TECHNISCHE UNIVERSITÄT DARMSTADT

Fachbereich Mathematik

Masterthesis

On Resolvent Estimates in L^p for the Stokes Operator in Lipschitz Domains

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Introduction

Chapter 1

Fundamentals

The purpose of this chapter is to collect basic definitions that will be used throughout the subsequent chapters. Furthermore we want to formulate the main problem regarding the resolvent estimates of the Stokes operator.

1.1 Lipschitz-Domains

In this first section we will establish the fundamental notions regarding bounded Lipschitz domains.

defn:lipschitzDomain

Definition 1.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded open connected set. We call Ω a *bounded Lipschitz domain* if there exist $r_0, M > 0$ such that for all $x \in \partial\Omega$ there exists a function $\eta_x \colon \mathbb{R}^{d-1} \to \mathbb{R}$ which is Lipschitz continuous and fulfills $\eta_x(0) = 0$ and $\|\nabla \eta_x\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} \leq M$, and a rotation $R_x \colon \mathbb{R}^d \to \mathbb{R}^d$ such that for all $0 < r \leq r_0$

$$\begin{aligned} R_x[\Omega - \{x\}] \cap D(r) &= D_{\eta_x}(r) \\ R_x[\partial\Omega - \{x\}] \cap D(r) &= I_{\eta_x}(r), \end{aligned}$$

where

$$\begin{aligned} D(r) &:= \{(x', x_d) \colon |x'| < r, |x_d| < 10d(M+1)r\} \\ D_{\eta_x}(r) &:= \{(x', x_d) \colon |x'| < r, \eta_x(x') < x_d < 10d(M+1)r\} \\ I_{\eta_x}(r) &:= \{(x', x_d) \colon |x'| < r, \eta_x(x') = x_d\}. \end{aligned}$$

It is common to refer to sets of the form D_{η_x} as *Lipschitz cylinders*.

If Ω is a bounded Lipschitz domain, $x \in \partial\Omega$ and $0 < r \leq r_0$, then we may define $U_{x,r} := \{x\} + R_x^{-1} D(r)$, where R_x is the rotation corresponding to x from Definition 1.1.

This is all we need to define the Lipschitz character of a bounded Lipschitz domain Ω as suggested by Pipher and Verchota in [**?**].

Definition 1.2. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and $x_1, \dots, x_n \in \partial\Omega$ be such that $\{U_{x_i,r_0}\}_{i=1}^n$ covers $\partial\Omega$. Furthermore let M be the constant from Definition 1.1. Then a constant $C > 0$ is said to depend on the *Lipschitz character of* Ω if it depends on M and n.

That the Lipschitz character indeed a fruitful concept will be emphasized by the following theorem. This result is a crucial ingredient in the proof of the Rellich estimates in Chapter 4 as it provides a useful approximating property of Lipschitz domains. In short it enables us to approximate a bounded Lipschitz domain Ω by a sequence (Ω_j) of C^∞ domains in such a way that estimates on Ω_j with bounding constants that only depend on the Lipschitz characters may be extended to Ω when taking the limit. The original proof of this Theorem goes back to Verchota [**?**] and Necas [**?**]. The presented version of this theorem appeared in Brown [**?**].

`thm:smoothApproximation`

Theorem 1.1 (Verchota, Necas)**.** *Let Ω be a Lipschitz domain. Then there exists a sequence of C^∞-domains Ω_k with uniform Lipschitz characters, homeomorphisms $\Lambda_k\colon \partial\Omega \to \partial\Omega_k$, functions $\vartheta_k\colon \partial\Omega \to \mathbb{R}^+$ and a smooth compactly supported vector field $h\colon \mathbb{R}^d \to \mathbb{R}^d$ which satisfy the following properties:*

i) There exists a covering of $\partial\Omega$ by coordinate cylinders which also serve as coordinate cylinders for $\partial\Omega_k$.

ii) The homeomorhpisms $\Lambda_k\colon \partial\Omega \to \partial\Omega_k$ satisfy

$$\sup_{Q\in\partial\Omega} |Q - \Lambda_k(Q)| \to 0 \quad \text{as } k \to \infty$$

and $\Lambda_k(P)$ approaches P nontangentially meaning that for all $k \in \mathbb{N}$

$$|P - \Lambda_k(P)| < (1+\beta)\,\mathrm{dist}(\Lambda_k(P), \partial\Omega)$$

for some constant β depending only on d and the Lipschitz character of Ω.

iii) The normals ν_k of $\partial\Omega_k$ satisfy $\lim_{k\to\infty} \nu_k(\Lambda_k(P)) = \nu(P)$ a. e. for all $P \in \partial\Omega$

iv) The functions ϑ_k satisfy $\delta \leq \vartheta_k \leq \delta^{-1}$ for some $\delta > 0$, $\vartheta^k \to 1$ pointwise a. e. and

$$\int_E \vartheta_k(Q)\,\mathrm{d}\sigma(Q) = \int_{\Lambda_k(E)} \mathrm{d}\sigma_k(Q),$$

where $E \subset \partial\Omega$ is measurable and σ_k denotes the surface measure on Ω_k.

v) The vector field α satisfies $\langle \alpha, \nu_k \rangle \geq c > 0$ a.e. on each $\partial\Omega_k$ where ν_k denotes the unit inner normal to $\partial\Omega_k$.

The next concept we introduce, will allow us to talk about boundary values of functions which are defined on Ω, by considering their nontangential behavior. The first step will be to introduce nontangential approach regions. Unaffortunately in the literature there exist at least two different concepts. In the following, by a cone we mean an open, circular, truncated cone with only one convex component.

`defn:regularFamilyOfCones`

Definition 1.3 (Regular family of cones, Verchota)**.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. If $q \in \partial\Omega$, then $\Gamma(q)$ will denote a cone with vertex q and one component in Ω. Assigning to each $q \in \partial\Omega$ one cone $\Gamma(q)$ the family $\{\Gamma(q) \colon q \in \partial\Omega\}$ will be called *regular* if there exist $x_1, \ldots, x_{n_0} \in \partial\Omega$, $\tilde{r} > 0$ and rotations $\tilde{R}_{x_1}, \ldots, \tilde{R}_{x_{n_0}}$ such that

$$\partial\Omega \subset \bigcup_{i=1}^{n_0} \{x_i\} + \tilde{R}_{x_i}^{-1} D(4\tilde{r}/5),$$

and such that there exist Lipschitz continuous functions $\tilde{\eta}_{x_i} \colon \mathbb{R}^{d-1} \to \mathbb{R}$ such that for all $\tilde{r} \leq r \leq \nu\tilde{r}$ with

$$\nu := 1 + [1 + [10d(M+1)]^2]^{1/2}$$

we have

$$\begin{aligned} \tilde{R}_{x_i}[\Omega - \{x_i\}] \cap D(r) &= D_{\tilde{\eta}_{x_i}}(r) \\ \tilde{R}_{x_i}[\partial\Omega - \{x_i\}] \cap D(r) &= I_{\tilde{\eta}_{x_i}}(r). \end{aligned}$$

In addition for all i there exist cones α_i, β_i and γ_i with vertex at the origin and axis along the x_d-axis such that

$$\alpha_i \subset \overline{\beta_i} \setminus \{0\} \subset \gamma_i$$

and such that for all $q \in [\{x_i\} + \tilde{R}_{x_i}^{-1} D(4\tilde{r}/5)] \cap \partial\Omega$, we have

$$\begin{aligned} \tilde{R}_{x_i}^{-1}\alpha_i + \{q\} \subset \Gamma(q) \subset \overline{\Gamma(q)} \setminus \{q\} &\subset \tilde{R}_{x_i}^{-1}\beta_i + \{q\}, \\ \tilde{R}_{x_i}^{-1}\gamma_i + \{q\} &\subset [\{x_i\} + \tilde{R}_{x_i}^{-1} D(\tilde{r})] \cap \Omega. \end{aligned}$$

We will sometimes denote a regular cone as above by $\Gamma_{\mathrm{V}}(q)$.

For the existence of such families of cones see the Appendix of Verchota [**?**].

In Verchota cones $\Gamma_V(q)$ we have the properties that for all Ω there exists a constant $C > 0$ depending only on the Lipschitz character such that for all $q,\ p \in \partial\Omega$ and any $x \in \Gamma_V(p)$ we have that

eq:verCone1

$$|x-q| \geq C|x-p| \tag{1.1}$$

eq:verCone2

$$|x-q| \geq C|p-q|. \tag{1.2}$$

For a proof see Verchota [**?**, p. 9f.]

defn:nontangentialApproachRegion

Definition 1.4 (Nontangential approach region, Shen)**.** For $\alpha > 1$ and $q \in \partial\Omega$ we define

$$\Gamma(q) := \left\{x \in \Omega \setminus \partial\Omega \colon |x-q| < \alpha \operatorname{dist}(x, \partial\Omega)\right\}$$

If α si chosen sufficiently large (see Shen [**?**]) we call $\{\Gamma(q) \colon q \in \partial\Omega\}$ a *family of non-tangential approach regions*.

Note that in Shen cones $\Gamma(p)$, we have that for $q, y \in \partial\Omega$ and $x \in \Gamma(q)$

eq:shenConeEstimate

$$\begin{aligned} |q-y| &\leq |q-x| + |x-y| \leq \alpha \operatorname{dist}(x, \partial\Omega) + |x-y| \\ &\leq (\alpha+1)|x-y| \end{aligned} \tag{1.3}$$

where α is the constant from 1.4. It is reasonable to choose α in a way that also condition (ii) from Theorem 1.1 holds.

Depending on the type of cones used one may introduce similar concepts of nontangential convergence and nontangential maximal functions.

Definition 1.5. For a function u in Ω and a fixed family of nontangential approach regions, we define the nontangential maximal function $(u)^*$ by

eq:defnNontangMaxFunction

$$(u)^*(q) = \sup\left\{|u(x)| \colon x \in \Gamma(q)\right\} \tag{1.4}$$

for $q \in \partial\Omega$.

Note that Tolksdorf [**?**] and Shen [**?**] show that the choice of α for the nontangential maximal function does not affect their p-norms in an unpredictable way. In fact their p-norms for different α_1 and α_2 stay comparable with a constant only depending on d, α_1, α_2 and the Lipschitz character.

The above mentioned constructions of cones are not limited to cones that lay in the interior of the domain Ω. In fact the same construction can be carried out for the exterior domain $\mathbb{R}^d \setminus \overline{\Omega}$ yielding cones that lay outside of Ω.

Definition 1.6 (Nontangential convergence)**.** Let Ω be a bounded Lipschitz domain and $\{\Gamma(q)\colon q \in \partial\Omega\}$ be a family of nontangential approach regions which lies either inside or outside Ω. Let furthermore u be a function on Ω and f a function on $\partial\Omega$. We say that $u = f$ *in the sense of nontangential convergence* if

$$\lim_{\substack{x\to q\\ x\in\Gamma q}} u(x) = f(q), \quad \text{for a. e. } q \in \partial\Omega.$$

Usually the nontangential limits taken from inside and outside the domain will differ. For functions u on Ω we will therefore often use the notation u_+ to denote the *inner* nontangential limit for a (implicitly) fixed family of nontangential approach regions and u_- for the respective *outer* nontangential limit.

1.2 The Stokes Operator

In this section we will introduce the Stokes operator on $\mathrm{L}^2(\Omega)$ and $\mathrm{L}^p(\Omega)$ for general p and establish a relation to the *Dirichlet problem for the Stokes resolvent system*

$$\begin{aligned} -\Delta u + \nabla\phi + \lambda u &= f \quad \text{in } \Omega,\\ \operatorname{div} u &= 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \qquad \texttt{eq:stokesResolventSystem} \tag{1.5}$$

where $\lambda \in \Sigma_\theta := \{z \in \mathbb{C}\colon \lambda \neq 0 \text{ and } |\arg(z)| < \pi - \theta\}$ and $\theta \in (0, \pi/2)$.

We beginn by defining the relevant function spaces. Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ be a bounded Lipschitz domain and $1 < p < \infty$. We define

$$\mathrm{C}^\infty_{c,\sigma}(\Omega) := \{\varphi \in \mathrm{C}^\infty_c(\Omega; \mathbb{C}^d)\colon \operatorname{div}(\varphi) = 0\},$$

which can serve as a suitable space of test functions. We can now close this space in $\mathrm{L}^p(\Omega)$ and the Sobolev Space $\mathrm{W}_{1,p}(\Omega)$ which gives

$$\mathrm{L}^p_\sigma := \overline{\mathrm{C}^\infty_{c,\sigma}(\Omega)}_{\mathrm{L}^p}$$

and

$$\mathrm{W}^{1,p}_{0,\sigma}(\Omega) := \overline{\mathrm{C}^\infty_{c,\sigma}(\Omega)}_{\mathrm{W}^{1,p}}.$$

If $p = 0$, we will use the symbol $\mathrm{H}^1_{0,\sigma}(\Omega)$ to denote $\mathrm{W}^{1,2}_{0,\sigma}(\Omega)$ to emphasize that this space is a Hilbert space.

In order to define the Stokes operator, we introduce the following sesquilinear form

$$\mathfrak{a}\colon \mathrm{H}^1_{0,\sigma}(\Omega)\times\mathrm{H}^1_{0,\sigma}(\Omega)\to\mathbb{C},\quad (u,v)\mapsto\int_\Omega \nabla u\cdot\overline{\nabla v}\,\mathrm{d}x.$$

Note that for $u\in\mathrm{H}^1_{0,\sigma}(\Omega)$ the gradient ∇u is a matrix and an element of the space $\mathrm{L}^2(\Omega;\mathbb{C}^{d\times d})$.

defn:stokes

Definition 1.7. The *Stokes operator* A_2 on $\mathrm{L}^2_\sigma(\Omega)$ is given via

$$\begin{aligned}\mathrm{D}(A_2) &:= \Big\{u\in\mathrm{H}^1_{0,\sigma}(\Omega)\colon \exists! f\in\mathrm{L}^2_\sigma(\Omega)\text{ s.t. }\forall v\in\mathrm{H}^1_{0,\sigma}(\Omega)\colon \mathfrak{a}(u,v)=\int_\Omega f\cdot\overline{v}\,\mathrm{d}x\Big\}\\ A_2u &:= f,\end{aligned}$$

where $u\in\mathrm{D}(A_2)$ and f comes from the definition of the domain.

The following theorem from Mitrea and Monniaux [**?**, Thm 4.7] shows that our definition of the Stokes operator and the one used in Shen's paper coincide. Another advantage of this characterization is the immediate link of the Stokes operator to the Stokes system.

Theorem 1.2. *If $\Omega\subseteq\mathbb{R}^d$, $d\geq 2$ is a bounded Lipschitz domain and A_2 is the Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$ then*

$$\mathrm{D}(A_2)=\big\{u\in H^1_{0,\sigma}(\Omega)\colon \exists\pi\in\mathrm{L}^2(\Omega)\text{ s.t. } -\Delta u+\nabla\pi\in\mathrm{L}^2_\sigma(\Omega)\big\},$$

where the expression $\Delta u+\nabla\pi\in\mathrm{L}^2_\sigma(\Omega)$ needs to be understood in the distributional sense. For $u\in\mathrm{D}(A_2)$ and the corresponding pressure π we have

$$A_2u=-\Delta u+\nabla\pi.$$

The following proposition summarizes some facts about the Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$.

Proposition 1.3. *Let $\Omega\subset\mathbb{R}^d$ be a bounded Lipschitz domain and A_2 the Stokes operator as in Definition 1.7. Then we have*

a) A_2 is closed with dense domain. Furthermore $0\in\rho(A_2)$.

b) $\sigma(A)\subset[0,\infty)$ and for all $\theta\in(0,\pi]$ there exists $C>0$ such that

eq:resolventEstimateL2

$$\|\lambda(\lambda+A)^{-1}\|_{\mathfrak{L}(\mathrm{L}^2_\sigma)}\leq C,\quad \textit{for all }\lambda\in\mathbb{C}\setminus\Sigma_\theta. \tag{1.6}$$

In particular $-A_2$ generates a bounded analytic semigroup on $\mathrm{L}^2_\sigma(\Omega)$.

With these results at hand we can now give a quick recap of the solution theory to (1.5). Let $f \in \mathrm{L}^2_\sigma$ and $\lambda \in \Sigma_\theta$, $\theta \in (0,\pi/2)$. By the previous proposition we know that there exists a unique $u \in \mathrm{D}(A_2) \subseteq \mathrm{H}^1_{0,\sigma}(\Omega)$ and some $\pi \in \mathrm{L}^2(\Omega)$ such that

$$-\Delta u + \nabla \pi + \lambda u = A_2 u = f.$$

For general $f = \mathrm{L}^2(\Omega)$ we use the *Helmholtz projection* $\mathbb{P}_2$ to get

$$\Delta u + \nabla \pi + \lambda u + (I - \mathbb{P}_2) f = \mathbb{P}_2 f + (I - \mathbb{P}_2) f = f,$$

where u and π now correspond to $\mathbb{P}_2 f \in \mathrm{L}^2_\sigma(\Omega)$. On bounded Lipschitz domains the orthogonal complement to $\mathbb{P}_2[\mathrm{L}^2(\Omega;\mathbb{C}^d)] = \mathrm{L}2_\sigma(\Omega)$ is characterized via

$$\mathrm{L}^2_\sigma(\Omega)^\perp = \left\{ f \in \mathrm{L}^2(\Omega;\mathbb{C}^d) \colon f = \nabla \phi, \text{ for some } \phi \in \mathrm{L}^2(\Omega) \right\}.$$

A proof of this fact can be found in the book of Sohr [**?**, Lemma 2.5.3]. Using this result we find $g \in \mathrm{L}^2(\Omega)$ such that $\nabla g = (I - \mathbb{P}_2) f$ in the sense of distributions and we see that

$$-\Delta u + \nabla(\pi + g) + \lambda u = f.$$

Furthermore we may deduce from the resolvent estimate (1.6) that the solution u which apparently is not affected by the additional part $(I - \mathbb{P}_2) f$ fulfills the inequality

$$|\lambda|^{-1} \|u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} = |\lambda|^{-1} \|(A_2 + \lambda)^{-1} \mathbb{P}_2 f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \leq C \|f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)},$$

where C depends only on θ.

In order to develop an L^p-theory for system (1.5), one way is to study the Stokes operator on subspaces of $\mathrm{L}^p(\Omega;\mathbb{C}^d)$.

Definition 1.8. Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ be a bounded Lipschitz domain and $1 < p < \infty$. If $p > 2$ we define the Stokes operator A_p via its *part of* A_2 *in* $\mathrm{L}^p_\sigma(\Omega)$.

$$\begin{aligned} \mathrm{D}(A_p) &:= \left\{ u \in \mathrm{D}(A_2) \cap \mathrm{L}^p_\sigma(\Omega) \colon A_2 u \in \mathrm{L}^p_\sigma(\Omega) \right\} \\ A_p u &:= A_2 u, \quad u \in \mathrm{D}(A_p). \end{aligned}$$

If $p < 2$ and A_2 is closable in $\mathrm{L}^p_\sigma(\Omega)$, then $A_p := \overline{A_2}$.

Chapter 2

Estimating Fundamental Solutions

The purpose of this section is to study fundamental solutions of the Stokes resolvent problem and to deduce related estimates which will be crucial for the next chapters.

Let $\lambda = r\mathrm{e}^{\mathrm{i}\tau}$ with $0 < r < \infty$ and $-\pi + \theta < \tau < \pi - \theta$ and set $k = \sqrt{r}\mathrm{e}^{\mathrm{i}(\pi+\tau)/2}$. Then

$$k^2 = -\lambda \quad \text{and} \quad \frac{\theta}{2} < \arg(k) < \pi - \frac{\theta}{2}$$

as

$$\arg(k) = \frac{\pi+\tau}{2} > \frac{\pi}{2} + \frac{-\pi+\theta}{2} = \frac{\theta}{2}$$

on the one hand and

$$< \frac{\pi}{2} + \frac{\pi-\theta}{2} = \pi - \frac{\theta}{2}$$

on the other hand. This gives rise to the following estimate

$$\operatorname{Im}(k) > \sin(\theta/2)\sqrt{|\lambda|}. \qquad \texttt{eq:imaginaryPartEstimate} \tag{2.1}$$

Indeed, we have

$$\operatorname{Im}(k) = \sqrt{r}\sin(\frac{\pi+\tau}{2}) = \sqrt{|\lambda|}\sin(\frac{\pi+\tau}{2}) \quad \text{and} \quad \frac{\theta}{2} < \frac{\pi+\tau}{2} < \pi - \frac{\theta}{2}$$

which gives for τ with $\frac{\pi+\tau}{2} \leq \frac{\pi}{2}$ that $\sin(\frac{\pi+\tau}{2}) \geq \sin(\frac{\theta}{2})$ and for τ with $\frac{\pi+\tau}{2} > \frac{\pi}{2}$ that $\sin(\frac{\pi+\tau}{2}) > \sin(\pi - \frac{\theta}{2}) = \sin(\frac{\theta}{2})$.

Before diving into fundamental solutions of the Stokes resolvent problem, we will first consider a fundamental solution for the (scalar) Helmoltz equation in $\mathbb{R}^d$

$$-\Delta u + \lambda u = 0.$$

One fundamental solution with pole at the origin is given by

eq:definitionFundamentalHelmholtz

$$G(x;\lambda) = \frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}} \cdot \frac{1}{|x|^{d-2}} \cdot (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|), \tag{2.2}$$

where $H^{(1)}_\nu(z)$ is the Hankel function of the first kind which can be written as

eq:integralRepresentationHankel

$$H^{(1)}_\nu(z) = \frac{2^{\nu+1}\mathrm{e}^{\mathrm{i}(z-\nu\pi)}z^\nu}{\mathrm{i}\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \int_0^\infty \mathrm{e}^{2z\mathrm{i}s} s^{\nu-\frac{1}{2}}(1+s)^{\nu-\frac{1}{2}}\,\mathrm{d}s. \tag{2.3}$$

This formula holds for $\nu > -\frac{1}{2}$ and $0 < \arg(z) < \pi$. We will usually set

$$\nu = \frac{d}{2} - 1 \quad \text{and} \quad z = k|x|.$$

Therefore, the formula (2.3) will hold for all dimensions $d \geq 2$ and all $x \in \mathbb{R}^d$. In the case $d = 2$ formula (2.2) simplifies to

eq:2dDefinitionFundamentalHelmholtz

$$G(x;\lambda) = \frac{\mathrm{i}}{4} H^{(1)}_0(k|x|), \tag{2.4}$$

in the case $d = 3$ one has an even easier formula, namely

eq:3dDefinitionFundamentalHelmholtz

$$G(x;\lambda) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\pi|x|}. \tag{2.5}$$

Our first estimate is concerned with derivatives of the fundamental solution for the (scalar) Helmholtz equation.

lem:estimateHelmholtzDerivatives

Lemma 2.1. *Let $\lambda \in \Sigma_\theta$. Then*

eq:estimateHelmholtzDerivatives

$$|\nabla^l_x G(x;\lambda)| \leq \frac{C_l \mathrm{e}^{-c\sqrt{|\lambda|}|x|}}{|x|^{d-2+l}} \tag{2.6}$$

for any integer $l \geq 0$ if $d \geq 3$ and for $l \geq 1$ if $d = 2$. Here, $c > 0$ depends only on θ and C_l depends only on d, l and θ.

