambda)\right| + \frac{1}{|\lambda|}\left|\nabla_x^{l+2} G(x;0)\right| \\ &\le \frac{C_l\, e^{-c\sqrt{|\lambda|}|x|}}{|x|^{d-2+l}} + \frac{1}{|\lambda|}\frac{C_{l+2}\, e^{-c\sqrt{|\lambda|}|x|}}{|x|^2|x|^{d-2+l}} + \frac{1}{|\lambda|}\frac{C}{|x|^2|x|^{d-2+l}} \\ &\le \frac{C}{1+|\lambda||x|^2}\,\frac{1}{|x|^{d-2+l}}. \end{aligned}$$

Now let $|\lambda||x|^2 \le (1/2)$. Then by Lemma 2.1 and Lemma 2.3 we get

$$\begin{aligned} \left|\nabla_x^l \Gamma(x;\lambda)\right| &\le \left|\nabla_x^l G(x;\lambda)\right| + \frac{1}{|\lambda|}\cdot\left|\nabla_x^{l+2}(G(x;\lambda) - G(x;0))\right| \\ &\le \frac{C}{|x|^{d-2+l}} + \frac{1}{|\lambda|}\cdot C|\lambda||x|^{4-d-(l+2)} \\ &\le \frac{C}{|x|^{d-2+l}}\frac{(1+|\lambda||x|^2)}{(1+|\lambda||x|^2)} \\ &\le \frac{C}{(1+|\lambda||x|^2)|x|^{d-2+l}} \end{aligned}$$

which gives the claim. □

If $\lambda = 0$, the Stokes resolvent problem becomes just the Stokes problem in $\mathbb{R}^d$

$$\begin{aligned} -\Delta u + \nabla \phi + \lambda u &= 0, \\ \operatorname{div} u &= 0. \end{aligned} \tag{2.31}$$

Whereas the fundamental solution for the pressure is maintained, the matrix of fundamental solutions to the Stokes problem in $\mathbb{R}^d$ with pole at the origin is given by $\Gamma(x; 0) = (\Gamma_{\alpha\beta}(x; 0))_{d\times d}$, where

$$\Gamma_{\alpha\beta}(x; 0) := \frac{1}{2\omega_d}\Big\{\frac{\delta_{\alpha\beta}}{(d-2)|x|^{d-2}} + \frac{x_\alpha x_\beta}{|x|^d}\Big\} \tag{2.32}$$

if $d \geq 3$ and

$$\Gamma_{\alpha\beta}(x; 0) := \frac{1}{2\omega_2}\Big\{ - \delta_{\alpha\beta}\log(|x|) + \frac{x_\alpha x_\beta}{|x|^2}\Big\} \tag{2.33}$$

for $d = 2$. Note that the given fundamental solution for the case $d = 2$ differs from the one given by Mitrea and Wright [22, Sec. 4.2] by having summands with alternating signs. Considering the structure of the fundamental solution for $d \geq 3$, our choice seems more natural with regard to the structure of the fundamental solutions to the Laplace equation (2.16). The alternating sign is necessary for $\Gamma_{\alpha\beta}$ to be divergence free.

One important technique in the following chapter will be to reduce problems formulated for $\Gamma(x; \lambda)$ to problems formulated in $\Gamma(x; 0)$ perturbed by the difference $\Gamma(x; \lambda) - \Gamma(x; 0)$. Under this aspect it seems reasonable to study estimates of the difference of fundamental solutions. To this end it is helpful to rewrite parts of the fundamental solution. Using the fact that for $d \geq 5$ or $d = 3$, we have

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{1}{|x|^{d-4}}\right) = -(d-4)\frac{\partial}{\partial x_\alpha}\frac{x_\beta}{|x|^{d-2}} = -(d-4)\frac{\delta_{\alpha\beta}}{|x|^{d-2}} + \frac{(d-4)(d-2)x_\alpha x_\beta}{|x|^d}.$$

This allows us to write

$$\frac{x_\alpha x_\beta}{|x|^d} = \frac{\delta_{\alpha\beta}}{(d-2)|x|^{d-2}} + \frac{1}{(d-4)(d-2)}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{1}{|x|^{d-4}}\right),$$

which, considering definition (2.32), gives

$$\Gamma_{\alpha\beta}(x; 0) = G(x; 0)\delta_{\alpha\beta} + \frac{1}{2\omega_d(d-4)(d-2)}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{1}{|x|^{d-4}}\right). \tag{2.34}$$

A similar trick works for $d = 4$: Since $\omega_4 = 2\pi^2$, we have

$$\Gamma_{\alpha\beta}(x; 0) = \frac{1}{2\omega_4}\frac{1}{|x|^2}\delta_{\alpha\beta} - \frac{1}{8\pi^2}\left(\frac{\delta_{\alpha\beta}}{|x|^2} - \frac{2x_\alpha x_\beta}{|x|^4}\right)$$

$$
\begin{aligned}
&= G(x;0)\delta_{\alpha\beta} - \frac{1}{8\pi^2}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}(\log(|x|)) \\
&= G(x;0)\delta_{\alpha\beta} - \frac{1}{4\omega_4}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}(\log(|x|))
\end{aligned} \tag{2.35}
$$

In the case $d = 2$, we use

$$
\frac{1}{8\pi}\frac{\partial^2}{\partial_\alpha \partial_\beta}(|x|^2 \log(|x|)) = \frac{\delta_{\alpha\beta}}{4\pi}\log(|x|) + \frac{1}{4\pi}\frac{x_\alpha x_\beta}{|x|^2} + \frac{\delta_{\alpha\beta}}{8\pi}
$$

to find the identity

$$
\Gamma_{\alpha\beta}(x;0) = G(x;0)\delta_{\alpha\beta} - \frac{\delta_{\alpha\beta}}{8\pi} - \frac{\partial^2}{\partial x_\alpha \partial x_\beta}(|x|^2 \log(|x|)). \tag{2.36}
$$

This ends the preparatory step and brings us to the next theorem.

Theorem 2.6. *Let $\lambda \in \Sigma_\theta$. Suppose that $|\lambda||x|^2 \leq (1/2)$. Then*

$$
\left|\nabla_x\{\Gamma(x;\lambda) - \Gamma(x;0)\}\right| \leq \begin{cases} C|\lambda||x|^{3-d} & \text{if } d \geq 7 \text{ or } d = 5, \\ C|\lambda||x|^{3-d}\left|\log(|\lambda||x|^2)\right| & \text{if } d = 4 \text{ or } 6, \\ C\sqrt{|\lambda|}|x|^{-1} & \text{if } d = 3, \\ C|\lambda||x|\left|\log(|\lambda||x|^2)\right| & \text{if } d = 2, \end{cases} \tag{2.37}
$$

where C depends only on d and θ.

Proof. We will split the proof in several parts. According to the preparatory step, for $d \geq 2$ and all $\alpha, \beta = 1, \dots, d$, the difference $\partial_\gamma\{\Gamma_{\alpha\beta}(x;\lambda) - \Gamma_{\alpha\beta}(x;0)\}, \gamma = 1, \dots, d$, is always of the form

$$
\begin{aligned}
&\frac{\partial}{\partial x_\gamma}\left\{\Gamma_{\alpha\beta}(x;\lambda) - \Gamma_{\alpha\beta}(x;0)\right\} \\
&\quad = \frac{\partial}{\partial x_\gamma}\left\{G(x;\lambda) - G(x;0)\right\}\delta_{\alpha\beta} - \frac{1}{\lambda}\frac{\partial^3}{\partial x_\gamma \partial x_\alpha \partial x_\beta}\left\{G(x;\lambda) - G(x;0) + \left[\dots\right]\right\}.
\end{aligned}
$$

But the first term on the right hand side of the above expression is already under control thanks to Lemma 2.3. It thus suffices to estimate the second term on the right hand side.

We start by considering the cases $d = 3$ and $d \geq 5$. Taking into account identity (2.34), we have for all $\alpha, \beta, \gamma = 1, \dots, d$:

$$
G(x;\lambda) - G(x;0) + \left[\dots\right] = \frac{1}{\lambda}\left\{G(x;\lambda) - G(x;0) + \frac{\lambda}{2\omega_d(d-4)(d-2)|x|^{d-4}}\right\}.
$$

If $d = 3$, a direct calculation will then yield the desired result: We start by noting that $\omega_3 = 4\pi$ gives

$$\begin{aligned}G(x;\lambda) - G(x;0) - \frac{\lambda}{2\omega_3|x|^{-1}} &= \frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\pi|x|} - \frac{1}{4\pi|x|} - \frac{(\mathrm{i}k)^2}{2\omega_3|x|^{-1}}\\ &= \frac{1}{4\pi|x|}\Big(\mathrm{e}^{\mathrm{i}k|x|} - 1 - \frac{(\mathrm{i}k)^2|x|^2}{2}\Big)\\ &= \frac{1}{4\pi|x|}\Big(\mathrm{i}k|x| + \sum_{n=3}^{\infty}\frac{(\mathrm{i}k|x|)^n}{n!}\Big)\\ &= \frac{1}{4\pi}\Big(\mathrm{i}k + \sum_{n=3}^{\infty}\frac{(\mathrm{i}k)^n|x|^{n-1}}{n!}\Big).\end{aligned}$$

Taking the first derivative of this expression we get

$$\frac{\partial}{\partial x_\beta}\Big\{\dots\Big\} = \frac{x_\beta}{4\pi}\sum_{n=3}^{\infty}\frac{(ik)^n(n-1)}{n!}|x|^{n-3}$$

and differentiating with respect to x_α yields

$$\frac{\partial}{\partial x_\alpha}\Big\{\dots\Big\} = \frac{\delta_{\alpha\beta}}{4\pi}\sum_{n=3}^{\infty}\frac{(ik)^n(n-1)}{n!}|x|^{n-3} + \frac{x_\beta x_\alpha}{4\pi}\sum_{n=4}^{\infty}\frac{(\mathrm{i}k)^n(n-1)(n-3)}{n!}|x|^{n-5}.$$

As we are interested in estimating the *gradient* of the difference of $\Gamma(x;\lambda)$ and $\Gamma(x;0)$, we have to consider one additional derivative. This leaves us with

$$\begin{aligned}\frac{\partial}{\partial x_\gamma}\Big\{\dots\Big\} = {}&\frac{\delta_{\alpha\beta}x_\gamma + \delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta}{4\pi}\sum_{n=4}^{\infty}\frac{(\mathrm{i}k)^n(n-1)(n-3)}{n!}|x|^{n-5}\\ &+ \frac{x_\beta x_\alpha x_\gamma}{4\pi}\sum_{n=4}^{\infty}\frac{(\mathrm{i}k)^n(n-1)(n-3)(n-5)}{n!}|x|^{n-7}.\end{aligned}$$

We can now prove the desired estimate via

$$\begin{aligned}\left|\frac{1}{\lambda}\frac{\partial^3}{\partial x_\gamma\partial x_\alpha\partial x_\beta}\Big\{\dots\Big\}\right| &\le \frac{1}{|k|^2\pi}\sum_{n=4}^{\infty}\frac{|k|^n(n-1)(n-3)(1+(n-5))}{n!}|x|^{n-4}\\ &\le \frac{1}{|k|^2|x|\pi}|k|^3\sum_{k=4}^{\infty}\frac{(n-1)(n-3)(1+(n-5))}{n!}|k|^{n-3}|x|^{n-3}\\ &\le C\frac{1}{|k||x|}.\end{aligned}$$

This gives the claim for $d = 3$. If $d \ge 5$, equation (2.21) gives

$$\begin{aligned}G(x;\lambda) - G(x;0) + \frac{\lambda}{2\omega_2(d-4)(d-2)|x|^{d-4}}&\\ = \frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}}\frac{1}{|x|^{d-2}}\Big\{z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2\Big\},&\end{aligned} \tag{2.38}$$

where $z = k|x|$, a_d was calculated in (2.20) and b_d is given by

$$b_d = -\frac{2\mathrm{i}(2\pi)^{\frac{d}{2}-1}}{\omega_d(d-4)(d-2)}.$$

Using relation (2.17) and the functional equation of the Gamma function (2.18) twice, we see that

$$\begin{aligned} b_d &= -\frac{2\mathrm{i}(2\pi)^{\frac{d}{2}-1}\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}(d-2)(d-4)} = \frac{2^{\frac{d}{2}-1}}{\pi\mathrm{i}(d-4)}\frac{\Gamma(\frac{d}{2})}{(d-2)} = \frac{2^{\frac{d}{2}-1}}{2\pi\mathrm{i}}\frac{\Gamma(\frac{d}{2}-1)}{(d-4)} \\ &= \frac{2^{\frac{d}{2}-1}}{4\pi\mathrm{i}}\frac{\Gamma(\frac{d}{2}-1)}{(\frac{d}{2}-1-1)} = \frac{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2}-1-1)}{4\pi\mathrm{i}} = \frac{2^{\nu_d}\Gamma(\nu_d-1)}{4\pi\mathrm{i}}. \end{aligned}$$

This shows that for $d \geq 5$, b_d is the second coefficient of the asymptotic expansions (2.13)-(2.15), respectively. Now we split the proof for $d \geq 5$ into (1) $d \geq 7$, (2) $d = 6$ and (3) $d = 5$. If $d \geq 7$, we use the asymptotic expansions (2.15) to estimate the part of (2.38) which involves the Hankel function as

$$\left|\frac{\mathrm{d}^l}{\mathrm{d}z^l}\left\{z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2\right\}\right| \leq C|z|^{4-l} \tag{2.39}$$

for $0 \leq l \leq 3$ and $z \in \mathbb{C} \setminus (-\infty, 0]$. For better readability we define the function

$$g(z) := z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2$$

and consider the function $f(x) := g(k|x|)$ on $\mathbb{R}^d \setminus \{0\}$. The derivatives of f read

$$\begin{aligned} \frac{\partial}{\partial x_\beta} f(x) &= \Big(\frac{\mathrm{d}}{\mathrm{d}z}g\Big)(k|x|) \cdot \frac{kx_\beta}{|x|}, \\ \frac{\partial^2}{\partial x_\alpha \partial x_\beta} f(x) &= \Big(\frac{\mathrm{d}^2}{\mathrm{d}z^2}g\Big)(k|x|) \cdot \frac{k^2 x_\alpha x_\beta}{|x|^2} + \Big(\frac{\mathrm{d}}{\mathrm{d}z}g\Big)(k|x|) \cdot k\left\{\frac{\delta_{\alpha\beta}}{|x|} - \frac{x_\beta x_\alpha}{|x|^3}\right\}, \\ \frac{\partial^3}{\partial x_\gamma \partial x_\alpha \partial x_\beta} f(x) &= \Big(\frac{\mathrm{d}^3}{\mathrm{d}z^3}g\Big)(k|x|) \cdot \frac{k^3 x_\alpha x_\beta x_\gamma}{|x|^3} \\ &\quad + \Big(\frac{\mathrm{d}^2}{\mathrm{d}z^2}g\Big)(k|x|) \cdot k^2\left\{\frac{x_\alpha\delta_{\beta\gamma} + x_\beta\delta_{\alpha\gamma} + x_\gamma\delta_{\alpha\beta}}{|x|^2} - \frac{x_\alpha x_\beta x_\gamma}{|x|^4}\right\} \\ &\quad + \Big(\frac{\mathrm{d}}{\mathrm{d}z}g\Big)(k|x|) \cdot k\left\{-\frac{x_\gamma\delta_{\alpha\beta} + x_\alpha\delta_{\beta\gamma} + x_\beta\delta_{\alpha\gamma}}{|x|^3} + \frac{3x_\alpha x_\beta x_\gamma}{|x|^5}\right\}. \end{aligned}$$

If we now look for estimates on the absolute value of the derivatives, we see that by (2.39)

$$\left|\nabla^l f(x)\right| \leq C|k|^4|x|^{4-l}, \quad 1 \leq l \leq 3,$$

where C only depends on l. We can now finally uncover the desired estimate via

$$\left|\frac{1}{\lambda}\nabla_x^3\left\{\frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}}\frac{1}{|x|^{d-2}}\left\{z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2\right\}\right\}\right|$$

$$\leq C\frac{1}{|k|^2}\sum_{l=0}^{3}\left|\nabla^{3-l}\Big(\frac{1}{|x|^{d-2}}\Big)\right|\left|\nabla^l f(x)\right| \leq C\sum_{l=0}^{3}|x|^{-d+2-3+l}|k|^2|x|^{4-l} = C|\lambda||x|^{3-d},$$

where C is a constant only depending on d.

If $d = 6$, the asymptotic expansion (2.14) gives us in analogy to (2.39) the estimate

$$\left|\frac{\mathrm{d}^l}{\mathrm{d}z^l}\left\{z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2\right\}\right| \leq C|z|^{4-l}\left|\log(z)\right|, \tag{2.40}$$

for $0 \leq l \leq 3$ and $z \in \mathbb{C} \setminus (-\infty, 0]$. Using as before the expressions for the derivatives of f, we estimate their absolute values as

$$\left|\nabla^l f(x)\right| \leq C|k|^4|x|^{4-l}\left|\log(|\lambda||x|^2)\right|,$$

which, by a calculation analogous to the case $d \geq 7$, yields the claim.

For $d = 5$, we differentiate (2.38) twice and use relation (2.34) for the fundamental solution of the Stokes problem to write

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left\{G(x;\lambda) - G(x;0) + \frac{\lambda}{6\,\omega_5|x|^{d-4}}\right\}$$
$$= \frac{\partial^2}{\partial x_\alpha x_\beta}\left\{\frac{\mathrm{i}}{4(2\pi)^{\frac{3}{2}}}\cdot\frac{1}{|x|^3}\left[z^{\frac{3}{2}}H^{(1)}_{\frac{3}{2}}(z) - a_5 - b_5 z^2 - wz^3\right]\right\},$$

where $w \in \mathbb{C}$ can be an arbitrary constant if we set $z = k|x|$. Now, for the appropriate choice of $w \in \mathbb{C}$ the asymptotic expansion (2.13) gives the same estimate as (2.39) which, like for $d \geq 7$, proves the claim for $d = 5$.

In the case $d = 4$ we use the respective relation for the fundamental solution (2.35) in order to simplify the difference

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left\{G(x;\lambda) - G(x;0) - \frac{\lambda\log(|x|)}{4\,\omega_4}\right\}$$
$$= \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left\{\frac{\mathrm{i}}{8\pi|x|^2}\left[zH^{(1)}_1(z) - a_4 - wz^2 - b_4 z^2\log(z)\right]\right\},$$

where $z = k|x|$, $b_4 = (\mathrm{i}/\pi)$ and $w \in \mathbb{C}$ is an arbitrary constant. Using the asymptotic expansion (2.12) and the appropriate constant $w \in \mathbb{C}$ we get the estimate

$$\left|\frac{\mathrm{d}^l}{\mathrm{d}z^l}\left\{zH^{(1)}_1(z) - a_4 - wz^2 - b_4 z^2\log(z)\right\}\right| \leq C|z|^{4-l}\left|\log(z)\right|.$$

The estimate has the same right hand side as (2.40) and the proof can be carried out just as in the previous cases.

For $d = 2$, the claimed estimate follows from a direct calculation which is postponed until appendix A.2. □

We can now use the assumption $|\lambda||x|^2 \leq (1/2)$ to unify the structure of the estimates from Theorem 2.6.

Corollary 2.7. *Let $\lambda \in \Sigma_\theta$. Suppose that $|\lambda||x|^2 \leq (1/2)$. Then for all $d \geq 2$*

$$\left|\nabla_x\{\Gamma(x;\lambda) - \Gamma(x;0)\}\right| \leq C\sqrt{|\lambda|}|x|^{2-d},$$

where C depends only on d and θ.

Proof. We just extend the estimates given in Theorem 2.6. Let $d \geq 7$ or $d = 5$. Since $\sqrt{|\lambda|} \leq C|x|^{-1}$, we have

$$C|\lambda||x|^{3-d} \leq C\sqrt{|\lambda|}|x|^{2-d}.$$

For $d = 2, 4, 6$, we have

$$|\lambda||x|^{3-d}\left|\log(|\lambda||x|^2)\right| = C\sqrt{|\lambda|}|x|^{2-d} \cdot \sqrt{|\lambda|}|x|\left|\log(|\lambda||x|^2)\right| \leq C\sqrt{|\lambda|}|x|^{2-d},$$

since $\sqrt{|\lambda|}|x|\left|\log(|\lambda||x|^2)\right|$ is bounded for $|\lambda||x|^2 \leq (1/2)$. $\square$

Chapter 3

Single and Double Layer Potentials

In this chapter, we will deal with *single* and *double layer potentials*. Both will serve as "representation formulas" for solutions to the Stokes resolvent problem. We will study their properties as they will serve as the crucial ingredient to solving the L^2 Dirichlet problem associated to the Stokes resolvent problem on bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$: For $\lambda \in \mathbb{C} \setminus (-\infty, 0)$ and

$$g \in \mathrm{L}^2_\nu(\partial\Omega) := \left\{ g \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d) \colon \int_{\partial\Omega} g \cdot \nu \, \mathrm{d}\sigma = 0 \right\}$$

we are looking for smooth functions u and ϕ that satisfy

$$(\mathrm{Dir}_\lambda) \begin{cases} -\Delta u + \nabla \phi + \lambda u = 0 & \in \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = g & \text{nontangentially on } \partial\Omega, \\ (u)^* \in \mathrm{L}^2(\partial\Omega). \end{cases}$$

In this chapter we will thus always assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$ with $d \geq 2$ and $1 < p < \infty$. We will also tacitly use the summation convention.

We note that due to the new two dimensional estimates on fundamental solutions in Chapter 2, namely the continuation of Theorem 2.5and Theorem 2.6 for the case $d = 2$, we could extend all results from Chapter 3 of Shen's seminal paper [26] that are relevant to the analysis of the L^2 Dirichlet problem in a straightforward way.

Let $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi/2)$. Furthermore, let $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$. The single layer potential $u = \mathcal{S}_\lambda(f)$ is defined by

$$(\mathcal{S}_\lambda(f))_j(x) := \int_{\partial\Omega} \Gamma_{jk}(x - y; \lambda) f_k(y) \, \mathrm{d}\sigma(y), \tag{3.1}$$

where Γ_{jk} is the fundamental solution to the Stokes resolvent problem given by (2.27). For the pressure, respectively, we define the single layer potential $\phi = \mathcal{S}_\Phi(f)$ by

$$\mathcal{S}_\Phi(f)(x) := \int_{\partial\Omega} \Phi_k(x-y) f_k(y) \, \mathrm{d}\sigma(y), \tag{3.2}$$

where Φ_k is given by (2.28). As we have already shown, (u, ϕ) defines a solution to the Stokes resolvent problem (2.26) in $\mathbb{R}^d \setminus \partial\Omega$.

We define two further integral operators that map to functions living on $\partial\Omega$:

$$T_\lambda^*(f)(q) = \sup_{t>0} \left| \int_{\substack{y\in\partial\Omega \\ |y-q|>t}} \nabla_x \Gamma(q-y;\lambda) f(y) \, \mathrm{d}\sigma(y) \right| \tag{3.3}$$

$$T_\lambda(f)(q) = \mathrm{p.\,v.} \int_{\partial\Omega} \nabla_x \Gamma(q-y;\lambda) f(y) \, \mathrm{d}\sigma(y) \tag{3.4}$$

for $q \in \partial\Omega$ which will be used to prove boundedness of maximal operators related to u and its gradient.

The following lemma will be a good companion for the forthcoming calculation of estimates.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain with corresponding numbers r_0 and M. Then there exists $C > 0$ depending only on d and M such that*

$$\sigma(\mathrm{B}(q,r) \cap \partial\Omega) \leq C r^{d-1}$$

for all $r > 0$ and $q \in \partial\Omega$. Furthermore, there exists a constant $C > 0$, depending only on d and the Lipschitz character of Ω, such that

$$\sigma(\partial\Omega) \leq C r_0^{d-1}.$$

Another cornerstone in the theory of the single and double layer potentials is the following lemma, see Tolksdorf [32, Lem. 4.3.2], as it will allow us to bring into play the estimates from Section 2.2.

Lemma 3.2. *Let $\Omega \subset R^d$, $d \geq 2$, be a bounded Lipschitz domain with corresponding numbers r_0 and M. Let $x \in \mathbb{R}^d$, $0 < \varepsilon \leq (r_0/4)$, and $l \in \mathbb{N}_0$ with $l < d-1$. Then there exists a constant $C > 0$ depending only on d, l and M such that*

$$\int_{\partial\Omega\cap\mathrm{B}(x,\varepsilon)} \frac{1}{|x-y|^l} \, \mathrm{d}\sigma(y) \leq C\varepsilon^{d-l-1}.$$

We are now in the position to prove our first lemma on the way to establish the single layer potential as a benevolent operator for tackling boundary value problems on bounded Lipschitz domains. The lemma deals with mapping properties of the aforementioned integral operators T_λ and T_λ^*.

Lemma 3.3. *Let* $1 < p < \infty$ *and* $T_\lambda(f), T_\lambda^*(f)$ *be defined by* (3.3) *and* (3.4). *Then* $T_\lambda(f)(P)$ *exists for almost everywhere* $P \in \partial\Omega$ *and*

$$\|T_\lambda(f)\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^{d\times d})} \leq \|T_\lambda^*(f)\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p \, \|f\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)}, \tag{3.5}$$

where C_p *depends only on* d, θ, p, *and the Lipschitz character of* Ω.