Proof. We start with the case $l = 0$ and $d \geq 3$. Let $\mathrm{Im}(z) > 0$ and $\nu - \frac{1}{2} \geq 0$. Then (2.3) gives

$$|H^{(1)}_\nu(z)| \leq C\mathrm{e}^{-\mathrm{Im}(z)}|z|^\nu \int_0^\infty \mathrm{e}^{-2s\,\mathrm{Im}(z)} s^{\nu-\frac{1}{2}}(1+s)^{\nu-\frac{1}{2}}\,\mathrm{d}s.$$

Since by the substitution rule

$$\begin{aligned}
\mathrm{e}^{\frac{-\mathrm{Im}(z)}{2}} \int_0^\infty \mathrm{e}^{-s\,\mathrm{Im}(z)} s^{\nu-\frac{1}{2}}(1+s)^{\nu-\frac{1}{2}}\,\mathrm{d}s &= \int_0^\infty \mathrm{e}^{-\mathrm{Im}(z)(s+\frac{1}{2})} s^{\nu-\frac{1}{2}}(1+s)^{\nu-\frac{1}{2}}\,\mathrm{d}s \\
&= \int_{\frac{1}{2}}^\infty \mathrm{e}^{-\mathrm{Im}(z)t}(t^2 - \frac{1}{4})^{\nu-\frac{1}{2}}\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&\le \int_0^\infty e^{-\operatorname{Im}(z)t} t^{2\nu-1}\,\mathrm{d}t \\
&= \int_0^\infty e^{-u} \left(\frac{u}{\operatorname{Im}(z)}\right)^{2\nu-1} (\operatorname{Im}(z))^{-1}\,\mathrm{d}u \\
&= (\operatorname{Im}(z))^{-2\nu} \int_0^\infty \mathrm{e}^{-u} u^{2\nu-1}\,\mathrm{d}u,
\end{aligned}$$

we can estimate

$$|z|^{\nu} |H_\nu^{(1)}(z)| \le C|z|^{2\nu} |\operatorname{Im}(z)|^{-2\nu} \mathrm{e}^{-\frac{\operatorname{Im}(z)}{2}},$$

which for $z = k|x|$ gives

$$|kx|^{\nu} |H_\nu^{(1)}(k|x|)| \le \sin(\theta/2)^{-2\nu} \mathrm{e}^{-\frac{1}{2}\sin(\theta/2)\sqrt{|\lambda|}|x|}, \tag{2.7}$$

eq:zHEstimate

where we used (2.1) to estimate

$$(|kx|)^{2\nu} \cdot |\operatorname{Im}(k|x|)|^{-2\nu} = |\lambda|^{\nu} \cdot |\operatorname{Im}(k)|^{-2\nu} \le \sin(\theta/2)^{-2\nu}.$$

Using (2.2), we estimate for $d \ge 3$ setting $\nu = \frac{d}{2} - 1$

$$|G(x;\lambda)| \le C|x|^{2-d} e^{-c\sqrt{|\lambda|}|x|}.$$

This gives the estimate for $l = 0$ and $d \ge 3$.

Using the relation

$$\frac{\mathrm{d}}{\mathrm{d}z}\{z^{-\nu} H_\nu^{(1)}(z)\} = -z^{-\nu} H_{\nu+1}^{(1)}(z),$$

we can inductively establish the estimate (2.6) for $l \ge 1$ and $d \ge 2$: For $1 \le j \le d$, we calculate

$$\begin{aligned}
|\nabla_x G(x;\lambda)| &\le C \cdot \left\{ |x|^{1-d} \cdot (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|) - |x|^{d-2} (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}}(k|x|) k \right\} \\
&\le C \cdot |x|^{1-d} \left\{ (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|) - (k|x|)^{\frac{d}{2}} H^{(1)}_{\frac{d}{2}}(k|x|) \right\},
\end{aligned}$$

where the first summand does not arise in the case $d = 2$ due to (2.4). The terms in the bracket can now be estimated individually by (2.7). The extension of this proof to orders of differentiation $l \ge 2$ is straightforward using the Leibniz product rule for higher derivatives. □

In the derivation of the next estimates we will use the following useful interior estimate for solutions of Poisson's equation which we write down for further use.

lem:interiorEstimatePoisson

Lemma 2.2. *Let w be a solution to $\Delta w = f$ in $\mathrm{B}(x,r)$. Then*

eq:interiorEstimatePoisson

$$|\nabla^l w(x)| \leq C r^{-l} \sup_{\mathrm{B}(x,r)} |w| + C \max_{0\leq j\leq l-1} \sup_{\mathrm{B}(x,r)} r^{j-l+2} |\nabla^j f|. \tag{2.8}$$

Proof. If $l = 1$, estimate (2.8) is a consequence of the comparison principle and a proof of this fact can be found in Gilbarg and Trudinger [**?**, 3.4]. We can now use this estimate to inductively deduce the estimates for higher derivatives. Note that by a translation and rescaling like

$$u_r(x) := u(rx) \quad \text{and} \quad f_r(x) := r^2 f(rx)$$

we may assume that $\Delta w = f$ in $\mathrm{B}(0,1)$ and that it suffices to prove

eq:interiorEstimatePoissonSimple

$$|\nabla^l w(0)| \leq C \sup_{\mathrm{B}(0,1)} |w| + C \max_{0\leq j\leq l-1} \sup_{\mathrm{B}(0,1)} |\nabla^j f|. \tag{2.9}$$

for $l > 1$. By the Schwartz Theorem we have that if w solves Poisson's equation with right hand side f, then $\nabla^l w$ solves Poisson's equation with right hand side $\nabla^l f$. We can thus estimate

$$\begin{aligned}
|\nabla^l w(0)| &\leq C_l \sup_{\mathrm{B}(0,1/(2^{(l-1)}))} |\nabla^{l-1} w| + C_l \sup_{\mathrm{B}(0,1/(2^{(l-1)}))} |\nabla^{l-1} f| \\
&\leq C_l \sup_{\mathrm{B}(0,1/(2^{(l-2)})} |\nabla^{l-2} w| + C_l \{ \sup_{\mathrm{B}(0,1)} |\nabla^{l-2} f| + \sup_{\mathrm{B}(0,1)} |\nabla^{l-1} f| \} \\
&\leq \dots \\
&\leq C_l \sup_{\mathrm{B}(0,1)} |w| + C_l \sum_{j=0}^{l-1} \sup_{B(0,1)} |\nabla^j f|
\end{aligned}$$

which readily yields the desired estimate. □

We will need the following asymptotic expansions for the function $z^\nu H_\nu^{(1)}(z)$ in $\mathbb{C} \setminus (-\infty, 0]$.

eq:asymptoticd4

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{\mathrm{i}}{\pi} z^2 \log(z) + \omega z^2 + O(|z|^4 |\log(z)) \text{ if } d = 4, \tag{2.10}$$

eq:asymptoticd5

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{2^\nu \Gamma(\nu - 1)}{4\pi \mathrm{i}} z^2 + \omega z^3 + O(|z|^4) \text{ if } d = 5, \tag{2.11}$$

eq:asymptoticd6

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{2^\nu \Gamma(\nu - 1)}{4\pi \mathrm{i}} z^2 + O(|z|^4 |\log z|) \text{ if } d = 6, \tag{2.12}$$

eq:asymptoticd7

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{2^\nu \Gamma(\nu - 1)}{4\pi \mathrm{i}} z^2 + O(|z|^4) \text{ if } d \geq 7. \tag{2.13}$$

The derivation of these asymptotic expansions is based on asymptotic expansions of the Bessel functions of the first and the second kind and can be found in Tolksdorf [**?**].

We will denote the fundamental solution for $-\Delta$ in $\mathbb{R}^d$ with pole at the origin by

$$G(x;0) := \begin{cases} -\frac{1}{2\pi}\log(|x|), & \text{for } d=2, \\ c_d \frac{1}{|x|^{d-2}}, & \text{for } d>2, \end{cases}$$

where

$$c_d = \frac{1}{(d-2)\omega_d}, \quad \text{with} \quad \omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = |\mathbb{S}^{d-1}|.$$

Since

$$(d-2)\omega_d = 2(\frac{d}{2}-1)\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = 2(\frac{d}{2}-1)\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}-1)(\frac{d}{2}-1)} = \frac{4\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}-1)},$$

we will also sometimes use

$$c_d := \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}.$$

The leading coefficient of the asymptotic expansions for $d \geq 3$ will be denoted as

eq:Defnad

$$a_d := \frac{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2}-1)}{\mathrm{i}\pi}. \tag{2.14}$$

The coefficients a_d and c_d are related in the following way

$$c_d = \frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}}\, a_d.$$

This allows us to write for $d \geq 3$

eq:HelmholtzLaplaceDifference

$$G(x;\lambda) - G(x;0) = \frac{i}{4(2\pi)^{\frac{d}{2}-1}} \cdot \frac{1}{|x|^{d-2}} \Big\{ (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|) - a_d \Big\}. \tag{2.15}$$

The following lemma helps us to estimate derivatives of (2.15) and the 2-dimensional counterpart.

lem:HelmholtzLaplaceDifference

Lemma 2.3. *Let $\lambda \in \Sigma_\theta$. Then*

eq:HelmholtzLaplaceDifferenceEstimate

$$|\nabla_x^l \{G(x;\lambda) - G(x;0)\}| \leq C|\lambda||x|^{4-d-l} \tag{2.16}$$

if $d \geq 5$ and $l \geq 0$, where C depends only on d, l and θ. If $d = 3$ or 4, estimate (2.16) *holds for $l \geq 1$ and if $d = 2$, the estimate holds for $l \geq 3$.*

Proof. (a) In this part we will show that the desired estimates (2.16) and (2.16) hold if we assume that $|\lambda||x|^2 > \frac{1}{2}$. In this case, Lemma 2.1 gives

$$|\nabla_x^l\{G(x;\lambda)-G(x;0)\}| \le C\Big\{\frac{e^{-c\sqrt{|\lambda|}|x|}}{|x|^{d-2+l}}+\frac{1}{|x|^{d-2+l}}\Big\} \le C\frac{|\lambda|}{|x|^{d-4+l}},$$

where C depends only on d, l and θ. Therefore, for the remaining proof we will suppose $|\lambda||x|^2 \le \frac{1}{2}$.

(b) In this step, we show that we can restrict ourselves to proving (2.16) in three cases: (1) $d \ge 5$ and $l = 0$; (2) $d = 3$ or 4 and $l = 1$; (3) $d = 2$ and $l = 3$.

Suppose (2.16) holds in case (1) and let $l > 1$. If we set $w(x) = G(x;\lambda) - G(x;0)$, we have $\Delta_x w = \lambda G(x;\lambda)$ in $\mathbb{R}^d \setminus \{0\}$. For $f = \lambda G(x;\lambda)$ estimate (2.8) now gives

$$\begin{aligned}|\nabla^l w(x)| &\le Cr^{-l}\sup_{B(x,r)}|w| + C\max_{0\le j\le l-1}\sup_{B(x,r)} r^{j-l+2}|\nabla^j f|\\ &\le Cr^{-l}\sup_{y\in B(x,r)}|\lambda||y|^{4-d} + C\sum_{j=0}^{l-1}\sup_{y\in B(x,r)} r^{j-l+2}|\lambda||y|^{2-d-j}\\ &= Cr^{-l}|\lambda||x-r\frac{x}{|x|}|^{4-d} + C\sum_{j=0}^{l-1} r^{j-l+2}|\lambda||x-r\frac{x}{|x|}|^{2-d-j},\end{aligned}$$

for all $0 < r < |x|$, where we used (2.16) with $l = 1$ for the first summand and (2.6) to estimate the second summand. Setting $r = \frac{|x|}{2}$ now gives

$$\begin{aligned}|\nabla^l w(x)| &\le C|\lambda|x|^{-l}|x|^{4-d} + C\sum_{j=0}^{l-1}|x|^{j-l+2}|\lambda||x|^{2-d-j}\\ &\le C|\lambda||x|^{4-d-l}.\end{aligned}$$

The proof for case (2) is completely analogous if one sets $w(x) = \nabla_x(G(x;\lambda) - G(x;0))$ and $f(x) = \lambda\nabla_x G(x;\lambda)$. Also case (3) is proven in a similar fashion. We will give the proof for the sake of completeness.

For w and f as in case (2) by (2.16) we get

$$\begin{aligned}|\nabla^l w(x)| &\le Cr^{-l}\sup_{B(x,r)}|w| + C\max_{0\le j\le l-1}\sup_{B(x,r)} r^{j-l+2}|\nabla^j f|\\ &\le Cr^{-l}\sup_{y\in B(x,r)}|\lambda||y|(|\log|\lambda||y|^2|+1) + C\sum_{j=0}^{l-1}\sup_{y\in B(x,r)} r^{j-l+2}|\lambda||y|^{-j-1}\\ &\le S_1 + S_2,\end{aligned}$$

wheras

$$\begin{aligned} S_1 &\leq Cr^{-l}|\lambda||x+r\frac{x}{|x|}|(|\log|\lambda||x-r\frac{x}{|x|}|^2|+|\log(|\lambda||x+r\frac{x}{|x|}|^2)|+1) \\ &\leq C|\lambda||x|^{1-l}(|\log(|\lambda||x|^2)|+1) \end{aligned}$$

if we choose $r=\frac{|x|}{2}$. For S_2 we calculate as before, using estimate (2.6)

$$\begin{aligned} S_2 &\leq C\sum_{j=0}^{l-1} C|x|^{j-l+2}|\lambda||x|^{-j-1} \\ &\leq C|\lambda||x|^{1-l}. \end{aligned}$$

(c) In this step we prove (2.16) for $d \geq 5$ and $l=0$. First, note that for the functions

$$\begin{aligned} g(x) &:= (k|x|)^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(k|x|), \quad g(0)=a_d, \\ h(z) &:= z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z), \quad h(0)=a_d, \end{aligned}$$

the mean value theorem yields the estimate

$$|g(x)-g(0)| \leq |x| \sup_{\mathrm{B}(0,|x|)} |\nabla f(y)| \leq |x||k| \sup_{\mathrm{B}(0,|x|)} \frac{\mathrm{d}}{\mathrm{d}z}|h(k|x|)|.$$

Due to (2.15) we estimate

eq:HelmholtzLaplaceDifferenceGt5

$$\begin{aligned} |G(x;\lambda)-G(x;0)| &\leq C|x|^{2-d}\cdot|k||x| \max_{\substack{|z|\leq|k||x| \\ \mathrm{Im}(z)>0}} \Big|\frac{\mathrm{d}}{\mathrm{d}z}\big\{z^{\frac{d}{2}-1}H_{\frac{d}{2}-1}(z)\big\}\Big| \\ &= C|x|^{2-d}\cdot|k||x| \max_{\substack{|z|\leq|k||x| \\ \mathrm{Im}(z)>0}} |z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-2}(z)|, \end{aligned} \tag{2.17}$$

where for the last equality we used the relation

$$\frac{\mathrm{d}}{\mathrm{d}z}\big\{z^{\nu}H^{(1)}_{\nu}(z)\big\} = z^{\nu}H^{(1)}_{\nu-1}(z). \tag{2.18}$$

Since the asymptotic expansions yield that $|z^{\nu}H^{(1)}_{\nu}(z)| \leq C_{\nu}$ for $\nu>0$ and $|z|\leq 1$ with $\mathrm{Im}(z)>0$ it follows from (2.17) that

$$|G(x;\lambda)-G(x;0)| \leq C|x|^{2-d}\cdot|k||x|\cdot|k||x| \max_{\substack{|z|\leq|k||x| \\ \mathrm{Im}(z)>0}} |z^{\frac{d}{2}-2}H^{(1)}_{\frac{d}{2}-2}(z)| \leq C|\lambda||x|^{4-d}$$

(d) Now we consider the case $d=4$ and $l=1$. The asymptotic expansion (2.10) gives that

eq:mwt4d

$$\Big|\frac{\mathrm{d}}{\mathrm{d}z}\Big\{\frac{zH^{(1)}_1(z)-a_4}{z^2}\Big\}\Big| \leq C|z|^{-1} \tag{2.19}$$

for all $|z| \leq \frac{1}{2}$ with $\mathrm{Im}(z) > 0$. Since

$$\frac{G(x;\lambda) - G(x;0)}{\lambda} = -\frac{C(zH_1^{(1)}(z) - a_4)}{z^2},$$

where $z = k|x|$, it follows from (2.19) and the mean value theorem that

$$\Big|\frac{\nabla_x\{G(x;\lambda) - G(x;0)\}}{\lambda}\Big| \leq C|k|\frac{\mathrm{d}}{\mathrm{d}z}\Big(\frac{zH_1^{(1)}(z) - a_4}{z^2}\Big)\Big|_{z=k|x|} \leq C|k||k|^{-1}|x|^{-1}.$$

Which after rearrangement of the involved terms gives the claim.

(e) For the case $d = 3$ and $l = 1$, equation (2.15) reads

$$G(x;\lambda) - G(x;0) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\pi|x|} - \frac{c_3}{|x|} = \frac{\mathrm{e}^{\mathrm{i}k|x|} - 1}{4\pi|x|}.$$

Now we calculate

$$\begin{aligned}\frac{\partial}{\partial x_j}\Big\{\frac{\mathrm{e}^{\mathrm{i}k|x|} - 1}{|x|}\Big\} &= \frac{\partial}{\partial x_j}\Big\{\frac{\mathrm{e}^{\mathrm{i}k|x|} - 1 - \mathrm{i}k|x|}{|x|}\Big\} = \frac{\partial}{\partial x_j}\Big\{\sum_{n=2}^{\infty}\frac{(\mathrm{i}k|x|)^n}{n!}\cdot\frac{1}{|x|}\Big\}\\ &= \sum_{n=2}^{\infty}\frac{(\mathrm{i}k)^n}{n!}(n-1)\cdot\frac{x_j}{|x|}|x|^{n-2}\end{aligned}$$

which in turn implies

$$\Big|\frac{\partial}{\partial x_j}\Big\{\frac{\mathrm{e}^{\mathrm{i}k|x|} - 1}{|x|}\Big\}\Big| \leq |\lambda|\sum_{n=2}^{\infty}\frac{n-1}{n!}|k|^{n-2}|x|^{n-2} \leq C|\lambda|$$

since $|\lambda||x| < \frac{1}{2}$.

(f) For the last case $d = 2$ and $l = 3$, we will directly calculate the estimate using the asymptotic expansion of $H_0^{(1)}(z)$ with $z = k|x|$:

$$\begin{aligned}H_0^{(1)}(z) &= J_0(z) + \mathrm{i}Y_0(z)\\ &= \sum_{l=0}^{\infty}\frac{(-1)^l}{(l!)^2 4^l}z^{2l}\Big(1 - \frac{2\mathrm{i}\log(2)}{\pi}\Big) - \frac{2\mathrm{i}}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{(l!)^2 4^l}\psi(l+1)\cdot z^{2l}\\ &\quad + \frac{2\mathrm{i}}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{(l!)^2 4^l}z^{2l}\log(z)\end{aligned}$$

The first complex derivative of $H_0^{(1)}(z)$ reads

$$\frac{\mathrm{d}}{\mathrm{d}z}H_0^{(1)}(z) =$$

$\square$

Remark 2.4. In the situation of Lemma 2.3 one can show by considering the asymptotic expansions and if $|\lambda||x|^2 \leq (1/2)$ that

$$|G(x;\lambda) - G(x;0)| \leq \begin{cases} C\sqrt{|\lambda|} & \text{if } d = 3, \\ C|\lambda|\{|\log(|\lambda||x|^2)| + 1\} & \text{if } d = 4. \end{cases}$$

Also using the asymptotic expansions it can be shown that if $d = 2$

$$|\nabla_x^l \{G(x;\lambda) - G(x;0)\}| \leq C|\lambda||x|^{2-l}\{|\log(|\lambda||x|^2)| + 1\}$$

for $l \in \{1, 2\}$.

The fundamental solution to the (scalar) Helmholtz equation and the Laplace equation form the main ingredient for the following matrix of fundamental solutions $(\Gamma(x;\lambda) = (\Gamma_{\alpha\beta}(x;\lambda))_{d\times d}$ with pole at the origin to the Stokes resolvent problem with $\lambda \in \Sigma_\theta$:

eq:fundamentalMatrixStokes

$$\Gamma_{\alpha\beta}(x;\lambda) = G(x;\lambda)\delta_{\alpha\beta} - \frac{1}{\lambda}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\{G(x;\lambda) - G(x;0)\}, \quad \alpha, \beta = 1, \ldots, d. \tag{2.20}$$

Having the formula at sight, the following observations are obvious:

$$\Gamma_{\alpha\beta}(x;\lambda) = \Gamma_{\beta\alpha}(x;\lambda), \quad \overline{\Gamma_{\alpha\beta}(x;\lambda)} = \Gamma_{\alpha\beta}(x;\bar{\lambda}) \quad \text{and} \quad \Gamma_{\alpha\beta}(x;\lambda) = \Gamma_{\alpha\beta}(-x;\lambda).$$

For the pressure, we define the vector of fundamental solutions

eq:fundamentalVectorPressure

$$\Phi_\beta(x) = -\frac{\partial}{\partial x_\beta}\{G(x;0)\} = \frac{x_\beta}{\omega_d|x|^d}. \tag{2.21}$$

We note that $\Phi_\beta(x) = \Phi_\beta(-x)$.

Using the fact that $\Delta_x G(x;\lambda) = \lambda G(x;\lambda)$ in $\mathbb{R}^d \setminus \{0\}$, one can see that on $\mathbb{R}^d \setminus \{0\}$ and for all $1 \leq \beta \leq d$

eq:solutionStokesSystem

$$\begin{cases} (-\Delta_x + \lambda)\Gamma_{\alpha\beta}(x;\lambda) + \frac{\partial}{\partial x_\alpha}\{\Phi_\beta(x)\} & = 0 \quad \text{for } 1 \leq \alpha \leq d, \\ \frac{\partial}{\partial x_\alpha}\{\Gamma_{\alpha\beta}(x;\lambda)\} & = 0. \end{cases} \tag{2.22}$$

Note that in the last equation the summation convention was used.