Proof. If $\lambda = 0$, the lemma is known due to Fabes, Kenig and Verchota [8] as a consequence of the seminal result of Coifman, McIntosh and Meyer [3]. One idea of the proof in the case $\lambda \in \Sigma_\theta$ will thus be to nourish from this result and to consider the difference $\Gamma(x-y;\lambda) - \Gamma(x-y;0)$.

We start with the second inequality of (3.5). To this end, let $t > 0$ and additionally assume that $t^2|\lambda| > (1/2)$. In this case, Theorem 2.5 gives us the estimate

$$\left| \int_{|y-q|>t} \nabla_x \Gamma(q-y;\lambda) f(y) \, \mathrm{d}\sigma(y) \right| \leq C \int_{|q-y|>t} \frac{|f(y)|}{|\lambda||q-y|^{d+1}} \, \mathrm{d}\sigma(y),$$

where C depends on d and θ. Choose now $N \in \mathbb{N}$ such that $2^N t \leq \mathrm{diam}(\Omega) < 2^{N+1} t$. We now exhaust the domain of integration by suitable annuli and use the inner radii to simplify the integrand and the outer radii to amplify the domain of integration:

$$\begin{aligned}
&\sum_{k=0}^{N} \int_{2^k t < |q-y| < 2^{k+1}} \frac{1}{|\lambda||q-y|^{d+1}} |f(y)| \, \mathrm{d}\sigma(y) \\
&\qquad \leq \sum_{k=0}^{N} \int_{2^k t < |q-y| < 2^{k+1}} \frac{1}{|\lambda| 2^{k(d+1)} t^{d+1}} |f(y)| \, \mathrm{d}\sigma(y) \\
&\qquad \leq \frac{1}{|\lambda| t^2} \frac{1}{2^{1-d}} \sum_{k=0}^{N} \frac{1}{2^{2k}} \frac{1}{(2^{k+1}t)^{d-1}} \int_{\mathrm{B}(q,2^{k+1}t)\cap\partial\Omega} |f(y)| \, \mathrm{d}\sigma(y).
\end{aligned} \tag{3.6}$$

Note that due to Lemma 3.1 we have that

$$\frac{1}{(2^{k+1}t)^{d-1}} \int_{\mathrm{B}(q,2^{k+1}t)\cap\partial\Omega} |f(y)| \, \mathrm{d}\sigma(y) \leq C M_{\partial\Omega}(f)(q), \quad k = 0, \dots, N, \tag{3.7}$$

with a constant C that depends on d and the Lipschitz character of Ω. Now we glue together (3.6) and (3.7), take $N \to \infty$ noting the geometric series and get the estimate

$$\left| \int_{|y-q|>t} \nabla_x \Gamma(q-y;\lambda) f(y) \, \mathrm{d}\sigma(y) \right| \leq C M_{\partial\Omega}(f)(q). \tag{3.8}$$

Now let $t^2|\lambda| \leq (1/2)$. We then split the integral as follows:

$$\begin{aligned}\left|\int_{|q-y|>t} \nabla_x \Gamma(q-y;\lambda) f(y)\,\mathrm{d}\sigma(y)\right| &\leq \left|\int_{|q-y|\geq(2|\lambda|)^{-1/2}} \nabla_x \Gamma(q-y;\lambda) f(y)\,\mathrm{d}\sigma(y)\right| \\ &\quad + \left|\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x \Gamma(q-y;\lambda) f(y)\,\mathrm{d}\sigma(y)\right|.\end{aligned}$$

For the first summand, note that estimate (3.8) holds for all $t > 0$ and thus in particular for $t = (2|\lambda|)^{-1/2}$. For the second term, we add a special zero and use the triangle inequality to estimate

$$\begin{aligned}&\left|\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x \Gamma(q-y;\lambda) f(y)\,\mathrm{d}\sigma(y)\right| \\ &\qquad \leq \int_{t<|q-y|<(2|\lambda|)^{-1/2}} \left|\nabla_x \Gamma(q-y;\lambda) - \nabla_x \Gamma(q-y;0)\right| |f(y)|\,\mathrm{d}\sigma(y) \\ &\qquad\quad + \left|\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x \Gamma(q-y;0) f(y)\,\mathrm{d}\sigma(y)\right|.\end{aligned}$$

We don't need to worry about the second summand here since the corresponding estimate is already covered by the $\lambda = 0$ case:

$$\begin{aligned}&\left|\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x \Gamma(q-y;0)\, f(y)\,\mathrm{d}\sigma\right| \\ &\qquad \leq \left|\int_{|q-y|>t} \nabla_x \Gamma(q-y;0)\, f(y)\,\mathrm{d}\sigma\right| \leq T_0^*(f)(q).\end{aligned} \quad 1 \tag{3.9}$$

For the first summand we make use of Theorem 2.6 and more precisely of Corollary 2.7 which unifies all estimates: We start by estimating

$$\begin{aligned}&\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \left|\nabla_x \Gamma(q-y;\lambda) - \nabla_x \Gamma(q-y;0)\right| |f(y)|\,\mathrm{d}\sigma(y) \\ &\qquad \leq C \int_{t<|q-y|<(2|\lambda|)^{-1/2}} \sqrt{|\lambda|}\, |q-y|^{2-d}\, |f(y)|\,\mathrm{d}\sigma(y),\end{aligned} \tag{3.10}$$

where C depends on d and θ. Now we choose N such that

$$2^{N+1}t > (2|\lambda|)^{-1/2} \geq 2^N t \tag{3.11}$$

holds. Once again we integrate over annuli, use the inner radii to loose the term $|q-y|^{2-d}$ and use the outer radii to expand the domain of integration to balls with this radius:

$$\begin{aligned}\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \frac{1}{|q-y|^{d-2}} |f(y)|\,\mathrm{d}\sigma &\leq \sum_{k=0}^{N} \int_{2^k t\leq|q-y|<2^{k+1}t} \frac{1}{|q-y|^{d-2}} |f(y)|\,\mathrm{d}\sigma \\ &\leq \sum_{k=0}^{N} \frac{1}{(2^k t)^{d-2}} \int_{\mathrm{B}(q,2^{k+1}t)\cap\partial\Omega} |f(y)|\,\mathrm{d}\sigma\end{aligned}$$

$$= 2^d \sum_{k=0}^{N} 2^k t \, \frac{1}{(2^{k+1}t)^{d-1}} \int_{\mathrm{B}(q,2^{k+1}t)\cap\partial\Omega} |f(y)| \, \mathrm{d}\sigma.$$

As before we use Lemma 3.1 to bring the Hardy-Littlewood maximal operator into the game like for inequality (3.7). This time we cannot take $N \longrightarrow \infty$ as the resulting geometric series wouldn't converge. But N was chosen wisely, see (3.11) and thus

$$\sum_{k=0}^{N} 2^k t \leq 2^{N+1} t \leq 2^{1/2} |\lambda|^{-1/2}.$$

which yields the inequality

$$\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \frac{1}{|q-y|^{d-2}} \, |f(y)| \, \mathrm{d}\sigma \leq C|\lambda|^{-1/2} M_{\partial\Omega}(f)(q),$$

where C depends on d and the Lipschitz character of Ω. Taking into account the foregoing calculations together with estimate (3.10) and (3.9) we derive

$$\left| \int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x \Gamma(q-y;0) f(y) \, \mathrm{d}\sigma \right| \leq C \Big\{ T_0^*(f)(q) + M_{\partial\Omega}(f)(q) \Big\}, \qquad (3.12)$$

with C depending only on d, θ and the Lipschitz character of Ω.

It is now the time to take the supremum over all $t > 0$, and considering estimates (3.8) and (3.12) we finally see that

$$T_\lambda^*(f)(q) \leq C \Big\{ T_0^*(f)(q) + M_{\partial\Omega}(f)(q) \Big\},$$

for all $q \in \partial\Omega$. Once again using the result for $\lambda = 0$ and the L^p boundedness of the Hardy-Littlewood maximal operator, we conclude the first part of the claimed inequality

$$\|T_\lambda^*(f)\|_{\mathrm{L}^p(\partial\Omega)} \leq C \, \|f\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)}$$

To conclude the left inequality in (3.5), we want to use a standard result argument harmonic analysis. To this end we define the operators

$$T_\lambda^{(t)}(f)(q) := \int_{\substack{y\in\partial\Omega \\ |y-q|>t}} \nabla_x \Gamma(q-y;\lambda) f(y) \, \mathrm{d}\sigma(y).$$

Suppose we can show that

$$T_\lambda(f)(q) = \lim_{t\to 0} T_\lambda^{(t)}(f)(q) \qquad (3.13)$$

exists for almost every $q \in \partial\Omega$ and all $f \in \mathrm{C}(\partial\Omega;\mathbb{C}^d)$. Now, note that $\mathrm{C}(\partial\Omega;\mathbb{C}^d)$ is dense in $\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ and that $T_\lambda^*(f)$ is bounded on $\mathrm{L}^p(\partial\Omega;C^d)$ as we showed earlier. Then Grafakos [14, Thm. 2.1.14] gives that T_λ is bounded on $\mathrm{L}^p(\partial\Omega;\mathbb{C}^{d\times d})$.

In order to prove the existence of the pointwise limit (3.13), we split the operator T_λ as follows:

$$T_\lambda(f)(q) = T_0(f)(q) + \lim_{t\to 0} \int_{\substack{y\in\partial\Omega \\ |y-q|>t}} \nabla_x \big\{\Gamma(q-y;\lambda) - \Gamma(q-y;0)\big\} f(y)\,\mathrm{d}\sigma(y).$$

The right summand is well defined for $f \in \mathrm{C}(\partial\Omega;\mathbb{C}^d)$, once we prove integrability of the integral kernel $\big|\nabla_x\big\{\Gamma(q-y;\lambda) - \Gamma(q-y;0)\big\}\big|$ on $\partial\Omega$. To this end, we first note that it suffices to consider the integral

$$\int_{|q-y|\le\varepsilon} \big|\nabla_x\big\{\Gamma(q-y;\lambda) - \Gamma(P-y;0)\big\}\big|\,\mathrm{d}\sigma(y),$$

for $\varepsilon \le \min(2|\lambda|^{-1/2}, r_0/4)$ as the integrand is bounded away from 0 and the domain of integration is bounded. Now Corollary 2.7 and Lemma 3.2 give that the integrand can be estimated by

$$C\int_{|q-y|\le\varepsilon} \sqrt{|\lambda|}|q-y|^{2-d}\,\mathrm{d}\sigma(y) \le C\sqrt{|\lambda|}\varepsilon \le C,$$

where C is a constant depending on d, θ and the Lipschitz character of Ω. Based on the preceding calculation we conclude that for all $f \in \mathrm{C}(\partial\Omega, \mathbb{C}^d)$ the operator $T_\lambda(f)(q)$ exists whenever $T_0(f)(q)$ exists. $T_0(f)(q)$ exists for almost everywhere $q \in \partial\Omega$ because of Fabes, Kenig and Verchota [8]. As furthermore $T^*_\lambda(f)(q)$ is bounded on $\mathrm{L}^p(\partial\Omega)$ we may now apply Theorem 2.1.14 from Grafakos [14] to conclude that $T_\lambda(f)(q)$ exists now for all $f \in \mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ and almost everywhere $q \in \partial\Omega$. The desired L^p estimate for $T_\lambda(f)$ now follows from the observation that $|T_\lambda(f)(q)| \le T^*_\lambda(f)(q)$ for almost everywhere $q \in \partial\Omega$. □

For further use, we state a very useful lemma which can be considered a *Young*-type inequality for L^p spaces on boundaries of Lipschitz domains. A proof can be found in Tolksdorf [32, Prop 1.1.4].

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^d$, μ a σ-finite measure on Ω, and $1 \le p < \infty$. Let $g\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a function such that the function $\Omega\times\Omega \ni (x,y) \mapsto g(x-y)$ is measurable with respect to the product measure $\mu\times\mu$ and such that*

$$A + B := \sup_{x\in\Omega} \|g(x-\cdot)\|_{\mathrm{L}^1(\Omega,\mu)} + \sup_{y\in\Omega} \|g(\cdot-y)\|_{\mathrm{L}^1(\Omega,\mu)} < \infty.$$

If $f \in \mathrm{L}^p(\Omega,\mu)$, then $x \mapsto \int_\Omega g(x-y)f(y)\,\mathrm{d}\mu(y) \in \mathrm{L}^p(\Omega,\mu)$ and

$$\Big\| \int_\Omega g(\cdot - y) f(y)\,\mathrm{d}\mu(y)\Big\|_{\mathrm{L}^p(\Omega,\mu)} \le A^{1-1/p}B^{1/p}\|f\|_{\mathrm{L}^p(\Omega,\mu)}.$$

For us, Lemma 3.4 will be applied often to integral kernels g that result from an application of the Theorems in Chapter 2. The following Lemma shows that these integral kernels fulfill the requirements from Lemma 3.4.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then*

$$\sup_{q\in\partial\Omega} \int_{\partial\Omega} \frac{1}{|q-y|^{d-2}} \le Cr_0,$$

where C is a constant depending only on d and the Lipschitz character of Ω.

Proof. Let r_0 be the radius from the definition of Lipschitz cylinders and $q \in \partial\Omega$. Splitting the domain of integration and applying Lemma 3.2, we get

$$\begin{aligned}
&\int_{\partial\Omega} \frac{1}{|q-y|^{d-2}}\, \mathrm{d}\sigma(y) \\
&\quad \le \int_{\partial\Omega\cap \mathrm{B}(q;r_0/4)} \frac{1}{|q-y|^{d-2}}\, \mathrm{d}\sigma(y) + \int_{\partial\Omega\setminus \mathrm{B}(q;r_0/4)} \frac{1}{|q-y|^{d-2}}\, \mathrm{d}\sigma(y) \\
&\quad \le Cr_0 + r_0^{2-d} 4^{d-2} \sigma(\partial\Omega) \le C(r_0 + r_0^{2-d} r_0^{d-1}),
\end{aligned}$$

where C depends only on d and the Lipschitz character of Ω. This proves the claim. □

We can now prove the boundedness of certain nontangential maximal operators.

Lemma 3.6. *Let $1 < p < \infty$ and (u, ϕ) be given by* (3.1) *and* (3.2). *Then*

$$\|(\nabla u)^*\|_{\mathrm{L}^p(\partial\Omega)} + \|(\phi)^*\|_{\mathrm{L}^p(\partial\Omega)} \le C_p \|f\|_{\mathrm{L}^p(\partial\Omega)}, \tag{3.14}$$

where C_p depends only on d, θ, p and the Lipschitz character of Ω. Let furthermore $d \ge 3$. Then

$$\|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \le C_p \|f\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)}, \tag{3.15}$$

where C_p depends only on d, θ, p and the Lipschitz character of Ω.

Proof. A proof of the estimate $\|(\phi)^*\|_{\mathrm{L}^p(\partial\Omega)} \le C_p \|f\|_{\mathrm{L}^p(\partial\Omega,\mathbb{C}^d)}$ can be found in Verchota's dissertation [34, Lem. 1.3]. The proof for $\|(\nabla u)^*\|_{\mathrm{L}^p(\partial\Omega)}$ works in the same way. We will provide a proof for the sake of completeness. To imitate the proof of Verchota, we will work with the corresponding type of cones. Therefore the results for ∇u and ϕ will at first only be established for the type of maximal operators defined by Verchota. The transferability to Shen's maximal operators is given by Tolksdorf [32, p. 90ff.] as the solution (u, ϕ) has a representation as a single layer potential.

Let $q \in \partial\Omega$, $x \in \Gamma_{\mathrm{V}}(q)$ and set $t = |x-q|$. Then,

$$\begin{aligned}
|(\nabla u)(x)| &= \left| \int_{\partial\Omega} \nabla_x \Gamma_{jk}(x-y;\lambda) f_k \,\mathrm{d}\sigma(y) \right| \\
&\leq \left| \int_{|y-q|>t} \nabla_x \Gamma_{jk}(x-y;\lambda) f_k \,\mathrm{d}\sigma(y) \right| + \left| \int_{|y-q|\leq t} \nabla_x \Gamma_{jk}(x-y;\lambda) f_k \,\mathrm{d}\sigma(y) \right| \\
&=: I_1 + I_2.
\end{aligned}$$

We will now estimate I_1 and I_2 separately. Note that in Verchota cones $\Gamma_{\mathrm{V}}(q)$ we have that for all $s \in \partial\Omega$ we have $|x-s| \geq C|x-q|$, where C is a constant only depending on d and the Lipschitz character of Ω, see inequality 1.1. By Theorem 2.5 we know that

$$\begin{aligned}
I_2 &\leq C \int_{|y-q|\leq t} \frac{1}{|x-y|^{d-1}} |f(y)| \,\mathrm{d}\sigma(y) \\
&\leq \frac{C}{t^{d-1}} \int_{|y-q|\leq t} |f(y)| \,\mathrm{d}\sigma(y) \leq C M_{\partial\Omega}(f)(q),
\end{aligned}$$

where we used also Lemma 3.1 to bring the Hardy-Littlewood maximal operator into play. Here, C depends on d, θ and the Lipschitz character of Ω. For I_1, we calculate

$$\begin{aligned}
&\left| \int_{|y-q|>t} \nabla_x \Gamma_{jk}(q-y;\lambda) f_k(y) - \nabla_x \Gamma_{jk}(q-y;\lambda) f_k(y) + \nabla_x \Gamma_{jk}(q-y;\lambda) f_k(y) \,\mathrm{d}\sigma(y) \right| \\
&\quad \leq \left| \int_{|y-q|>t} \nabla_x \Big\{ \Gamma_{jk}(x-y;\lambda) - \Gamma_{jk}(q-y;\lambda) \Big\} f_k(y) \,\mathrm{d}\sigma(y) \right| \\
&\qquad + \left| \int_{|y-q|>t} \nabla_x \Gamma_{jk}(q-y;\lambda) f_k(y) \,\mathrm{d}\sigma(y) \right|.
\end{aligned}$$

The second summand can directly be estimated by $T_\lambda^*(f)(q)$. For the first one we apply the mean value theorem and use Theorem 2.5 to derive the following estimation:

$$\begin{aligned}
\int_{|y-q|>t} & \big|\nabla_x \Gamma_{jk}(x-y;\lambda) - \nabla_x \Gamma_{jk}(P-y;\lambda)\big| \, |f(y)| \,\mathrm{d}\sigma(y) \\
&\leq \int_{|y-q|>t} \big|\nabla^2 \Gamma_{jk}(s-y;\lambda)\big| |x-q| \, |f(y)| \,\mathrm{d}\sigma(y) \\
&\leq C \int_{|y-q|>t} \frac{t}{|s-y|^d} |f(y)| \,\mathrm{d}\sigma(y) \\
&\leq C \int_{|y-q|>t} \frac{t}{|y-q|^d} |f(y)| \,\mathrm{d}\sigma(y) \\
&\leq C \int_{\partial\Omega} \frac{t}{(t+|y-q|)^d} |f(y)| \,\mathrm{d}\sigma(y),
\end{aligned}$$

where s is an element on the line connecting x and q and we used the property of Verchota cones that $|s-y| \geq C|y-q|$, see inequality (1.2). Note that Verchota cones are convex.

As in Verchota [34, Lem 1.3], the integral may now be bounded by the Hardy-Littlewood maximal operator due to an application of a suitable result from Grafakos [14, Thm. 2.1.10] as the kernel $t(t + |y - P|)^{-d}$ is uniformly integrable and radially decreasing on $\partial\Omega$. Summing up we have shown that

$$|(\nabla u)(x)| \leq C\Big\{M_{\partial\Omega}f(P) + T_\lambda^*(f)(P)\Big\},$$

where C only depends on d, θ and the Lipschitz character of Ω. We thus may take the supremum over all $x \in \Gamma_{\mathrm{V}}(q)$ and conclude the desired estimate by the well known mapping properties of the Hardy-Littlewood maximal operator and the respective results from Lemma 3.3.

We will now work on the proof of the estimate for $(u)^*$ for $d \geq 3$. In order to derive L^p bounds on this maximal operator, we will work directly with the Definition of the single layer potential (3.1). For $q \in \partial\Omega$, estimate (2.30) together with the estimate for Shen cones (1.3) gives that for all $x \in \Gamma(q)$

$$|u^*(x)| \leq C\int_{\partial\Omega} \frac{1}{|x - y|^{d-2}}|f(y)|\,\mathrm{d}\sigma(y) \leq C\int_{\partial\Omega} \frac{1}{|q - y|^{d-2}}|f(y)|\,\mathrm{d}\sigma(y),$$

where C only depends on d, θ and the Lipschitz character of Ω. Passing to the maximal operator yields the inequality

$$u^*(q) \leq C\int_{\partial\Omega} \frac{1}{|q - y|^{d-2}}|f(y)|\,\mathrm{d}\sigma(y).$$

Estimating the kernel via Lemma 3.5 and applying the Young inequality for convolutions from Lemma 3.4 the claim follows. □

Remark 3.7. We note that in addition to the consideration of $d = 2$, Lemma 3.6 differs in the form of estimate (3.15) from the original statement in Shen's work [26, Lem. 3.2]. There, the author derives an estimate of the form $|\lambda|^{1/2}\|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p\|f\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)}$ which is based on Shen's version of Lemma 2.1, namely [26, Lem. 2.1]. As we could not follow the proof in [26], we provided a similar estimate and since the estimate won't be needed in the course of this thesis, we will not pursue the verification of Shen's estimate further. Another approach to the integrability of $(u)^*$ for the L^2 case will be given in Chapter 4.

The next lemma deals with *trace formulas* for ∇u and ϕ. We will then finally be able to talk about boundary values since the existence of nontangential limits guarantees that there exists something on $\partial\Omega$ that is related to the function inside Ω or inside $\mathbb{R}^d \setminus \overline{\Omega}$, respectively.

Lemma 3.8. *Let* (u, ϕ) *be given by* (3.1) *and* (3.2) *with* $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ *and* $1 < p < \infty$. *Then*

$$\begin{aligned}\Big(\frac{\partial u_i}{\partial x_j}\Big)_{\pm}(x) &= \pm\frac{1}{2}\big\{n_j(x)f_i(x) - n_i(x)n_j(x)n_k(x)f_k(x)\big\} \\ &\quad + \mathrm{p.\,v.}\int_{\partial\Omega}\frac{\partial}{\partial x_j}\Big\{\Gamma_{ik}(x-y;\lambda)\Big\}f_k(y)\,\mathrm{d}\sigma(y), \qquad (3.16)\\ \phi_{\pm}(x) &= \mp\frac{1}{2}n_k(x)f_k(x) + \mathrm{p.\,v.}\int_{\partial\Omega}\Phi_k(x-y)f_k(y)\,\mathrm{d}\sigma(y)\end{aligned}$$

for almost everywhere $x \in \partial\Omega$. *The subscripts* $+$ *and* $-$ *indicate nontangential limits taken inside* Ω *and outside* $\overline{\Omega}$, *respectively.*

Proof. The correctness of the trace formulas (3.16) is known for the case $\lambda = 0$ due to Mitrea and Write [22, Prop 4.4]. This fact will now be reused for $\lambda \in \Sigma_\theta$. We insert a 0 to the nontangential limit such that

$$(\nabla u_j)_{\pm}(x) = (\nabla v_j)_{\pm}(x) + (\nabla u_j - \nabla v_j)_{\pm}(x),$$

where $v_j(x) = \int_{\partial\Omega}\Gamma_{jk}(x-y;0)f_k(y)\,\mathrm{d}\sigma(y)$. Because of [22] we know that the first nontangential limit exists and is given by (3.16) with $\lambda = 0$. It therefore remains to show the identity

$$(\nabla u_j - \nabla v_j)_{\pm}(x) = \int_{\partial\Omega}\nabla_x\Big\{\Gamma_{jk}(x-y;\lambda) - \Gamma_{jk}(x-y;0)\Big\}f_k(y)\,\mathrm{d}\sigma(y)$$

for all $x \in \partial\Omega$. To this end let $(x_l)_{l\in\mathbb{N}}$ a sequence in $\Gamma(x)$ with $\lim_{l\to\infty} x_l = x$. Furthermore let us note that for almost everywhere $x \in \partial\Omega$ we have that

$$\int_{\partial\Omega}\frac{1}{|x-y|^{d-2}}|f(y)|\,\mathrm{d}\sigma(y) < \infty.$$

This is a consequence of Lemma 3.5 and Young's inequality from Lemma 3.4. Now, we will show that

$$\frac{1}{|x-y|^{d-2}}|f(y)|$$

gives a suitable function for dominated convergence. Set $\varepsilon = (4|\lambda|^2)^{-1}$ and without loss of generality assume that $\operatorname{supp} f \subseteq \mathrm{B}(x,\varepsilon)$. Furthermore assume that $|x_l - x| < \varepsilon$ for all $l \in \mathbb{N}$. Then $|x_l - y| \le (2|\lambda|^2)^{-1}$ and Corollary 2.7 give

$$\begin{aligned}&\Big|\int_{\partial\Omega}\nabla_x\Big\{\Gamma_{jk}(x_l-y;\lambda) - \Gamma_{jk}(x_l-y;0)\Big\}f_k(y)\,\mathrm{d}\sigma(y)\Big| \\ &\quad\le \int_{\partial\Omega}\sqrt{|\lambda|}\frac{1}{|x_l-y|^{d-2}}\,|f(y)|\,\mathrm{d}\sigma(y)\end{aligned}$$

$$\leq C\sqrt{|\lambda|}\int_{\partial\Omega}\frac{1}{|x_l-y|^{d-2}}\,|f(y)|\,\mathrm{d}\sigma(y)<\infty.$$

Now dominated convergence gives the claim for $x_l \longrightarrow x$. Note that it does not affect the proof if the sequence x_l lays inside Ω or outside $\overline{\Omega}$ and thus the same proof holds for a sequence (x_l) in $\Gamma^{\mathrm{ext}}(x)$. □

The previous lemma enables us to talk about boundary values of partial derivatives. The next theorem will now give a similar result but for *conormal derivatives*, which are defined for solutions (u,ϕ) to the Stokes (resolvent) system via

$$\frac{\partial u}{\partial \nu} \coloneqq \frac{\partial u}{\partial n} - \phi n, \tag{3.17}$$

see Mitrea and Wright [22, Eq. (1.2)], where n denotes the outer unit normal vector. We will also be working with the tangential gradient which is defined via

$$\nabla_{\tan} u \coloneqq \nabla u - \langle \nabla u, n\rangle n, \tag{3.18}$$

see Mitrea and Wright [22, p. 17].