We now keep up to the spirit of this exhausting section by proving further estimates, this time for the fundamental solutions to the Stokes resolvent problem.

thm:fundamentalMatrixEstimate

Theorem 2.5. *Let $\lambda \in \Sigma_\theta$. Then for any $d \geq 3$ and $l \geq 0$*

eq:fundamentalMatrixEstimate

$$|\nabla_x^l \Gamma(x;\lambda)| \leq \frac{C}{(1 + |\lambda||x|^2)|x|^{d-2+l}} \tag{2.23}$$

where C depends only on d, l and θ. For $d = 2$ and $l \geq 1$ the same estimate holds.

Proof. Let $|\lambda||x|^2 > \frac{1}{2}$. Then there exist constants C_a, C_b, C_c such that

$$\begin{aligned}
\mathrm{e}^{-c\sqrt{|\lambda|}|x|}(1+|\lambda||x|^2) &\leq C_a \\
1 &\leq \frac{C_b|\lambda||x|^2}{1+|\lambda||x|^2} \\
\mathrm{e}^{-c\sqrt{|\lambda|}|x|} &\leq \frac{C_c|\lambda||x|^2}{1+|\lambda||x|^2},
\end{aligned}$$

where c is the constant from Lemma 2.1. Using these estimates and Lemma 2.1 gives

$$\begin{aligned}
|\nabla_x^l \Gamma(x;\lambda)| &\leq |\nabla_x^l G(x;\lambda)| + \frac{1}{|\lambda|}|\nabla_x^{l+2} G(x;\lambda)| + \frac{1}{|\lambda|}|\nabla_x^{l+2} G(x;0)| \\
&\leq \frac{C_l \mathrm{e}^{-c\sqrt{|\lambda|}|x|}}{|x|^{d-2+l}} + \frac{1}{|\lambda|}\frac{C_{l+2}\mathrm{e}^{-\sqrt{|\lambda|}|x|}}{|x|^2|x|^{d-2+l}} + \frac{1}{|\lambda|}\frac{C}{|x|^2|x|^{d-2+l}} \\
&\leq \frac{C}{1+|\lambda||x|^2}\,\frac{1}{|x|^{d-2+l}}.
\end{aligned}$$

Now let $|\lambda||x|^2 \leq \frac{1}{2}$. Then by 2.1 and 2.3 we get

$$\begin{aligned}
|\nabla_x^l \Gamma(x;\lambda)| &\leq |\nabla_x^l G(x;\lambda)| + \frac{1}{|\lambda|}|\nabla_x^{l+2}(G(x;\lambda) - G(x;0))| \\
&\leq \frac{C}{|x|^{d-2+l}} + \frac{1}{|\lambda|}\cdot C|\lambda||x|^{4-d-(l+2)} \\
&\leq \frac{C}{|x|^{d-2+l}}\frac{(1+|\lambda||x|^2)}{(1+|\lambda||x|^2)} \\
&\leq \frac{C}{(1+|\lambda||x|^2)|x|^{d-2+l}}
\end{aligned}$$

which gives the claim. □

If $\lambda = 0$, the matrix of fundamental solutions to the Stokes problem in $\mathbb{R}^d$ with pole at the origin is given by $\Gamma(x;0) = (\Gamma_{\alpha\beta}(x;0))_{d\times d}$, where

eq:fundamentalSolutionStokes

$$\Gamma_{\alpha\beta}(x;0) = \frac{1}{2\omega_d}\Big\{\frac{\delta_{\alpha\beta}}{(d-2)|x|^{d-2}} + \frac{x_\alpha x_\beta}{|x|^d}\Big\} \tag{2.24}$$

if $d \geq 3$ and

eq:fundamentalSolutionStokes2d

$$\Gamma_{\alpha\beta}(x;0) = \frac{1}{2\omega_2}\Big\{-\delta_{\alpha\beta}\log(|x|) + \frac{x_\alpha x_\beta}{|x|^2}\Big\} \tag{2.25}$$

for $d = 2$. Note that the given fundamental solution for the case $d = 2$ differs from the one given by Mitrea and Wright [**?**] by having summands with alternating signs. The alternatig sign is necessary for $\Gamma_{\alpha\beta}$ to be divergence free.

One important technique in the following chapter will be to reduce problems formulated for $\Gamma(x;\lambda)$ to problems formulated in $\Gamma(x;0)$ and the difference $\Gamma(x;\lambda) - \Gamma(x;0)$.

Under this aspect it seems reasonable to study estimates of the difference of fundamental solutions. To this end it is helpful to rewrite parts of the fundamental solution. Using the fact that for $d \geq 5$ or $d = 3$ we have

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\Big(\frac{1}{|x|^{d-4}}\Big) = -(d-4)\frac{\partial}{\partial x_\alpha}\frac{x_\beta}{|x|^{d-2}} = -(d-4)\frac{\delta_{\alpha\beta}}{|x|^{d-2}} + \frac{(d-4)(d-2)x_\alpha x_\beta}{|x|^d}$$

This allows us to express

$$\frac{x_\alpha x_\beta}{|x|^d} = \frac{\delta_{\alpha\beta}}{(d-2)|x|^{d-2}} + \frac{1}{(d-4)(d-2)}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\Big(\frac{1}{|x|^{d-4}}\Big)$$

which considering (2.24) gives

eq:fundamentalSolutionStokes35

$$\Gamma_{\alpha\beta}(x;0) = G(x;0)\delta_{\alpha\beta} + \frac{1}{2\omega_d(d-4)(d-2)}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\Big(\frac{1}{|x|^{d-4}}\Big). \tag{2.26}$$

A similar trick works for $d = 4$. Since $\omega_4 = 2\pi^2$, we have

eq:fundamentalSolutionStokes4

$$\begin{aligned}\Gamma_{\alpha\beta}(x;0) &= \frac{1}{2\omega_4}\frac{1}{|x|^2}\delta_{\alpha\beta} - \frac{1}{8\pi^2}\Big(\frac{\delta_{\alpha\beta}}{|x|^2} - \frac{2x_\alpha x_\beta}{|x|^4}\Big)\\ &= G(x;0)\delta_{\alpha\beta} - \frac{1}{8\pi^2}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}(\log(|x|))\end{aligned} \tag{2.27}$$

In the case $d = 2$ this game shows that since

$$\frac{1}{8\pi}\frac{\partial^2}{\partial_\alpha \partial_\beta}(|x|^2\log(|x|)) = \frac{\delta_{\alpha\beta}}{4\pi}\log(|x|) + \frac{1}{4\pi}\frac{x_\alpha x_\beta}{|x|^2} + \frac{\delta_{\alpha\beta}}{8\pi},$$

we can write

eq:fundamentalSolutionStokes2

$$\Gamma_{\alpha\beta}(x;0) = G(x;0)\delta_{\alpha\beta} - \frac{\delta_{\alpha\beta}}{8\pi} - \frac{\partial^2}{\partial x_\alpha \partial x_\beta}(|x|^2\log(|x|)) \tag{2.28}$$

This ends the preparatory step and brings us to the next theorem.

thm:differenceFundamentalSolutionStokes

Theorem 2.6. *Let $\lambda \in \Sigma_\theta$. Suppose that $|\lambda||x|^2 \leq \frac{1}{2}$. Then*

$$|\nabla_x\{\Gamma(x;\lambda) - \Gamma(x;0)\}| \leq \begin{cases} C|\lambda||x|^{3-d} & \text{if } d \geq 7 \text{ or } d = 5,\\ C|\lambda||x|^{3-d}|\log(|\lambda||x|^2)| & \text{if } d = 4 \text{ or } 6,\\ C\sqrt{|\lambda|}|x|^{-1} & \text{if } d = 3,\\ C|\lambda||x|(|\log(|\lambda||x|^2)| + 1) & \text{if } d = 2,\end{cases} \tag{2.29}$$

where C depends only on d and θ.

Proof. We will split the proof in several parts. We start by considering the cases $d = 3$ and $d \geq 5$. Taking into account (2.26) we have

$$\begin{aligned}
&\Gamma_{\alpha\beta}(x;\lambda) - \Gamma_{\alpha\beta}(x;0) \\
&\quad = \{G(x;\lambda) - G(x;0)\}\delta_{\alpha\beta} \\
&\qquad - \frac{1}{\lambda}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\Big\{G(x;\lambda) - G(x;0) + \frac{\lambda}{2\omega_d(d-4)(d-2)|x|^{d-4}}\Big\}
\end{aligned}$$

As the first term can already be estimated via Lemma 2.3, we will only be concerned about the second one. If $d = 3$, a direct calculation will yield the desired result: We start by noting that $\omega_3 = 4\pi$ gives

$$\begin{aligned}
\frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\pi|x|} - \frac{1}{4\pi|x|} - \frac{(\mathrm{i}k)^2}{2\omega_3|x|^{-1}} &= \frac{1}{4\pi|x|}\Big(\mathrm{e}^{\mathrm{i}k|x|} - 1 - \frac{(\mathrm{i}k)^2|x|^2}{2}\Big) \\
&= \frac{1}{4\pi|x|}\Big(\mathrm{i}k|x| + \sum_{n=3}^{\infty}\frac{(\mathrm{i}k|x|)^n}{n!}\Big) \\
&= \frac{1}{4\pi}\Big(\mathrm{i}k + \sum_{n=3}^{\infty}\frac{(\mathrm{i}k)^n|x|^{n-1}}{n!}\Big).
\end{aligned}$$

Taking the first derivative of this expression we get

$$\frac{\partial}{\partial x_\beta}\cdots = \frac{x_\beta}{4\pi}\sum_{n-3}^{\infty}\frac{(ik)^n(n-1)}{n!}|x|^{n-3}$$

and differentiating with respect to x_α yields

$$\frac{\partial}{\partial x_\alpha}\cdots = \frac{\delta_{\alpha\beta}}{4\pi}\sum_{n-3}^{\infty}\frac{(ik)^n(n-1)}{n!}|x|^{n-3} + \frac{x_\beta x_\alpha}{4\pi}\sum_{n=4}^{\infty}\frac{(\mathrm{i}k)^n(n-1)(n-3)}{n!}|x|^{n-5}.$$

As we are interested in estimating the gradient of the difference of $\Gamma(x;\lambda)$ and $\Gamma(x;0)$ we have to consider one additional derivative. This leaves us with

$$\begin{aligned}
\frac{\partial}{\partial x_\gamma}\cdots = {}& \frac{\delta_{\alpha\beta}x_\gamma + \delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta}{4\pi}\sum_{k=4}^{\infty}\frac{(\mathrm{i}k)^n(n-1)(n-3)}{n!}|x|^{n-5} \\
&+ \frac{x_\beta x_\alpha x_\gamma}{4\pi}\sum_{n=4}^{\infty}\frac{(\mathrm{i}k)^n(n-1)(n-3)(n-5)}{n!}|x|^{n-7}.
\end{aligned}$$

We can now prove the stated estimate

$$\begin{aligned}
|\frac{1}{\lambda}\frac{\partial^3}{\partial x_\gamma \partial x_\alpha \partial x_\beta}\cdots| &\leq \frac{1}{|k|^2\pi}\sum_{k=4}^{\infty}\frac{|k|^n(n-1)(n-3)(1+(n-5))}{n!}|x|^{n-4} \\
&\leq \frac{1}{|k|^2|x|\pi}|k|^3\sum_{k=4}^{\infty}\frac{(n-1)(n-3)(1+(n-5))}{n!}|k|^{n-3}|x|^{n-3}
\end{aligned}$$

$$\leq C\frac{1}{|k||x|}.$$

This gives the claim for $d=3$. If $d\geq 5$, equation (2.15) gives

$$\begin{aligned}G(x;\lambda)-G(x;0)&+\frac{\lambda}{2\omega_2(d-4)(d-2)|x|^{d-4}}\\&=\frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}}\frac{1}{|x|^{d-2}}\{z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z)-a_d-b_dz^2\},\end{aligned}$$

where $z=k|x|$, a_d was calculated in (2.14) and b_d is given by

$$\begin{aligned}b_d&=\frac{2\mathrm{i}(2\pi)^{\frac{d}{2}-1}}{\omega_d(d-4)(d-2)}=-\frac{2\mathrm{i}(2\pi)^{\frac{d}{2}-1}\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}(d-2)(d-4)}=\frac{2^{\frac{d}{2}-1}}{\pi\mathrm{i}(d-4)}\frac{\Gamma(\frac{d}{2})}{(d-2)}\\&=\frac{2^{\frac{d}{2}-1}}{2\pi\mathrm{i}}\frac{\Gamma(\frac{d}{2}-1)}{(d-4)}=\frac{2^{\frac{d}{2}-1}}{4\pi\mathrm{i}}\frac{\Gamma(\frac{d}{2}-1)}{(\frac{d}{2}-1-1)}=\frac{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2}-2)}{4\pi\mathrm{i}}.\end{aligned}$$

If $d\geq 7$ this shows that b_d is the second coefficient of the asymptotic expansion (2.13) and thus we can estimate

eq:estimateDerivativesd7

$$\Big|\frac{\mathrm{d}^l}{\mathrm{d}z^l}\Big\{z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z)-a_d-b_dz^2\Big\}\Big|\leq C|z|^{4-l} \tag{2.30}$$

for $0\leq l\leq 3$ and $z\in\mathbb{C}\setminus(-\infty,0]$. For better readability we set

$$g(z)=z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z)-a_d-b_dz^2$$

and consider the function $f(x)=g(k|x|)$ on $\mathbb{R}^d\setminus\{0\}$. The derivatives of f read

$$\begin{aligned}\frac{\partial}{\partial x_\beta}f(x)&=(\frac{\mathrm{d}}{\mathrm{d}z}g)(k|x|)\frac{kx_\beta}{|x|}\\ \frac{\partial^2}{\partial x_\alpha\partial x_\beta}f(x)&=(\frac{\mathrm{d}^2}{\mathrm{d}z^2}g)(k|x|)\frac{k^2x_\alpha x_\beta}{|x|^2}+(\frac{\mathrm{d}}{\mathrm{d}z})g(k|x|)k\Big\{\frac{\delta_{\alpha\beta}}{|x|}-\frac{x_\beta x_\alpha}{|x|^3}\Big\}\\ \frac{\partial^3}{\partial x_\gamma\partial x_\alpha\partial x_\beta}f(x)&=(\frac{\mathrm{d}^3}{\mathrm{d}z^3}g)(k|x|)\frac{k^3x_\alpha x_\beta x_\gamma}{|x|^3}\\&\quad+(\frac{\mathrm{d}^2}{\mathrm{d}z^2}g)(k|x|)k^2\Big\{\frac{x_\alpha\delta_{\beta\gamma}+x_\beta\delta_{\alpha\gamma}+x_\gamma\delta_{\alpha\beta}}{|x|^2}-\frac{3x_\alpha x_\beta x_\gamma}{|x|^4}\Big\}\\&\quad+(\frac{\mathrm{d}}{\mathrm{d}z}g)(k|x|)k\Big\{-\frac{\delta_{\alpha\beta}x_\gamma}{|x|^3}-\frac{x_\alpha\delta_{\beta\gamma}+x_\beta\delta_{\alpha\gamma}}{|x|^3}+\frac{3x_\alpha x_\beta x_\gamma}{|x|^5}\Big\}.\end{aligned}$$

If we now look for estimates on the absolute value of the derivatives, we see that by (2.30)

$$|\nabla^l_xf(x)|\leq C|k|^4|x|^{4-l},\quad 1\leq l\leq 3,$$

where C only depends on l. We can now finally uncover the desired estimate via

$$\begin{aligned}
&\Big|\frac{1}{\lambda}\nabla_x^3\Big\{\frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}}\frac{1}{|x|^{d-2}}\{z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z)-a_d-b_dz^2\}\Big\}\Big| \\
&\quad\le C\frac{1}{|k|^2}\sum_{l=0}^{3}|\nabla_x^{3-l}(\frac{1}{|x|^{d-2}})||\nabla_x^l f(x)|\le C\sum_{l=0}^{3}|x|^{-d+2-3+l}|k|^2|x|^{4-l}=C|\lambda||x|^{3-d},
\end{aligned}$$

where C is a constant only depending on d. If $d=6$, this shows that the asymptotic expansion (2.12) gives us similar to (2.30)

eq:estimateDerivativesd6

$$|\frac{\mathrm{d}^l}{\mathrm{d}z^l}\Big\{z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z)-a_d-b_dz^2\Big\}|\le C|z|^{4-l}|\log(z)|, \tag{2.31}$$

for $0\le l\le 3$ and $z\in\mathbb{C}\setminus(-\infty,0]$. Using as before the expressions for derivatives of f, we can estimate

$$|\nabla_x^l f(x)|\le C|k|^4|x|^{4-l}|\log(|\lambda||x|^2)|,$$

which by a calculation analogous to the case $d\ge 7$ yields

$$|\nabla_x\{\Gamma(x;\lambda)-\Gamma(x;0)\}|\le C|\lambda||x|^{3-d}|\log(|\lambda||x|^2)|.$$

For $d=5$ write

$$\begin{aligned}
&\frac{\partial^2}{\partial x_\alpha\partial x_\beta}\Big\{G(x;\lambda)-G(x;0)+\frac{\lambda}{2\omega_d(d-4)(d-2)|x|d-4}\Big\} \\
&\quad=\frac{\partial^2}{\partial x_\alpha x_\beta}\Big\{\frac{\mathrm{i}}{4(2\pi)^{\frac{3}{2}}}\cdot\frac{1}{|x|^3}\big[z^{\frac{3}{2}}H^{(1)}_{\frac{3}{2}}(z)-a_5-b_5z^2-wz^3\big]\Big\},
\end{aligned}$$

where $w\in\mathbb{C}$ can be an any constant if we set $z=k|x|$. Now, for the appropriate choice of $w\in\mathbb{C}$ the asymptotic expansion (2.11) gives the same estimate as (2.30) which proves the claim for $d=5$.

In the case $d=4$ we use (2.27) to reformulate

$$\begin{aligned}
&\Gamma_{\alpha\beta}(x;\lambda)-\Gamma_{\alpha\beta}(x;0) \\
&\quad=\{G(x;\lambda)-G(x;0)\}\delta_{\alpha\beta}-\frac{1}{\lambda}\frac{\partial^2}{\partial x_\alpha\partial x_\beta}\Big\{G(x;\lambda)-G(x;0)-\frac{\lambda\log(|x|)}{8\pi^2}\Big\} \\
&\quad=\{G(x;\lambda)-G(x;0)\}\delta_{\alpha\beta} \\
&\qquad\frac{\mathrm{i}}{\lambda}\frac{\partial^2}{\partial x_\alpha\partial x_\beta}\Big\{\frac{1}{8\pi|x|^2}[zH^{(1)}_1(z)-a_4-wz^2-b_4z^2\log(z)]\Big\},
\end{aligned}$$

where $z=k|x|$, $b_4=\frac{\mathrm{i}}{\pi}$ and $w\in\mathbb{C}$ is an arbitrary constant. Using the asymptotic expansion (2.10) and the appropriate constant $w\in\mathbb{C}$ we get the estimate

$$|\frac{\mathrm{d}^l}{\mathrm{d}z^l}\Big\{zH^{(1)}_1(z)-a_4-wz^2-b_4z^2\log(z)\Big\}|\le C|z|^{4-l}|\log(z)|.$$

For $d=2$ the estimate follows from a direct calculation. $\square$

We can now use the assumption $|\lambda||x|^2 \leq \frac{1}{2}$ to unify the structure of the estimates for $d \geq 2$.

`cor:differenceFundamentalSolutionStokes`

Corollary 2.7. *Let $\lambda \in \Sigma_\theta$. Suppose that $|\lambda||x|^2 \leq \frac{1}{2}$. Then for all $d \geq 2$*

$$|\nabla_x\{\Gamma(x;\lambda) - \Gamma(x;0)\}| \leq C\sqrt{|\lambda|}|x|^{2-d},$$

where C depends only on d and θ.

Proof. We just extend the estimates given in Theorem 2.6. Let $d \geq 7$ or $d = 5$. Since $|\lambda|^{\frac{1}{2}} \leq C|x|$ we have

$$C|\lambda||x|^{3-d} \leq C|\lambda|^{\frac{1}{2}}|x|^{2-d}.$$

For $d = 4, 6$ we have

$$C|\lambda||x|^{3-d}|\log(|\lambda||x|^2)| = C|\lambda|^{\frac{1}{2}}|x|^{2-d} \cdot |\lambda|^{\frac{1}{2}}|x||\log(|\lambda||x|^2)| \leq C|\lambda|^{\frac{1}{2}}|x|^{2-d},$$

since $|\lambda|^{\frac{1}{2}}|x||\log(|\lambda||x|^2)|$ is bounded for $|\lambda||x|^2 \leq \frac{1}{2}$. For $d = 2$ the same argument applies to the expression $|\lambda|^{\frac{1}{2}}|x|(|\log(|\lambda||x|^2)| + 1)$. $\square$

Chapter 3

Single and Double Layer Potentials

In this chapter, we will deal with *single* and *double layer potentials*. Both will serve as "representation formulas" for solutions to the Stokes resolvent problem. We will study their properties as they will serve as the crucial ingredient to solving the Neumann and Dirichlet boundary problems associated to the Stokes resolvent problem. In this chapter we will always assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$ with $d \geq 2$ and $1 < p < \infty$. We will also tacitly use the summation convention.

Let $\lambda \in \Sigma_\theta$. For $f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$, the single layer potential $u = \mathscr{S}_\lambda(f)$ is defined by

eq:defSingleLayer

$$u_j(x) = \int_{\partial\Omega} \Gamma_{jk}(x-y;\lambda) f_k(y) \,\mathrm{d}\sigma(y), \tag{3.1}$$

where Γ_{jk} is the fundamental solution to the Stokes reolvent problem given by (2.20). For the pressure, respectively, we define the single layer potential $\phi = \mathscr{S}_\Phi(f)$ by

eq:defSingleLayerPressure

$$\phi(x) = \int_{\partial\Omega} \Phi_k(x-y) f_k(y) \,\mathrm{d}\sigma(y), \tag{3.2}$$

where Φ_k is given by (2.21). As we have already shown, (u,ϕ) defines a solution to the Stokes resolvent problem (**??**).

We define two further integral operators

eq:supTOperator

$$T^*_\lambda(f)(P) = \sup_{t>0} \Big| \int_{\substack{y\in\partial\Omega \\ |y-P|>t}} \nabla_x \Gamma(P-y;\lambda) f(y) \,\mathrm{d}\sigma(y) \Big| \tag{3.3}$$

eq:pvTOperator

$$T_\lambda(f)(P) = \mathrm{p.\,v.} \int_{\partial\Omega} \nabla_x \Gamma(P-y;\lambda) f(y) \,\mathrm{d}\sigma(y) \tag{3.4}$$

for $P \in \partial\Omega$ which will be used to prove boundedness of maximal operators related to u.

lem:lpBoundednessT

Lemma 3.1. *Let $1 < p < \infty$ and $T_\lambda(f), T^*_\lambda(f)$ be defined by* (3.3) *and* (3.4). *Then $T_\lambda(f)(P)$ exists for almost everywhere $P \in \partial\Omega$ and*

eq:lpBoundednessT

$$\|T_\lambda(f)\|_{\mathrm{L}^p(\partial\Omega)} \leq \|T^*_\lambda(f)\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p \|f\|_{\mathrm{L}^p(\partial\Omega)}, \tag{3.5}$$

where C_p depends only on d, θ, p, and the Lipschitz character of Ω.