Theorem 3.9. *Let $\lambda \in \Sigma_\theta$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. Let (u,ϕ) be given by* (3.1) *and* (3.2) *with $f \in \mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ and $1 < p < \infty$. Then $\nabla_{\tan}u_+ = \nabla_{\tan}u_-$ and*

$$\Big(\frac{\partial u}{\partial \nu}\Big)_{\pm} = \Big(\pm\frac{1}{2}I + \mathcal{K}_\lambda\Big)f \tag{3.19}$$

on $\partial\Omega$, with $\mathcal{K}_\lambda$ a bounded operator on $\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ satisfying

$$\|\mathcal{K}_\lambda f\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p\|f\|_{\mathrm{L}^p(\partial\Omega)},$$

where C_p depends only on d, θ, p and the Lipschitz character of Ω.

Proof. For the jth component of the tangential derivative of u_i, $1 \leq i,j \leq d$, we calculate using the results from Lemma 3.8

$$\begin{aligned}((\nabla_{\tan}u_i)_+)_j &= \Big(\frac{\partial u_i}{\partial x_j}\Big)_+ - \langle(\nabla u_i)_+, n\rangle n_j\\ &= \Big(\frac{\partial u_i}{\partial x_j}\Big)_+ - \Big(\frac{\partial u_i}{\partial x_k}\Big)_+ n_k n_j\\ &= \frac{1}{2}\big\{n_j f_i - n_i n_j n_k f_k\big\} - \frac{1}{2}\big\{n_k f_i - n_i n_k n_l f_l\big\} n_k n_j\\ &\quad + \mathrm{p.\,v.}\int_{\partial\Omega}\frac{\partial}{\partial x_j}\Big\{\Gamma_{ik}(\,\cdot - y;\lambda)\Big\} f_k(y)\,\mathrm{d}\sigma(y)\end{aligned}$$

$$+\,\mathrm{p.\,v.}\int_{\partial\Omega}\frac{\partial}{\partial x_k}\Big\{\Gamma_{il}(\,\cdot-y;\lambda)\Big\}f_l(y)\,\mathrm{d}\sigma(y)\,n_k n_j,$$

for almost everywhere $x \in \partial\Omega$. As the first two summands add up to zero, the entire expression does not depend on the direction of the nontangential limit. This gives

$$(\nabla_{\tan}u)_+ = (\nabla_{\tan}u)_-,$$

for almost everywhere $x \in \partial\Omega$. We calculate for the jth component of the nontangential limit of the conormal derivative of u at $x \in \partial\Omega$ using the results from Lemma 3.8

$$\begin{aligned}
&\Big(\frac{\partial u_j}{\partial x_i}\Big)_+ n_i - \phi_+ n_j \\
&\quad = \frac{1}{2}\big\{n_i f_j - n_j n_i n_k f_k\big\}n_i + \mathrm{p.\,v.}\int_{\partial\Omega}\frac{\partial}{\partial x_i}\Big\{\Gamma_{jk}(\,\cdot-y;\lambda)\Big\}f_k(y)\,\mathrm{d}\sigma(y)n_i \\
&\qquad + \frac{1}{2}n_k f_k n_j - \mathrm{p.\,v.}\int_{\partial\Omega}\Phi_k(\,\cdot-y)f_k(y)\,\mathrm{d}\sigma(y)n_j \\
&\quad = \frac{1}{2}f_j + (\mathcal{K}_\lambda f)_j
\end{aligned}$$

almost everywhere and where $\mathcal{K}_\lambda$ is a singular integral operator defined via

$$\begin{aligned}
(\mathcal{K}_\lambda f)_j(x) := \mathrm{p.\,v.}&\int_{\partial\Omega}\frac{\partial}{\partial x_i}\Big\{\Gamma_{jk}(x-y;\lambda)\Big\}f_k(y)\,\mathrm{d}\sigma(y)\,n_i(x) \\
-\,\mathrm{p.\,v.}&\int_{\partial\Omega}\Phi_k(x-y)f_k(y)\,\mathrm{d}\sigma(y)\,n_j(x).
\end{aligned}\tag{3.20}$$

We note that $\mathcal{K}_\lambda$ essentially consists of two boundary layer potentials. The L^p boundedness of the first one was proven in Lemma 3.3. The L^p boundedness of the second boundary layer potential follows in an analogous way using the fact that the operators

$$A^*(f)(q) = \sup_{t>0}\left|\int_{\substack{y\in\partial\Omega\\|y-q|>t}}\frac{q-y}{|q-y|^d}f(y)\,\mathrm{d}\sigma(y)\right|,\qquad q \in \partial\Omega,$$

are bounded by the corresponding result from Verchota [34, Lem. 1.2]. □

Similar to $\mathcal{K}_\lambda$, for $\lambda = 0$ we have

$$\begin{aligned}
(\mathcal{K}_0 f)_j(x) = \mathrm{p.\,v.}&\int_{\partial\Omega}\frac{\partial}{\partial x_i}\Big\{\Gamma_{jk}(x-y;0)\Big\}f_k(y)\,\mathrm{d}\sigma(y)\,n_i(x) \\
-\,\mathrm{p.\,v.}&\int_{\partial\Omega}\Phi_k(x-y)f_k(y)\,\mathrm{d}\sigma(y)\,n_j(x),
\end{aligned}\tag{3.21}$$

as was shown by Mitrea and Wright [22, Prop. 4.4]. If one compares (3.20) with (3.21), then the only difference lies withing the boundary integral involving the fundamental solutions Γ_{jk}.

The next result will be crucial for solving the L^2 Dirichlet problem in Chapter 5 and will fortify the hopes of translating results for $\lambda = 0$ to $\lambda \in \Sigma_\theta$.

Lemma 3.10. *Let $\lambda \in \Sigma_\theta$ and $d \geq 2$ and let $\mathcal{K}_\lambda$ and $\mathcal{K}_0$ be defined by* (3.20) *and* (3.21)*, respectively. Then the operator $\mathcal{K}_\lambda - \mathcal{K}_0$ on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ is compact.*

Proof. The idea of this proof is similar to the one in Tolksdorf [32, Lemma 4.3.5]. Let $f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$and let's denote $\mathcal{K} := \mathcal{K}_\lambda - \mathcal{K}_0$. We will now try to approximate $\mathcal{K}$ by compact operators in the operator norm. To this end, we define for all $\varepsilon > 0$

$$(\mathcal{K}^{(\varepsilon)} f)(x) := \int_{\partial\Omega \setminus \mathrm{B}(x,\varepsilon)} \nabla_x \Big\{ \Gamma(x-y;\lambda) - \Gamma(x-y;0) \Big\} f(y) \, \mathrm{d}\sigma(y) \, n, \qquad x \in \partial\Omega.$$

We can now estimate by Young's inequality 3.4

$$\begin{aligned} &\Big\| \mathcal{K} f - \mathcal{K}^{(\varepsilon)} f \Big\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \\ &\quad \leq \sup_{p \in \partial\Omega} \Big\| \nabla_x \Big\{ \Gamma(p-\cdot;\lambda) - \Gamma(p-\cdot;0) \Big\} 1_{\mathrm{B}(p,\varepsilon)} \Big\|_{\mathrm{L}^1(\partial\Omega;\mathbb{C}^{d\times d})} \|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}. \end{aligned}$$

Our goal is to show that

$$\sup_{p \in \partial\Omega} \Big\| \nabla_x \Big\{ \Gamma(p-\cdot;\lambda) - \Gamma(p-\cdot;0) \Big\} 1_{\mathrm{B}(p,\varepsilon)} \Big\|_{\mathrm{L}^1(\partial\Omega;\mathbb{C}^{d\times d})} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

To this end, let ε be small enough such that we can apply the estimates from Corollary 2.7 to calculate for some $p \in \partial\Omega$

$$\begin{aligned} &\Big\| \nabla_x \Big\{ \Gamma(p-\cdot;\lambda) - \Gamma(p-\cdot;0) \Big\} 1_{\mathrm{B}(p,\varepsilon)} \Big\|_{\mathrm{L}^1(\partial\Omega;\mathbb{C}^{d\times d})} \\ &\qquad \leq C \int_{\partial\Omega \cap \mathrm{B}(p,\varepsilon)} \sqrt{|\lambda|} |p-y|^{2-d} \, \mathrm{d}\sigma(y) \leq C\sqrt{|\lambda|}\varepsilon \end{aligned}$$

where for the last step we applied Lemma 3.1. For $\varepsilon \to 0$ this gives us $\mathcal{K}^{(\varepsilon)} \to \mathcal{K}$ in the operator norm.

The last step is to verify the compactness of $\mathcal{K}^{(\varepsilon)}$. We note that the integral kernel of $\mathcal{K}^{(\varepsilon)}$ is bounded which gives us that in particular the kernel is an element of the space $\mathrm{L}^2(\partial\Omega \times \partial\Omega; \mathbb{C}^{d\times d})$. The compactness of $\mathcal{K}^{(\varepsilon)}$ now follows from Weidmann [36, Thm. 6.11].

As a consequence, $\mathcal{K}$ is compact since the limit of compact operators with respect to the operator norm gives again a compact operator. □

Our next step is to introduce the *double layer potential* $u(x) = \mathcal{D}_\lambda(f)(x)$ for the Stokes resolvent problem via

$$(\mathcal{D}_\lambda(f))_j(x) := \int_{\partial\Omega} \Big\{ \frac{\partial}{\partial y_i} \{\Gamma_{jk}(y-x;\lambda)\} n_i(y) - \Phi_j(y-x) n_k(y) \Big\} f_k(y) \, \mathrm{d}\sigma(y). \quad (3.22)$$

The corresponding pressure $\phi(x) = \mathcal{D}_\Phi(f)(x)$ is defined as

$$\mathcal{D}_\Phi(f)(x) := \frac{\partial^2}{\partial x_i \partial x_k} \int_{\partial\Omega} G(y-x;0)\, n_i(y) f_k(y)\, \mathrm{d}\sigma(y) \\ + \lambda \int_{\partial\Omega} G(y-x;0)\, n_k(y) f_k(y)\, \mathrm{d}\sigma(y). \tag{3.23}$$

Using (2.28) and (2.29) one can show that (u, ϕ) defines again a solution to the Stokes resolvent problem in $\mathbb{R}^d \setminus \partial\Omega$.

The following theorem will give us a suitable operator which maps a given function $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ to boundary values of $u = \mathcal{D}_\lambda(f)$ in the form of nontangential limits. It will then be the task of the following chapters to prove the invertibility of this operator.

Theorem 3.11. *Let $\lambda \in \Sigma_\theta$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. Let u be given by* (3.22) *for $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$, $1 < p < \infty$. Then*

$$\|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p \|f\|_{\mathrm{L}^p(\partial\Omega)}, \tag{3.24}$$

where C_p depends only on d, p, θ and the Lipschitz character of Ω. Furthermore

$$u_\pm = \Big(\mp \frac{1}{2} I + \mathcal{K}^*_{\bar{\lambda}} \Big) f, \tag{3.25}$$

where $\mathcal{K}^*_{\bar{\lambda}}$ is the adjoint of the operator $\mathcal{K}_{\bar{\lambda}}$ in (3.19)

Proof. The estimate for $(u)^*$ is a direct consequence of Lemma 3.6 and the estimates on the nontangential maximal functions for the single layer potentials $(\nabla \mathcal{S}_\lambda(f))^*$ and $(\mathcal{S}_\Phi(f))^*$: We have on the one hand

$$\int_{\partial\Omega} \frac{\partial}{\partial y_i} \Big\{ \Gamma_{jk}(y-x;\lambda) \Big\}\, n_i(y) f_k(y)\, \mathrm{d}\sigma(y) \\ = - \int_{\partial\Omega} \frac{\partial}{\partial x_i} \Big\{ \Gamma_{jk}(x-y;\lambda) \Big\}\, n_i(y) f_k(y)\, \mathrm{d}\sigma(y) = -\frac{\partial}{\partial x_i} \mathcal{S}_\lambda(n_i f)_j(x) \tag{3.26}$$

and on the other hand

$$- \int_{\partial\Omega} \Phi_j(y-x) n_k(y) f_k(y)\, \mathrm{d}\sigma(y) \\ = \int_{\partial\Omega} \Phi_l(x-y)\, \delta_{lj} n_k(y) f_k(y)\, \mathrm{d}\sigma(y) = \mathcal{S}_\Phi(\tilde{f}^j)(x), \tag{3.27}$$

where $\tilde{f}^j_l = \delta_{lj} n_k f_k$. This shows that

$$u_j(x) = (\mathcal{D}_\lambda(f))_j(x) = -\frac{\partial}{\partial x_i} \mathcal{S}_\lambda(n_i f)_j(x) + \mathcal{S}_\Phi(\tilde{f}^j)(x).$$

Therefore we have for $x \in \Gamma(q)$, $q \in \partial\Omega$,

$$|u(x)| \leq C\Big\{|\nabla_x \mathcal{S}_\lambda(n_i f)(x)| + |\mathcal{S}_\Phi(\tilde{f}^j)|\Big\},$$

with C depending only on d. Hence by Lemma 3.6 we derive the estimation

$$\|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C\Big\{\sum_{i=1}^d \|n_i f\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)} + \sum_{i=1}^d \|\mathcal{S}_\Phi(\tilde{f}^j)\|_{\mathrm{L}^p(\partial\Omega)}\Big\} \leq C\|f\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)},$$

where C depends on d, p, θ and the Lipschitz character of Ω.

For the proof of (3.25), we begin by determining the adjoint of the operator $\mathcal{K}_{\bar{\lambda}}$. To this end we will first work with truncated operators $\mathcal{K}_\lambda^{(\varepsilon)}\colon \mathrm{L}^2(\partial\Omega;\mathbb{C}^d) \to \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ which are defined via

$$\begin{aligned}(\mathcal{K}_\lambda^{(\varepsilon)} f)_j(x) := &\int_{\partial\Omega} 1_{\mathrm{E}(x;\varepsilon)}(y)\frac{\partial}{\partial x_i}\Gamma_{jk}(x-y;\lambda) f_k(y)\,\mathrm{d}\sigma(y)\, n_i(x)\\ &- \int_{\partial\Omega} 1_{\mathrm{E}(x;\varepsilon)}(y)\,\Phi_k(x-y) f_k(y)\,\mathrm{d}\sigma(y)\, n_j(x),\end{aligned}$$

for $x \in \partial\Omega$ and $\mathrm{E}(x,\varepsilon) := \mathbb{R}^d \setminus \mathrm{B}(x;\varepsilon)$. Now for $f \in \mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ and $g \in \mathrm{L}^q(\partial\Omega;\mathbb{C}^d)$ with $1/p + 1/q = 1$ we calculate

$$\begin{aligned}\Big\langle \mathcal{K}_{\bar{\lambda}}^{(\varepsilon)} f, g\Big\rangle &= \int_{\partial\Omega} (\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)} f)_j(x)\,\overline{g_j(x)}\,\mathrm{d}\sigma(x)\\ &= \int_{\partial\Omega}\int_{\partial\Omega} \frac{\partial}{\partial x_i}\Big\{\Gamma_{jk}(x-y;\overline{\lambda})\Big\} f_k(y) 1_{\mathrm{E}(x;\varepsilon)}(y)\,\mathrm{d}\sigma(y)\, n_i(x)\,\overline{g_j(x)}\,\mathrm{d}\sigma(x)\\ &\quad + \int_{\partial\Omega}\int_{\partial\Omega} \Phi_k(x-y) f_k(y) 1_{\mathrm{E}(x;\varepsilon)}(y)\,\mathrm{d}\sigma(y)\, n_j(x)\,\overline{g_j(x)}\,\mathrm{d}\sigma(x).\end{aligned}$$

Note that $1_{\mathrm{E}(x;\varepsilon)}(y) = 1_{\mathrm{E}(y;\varepsilon)}(x)$ for all $x, y \in \partial\Omega$. Now an application of Fubini's theorem and factoring out $f_k(y)$ gives that

$$\begin{aligned}\Big\langle \mathcal{K}_{\bar{\lambda}}^{(\varepsilon)} f, g\Big\rangle = \int_{\partial\Omega} f_k(y) \int_{\partial\Omega} \Big\{&\frac{\partial}{\partial x_i}\{\Gamma_{jk}(x-y;\overline{\lambda})\}\, n_i(x)\\ &- \Phi_k(x-y) n_j(x) \qquad \Big\}\, 1_{\mathrm{E}(y,\varepsilon)}(x)\,\overline{g_j(x)}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y).\end{aligned}$$

Therefore we see that the adjoint of the truncated operator $\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)}$ is given by

$$\begin{aligned}\big((\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)})^* g\big)_k(y) = \int_{\partial\Omega} \Big\{&\frac{\partial}{\partial x_i}\{\Gamma_{jk}(x-y;\lambda)\} n_i(x)\\ &- \Phi_k(x-y) n_j(x) \qquad \Big\}\, 1_{\mathrm{E}(y,\varepsilon)}(x)\, g_j(x)\,\mathrm{d}\sigma(x),\end{aligned}$$

for $y \in \partial\Omega$ since $\overline{\Gamma_{jk}(x-y;\lambda)} = \Gamma_{jk}(x-y;\overline{\lambda})$.

In the next step we, will go from truncated operators to principal value operators through the dominated convergence theorem. For this to work we will look for suitable majorants. For $x \in \partial\Omega$, we estimate

$$\begin{aligned} \left|(\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)} f)_j(x)\right| &= \bigg| \int_{|x-y|>\varepsilon} \frac{\partial}{\partial x_i} \Big\{ \Gamma_{jk}(x-y;\lambda) \Big\} f_k(y)\, \mathrm{d}\sigma(y)\, n_i(x) \\ &\qquad - \int_{|x-y|>\varepsilon} \Phi_k(x-y) f_k(y)\, n_j(x)\, \mathrm{d}\sigma(y) \bigg| \\ &\leq T_\lambda^*(f)(x) + A^*(f n_j)(x). \end{aligned} \tag{3.28}$$

We know from Lemma 3.3 and the respective result for A^* that the right hand side of inequality (3.28) is p-integrable and hence we get from dominated convergence

$$\lim_{\varepsilon\to 0} \left\langle \mathcal{K}_{\bar{\lambda}}^{(\varepsilon)} f, g \right\rangle = \left\langle \mathcal{K}_{\bar{\lambda}} f, g \right\rangle.$$

With a similar argument we get

$$\begin{aligned} \left|\big((\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)})^* g\big)_k(y)\right| &= \bigg| \int_{|x-y|>\varepsilon} \frac{\partial}{\partial x_i} \Big\{ \Gamma_{jk}(x-y;\overline{\lambda}) \Big\}\, n_i(x) g_j(x)\, \mathrm{d}\sigma(x) \\ &\qquad - \int_{|x-y|>\varepsilon} \Phi_k(x-y) n_j(x) g_j(x)\, \mathrm{d}\sigma(x) \bigg| \\ &\leq \sum_{i=1}^d T_\lambda^*(n_i g)(y) + \sum_{i=1}^d A^*(\tilde{g}^j)(y), \end{aligned}$$

where $\tilde{g}_l^j = \delta_{lj} n_k f_k$ and therefore the dominated convergence theorem yields

$$\lim_{\varepsilon\to 0} \left\langle f, \left(\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)}\right)^* g \right\rangle = \left\langle f, \mathcal{K}_{\bar{\lambda}}^{(*)} g \right\rangle,$$

where the limit operator $\mathcal{K}_{\bar{\lambda}}^{(*)}$ is defined via

$$\big((\mathcal{K}_{\bar{\lambda}}^{(*)} g\big)_k(y) := \mathrm{p.\,v.} \int_{\partial\Omega} \Big\{ \frac{\partial}{\partial x_i} \big\{ \Gamma_{kj}(x-y;\lambda) \big\} n_i(x) - \Phi_k(x-y) n_j(x) \Big\} g_j(x)\, \mathrm{d}\sigma(x).$$

Of course by the uniqueness of the adjoint operator, the identity $\langle \mathcal{K}_{\bar{\lambda}} f, g\rangle = \langle f, \mathcal{K}_{\bar{\lambda}}^{(*)} g\rangle$ shows that $\mathcal{K}_{\bar{\lambda}}^* = \mathcal{K}_{\bar{\lambda}}^{(*)}$. Note that we have used the symmetry of (Γ_{jk}).

In the last part of this proof we will show that the equality (3.25) holds. Note that by (3.26) and (3.27) we have made Lemma 3.8 accessible. For $x \in \partial\Omega$ we can now calculate

$$\begin{aligned} &\Big(\int_{\partial\Omega} \frac{\partial}{\partial y_i} \Big\{ \Gamma_{jk}(y-\cdot\,;\lambda) \Big\} n_i(y) f_k(y)\, \mathrm{d}\sigma(y) \Big)_\pm (x) \\ &\qquad = -\big(\frac{\partial}{\partial x_i} \mathcal{S}_\lambda(n_i f)_j \big)_\pm (x) \end{aligned}$$

$$
\begin{aligned}
&= \mp\frac{1}{2}\big\{n_i(x)n_i(x)f_j(x) - n_j(x)n_i(x)n_k(x)n_i(x)f_k(x)\big\} \\
&\qquad - \text{p. v.} \int_{\partial\Omega} \frac{\partial}{\partial x_i}\Big\{\Gamma_{jk}(x-y;\lambda)\Big\}\, n_i(y)f_k(y)\,\mathrm{d}\sigma(y) \\
&= \mp\frac{1}{2}\big\{f_j(x) - n_j(x)n_k(x)f_k(x)\big\} \\
&\qquad + \text{p. v.} \int_{\partial\Omega} \frac{\partial}{\partial y_i}\Big\{\Gamma_{jk}(y-x;\lambda)\Big\}\, n_i(y)f_k(y)\,\mathrm{d}\sigma(y),
\end{aligned}
$$

where we used trace formula (3.16). A similar procedure for the second integral part of the double layer potential gives

$$
\begin{aligned}
&-\Big(\int_{\partial\Omega} \Phi_j(y-\cdot\,)\, n_k(y)f_k(y)\,\mathrm{d}\sigma(y)\Big)_{\pm}(x) \\
&\qquad = \big(\mathcal{S}_\Phi(\tilde{f}^j)\big)_{\pm}(x) \\
&\qquad = \mp\frac{1}{2}n_k(x)\tilde{f}^j_k(x) - \text{p. v.} \int_{\partial\Omega} \Phi_k(x-y)\tilde{f}^j_k(x)\,\mathrm{d}\sigma(y) \\
&\qquad = \mp\frac{1}{2}n_j(x)n_k(x)f_k(x) - \text{p. v.} \int_{\partial\Omega} \Phi_j(x-y)\, n_k(x)f_k(x)\,\mathrm{d}\sigma(y).
\end{aligned}
$$

Putting everything together we get

$$
(u_j)_{\pm}(x) = \mp\frac{1}{2}f_j(x) + (\mathcal{K}^*_{\bar{\lambda}}f)_j(x)
$$

and the proof is finished. □

Chapter 4

Rellich Estimates

In this section, we will establish Rellich-type estimates for solutions of the Stokes resolvent problem (1.9) which will be used in the following chapter to prove the invertibility of the operators $\pm(1/2)I + \mathcal{K}_\lambda$ and their adjoints from Theorems 3.9 and 3.11.

We will for this entire section always assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with connected boundary. Furthermore, we will use the shorthand notation

$$\|\cdot\|_\partial := \|\cdot\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^k)}, \quad k \in \mathbb{N},$$

and we will tacitly use the summation convention whenever it is applicable.