Proof. If $\lambda = 0$, the Lemma is known [**?**] as a consequence of the seminal result of Coifman et al. [**?**]. One idea of the proof in the case $\lambda \in \Sigma_\theta$ will thus be to nourish from this result and to consider the difference $\Gamma(x-y;\lambda) - \Gamma(x-y;0)$ as a well-disposed integral kernel.

We start with the second inequality of 3.5. To this end, let $t > 0$ and additionally assume that $t^2|\lambda| \geq \frac{1}{2}$. Theorem 2.5 gives

$$\Big|\int_{|y-P|>t} \nabla\Gamma(P-y;\lambda) f(y)\,\mathrm{d}\sigma(y)\Big| \leq C \int_{|P-y|>t} \frac{|f(y)|}{|\lambda||P-y|^{d+1}}\,\mathrm{d}\sigma(y)$$

Choose now $N \in \mathbb{N}$ such that $2^N t \leq \operatorname{diam}(\Omega) < 2^{N+1}t$. We now exhaust the domain of integration by suitable annuli and calculate

$$\begin{aligned}
&\sum_{k=0}^{N} \int_{\mathrm{B}(P,2^{k+1}t)\cap\partial\Omega} \frac{1}{|\lambda| 2^{k(d+1)} t^{d+1}} |f(y)|\,\mathrm{d}\sigma(y) \\
&\quad\leq \frac{1}{|\lambda| t^2} \frac{1}{2^{1-d}} \sum_{k=0}^{N} \frac{1}{2^{2k}} \frac{1}{(2^{k+1}t)^{d-1}} \int_{\mathrm{B}(P,2^{k+1}t\cap\partial\Omega)} |f(y)|\,\mathrm{d}\sigma(y) \\
&\quad\leq C \sum_{k=0}^{N} \frac{1}{2^{2k}} M_{\partial\Omega}(f)(P) \\
&\quad\leq C M_{\partial\Omega}(f)(P)
\end{aligned}$$

where for the second inequality we used Lemma **??** to estimate

$$\frac{1}{(2^{k+1}t)^{d-1}} \leq C(\sigma(\mathrm{B}(P,2^{k+1}d) \cap \partial\Omega))^{-1}.$$

which gives the claimed estimate with a constant C that depends on d, θ and the Lipschitz character of Ω. Now let $t^2|\lambda| < \frac{1}{2}$. We then split the integral as follows

$$\begin{aligned}
&\Big|\int_{|y-P|>t} \nabla\Gamma(P-y;\lambda) f(y)\,\mathrm{d}\sigma(y)\Big| \\
&\quad\leq \Big|\int_{|y-P|\geq(2|\lambda|)^{-1/2}} \nabla\Gamma(P-y;\lambda) f(y)\,\mathrm{d}\sigma\Big| + \Big|\int_{t<|y-P|<(2|\lambda|)^{-1/2}} \nabla\Gamma(P-y;\lambda) f(y)\,\mathrm{d}\sigma\Big|.
\end{aligned}$$

The first summand can be estimated like in the step before, if we substitute t by $(2|\lambda|)^{-1/2}$ For the second term we use the principle of the nutrient zero and estimate

$$\begin{aligned}
&\Big|\int_{t<|y-P|<(2|\lambda|)^{-1/2}} \nabla\Gamma(P-y;\lambda) f(y)\,\mathrm{d}\sigma\Big| \\
&\quad\leq \int_{t<|y-P|<(2|\lambda|)^{-1/2}} |\nabla\Gamma(P-y;\lambda) - \nabla\Gamma(P-y;0)| f(y)|\,\mathrm{d}\sigma
\end{aligned}$$

$$+\Big|\int_{t<|y-P|<(2|\lambda|)^{-1/2}} \nabla\Gamma(P-y;0)f(y)\,\mathrm{d}\sigma\Big|.$$

We don't need to worry about the second summand here since the corresponding estimate is already coverd by the case of $\lambda = 0$ and therefore

$$\begin{aligned}
&\Big|\int_{t<|y-P|<(2|\lambda|)^{-1/2}} \nabla\Gamma(P-y;0)f(y)\,\mathrm{d}\sigma\Big| \\
&\quad\leq \Big|\int_{|y-P|>t} \nabla\Gamma(P-y;0)f(y)\,\mathrm{d}\sigma\Big| + \Big|\int_{|y-P|>(2|\lambda|)^{-1/2}} \nabla\Gamma(P-y;0)f(y)\,\mathrm{d}\sigma\Big| \\
&\quad\leq 2T_0^*(f)(P).
\end{aligned}$$

For the first summand we make use of Theorem 2.6 and more precisely of Corollary 2.7 which unifies all estimates. We then calculate

$$\begin{aligned}
&\int_{t<|y-P|<(2|\lambda|)^{-1/2}} |\nabla\Gamma(P-y;\lambda)-\nabla\Gamma(P-y;0)||f(y)|\,\mathrm{d}\sigma \\
&\qquad\leq \int_{t<|y-P|<(2|\lambda|)^{-1/2}} |\lambda|^{\frac{1}{2}}|y-P|^{2-d}|f(y)|\,\mathrm{d}\sigma,
\end{aligned}$$

and as before we choose adequate N such that $2^{N+1}t > (2|\lambda|)^{-1/2} \geq 2^N t$ which leads to

$$\begin{aligned}
&\leq |\lambda|^{\frac{1}{2}} \sum_{k=0}^{N} \int_{2^k t<|y-P|<2^{k+1}t} |y-P|^{2-d}|f(y)|\,\mathrm{d}\sigma \\
&\leq |\lambda|^{\frac{1}{2}} t^{2-d} \sum_{k=0}^{N} 2^{k(2-d)} \int_{\mathrm{B}(P,2^{k+1}t)} |f(y)|\,\mathrm{d}\sigma \\
&\leq 2^d|\lambda|^{\frac{1}{2}} t \sum_{k=0}^{N} 2^{k-1}\, 2^{(k+1)(1-d)} t^{1-d} \int_{\mathrm{B}(P,2^{k+1}t)} |f(y)|\,\mathrm{d}\sigma \\
&\leq C|\lambda|^{\frac{1}{2}} t \frac{2^N-1}{1} M_{\partial\Omega}(f)(P) \\
&\leq C|\lambda|^{\frac{1}{2}}(2|\lambda|)^{-\frac{1}{2}} M_{\partial\Omega}(f)(P).
\end{aligned}$$

Taking now the supremum over all $t > 0$ we see that

$$T_\lambda^*(f)(P) \leq C(M_{\partial\Omega}(f)(P) + T_0^*(f)(P)),$$

for all $P \in \partial\Omega$. Once again using the result for $\lambda = 0$ and the L^p-boundedness of the Hardy-Littlewood maximal operator we see that

$$\|T_\lambda^*(f)\|_{\mathrm{L}^p(\partial\Omega)} \leq C\,\|f\|_{\mathrm{L}^p(\partial\Omega)}$$

To conclude the first inequality in (3.5), we want to use a standard result from harmonic analysis [**?**, 2.1.14]. First we will show that the integral operator

$$T_\lambda(f)(P) = \lim_{t\to 0} \int_{\substack{y\in\partial\Omega \\ |y-P|>t}} \nabla_x\Gamma(P-y;\lambda)f(y)\,\mathrm{d}\sigma(y)$$

exists for almost every $P \in \partial\Omega$ and all $f \in \mathrm{C}(\partial\Omega;\mathbb{C}^d)$. In a first step, we can split this operator formally in

$$T_\lambda(f)(P) = T_0(f)(P) + \lim_{t\to 0} \int_{\substack{y\in\partial\Omega \\ |y-P|>t}} \nabla_x\{\Gamma(P-y;\lambda) - \Gamma(P-y;0)\} f(y)\,\mathrm{d}\sigma(y)$$

The right expression is well defined for $f \in \mathrm{C}_0^\infty$, once we prove integrability of

$$|\nabla\{\Gamma(P-y;\lambda) - \Gamma(P-y;0)\}|$$

on $\partial\Omega$. To this end we first note that it suffices to consider the integral

$$\int_{|P-y|\leq\varepsilon} |\nabla\{\Gamma(P-y;\lambda) - \Gamma(P-y;0)\}|\,\mathrm{d}\sigma(y),$$

for $\varepsilon \leq \min(2|\lambda|^{-1/2}, r_0/4)$ as the integrand is smooth away from 0 and the domain of integration is bounded. Now Corollary 2.7 and Tolksdorf 4.3.2 give that this can be estimated by

$$\int_{|P-y|\leq\varepsilon} |\lambda|^{1/2}|P-y|^{2-d}\,\mathrm{d}\sigma(y) \leq C|\lambda|^{1/2}\varepsilon \leq C.$$

Based on the preceding calculation we conclude that for all $f \in \mathrm{C}(\partial\Omega,\mathbb{C}^d)$ the operator $T_\lambda(f)(P)$ exists whenever $T_0(f)(P)$ exists. $T_0(f)(P)$ exists for almost everywhere $P \in \partial\Omega$ because of Fabes, Kenig and Verchota [**?**]. As furthermore $T_\lambda^*(f)(P)$ is bounded on $\mathrm{L}^p(\partial\Omega)$ we may now apply Theorem 2.1.14 from Grafakos [**?**] to conclude that $T_\lambda(f)(P)$ exists now for all $f \in \mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ and almost everywhere $P \in \partial\Omega$. The desired L^p estimate on $T_\lambda(f)$ now follows from the observation that

$$|T_\lambda(f)(P)| \leq T_\lambda^*(f)(P)$$

for almost everywhere $P \in \partial\Omega$. □

We can now prove the boundedness of certain nontangential maximal operators.

`lem:nontangentialMaximalFunctions`

Lemma 3.2. *Let* $1 < p < \infty$ *and* (u,ϕ) *be given by* (3.1) *and* (3.2). *Then*

$$\|(\nabla u)^*\|_{\mathrm{L}^p(\partial\Omega)} + \|(\phi)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p\|f\|_{\mathrm{L}^p(\partial\Omega)}, \tag{3.6}$$

where C_p *depends only on* d, θ, p *and the Lipschitz character of* Ω. *Let furthermore* $d \geq 3$. *Then*

$$\|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p\|f\|_{\mathrm{L}^p(\partial\Omega)},$$

where C_p *depends only on* d, θ, p *and the Lipschitz character of* Ω.

Proof. A proof of the estimate $\|(\phi)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p\|f\|_{\mathrm{L}^p(\partial\Omega)}$ can be found in Verchota [**?**]. The proof for $\|(\nabla u)^*\|_{\mathrm{L}^p(\partial\Omega)}$ works in the same way. We will provide a proof for the sake of completeness. To immitate the proof of Verchota, we will work with the corresponding type of cones. Therefore the results for ∇u and ϕ will at first only be established for the type of maximal operators defined by Verchota. The transferability to Shen's maximal operators is given by Tolksdorf [**?**] as the solution (u,ϕ) has a representation as a single layer potential.

Let $x \in \Gamma_{\mathrm{V}}(P)$ and set $t = |x-P|$. Then,

$$\begin{aligned}
|(\nabla u)(x)| &= \Big|\int_{\partial\Omega} \nabla\Gamma_{jk}(x-y;\lambda) f_k \,\mathrm{d}\sigma(y)\Big| \\
&\leq \Big|\int_{|y-P|>t} \nabla\Gamma_{jk}(x-y;\lambda) f_k \,\mathrm{d}\sigma(y)\Big| + \Big|\int_{|y-P|\leq t} \nabla\Gamma_{jk}(x-y;\lambda) f_k \,\mathrm{d}\sigma(y)\Big| \\
&= I_1 + I_2.
\end{aligned}$$

We will now estimate I_1 and I_2 seperately. Note that in Verchota cones we have that for all $Q \in \partial\Omega$ we have $|x-Q| > C|x-P|$, where C is a constant only depending on d and the Lipschitz character of Ω. By Theorem 2.5 we know that

$$\begin{aligned}
I_2 &\leq C\int_{|y-P|\leq t} \frac{1}{|x-y|^{d-1}} |f(y)| \,\mathrm{d}\sigma(y) \\
&\leq \frac{C}{t^{n-1}} \int_{|y-P|\leq t} |f(y)| \,\mathrm{d}\sigma(y) \leq C M_{\partial\Omega}(f)(P).
\end{aligned}$$

For I_1, we calculate

$$\begin{aligned}
&\Big|\int_{|y-P|>t} \nabla\Gamma_{jk}(x-y;\lambda) f_k(y) - \nabla\Gamma_{jk}(x-y;0) f_k(y) + \nabla\Gamma_{jk}(x-y;0) f_k(y) \,\mathrm{d}\sigma(y)\Big| \\
&\quad\leq \Big|\int_{|y-P|>t} \nabla\Gamma_{jk}(x-y;\lambda) f_k(y) - \nabla\Gamma_{jk}(P-y;\lambda) f_k(y) \,\mathrm{d}\sigma(y)\Big| \\
&\qquad + \Big|\int_{|y-P|>t} \nabla\Gamma_{jk}(P-y;\lambda) f_k(y) \,\mathrm{d}\sigma(y)\Big|.
\end{aligned}$$

The second summand can directly be estimated by $T_\lambda^*(f)(P)$. For the second one we apply the mean value theorem and derive using once again Theorem 2.5

$$\begin{aligned}
&\int_{|y-P|>t} \big|\nabla\Gamma_{jk}(x-y;\lambda) - \nabla\Gamma_{jk}(P-y;\lambda)\big| |f(y)| \,\mathrm{d}\sigma(y) \\
&\quad\leq \int_{|y-P|>t} |\nabla^2\Gamma_{jk}(s-y;\lambda)| |x-P| |f(y)| \,\mathrm{d}\sigma(y) \\
&\quad\leq Ct \int_{|y-P|>t} \frac{1}{|s-y|^d} |f(y)| \,\mathrm{d}\sigma(y) \\
&\quad\leq Ct \int_{|y-P|>t} \frac{1}{|y-P|^d} |f(y)| \,\mathrm{d}\sigma(y)
\end{aligned}$$

$$\leq C\int_{\partial\Omega}\frac{t}{(t+|y-P|)^d}|f(y)|\,\mathrm{d}\sigma(y).$$

where s is an element on the line connecting x and P and we used the property of Verchota-cones that $|s-y|\geq C|y-P|$. Note that Verchota cones are convex. As in Verchota [?] the integral may now be bounded by the Hardy-Littlewood maximal operator due to an application of a suitable Lemma from Dahlberg [?] as the kernel $\frac{t}{(t+|y-P|)^d}$ is uniformly integrable on $\partial\Omega$. Summing up we have shown that

$$|(\nabla u)(x)|\leq C\{M_{\partial\Omega}f(P)+T^*_\lambda(f)(P)\}$$

where C only depends on d, θ and the Lipschitz character of Ω. We thus may take the supremum over all $x\in\Gamma_{\mathrm{V}}$ and conclude the desired estimate by the well known mapping properties of the Hardy-Littlewood maximal operator and the respective results from Lemma **??**.

We will now work on the proof of the estimate for $(u)^*$ for $d\geq 3$. In order to derive L^p estimates on this maximal operator we will work directly with the Definition of the single layer potential (3.1). For $P\in\partial\Omega$, estimate (2.23) together with the estimate for Shen cones (1.3) gives that for all $x\in\Gamma(P)$

$$|u^*(x)|\leq C\int_{\partial\Omega}\frac{1}{|x-y|^{d-2}}|f(y)|\,\mathrm{d}\sigma(y)\leq C\int_{\partial\Omega}\frac{1}{|P-y|^{d-2}}|f(y)|\,\mathrm{d}\sigma(y),$$

where C only depends on d, θ and the Lipschitz character of Ω. Passing to the maximal operator yields the inequality

$$u^*(P)\leq C\int_{\partial\Omega}\frac{1}{|P-y|^{d-2}}|f(y)|\,\mathrm{d}\sigma(y),$$

We are now left with the task to estimate the integral

$$\int_{\partial\Omega}\frac{1}{|P-y|^{d-2}}\,\mathrm{d}\sigma(y)$$

uniformly for all $P\in\partial\Omega$, as the rest can be handled using the Young inequality. Let r_0 be the Radius from the definition of Lipschitz cylinders. Then

$$\int_{\partial\Omega}\frac{1}{|P-y|^{d-2}}\,\mathrm{d}\sigma(y)\leq\int_{\partial\Omega\cap\mathrm{B}(P;r_0/4)}\frac{1}{|P-y|^{d-2}}\,\mathrm{d}\sigma(y)+\int_{\partial\Omega\setminus\mathrm{B}(P;r_0/4)}\frac{1}{|P-y|^{d-2}}\,\mathrm{d}\sigma(y).$$
$$\leq Cr_0/4+\sigma(\partial\Omega)r_0^{2-d}4^{d-2}.$$

where C only depends on d and the Lipschitz character of Ω. □

The next Lemma deals with *trace formulas* for ∇u and ϕ. We can now finally talk about boundary values as the existence of nontangential limits guarantees that there exists something on $\partial\Omega$ that is related to the function inside Ω.

lem:traceFormulas

Lemma 3.3. *Let (u,ϕ) be given by* (3.1) *and* (3.2) *with $f \in \mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ and $1 < p < \infty$. Then*

$$\Big(\frac{\partial u_i}{\partial x_j}\Big)_\pm(x) = \pm\frac{1}{2}\{n_j(x)f_i(x) - n_i(x)n_j(x)n_k(x)f_k(x)\} + \mathrm{p.v.}\int_{\partial\Omega}\frac{\partial}{\partial x_j}\{\Gamma_{ik}(x-y;\lambda)\}f_k(y)\,\mathrm{d}\sigma(y), \quad \text{eq:traceFormula} \tag{3.7}$$

$$\phi_\pm(x) = \mp\frac{1}{2}n_k(x)f_k(x) + \mathrm{p.v.}\int_{\partial\Omega}\Phi_k(x-y)f_k(y)\,\mathrm{d}\sigma(y)$$

for almost everywhere $x \in \partial\Omega$. The subscripts $+$ and $-$ indicate nontangential limits taken inside Ω and outside $\overline{\Omega}$, respectively.

Proof. The correctness of the trace formulas (3.7) is known for the case $\lambda = 0$ since Fabes, Kenig and Verchota [**?**]. This fact will now be reused for $\lambda \in \Sigma_\theta$. We insert a 0 to the nontangential limit as

$$(\nabla u_j)_\pm(x) = (\nabla v_j)_\pm(x) + (\nabla u_j - \nabla v_j)_\pm(x),$$

where $v_j(x) = \int_{\partial\Omega}\Gamma_{jk}(x-y;0)f_k(y)\,\mathrm{d}\sigma(y)$. Because of [**?**] we know that the first nontangential limit exists and is given by (3.7) with $\lambda = 0$. It therefore remains to show that

$$(\nabla u_j - \nabla v_j)_\pm(x) = \int_{\partial\Omega}\nabla\{\Gamma_{jk}(x-y;\lambda) - \Gamma_{jk}(x-y;0)\}f_k(y)\,\mathrm{d}\sigma(y)$$

for all $x \in \partial\Omega$. To this end let $(x_l)_{l\in\mathbb{N}}$ a sequence in $\Gamma(x)$ with $\lim_{l\to\infty}x_l = x$. Furthermore let us note that for almost everywhere $x \in \partial\Omega$ we have that

$$\int_{\partial\Omega}\frac{1}{|x-y|^{d-2}}|f(y)|\,\mathrm{d}\sigma(y) < \infty.$$

This is a consequence of the fact that

$$\sup_{x\in\partial\Omega}\Big|\int_{\partial\Omega}\frac{1}{|x-y|^{d-2}}\,\mathrm{d}\sigma(y)\Big| < \infty$$

and an application of Young's inequality which can be found in Tolksdorf [**?**]: Let $x \in \partial\Omega$. Then

$$\begin{aligned}\int_{\partial\Omega}&\frac{1}{|x-y|^{d-2}}\,\mathrm{d}\sigma(y)\\ &\le \int_{\partial\Omega\cap\mathrm{B}(x,r_0/4)}\frac{1}{|x-y|^{d-2}}\,\mathrm{d}\sigma(y) + \int_{\partial\Omega\setminus\mathrm{B}(x,r_0/4)}\frac{1}{|x-y|^{d-2}}\,\mathrm{d}\sigma(y)\\ &\le Cr_0 + r^{2-d}4^{d-2}\sigma(\partial\Omega)\end{aligned}$$

by Lemma **??**. Now Young's inequality gives us the desired result. In the next step we will show that

$$\frac{1}{|x-y|^{d-2}}|f(y)|$$

gives a suitable function for dominated convergence. Set $\varepsilon = (4|\lambda|^2)^{-1}$ and without loss of generality assume that $\operatorname{supp} f \subseteq \mathrm{B}(x,\varepsilon)$. Furthermore assume that $|x_l - x| < \varepsilon$ for all $l \in \mathbb{N}$. Then $|x_l - y| \leq (2|\lambda|^2)^{-1}$ and Corollary 2.7 give

$$\begin{aligned}
&(\nabla u_j - \nabla v_j)(x_l) \int_{\partial\Omega} \nabla\{\Gamma_{jk}(x_l - y;\lambda) - \Gamma_{jk}(x_l - y;0)\} f_k(y)\,\mathrm{d}\sigma(y) \\
&\leq \int_{\partial\Omega} \frac{1}{\sqrt{|\lambda|}|x_l - y|^{d-2}}|f(y)|\,\mathrm{d}\sigma(y) \\
&\leq \frac{C}{\sqrt{|\lambda|}} \int_{\partial\Omega} \frac{1}{|x-y|^{d-2}}|f(y)|\,\mathrm{d}\sigma(y) < \infty.
\end{aligned}$$

Now dominated convergence gives the claim for $x_l \to x$. Note that it does not affect the proof if the sequence x_l lays inside Ω or outside $\overline{\Omega}$. □

The previous Lemma enables us to talk about boundary values of partial derivatives. The next theorem will now give a similar result but for conormal derivatives which are defined by

$$\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial n} - \phi n.$$

We will also be working with tangential derivatives which are defined via

DEFINE

thm:jumpConditions

Theorem 3.4. *Let $\lambda \in \Sigma_\theta$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$. Let (u,ϕ) be given by* (3.1) *and* (3.2) *with $f \in \mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ and $1 < p < \infty$. Then $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$ and*

eq:nontangentialConormalDerivative

$$\left(\frac{\partial u}{\partial \nu}\right)_\pm = \left(\pm\frac{1}{2}I + \mathscr{K}_\lambda\right)f \tag{3.8}$$

on $\partial\Omega$, with $\mathscr{K}_\lambda$ a bounded operator on $\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ with

$$\|\mathscr{K}_\lambda f\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p \|f\|_{\mathrm{L}^p(\partial\Omega)},$$

where C_p depends only on d, θ, p and the Lipschitz character of Ω.

Proof. For the the jth component of the tangential derivative of u_i, $1 \le i, j \le d$, we calculate using the results from Lemma 3.3

$$\begin{aligned}
((\nabla_{\tan} u_i)_+)_j &= (\frac{\partial u_i}{\partial x_j})_+ - \langle (\nabla u_i)_+, n\rangle n_j \\
&= (\frac{\partial u_i}{\partial x_j})_+ - (\frac{\partial u_i}{\partial x_k})_+ n_k n_j \\
&= \frac{1}{2}\{n_j f_i - n_i n_j n_k f_k\} - \frac{1}{2}\{n_k f_i - n_i n_k n_l f_l\} n_k n_j \\
&\quad + \mathrm{p.v.} \int_{\partial\Omega} \frac{\partial}{\partial x_j}\{\Gamma_{ik}(x-y;\lambda)\} f_k(y)\,\mathrm{d}\sigma(y) \\
&\quad + \mathrm{p.v.} \int_{\partial\Omega} \frac{\partial}{\partial x_k}\{\Gamma_{il}(x-y;\lambda)\} f_l(y)\,\mathrm{d}\sigma(y) n_k n_j.
\end{aligned}$$

As the first two summands add up to zero, the entire expression does not depend on the direction of the nontangential limit. This gives

$$(\nabla_{\tan} u)_+ = (\nabla_{\tan} u)_-$$

We calculate for the jth component of the nontangential limit of the conormal derivative of u at $x \in \partial\Omega$ using the results from Lemma 3.3

$$\begin{aligned}
&(\frac{\partial u_j}{\partial x_i})_+ (x) n_i - \phi_+(x) n_j \\
&= \frac{1}{2}\{n_i f_j(x) - n_j n_i n_k f_k(x)\} n_i + \mathrm{p.v.} \int_{\partial\Omega} \frac{\partial}{\partial x_i}\{\Gamma_{jk}(x-y;\lambda)\} f_k(y)\,\mathrm{d}\sigma(y) n_i \\
&\quad + \frac{1}{2} n_k f_k(x) n_j - \mathrm{p.v.} \int_{\partial\Omega} \Phi_k(x-y) f_k(y)\,\mathrm{d}\sigma(y) n_j \\
&= \frac{1}{2} f_j(x) + (\mathscr{K}_\lambda f)_j(x),
\end{aligned}$$

where n denotes the normal vector at x and

$$(\mathscr{K}_\lambda f)(x) = \mathrm{p.v.} \int_{\partial\Omega} \nabla_x \Gamma(x-y;\lambda) f(y)\,\mathrm{d}\sigma(y) n - \mathrm{p.v.} \int_{\partial\Omega} \Phi_k(x-y) f_k(y)\,\mathrm{d}\sigma(y) n. \tag{3.9}$$

eq:defnKlambda

We note that $\mathscr{K}_\lambda$ essentially consists of two boundary layer potentials. The L^p-boundedness of the first one was proven in Lemma 3.1. The L^p-boundedness of the second boundary layer potential follows in an analogous way using the fact that the operators

$$A^*(f)(P) = \sup_{t>0} \Big| \int_{\substack{y\in\partial\Omega \\ |y-P|>t}} \frac{P-y}{|P-y|^n} f(y)\,\mathrm{d}\sigma(y) \Big|, \quad P \in \partial\Omega$$

are bounded by Lemma 1.2 of Verchota [**?**]. □

Similar to $\mathscr{K}_\lambda$ for $\lambda = 0$ we have

$$(\mathscr{K}_0 f)(x) = \mathrm{p.v.}\int_{\partial\Omega} \nabla_x \Gamma(x-y;0) f(y)\,\mathrm{d}\sigma(y) n - \mathrm{p.v.}\int_{\partial\Omega} \Phi_k(x-y) f_k(y)\,\mathrm{d}\sigma(y) n, \tag{3.10}$$

eq:defnK0

as was shown by Fabes, Kenig and Verchota [**?**, (0.12)].

We now note a fact that will be crucial for solving the L^2 Dirichlet problem in Chapter 5 and will fortify the hopes of translating results for $\lambda = 0$ to $\lambda \in \Sigma_\theta$.

lem:compactness

Lemma 3.5. *Let $\lambda \in \Sigma_\theta$ and $d \geq 3$. Then the operator $\mathscr{K}_\lambda - \mathscr{K}_0$ on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ is compact.*

Proof. The idea of this proof is similar to the one in Tolksdorf [**?**, Lemma 4.3.5]. Let $f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$. Let's denote $\mathscr{K} := \mathscr{K}_\lambda - \mathscr{K}_0$. We will now try to approximate $\mathscr{K}$ by compact operators in the operator norm. To this end we define for all $\varepsilon > 0$

$$(\mathscr{K}^{(\varepsilon)} f)(x) := \int_{\partial\Omega\setminus \mathrm{B}(x,\varepsilon)} \nabla\{\Gamma(x-y;\lambda) - \Gamma(x-y;0)\} f(y)\,\mathrm{d}\sigma(y), \quad x \in \partial\Omega.$$

We can now estimate by Young's inequality **??**

$$\|(\mathscr{K}^{(\varepsilon)} f)(x)\|_{\mathrm{L}^2(\partial\Omega)} \leq \sup_{p\in\partial\Omega} \|\nabla\{\Gamma(p-\cdot;\lambda) - \Gamma(p-\cdot;0)\} 1_{\mathrm{B}(p,\varepsilon)}\|_{\mathrm{L}^1(\partial\Omega)} \|f\|_{\mathrm{L}^2(\partial\Omega)}.$$

Our goal is to show that

$$\sup_{p\in\partial\Omega} \|\nabla\{\Gamma(p-\cdot;\lambda) - \Gamma(p-\cdot;0)\} 1_{\mathrm{B}(p,\varepsilon)}\|_{\mathrm{L}^1(\partial\Omega)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

To this end, let ε be small enough such that we can apply the estimates from Corollary 2.7 to calculate for some $p \in \partial\Omega$

$$\begin{aligned} &\|\nabla\{\Gamma(p-\cdot;\lambda) - \Gamma(p-\cdot;0)\} 1_{\mathrm{B}(p,\varepsilon)}\|_{\mathrm{L}^1(\partial\Omega)} \\ &\qquad \leq C \int_{\partial\Omega\cap \mathrm{B}(p,\varepsilon)} \sqrt{|\lambda|}|p-y|^{2-d}\,\mathrm{d}\sigma(y) \leq C\sqrt{|\lambda|}\varepsilon \end{aligned}$$

where for the last step we applied Lemma **??**. For $\varepsilon \to 0$ this gives us $\mathscr{K}^{(\varepsilon)} \to \mathscr{K}$ in the operator norm. The last step is to verify the compactness of $\mathscr{K}^(\varepsilon)$. We note that the integral kernel of $\mathscr{K}^{(\varepsilon)}$ is bounded which gives us that in particular the kernel is an element of $\mathrm{L}^2(\partial\Omega\times\partial\Omega;\mathbb{C}^{d\times d}$. The compactness of $\mathscr{K}^{(\varepsilon)}$ now follows from Weidmann [**?**, Thm. 6.11]. $\square$

Our next step is to introduce the *double layer potential* $u(x) = \mathscr{D}_\lambda(f)(x)$ for the Stokes resolvent problem, where

eq:defDoubleLayer

$$u_j(x) = \int_{\partial\Omega} \Big\{ \frac{\partial}{\partial y_i} \{\Gamma_{jk}(y-x;\lambda)\} n_i(y) - \Phi_j(y-x) n_k(y) \Big\} f_k(y) \,\mathrm{d}\sigma(y). \tag{3.11}$$

The corresponding pressure $\phi(x) = \mathscr{D}_\phi(f)(x)$ is defined via

$$\phi(x) = \frac{\partial^2}{\partial x_i \partial x_k} \int_{\partial\Omega} G(y-x;0) n_i(y) f_k(y) \,\mathrm{d}\sigma(y) + \lambda \int_{\partial\Omega} G(y-x;0) n_k(y) f_k(y) \,\mathrm{d}\sigma(y).$$

eq:defDoubleLayerPressure

$$\tag{3.12}$$

Using 2.21 and 2.22 one can show that (u, ϕ) defines again a solution to the Stokes resolvent problem in $\mathbb{R}^d \setminus \partial\Omega$.

The next theorem will give us a suitable operator which maps a given function $f \in \mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ to boundary values of $u = \mathscr{D}_\lambda(f)$.

thm:nontangentialLimitDoubleLayer

Theorem 3.6. *Let $\lambda \in \Sigma_\theta$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$. Let u be given by* (3.11) *for $f \in \mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$, $1 < p < \infty$. Then*

eq:lpBoundednessUNontangentialMax

$$\|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p \|f\|_{\mathrm{L}^p(\partial\Omega)} \tag{3.13}$$

where C_p depends only on d, p, θ and the Lipschitz character of Ω. Furthermore

eq:nontangentialLimitDoubleLayer

$$u_\pm = \Big(\mp \frac{1}{2} I + \mathscr{K}^*_{\bar{\lambda}} \Big) f, \tag{3.14}$$

where $K^*_{\bar{\lambda}}$ is the adjoint of the operator $K_{\bar{\lambda}}$ in (3.8)

Proof. The estimate for $(u)^*$ is a direct consequence of Lemma 3.2, in particular of the estimates on $(\nabla u)^*$ and $(\phi)^*$.

For the proof of (3.14), we begin by determining the adjoint of the operator $\mathscr{K}_{\bar{\lambda}}$. To this end we fill first work with truncated operators $\mathscr{K}^{(\varepsilon)}_\lambda$ which are defined as

$$(\mathscr{K}^{(\varepsilon)}_\lambda f)(x) = \int_{\partial\Omega} 1_{\mathrm{E}(x,\varepsilon)} \nabla_x \Gamma(x-y;\lambda) f(y) \,\mathrm{d}\sigma(y) n - \int_{\partial\Omega} 1_{\mathrm{E}(x,\varepsilon)} \Phi_k(x-y) f_k(y) \,\mathrm{d}\sigma(y) n,$$

for $x \in \partial\Omega$ and $\mathrm{E}(x,\varepsilon) := \mathbb{R}^d \setminus \mathrm{B}(x,\varepsilon)$. Now for $f \in \mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ and $g \in \mathrm{L}^q(\partial\Omega;\mathbb{C}^d)$ with $1/p + 1/q = 1$ we calculate

$$\begin{aligned}
\langle \mathscr{K}^{(\varepsilon)}_{\bar{\lambda}} f, g \rangle &= \int_{\partial\Omega} (\mathscr{K}^{(\varepsilon)}_{\bar{\lambda}} f_j)(x) \overline{g_j(x)} \,\mathrm{d}\sigma(x) \\
&= \int_{\partial\Omega} \int_{\partial\Omega} \frac{\partial}{\partial x_i} \{\Gamma_{jk}(x-y;\overline{\lambda})\} f_k(y) 1_{\mathrm{E}(x,\varepsilon}(y) \,\mathrm{d}\sigma(y) n_i(x) \overline{g_j(x)} \,\mathrm{d}\sigma(x) \\
&\quad + \int_{\partial\Omega} \int_{\partial\Omega} \Phi_k(x-y) f_k(y) 1_{\mathrm{E}(x,\varepsilon)}(y) \,\mathrm{d}\sigma(y) n_j(x) \overline{g_j(x)} \,\mathrm{d}\sigma(x).
\end{aligned}$$

Note that $1_{\mathrm{E}(x,\varepsilon)}(y) = 1_{\mathrm{E}(y,\varepsilon)}(x)$. Now an application of Fubini and factoring out $f_k(y)$ gives that the lengthy expression is equal to

$$\int_{\partial\Omega} f_k(y) \int_{\partial\Omega} \Big\{ \frac{\partial}{\partial x_i}\{\Gamma_{jk}(x-y;\bar{\lambda})\} n_i(x) - \Phi_k(x-y) n_j(x) \Big\} 1_{\mathrm{E}(y,\varepsilon)}(x) \overline{g_j(x)}\, \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(y).$$

Therefore we see that the adjoint of the truncated operator $\mathscr{K}_{\bar{\lambda}}^{(\varepsilon)}$ is given by

$$\big((K_{\bar{\lambda}}^{(\varepsilon)})^* g\big)_k(y) = \int_{\partial\Omega} \Big\{ \frac{\partial}{\partial x_i}\{\Gamma_{jk}(x-y;\lambda)\} n_i(x) - \Phi_k(x-y) n_j(x) \Big\} 1_{\mathrm{E}(y,\varepsilon)}(x) g_j(x)\, \mathrm{d}\sigma(x),$$

for $y \in \partial\Omega$ since $\overline{\Gamma_{jk}(x-y;\lambda)} = \Gamma_{jk}(x-y;\bar{\lambda})$.

In the next step we will go from truncated operators to principal value operators. For this to work we will look for suitable majorants. If $x \in \partial\Omega$ we estimate

$$\begin{aligned} |(\mathscr{K}_{\bar{\lambda}}^{(\varepsilon)} f)_j(x)| &= \Big| \int_{|x-y|>\varepsilon} \frac{\partial}{\partial x_i}\{\Gamma_{jk}(x-y;\lambda)\} f_k(y)\, \mathrm{d}\sigma(y) n_i(x) \\ &\qquad - \int_{|x-y|>\varepsilon} \Phi_k(x-y) f_k(y) n_j(x)\, \mathrm{d}\sigma(y) \Big| \\ &\le T_\lambda^*(f)(x) + A^*(f)(x). \end{aligned}$$

Now dominated convergence gives

$$\lim_{\varepsilon\to 0} \langle K_{\bar{\lambda}}^{(\varepsilon)} f, g\rangle = \langle K_{\bar{\lambda}} f, g\rangle.$$

A similar argument gives

$$\lim_{\varepsilon\to 0} \langle f, K_{\bar{\lambda}}^{(\varepsilon)^{(*)}} g\rangle = \langle f, K_{\bar{\lambda}}^{(*)} g\rangle,$$

where

$$\big((K_{\bar{\lambda}}^* g\big)_k(y) = \mathrm{p.\,v.} \int_{\partial\Omega} \Big\{ \frac{\partial}{\partial x_i}\{\Gamma_{kj}(x-y;\lambda)\} n_i(x) - \Phi_k(x-y) n_j(x) \Big\} g_j(x)\, \mathrm{d}\sigma(x).$$

Note that we have used the symmetry of $(\Gamma_{\alpha\beta})$.

The last part now consists of proving that the equality (3.14) holds. To simplify the calculations and make Lemma 3.3 more accessible note that on the one hand

$$\begin{aligned} \int_{\partial\Omega} \frac{\partial}{\partial y_i}\{\Gamma_{jk}(y-x;\lambda)\} n_i(y) f_k(y)\, \mathrm{d}\sigma(y) &= -\int_{\partial\Omega} \frac{\partial}{\partial x_i}\{\Gamma_{jk}(x-y;\lambda)\} n_i(y) f_k(y)\, \mathrm{d}\sigma(y) \\ &= -\frac{\partial}{\partial x_i} \mathscr{S}(n_i f)_j(x) \end{aligned}$$

and on the other hand

$$-\int_{\partial\Omega} \Phi_j(y-x) n_k(y) f_k(y)\, \mathrm{d}\sigma(y) = \int_{\partial\Omega} \Phi_l(x-y) \delta_{lj} n_k(y) f_k(y)\, \mathrm{d}\sigma(y) = \mathscr{S}_\Phi(\tilde{f}^j)(x),$$

where $\tilde{f}^j_l = \delta_{lj} n_k f_k$. For $x \in \partial\Omega$ we can now calculate

$$\begin{aligned}
&\Big(\int_{\partial\Omega} \frac{\partial}{\partial y_i}\{\Gamma_{jk}(y-\cdot\,;\lambda)\} n_i(y) f_k(y)\,\mathrm{d}\sigma(y)\Big)_{\pm}(x) \\
&\quad = -\big(\frac{\partial}{\partial x_i}\mathscr{S}_\lambda(n_i f)_j\big)_{\pm}(x) \\
&\quad = \mp\frac{1}{2}\{n_i(x) n_i(x) f_j(x) - n_j(x) n_i(x) n_k(x) n_i(x) f_k(x)\} \\
&\qquad - \mathrm{p.v.}\int_{\partial\Omega} \frac{\partial}{\partial x_i}\{\Gamma_{jk}(x-y;\lambda)\} n_i(y) f_k(y)\,\mathrm{d}\sigma(y) \\
&\quad = \mp\frac{1}{2}\{f_j(x) - n_j(x) n_k(x) f_k(x)\} \\
&\qquad + \mathrm{p.v.}\int_{\partial\Omega} \frac{\partial}{\partial y_i}\{\Gamma_{jk}(x-y;\lambda)\} n_i(y) f_k(y)\,\mathrm{d}\sigma(y),
\end{aligned}$$

where we used trace formula (3.7). A similar procedure for the second integral part of the double layer potential gives

$$\begin{aligned}
&-\Big(\int_{\partial\Omega} \Phi_j(y-\cdot\,) n_k(y) f_k(y)\,\mathrm{d}\sigma(y)\Big)_{\pm}(x) \\
&\quad = \big(\mathscr{S}_\Phi(\tilde{f}^j)\big)_{\pm}(x) \\
&\quad = \mp\frac{1}{2} n_k(x)\tilde{f}^j_k(x) - \mathrm{p.v.}\int_{\partial\Omega} \Phi_k(x-y)\tilde{f}^j_k(x)\,\mathrm{d}\sigma(y) \\
&\quad = \mp\frac{1}{2} n_j(x) n_k(x) f_k(x) - \mathrm{p.v.}\int_{\partial\Omega} \Phi_j(x-y) n_k(x) f_k(x)\,\mathrm{d}\sigma(y)
\end{aligned}$$

Putting everything together we get

$$(u_j)_{\pm}(x) = \mp\frac{1}{2} f_j(x) + (K^*_{\bar{\lambda}} f)_j(x)$$

which proves the claim. $\square$

Chapter 4

Rellich Estimates

In this section we will establish Rellich type estimates for the Stokes resolvent problem. We will for this entire section always assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$ with connected boundary and $|\partial\Omega| = 1$. Furthermore we will use the shorthand notation

$$\|\cdot\|_\partial \coloneqq \|\cdot\|_{\mathrm{L}^2(\partial\Omega)}.$$

The goal of this section is to derive an Rellich type inequality which will be used to prove the invertibility of the operators $\pm(1/2)I + \mathscr{K}_\lambda$. This inequality is part of the following theorem.

thm:rellich

Theorem 4.1. *Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0,1)$. Let (u,ϕ) be a solution to the Stokes resolvent problem in Ω and suppose that $(\nabla u)^* \in \mathrm{L}^2(\partial\Omega)$ and $(\phi)^* \in \mathrm{L}^(\partial\Omega)$. Furthermore, assume that ∇u, ϕ have nontangential limits almost everywhere on $\partial\Omega$. Then*

eq:rellich1

$$\|\nabla u\|_\partial + \|\phi - \fint_{\partial\Omega} \phi\|_\partial \leq C\{\|\nabla_{\tan} u\|_\partial + |\lambda|^{1/2}\|u\|_\partial + |\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}\} \tag{4.1}$$

and

eq:rellich2

$$\|\nabla u\|_\partial + |\lambda|^{1/2}\|u\|_\partial + |\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\phi\|_\partial \leq C\|\frac{\partial u}{\partial \nu}\|_\partial, \tag{4.2}$$

where $\frac{\partial}{\partial\nu}$ denotes the conormal derivative, and C depends only on d, τ, θ and the Lipschitz character of Ω.

We will now prepare the proof of this theorem by proving several helpful lemmata.

lem:rellichIdentity

Lemma 4.2. *Under the same conditions on (u,ϕ) as in Theorem 4.1, we have*

$$\int_{\partial\Omega} h_k n_k |\nabla u|^2 \,\mathrm{d}\sigma = 2\,\mathrm{Re}\int_{\partial\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k}\Big(\frac{\partial u}{\partial \nu}\Big)_i \,\mathrm{d}\sigma + \int_\Omega \mathrm{div}\,(h)|\nabla u|^2 \,\mathrm{d}x - 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_j}\cdot\frac{\partial u_i}{\partial x_k}\cdot\frac{\partial \bar{u}_i}{\partial x_j}\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_i}\cdot\frac{\partial u_i}{\partial x_k}\bar{\phi}\,\mathrm{d}x - 2\,\mathrm{Re}\int_\Omega h_k\frac{\partial u_i}{\partial x_k}\cdot\bar{\lambda}\bar{u}_i\,\mathrm{d}x \tag{4.3}$$

eq:rellichIdentity

and

$$\int_{\partial\Omega} h_k n_k |\nabla u|^2 \,\mathrm{d}\sigma = 2\,\mathrm{Re}\int_{\partial\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_j}\Big\{n_k\frac{\partial u_i}{\partial x_j} - n_j\frac{\partial u_i}{\partial x_k}\Big\}\,\mathrm{d}\sigma + 2\,\mathrm{Re}\int_{\partial\Omega} h_k\bar{\phi}\Big\{n_i\frac{\partial u_i}{\partial x_k} - n_k\frac{\partial u_i}{\partial x_i}\Big\}\,\mathrm{d}\sigma - \int_\Omega \mathrm{div}\,(h)|\nabla u|^2\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_j}\cdot\frac{\partial u_i}{\partial x_k}\cdot\frac{\partial \bar{u}_i}{\partial x_j}\,\mathrm{d}x - 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_i}\cdot\frac{\partial u_i}{\partial x_k}\bar{\phi}\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k\frac{\partial u_i}{\partial x_k}\cdot\bar{\lambda}\bar{u}_i\,\mathrm{d}x, \tag{4.4}$$

eq:rellichIdentity2

where $h=(h_1,\dots,h_d)\in \mathrm{C}_0^1(\mathbb{R}^d,\mathbb{R}^d)$.

Proof. The proof of the stated identities reduces to several integrations by part once we establish its applicability. Approximating $\partial\Omega$ by a sequence of C^∞ domains with uniform Lipschitz characters as described in Verchota [**?**] and the facts that on the one hand $(\nabla u)^*, (\phi)^* \in \mathrm{L}^2(\partial\Omega)$ and on the other hand ∇u and ϕ have nontantential limits almost everywhere, the integration by parts is justified. Details on how the approximation argument works can be found in Brown **??**. For the sake of completeness we show the approximation argument once for a solid integral and once for a boundary integral.