The following Theorem formulates the aforementioned Rellich estimates and is the central result of this chapter:

Theorem 4.1. *Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0,1)$. Let (u,ϕ) be a smooth solution to the Stokes resolvent problem* (1.9) *in Ω and suppose that $(\nabla u)^* \in \mathrm{L}^2(\partial\Omega)$ and $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$. Furthermore, assume that ∇u, ϕ have nontangential limits almost everywhere on $\partial\Omega$. Then*

$$\begin{aligned} &\|\nabla u\|_\partial + \left\| \phi - \left\{ \frac{1}{{r_0}^{d-1}} \int_{\partial\Omega} \phi \, \mathrm{d}\sigma \right\} \right\|_\partial \\ &\qquad \leq C \left\{ \left\|\nabla_{\tan} u\right\|_\partial + |\lambda|^{1/2} \|u\|_\partial + |\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \right\} \end{aligned} \tag{4.1}$$

and

$$\|\nabla u\|_\partial + |\lambda|^{1/2} \|u\|_\partial + |\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\phi\|_\partial \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_\partial, \tag{4.2}$$

where $\frac{\partial}{\partial \nu}$ denotes the conormal derivative, and C depends only on d, τ, θ and the Lipschitz character of Ω.

Remark 4.2. The assumptions on u in Theorem 4.1 are sufficient for u to have a nontangential limit and a square integrable maximal function $(u)^*$. Indeed for $d = 2$ we have $(u)^* \in \mathrm{L}^\infty(\partial\Omega)$, for $d = 3$ we have $(u)^* \in \mathrm{L}^p(\partial\Omega), p \in (1,\infty)$, and for $d \geq 3$ we have $(u)^* \in \mathrm{L}^p(\partial\Omega)$, $p \in \big(1, 2(d-1)/(d-3)\big)$. A proof of these facts can be found in Shen's notes [27, Prop. 7.1.3].

We will now prepare the proof of Theorem 4.1 by proving several helpful lemmata. The first lemma deals with so called *Rellich identities* for solutions of the Stokes resolvent system (1.9).

Lemma 4.3. *Under the same conditions on (u, ϕ) as in Theorem 4.1, we have*

$$\begin{aligned}\int_{\partial\Omega} h_k n_k |\nabla u|^2 \,\mathrm{d}\sigma = 2\,\mathrm{Re}\int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k}\Big(\frac{\partial u}{\partial \nu}\Big)_i \,\mathrm{d}\sigma + \int_\Omega \mathrm{div}\,(h)\,|\nabla u|^2 \,\mathrm{d}x \\ - 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_j}\cdot\frac{\partial u_i}{\partial x_k}\cdot\frac{\partial \overline{u}_i}{\partial x_j}\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_i}\cdot\frac{\partial u_i}{\partial x_k}\,\overline{\phi}\,\mathrm{d}x \\ - 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial u_i}{\partial x_k}\cdot\overline{\lambda u_i}\,\mathrm{d}x \qquad (4.3)\end{aligned}$$

and

$$\begin{aligned}\int_{\partial\Omega} h_k n_k |\nabla u|^2 \,\mathrm{d}\sigma = 2\,\mathrm{Re}\int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_j}\Big\{n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k}\Big\}\,\mathrm{d}\sigma \\ + 2\,\mathrm{Re}\int_{\partial\Omega} h_k\,\overline{\phi}\,\Big\{n_i \frac{\partial u_i}{\partial x_k} - n_k \frac{\partial u_i}{\partial x_i}\Big\}\,\mathrm{d}\sigma - \int_\Omega \mathrm{div}\,(h)\,|\nabla u|^2\,\mathrm{d}x \\ + 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_j}\cdot\frac{\partial u_i}{\partial x_k}\cdot\frac{\partial \overline{u}_i}{\partial x_j}\,\mathrm{d}x - 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_i}\cdot\frac{\partial u_i}{\partial x_k}\,\overline{\phi}\,\mathrm{d}x \\ + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial u_i}{\partial x_k}\cdot\overline{\lambda u_i}\,\mathrm{d}x, \qquad (4.4)\end{aligned}$$

where $h = (h_1, \dots, h_d) \in \mathrm{C}_0^1(\mathbb{R}^d, \mathbb{R}^d)$.

Proof. The proof of the stated identities reduces to several applications of the divergence theorem once we establish its applicability. To this end, we want to make Proposition 1.8 available. We note that the assumptions given in Theorem 4.1 are sufficient for this purpose and we will verify them, once they are used.

Let's expand the first summand in (4.3) using the definition of conormal derivatives

$$\begin{aligned}2\,\mathrm{Re}\int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k}\Big(\frac{\partial u}{\partial \nu}\Big)_i \,\mathrm{d}\sigma = 2\,\mathrm{Re}\int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial u_i}{\partial x_j} n_j \,\mathrm{d}\sigma - 2\,\mathrm{Re}\int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k}\,\phi\, n_i\,\mathrm{d}x \\ =: I_1 - I_2.\end{aligned}$$

The divergence theorem 1.8 is applicable for I_1 as h is bounded and defined everywhere and the integrand has nontangential limits that can be dominated by $|(\nabla u)^*|^2 \in \mathrm{L}^2(\partial\Omega)$. Therefore, we find using the divergence theorem and the product rule:

$$\begin{aligned}
I_1 &= 2\,\mathrm{Re}\int_\Omega \frac{\partial}{\partial x_j}\left\{h_k \frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial u_i}{\partial x_j}\right\}\mathrm{d}x \\
&= 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_j}\cdot\frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial u_i}{\partial x_j}\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_k}\cdot\frac{\partial u_i}{\partial x_j}\,\mathrm{d}x \\
&\qquad + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial^2 u_i}{\partial x_j^2}\,\mathrm{d}x \\
&=: I_3 + I_4 + I_5.
\end{aligned}$$

For I_5, we use the fact that u solves the Stokes resolvent problem which gives

$$\begin{aligned}
I_5 &= 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial \phi}{\partial x_i}\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k}\lambda u_i\,\mathrm{d}x \\
&=: I_6 + I_7.
\end{aligned}$$

Now we want to apply the divergence theorem, i.e. Proposition 1.8, to integral I_2. This is possible since h is defined everywhere and bounded, $(\partial_k u_i)\cdot\phi$ has a nontangential limit and can be bounded by $\big(|(\nabla u)^*||(\phi)^*|\big)$ which is integrable due to Hölder's inequality as $(\nabla u)^*$ and $(\phi)^*$ are square integrable by assumption. Thus, the divergence theorem is applicable and yields together with the product rule:

$$\begin{aligned}
I_2 &= 2\,\mathrm{Re}\int_\Omega \frac{\partial}{\partial x_i}\left\{h_k \frac{\partial \overline{u}_i}{\partial x_k}\,\phi\right\}\mathrm{d}x \\
&= 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_i}\cdot\frac{\partial \overline{u}_i}{\partial x_k}\,\phi\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_k}\,\phi\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial \phi}{\partial x_i}\,\mathrm{d}x \\
&=: I_8 + I_9 + I_6.
\end{aligned}$$

One term that hasn't come up so far, the second summand of the right hand side in (4.3), will now be expanded:

$$\begin{aligned}
\int_\Omega \mathrm{div}\,(h)\,|\nabla u|^2\,\mathrm{d}x &= \int_\Omega \mathrm{div}\,(h\,|\nabla u|^2)\,\mathrm{d}x - \int_\Omega h_k \frac{\partial}{\partial x_i}\left\{|\nabla u|^2\right\}\mathrm{d}x \\
&=: I_{10} - I_{11}.
\end{aligned}$$

Expanding the Integral I_{11} gives us the identity

$$I_{11} = \int_\Omega h_i \frac{\partial}{\partial x_i}\left\{\frac{\partial u_k}{\partial x_j}\cdot\frac{\partial \overline{u}_k}{\partial x_j}\right\}\mathrm{d}x = \int_\Omega h_i\left\{\frac{\partial^2 u_k}{\partial x_i \partial x_j}\cdot\frac{\partial \overline{u}_k}{\partial x_j} + \frac{\partial u_k}{\partial x_j}\cdot\frac{\partial^2 \overline{u}_k}{\partial x_i \partial x_j}\right\}\mathrm{d}x = I_4.$$

If we now put everything together, the right hand side of (4.3) reads

$$
\begin{aligned}
&(I_1 - I_2) + (I_{10} - I_{11}) - I_3 + I_8 - I_7 \\
&\quad = (I_3 + I_4 + I_6 + I_7) - (I_8 + I_9 + I_6) + I_{10} - I_{11} - I_3 + I_8 - I_7 = I_{10}.
\end{aligned}
$$

Noting that by the divergence theorem, which is applicable with the same justification as for the integral I_1, we have

$$
I_{10} = \int_{\partial\Omega} h_k n_k |\nabla u|^2 \, \mathrm{d}\sigma.
$$

Thus, the first identity is proven.

In order to prove identity (4.4), we show that the expression we get from considering ((4.3) + (4.4)) holds, i.e. we show the identity

$$
\begin{aligned}
2\int_{\partial\Omega} h_k n_k |\nabla u|^2 \, \mathrm{d}\sigma = 2\,\mathrm{Re} \int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k} \left(\frac{\partial u}{\partial \nu} \right)_i \mathrm{d}\sigma \\
+ 2\,\mathrm{Re} \int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_j} \left\{ n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k} \right\} \mathrm{d}\sigma \\
+ 2\,\mathrm{Re} \int_{\partial\Omega} h_k \, \overline{\phi} \left\{ n_i \frac{\partial u_i}{\partial x_i} - n_k \frac{\partial u_i}{\partial x_i} \right\} \mathrm{d}\sigma.
\end{aligned}
$$

To this end, note that the left side of the identity equals $2\,I_{10}$, whereas the right hand side can be written as

$$
(I_1 - I_2) + (2\,I_{10} - I_1) + (I_2 - 0),
$$

where we also used the fact that $\operatorname{div} u = \partial_i u_i = 0$. $\square$

Consider the operators $\partial_{\tau_{jk}}$ which act on compactly supported continuously differentiable functions ψ in the neighborhood of $\partial\Omega$ by

$$
\partial_{\tau_{jk}} \psi := n_j \frac{\partial \psi}{\partial x_k}\bigg|_{\partial\Omega} - n_k \frac{\partial \psi}{\partial x_j}\bigg|_{\partial\Omega}, \qquad j, k = 1, \ldots, d. \tag{4.5}
$$

These operators show up in identity (4.4) as

$$
\partial_{\tau_{kj}} u_i = \left\{ n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k} \right\} \quad \text{and} \quad \partial_{\tau_{ik}} u_i = \left\{ n_i \frac{\partial u_i}{\partial x_k} - n_k \frac{\partial u_i}{\partial x_i} \right\}
$$

and have been introduced by Mitrea and Wright [22, p. 16]. These operators are called *first-order tangential derivative operators* and relate to the tangential gradient, which has been introduced in (3.18), in the following way:

$$
\left(\nabla_{\tan} \psi \right)_j = \frac{\partial \psi}{\partial x_j} - n_k n_j \frac{\partial \psi}{\partial x_k} = n_k \, \partial_{\tau_{kj}} \psi. \tag{4.6}
$$

The tangential derivative operators come with a helpful "integration by parts" rule and can be used to define Sobolev spaces on the boundary $\partial\Omega$: For $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, we start by defining antilinear functionals on $\mathrm{C}_0^\infty(\mathbb{R}^d)$ by setting

$$\partial_{\tau_{kj}} f \colon \mathrm{C}_0^\infty(\mathbb{R}^d) \ni \psi \mapsto \int_{\partial\Omega} f\, \overline{\partial_{\tau_{jk}} \psi}\, \mathrm{d}\sigma.$$

Now the weak tangential derivatives of f are given by those functionals which are regular, i.e. elements of $\mathrm{L}^1_{\mathrm{loc}}(\partial\Omega)$. In this case, the following integration by parts formula holds:

$$\int_{\partial\Omega} f\, \overline{(\partial_{\tau_{jk}} \psi)}\, \mathrm{d}\sigma = \int_{\partial\Omega} (\partial_{\tau_{kj}} f)\, \overline{\psi}\, \mathrm{d}\sigma. \tag{4.7}$$

Note that the minus sign that usually comes with the integration by parts is hidden in the new order of indices. For $p \in (1, \infty)$, we define the corresponding Sobolev space via

$$\mathrm{W}^{1,p}(\partial\Omega) = \Big\{ f \in \mathrm{L}^p(\partial\Omega) \colon \partial_{\tau_{jk}} f \in \mathrm{L}^p(\partial\Omega),\ j,k = 1, \dots, d \Big\}.$$

We extend our detour by the following basic lemma on a reverse triangle inequality for elements of the sector Σ_θ. A powerful generalization of this Lemma can be found in Tolksdorf [32, Lem. 5.2.4].

Lemma 4.4. *Let $\theta \in (0, \pi/2)$. Then there exists α depending only on θ such that for all $\lambda \in \Sigma_\theta$ the following inequality holds:*

$$\mathrm{Re}(\lambda) + \alpha \,\big|\mathrm{Im}(\lambda)\big| \geq |\lambda|.$$

Proof. For the moment being, suppose $|\lambda| = 1$. Then, we have $\mathrm{Re}(\lambda) = \cos(\varphi)$ and $\mathrm{Im}(\lambda) = \sin(\varphi)$ with $|\varphi| \in (0, \pi - \theta)$. Set

$$\alpha = \frac{1 - \cos(\pi - \theta)}{\sin(\pi - \theta)} \geq \frac{1 - \cos(|\varphi|)}{\sin(|\varphi|)}, \tag{4.8}$$

where the estimation is a consequence of the fact that tan is strictly increasing on $(0, \pi/2)$ and the identity

$$\frac{1 - \cos(x)}{\sin(x)} = \tan(x/2)$$

which is readily derived using trigonometric identities for $\sin(2x)$ and $\cos(2x)$.

If $\varphi = |\varphi|$, we use (4.8) and derive the inequality

$$\mathrm{Re}(\lambda) + \alpha \,|\, \mathrm{Im}(\lambda) \,| = \cos(\varphi) + \alpha \sin(\varphi) \geq 1.$$

Conversely, if $\varphi = -|\varphi|$, then we have by the symmetry properties of sin and cos that

$$\mathrm{Re}(\lambda) + \alpha \,|\, \mathrm{Im}(\lambda) \,| = \cos(-\varphi) + \alpha \sin(-\varphi) \geq 1.$$

For arbitrary λ, the claim follows by considering the normalized value $(\lambda/|\lambda|)$. $\square$

The next lemma enables us to handle the solid integrals in (4.3) and (4.4).

Lemma 4.5. *Under the same assumptions on (u, ϕ) and λ as in Theorem 4.1, we have*

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + |\lambda| \int_\Omega |u|^2 \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_\partial \|u\|_\partial, \tag{4.9}$$

where C depends only on θ.

Proof. Inserting the solution u into the variational problem of the Stokes resolvent problem gives us

$$\int_\Omega -\Delta u \cdot \overline{u} \,\mathrm{d}x + \lambda \int_\Omega u \cdot \overline{u} \,\mathrm{d}x = - \int_\Omega \nabla \phi \cdot \overline{u} \,\mathrm{d}x. \tag{4.10}$$

Rewriting the first term of equation (4.10) and using the product rule leads to

$$- \int_\Omega \frac{\partial^2 u_j}{\partial x_i \partial x_i} \,\overline{u}_j \,\mathrm{d}x = - \int_\Omega \frac{\partial}{\partial x_i} \left\{ \overline{u}_j \frac{\partial u_j}{\partial x_i} \right\} \mathrm{d}x + \int_\Omega \frac{\partial u_j}{\partial x_i} \cdot \frac{\partial \overline{u}_j}{\partial x_i} \,\mathrm{d}x.$$

Note that since u is solenoidal, we have for the third term of equation (4.10)

$$- \int_\Omega \frac{\partial \phi}{\partial x_i} \,\overline{u}_i \,\mathrm{d}x = - \int_\Omega \frac{\partial}{\partial x_i} \left\{ \phi \,\overline{u}_i \right\} \mathrm{d}x.$$

Now we want to transform the first and third of the above solid integrals into boundary integrals through Proposition 1.8. By the assumptions formulated in Theorem 4.1, ϕ and ∇u have a nontangential limit and for both nontangential maximal functions the inclusion $(\phi)^*, (\nabla u)^* \in \mathrm{L}^2(\partial\Omega)$ holds. Furthermore, according to Remark 4.2, also u has a nontangential limit and the nontangential maximal function satisfies $(u)^* \in \mathrm{L}^2(\partial\Omega)$. Therefore, the function $|\phi \,\overline{u}_i|$ may be dominated by $|(\phi)^*(u)^*| \in \mathrm{L}^2(\partial\Omega)$ and the function $|(\partial_j u_i) u_i|$ may be dominated by $|(\nabla u)^*(u)^*|$, respectively. Thus, the door to Proposition 1.8 has been opened which allows to transform equation (4.10) into

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot \overline{u} \,\mathrm{d}\sigma + \lambda \int_\Omega |u|^2 \,\mathrm{d}x = - \int_{\partial\Omega} \phi \, n \cdot \overline{u} \,\mathrm{d}\sigma.$$

We can rearrange the terms of this identity and use the definition of conormal derivatives, see equation (3.17), to derive

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \lambda \int_\Omega |u|^2 \,\mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \overline{u} \,\mathrm{d}\sigma. \tag{4.11}$$

If we now take the real and imaginary part of (4.11) and sum them up with the prefactor $\alpha(\theta) > 0$ from Lemma 4.4, we get

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \Big\{ \mathrm{Re}(\lambda) + \alpha \left| \mathrm{Im}(\lambda) \right| \Big\} \int_\Omega |u|^2 \,\mathrm{d}x \leq (1 + \alpha) \left| \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \overline{u} \,\mathrm{d}\sigma \right|.$$

Lemma 4.4 now gives

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x + |\lambda| \int_\Omega |u|^2\,\mathrm{d}x \le C \left| \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \bar{u}\,\mathrm{d}\sigma \right|,$$

with $C = (1+\alpha)$ from which we readily derive estimate (4.9) after applying the Cauchy-Schwartz inequality. □

The next lemma combines Rellich identities (4.3) and (4.4) with estimate (4.9).

Lemma 4.6. *Under the same assumptions on* (u, ϕ) *and* λ *as in Theorem 4.1, we have*

$$\|\nabla u\|_\partial \le C_\varepsilon \left\| \frac{\partial u}{\partial \nu} \right\|_\partial + \varepsilon \left\{ \|\nabla u\|_\partial + \|\phi\|_\partial + |\lambda|^{1/2} \|u\|_\partial \right\} \tag{4.12}$$

and

$$\|\nabla u\|_\partial \le C_\varepsilon \left\{ \left\|\nabla_{\tan} u\right\|_\partial + |\lambda|^{1/2} \|u\|_\partial \right\} + \varepsilon \left\{ \|\nabla u\|_\partial + \|\phi\|_\partial \right\} \tag{4.13}$$

for all $\varepsilon \in (0,1)$*, where* C_ε *depends only on* d*,* θ*,* τ*,* ε *and the Lipschitz character of* Ω*.*

Proof. Let $h = (h_1, \dots, h_d) \in \mathrm{C}_0^1(\mathbb{R}^d, \mathbb{R}^d)$ with $h_k n_k \ge c > 0$ on $\partial\Omega$ as given by Therorem 1.3 v). The idea of the proof of the desired estimates (4.12) and (4.13) is to first use the Rellich identities from Lemma 4.3 with this particular h to estimate $\|\nabla u\|_\partial$ and then to bound the resulting right hand side by providing individual estimates.

Before we start, note that we have $\Delta\phi = 0$ on the one hand and for the nontangential maximal function $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$ on the other hand. According to Shen [26, p. 410], a result from Dahlberg [4] gives the estimation

$$\int_\Omega |\phi|^2\,\mathrm{d}x \le C\,\|(\phi)^*\|_\partial^2 \le C\,\|\phi\|_\partial^2. \tag{4.14}$$

We will now prove the first estimate (4.12). In view of identity (4.3), we have

$$\begin{aligned} \|\nabla u\|_\partial^2 \le C \Bigg\{ &\|\nabla u\|_\partial \left\|\frac{\partial u}{\partial \nu}\right\|_\partial + \int_\Omega |\nabla u|^2\,\mathrm{d}x \\ &+ \int_\Omega |\nabla u|\,|\phi|\,\mathrm{d}x + |\lambda| \int_\Omega |\nabla u|\,|u|\,\mathrm{d}x \Bigg\}, \end{aligned} \tag{4.15}$$

where the first term follows from the Cauchy-Schwartz inequality and C only depends on d and the Lipschitz character of Ω.

For now, we keep the first term of (4.15) as it is, the second term can be handled via Lemma 4.5. The goal for the remaining two integrals will be to bound each of them by a product of norms $\|\cdot\|_\partial$. To this end, for the third integral we calculate

$$\int_\Omega |\nabla u||\phi|\,\mathrm{d}x \le \left(\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)^{1/2} \left(\int_\Omega |\phi|^2\,\mathrm{d}x\right)^{1/2} \le C \left\|\frac{\partial u}{\partial \nu}\right\|_\partial^{1/2} \|u\|_\partial^{1/2}\,\|\phi\|_\partial, \tag{4.16}$$

where the first step is due to the Cauchy-Schwartz inequality and the second step combines estimate (4.9) with estimate (4.14).

The last integral of (4.15) can be estimated as follows:

$$|\lambda| \int_\Omega |\nabla u|\, |u| \,\mathrm{d}x \le \frac{|\lambda|^{3/2}}{2} \int_\Omega |u|^2 \,\mathrm{d}x + \frac{|\lambda|^{1/2}}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x \le C \left\| \frac{\partial u}{\partial \nu} \right\|_\partial |\lambda|^{1/2} \|u\|_\partial, \tag{4.17}$$

where in the first step we used the weighted Young inequality and in the second step we applied estimate (4.9). Putting everything together, we calculate

$$\|\nabla u\|_\partial^2 \le C \left\{ \|\nabla u\|_\partial \left\| \frac{\partial u}{\partial \nu} \right\|_\partial + \left\| \frac{\partial u}{\partial \nu} \right\|_\partial \|u\|_\partial + \left\| \frac{\partial u}{\partial \nu} \right\|_\partial^{1/2} \|u\|_\partial^{1/2} \|\phi\|_\partial + \left\| \frac{\partial u}{\partial \nu} \right\|_\partial |\lambda|^{1/2} \|u\|_\partial \right\}.$$

If we now use the assumption $|\lambda| \ge \tau$ which allows us to bound $\|u\|_\partial$ via

$$\|u\|_\partial \le \frac{|\lambda|^{1/2}}{\tau^{1/2}} \|u\|_\partial = C\, |\lambda|^{1/2} \|u\|_\partial,$$

the desired estimate (4.12) now follows applying Young's weighted inequality with an ε and the norm equivalence on finite dimensional vector spaces. Note that for the product of three norms from inequality (4.16) we need to apply the Young inequality twice:

$$\begin{aligned}
\left\| \frac{\partial u}{\partial \nu} \right\|_\partial^{1/2} \|u\|_\partial^{1/2} \|\phi\|_\partial &\le \left\{ \frac{1}{4\varepsilon} \left\| \frac{\partial u}{\partial \nu} \right\|_\partial + \varepsilon \|u\|_\partial \right\} \|\phi\|_\partial \\
&\le \frac{1}{32\,\varepsilon^3} \left\| \frac{\partial u}{\partial \nu} \right\|_\partial^2 + \frac{\varepsilon}{2} \|\phi\|_\partial^2 + \frac{1}{2} \|u\|_\partial^2 + \frac{\varepsilon^2}{2} \|\phi\|_\partial^2 \\
&\le C_\varepsilon \left\{ \left\| \frac{\partial u}{\partial \nu} \right\|_\partial^2 + \|u\|_\partial^2 \right\} + \varepsilon \|\phi\|_\partial^2,
\end{aligned}$$

where for the last inequality we used the fact that $\varepsilon < 1$.

For inequality (4.13), we use the Rellich identity (4.4) and the relation (4.6) to obtain the estimate

$$\begin{aligned}
\|\nabla u\|_\partial^2 \le C \Big\{ &\left\|\nabla_{\tan} u\right\|_\partial \Big\{ \|\nabla u\|_\partial + \|\phi\|_\partial \Big\} \\
&+ \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega |\nabla u| |\phi| \,\mathrm{d}x + |\lambda| \int_\Omega |\nabla u| |u| \,\mathrm{d}x \Big\},
\end{aligned} \tag{4.18}$$

where C only depends on d and the Lipschitz character of Ω. As before, we estimate the three terms on the right side of (4.18) using (4.9), (4.16) and (4.17), respectively, and obtain the estimate

$$\|\nabla u\|_\partial^2 \le C \left\{ \left\|\nabla_{\tan} u\right\|_\partial \Big\{ \|\nabla u\|_\partial + \|\phi\|_\partial \Big\} \right.$$

$$+ \left\| \frac{\partial u}{\partial \nu} \right\|_\partial \|u\|_\partial + \left\| \frac{\partial u}{\partial \nu} \right\|_\partial^{1/2} \|u\|_\partial^{1/2} \|\phi\|_\partial + \left\| \frac{\partial u}{\partial \nu} \right\|_\partial |\lambda|^{1/2} \|u\|_\partial \Big\}.$$

If we now use the Young inequality with an ε, we get

$$\|\nabla u\|_\partial^2 \leq C_\varepsilon \left\{ \|\nabla_{\tan} u\|_\partial^2 + |\lambda| \, \|u\|_\partial^2 \right\} + \varepsilon \left\{ \|\nabla u\|_\partial^2 + \|\phi\|_\partial^2 + \frac{1}{4} \left\| \frac{\partial u}{\partial \nu} \right\|_\partial^2 \right\}.$$

The claim now follows if we use the definition of the conormal derivative and the norm equivalence on finite dimensional vector spaces. □

We prove one last lemma before we tackle the central theorem of this chapter. The following lemma will not depend on the lemmata which were proven in the preceding part of this chapter, as the approach to derive the desired boundary estimates will be different: We will not rely upon the Rellich identities from Lemma 4.3 but directly part from a variational formulation of the Stokes resolvent problem on the boundary. Furthermore, we will work with the following Theorem about the regularity problem and the Neumann problem for the Laplacian on bounded Lipschitz domains:

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. Then the following statements hold:*

a) Given $f \in \mathrm{H}^1(\partial\Omega)$, there exists a unique $\psi \in$ with $(\nabla\psi)^* \in \mathrm{L}^2(\partial\Omega)$ such that $\Delta u = 0$ in Ω and ψ converges nontangentially to f a.e.. Furthermore, the estimate

$$\|\nabla\psi\|_\partial \leq C \, \|f\|_{\mathrm{H}^1(\partial\Omega)}$$

holds with a constant only depending on the Lipschitz character of Ω.

b) Given $f \in \mathrm{L}^2(\partial\Omega)$ with $\int_{\partial\Omega} f \, \mathrm{d}\sigma = 0$, there exists a harmonic function ψ on Ω with $\frac{\partial\psi}{\partial n} = f$ a.e.. Furthermore, the estimate

$$\|\psi\|_{\mathrm{H}^1(\partial\Omega)} \leq C \, \|f\|_\partial$$

holds with C only depending on the Lipschitz character of Ω.