Let $(\Omega_l)_{l\in\mathbb{N}}$ denote the approximating sequence of C^∞ domains with outer normal $n^{(l)}$. Then

$$\int_{\partial\Omega_l} h_k n_k^{(l)}|\nabla u|^2\,\mathrm{d}\sigma_l = \int_{\partial\Omega} w_l(x)h_k(\Lambda_l(x))n_k^{(l)}(\Lambda_l(x))|\nabla u|^2(\Lambda_l(x))\,\mathrm{d}\sigma$$

Now we know that $\lim_{l\to\infty} w_l(x)=1$ and $\lim_{l\to\infty}\Lambda_l(x)=x$ almost everywhere and $\Lambda_l(x)\in\Gamma(x)$ for all $l\in\mathbb{N}$. Furthermore we know that $\lim_{l\to\infty} n_k^{(l)} = n_k$ almost everywhere and that ∇u has a nontangential limit almost everywhere. This gives us that the integrand converges almost everywhere to $h_k(x)n_k(x)|\nabla u|^2(x)$. Now furthermore we have that the integrand is dominated by $\delta\|h\|_\infty((\nabla u)^*)^2$, where δ is a uniform bound to w_l. Since by assumption $(\nabla u)^*\in\mathrm{L}^2(\partial\Omega)$, the dominated convergence theorem gives us

$$\lim_{l\to\infty}\int_{\partial\Omega} w_l(x)h_k(\Lambda_l(x))n_k^{(l)}(\Lambda_l(x))|\nabla u|^2(\Lambda_l(x))\,\mathrm{d}\sigma = \int_{\partial\Omega} h_k n_k|\nabla u|^2\,\mathrm{d}\sigma.$$

It is easy to check, that the only differences in the approximation argument when applied to the other boundary integrals lie in the choice of the majorant. Now consider for instance the solid integral

We now start a formal calculation on Ω and $\partial\Omega$ keeping in mind that the stated equalities sometimes hold only after the application of the former approximation argument.

Let's expand the first summand in (4.3) using the definition of conormal derivatives

$$\begin{aligned}2\operatorname{Re}\int_{\partial\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k}\Big(\frac{\partial u}{\partial \nu}\Big)_i \,\mathrm{d}\sigma &= 2\operatorname{Re}\int_{\partial\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k}\big(\frac{\partial u_i}{\partial x_j}\big) n_j \,\mathrm{d}\sigma - 2\operatorname{Re}\int_{\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k}\phi n_i \,\mathrm{d}x \\ &=: I_1 - I_2.\end{aligned}$$

For I_1 we find using the divergence theorem

$$\begin{aligned}I_1 &= 2\operatorname{Re}\int_{\Omega} \frac{\partial}{\partial x_j}\big(h_k \frac{\partial}{\partial x_j}\frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial u_i}{\partial x_j}\big)\,\mathrm{d}x \\ &= 2\operatorname{Re}\int_{\Omega} \frac{\partial h_k}{\partial x_j}\frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial u_i}{\partial x_j} + h_k \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_k}\frac{\partial u_i}{\partial x_j} + h_k \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial^2 u_i}{\partial x_j^2}\,\mathrm{d}x \\ &=: I_3 + I_4 + I_5.\end{aligned}$$

For I_5 we use the fact that u solves the Stokes resolvent problem which gives

$$\begin{aligned}I_5 &= 2\operatorname{Re}\int_{\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \phi}{\partial x_i}\,\mathrm{d}x + 2\operatorname{Re}\int_{\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k}\lambda u_i \,\mathrm{d}x \\ &=: I_6 + I_7.\end{aligned}$$

Another application of the divergence theorem gives

$$\begin{aligned}I_2 &= 2\operatorname{Re}\int_{\Omega} \frac{\partial}{\partial x_i}\big(h_k \frac{\partial \bar{u}_i}{\partial x_k}\phi\big)\,\mathrm{d}x \\ &= 2\operatorname{Re}\int_{\Omega} \frac{\partial h_k}{\partial x_i}\frac{\partial \bar{u}_i}{\partial x_k}\phi + h_k \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_k}\phi + h_k \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \phi}{\partial x_i}\,\mathrm{d}x \\ &=: I_8 + I_9 + I_6.\end{aligned}$$

One term that hasn't come up so far, the second summand of the right side in (4.3), will now be expanded

$$\begin{aligned}\operatorname{div}(h)|\nabla u|^2 \,\mathrm{d}x &= \int_{\Omega} \operatorname{div}(h|\nabla u|^2)\,\mathrm{d}x - \int_{\Omega} h_k \frac{\partial}{\partial x_i}|\nabla u|^2 \,\mathrm{d}x \\ &=: I_{10} - I_{11}.\end{aligned}$$

Expanding this further gives us

$$I_{11} = \int_{\Omega} h_i \frac{\partial}{\partial x_i}\big(\frac{\partial u_k}{\partial x_j}\frac{\partial \bar{u}_k}{\partial x_j}\big)\,\mathrm{d}x = \int_{\Omega} h_i \frac{\partial^2 u_k}{\partial x_i \partial x_j}\frac{\partial \bar{u}_k}{\partial x_j} + \frac{\partial u_k}{\partial x_j}\frac{\partial^2 \bar{u}_k}{\partial x_i \partial x_j} = I_4.$$

If we now put everything together, the right side of (4.3) reads

$$\begin{aligned}&(I_1 - I_2) + (I_{10} - I_{11}) - I_3 + I_8 - I_7\\&\quad = (I_3 + I_4 + I_6 + I_7) - (I_8 + I_9 + I_6) + I_{10} - I_{11} - I_3 + I_8 - I_7 = I_{10}.\end{aligned}$$

Noting that by the divergence theorem we have

$$I_{10} = \int_{\partial\Omega} h_k n_k |\nabla u|^2 \,\mathrm{d}\sigma,$$

the first identity is proven.

In order to prove identity (4.4), we show that the expression we get from considering ((4.3) + (4.4)) holds, i.e. we show the identity

$$\begin{aligned}2\int_{\partial\Omega} h_k n_k |\nabla u|^2 \,\mathrm{d}\sigma = 2\,\mathrm{Re}\int_{\partial\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k}\Big(\frac{\partial u}{\partial \nu}\Big)_i&\\ + 2\,\mathrm{Re}\int_{\partial\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_j}\Big\{n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k}\Big\}\,\mathrm{d}\sigma&\\ + 2\,\mathrm{Re}\int_{\partial\Omega} h_k \bar{\phi}\Big\{n_i \frac{\partial u_i}{\partial x_i} - n_k \frac{\partial u_i}{\partial x_i}\Big\}\,\mathrm{d}\sigma.&\end{aligned}$$

To this end, note that the left side of the identity equals $2I_10$, whereas the right side can be written as

$$I_1 - I_2 + 2I_{10} - I_1 + I_2 - 0,$$

where we used the fact that $\operatorname{div} u = 0$. □

We note that the operators

$$\Big\{n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k}\Big\} \quad \text{and} \quad \Big\{n_i \frac{\partial u_i}{\partial x_k} - n_k \frac{\partial u_i}{\partial x_i}\Big\}$$

are the *first-order tangential derivative operators* which can be found in Mitrea and Wrigth [**?**].

We make a quick detour that gives us the following lemma.

lem:lambdaIneq

Lemma 4.3. *Let $\theta \in (0, \pi/2)$. Then there exists α depending only on θ such that*

$$\mathrm{Re}(\lambda) + \alpha|\mathrm{Im}(\lambda)| \geq |\lambda|$$

for all $\lambda \in \Sigma_\theta$.

Proof. For $|\lambda| = 1$ we have $\mathrm{Re}(\lambda) = \cos(\varphi)$ and $\mathrm{Im}(\lambda) = \sin(\varphi)$ for some $\varphi \in (0, \pi - \theta)$. Set

$$\alpha = \frac{1-\cos(\pi-\theta)}{\sin(\pi-\theta)} \geq \frac{1-\cos(\varphi)}{\sin(\varphi)}.$$

Then we have

$$\mathrm{Re}(\lambda) + \alpha|\mathrm{Im}(\lambda)| = \cos(\varphi) + \alpha\sin(\varphi) \geq 1.$$

For arbitrary λ the claim follows by considering the normalized value $\lambda/|\lambda|$. □

The next lemma enables us to handle the solid integrals in (4.3) and (4.4).

lem:laxMilgramIneq

Lemma 4.4. *Under the same assumptions on (u, ϕ) and λ as in Theorem 4.1, we have*

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + |\lambda| \int_\Omega |u|^2 \leq C \|\frac{\partial u}{\partial \nu}\|_\partial \|u\|_\partial, \tag{4.5}$$

eq:laxMilgramIneq

where C depends only on θ.

Proof. Testing the Stokes resolvent problem against the solution u gives us

$$\int_\Omega -\Delta u \cdot \bar{u} \,\mathrm{d}x + \lambda \int_\Omega u \cdot \bar{u} \,\mathrm{d}x = \int_\Omega -\nabla\phi \cdot \bar{u} \,\mathrm{d}x$$

Using integration by parts which may, as in the proof of the previous lemma, be justified by an approximation argument, we get

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot \bar{u} \,\mathrm{d}\sigma + \lambda \int_\Omega |u|^2 \,\mathrm{d}x = \int_{\partial\Omega} \Phi n \cdot \bar{u} \,\mathrm{d}\sigma$$

or with the definition of conormal derivatives

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \lambda \int_\Omega |u|^2 \,\mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \bar{u} \,\mathrm{d}\sigma. \tag{4.6}$$

eq:testedStokes

If we now take the real and imaginary part of (4.6) and sum them up with a prefactor $\alpha > 0$, we get

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \{\mathrm{Re}(\lambda) + \alpha|\mathrm{Im}(\lambda)|\} \int_\Omega |u|^2 \,\mathrm{d}x \leq (1+\alpha)\Big|\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \bar{u} \,\mathrm{d}\sigma\Big|.$$

Lemma 4.3 now gives

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + |\lambda| \int_\Omega |u|^2 \,\mathrm{d}x \leq C\Big|\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \bar{u} \,\mathrm{d}\sigma\Big|,$$

from which we ge estimate (4.5) after applying the Cauchy-Schwartz inequality. □

The next lemma combines Rellich identities (4.3) and (4.4) with estimate (4.5).

Lemma 4.5. *Under the same assumptions on (u,ϕ) and λ as in Theorem 4.1, we have*

$$\|\nabla u\|_\partial \le C_\varepsilon \|\frac{\partial u}{\partial \nu}\|_\partial + \varepsilon\Big\{\|\nabla u\|_\partial + \|\phi\|_\partial + \||\lambda|^{1/2}u\|_\partial\Big\} \quad \texttt{eq:gradEstimateRellich} \tag{4.7}$$

and

$$\|\nabla u\|_\partial \le C_\varepsilon\Big\{\|\nabla_{\tan} u\|_\partial + \||\lambda|^{1/2}u\|_\partial\Big\} + \varepsilon\{\|\nabla u\|_\partial + \|\phi\|_\partial\} \quad \texttt{eq:gradEstimateRellich2} \tag{4.8}$$

for all $\varepsilon \in (0,1)$, where C_ε depends only on d, θ, τ, ε and the Lipschitz character of Ω.

Proof. Let $h = (h_1,\dots,h_d) \in \mathrm{C}^1_0(\mathbb{R}^d,\mathbb{R}^d)$ with $h_k n_k \ge c > 0$ on $\partial\Omega$. The existence of this vector field follows from Verchota. Now in view of identity (**??**), we have

$$\|\nabla u\|^2_\partial \le C\Big\{\|\nabla u\|_\partial \|\frac{\partial u}{\partial \nu}\|_\partial + \int_\Omega |\nabla u|^2\,\mathrm{d}x + \int_\Omega |\nabla u||\phi|\,\mathrm{d}x + |\lambda \int_\Omega |\nabla u||u|\,\mathrm{d}x\Big\}, \quad \texttt{eq:normRellich} \tag{4.9}$$

where the first term follows from the Cauchy-Schwartz inequality. Since $\Delta\phi = 0$ and the nontangential maximal function $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$ a result from Dahlberg [**?**] gives

$$\int_\Omega |\phi|^2\,\mathrm{d}x \le C\|(\phi)^*\|^2_\partial \le C\|\phi\|^2_\partial. \quad \texttt{eq:dahlbergEstimate} \tag{4.10}$$

The last summand of (4.9) can be estimated as follows

$$\begin{aligned} |\lambda|\int_\Omega |\nabla u||u|\,\mathrm{d}x &\le |\lambda|\{\frac{|\lambda|^{1/2}}{2}\int_\Omega |u|^2\,\mathrm{d}x + \frac{1}{2|\lambda|^{1/2}}\int_\Omega |\nabla u|^2\,\mathrm{d}x \\ &\le C\|\frac{\partial u}{\partial \nu}\|_\partial \||\lambda|^{1/2}u\|_\partial, \end{aligned} \quad \texttt{eq:lambdaNablaU} \tag{4.11}$$

where in the first step we used the weigthed Young inequality and in the second step we applied estimate (4.5). Similarly we calculate

$$\int_\Omega |\nabla u||\phi|\,\mathrm{d}x \le \Big(\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega |\phi|^2\,\mathrm{d}x\Big)^{1/2} \le C\|\frac{\partial u}{\partial \nu}\|^{1/2}_\partial \|u\|^{1/2}_\partial \|\phi\|_\partial, \quad \texttt{eq:nablaPhi} \tag{4.12}$$

where the first step is just the Cauchy-Schwartz inequality and the second step combines estimate (**??**) with estimate (4.10). Putting everything together, we calculate

$$\begin{aligned} \|\nabla u\|^2_\partial \le{}& C\|\nabla u\|_\partial \|\frac{\partial u}{\partial \nu}\|_\partial + C\|\frac{\partial u}{\partial \nu}\|_\partial \|u\|_\partial + C\|\frac{\partial u}{\partial \nu}\|^{1/2}_\partial \|u\|^{1/2}_\partial \|\phi\|_\partial \\ &+ C\|\frac{\partial u}{\partial \nu}\|_\partial \||\lambda|^{1/2}u\|_\partial. \end{aligned}$$

Note that we used the fact that $|\lambda| \ge \tau$ to bound $\|u\|_\partial$ as

$$\|u\|_\partial \le \frac{|\lambda|^{1/2}}{\tau^{1/2}}\|u\|_\partial = C|\lambda|^{1/2}\|u\|_\partial.$$

The desired estimate (4.7) now follows applying Young's weighted inequality and the norm equivalence on finite dimensional vector spaces.

For inequality (4.8) we use the identity (4.4) and obtain

$$\begin{aligned}\|\nabla u\|_\partial^2 \leq C\|\nabla_{\tan} u\|_\partial\{\|\nabla u\|_\partial + \|\phi\|_\partial\} + C\int_\Omega |\nabla u|^2 \,\mathrm{d}x \\ + C\int_\Omega |\nabla u||\phi| \,\mathrm{d}x + C|\lambda|\int_\Omega |\nabla u||u| \,\mathrm{d}x.\end{aligned}$$

Using estimates (4.5), (4.10), (4.11) and (4.12) together with the weighted Young inequality gives us

$$\|\nabla u\|_\partial^2 \leq C_\varepsilon\{\|\nabla_{\tan} u\|_\partial^2 + \||\lambda|^{1/2}u\|_\partial^2\} + \varepsilon\{\|\nabla u\|_\partial^2 + \|\phi\|_\partial^2 + \frac{1}{4}\|\frac{\partial u}{\partial \nu}\|_\partial^2\}.$$

The claim now follows if we use the definition of the conormal derivative and the norm equivalence on finite dimensional vector spaces. □

We prove one last lemma befor we tackle the central theorem of this chapter.

Lemma 4.6. *Assume that (u,ϕ) satisfies the same conditions as in Theorem 4.1. Then*

eq:phiDashintPhi

$$\|\phi - \fint_{\partial\Omega} \phi\|_\partial \leq C\{\|\nabla u\|_\partial + |\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega} \tag{4.13}$$

and

eq:lambdaun

$$|\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C\{\|\phi\|_\partial + \|\nabla u\|_\partial, \tag{4.14}$$

where C depends only on d and the Lipschitz character of Ω.

Proof. By Verchota's approximation argument [**?**] we may assume that $\Delta u = \nabla\phi + \lambda u$ on $\partial\Omega$. Multiplying the Stokes resolvent equation on $\partial\Omega$ with n and using the triangle inequality gives

eq:stokesEquationH1

$$\begin{aligned}\|\nabla\phi\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} &\leq \|\Delta u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + |\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}, \\ |\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} &\leq \|\Delta u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\nabla\phi\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}.\end{aligned} \tag{4.15}$$

We will now show that

eq:deltaun

$$\|\Delta u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C\|\nabla u\|_\partial \tag{4.16}$$

and

eq:nablaPhin

$$c\|\phi - \fint_{\partial\Omega} \phi \,\mathrm{d}\sigma\|_\partial \leq \|\nabla\phi\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C\|\phi\|_\partial \tag{4.17}$$

Using these two estimates applied to (4.15), we can directly derive (4.13) and (4.14).

In order to prove (4.16), note that

$$\Delta u \cdot n = n_i \frac{\partial^2 u_i}{\partial x_j^2} = \Big(n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i} \Big) \frac{\partial u_i}{\partial x_j}$$

since $\operatorname{div} u = 0$ in $\overline{\Omega}$. As the expression in between the brackets is a tangential derivative we derive estimate (4.16) from

$$|\langle \Delta u \cdot n, u \rangle| = |\langle \nabla u, \nabla_{\tan} u \rangle| \leq \|\nabla u\|_\partial^2$$

since this implies

$$\|\nabla u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq \|\nabla u\|_\partial.$$

Now for the proof of estimate (4.17) we will use L^2-estimates for the Neumann and regularity problems for the Laplace equation in Lipschitz domains. For $g \in \mathrm{L}^2(\partial\Omega)$ with mean value zero, by Jerison and Kenig [**?**] the Neumann problem for Laplace's equation on the Lipschitz domain Ω has a solution ψ with $(\nabla\psi)^* \in \mathrm{L}^2(\partial\Omega)$ and $\frac{\partial\psi}{\partial n} = g$ on $\partial\Omega$. Green's identity we have that since ϕ and ψ are harmonic

$$\begin{aligned} \Big|\int_{\partial\Omega} \phi g \, \mathrm{d}\sigma\Big| &= \Big|\int_{\partial\Omega} \phi \frac{\partial\psi}{\partial n} \mathrm{d}\sigma\Big| = \Big|\int_{\partial\Omega} \frac{\partial\phi}{\partial n} \psi \, \mathrm{d}\sigma\Big| \\ &\leq \|\frac{\partial\phi}{\partial n}\|_{\mathrm{H}^{-1}(\partial\Omega)} \|\psi\|_{\mathrm{H}^1(\partial\Omega)} \leq C \|\frac{\partial\phi}{\partial n}\|_{\mathrm{H}^{-1}(\partial\Omega)} \|g\|_\partial, \end{aligned} \quad \text{eq:dualityPhi} \tag{4.18}$$

where in the last step we used the estimate $\|\psi\|_{\mathrm{H}^1(\partial\Omega)} \leq C\|g\|_\partial$ for the L^2 Neumann problem which can be found in Jerison and Kenig [**?**]. Now if we set $\bar{g} = \phi - \tilde{\phi}$, with $\tilde{\phi} = ⨍_{\partial\Omega} \phi \, \mathrm{d}\sigma$ and use that $\int_{\partial\Omega} (\phi - \tilde{\phi})\overline{(\phi - \tilde{\phi})} \, \mathrm{d}\sigma = \int_{\partial\Omega} \phi \overline{(\phi - \tilde{\phi})} \, \mathrm{d}\sigma$, we get from (4.18)

$$\|\phi - \tilde{\phi}\|_\partial^2 \leq C \|\frac{\partial\phi}{\partial n}\|_{\mathrm{H}^{-1}(\partial\Omega)} \|\phi - \tilde{\phi}\|_\partial$$

or, after rearranging and expanding

$$\|\phi - ⨍_{\partial\Omega} \phi \, \mathrm{d}\sigma\|_\partial \leq C \|\frac{\partial\phi}{\partial n}\|_{\mathrm{H}^{-1}(\partial\Omega)}$$

We work in a similar way with results from the regularity problem of Laplace's equation by Jerison and Kenig [**?**]. Given $f \in \mathrm{H}^1(\partial\Omega)$, there exists a harmonic function ψ in Ω such that $(\nabla\psi)^* \in \mathrm{L}^2(\partial\Omega)$ and $\psi = f$ on $\partial\Omega$. As for (4.18), we calculate

$$\begin{aligned} \Big|\int_{\partial\Omega} \frac{\partial\phi}{f} \mathrm{d}\sigma\Big| &= \Big|\int_{\partial\Omega} \frac{\partial\phi}{\psi} \mathrm{d}\sigma\Big| = \Big|\int_{\partial\Omega} \phi \frac{\partial\psi}{\partial n} \mathrm{d}\sigma\Big| \\ &\leq \|\phi\|_\partial \|\nabla\psi\|_\partial \leq C\|\phi\|_\partial \|f\|_{\mathrm{H}^1(\partial\Omega)}, \end{aligned}$$

where in the last step we used the estimate $\|\nabla\psi\|_\partial \leq C\|f\|_{\mathrm{H}^1(\partial\Omega)}$ for the L^2 regularity problem. By duality this gives that

$$\|\frac{\partial\phi}{\partial n}\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C\|\phi\|_\partial.$$

□

rem:harmonicEstimate

Remark 4.7. A careful look at the proof of inequality (4.17) reveals that the estimate

$$c\|\phi\|_\partial \leq \|\nabla\phi\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)},$$

holds for all harmonic functions ϕ with vanishing mean on $\partial\Omega$.

After all this preparation we are now able to prove Theorem 4.1.

Proof of Theorem 4.1. For the proof of estimate (4.1), without loss of generality we can assume that $\int_{\partial\Omega}\phi\,\mathrm{d}\sigma = 0$.

Using (4.13) for the second summand in (4.1) and and then (4.8) for the terms involving ∇u we get

$$\begin{aligned}\|\nabla u\|_\partial + \|\phi\|_\partial &\leq C\{\|\nabla u\|_\partial + |\lambda|\|u\cdot n\|_{\mathrm{H}^1(\partial\Omega)}\}\\ &\leq C_\varepsilon\Big\{\|\nabla_{\tan}u\|_\partial + |\lambda|^{1/2}\|u\|_\partial + |\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}\Big\}\\ &\quad + C\varepsilon\{\|\nabla u\|_\partial + \|\phi\|_\partial\}\end{aligned}$$

for all $\varepsilon \in (0,1)$. Chosing ε such that $C\varepsilon < (1/2)$ we can rearrange the above inequality and obtain estimate (4.1).

Estimate (4.2) will need more effort to be proven. We start with inequality (4.14) and derive

$$\|\nabla u\|_\partial + \|\phi\|_\partial + |\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C\{\|\nabla u\|_\partial + \|\phi\|_\partial\} \leq C\Big\{\|\frac{\partial u}{\partial\nu}\|_\partial + \|\nabla u\|_\partial\Big\},$$

where in the last step we used the definition of conormal derivatives. If we now apply (4.7) we get

$$\|\nabla u\|_\partial + \|\phi\|_\partial + |\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C_\varepsilon\|\frac{\partial u}{\partial\nu}\|_\partial + \varepsilon\big\{\|\nabla u\|_\partial + \|\phi\|_\partial + \||\lambda|^{1/2}u\|_\partial\big\}$$

for all $\varepsilon \in (0,1)$. Choosing ε appropriately yields

eq:partOfRellich2

$$\|\nabla u\|_\partial + \|\phi\|_\partial + |\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C\|\frac{\partial u}{\partial\nu}\|_\partial + C|\lambda|^{1/2}\|u\|_\partial. \tag{4.19}$$

Now we need to estimate $|\lambda|^{1/2}\|u\|_\partial$. Green's identity yields

eq:hknkgreen

$$\int_{\partial\Omega} h_k n_k |u|^2 \,\mathrm{d}\sigma = \int_\Omega \frac{\partial}{\partial x_k}\big(h_k|u|^2\big)\,\mathrm{d}x = \int_\Omega \frac{\partial h_k}{\partial x_k}|u|^2\,\mathrm{d}x + \int_\Omega h_k \frac{\partial |u|^2}{\partial x_k}\,\mathrm{d}x$$
$$= \int_\Omega \operatorname{div}(h)|u|^2\,\mathrm{d}x + 2\operatorname{Re}\int_\Omega h_k \frac{\partial \bar{u}_i}{\partial x_k} u_i \,\mathrm{d}x. \tag{4.20}$$

We choose $h \in \mathrm{C}_0^1(\mathbb{R}^d,\mathbb{R}^d)$ with $h_k n_k \geq c > 0$ on $\partial\Omega$. The existence of such a function h was proven by Verchota [**?**]. Using this, we can continue the estimate (4.20) as

eq:estupartial

$$\|u\|_\partial^2 \leq C\int_\Omega |u|^2\,\mathrm{d}x + C\int_\Omega |u||\nabla u|\,\mathrm{d}x. \tag{4.21}$$

The next estimate uses (4.21) and (4.5) which gives

$$\begin{aligned}
|\lambda|\|u\|_\partial^2 &\leq |\lambda| C\int_\Omega |u|^2\,\mathrm{d}x + |\lambda| C\int_\Omega |u||\nabla u|\,\mathrm{d}x\\
&\leq C\|\frac{\partial u}{\partial \nu}\|_\partial \|u\|_\partial + |\lambda|^{1/2} C\int_\Omega (|\lambda|^{1/2}|u|)|\nabla u|\,\mathrm{d}x\\
&\leq C\|\frac{\partial u}{\partial \nu}\|_\partial \|u\|_\partial + |\lambda|^{1/2} C\Big(\int_\Omega |\lambda||u|^2\Big)^{1/2}\Big(\int_\Omega (|\nabla u|^2\,\mathrm{d}x\Big)^{1/2}\\
&\leq C\|\frac{\partial u}{\partial \nu}\|_\partial \||\lambda|^{1/2} u\|_\partial.
\end{aligned}$$

Note that for the last estimate we also used the fact that $|\lambda| \geq \tau$ helps us to bound $\|u\|_\partial$ by $C|\lambda|^{1/2}\|u\|_\partial$. Rearranging terms in the last estimate, we now derive

eq:lambda12u

$$\||\lambda|^{1/2} u\|_\partial \leq C\|\frac{\partial u}{\partial \nu}\|_\partial. \tag{4.22}$$

Estimate (4.2) follows directly from (4.19) in combination with (4.22) and this concludes our proof. □

Shen proved that under reasonable assumptions a theorem similar to 4.1 also holds for exterior domains. It is important to note that in the case $d = 2$ solutions u that are given as a single layer potential do not fulfill the stated requirements on the decay.

thm:rellichExterior

Theorem 4.8. *Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0,1)$. Let (u,ϕ) be a solution of the Stokes resolvent Problem in $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$. Suppose additionally that $(\nabla u)^*$, $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$ and that ∇u, ϕ have nontangential limits almost everywhere on $\partial\Omega$. Furthermore let for $|x| \to \infty$*

$$|\phi(x)| + |\nabla u(x)| = O(|x|^{1-d}) \quad \textit{and} \quad u(x) = \begin{cases} O(|x|^{2-d}) & \textit{if } d \geq 3\\ o(1) & \textit{if } d = 2.\end{cases}$$

Then

eq:rellich1ext

$$\|\nabla u\|_\partial + \|\phi\|_\partial \le C\big\{\|\nabla_{\tan} u\|_\partial + |\lambda|^{1/2}\|u\|_\partial + |\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}\big\} \tag{4.23}$$

and

eq:rellich2ext

$$\|\nabla u\|_\partial + |\lambda|^{1/2}\|u\|_\partial + |\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\phi\|_\partial \le C\|\frac{\partial u}{\partial \nu}\|_\partial, \tag{4.24}$$

where C depends only on d, τ, θ and the Lipschitz character of Ω.

Chapter 5

Solving the L^2-Dirichlet Problem

This section is all about the application of the method of layer potentials to solve the L^2-Dirichlet problem for the Stokes resolvent system. Furthermore we will establish a uniform L^p-estimate for the nontangential maximal function which will be important for the proof of our central theorem.

For the remainder of this chapter let Ω always denote a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$ with connected boundary. We will use $\mathrm{L}^2_n(\partial\Omega)$ to denote the function space

$$\mathrm{L}^2_n(\partial\Omega) := \Big\{ f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d) \colon \int_{\partial\Omega} f \cdot n \,\mathrm{d}\sigma = 0 \Big\},$$

and $\mathrm{L}^2_0(\partial\Omega;\mathbb{C}^d)$ to denote the function space of L^2 functions with mean value zero. As before $\|\cdot\|_\partial$ stands for the norm of $\mathrm{L}^2(\partial\Omega)$.

We will first derive bounds on the inverse operator of $(1/2)I + \mathscr{K}_\lambda$ from Chapter 3.

Lemma 5.1. *Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0,1)$. Suppose that $|\partial\Omega| = 1$. Then $(1/2)I + \mathscr{K}_\lambda$ is an isomorphism on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ and*

eq:inverseEstimate

$$\|f\|_\partial \leq C \big\| \big((1/2)I + \mathscr{K}_\lambda\big) f \big\|_\partial \quad \textit{for any } f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d), \tag{5.1}$$

where C depends only on d, θ, τ and the Lipschitz character of Ω.

Proof. We start with $f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ and the corresponding single layer potentials $u = \mathscr{S}_\lambda(f)$ and $\phi = \mathscr{S}_\Phi(f)$ given by (3.1) and (3.2). We saw in Chapter 3 that (u,ϕ) solves the Stokes resolvent problem in $\mathbb{R}^d \setminus \partial\Omega$ and got from Lemma 3.2 with $p = 2$ for the nontangential maximal functions that $(\nabla u)^*$, $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$. We furthermore saw in Lemma 3.3 that ∇u and ϕ have nontangential limits almost everywhere on $\partial\Omega$. Finally in Theorem 3.4 we saw that $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$ and derived the jump condition $\big(\frac{\partial u}{\partial \nu}\big)_\pm = (\pm(1/2)I + \mathscr{K}_\lambda) f$.

Our next step will be to show the estimate

eq:negNablaPhi

$$\|\nabla u_-\|_\partial + \|\phi_-\|_\partial \le C\|\big(\frac{\partial u}{\partial \nu}\big)\|_\partial. \tag{5.2}$$

Assuming that (5.2) holds we can prove (5.1): Set $f = \big(\frac{\partial u}{\partial \nu}\big)_+ - \big(\frac{\partial u}{\partial \nu}\big)_-$. Then this gives with the definition of the conormal derivative and estimate (5.2) that

$$\begin{aligned}\|f\|_\partial &\le \|\big(\frac{\partial u}{\partial \nu}\big)_+\|_\partial + \|\big(\frac{\partial u}{\partial \nu}\big)_-\|_\partial \\ &\le \|\big(\frac{\partial u}{\partial \nu}\big)_+\|_\partial + \|\big(\frac{\partial u}{\partial n}\big)_-\|_\partial + \|\phi_- n\|_\partial \\ &\le C\|\big(\frac{\partial u}{\partial \nu}\big)_+\|_\partial = C\|(1/2)I + K_\lambda)f\|_\partial.\end{aligned}$$

In order to prove (5.2), note that since $|u(x)| + |\nabla u(x)| = O(|x|^{-N})$ for all $N > 0$ and $\phi(x) = O(|x|^{1-d})$ as $|x| \to \infty$ we can use Theorem 4.8 to derive

eq:nablauminus

$$\begin{aligned}\|\nabla u_-\|_\partial + \|\phi_-\|_\partial &\le C\Big\{\|\nabla_{\tan} u_-\|_\partial + |\lambda|^{1/2}\|u_-\|_\partial + |\lambda|\|n\cdot u_-\|_{\mathrm{H}^{-1}(\partial\Omega)}\Big\} \\ &= C\Big\{\|\nabla_{\tan} u_+\|_\partial + |\lambda|^{1/2}\|u_+\|_\partial + |\lambda|\|n\cdot u_+\|_{\mathrm{H}^{-1}(\partial\Omega)}\Big\},\end{aligned} \tag{5.3}$$

where we used the fact that $u_+ = u_-$ and $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$ on $\partial\Omega$. Inequality (4.2) of Theorem 4.1 now allows us to estimate the right hand side of (**??**) by $C\|(\frac{\partial u}{\partial \nu})_+\|_\partial$ and thus the desired estimate (**??**) follows.

Let's now work on the invertibility of $(1/2)I + \mathscr{K}_\lambda$. In the case $\lambda = 0$, Fabes, Kenig and Verchota showed in [**?**] that $(1/2)I + \mathscr{K}_0$ as an operator on $\mathrm{L}^2(\partial\Omega;\mathbb{R}^d)$ has a one dimensional null space and as range the space $\mathrm{L}^2_0(\partial\Omega;\mathbb{R}^d)$. Thus $(1/2)I + \mathscr{K}_0$ has Fredholm index 0. This remains true if we replace $\mathrm{L}^2(\partial\Omega;\mathbb{R}^d)$ by $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ as this just corresponds to a complexification of the vector space and the operator. Since the operator $\mathscr{K}_\lambda - \mathscr{K}_0$ is compact on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ (see Toksdorf [**?**]) we deduce that the operator

$$(1/2)I + \mathscr{K}_\lambda = (1/2)I + \mathscr{K}_0 + (\mathscr{K}_\lambda - \mathscr{K}_0)$$

has the Fredholm index zero as well for all $\lambda \in \Sigma_\theta$. Now inequality (5.1) gives that $(1/2)I + \mathscr{K}_\lambda$ is injective and thus the Fredholm index of zero implies that it is also surjective and hence an isomorphism. □

The next lemma works with the counterpart of $(1/2)I + \mathscr{K}_\lambda$.

lem:inverseEstimate

Lemma 5.2. *Let $\lambda \in \Sigma_\theta$. Then $-(1/2)I + \mathscr{K}_\lambda$ is a Fredholm operator on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ with index zero and*

eq:inverseEstimate2

$$\|f\|_\partial \le C\|\big(-(1/2)I + \mathscr{K}_\lambda\big)f\|_\partial \quad \textit{for all } f \in \mathrm{L}^2_n(\partial\Omega). \tag{5.4}$$

Proof. In the case $\lambda = 0$, Fabes Kenig and Verchota proved in [?] that the Fredholm index of the operator $-(1/2)I + \mathscr{K}_0$ on $\mathrm{L}^2(\partial\Omega;\mathbb{R}^d)$ is zero and estimate (5.4) holds. As in the previous proof, this still remains true if we complexify the operator making it a Fredholm operator with index zero on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$. Since $\mathscr{K}_\lambda - \mathscr{K}_0$ is compact on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ and the Fredholm index remains unchanges under compact perturbations, we know that the Fredholm index of $-(1/2)I + \mathscr{K}_\lambda$ on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ is zero for all $\lambda \in \Sigma_\theta$. This proves the first claim of the lemma.

Now let $\tau < \frac{1}{2\,\mathrm{diam}(\Omega)^2+1}$ and $|\lambda| < \tau$. Then

$$\|(\mathscr{K}_\lambda - \mathscr{K}_0 f\|_\partial \le C|\lambda|^{1/2}\|f\|_\partial.$$

In order to prove this inequality we once again apply Young's inequality, i.e. we start by estimating

$$\|(\mathscr{K}_\lambda - \mathscr{K}_0 f\|_\partial \le \sup_{p\in\partial\Omega} \|\nabla_x\{\Gamma(p-\cdot;\lambda) - \Gamma(p-\cdot;0)\}\|_{\mathrm{L}^1(\partial\Omega)}\|f\|_{\mathrm{L}^2(\partial\Omega)}.$$

In the next step we prove that for $p \in \partial\Omega$ the integral over the gradients of Γ can be estimated independent of p. This is straightforward using Lemma **??** as Corollary 2.7 gives us

$$\begin{aligned}
&\int_{\partial\Omega} |\nabla_x\{\Gamma(p-y;\lambda) - \Gamma(p-y;0)\}|\,\mathrm{d}\sigma(y) \\
&\quad \le C|\lambda|^{1/2}\int_{\partial\Omega} \frac{1}{|p-y|^{d-2}}\,\mathrm{d}\sigma(y) \\
&\quad = |\lambda|^{1/2}C\int_{\partial\Omega\cap \mathrm{B}(p,r_0/4)} \frac{1}{|p-y|^{d-2}}\,\mathrm{d}\sigma(y) + |\lambda|^{1/2}C\int_{\partial\Omega\setminus \mathrm{B}(p,r_0/4)} \frac{1}{|p-y|^{d-2}}\,\mathrm{d}\sigma(y) \\
&\quad \le C|\lambda|^{1/2}(r_0/4 + 4^{2-d}r_0^{d-2}|\partial\Omega|),
\end{aligned}$$

where r_0 is the radius from the definition of Lipschitz domains. Note that by the choice of τ the estimate from Corollary 2.7 applies on the whole domain of integration.

For $f \in \mathrm{L}^2_n(\partial\Omega)$ we can now estimate

$$\begin{aligned}
\|f\|_\partial &\le C\|(-(1/2)I + \mathscr{K}_0)f\|_\partial \\
&\le C\|(-(1/2)I + \mathscr{K}_\lambda)f\|_\partial + \|(\mathscr{K}_\lambda - \mathscr{K}_0)f\|_\partial \\
&\le C\|(-(1/2)I + \mathscr{K}_\lambda)f\|_\partial + C|\lambda|^{1/2}\|f\|_\partial,
\end{aligned}$$

with a constant C depending only on d, θ and the Lipschitz character of $\partial\Omega$. Choosing τ even smaller allows us to rearrange the terms in the above estimate such that estimate (5.4) holds for $\lambda \in \Sigma_\theta$ and $|\lambda| < \tau$, with τ depending on d, θ and the Lipschitz character of Ω.

Now leave τ fixed and consider the case $|\lambda| \geq \tau$. This case will be handled using the Rellich estimates from Section 4. We use the facts that for $\nabla_{\tan} u$ and u the inner and outer nontangential limits coincide and apply Theorems 4.1 and 4.8 to conclude that

$$\begin{aligned}
&\|\nabla u_+\|_\partial + \|\phi_+ - \fint_{\partial\Omega} \phi_+\| \\
&\quad \leq C\{\|(\nabla_{\tan} u)_+\|_\partial + |\lambda|^{1/2}\|u_+\|_\partial + |\lambda|\|u_+ \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}\} \\
&\quad = C\{\|(\nabla_{\tan} u)_-\|_\partial + |\lambda|^{1/2}\|u_-\|_\partial + |\lambda|\|u_- \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}\} \\
&\quad \leq C\|\big(\frac{\partial u}{\partial \nu}\big)_-\|_\partial.
\end{aligned}$$

We can now use this inequality to estimate $\|\big(\frac{\partial u}{\partial \nu}\big)_+\|_\partial$ since

$$\begin{aligned}
\|\big(\frac{\partial u}{\partial \nu}\big)_+\|_\partial &\leq \|\big(\frac{\partial u}{\partial n}\big)_+\|_\partial + C\|\phi_+\|_\partial \\
&\leq C\|(\nabla u)_+\|_\partial + C\|\phi_+ - \fint_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma\|_\partial + C\big|\fint_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma\big| \\
&\leq C\|\big(\frac{\partial u}{\partial \nu}\big)_-\|_\partial + C\big|\int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma\big|
\end{aligned}$$

Considering the jump relation (3.8) and the previous estimate we get that

$$\begin{aligned}
\|f\|_\partial &\leq \|\big(\frac{\partial u}{\partial \nu}\big)_+\|_\partial + \|\big(\frac{\partial u}{\partial \nu}\big)_-\|_\partial \\
&\leq C\|\big(\frac{\partial u}{\partial \nu}\big)_-\|_\partial + C\big|\int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma\big| \\
&\leq C\|(-(1/2)I + \mathscr{K}_\lambda)f\|_\partial + C\big|\int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma\big|. \qquad \texttt{eq:estimatef}
\end{aligned} \tag{5.5}$$

We now are left with the term $\int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma$ that needs to be estimated. To this end, note that multiplying the conormal derivatives of u by n gives

$$\big(\frac{\partial u}{\partial \nu}\big)_+ \cdot n = \big(\frac{\partial u_i}{\partial x_j}\big)_+ n_i n_j - \phi_+ = n_j\big(n_i \frac{\partial}{\partial x_j} - n_j\big(\frac{\partial}{\partial x_i}\big)u_i\big)_+ - \phi_+,$$

where for the second equality we used that $\operatorname{div}(u) = 0$ in Ω and thus this also holds for the nontangential limit. This identity now implies

$$\begin{aligned}
\big|\int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma\big| &\leq \big|\int_{\partial\Omega} \big(\frac{\partial u}{\partial \nu}\big)_+ \cdot n \,\mathrm{d}\sigma\big| + C\|\nabla_{\tan} u\|_\partial \\
&\leq \big|\int_{\partial\Omega} \big(\frac{\partial u}{\partial \nu}\big)_- \cdot n \,\mathrm{d}\sigma\big| + C\|\nabla_{\tan} u\|_\partial \\
&\leq C\|\big(\frac{\partial u}{\partial \nu}\big)_-\|_\partial, \qquad \texttt{eq:estimatephiplus}
\end{aligned} \tag{5.6}$$

where in the second step, we used the jump relation to exchange $\big(\frac{\partial u}{\partial \nu}\big)_+ \cdot n$ by $\big(\frac{\partial u}{\partial \nu}\big)_- + f \cdot n$ and then used the fact $f \in \mathrm{L}^2_n(\partial\Omega)$. The third step follows from Theorem 4.8 considering that $\|\nabla_{\tan} u\|_\partial \leq C\|\nabla u\|_\partial$. Now extending estimate (5.5) by (5.6) gives

$$\|f\|_\partial \leq C\|\big(-(1/2)I+\mathscr{K}_\lambda\big)f\|_\partial + C\|\big(\frac{\partial u}{\partial \nu}\big)_-\|_\partial \leq C\|\big(-(1/2)I+\mathscr{K}_\lambda\big)f\|_\partial,$$

where we used the jump relation (3.8) again. This proves estimate (5.4) in the case $|\lambda| \geq \tau$ and thus concludes the proof. □

In the following lemma we will show the uniqueness of solutions to the L^2 Dirichlet problem to the Stokes resolvent system.

lem:l2unique

Lemma 5.3. *Let $\lambda \in \Sigma_\theta$ and (u,ϕ) be a solution to the Stokes resolvent problem in Ω. Furthermore suppose that the nontangential limit of u exists almost everywhere on $\partial\Omega$ and that $(u)^* \in \mathrm{L}^2(\partial\Omega)$. Then*

$$\int_\Omega |u|^2 \, \mathrm{d}x \leq C \int_{\partial\Omega} |u|^2 \, \mathrm{d}\sigma, \tag{5.7}$$

eq:OmegaBoundaryEstimate

where C depends only on d, θ and the Lipschitz character of Ω.