Proof. According to Kenig, for the differential operator Δ the *regularity problem*, $(R)_2$, and the *Neumann problem*, $(N)_2$, are solvable for data $f \in \mathrm{L}^2(\partial\Omega)$, see [18, Thm. 2.1.10]. Checking the definitions of $(R)_2$, see [18, Defn. 1.7.10], and $(N)_2$, see [18, Defn. 1.7.9], the claimed properties of follow from Theorems 1.8.2 and 1.8.3 in [18, Chap. 1]. We also refer to the works of Jerison and Kenig on the L^2 regularity problem [16] and the Neumann problem [17]. □

Lemma 4.8. *Assume that* (u, ϕ) *satisfies the same conditions as in Theorem 4.1. Then,*

$$\left\| \phi - \left\{ \frac{1}{{r_0}^{d-1}} \int_{\partial\Omega} \phi \, \mathrm{d}\sigma \right\} \right\|_{\partial} \leq C \left\{ \|\nabla u\|_{\partial} + |\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \right\} \tag{4.19}$$

and

$$|\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C \left\{ \|\phi\|_{\partial} + \|\nabla u\|_{\partial} \right\}, \tag{4.20}$$

where C *depends only on* d *and the Lipschitz character of* Ω.

Proof. Our first goal will be to show that without loss of generality we may assume that $\Delta u = \nabla\phi + \lambda u$ on $\partial\Omega$. The central player will once again be Theorem 1.3. To this end, let (Ω_j) be a sequence of approximating C^∞ domains as in Theorem 1.3 and suppose that 4.19 holds for all Ω_j, i.e. we have proven the inequality

$$\left\| \phi - \left\{ \frac{1}{{r_0}^{d-1}} \int_{\partial\Omega} \phi \, \mathrm{d}\sigma \right\} \right\|_{\mathrm{L}^2(\partial\Omega_j;\mathbb{C}^d)} \leq C \Big\{ \|\nabla u\|_{\mathrm{L}^2(\partial\Omega_j;\mathbb{C}^{d\times d})} + |\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega_j)} \Big\} \tag{4.21}$$

and an inequality analogous to (4.20). Note that the mean value integral of ϕ should remain unchanged. It is now crucial that the constants in these inequalities only depend on the Lipschitz character of Ω and not on other geometric properties of the domain. Thus C does not depend on j and we may take the limit $j \to \infty$ using dominated convergence.

For the rest of the proof we will thus assume that (u, ϕ) satisfies the Stokes resolvent problem in a domain Ω' for some $\overline{\Omega} \subseteq \Omega'$. In particular we have $\Delta u = \nabla\phi + \lambda u$ on $\partial\Omega$. Multiplying this identity on $\partial\Omega$ with the outer normal vector n and using the triangle inequality gives the following set of estimates:

$$\begin{aligned} \|\nabla\phi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} &\leq \|\Delta u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + |\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}, \\ |\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} &\leq \|\Delta u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\nabla\phi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}. \end{aligned} \tag{4.22}$$

This looks almost like the desired pair of inequalities. We will now show that on the one hand the estimate

$$\|\Delta u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C \, \|\nabla u\|_{\partial} \tag{4.23}$$

and on the other hand the estimate

$$c \left\| \phi - \left\{ \frac{1}{{r_0}^{d-1}} \int_{\partial\Omega} \phi \, \mathrm{d}\sigma \right\} \right\|_{\partial} \leq \|\nabla\phi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C \, \|\phi\|_{\partial} \tag{4.24}$$

holds for constants c, C that only depend on d and the Lipschitz character of Ω. Using these two estimates applied to the respective terms of (4.22), we can directly verify (4.19) and (4.20).

In order to prove (4.23), we note the following identity of differential operators

$$\Delta u \cdot n = n_i \frac{\partial^2 u_i}{\partial x_j^2} = \left\{ n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i} \right\} \frac{\partial u_i}{\partial x_j},$$

where we used the fact that $\operatorname{div} u = 0$ in $\overline{\Omega}$. The expression between the brackets is the first-order tangential derivative $\partial_{\tau_{ij}}$, see (4.5). We can thus see that

$$\left| \langle \Delta u \cdot n, u \rangle \right| = \left| \langle \nabla u, \nabla_{\tan} u \rangle \right| \leq 2 \, \|\nabla u\|_{\partial}^2 ,$$

where we compared the tangential gradient to the usual gradient via the triangle inequality and applied the Cauchy-Schwartz estimate. Identifying $\Delta u \cdot n$ with an element of $\mathrm{H}^{-1}(\partial\Omega)$, the estimate (4.23) follows.

For the proof of estimate (4.24), we will use L^2 estimates for the Neumann and regularity problems for the Laplace equation in Lipschitz domains. Jerison and Kenig showed that for $g \in \mathrm{L}^2(\partial\Omega)$ with mean value zero the *Neumann problem* for Laplace's equation on the Lipschitz domain Ω has a unique solution ψ with $(\nabla\psi)^* \in \mathrm{L}^2(\partial\Omega)$, $\frac{\partial\psi}{\partial n} = g$ a.e. on $\partial\Omega$ and the solution fulfills the estimate $\|\psi\|_{\mathrm{H}^1(\partial\Omega)} \leq C \|g\|_{\partial}$, see Theorem 4.7. By Green's theorem, which is applicable since $\partial\Omega$ is assumed to be smooth, we see that

$$\int_{\partial\Omega} \phi \, \frac{\partial\psi}{\partial x_i} n_i \, \mathrm{d}\sigma = \int_{\Omega} \frac{\partial}{\partial x_i} \left\{ \phi \frac{\partial\psi}{\partial x_i} \right\} dx = \int_{\Omega} \frac{\partial\phi}{\partial x_i} \cdot \frac{\partial\psi}{\partial x_i} \, \mathrm{d}x = \int_{\partial\Omega} \psi \, \frac{\partial\phi}{\partial x_i} n_i \, \mathrm{d}\sigma.$$

We can then use this identity and the estimate of ψ against the data g to derive

$$\begin{aligned} \left| \int_{\partial\Omega} \phi \, g \, \mathrm{d}\sigma \right| = \left| \int_{\partial\Omega} \frac{\partial\phi}{\partial n} \, \psi \, \mathrm{d}\sigma \right| &\leq \left\| \frac{\partial\phi}{\partial n} \right\|_{\mathrm{H}^{-1}(\partial\Omega)} \|\psi\|_{\mathrm{H}^1(\partial\Omega)} \\ &\leq C \left\| \frac{\partial\phi}{\partial n} \right\|_{\mathrm{H}^{-1}(\partial\Omega)} \|g\|_{\partial}. \end{aligned} \tag{4.25}$$

Now, if we set $\overline{g} = \phi - \tilde{\phi}$, with $\tilde{\phi} := |\partial\Omega|^{-1} \int_{\partial\Omega} \phi \, \mathrm{d}\sigma$, we arrive at the following estimate:

$$\|\phi - \tilde{\phi}\|_{\partial}^2 = \int_{\partial\Omega} (\phi - \tilde{\phi}) \, \overline{(\phi - \tilde{\phi})} \, \mathrm{d}\sigma = \int_{\partial\Omega} \phi \, \overline{(\phi - \tilde{\phi})} \, \mathrm{d}\sigma \leq C \left\| \frac{\partial\phi}{\partial n} \right\|_{\mathrm{H}^{-1}(\partial\Omega)} \|\phi - \tilde{\phi}\|_{\partial},$$

where in the last step we used (4.25). This together with Lemma 3.1 proves the left side of inequality (4.24).

For the right side of inequality (4.24), we work in a similar way. We will use results for the *regularity problem* of Laplace's equation by Jerison and Kenig, see Theorem 4.7: Given $f \in \mathrm{H}^1(\partial\Omega)$, there exists a harmonic function ψ in Ω such that $(\nabla\psi)^* \in \mathrm{L}^2(\partial\Omega)$ and $\psi = f$ on $\partial\Omega$ nontangentially. Furthermore, the estimate $\|\nabla\psi\|_{\partial} \leq C \, \|f\|_{\mathrm{H}^1(\partial\Omega)}$ holds. As for (4.25), we calculate

$$\left| \int_{\partial\Omega} \frac{\partial\phi}{\partial n} f \, \mathrm{d}\sigma \right| = \left| \int_{\partial\Omega} \phi \, \frac{\partial\psi}{\partial n} \, \mathrm{d}\sigma \right| \leq \|\phi\|_{\partial} \|\nabla\psi\|_{\partial} \leq C \, \|\phi\|_{\partial} \|f\|_{\mathrm{H}^1(\partial\Omega)},$$

Interpreting the function $\frac{\partial \phi}{\partial n} \in \mathrm{L}^2(\partial\Omega)$ as a functional on $\mathrm{H}^1(\partial\Omega)$, we obtain

$$\left\| \frac{\partial \phi}{\partial n} \right\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C \, \|\phi\|_\partial .$$

This finishes the proof of inequality (4.24). $\square$

Remark 4.9. Throughout the proof of Lemma 4.8, we derived several estimates that prove to hold under more general conditions if we study them closely. A careful look at the proof of inequality (4.23) reveals that the estimate

$$\|\Delta u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C \, \|\nabla u\|_\partial$$

holds for *all* smooth vector fields u that fulfill the condition $\operatorname{div}(u) = 0$ and not only for velocity fields for the Stokes resolvent problem.

Additionally, examining the proof of inequality (4.24) shows us that the estimate

$$c \, \|\phi\|_\partial \leq \|\nabla \phi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)},$$

holds for *all* harmonic functions ϕ with vanishing mean on $\partial\Omega$ and not only those that correspond to the pressure term of the Stokes resolvent problem.

After all this preparation we have acquainted enough tools and are now able to prove Theorem 4.1.

Proof of Theorem 4.1. For the proof of estimate (4.1), we can assume without loss of generality that $\int_{\partial\Omega} \phi \, \mathrm{d}\sigma = 0$.

We start by proving estimate (4.1). Using (4.19) to bound the second summand in (4.1) and then (4.13) for ∇u, we get

$$\begin{aligned} \|\nabla u\|_\partial + \|\phi\|_\partial &\leq C \left\{ \|\nabla u\|_\partial + |\lambda| \, \|u \cdot n\|_{\mathrm{H}^1(\partial\Omega)} \right\} \\ &\leq C_\varepsilon \left\{ \left\|\nabla_{\tan} u\right\|_\partial + |\lambda|^{1/2} \|u\|_\partial + |\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \right\} + C \, \varepsilon \left\{ \|\nabla u\|_\partial + \|\phi\|_\partial \right\} \end{aligned}$$

for all $\varepsilon \in (0, 1)$. Choosing ε such that $C \, \varepsilon < (1/2)$, we can rearrange the above inequality and obtain estimate (4.1).

Estimate (4.2) will need more effort to be proven. In order to obtain the desired estimate, we will divide the left hand side of inequality (4.2) into two groups as the following line suggests and then bound both groups separately.

$$\left\{ \|\nabla u\|_\partial + |\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\phi\|_\partial \right\} + |\lambda|^{1/2} \|u\|_\partial =: G_1 + G_2 .$$

We start with inequality (4.20) and derive

$$G_1 \leq C\left\{\|\nabla u\|_\partial + \|\phi\|_\partial\right\} \leq C\left\{\Big\| \frac{\partial u}{\|\nabla u\|_\partial + \partial \nu} \Big\|_\partial\right\},$$

where in the last step we used the definition of conormal derivatives to bound the pressure term. If we now apply (4.12), we get

$$G_1 \leq C_\varepsilon \Big\| \frac{\partial u}{\partial \nu} \Big\|_\partial + \varepsilon\left\{\|\nabla u\|_\partial + \|\phi\|_\partial + |\lambda|^{1/2}\|u\|_\partial\right\}$$

for all $\varepsilon \in (0,1)$. Choosing ε appropriately yields

$$\|\nabla u\|_\partial + |\lambda|\,\|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\phi\|_\partial \leq C\left\{\Big\| \frac{\partial u}{\partial \nu} \Big\|_\partial + |\lambda|^{1/2}\|u\|_\partial\right\}, \tag{4.26}$$

and the first group has been successfully bounded.

Now we need to estimate G_2. For this, we will work with the next identity which is a consequence of Green's theorem:

$$\begin{aligned}\int_{\partial\Omega} h_k n_k |u|^2 \,\mathrm{d}\sigma &= \int_\Omega \frac{\partial h_k}{\partial x_k} |u|^2 \,\mathrm{d}x + \int_\Omega h_k \frac{\partial |u|^2}{\partial x_k} \,\mathrm{d}x \\ &= \int_\Omega \operatorname{div}(h)\, |u|^2 \,\mathrm{d}x + 2\operatorname{Re}\int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k} u_i \,\mathrm{d}x, \end{aligned} \tag{4.27}$$

where $h \in \mathrm{C}_0^1(\mathbb{R}^d, \mathbb{R}^d)$. This calculation is valid since Remark 4.2 assures the existence of nontangential limits of u and furthermore gives $(u)^* \in \mathrm{L}^2(\partial\Omega)$. Thus, $(h_k n_k |u|^2)$ can be dominated by the integrable function $(\|h\|_\infty |(u)^*|^2)$ and the claim follows from Proposition 1.8.

Next, for identity (4.27), we choose $h \in \mathrm{C}_0^1(\mathbb{R}^d, \mathbb{R}^d)$ with $h_k n_k \geq c > 0$ on $\partial\Omega$, which is possible due to Theorem 1.3. Then we take absolute values and estimate

$$\|u\|_\partial^2 \leq C\left\{\int_\Omega |u|^2 \,\mathrm{d}x + \int_\Omega |u||\nabla u| \,\mathrm{d}x\right\}. \tag{4.28}$$

Multiplying (4.28) with $|\lambda|$ and using (4.9) three times, once directly and twice after an application of the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}|\lambda|\,\|u\|_\partial^2 &\leq C\left\{|\lambda| \int_\Omega |u|^2 \,\mathrm{d}x + |\lambda|^{1/2} \int_\Omega (|\lambda|^{1/2}|u|)|\nabla u| \,\mathrm{d}x\right\} \\ &\leq C\left\{\Big\| \frac{\partial u}{\partial \nu} \Big\|_\partial \|u\|_\partial + |\lambda|^{1/2}\Big(|\lambda| \int_\Omega |u|^2\Big)^{1/2}\Big(\int_\Omega (|\nabla u|^2 \,\mathrm{d}x\Big)^{1/2}\right\} \\ &\leq C\Big\| \frac{\partial u}{\partial \nu} \Big\|_\partial |\lambda|^{1/2}\|u\|_\partial.\end{aligned}$$

Note that for the last estimate we also used the fact that $|\lambda| \geq \tau$ helps us to bound $\|u\|_\partial$ by $C\,|\lambda|^{1/2}\|u\|_\partial$. Rearranging terms in the last estimate, we now derive

$$G_2 = |\lambda|^{1/2}\|u\|_\partial \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_\partial. \tag{4.29}$$

This bounds the second group and concludes our proof. □

Shen proved that under reasonable assumptions a theorem similar to Theorem 4.1 also holds for exterior domains, see [26, Thm. 4.6].

Theorem 4.10. *Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0,1)$. Let (u, ϕ) be a solution of the Stokes resolvent problem in $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$. Suppose additionally that $(\nabla u)^*$, $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$ and that ∇u, ϕ have nontangential limits almost everywhere on $\partial\Omega$. Furthermore, let for $|x| \to \infty$*

$$|\phi(x)| + |\nabla u(x)| = O(|x|^{1-d}) \quad \text{and} \quad u(x) = \begin{cases} O(|x|^{2-d}) & \text{if } d \geq 3, \\ o(1) & \text{if } d = 2. \end{cases}$$

Then the estimates

$$\|\nabla u\|_\partial + \|\phi\|_\partial \leq C \left\{ \|\nabla_{\tan} u\|_\partial + |\lambda|^{1/2}\|u\|_\partial + |\lambda|\, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \right\} \tag{4.30}$$

and

$$\|\nabla u\|_\partial + |\lambda|^{1/2}\|u\|_\partial + |\lambda|\, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\phi\|_\partial \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_\partial \tag{4.31}$$

hold, where C depends only on d, τ, θ and the Lipschitz character of Ω.

Chapter 5

Solving the L^2 Dirichlet Problem

This section is all about the application of the method of layer potentials to solve the L^2 Dirichlet problem (Dir_λ) for the Stokes resolvent system. The results from Chapter 3 on the jump relations and the corresponding operators, see Theorem 3.9 and Theorem 3.11, together with the Rellich estimates from Chapter 4 will be the essential ingredients.

In the first part of this chapter, we will show that the operators $\pm(1/2)I + \mathcal{K}_\lambda$ and their adjoints are isomorphisms on suitable Hilbert spaces. Then, in the second part of this chapter, we will build on this information to construct solutions to (Dir_λ) via double layer potentials.

For the remainder of this chapter, let Ω always denote a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with connected boundary. We will use $\mathrm{L}^2_n(\partial\Omega)$ to denote the function space

$$\mathrm{L}^2_n(\partial\Omega) := \Big\{ f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d) \colon \int_{\partial\Omega} f \cdot n \, \mathrm{d}\sigma = 0 \Big\},$$

and $\mathrm{L}^2_0(\partial\Omega; \mathbb{C}^d)$ to denote the function space of L^2 functions with mean value zero. As before, $\|\cdot\|_\partial$ stands for the norms of the spaces $\mathrm{L}^2(\partial\Omega; \mathbb{C}^k)$, $k \in \mathbb{N}$.

The following results will build on the application of Theorem 4.10. Therefore, the next lemma shows that solutions to the Stokes resolvent problem (1.9) that are given by single layer potentials fulfill the requirements of the theorem.

Lemma 5.1. *Let $\lambda \in \Sigma_\theta$ and (u, ϕ) be given by* (3.1) *and* (3.2)*, respectively. Then the following holds for $|x| \to \infty$:*

$$|\phi(x)| + |\nabla u(x)| = O(|x|^{1-d}) \quad \textit{and} \quad u(x) = \begin{cases} O(|x|^{2-d}) & \textit{if } d \geq 3, \\ o(1) & \textit{if } d = 2. \end{cases}$$

Proof. If $d \geq 2$, then an application of the dominated convergence theorem gives that $|\phi(x)| + |\nabla u(x)| = O(|x|^{1-d})$ as $|\phi(x)| = O(|\Phi_k(x)|) = O(|x|^{1-d})$ by (2.28). Furthermore, we have $|\nabla u(x)| = O(|x|^{1-d})$ by estimate (2.30).

If $d \geq 3$, then the first part of Lemma 2.1 and an application of the dominated convergence theorem give $u(x) = O(|x|^{2-d})$. If $d = 2$, consider the definition of the fundamental matrix (2.27). According to the first part of Lemma 2.1 and the asymptotic behavior of the second derivative of $\log(|x|)$, we only have to worry about the first summand in (2.27). But the asymptotic behavior of the fundamental solution to the scalar Helmholtz equation is already available thanks to the second part of Lemma 2.1. Through dominated convergence, the same asymptotic behavior holds for $u(x)$. $\square$

As announced in the introduction to this chapter, we will study the invertibility of the operators $\pm(1/2)I + \mathcal{K}_\lambda$from Chapter 3, starting with the one corresponding to +. We will furthermore be concerned about bounds on the inverse of the operator $(1/2)I + \mathcal{K}_\lambda$.

Lemma 5.2. *Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0, 1)$. Suppose that $|\partial\Omega| = 1$. Then $(1/2)I + \mathcal{K}_\lambda$ is an isomorphism on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ and*

$$\|f\|_\partial \leq C \left\| \left((1/2)I + \mathcal{K}_\lambda\right) f \right\|_\partial \quad \textit{for any } f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d), \tag{5.1}$$

where C depends only on d, θ, τ and the Lipschitz character of Ω.

Proof. We start with $f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ and the corresponding single layer potentials $u = \mathcal{S}_\lambda(f)$ and $\phi = \mathcal{S}_\Phi(f)$ given by (3.1) and (3.2). We saw in Chapter 3 that (u, ϕ) solves the Stokes resolvent problem in $\mathbb{R}^d \setminus \partial\Omega$ and got from Lemma 3.6 with $p = 2$ for the nontangential maximal functions that $(\nabla u)^*$, $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$. We furthermore saw in Lemma 3.8 that ∇u and ϕ have nontangential limits almost everywhere on $\partial\Omega$. Finally, in Theorem 3.9 we saw that $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$ and derived the jump relation $\left(\frac{\partial u}{\partial \nu}\right)_\pm = (\pm(1/2)I + \mathcal{K}_\lambda) f$.

Let us assume for a moment that the estimate

$$\|\nabla u_-\|_\partial + \|\phi_-\|_\partial \leq C \left\| \left(\frac{\partial u}{\partial \nu}\right)_+ \right\|_\partial. \tag{5.2}$$

holds for a constant C depending only on d, θ, τ and the Lipschitz character of Ω. Using (5.2), we can prove (5.1): Note that the jump relation (3.19) gives us $f = \left(\frac{\partial u}{\partial \nu}\right)_+ - \left(\frac{\partial u}{\partial \nu}\right)_-$. With the definition of the conormal derivative and estimate (5.2), we calculate that

$$\begin{aligned}
\|f\|_\partial &\leq \left\| \left(\frac{\partial u}{\partial \nu}\right)_+ \right\|_\partial + \left\| \left(\frac{\partial u}{\partial \nu}\right)_- \right\|_\partial \\
&\leq \left\| \left(\frac{\partial u}{\partial \nu}\right)_+ \right\|_\partial + \left\| \left(\frac{\partial u}{\partial n}\right)_- \right\|_\partial + \|\phi_-\|_\partial \\
&\leq C \left\| \left(\frac{\partial u}{\partial \nu}\right)_+ \right\|_\partial = C \left\| \left((1/2)I + K_\lambda\right) f \right\|_\partial.
\end{aligned}$$

In order to prove (5.2), note that due to Lemma 5.1 we can use Theorem 4.10 to derive the following inequality for the outer nontangential limit:

$$\begin{aligned}\|\nabla u_-\|_\partial + \|\phi_-\|_\partial &\le C\left\{\left\|\nabla_{\tan} u_-\right\|_\partial + |\lambda|^{1/2}\|u_-\|_\partial + |\lambda|\,\|n\cdot u_-\|_{\mathrm{H}^{-1}(\partial\Omega)}\right\}\\ &= C\left\{\left\|\nabla_{\tan} u_+\right\|_\partial + |\lambda|^{1/2}\|u_+\|_\partial + |\lambda|\,\|n\cdot u_+\|_{\mathrm{H}^{-1}(\partial\Omega)}\right\},\end{aligned} \tag{5.3}$$

where we used the fact that $u_+ = u_-$ and $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$ on $\partial\Omega$. The former fact is a consequence of the continuity of the single layer potential across $\partial\Omega$, see Mitrea and Wright [22, Prop. 4.7]. Inequality (4.2) of Theorem 4.1 now allows us to estimate the right hand side of (5.3) by $C\,\|\big(\frac{\partial u}{\partial \nu}\big)_+\|_\partial$ and thus the desired estimate (5.2) follows.

Let's now work on the invertibility of $(1/2)I + \mathcal{K}_\lambda$. In the case $\lambda = 0$, Mitrea and Wright showed in [22, Eq. (5.166)] that $(1/2)I + \mathcal{K}_0$ as an operator on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ has a one dimensional null space and as range the space $\mathrm{L}^2_0(\partial\Omega;\mathbb{C}^d)$. In other words, $(1/2)I+\mathcal{K}_0$ has Fredholm index 0 as the orthogonal complement of $\mathrm{L}^2_0(\partial\Omega)$ is the span of the normal vector n which is one dimensional. Since the operator $\mathcal{K}_\lambda - \mathcal{K}_0$ is compact on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ by Lemma 3.10, we deduce that for all $\lambda \in \Sigma_\theta$ the operator

$$(1/2)I + \mathcal{K}_\lambda = (1/2)I + \mathcal{K}_0 + (\mathcal{K}_\lambda - \mathcal{K}_0)$$

has the Fredholm index zero as well. Now inequality (5.1) gives that $(1/2)I + \mathcal{K}_\lambda$ is injective and thus the Fredholm index of zero implies that it is also surjective and hence an isomorphism. □

The next lemma is the counterpart to Lemma 5.2 and proves a similar result for the operator $-(1/2)I + \mathcal{K}_\lambda$ on the slightly smaller space $\mathrm{L}^2_n(\partial\Omega)$.

Lemma 5.3. *Let $\lambda \in \Sigma_\theta$. Then $-(1/2)I + \mathcal{K}_\lambda$ is a Fredholm operator on* $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ *with index zero and*

$$\|f\|_\partial \le C\left\|\big(-(1/2)I + \mathcal{K}_\lambda\big)f\right\|_\partial \quad \text{for all } f \in \mathrm{L}^2_n(\partial\Omega), \tag{5.4}$$

where C depends only on d, θ and the Lipschitz character of Ω.

Proof. Let us assume without loss of generality that $|\partial\Omega| = 1$. In the case $\lambda = 0$, Mitrea and Wright showed in [22, Eq. (5.166)] that the Fredholm index of the operator $-(1/2)I + \mathcal{K}_0$ on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ is zero and estimate (5.4) holds. Since $\mathcal{K}_\lambda - \mathcal{K}_0$ is compact on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ and the Fredholm index remains unchanged under compact perturbations, we know that the Fredholm index of $-(1/2)I + \mathcal{K}_\lambda$ on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ is zero for all $\lambda \in \Sigma_\theta$. This proves the first claim of the lemma.

Now let $\tau < (2\,\mathrm{diam}(\Omega)^2 + 1)^{-1}$ and $|\lambda| < \tau$. We claim that

$$\big\| (\mathcal{K}_\lambda - \mathcal{K}_0)f \big\|_\partial \leq C\,|\lambda|^{1/2} \|f\|_\partial.$$

In order to prove this inequality, we want to apply Young's inequality from Lemma 3.4. To this end, we start by estimating

$$\big\| (\mathcal{K}_\lambda - \mathcal{K}_0)f \big\|_\partial \leq \sup_{\substack{p \in \partial\Omega \\ i,j=1,\dots,d}} \Big\| \nabla_x \Big\{ \Gamma_{ij}(p - \cdot\,; \lambda) - \Gamma_{ij}(p - \cdot\,; 0) \Big\} \Big\|_{\mathrm{L}^1(\partial\Omega;\mathbb{C}^d)} \|f\|_{\mathrm{L}^2(\partial\Omega)}.$$

In the next step, we prove that for $p \in \partial\Omega$ the integral over the gradients of Γ can be estimated independent of p and of course i and j. This is straightforward using Lemma 3.2 as Corollary 2.7 gives us

$$\begin{aligned}
&\int_{\partial\Omega} \Big| \nabla_x \big\{ \Gamma_{ij}(p - y; \lambda) - \Gamma_{ij}(p - y; 0) \big\} \Big| \,\mathrm{d}\sigma(y) \\
&\quad \leq C\,|\lambda|^{1/2} \int_{\partial\Omega} \frac{1}{|p - y|^{d-2}} \,\mathrm{d}\sigma(y) \\
&\quad = C\,|\lambda|^{1/2} \int_{\partial\Omega \cap \mathrm{B}(p,\, r_0/4)} \frac{1}{|p - y|^{d-2}} \,\mathrm{d}\sigma(y) + C|\lambda|^{1/2} \int_{\partial\Omega \setminus \mathrm{B}(p,\, r_0/4)} \frac{1}{|p - y|^{d-2}} \,\mathrm{d}\sigma(y) \\
&\quad \leq C\,|\lambda|^{1/2} \big(r_0/4 + 4^{2-d} r_0^{d-2} |\partial\Omega| \big),
\end{aligned}$$

where r_0 is the radius from the definition of Lipschitz domains. But as $|\partial\Omega| = 1$ and r_0 can be related to $|\partial\Omega|$ by Lemma 3.1, we get

$$\int_{\partial\Omega} \Big| \nabla_x \big\{ \Gamma_{ij}(p - y; \lambda) - \Gamma_{ij}(p - y; 0) \big\} \Big| \,\mathrm{d}\sigma(y) \leq C\,|\lambda|^{1/2}$$

with a constant C that only depends on d, θ and the Lipschitz character of Ω. Note that by the choice of τ the estimate from Corollary 2.7 applies on the whole domain of integration.

For $f \in \mathrm{L}^2_n(\partial\Omega)$, we can now estimate

$$\begin{aligned}
\|f\|_\partial &\leq C\, \big\| (-(1/2)I + \mathcal{K}_0)f \big\|_\partial \\
&\leq C\, \big\| (-(1/2)I + \mathcal{K}_\lambda)f \big\|_\partial + \big\| (\mathcal{K}_\lambda - \mathcal{K}_0)f \big\|_\partial \\
&\leq C\, \big\| (-(1/2)I + \mathcal{K}_\lambda)f \big\|_\partial + C\,|\lambda|^{1/2} \|f\|_\partial,
\end{aligned}$$

with a constant C which depends only on d, θ and the Lipschitz character of $\partial\Omega$. Choosing τ smaller than $(2C)^{-1}$ allows us to rearrange the terms in the above estimate such that estimate (5.4) holds for $\lambda \in \Sigma_\theta$ and $|\lambda| < \tau$, with τ depending on d, θ and the Lipschitz character of Ω.

Now leave τ fixed and consider the case $|\lambda| \geq \tau$. This case will be handled using the Rellich estimates from Section 4. We use the facts that for $\nabla_{\tan} u$ and u the inner and outer nontangential limits coincide and apply Theorems 4.1 and 4.10 to conclude that

$$\begin{aligned}
&\|\nabla u_+\|_\partial + \left\| \phi_+ - \frac{1}{r_0^{d-1}} \int_{\partial\Omega} \phi_+ \, \mathrm{d}\sigma \right\|_\partial \\
&\quad \leq C \left\{ \left\| (\nabla_{\tan} u)_+ \right\|_\partial + |\lambda|^{1/2} \|u_+\|_\partial + |\lambda| \, \|u_+ \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \right\} \\
&\quad = C \left\{ \left\| (\nabla_{\tan} u)_- \right\|_\partial + |\lambda|^{1/2} \|u_-\|_\partial + |\lambda| \, \|u_- \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \right\} \leq C \left\| \left(\frac{\partial u}{\partial \nu} \right)_- \right\|_\partial .
\end{aligned}$$

We can now use this inequality to estimate $\|\left(\frac{\partial u}{\partial \nu}\right)_+\|_\partial$ via

$$\begin{aligned}
\left\| \left(\frac{\partial u}{\partial \nu} \right)_+ \right\|_\partial &\leq \left\| \left(\frac{\partial u}{\partial n} \right)_+ \right\|_\partial + C \, \|\phi_+\|_\partial \\
&\leq C \left\{ \|(\nabla u)_+\|_\partial + \left\| \phi_+ - \frac{1}{r_0^{d-1}} \int_{\partial\Omega} \phi_+ \, \mathrm{d}\sigma \right\|_\partial + \left| \frac{1}{r_0^{d-1}} \int_{\partial\Omega} \phi_+ \, \mathrm{d}\sigma \right| \right\} \\
&\leq C \left\{ \left\| \left(\frac{\partial u}{\partial \nu} \right)_- \right\|_\partial + \left| \int_{\partial\Omega} \phi_+ \, \mathrm{d}\sigma \right| \right\}.
\end{aligned}$$

Furthermore, considering the jump relation (3.19) and the previous estimate, we get that

$$\begin{aligned}
\|f\|_\partial &\leq \left\| \left(\frac{\partial u}{\partial \nu} \right)_+ \right\|_\partial + \left\| \left(\frac{\partial u}{\partial \nu} \right)_- \right\|_\partial \\
&\leq C \left\{ \left\| \left(\frac{\partial u}{\partial \nu} \right)_- \right\|_\partial + \left| \int_{\partial\Omega} \phi_+ \, \mathrm{d}\sigma \right| \right\} \\
&\leq C \left\{ \left\| (-(1/2)I + \mathcal{K}_\lambda) f \right\|_\partial + \left| \int_{\partial\Omega} \phi_+ \, \mathrm{d}\sigma \right| \right\}. \qquad (5.5)
\end{aligned}$$

Now we are left with the term $\int_{\partial\Omega} \phi_+ \, \mathrm{d}\sigma$ that needs to be estimated. To this end, note that multiplying the conormal derivatives of u by n gives

$$\left(\frac{\partial u}{\partial \nu} \right)_+ \cdot n = \left(\frac{\partial u_i}{\partial x_j} \right)_+ n_i n_j - \phi_+ = n_j \left(n_i \frac{\partial u_i}{\partial x_j} - n_j \left(\frac{\partial u_i}{\partial x_i} \right) \right)_+ - \phi_+ ,$$

where for the second equality we used that $\operatorname{div}(u) = 0$ in Ω and thus this also holds for the nontangential limit. Note that the expression on the right hand side involves a first-order tangential derivative operator, see (4.5) and thus we can also write

$$\left(\frac{\partial u}{\partial \nu} \right)_+ \cdot n = n_j \left(\partial_{\tau_{ij}} u_i \right)_+ - \phi_+ .$$

Using the above identity and relation (4.6) to bring the tangential gradient into the game, it follows that

$$\begin{aligned}\left|\int_{\partial\Omega}\phi_+\,\mathrm{d}\sigma\right| &\leq \left|\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)_+\cdot n\,\mathrm{d}\sigma\right| + C\left\|(\nabla_{\tan}u)_+\right\|_\partial \\ &\leq \left|\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)_-\cdot n\,\mathrm{d}\sigma\right| + C\left\|(\nabla_{\tan}u)_-\right\|_\partial \\ &\leq C\left\|\left(\frac{\partial u}{\partial\nu}\right)_-\right\|_\partial, \end{aligned} \tag{5.6}$$

where in the second step, we used the jump relation to exchange $\left(\frac{\partial u}{\partial\nu}\right)_+\cdot n$ by $\left(\frac{\partial u}{\partial\nu}\right)_- + f\cdot n$ and then used the fact $f\in\mathrm{L}^2_n(\partial\Omega)$. The third step follows from Theorem 4.10 considering that $\left\|(\nabla_{\tan}u)_-\right\|_\partial \leq C\,\|(\nabla u)_-\|_\partial$ with a constant that only depends on d. Now, extending estimate (5.5) by (5.6) gives

$$\|f\|_\partial \leq C\left\|\left(-(1/2)I+\mathcal{K}_\lambda\right)f\right\|_\partial + C\left\|\left(\frac{\partial u}{\partial\nu}\right)_-\right\|_\partial \leq C\left\|\left(-(1/2)I+\mathcal{K}_\lambda\right)f\right\|_\partial,$$

where we used the jump relation (3.19) again. This proves estimate (5.4) in the case $|\lambda|\geq\tau$ and thus concludes the proof. □

With the following lemma that looks like a reverse trace theorem, we will later show the uniqueness of solutions to the L^2 Dirichlet problem (Dir_λ) for the Stokes resolvent system.

Lemma 5.4. *Let $\lambda\in\Sigma_\theta$ and (u,ϕ) be a solution to the Stokes resolvent problem* (1.9) *in Ω. Furthermore, suppose that the nontangential limit of u exists almost everywhere on $\partial\Omega$ and that $(u)^*\in\mathrm{L}^2(\partial\Omega)$. Then*

$$\int_\Omega |u|^2\,\mathrm{d}x \leq C\int_{\partial\Omega}|u|^2\,\mathrm{d}\sigma, \tag{5.7}$$

where C depends only on d, θ and the Lipschitz character of Ω.

Proof. We use the approximation theorem 1.3 and approximate Ω by a sequence of smooth domains with uniform Lipschitz characters from inside. It suffices to prove (5.7) for elements of this sequence of domains as $(u)^*\in\mathrm{L}^2(\partial\Omega)$. As a consequence, we will assume for the rest of the proof that Ω is smooth and that u, ϕ are smooth in $\overline{\Omega}$. Let $(w,\psi)\in\mathrm{H}^1_0(\Omega;\mathbb{C}^d)\times\mathrm{H}^1(\Omega)$ be a solution to the inhomogenous system

$$\begin{aligned} -\Delta w + \lambda w + \nabla\psi &= \overline{u} \text{ in } \Omega, \\ \operatorname{div}(w) &= 0 \text{ in } \Omega. \end{aligned} \tag{5.8}$$

In fact, the regularity theory for the Stokes equation gives us that w and ψ are even smooth in Ω as $\overline{u}$ is smooth. It follows from testing (5.8) against u that

$$\int_\Omega |u|^2 \,\mathrm{d}x = \int_\Omega u \cdot \big\{ -\Delta w + \lambda w + \nabla\psi \big\} \,\mathrm{d}x. \tag{5.9}$$

The left hand side of (5.9) gives the starting point for the proof of inequality (5.7).

Using one of Green's identities, see [6, Thm. 3, App. C.2], on the first summand and the fact that u is the solution to the Stokes resolvent problem gives that

$$\begin{aligned}\int_\Omega -u \cdot \Delta w \,\mathrm{d}x &= \int_\Omega -w \cdot \Delta u \,\mathrm{d}x - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} \,\mathrm{d}\sigma, \\ &= \int_\Omega w \cdot (-\lambda u - \nabla\phi) \,\mathrm{d}x - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} \,\mathrm{d}\sigma \\ &= \int_\Omega -\lambda w \cdot u \,\mathrm{d}x - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} \,\mathrm{d}\sigma,\end{aligned}$$

where in the last step we used integration by parts and the fact that w vanishes on $\partial\Omega$ and is divergence free:

$$\int_\Omega w \cdot \nabla\phi \,\mathrm{d}x = -\int_\Omega \operatorname{div}(w)\, \phi \,\mathrm{d}x + \int_{\partial\Omega} \phi\, w \cdot n \,\mathrm{d}\sigma = 0.$$

For the third summand in (5.9), we do the same with the only difference that the second integral does not vanish:

$$\int_\Omega u \cdot \nabla\psi \,\mathrm{d}x = -\int_\Omega \operatorname{div}(u)\, \psi \,\mathrm{d}x + \int_{\partial\Omega} u \cdot n\, \psi \,\mathrm{d}\sigma = \int_{\partial\Omega} u \cdot n\, \psi \,\mathrm{d}\sigma.$$

Putting everything together gives

$$\int_\Omega |u|^2 \,\mathrm{d}x = \left| \int_{\partial\Omega} u \cdot \left\{ -\frac{\partial w}{\partial n} + n\psi \right\} \mathrm{d}\sigma \right| \le \|u\|_\partial \left\{ \|\nabla w\|_\partial + \|\psi\|_\partial \right\} \tag{5.10}$$

by the Cauchy-Schwartz inequality. As the pressure ψ is only specified modulo additive constants, we may as well assume that $\int_{\partial\Omega} \psi \,\mathrm{d}\sigma = 0$. Furthermore, by the Schwartz theorem, we see from (5.8) that $\Delta\psi = \operatorname{div}(\overline{u}) = 0$ in Ω. As stated in Remark 4.9, this allows us to use the results from the proof of (4.24) with $\phi = \psi$ to conclude that

$$\|\psi\|_\partial \le C \, \|\nabla\psi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}$$

and since w has vanishing trace on $\partial\Omega$ we can use that property together with the fact that (w, ψ) solves (5.8) to further estimate

$$\le C \left\{ \|\Delta w \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\overline{u} \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \right\}$$

$$\leq C \left\{ \|\nabla w\|_\partial + \|u\|_\partial \right\}, \tag{5.11}$$

where for the last estimate we used (4.23) which is applicable since $\operatorname{div} w = 0$ on Ω, see Remark 4.9. If we combine inequalities (5.10) and (5.11), we get

$$\int_\Omega |u|^2 \, \mathrm{d}x \leq C \left\{ \|u\|_\partial \|\nabla w\|_\partial + \|u\|_\partial^2 \right\}. \tag{5.12}$$

We are left with estimating the first term on the right hand side of (5.12). To this end, it will suffice to show the following inequality

$$\int_{\partial\Omega} |\nabla w|^2 \, \mathrm{d}\sigma \leq C \left\{ \int_\Omega |u|^2 \, \mathrm{d}x + \int_{\partial\Omega} |u|^2 \, \mathrm{d}\sigma \right\} \tag{5.13}$$

with a constant C depending only on d, θ and the Lipschitz character of Ω since by the weighted Young inequality for real numbers this would make the estimate

$$C \, \|u\|_\partial \|\nabla w\|_\partial \leq \frac{1}{2} \int_\Omega |u|^2 \, \mathrm{d}x + C \int_{\partial\Omega} |u|^2 \, \mathrm{d}\sigma$$

available which after rearranging terms in (5.12) yields (5.7).

In order to derive estimate (5.13), we will need the Rellich-type identity

$$\begin{aligned} \int_{\partial\Omega} h_k n_k |\nabla w|^2 \, \mathrm{d}\sigma = &- \int_\Omega \operatorname{div}(h) \, |\nabla w|^2 \, \mathrm{d}x + 2 \operatorname{Re} \int_\Omega \frac{\partial h_k}{\partial x_j} \cdot \frac{\partial w_i}{\partial x_k} \cdot \frac{\partial \overline{w_i}}{\partial x_j} \, \mathrm{d}x \\ &- 2 \operatorname{Re} \int_\Omega \frac{\partial h_k}{\partial x_i} \cdot \frac{\partial w_i}{\partial x_k} \overline{\psi} \, \mathrm{d}x + 2 \operatorname{Re} \int_\Omega h_k \frac{\partial w_i}{\partial x_k} \cdot \overline{\lambda w_i} \, \mathrm{d}x \\ &+ 2 \operatorname{Re} \int_\Omega h_k \frac{\partial \overline{w_i}}{\partial x_k} \overline{u_i} \, \mathrm{d}x , \end{aligned} \tag{5.14}$$

where $h = (h_1, \ldots, h_d) \in \mathrm{C}_0^1(\mathbb{R}^d, \mathbb{R}^d)$. Note that since all involved quantities are smooth up to the boundary, integration by parts is allowed and the proof of the stated Rellich identity boils down to a formal calculation. The proof is analogous to the proof of Rellich identity (4.4) and one can prove that for (w, ψ) an equality like (4.4) holds with one extra term, namely

$$-2 \operatorname{Re} \int_\Omega h_k \frac{\partial \overline{w_i}}{\partial x_k} \overline{u_i} \, \mathrm{d}x.$$

This term will show up in the proof of Lemma 4.3 in the calculation of I_5 when one uses the fact that (w, ψ) solves problem (5.8). Finally, the first two terms on the right hand side of (4.4) vanish due to the integration by parts rule for tangential derivatives, see (4.7), and the fact that w is equal to 0 on $\partial\Omega$.

Now, let $h \in C_0^1(\mathbb{R}^d, \mathbb{R}^d)$ with $h_k n_k \geq c > 0$ on $\partial\Omega$ be the function from Theorem 1.3. We apply the triangle inequality to the Rellich-type identity (5.14) to obtain

$$\begin{aligned}\int_{\partial\Omega} |\nabla w|^2 \,\mathrm{d}\sigma \leq C \Big\{ \int_\Omega |\nabla w|^2 \,\mathrm{d}x + \int_\Omega |\nabla w||\psi| \,\mathrm{d}x \\ + |\lambda| \int_\Omega |\nabla w||w| \,\mathrm{d}x + \int_\Omega |\nabla w||u| \,\mathrm{d}x \Big\}\end{aligned} \tag{5.15}$$

with a constant C that only depends on d and the Lipschitz character of Ω. The left hand side of this estimation establishes the starting point for the proof of inequality 5.13.

The next step consists in deriving estimates which are compatible with the right hand side of (5.15). Testing the first equation of (5.8) with $\overline{w}$, integration by parts and Lemma 4.4 give us as in the proof of Lemma 4.5 the estimation

$$\int_\Omega |\nabla w|^2 \,\mathrm{d}x + |\lambda| \int_\Omega |w|^2 \,\mathrm{d}x \leq C \int_\Omega |w||u| \,\mathrm{d}x,$$

where C depends only on θ. The next step consists in using the previous inequality and the Poincaré inequality to estimate

$$\begin{aligned}\int_\Omega |\nabla w|^2 \,\mathrm{d}x + (1 + |\lambda|) \int_\Omega |w|^2 \,\mathrm{d}x &\leq (1 + C) \int_\Omega |\nabla w|^2 \,\mathrm{d}x + |\lambda| \int_\Omega |w|^2 \,\mathrm{d}x \\ &\leq C \int_\Omega |w||u| \,\mathrm{d}x \\ &\leq C \left(\int_\Omega |w|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_\Omega |u|^2 \,\mathrm{d}x \right)^{1/2}\end{aligned}$$

where for the last step we used the Cauchy-Schwartz inequality. The weighted Young inequality for real numbers allows us to further estimate

$$\begin{aligned}&\leq \frac{C}{4\varepsilon} \int_\Omega |u|^2 \,\mathrm{d}x + C\,\varepsilon \int_\Omega |w|^2 \,\mathrm{d}x \\ &= \frac{\tilde{C}}{1 + |\lambda|} \int_\Omega |u|^2 \,\mathrm{d}x + \frac{1}{2}(1 + |\lambda|) \int_\Omega |w|^2 \,\mathrm{d}x\end{aligned}$$

if we set $\varepsilon = \frac{(1+|\lambda|)}{2C}$. Rearranging terms, we can produce our next estimate

$$\int_\Omega |\nabla w|^2 \,\mathrm{d}x + (1 + |\lambda|) \int_\Omega |w|^2 \,\mathrm{d}x \leq \frac{C}{1 + |\lambda|} \int_\Omega |u|^2 \,\mathrm{d}x, \tag{5.16}$$

where C depends on d, θ and the Lipschitz character of Ω.

Now it's time to harvest: Using the weighted Young inequality, we see that we can simplify the right hand side of (5.15) via the estimation

$$\int_{\partial\Omega} |\nabla w|^2 \,\mathrm{d}\sigma \leq C_\varepsilon (1 + |\lambda|) \int_\Omega |\nabla w|^2 \,\mathrm{d}x + C\,|\lambda| \int_\Omega |w|^2 \,\mathrm{d}x$$

$$
\begin{aligned}
&+ C \int_\Omega |u|^2 \,\mathrm{d}x + \varepsilon \int_\Omega |\psi|^2 \,\mathrm{d}x \\
\leq C_\varepsilon \int_\Omega |u|^2 \,\mathrm{d}x + \varepsilon \int_\Omega |\psi|^2 \,\mathrm{d}x,
\end{aligned}
$$

where the second inequality is thanks to estimate (5.16). The first term on the right hand side is already fine for (5.13). For the second one we use the estimate $\|\psi\|_{\mathrm{L}^2(\Omega)} \leq C \|\psi\|_\partial$ and inequality (5.11) and arrive at

$$
\varepsilon \int_\Omega |\psi|^2 \,\mathrm{d}x \leq \varepsilon C \int_{\partial\Omega} |\nabla w|^2 \,\mathrm{d}\sigma + C_\varepsilon \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma.
$$

Choosing $\varepsilon = \frac{1}{2C}$ and rearranging finally gives the desired estimate (5.13). This concludes our proof. □

The next Theorem states the important fact that in L^2 the Dirichlet Stokes resolvent problem has a unique solution.

Theorem 5.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with connected boundary and let $\lambda \in \Sigma_\theta$. For all $g \in \mathrm{L}^2_n(\partial\Omega)$ there exists a unique u and harmonic function ϕ which is unique up to constants such that (u, ϕ) satisfies* (2.26)*, $(u)^* \in \mathrm{L}^2(\partial\Omega)$ and $u = g$ on $\partial\Omega$ in the sense of nontangential convergence. Moreover, the estimate $\|(u)^*\|_\partial \leq C \|g\|_\partial$ holds and u may be represented by the double layer potential $\mathcal{D}_\lambda(f)$ with $\|f\|_\partial \leq C \|g\|_\partial$, where in both cases C depends only on d, θ and the Lipschitz character of Ω.*

Proof. By Lemma 5.4, we already know that the problem under consideration admits at most one solution. Therefore, we only have to worry about the existence of a solution. In Chapter 3, it was already established that a solution to the Stokes resolvent problem is given by the double layer potentials $u := \mathcal{D}_\lambda(f)$ and $\phi := \mathcal{D}_\Phi(f)$, $f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$. It was also shown that in this way one also solves the L^2 Dirichlet problem with boundary data $\mathcal{D}_\lambda(f)_+$. From Theorem 3.11 we know that we find this nontangential limit as $\big((-1/2)I + \mathcal{K}^*_\lambda\big) f$. Thus, the central idea of this proof will be to invert the operator $(-1/2)I + \mathcal{K}^*_\lambda$ in order to find the *right* f to plug into the double layer potentials in order to attain the given boundary data $g \in \mathrm{L}^2_n(\partial\Omega)$ as a nontangential limit.

We first note that due to Lemma 5.3 the operator

$$
T\colon \mathrm{L}^2(\partial\Omega; \mathbb{C}^d) \to \mathrm{L}^2(\partial\Omega; \mathbb{C}^d), \quad \mathrm{x} \mapsto -(1/2)x + \mathcal{K}_{\bar\lambda} x,
$$

is a Fredholm operator on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ with index 0 and thus the same is true for its adjoint

$$
T^*\colon \mathrm{L}^2(\partial\Omega; \mathbb{C}^d) \to \mathrm{L}^2(\partial\Omega; \mathbb{C}^d), \quad x \mapsto -(1/2)x + \mathcal{K}^*_{\bar\lambda} x.
$$

In the following paragraphs, we will show that T^* has a bounded inverse.

We know that for all $f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ we have $\operatorname{div}(\mathcal{D}_\lambda(f)) = 0$ and therefore

$$\int_{\partial\Omega} T^* f \cdot n \, \mathrm{d}\sigma = \int_{\partial\Omega} u_+ \cdot n \, \mathrm{d}\sigma = 0$$

holds, where for the first equality we applied Theorem 3.11. The second equality uses the fact that since $(u)^*$ is integrable, Proposition 1.8 and hence the divergence theorem are available. This gives $\operatorname{Im}(T^*) \subseteq \mathrm{L}^2_n(\partial\Omega)$. Now on the one hand we have

$$\operatorname{span}(n) = \mathrm{L}^2_n(\partial\Omega)^\perp \subseteq \operatorname{Im}(T^*)^\perp = \ker(T)$$

and on the other hand, as T is injective on $\mathrm{L}^2_n(\partial\Omega)$ by (5.4), we have that

$$\operatorname{span}(n) \supseteq \ker(T).$$

This yields $\operatorname{span}(n) = \ker(T)$. We can use this equality and show that

$$\mathrm{L}^2_n(\partial\Omega) = \ker(T)^\perp = \overline{\operatorname{Im}(T^*)} = \operatorname{Im}(T^*),$$

where for the last equality we used the fact that the range of T^* is closed, as usual for Fredholm operators. With the same argument we can show for T that

$$\ker(T^*)^\perp = \overline{\operatorname{Im}(T)} = \operatorname{Im}(T).$$

Now we want to consider restrictions of the operators T and T^* and derive estimates on the operator norms of their inverses. To make the following proof more readable let

$$X = \mathrm{L}^2_n(\partial\Omega) \quad \text{and} \quad Y = \operatorname{Im}(T).$$

Both spaces are closed subspaces of the Hilbert space $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ and therefore again Hilbert spaces. Consequently, the operator

$$K'_{Y,X} \colon Y \to X, \quad x \mapsto T^* x$$

is invertible by the continuous inverse theorem. At this point we could already establish the solvability of the L^2 Dirichlet problem. But before we do this, in order to derive the additional estimates which were stated in the theorem, we want to bound the operator norm of $\left(K'_{Y,X}\right)^{-1}$ by a constant that does not depend on λ but on the sectoriality parameter θ. To this end, let us introduce the operator

$$K_{X,Y} \colon X \to Y, \quad x \mapsto Tx\,.$$

Now, for $x \in X$ and $y \in Y$ we have that

$$\left\langle x, K^*_{X,Y} y \right\rangle_X = \left\langle K_{X,Y} x, y \right\rangle_Y = \left\langle T x, y \right\rangle_Y = \left\langle x, T^* y \right\rangle_X = \left\langle x, K'_{Y,X} y \right\rangle_X$$

which shows that $K'_{Y,X} = K^*_{X,Y}$ on Y. With the above definitions at hand, Lemma 5.3 states that $K_{X,Y}$ is an invertible operator with operator norm of the inverse bounded from above by some $C > 0$ and C depends at most on d, θ and the Lipschitz character of Ω. As $\|K^*_{X,Y}\|_{\mathfrak{L}(Y,X)} = \|K_{X,Y}\|_{\mathfrak{L}(X,Y)}$ the same hold for the adjoint operator $K^*_{X,Y}$. In particular, we have that $\|(K^*_{X,Y})^{-1}\|_{\mathfrak{L}(X,Y)} = \|K^{-1}_{X,Y}\|_{\mathfrak{L}(Y,X)}$ Therefore, for all $f \in Y = \mathrm{Im}(T) = \mathrm{Im}\left((1/2)I + \mathcal{K}_{\bar{\lambda}}\right)$ we have

$$\begin{aligned} \|f\|_\partial &= \left\|(K'_{Y,X})^{-1} K'_{Y,X} f\right\|_\partial \\ &\leq \left\|(K^*_{X,Y})^{-1}\right\|_{\mathfrak{L}(X,Y)} \left\|K^*_{X,Y} f\right\|_\partial \leq C \left\|(-(1/2)I + \mathcal{K}^*_{\bar{\lambda}}) f\right\|_\partial. \end{aligned} \tag{5.17}$$

We are now in position to derive the missing estimates which were stated in the theorem. For $g \in \mathrm{L}^2_n(\partial\Omega)$, let $f \in \mathrm{Im}(T)$ with $T^* f = g$. Fix this f and let (u, ϕ) be the respective double layer potentials which were defined in equations (3.22) and (3.23). Then $u_+ = T^* f = g$ on $\partial\Omega$ by Theorem 3.11. Additionally, we have that

$$\|(u)^*\|_\partial \leq C \|f\|_\partial \leq C \|g\|_\partial$$

where we used inequality (3.24) and (5.17). In particular, this gives $(u)^* \in \mathrm{L}^2(\partial\Omega)$. Consequently, all claims of the Theorem have been proven. □

The next theorem can in some sense be regarded as a reverse trace theorem and will play an important role for the proof of the needed reverse Hölder inequality in the forthcoming chapter.

Theorem 5.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with connected boundary. Let $u \in \mathrm{H}^1(\Omega; \mathbb{C}^d)$ and $\pi \in \mathrm{L}^2(\Omega)$ satisfy the Stokes resolvent problem in Ω for some $\lambda \in \Sigma_\theta$. Then*

$$\left(\int_\Omega |u|^{p_d} \, \mathrm{d}x \right)^{1/p_d} \leq C \left(\int_{\partial\Omega} |u|^2 \, \mathrm{d}\sigma \right)^{1/2}, \tag{5.18}$$

where $p_d = \frac{2d}{d-1}$ and C depends only on d, θ and the Lipschitz character of Ω.

Proof. We start by citing the following central inequality

$$\left(\int_\Omega |u|^{p_d} \, \mathrm{d}x \right)^{1/p_d} \leq C \left(\int_{\partial\Omega} |(u)^*|^2 \, \mathrm{d}\sigma \right)^{1/2}, \tag{5.19}$$

where C only depends on d and the Lipschitz constant of Ω. The proof of (5.19) was carried out by Wei and Zhang in [35, Lem. 3.3] and can also be found in Shen's paper [26, p. 418f.].

Up to now, we do not know whether the right hand side of inequality (5.19) equals infinity. We only know that (u, ϕ) solves the Stokes resolvent problem and that it implicitly solves a Dirichlet problem with boundary data given as $f = \mathrm{Tr}_{\partial\Omega}(u) \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$. The following part of this proof will show that u coincides with the solution $w := \mathcal{D}_\lambda(g)$, $g \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$, of the L^2 Dirichlet problem (Dir_λ) with boundary data f as given by Theorem 5.5. If this is the case, then the knowledge about $(w)^*$ that is contained in Theorem 5.5 will help us to complete estimate (5.19).

In order to show that $u = w$ on Ω, consider a sequence $(\Omega_j)_{j\in\mathbb{N}}$ of smooth domains that approximates Ω from inside as described by Theorem 1.3. Then, an application of Lemma 5.4 shows

$$\int_{\Omega_j} |u - w|^2 \,\mathrm{d}x \le C \int_{\partial\Omega_j} |u - w|^2 \,\mathrm{d}\sigma_j, \tag{5.20}$$

where C does not depend on j but on the Lipschitz character of Ω. Furthermore, the trace theorem on bounded Lipschitz domains gives us for all $h \in \mathrm{H}^1(\Omega; \mathbb{C}^d)$ that

$$\|h\|_\partial^2 \le C \,\|h\|^2_{\mathrm{H}^1(\Omega;\mathbb{C}^d)}, \tag{5.21}$$

where C only depends on d and the Lipschitz character of Ω, see Wei and Zhang [35, Lem. 2.2]. Now let $\varepsilon > 0$ be given. From the theory of Sobolev spaces it is known that there exists $\varphi_\varepsilon \in \mathrm{C}^\infty(\overline{\Omega}; \mathbb{C}^d)$ such that $\|\varphi_\varepsilon - u\|^2_{\mathrm{H}^1(\Omega;\mathbb{C}^d)} \le \varepsilon/(6\tilde{C})$, see Adams and Fournier [1, Thm. 3.18]. Here, $\tilde{C}$ denotes the fixed constant from (5.21) which is uniform for all Ω_j and Ω. Thanks to Theorem 1.3, we know that

$$\int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \,\mathrm{d}\sigma_j \to \int_{\partial\Omega} |\varphi_\varepsilon - u|^2 \,\mathrm{d}\sigma, \qquad \text{as } j \to \infty,$$

since $w = f$ on $\partial\Omega$ in the sense of nontangential convergence. Therefore, we choose J large enough such that for all $j \ge J$ we have

$$\int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \,\mathrm{d}\sigma_j \le \int_{\partial\Omega} |\varphi_\varepsilon - u|^2 \,\mathrm{d}\sigma + \frac{\varepsilon}{6}\,.$$

Plugging everything together, this gives us the estimation

$$\begin{aligned}
&\int_{\partial\Omega_j} |u - w|^2 \,\mathrm{d}\sigma_j \\
&\quad\le 2\left\{ \int_{\partial\Omega_j} |u - \varphi_\varepsilon|^2 \,\mathrm{d}\sigma_j + \int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \,\mathrm{d}\sigma_j \right\} \le 4\,\tilde{C}\,\|u - \varphi_\varepsilon\|^2_{\mathrm{H}^1(\Omega;\mathbb{C}^d)} + \frac{\varepsilon}{3} \le \varepsilon.
\end{aligned}$$

Gluing together this inequality and (5.20), we see that $w = u$ holds a.e. in Ω once we take the limit $j \to \infty$ and remember that ε was chosen arbitrarily. In particular we have $(u)^* = (w)^*$ a.e. on $\partial\Omega$. Now, Theorem 5.5 gives

$$\int_{\partial\Omega} |(u)^*|^2 \,\mathrm{d}\sigma = \int_{\partial\Omega} |(w)^*|^2 \,\mathrm{d}\sigma \le C \int_{\partial\Omega} |f|^2 \,\mathrm{d}\sigma = C \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma$$

where C depends on d, θ, and the Lipschitz character of Ω. Using this inequality to continue estimate 5.19 concludes our proof. $\square$

Remark 5.7. At this point, the choice $p = 2d(d-1)^{-1}$ in Theorem 5.6 may seem arbitrary. Taking a closer look at the proof by Wei and Zhang in [35, Lem. 3.3], one sees that the choice of p is due to 2 facts: (1) For the dual exponent we have $p' = \frac{2d}{d+1}$. (2) It holds $\frac{1}{p} - \frac{1}{p'} = \frac{1}{d}$ and thus the *Hardy-Littlewood-Sobolev theorem on fractional integration* may be applied to estimate the p'-norm of the *Riesz potential* $I_1(f)$ of a function $f \in \mathrm{L}^p(\mathbb{R}^d)$ by the p-norm of f, see Grafakos [13, Thm 6.1.3].

In the following remark and the forthcoming chapter, we will make use of an integration argument which can be considered an application of the following Theorem on *integration along slices*. A proof of this result can be found in Federer [9, Thm. 3.2.12].

Theorem 5.8 (Co-area formula)**.** *If $f : \mathbb{R}^d \to \mathbb{R}$, $d \ge 2$, is Lipschitz continuous and $\mathfrak{g}$ the representative of a function $g \in \mathrm{L}^2(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} \mathfrak{g}(x) \Big[\sum_{i=1}^{d} \Big| \frac{\partial f}{\partial x_i}(x) \Big|^2 \Big]^{1/2} \,\mathrm{d}x = \int_{\mathbb{R}} \int_{f^{-1}(y)} \mathfrak{g}(x) \,\mathrm{d}m_{d-1}(x) \,\mathrm{d}y, \tag{5.22}$$

where m_{d-1} denotes the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$.

Note that the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$ is comparable to the surface measure σ of a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$. When using Theorem 5.8 in estimations, we will therefore always be working with the surface measure only.

Remark 5.9. Let (u, ϕ) be a solution of the Stokes resolvent system in the domain $\mathrm{B}(x_0, r) \subseteq \mathbb{R}^d$. Then the interior estimate

$$|\nabla^l u(x_0)| \le \frac{C_l}{r^{d-1+l}} \Big(\int_{\mathrm{B}(x_0,\, r)} |u(x)|^2 \,\mathrm{d}x \Big)^{1/2} \tag{5.23}$$

holds for all $l \in \mathbb{N}_0$, where C_l only depends on d, l and θ: Without loss of generality, we may rescale and translate and thus assume that $x_0 = 0$ and $r = 2$. Let $t \in (1, 2)$. By Theorem 5.5, we know that a solution to the Stokes resolvent system on $\mathrm{B}(0,\, t) \subsetneq \mathrm{B}(0,\, 2)$

with boundary values $g_t \coloneqq \mathrm{Tr}_{\partial \mathrm{B}(0,\,t)}(u) \in \mathrm{L}^2(\partial \mathrm{B}(0,\, t); \mathbb{C}^d)$ is given by a boundary layer potential $\mathcal{D}_\lambda(f_t)$, $f_t \in \mathrm{L}^2(\mathrm{B}(0,\, t); \mathbb{C}^d)$. We can use this fact to derive the desired estimate via

$$
\begin{aligned}
|\nabla^l u(0)|^2 &\leq C \left(\int_{\partial \mathrm{B}(0,\,t)} \Big\{ \big|\nabla_x^{l+1} \Gamma(y; \lambda)\big| + \big|\nabla_x^l \Phi(y)\big| \Big\} \, |f_t(y)| \, \mathrm{d}\sigma(y) \right)^2 \\
&\leq C \left(\int_{\partial \mathrm{B}(0,\,t)} \frac{|f_t(y)|}{t^{d-1+l}} \, \mathrm{d}\sigma(y) \right)^2 \\
&\leq C \int_{\partial \mathrm{B}(0,\,t)} |f_t(y)|^2 \, \mathrm{d}\sigma(y) \\
&\leq C \int_{\partial \mathrm{B}(0,\,t)} |u(y)|^2 \, \mathrm{d}\sigma(y),
\end{aligned}
$$

where in the last step we used the estimate of f_t against the "data" from Theorem 5.5. Integrating this inequality in t over the interval $(1, 2)$ and using the co-area formula (5.22) with Lipschitz function $|\cdot|$ gives

$$
|\nabla^l u(0)|^2 \leq C \int_{\mathrm{B}(0,\,2)} |u(x)|^2 \, \mathrm{d}x.
$$

Note that for this argument to work it is crucial that C does only depend on d and the Lipschitz character of Ω. Now the claim follows readily.

Chapter 6

Derivation of Resolvent Estimates

In this final chapter, we will prove that the Stokes semigroup is analytic on $\mathrm{L}^p_\sigma(\Omega)$ for bounded Lipschitz domains $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$.

The first step will be to establish a weak reverse Hölder estimate for local solutions of the Stokes resolvent problem. We start with a similar result on Lipschitz cylinders.

Lemma 6.1. *Let $\eta \colon \mathbb{R}^{d-1} \to \mathbb{R}$ be a Lipschitz function. Furthermore, let the functions $u \in \mathrm{H}^1(D_\eta(r); \mathbb{C}^d)$ and $\phi \in \mathrm{L}^2(D_\eta(2r))$ solve the Stokes resolvent problem in $D_\eta(2r)$ with $u = 0$ on $I_\eta(2r)$ for some $0 < r < \infty$ and $\lambda \in \Sigma_\theta$. Let $p_d = \frac{2d}{d-1}$. Then,*

$$\left(\frac{1}{r^{d-1}} \int_{D_\eta(r)} |u|^{p_d} \, \mathrm{d}x \right)^{1/p_d} \leq C \left(\frac{1}{r^{d-1}} \int_{D_\eta(2r)} |u|^2 \, \mathrm{d}x \right)^{1/2}, \tag{6.1}$$

where C only depends on d, M and θ.

Proof. Without loss of generality, we rescale and assume that $r = 1$. Let $t \in (1, 2)$. We note that thanks to a thorough investigation carried out by Tolksdorf, see [32, Lemma 1.3.25], a Lipschitz cylinder is itself a Lipschitz domain. It is therefore admissible to apply Theorem 5.18 to u in $D_\eta(t)$ which yields

$$\left(\int_{D_\eta(t)} |u|^{p_d} \, \mathrm{d}x \right)^{2/p_d} \leq C \int_{\partial D_\eta(t)} |u|^2 \, \mathrm{d}\sigma,$$

where C depends only on d, θ and the Lipschitz character of Ω. In particular, C does not depend on t. Since u vanishes on $I(2)$, we have that

$$\begin{aligned} \left(\int_{D_\eta(1)} |u|^p \, \mathrm{d}x \right)^{2/p} &\leq C \int_1^2 \left\{ \int_{\partial D(t)} |u(y)|^2 \chi_{D_\eta(2)}(y) \, \mathrm{d}\sigma(y) + \int_{I_\eta(2)} |u(y)|^2 \, \mathrm{d}\sigma(y) \right\} \mathrm{d}t \\ &= C \int_1^2 \int_{\partial D(y)} |u(y)|^2 \chi_{D_\eta(2)}(y) \, \mathrm{d}\sigma(y) \, \mathrm{d}t. \end{aligned}$$

We define the function

$$f\colon \mathbb{R}^d \to \mathbb{R}, \quad x \mapsto t \quad \text{iff} \quad x \in \partial D(t).$$

Since $x \in \partial D(t)$ and $y \in \partial D(s)$, $s, t > 0$, we have that

$$|f(x) - f(y)| = |t - s| \leq \operatorname{dist}(\partial D(t), \partial D(s)) \leq |x - y|$$

and consequently the function f is Lipschitz continuous with Lipschitz constant 1. Applying the co-area formula to integrate both sides over the interval $(1, 2)$ gives

$$\Big(\int_{D_\eta(1)} |u|^p \, \mathrm{d}x \Big)^{2/p} \leq C \int_{D_\eta(2)} |u|^2 \, \mathrm{d}x,$$

see Theorem 5.8. Estimate (6.1) now follows after dividing by $|D_\eta(1)|$. □

The next step is to extend the previous result to arbitrary Lipschitz domains. The following Lemma, which appeared in Tolksdorf [31, Lem. 4.2], reduces the amount of work that needs to be done to a few special cases.

Lemma 6.2. *Let $\Omega \subseteq \mathbb{R}^d$ be Lebesgue-measurable, $f, g \in \mathrm{L}^2(\Omega)$, $\alpha_2 > \alpha_1 > 1$, $p > 2$, $r > 0$ and $x_0 \in \mathbb{R}^d$ be such that $\mathrm{B}(x_0,\ r) \cap \Omega \neq \emptyset$. If there exists $C > 0$ such that*

$$\begin{aligned}
&\Big(\frac{1}{s^d} \int_{\Omega \cap \mathrm{B}(y,\, s)} |f|^p \, \mathrm{d}x \Big)^{1/p} \\
&\qquad \leq C \left\{ \Big(\frac{1}{s^d} \int_{\Omega \cap \alpha_1 \mathrm{B}(y,\, s)} |f|^2 \, \mathrm{d}x \Big)^{1/2} + \sup_{B' \cap \mathrm{B}(y,\, s)} \Big(\frac{1}{|B'|} \int_{\Omega \cap B'} |g|^2 \, \mathrm{d}x \Big)^{1/2} \right\}
\end{aligned}$$

holds for all balls $\mathrm{B}(y,\ s)$ with $\mathrm{B}(y,\ \alpha_2 s) \subseteq \mathrm{B}(x_0,\ \alpha_2 r)$ and which are either centered on $\partial\Omega$ or satisfy $\alpha_2 \mathrm{B}(y,\ s) \subseteq \Omega$, then for each $\alpha \in (1, \alpha_2)$ there exists a constant C' such that

$$\begin{aligned}
&\Big(\frac{1}{r^d} \int_{\Omega \cap \mathrm{B}(x_0,\, r)} |f|^p \, \mathrm{d}x \Big)^{1/p} \\
&\qquad \leq C' \left\{ \Big(\frac{1}{r^d} \int_{\Omega \cap \alpha \mathrm{B}(x_0,\, r)} |f|^2 \, \mathrm{d}x \Big)^{1/2} + \sup_{B' \cap \mathrm{B}(x_0,\, r)} \Big(\frac{1}{|B'|} \int_{\Omega \cap B'} |g|^2 \, \mathrm{d}x \Big)^{1/2} \right\}.
\end{aligned}$$

This constant C' only depends on d, α, α_1, α_2, p and C.

As of now, our toolbox comprises enough tools to prove that solutions to the Stokes resolvent system satisfy a weak reverse Hölder inequality.

Lemma 6.3. *Let $x_0 \in \overline{\Omega}$ and $0 < 2r < r_0$ and set $\alpha_1 = \sqrt{d^2 10^2 (1 + M)^2 + 4}$ and $\alpha_2 = \alpha_1 + 1$. Let $u \in \mathrm{H}^1(\mathrm{B}(x_0,\ \alpha_2 r) \cap \Omega; \mathbb{C}^d)$ and $\phi \in \mathrm{L}^2(\mathrm{B}(x_0,\ \alpha_2 r) \cap \Omega)$ satisfy the*

Stokes resolvent system in $\mathrm{B}(x_0,\ \alpha_2 r)\cap\Omega$. *If* $\mathrm{B}(x_0,\ \alpha_2 r)\cap\partial\Omega \neq \emptyset$, *we additionally assume* $u = 0$ *on* $\mathrm{B}(x_0,\ \alpha_2 r)\cap\partial\Omega$. *Then,*

$$\left(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, r)\cap\Omega} |u|^p\right)^{1/p} \leq C\left(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, 2r)\cap\Omega} |u|^2\right)^{1/2} \tag{6.2}$$

holds, where $p = p_d$. *Here,* $C > 0$ *only depends on* d, θ *and the Lipschitz character of* Ω.

Proof. Due to Lemma 6.2, it suffices to consider only two cases: (1) $x_0 \in \Omega$ with $\alpha_2\mathrm{B}(x_0,\ r) \subseteq \Omega$ and (2) $x_0 \in \partial\Omega$.

In order to prove (1), let $x_0 \in \Omega$ with $\alpha_2\mathrm{B}(x_0,\ r) \subseteq \Omega$. We may deploy the interior estimate (5.23) to derive that for all $x \in \mathrm{B}(x_0,\ r)$ the estimate

$$|u(x)|^p \leq C\left(\frac{1}{r^d}\int_{\mathrm{B}(x,\, r)} |u(y)|^2\,\mathrm{d}y\right)^{p/2}$$

holds which after integrating x over $\mathrm{B}(x_0,\ r)$ yields

$$\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, r)} |u(x)|^p\,\mathrm{d}x \leq C\left(\frac{1}{r^{d-1}}\int_{\mathrm{B}(x_0,\, \alpha_1 r)} |u(z)|^2\,\mathrm{d}z\right)^{p/2},$$

where we used the fact that $\alpha_1 > 2$. Here, C depends only on d and θ.

For (2), note that if $x_0 \in \partial\Omega$, then by Lemma 6.1 and Pythagoras' theorem we have

$$\begin{aligned}\left(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, r)\cap\Omega} |u|^p\right)^{1/p} &\leq \left(\frac{1}{r^d}\int_{D_{\eta x_0}(r)} |u|^p\right)^{1/p} \\ &\leq C\left(\frac{1}{r^d}\int_{D_{\eta x_0}(2r)} |u|^p\right)^{1/p} \\ &\leq C\left(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, \alpha_1 r)\cap\Omega} |u|^2\right)^{1/2}.\end{aligned}$$

Now, the claim follows readily from an application of Lemma 6.2 with the parameter $\alpha = 2 \in (1, \alpha_2)$. □

We note that estimate (6.2) is a weak reverse Hölder inequality and thus possesses a self-improving property, see Giaquinta and Martinazzi [10, Thm. 6.38] or Giaquinta and Modica [11, Prop. 5.1].

Proposition 6.4 (Giaquinta, Modica)**.** *Let* $\Omega \subseteq \mathbb{R}^d$ *be open,* $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, $q > 1$, *be a non-negative function. If there exist constants* $b > 0, R_0 > 0$ *such that*

$$\left(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, r)} f^q\,\mathrm{d}x\right)^{1/q} \leq \frac{b}{r^d}\int_{\mathrm{B}(x_0,\, 2r)} f\,\mathrm{d}x$$

for all $x_0 \in \Omega$ *and* $0 < r < \min\left\{R_0, \operatorname{dist}(x_0, \partial\Omega)/2\right\}$, *then* $f \in \mathrm{L}^{q+\varepsilon}_{\mathrm{loc}}(\Omega)$ *for some* $\varepsilon > 0$, *depending only on* d, q, *and* b *and there is a constant* $\tilde{C}$ *depending only on* d, q, ε *and* b *such that*

$$\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, r)} f^{q+\varepsilon}\,\mathrm{d}x\Big)^{1/(q+\varepsilon)} \leq \tilde{C}\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, 2r)} f^{q}\,\mathrm{d}x\Big)^{1/q}$$

for all $x_0 \in \Omega$ *and* $0 < r < \min\{R_0, \operatorname{dist}(x_0, \partial\Omega)/2\}$.

Remark 6.5. The self-improving property of reverse Hölder estimates can now be used to make the result of Lemma 6.3 a little bit better. Let $0 < 2r < r_0$. We are aiming to apply Proposition 6.4 for $x_0 \in \overline{\Omega}$ on the open set $\Omega \cap \mathrm{B}(x_0, \alpha_2 r)$, with α_2 as in Lemma 6.3. Let also u be as in Lemma 6.3 and set $R_0 = r_0/2$. Then, for $f = |u|^2 \chi_{\mathrm{B}(x_0,\, \alpha_2 r)\cap\Omega}$ which can be considered as a partial extension of u by 0 to $\mathbb{R}^d$ and $q = p/2$, inequality (6.2) reads

$$\begin{aligned}\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, r)} f^{q}\,\mathrm{d}x\Big)^{1/q} &= \Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, r)\cap\Omega} |u|^{p}\,\mathrm{d}x\Big)^{2/p} \\ &\leq C^2\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, 2r)\cap\Omega} |u|^{2}\,\mathrm{d}x = C^2\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, 2r)} f\,\mathrm{d}x.\end{aligned}$$

Consequently, Proposition 6.4 gives us that there exists some $\varepsilon > 0$ which depends only on d, q and C^2 and a constant $\tilde{C}$ depending only on d, q, ε and C^2 such that

$$\begin{aligned}\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, r/2)\cap\Omega} |u|^{p+\varepsilon'}\,\mathrm{d}x\Big)^{2/(p+\varepsilon')} &= \Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, r/2)} f^{q+\varepsilon}\,\mathrm{d}x\Big)^{1/(q+\varepsilon)} \\ &\leq \tilde{C}\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, r)\cap\Omega} |u|^{p}\,\mathrm{d}x\Big)^{1/p} \\ &\leq \tilde{C}C\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, 2r)\cap\Omega} |u|^{2}\,\mathrm{d}x\Big)^{1/2}.\end{aligned}$$

Another application of Lemma 6.2 gives us that for all $r < (r_0/4)$ it holds that

$$\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, r)\cap\Omega} |u|^{p+\varepsilon'}\,\mathrm{d}x\Big)^{2/(p+\varepsilon')} \leq C\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\, 2r)\cap\Omega} |u|^{2}\,\mathrm{d}x.\Big)^{1/2} \tag{6.3}$$

and we have succeeded in improving our original estimate (6.2).

The following extrapolation theorem by Shen [25, Thm. 3.3] will be necessary in order to derive L^p bounds on the solution of the Stokes resolvent system. Essentially, this theorem states that if the non locality of an L^2 bounded operator T can be quantified via a reverse Hölder estimate, then this operator extends to an L^p bounded operator for certain values p. In this sense, this extrapolation theorem can also be considered a p-sensitive version of the famous Calderón-Zygmund Lemma. Note that a more recent result from Tolksdorf [31, Thm. 4.1] generalizes this result to operators which are defined on spaces of Banach space valued functions.

Theorem 6.6. *Let T be a bounded sublinear operator on $\mathrm{L}^2(\Omega; \mathbb{C}^d)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and $\|T\|_{\mathfrak{L}(\mathrm{L}^2(\Omega;\mathbb{C}^d))} \leq C_0$. Let $p > 2$. Suppose that there exist constants $R_0 > 0$, $N > 1$ and $\alpha_2 > \alpha_1 > 1$ such that for any bounded measurable function f with $\operatorname{supp}(f) \subseteq \Omega \setminus \alpha_2 B$,*

$$\left\{\frac{1}{r^d}\int_{\Omega\cap B} |Tf|^p \,\mathrm{d}x\right\}^{1/p} \leq N\left\{\left(\frac{1}{r^d}\int_{\Omega\cap\alpha_1 B} |Tf|^2 \,\mathrm{d}x\right)^{1/2} + \sup_{B'\supset B}\left(\frac{1}{|B'|}\int_{B'} |f|^p \,\mathrm{d}x\right)^{1/p}\right\},$$

where $B = B(x_0, r)$ is a ball with $0 < r < R_0$ and either $x_0 \in \partial\Omega$ or $B(x_0, \alpha_2 r) \subseteq \Omega$. Then, the restriction of T to $\mathrm{L}^q(\Omega; \mathbb{C}^d)$ yields a bounded operator on $\mathrm{L}^q(\Omega; \mathbb{C}^d)$ for any $2 < q < p$. Moreover, the operator norm $\|T\|_{\mathfrak{L}(\mathrm{L}^q(\Omega;\mathbb{C}^d))}$ is bounded by a constant depending at most on d, N, C_0, p, q and the Lipschitz character of Ω.

We are now in the position to prove Theorem 1.17, the main theorem of this thesis. For this, the improved weak reverse Hölder inequality derived in Remark 6.5 will serve as the crucial ingredient, enabling us to apply the extrapolation theorem 6.6 to a suitable family of operators. As we want to prove resolvent estimates of the Stokes operator on L^p, the family of operators under consideration will basically consist of resolvent operators. To this end, note that from the last sentence of Theorem 6.6 we get uniform bounds on our operator family on L^p provided that we start with a uniform bound C_0 on L^2. This aspect regarding the uniformity of the estimates and the p-sensitivity of the extrapolation theorem are the distinguished properties of this theorem compared to classic results from the Calderón-Zygmund theory of convolution operators, see Grafakos [14, Sec. 5.3] or Stein [29, Ch. 2]

Proof of Theorem 1.17. Consider a family of scaled solution operators to the Stokes resolvent system (1.9), more precisely consider the family

$$T_\lambda\colon \mathrm{L}^2(\Omega; \mathbb{C}^d) \to \mathrm{L}^2(\Omega; \mathbb{C}^d), \quad f \mapsto (|\lambda| + 1)(A_2 + \lambda)^{-1}\mathbb{P}_2 f,$$

where $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi/2)$. Let us first verify that $u := (|\lambda| + 1)^{-1} T_\lambda(f)$ does indeed solve (1.9). To this end, note that since $\mathbb{P}_2 f \in \mathrm{L}^2_\sigma(\Omega)$ we know that by the mapping properties of the Stokes resolvent we have $u \in \mathrm{H}^1_{0,\sigma}(\Omega)$ and

$$A_2 u + \lambda u = \mathbb{P}_2 f.$$

Therefore, u is a weak solution to

$$-\Delta u + \lambda u = \mathbb{P}_2 f.$$

By the usual arguments (c.f. Section 1.2), there exists a pressure $\pi \in \mathrm{L}^2(\Omega)$ such that

$$-\Delta u + \nabla \pi + \lambda u = f$$

holds in the sense of distributions. Furthermore, by testing this identity with u and then using the Poincaré inequality together with Lemma 4.5, we derive the resolvent-type estimate

$$\|T_\lambda(f)\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} = (|\lambda| + 1)\|u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \leq C_0 \|f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)},$$

where C_0 only depends on d, θ and the Lipschitz character of Ω. Accordingly, the family T_λ is bounded on $\mathrm{L}^2(\Omega; \mathbb{C}^d)$ and C_0 is a uniform bound on the operator norms $\|T_\lambda\|_{\mathfrak{L}(\mathrm{L}^2(\Omega;\mathbb{C}^d))}$.

We will now show that the operators T_λ fulfill the estimate in Theorem 6.6, in order to deduce their L^p boundedness. To this end, let $x_0 \in \overline{\Omega}$ and $0 < 4r < r_0$ such that $3\mathrm{B}(x_0, r) \subseteq \Omega$ or $\mathrm{B}(x_0, r)$ is centered on $\partial\Omega$. Furthermore, let $f \in \mathrm{L}^\infty(\Omega; \mathbb{C}^d)$ with support in $\Omega \setminus 3\mathrm{B}(x_0, r)$. By construction, (u, π) does not only solve (2.26) in Ω, the pair also solves the Dirichlet problem

$$\begin{aligned} -\Delta u + \nabla \phi + \lambda u &= 0 \\ \operatorname{div}(u) &= 0 \end{aligned}$$

in $\Omega \cap 3\mathrm{B}(x_0, r)$ where $u = 0$ on $\partial\Omega \cap 3\mathrm{B}(x_0, r)$. Therefore, Remark 6.5 and more precisely inequality (6.3) give that

$$\left(\frac{1}{r^d} \int_{\Omega \cap \mathrm{B}(x_0, r)} |u|^p \,\mathrm{d}x\right)^{1/p} \leq C \left(\frac{1}{r^d} \int_{\Omega \cap 2\mathrm{B}(x_0, r)} |u|^2 \,\mathrm{d}x\right)^{1/2},$$

where $p = p_d + \varepsilon$. Multiplying this inequality on both sides with $(|\lambda| + 1)$ gives

$$\left(\frac{1}{r^d} \int_{\Omega \cap \mathrm{B}(x_0, r)} |T_\lambda(f)|^p \,\mathrm{d}x\right)^{1/p} \leq C \left(\frac{1}{r^d} \int_{\Omega \cap 2\mathrm{B}(x_0, r)} |T_\lambda(f)|^2 \,\mathrm{d}x\right)^{1/2}, \tag{6.4}$$

where C depends only on d, θ and the Lipschitz character of Ω. Now, Shen's extrapolation theorem 6.6 gives that T_λ is bounded on $\mathrm{L}^q(\Omega; \mathbb{C}^d)$ for all $2 < q < p_d + \varepsilon$ and that the operator norms $\|T_\lambda\|_{\mathfrak{L}(\mathrm{L}^q(\Omega;\mathbb{C}^d))}$ are uniformly bounded by a constant C_q depending only on d, θ, q and the Lipschitz character of Ω.

In the next step of the proof, we study the relationship between the operator T_λ and the resolvent of the Stokes operator A_q on $\mathrm{L}^q_\sigma(\Omega)$ for $q \in (2, p_d{+}\varepsilon)$. To this end, let $f \in \mathrm{L}^q_\sigma(\Omega)$.

We already know that $u = (1 + |\lambda|)^{-1} T_\lambda(f) = (A_2 + \lambda)^{-1} \mathbb{P}_2(f) \in \mathrm{L}^q_\sigma(\Omega) \cap \mathcal{D}(A_2)$ by the mapping properties of $T_\lambda(f)$. As $\mathrm{L}^q_\sigma(\Omega) \subseteq \mathrm{L}^2_\sigma(\Omega)$, we have furthermore that

$$\lambda u + A_2 u = f \in \mathrm{L}^q_\sigma(\Omega)$$

and thus $A_2 u \in \mathrm{L}^q_\sigma(\Omega)$. Appealing to Definition 1.12, we showed that $u \in \mathcal{D}(A_q)$ and that $A_2 u = A_q u$. Therefore, we have that

$$\lambda u + A_q u = f \in \mathrm{L}^q_\sigma(\Omega)$$

By the uniqueness of u, which follows from the L^2 theory of the Stokes resolvent problem, we have that $u = (\lambda + A_q)^{-1} f$. Hence, estimate 6.4 gives

$$\|u\|_{\mathrm{L}^q(\Omega;\mathbb{C}^d)} = \|(\lambda + A_q)^{-1} f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^d)} \leq \frac{C_q}{1 + |\lambda|} \|f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^d)}$$

and thus A_q is sectorial on $\mathrm{L}^q_\sigma(\Omega)$. If necessary, we take ε to be the minimum of the parameter ε used in the first part of this proof and the one from Theorem 1.16. It was also shown in Chapter 1 that the spaces $\mathrm{L}^q_\sigma(\Omega)$ are reflexive and that $\mathrm{L}^q_\sigma(\Omega)^* = \mathrm{L}^{q'}_\sigma(\Omega)$ where q' denotes the dual exponent $q' = q(q-1)^{-1}$, see Lemma 1.14. By a general result about sectorial operators on reflexive Banach spaces, which can be found in Haase's book [15, Prop. 2.1.1], we get that A_q is indeed densely defined and that $A_q^* = A_{q'}$. Therefore,

$$\|(A_q + \lambda)^{-1}\|_{\mathfrak{L}(\mathrm{L}^q_\sigma(\Omega))} = \|\big(A_q + \lambda)^{-1}\big) * \|_{\mathfrak{L}(\mathrm{L}^{q'}_\sigma(\Omega))} = \|(A_{q'} + \lambda)^{-1}\|_{\mathfrak{L}(\mathrm{L}^{q'}_\sigma(\Omega))}.$$

Consequently, also the operators $A_{q'}$ are sectorial, densely defined and closed. This completes the proof. $\square$

Appendix

We first collect expressions for the derivatives of the fundamental solutions to the scalar Helmholtz equation and the Laplace equation in $d = 2$.

For the fundamental solution to the Laplace equation $G(x; 0) = -\frac{1}{2\pi}\log(|x|)$, we have the following expressions for the partial derivatives:

$$\begin{aligned}
\partial_\gamma G(x; 0) &= -\frac{1}{2\pi}\frac{x_\gamma}{|x|^2} \\
\partial_\alpha\partial_\gamma G(x; 0) &= -\frac{1}{2\pi}\delta_{\alpha\gamma}\cdot\frac{1}{|x|^2} + \frac{1}{\pi}\cdot\frac{x_\alpha x_\gamma}{|x|^4} \\
\partial_\beta\partial_\alpha\partial_\gamma G(x; 0) &= \frac{1}{\pi}\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^4} - \frac{4}{\pi}\frac{x_\alpha x_\beta x_\gamma}{|x|^6}.
\end{aligned}$$

The fundamental solution for the scalar Helmholtz equation is given via

$$G(x; \lambda) = \frac{\mathrm{i}}{4}H_0^{(1)}(k|x|).$$

In the following, we will always have $z = k|x|$. Then, applications of the chain rule and the product rule of differentiation give

$$\begin{aligned}
\partial_\gamma G(x; \lambda) &= \frac{\mathrm{i}}{4}k\ \frac{x_\gamma}{|x|}\frac{\mathrm{d}}{\mathrm{d}z}H_0^{(1)}(z) \\
\partial_\alpha\partial_\gamma G(x; \lambda) &= \frac{\mathrm{i}}{4}k^2\ \frac{x_\alpha x_\gamma}{|x|^2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}H_0^{(1)}(z) + \frac{\mathrm{i}}{4}k\left(\frac{\delta_{\alpha\gamma}}{|x|} - \frac{x_\alpha x_\gamma}{|x|^3}\right)\frac{\mathrm{d}}{\mathrm{d}z}H_0^{(1)}(z) \\
\partial_\beta\partial_\alpha\partial_\gamma G(x; \lambda) &= \frac{\mathrm{i}}{4}k^3\ \frac{x_\alpha x_\beta x_\gamma}{|x|^3}\frac{\mathrm{d}^3}{\mathrm{d}z^3}H_0^{(1)}(z) \\
&\quad + \frac{\mathrm{i}}{4}k^2\left(\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^2} - 3\frac{x_\alpha x_\beta x_\gamma}{|x|^4}\right)\frac{\mathrm{d}^2}{\mathrm{d}z^2}H_0^{(1)}(z) \\
&\quad + \frac{\mathrm{i}}{4}k\left(3\frac{x_\alpha x_\beta x_\gamma}{|x|^5} - \frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^3}\right)\frac{\mathrm{d}}{\mathrm{d}z}H_0^{(1)}(z).
\end{aligned}$$

The series expansions for the Hankel function $H_0^{(1)}(z)$ read according to Lebedev [19]

$$
\begin{aligned}
\frac{\pi}{2\mathrm{i}} H_0^{(1)}(z) &= \frac{\pi}{2\mathrm{i}} J_0(z) + \frac{\pi}{2} Y_0(z) \\
&= \sum_{l=0}^{\infty} \frac{(-1)^l}{(l!)^2\, 4^l}\, z^{2l} \Big(-\frac{\mathrm{i}\pi}{2} - \log(2) - \psi(l+1) \Big) + \sum_{l=0}^{\infty} \frac{(-1)^l}{(l!)^2\, 4^l}\, z^{2l} \log(z) \\
&= \sum_{l=0}^{\infty} a_l\, z^{2l}\, C_l + \sum_{l=0}^{\infty} a_l z^{2l} \log(z)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\pi}{2\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z} H_0^{(1)}(z) &= \sum_{l=1}^{\infty} a_l\, (2l)\, z^{2l-1}\, C_l + \sum_{l=1}^{\infty} a_l\, (2l)\, z^{2l-1} \log(z) \; + \sum_{l=0}^{\infty} a_l z^{2l-1} \\
&= \sum_{l=1}^{\infty} b_l\, z^{2l-1}\, C_l + \sum_{l=1}^{\infty} b_l\, z^{2l-1} \log(z) \; + \sum_{l=0}^{\infty} a_l\, z^{2l-1}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\pi}{2\mathrm{i}} \frac{\mathrm{d}^2}{\mathrm{d}z^2} H_0^{(1)}(z) &= \sum_{l=1}^{\infty} b_l\, (2l-1)\, z^{2l-2}\, C_l + \sum_{l=1}^{\infty} b_l\, (2l-1)\, z^{2l-2} \log(z) + \sum_{l=1}^{\infty} b_l\, z^{2l-2} \\
&\quad + \sum_{l=0}^{\infty} a_l\, (2l-1)\, z^{2l-2} \\
&= \sum_{l=1}^{\infty} c_l\, z^{2l-2}\, C_l + \sum_{l=1}^{\infty} c_l\, z^{2l-2} \log(z) + \sum_{l=1}^{\infty} b_l\, z^{2l-2} + \sum_{l=0}^{\infty} a_l (2l-1)\, z^{2l-2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\pi}{2\mathrm{i}} \frac{\mathrm{d}^3}{\mathrm{d}z^3} H_0^{(1)}(z) &= \sum_{l=2}^{\infty} c_l\, (2l-2)\, z^{2l-3}\, C_l + \sum_{l=2}^{\infty} c_l\, (2l-2)\, z^{2l-3} \log(z) + \sum_{l=1}^{\infty} c_l\, z^{2l-3} \\
&\quad + \sum_{l=2}^{\infty} b_l (2l-2)\, z^{2l-3} + \sum_{l=0}^{\infty} a_l (2l-1)(2l-2)\, z^{2l-3} \\
&= \sum_{l=2}^{\infty} d_l\, z^{2l-3}\, C_l + \sum_{l=2}^{\infty} d_l\, z^{2l-3} \log(z) + \sum_{l=1}^{\infty} c_l\, z^{2l-3} \\
&\quad + \sum_{l=2}^{\infty} b_l (2l-2)\, z^{2l-3} + \sum_{l=0}^{\infty} a_l (2l-1)(2l-2)\, z^{2l-3},
\end{aligned}
$$

where, in order to increase readability, we introduced the following coefficients

$$
\begin{aligned}
C_l &:= -\frac{\mathrm{i}\pi}{2} - \log(2) - \psi(l+1) \\
a_l &:= \frac{(-1)^l}{(l!)^2\, 4^l}, \quad b_l := a_l \cdot 2l, \quad c_l := b_l \cdot (2l-1), \quad d_l := c_l \cdot (2l-2).
\end{aligned}
$$

A.1 Proof of Lemma 2.3 for $d = 2$

For $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi/2)$, we need to show that for $|\lambda||x|^2 \leq (1/2)$

$$\left|\partial_\beta \partial_\alpha \partial_\gamma \big\{G(x;\lambda) - G(x;0)\big\}\right| \leq C\,|\lambda||x|^{-1}$$

for a constant C that only depends on d and θ.

In order to make the next calculations better to digest, we decompose the third derivative of $G(\cdot;\lambda)$ as

$$\partial_\beta \partial_\alpha \partial_\gamma \big\{G(x;\lambda)\big\} = A_1 + A_2 + A_3,$$

where each A_i, $i = 1, \dots, 3$, corresponds to the term involving the ith derivative of $H_0^{(1)}$.

$$\begin{aligned} A_1 = -\frac{1}{2\pi} k^3 \frac{x_\alpha x_\beta x_\gamma}{|x|^3} \bigg\{ &\sum_{l=2}^{\infty} d_l\,(k|x|)^{2l-3}\,C_l + \sum_{l=2}^{\infty} d_l\,(k|x|)^{2l-3} \log(k|x|) \\ &+ \sum_{l=1}^{\infty} c_l\,(k|x|)^{2l-3} + \sum_{l=2}^{\infty} b_l (2l-2)\,(k|x|)^{2l-3} \\ &+ \sum_{l=0}^{\infty} a_l (2l-1)(2l-2)\,(k|x|)^{2l-3} \bigg\} \end{aligned}$$

We see that using the fact $|\lambda||x|^2 \leq (1/2)$ the only problematic term is

$$-\frac{1}{2\pi} k^3 \frac{x_\alpha x_\beta x_\gamma}{|x|^3} \sum_{l=0}^{0} a_l (2l-1)(2l-2)\,(k|x|)^{2l-3} \tag{P1}$$

For A_2, we calculate

$$\begin{aligned} A_2 = -\frac{1}{2\pi} k^2 &\bigg(\frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^2} - 3\frac{x_\alpha x_\beta x_\gamma}{|x|^4} \bigg) \\ &\cdot \bigg\{ \sum_{l=1}^{\infty} c_l\,(k|x|)^{2l-2}\,C_l + \sum_{l=1}^{\infty} c_l\,(k|x|)^{2l-2} \log(k|x|) \\ &\quad + \sum_{l=1}^{\infty} b_l\,(k|x|)^{2l-2} + \sum_{l=0}^{\infty} a_l (2l-1)\,(k|x|)^{2l-2} \bigg\}. \end{aligned}$$

As the prefactor already behaves like $|\lambda||x|^{-1}$, the problematic terms are given by

$$\begin{aligned} &-\frac{1}{2\pi} k^2 \bigg(\frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^2} - 3\frac{x_\alpha x_\beta x_\gamma}{|x|^4} \bigg) \cdot \sum_{l=1}^{1} c_l\,(k|x|)^{2l-2} \log(k|x|) \\ &-\frac{1}{2\pi} k^2 \bigg(\frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^2} - 3\frac{x_\alpha x_\beta x_\gamma}{|x|^4} \bigg) \cdot \sum_{l=0}^{0} a_l (2l-1)\,(k|x|)^{2l-2}. \end{aligned} \tag{P2}$$

For the last component, we have the following identity:

$$A_3 = -\frac{1}{2\pi}k\Big(3\frac{x_\alpha x_\beta x_\gamma}{|x|^5} - \frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^3}\Big)$$
$$\Big\{\sum_{l=1}^{\infty} b_l\,(k|x|)^{2l-1}\,C_l + \sum_{l=1}^{\infty} b_l\,(k|x|)^{2l-1}\log(k|x|) \; + \sum_{l=0}^{\infty} a_l\,(k|x|)^{2l-1}\Big\}$$

In this case, the prefactor behaves like $\sqrt{|\lambda|}|x|^{-2}$. Therefore, the problematic terms are only found in the last two sums

$$\begin{aligned}
&-\frac{1}{2\pi}k\left(3\frac{x_\alpha x_\beta x_\gamma}{|x|^5} - \frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^3}\right)\cdot\sum_{l=1}^{1} b_l\,(k|x|)^{2l-1}\log(k|x|)\\
&-\frac{1}{2\pi}k\left(3\frac{x_\alpha x_\beta x_\gamma}{|x|^5} - \frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^3}\right)\cdot\sum_{l=0}^{0} a_l\,(k|x|)^{2l-1}.
\end{aligned} \tag{P3}$$

Now we observe the astonishing fact that if we sum up all problematic terms (P1)-(P3) and subtract $\partial_\beta\partial_\alpha\partial_\gamma G(x;0)$ everything adds up to zero. Note that we have

$$a_0 = 1, \qquad c_1 = b_1 = -\frac{1}{2}$$

and therefore we get the following:

$$\begin{aligned}
&(\mathrm{P1}) + (\mathrm{P2}) + (\mathrm{P3}) - \partial_\beta\partial_\alpha\partial_\gamma G(x;0)\\
&\qquad= -\frac{1}{\pi}\frac{x_\alpha x_\beta x_\gamma}{|x|^6}\\
&\qquad\quad+ \frac{1}{4\pi}k^2\left(\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^2} - 3\frac{x_\alpha x_\beta x_\gamma}{|x|^4}\right)\log(k|x|)\\
&\qquad\quad+ \frac{1}{2\pi}\left(\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^4} - 3\frac{x_\alpha x_\beta x_\gamma}{|x|^6}\right)\\
&\qquad\quad+ \frac{1}{4\pi}k^2\left(3\frac{x_\alpha x_\beta x_\gamma}{|x|^4} - \frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^2}\right)\log(k|x|)\\
&\qquad\quad- \frac{1}{2\pi}\left(3\frac{x_\alpha x_\beta x_\gamma}{|x|^6} - \frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^4}\right)\\
&\qquad\quad- \frac{1}{\pi}\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^4} + \frac{4}{\pi}\frac{x_\alpha x_\beta x_\gamma}{|x|^6}\\
&\qquad= \;\; 0.
\end{aligned}$$

A.2 Proof of Lemma 2.6

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Fabian Gabel