Proof. We use Verchota's approximation theorem [**?**] and approximate Ω by a sequence of smooth domains with uniform Lipschitz characters from inside. As a consequence we may assume that Ω is smooth and u, ϕ are smooth in $\overline{\Omega}$. Let $(w,\psi) \in \mathrm{H}^1_0(\Omega;\mathbb{C}^d) \times \mathrm{H}^1(\Omega)$ be a solution to the inhomogenous system

$$\begin{cases} -\Delta w + \lambda w + \nabla\psi = \bar{u} & \text{in } \Omega, \\ \operatorname{div}(w) = 0 & \text{in } \Omega. \end{cases} \tag{5.8}$$

eq:inhomogenousStokes

In fact the regularity theory for the Stokes equation gives us that w and ψ are smooth. It follows from testing (5.8) against u that

$$\int_\Omega |u|^2 \, \mathrm{d}x = \int_\Omega u \cdot \{-\Delta w + \lambda w + \nabla\psi\} \, \mathrm{d}x. \tag{5.9}$$

eq:testingInhomogenousStokes

Using one of Green's identities on the first summand and the fact that u is the solution to the Stokes resolvent problem gives that

$$\begin{aligned} \int_\Omega -u \cdot \Delta w \, \mathrm{d}x &= \int_\Omega -w \cdot \Delta u \, \mathrm{d}x - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} \, \mathrm{d}\sigma, \\ &= \int_\Omega w \cdot (-\lambda u - \nabla\phi) \, \mathrm{d}x - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} \, \mathrm{d}\sigma \\ &= \int_\Omega -\lambda w \cdot u \, \mathrm{d}x - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} \, \mathrm{d}\sigma, \end{aligned}$$

where in the last step we used partial integration and the fact that w vanishes on $\partial\Omega$ and is divergence free:

$$\int_\Omega w\cdot\nabla\phi\,\mathrm{d}x = -\int_\Omega \operatorname{div}(w)\phi\,\mathrm{d}x + \int_{\partial\Omega}\phi w\cdot n\,\mathrm{d}\sigma = 0.$$

For the third summand in (5.9) we do the same with the only difference that the second integral does not vanish. Putting everything together gives

eq:u2estimate

$$\begin{aligned}\int_\Omega |u|^2\,\mathrm{d}x &= -\int_{\partial\Omega} u\cdot\Big\{\frac{\partial w}{\partial n} - \psi n\Big\}\,\mathrm{d}\sigma\\ &\le \|u\|_\partial\{\|\nabla w\|_\partial + \|\psi\|_\partial\}\end{aligned}\tag{5.10}$$

by the Cauchy-Schwartz inequality. As the pressure ψ is only specified modulo additive constants, we may as well assume that $\int_{\partial\Omega}\psi\,\mathrm{d}\sigma = 0$. Furthermore by the Schwartz theorem we see from (5.8) that $\Delta\psi = \operatorname{div}(\bar{u}) = 0$ in Ω. As stated in Remark 4.7, this allows us to use the results from the proof of (4.17) by setting $\phi = \psi$ to conclude that

$$\|\psi\|_\partial \le C\|\nabla\psi\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}$$

and since w has vanishing trace on $\partial\Omega$ we can use that fact that (w,ψ) solves (5.8) to further estimate

eq:psiEstimate

$$\begin{aligned}&\le C\{\|\Delta w\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}\}\\ &\le C\{\|\nabla w\|_\partial + \|u\|_\partial\},\end{aligned}\tag{5.11}$$

where for the last estimate we used (4.16) which is applicable since $\operatorname{div} w = 0$ on Ω. If we combine inequalities (5.10) and (5.11), we get

eq:u2estimate2

$$\int_\Omega |u|^2\,\mathrm{d}x \le C\|u\|_\partial\|\nabla w\|_\partial + C\|u\|_\partial^2.\tag{5.12}$$

We are left with estimating the first term in (5.12). In fact it will suffice to show the following inequality

eq:w2estimate

$$\int_{\partial\Omega}|\nabla w|^2\,\mathrm{d}\sigma \le C\int_\Omega |u|^2\,\mathrm{d}x + C\int_{\partial\Omega}|u|^2\,\mathrm{d}\sigma\tag{5.13}$$

since by the weighted Young inequality this would make the estimate

$$C\|u\|_\partial\|\nabla w\|_\partial \le \frac{1}{2}\int_\Omega |u|^2\,\mathrm{d}x + C\int_{\partial\Omega}|u|^2\,\mathrm{d}\sigma$$

available which after rearranging terms yields (5.7).

To this end, we will first prove the Rellich type identity

$$\int_{\partial\Omega} h_k n_k |\nabla w|^2\,\mathrm{d}\sigma = \int_\Omega \operatorname{div}(h)|\nabla w|^2\,\mathrm{d}x + 2\operatorname{Re}\int_\Omega h_j\frac{\partial\psi}{\partial x_j}\frac{\bar{w}_j}{\partial x_k}\,\mathrm{d}x$$

$$+2\,\mathrm{Re}\int_\Omega h_k\lambda w_j\frac{\partial\bar w_j}{\partial x_k}+2\,\mathrm{Re}\int_\Omega h_k\bar u_j\frac{\partial\bar w_j}{\partial x_k}\,\mathrm{d}x. \tag{5.14}$$

eq:rellichIdentity3

Note that since all involved quantities are smooth up to the boundary, integration by parts is allowed and the proof the stated Rellich identity boils down to a formal calculation. Let $h\in C_0^1(\mathbb{R}^d,\mathbb{R}^d)$ with $h_kn_k\ge c>0$ on $\partial\Omega$, see Verchota [**?**]. Then the divergence theorem gives

$$\int_{\partial\Omega}h_kn_k|\nabla w|^2\,\mathrm{d}\sigma=\int_\Omega\mathrm{div}\,(h|\nabla w|^2)\,\mathrm{d}x=\int_\Omega\mathrm{div}\,(h)|\nabla w|^2\,\mathrm{d}x+\int_\Omega h_k\frac{\partial}{\partial x_k}\big(|\nabla w|^2\big)\,\mathrm{d}x$$

and we can rewrite the second summand as

$$\begin{aligned}\int_\Omega h_k\frac{\partial}{\partial x_k}(|\nabla w|^2)\,\mathrm{d}x&=\int_\Omega h_k\frac{\partial}{\partial x_k}\big(\frac{\partial w_j}{\partial x_i}\frac{\partial\bar w_j}{\partial x_i}\big)\,\mathrm{d}x\\&=2\,\mathrm{Re}\int_\Omega h_k\frac{\partial^2w_j}{\partial x_k\partial x_i}\frac{\partial\bar w_j}{\partial x_i}\,\mathrm{d}x\\&=-2\,\mathrm{Re}\int_\Omega h_k\frac{\partial}{\partial x_k}\big(\frac{\partial^2}{\partial x_i^2}w_j\big)\bar w_j\,\mathrm{d}x+\int_{\partial\Omega}h_k\big(\frac{\partial^2}{\partial x_k\partial x_i}w_j\big)\bar w_j\,\mathrm{d}x\\&=2\,\mathrm{Re}\int_\Omega h_k(\Delta w_j)\frac{\partial\bar w_j}{\partial x_k}\,\mathrm{d}x+0\\&=2\,\mathrm{Re}\int_\Omega h_k\big(\frac{\partial\psi}{\partial x_j}+\lambda w_j-\bar u_j\big)\frac{\partial\bar w_j}{\partial x_k}\,\mathrm{d}x,\end{aligned}$$

where in addition to partial integration we used (5.8) and the fact that $w=0$ on $\partial\Omega$. Now we can apply the triangle inequality to the Rellich type identity (5.14) to obtain

$$\begin{aligned}\int_{\partial\Omega}|\nabla w|^2\,\mathrm{d}\sigma\le C\Big\{&\int_\Omega|\nabla w|^2\,\mathrm{d}x+\int_\Omega|\nabla w||\psi|\,\mathrm{d}x\\&+|\lambda|\int_\Omega|\nabla w||w|\,\mathrm{d}x+\int_\Omega|\nabla w||u|\,\mathrm{d}x\Big\}.\end{aligned}\tag{5.15}$$

eq:rellichInequality3

The next step consists in deriving estimates which are compatible with the right hand side of (5.15). Testing (5.8) with $\bar w$, itegration by parts gives us as in the proof of Lemma 4.4

$$\int_\Omega|\nabla w|^2\,\mathrm{d}x+|\lambda|\int_\Omega|w|^2\,\mathrm{d}x\le C\int_\Omega|w||u|\,\mathrm{d}x.$$

The next step consists in using the previous inequality and the Poincaré inequality to estimate

$$\begin{aligned}\int_\Omega|\nabla w|^2\,\mathrm{d}x+(1+|\lambda|)\int_\Omega|w|^2\,\mathrm{d}x&\le(1+C)\int_\Omega|\nabla w|^2\,\mathrm{d}x+|\lambda|\int_\Omega|w|^2\,\mathrm{d}x\\&\le C\int_\Omega|w||u|\,\mathrm{d}x\\&\le C\|w\|_\partial\|u\|_\partial,\end{aligned}$$

where for the last step we used the Cauchy-Schwartz inequality. The weighted Young inequality allows us to further estimate

$$\begin{aligned} &\leq \frac{C}{4\varepsilon}\int_\Omega |u|^2\,\mathrm{d}x + C\varepsilon\int_\Omega |w|^2\,\mathrm{d}x \\ &= \frac{\tilde{C}}{1+|\lambda|}\int_\Omega |u|^2\,\mathrm{d}x + \frac{1}{2}(1+|\lambda|)\int_\Omega |w|^2\,\mathrm{d}x \end{aligned}$$

if we set $\varepsilon = \frac{(1+|\lambda|)}{2C}$. Rearranging terms, we can produce our next estimate

$$\int_\Omega |\nabla w|^2\,\mathrm{d}x + (1+|\lambda|)\int_\Omega |w|^2\,\mathrm{d}x \leq \frac{C}{1+|\lambda|}\int_\Omega |u|^2\,\mathrm{d}x. \tag{5.16}$$

eq:w2lambdaEstimate

Now it's time to harvest: Using estimate (5.15) together with (5.16) gives

$$\begin{aligned} &\int_{\partial\Omega} |\nabla w|^2\,\mathrm{d}\sigma \\ &\quad\leq C\Big\{\int_\Omega |\nabla w|^2\,\mathrm{d}x + \int_\Omega |\nabla w||\psi|\,\mathrm{d}x + |\lambda|\int_\Omega |\nabla w||w|\,\mathrm{d}x + \int_\Omega |\nabla w||u|\,\mathrm{d}x\Big\}. \end{aligned}$$

Using the weighted Young inequality, we see that we can simplify the right hand to

$$C_\varepsilon(1+|\lambda|)\int_\Omega |\nabla w|^2\,\mathrm{d}x + C|\lambda|\int_\Omega |w|^2\,\mathrm{d}x + C\int_\Omega |u|^2\,\mathrm{d}x + \varepsilon\int_\Omega |\psi|^2\,\mathrm{d}x,$$

which with (5.16) can be bounded in this way

$$\leq C_\varepsilon\int_\Omega |u|^2\,\mathrm{d}x + \varepsilon\int_\Omega |\psi|^2\,\mathrm{d}x.$$

Using the estimate $\|\psi\|_{\mathrm{L}^2(\partial\Omega)} \leq C\|\psi\|_\partial$ and inequality (5.11) gives

$$\varepsilon\int_\Omega |\psi|^2\,\mathrm{d}x \leq \varepsilon C\int_{\partial\Omega} |\nabla w|^2\,\mathrm{d}\sigma + C_\varepsilon\int_{\partial\Omega} |u|^2\,\mathrm{d}\sigma.$$

Choosing $\varepsilon = \frac{1}{2C}$ and rearranging gives the desired estimate (5.13). This concludes our proof. □

The next Theorem states the important fact that in L^2 the Dirichlet Stokes resolvent problem has a unique solution.

thm:exAndUniqueSolution

Theorem 5.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$ with connected boundary and let $\lambda \in \Sigma_\theta$. For all $g \in \mathrm{L}^2_n(\partial\Omega)$ there exists a unique u and harmonic function ϕ which is unique up to constants such that (u,ϕ) satisfies (**??**), $(u)^* \in \mathrm{L}^2(\partial\Omega)$ and $u = g$ on $\partial\Omega$ in the sense of nontangential convergence. Moreover the estimate $\|(u)^*\|_\partial \leq C\|g\|_\partial$ holds and u may be represented by the double layer potential $\mathscr{D}_\lambda(f)$ with $\|f\|_\partial \leq C\|g\|_\partial$, where C depends only on d, θ and the Lipschitz character of Ω.*

Proof. By Lemma 5.3 we already now that the problem under consideration admits at most one solution. Therefore we only have to worry about the existence of a solution. To this end we want to apply Lemma 5.2. We first note that since $T := -(1/2)I + \mathscr{K}_{\bar{\lambda}}$ is a Fredholm operator on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ with index 0 the same is true for its adjoint $T^* := (-1/2)I + \mathscr{K}_{\bar{\lambda}}^*$. We know that for all $f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ we have $\operatorname{div}(\mathscr{D}_\lambda f) = 0$ and therefore

$$\int_{\partial\Omega} T^* f \cdot n \, \mathrm{d}\sigma = \int_{\partial\Omega} u_+ \cdot n \, \mathrm{d}\sigma = 0,$$

where for the first inequality we applied Theorem 3.6. The second equality uses Verchota's approximation scheme in order to apply the divergence theorem and the fact that $(u)^*$ is integrable together with dominated convergence. This gives $\operatorname{Im}(T^*) \subseteq \mathrm{L}^2_n(\partial\Omega)$. This gives us that we have

$$\operatorname{span}(n) = \mathrm{L}^2_n(\partial\Omega)^\perp \subseteq \operatorname{Im}(T^*)^\perp = \ker(T)$$

on the one hand and on the other hand, as T is injective on $\mathrm{L}^2_n(\partial\Omega)$ by (5.4), we have that $\operatorname{span}(n) \supseteq \ker(T)$. This yields $\operatorname{span}(n) = \ker(T)$. We can use this equality and show that

$$\mathrm{L}^2_n(\partial\Omega) = \ker(T)^\perp = \overline{\operatorname{Im}(T^*)} = \operatorname{Im}(T^*),$$

where for the last equality we used the fact that the range of T^* is closed, as usual for Fredholm operators. With the same argument we can show for T that

$$\ker(T^*)^\perp = \overline{\operatorname{Im}(T)} = \operatorname{Im}(T).$$

Consequently the operator

$$T^* \colon \operatorname{Im}(T) \to \mathrm{L}^2_n(\partial\Omega)$$

is invertible by the continuous inverse theorem. Considering once again estimate (5.4) and a duality argument we get that

eq:dualityArgument

$$\|f\|_\partial \leq C \|T^* f\|_\partial \tag{5.17}$$

for all $f \in \operatorname{Im}(-(1/2)I + \mathscr{K}_\lambda)$, as

$$\|(T^*)^{-1}\|_{\mathrm{L}^2_n(\partial\Omega), R(-(1/2)I + \mathscr{K}_{\bar{\lambda}}),} = \|T^{-1}\|_{R(-(1/2)I + \mathscr{K}_{\bar{\lambda}}), \mathrm{L}^2_n(\partial\Omega)}.$$

We are now in position to derive the missing estimates which were stated in the theorem. For $g \in \mathrm{L}^2_n(\partial\Omega)$ let $f \in \operatorname{Im}(T)$ with $T^* f = g$. Furthermore let (u, ϕ) be the double layer potential defined in equations (3.11) and (3.12). Then $u_+ = T^* f = g$ on $\partial\Omega$ by Theorem 3.6. Additionally we have that

$$\|(u)^*\|_\partial \leq C\|f\|_\partial \leq C\|g\|_\partial$$

where we used inequality (3.13) and (5.17). □

The next theorem can in some sense be regarded as a reverse trace theorem and will play an important role for the proof of the needed reverse Hölder inequality.

eq:reverseTrace

Theorem 5.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$ with connected boundary. Let $u \in \mathrm{H}^1(\Omega;\mathbb{C}^d)$ and $\pi \in \mathrm{L}^2(\Omega)$ satisfy the Stokes resolvent problem in Ω for some $\lambda \in \Sigma_\theta$. Then*

$$\Big(\int_\Omega |u|^p \,\mathrm{d}x\Big)^{1/p} \leq C\Big(\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma\Big)^{1/2}, \qquad \text{eq:reverseTrace} \tag{5.18}$$

where $p = \frac{2d}{d-1}$ and C depends only on d, θ and the Lipschitz character of Ω.

Proof. Let us denote the trace of u on $\partial\Omega$ by f and let furthermore $w = \mathscr{D}(g)$, $g \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$, be the solution of the L^2 Dirichlet problem with boundary data f as given by Theorem 5.4. For the sequence $(\Omega_j)_{j\in\mathbb{N}}$ of smooth domains that approximates Ω from inside as described by Verchota [**?**] an application of Lemma 5.3 shows

$$\int_{\Omega_j} |u-w|^2 \,\mathrm{d}x \leq C\int_{\partial\Omega_j} |u-w|^2 \,\mathrm{d}\sigma, \tag{5.19}$$

where C does not depend on j but on the Lipschitz character of Ω. Now let $\varepsilon > 0$ be given. Then there exists $\varphi_\varepsilon \in \mathrm{C}^\infty(\overline{\Omega})$ such that $\|\varphi_\varepsilon - u\|^2_{\mathrm{H}^1(\Omega)} \leq \frac{\varepsilon}{3}$. Due to Verchota we know that

$$\int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \,\mathrm{d}\sigma \to \int_{\partial\Omega} |\varphi_\varepsilon - u|^2 \,\mathrm{d}\sigma, \text{ as } j \to \infty$$

since $w = f$ on $\partial\Omega$ in the sense of nontangential convergence. Therefore we choose J large enough such that for all $j \geq J$ we have

$$\int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \,\mathrm{d}\sigma \leq \frac{2}{3}\varepsilon.$$

This gives together with the trace theorem that

$$\begin{aligned}\int_{\partial\Omega_j} |u-w|^2 \,\mathrm{d}\sigma &\leq \int_{\partial\Omega_j} |u-\varphi_\varepsilon|^2 \,\mathrm{d}\sigma + \int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \,\mathrm{d}\sigma \\ &\leq C\|u-\varphi_\varepsilon\|^2_{\mathrm{H}^1(\Omega)} + \int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \,\mathrm{d}\sigma,\end{aligned}$$

where once again C only depends on the Lipschitz character of Ω and is thus independent of j. Due to Verchota [**?**] we may take j large enough such that also the second summand of the previous inequality is smaller than $\varepsilon/2$. As a consequence we get that $w = u$ in Ω. Now Theorem 5.4 gives

$$\|(u)^*\|_\partial = \|(w)^*\|_\partial \leq C\|f\|_\partial = C\|u\|_\partial,$$

where C depends on d, θ, and the Lipschitz character of Ω. The claimed inequality now follows from estimate

weizhangestimate

$$\Big(\int_\Omega |u|^p \,\mathrm{d}x\Big)^{1/p} \leq C\Big(\int_{\partial\Omega} |(u)^*|^2 \,\mathrm{d}\sigma\Big)^{1/2}, \tag{5.20}$$

where C only depends on d and the Lipschitz constant of Ω. The proof of (5.20) was carried out by Wei and Zhang in [**?**]. $\square$

Remark 5.6. Let (u,ϕ) be a solution of the Stokes resolvent system in the domain $\mathrm{B}(x_0,r) \subset \mathbb{R}^d$.

eq:interiorEstimateDoubleLayer

$$|\nabla^l u(x_0)| \leq \frac{C_l}{r^l}\Big(⨍_{\mathrm{B}(x_0,r)} |u(x)|^2 \,\mathrm{d}x\Big)^{1/2} \tag{5.21}$$

for all $l \geq 0$, where C_l only depends on d, l and θ. Without loss of generality we may rescale and translate and assume that $x_0 = 0$ and $r = 2$. Let $t \in (1,2)$. By Theorem 5.4 we know that a solution to the Stokes resolvent system on $\mathrm{B}(0,t) \subsetneq \mathrm{B}(0,2)$ with boundary values $g_t := \mathrm{Tr}_{\partial\mathrm{B}(0,t)}\, u \in \mathrm{L}^2(\partial\mathrm{B}(0,t))$ is given by a boundary layer potential $\mathscr{D}_\lambda(f_t)$. We can use this fact to derive the desired estimate via

$$\begin{aligned}
|\nabla^l u(0)|^2 &\leq C\Big(\int_{\partial\mathrm{B}(0,t)} \big\{|\nabla_x^{l+1}\Gamma(y;\lambda)| + |\nabla_x^l\Phi(y)|\big\}|f_t(y)| \,\mathrm{d}\sigma(y)\Big)^2 \\
&\leq C\Big(\int_{\partial\mathrm{B}(0,t)} \frac{|f_t(y)|}{t^{d-1+l}} \,\mathrm{d}\sigma(y)\Big)^2 \\
&\leq C\int_{\partial\mathrm{B}(0,t)} |f_t(y)|^2 \,\mathrm{d}\sigma(y) \\
&\leq C\int_{\partial\mathrm{B}(0,t)} |u(y)|^2 \,\mathrm{d}\sigma(y),
\end{aligned}$$

where we also used the Cauchy inequality in the estimate of f_t against the "data" from Theorem 5.4. Integrating this inequality in t over the interval $(1,2)$ and using the co-area formula **??** gives

$$|\nabla^l u(0)|^2 \leq C\int_{\mathrm{B}(0,2)} |u(x)|^2 \,\mathrm{d}x.$$

Now the claim follows readily.

Chapter 6

Derivation of Resolvent Estimates

In this final chapter we will prove that the Stokes semigroup is analytic on $\mathrm{L}^p_\sigma(\Omega,\mathbb{C}^d)$ for bounded Lipschitz domains $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$.

The first step will be to establich a weak reverse Hölder estimate for local solutions of the Stokes resolvent problem. We start with a similar result on Lipschitz cylinders.

lem:reverseHoelderCylinder

Lemma 6.1. *Let $\eta : \mathbb{R}^{d-1} \to \mathbb{R}$ be a Lipschitz function. Furthermore, let $u \in \mathrm{H}^1(D_\eta(r);\mathbb{C}^d)$ and $\phi \in \mathrm{L}^2(D_\eta(2r))$ solve the Stokes resolvent problem in $D_\eta(2r)$ with $u = 0$ on $I_\eta(2r)$ for some $0 < r < \infty$ and $\lambda \in \Sigma_\theta$. Let $p_d = \frac{2d}{d-1}$. Then*

eq:reverseHoelderCylinder

$$\Big(\fint_{D_\eta(r)} |u|^{p_d}\,\mathrm{d}x\Big)^{1/p_d} \leq C\Big(\fint_{D_\eta(2r)} |u|^2\,\mathrm{d}x\Big)^{1/2}, \tag{6.1}$$

where C only depends on d, M and θ.

Proof. Without loss of generality we rescale and assume that $r = 1$. Let $t \in (1,2)$. We note that by [**?**, Lemma 1.3.25] a Lipschitz cylinder is itself a Lipschitz domain. It is therefore admissible to apply Theorem 5.18 to u in $D_\eta(t)$ which yields

$$\Big(\int_{D_\eta(t)} |u|^{p_d}\,\mathrm{d}x\Big)^{2/p_d} \leq C\int_{\partial D_\eta(t)} |u|^2\,\mathrm{d}\sigma,$$

where C depends only on d, θ and the Lipschitz character of Ω. In particular C does not depend on t. Since u vanishes on $I(2)$ we have that

$$\Big(\int_{D_\eta(1)} |u|^{p}\,\mathrm{d}x\Big)^{2/p} \leq C\int_{\partial D_\eta(t)\setminus I(2)} |u|^2\,\mathrm{d}\sigma.$$

Applying the coarea formula to integrate both sides over the interval $(1,2)$ gives

$$\Big(\int_{D_\eta(1)} |u|^{p}\,\mathrm{d}x\Big)^{2/p} \leq C\int_{D_\eta(2)} |u|^2\,\mathrm{d}x.$$

Estimate (6.1) now follows after dividing by $|D_\eta(1)|$. $\square$

The next step is to extend the result to arbitrary Lipschitz domains. The following Lemma reduces the amount of work to a few special cases.

lem:ballsforballs

Lemma 6.2 (Tolksdorf)**.** *Let $\Omega \subset \mathbb{R}^d$ be Lebesgue-measurable, $f, g \in \mathrm{L}^2(\Omega)$, $\alpha_2 > \alpha_1 > 1$, $p > 2$, $r > 0$ and $x_0 \in \mathbb{R}^d$ be such that $\mathrm{B}(x_0, r) \cap \Omega \neq \emptyset$. If there exists $C > 0$ such that*

$$\Big(\frac{1}{s^d}\int_{\Omega\cap \mathrm{B}(y,s)} |f|^p \,\mathrm{d}x\Big)^{1/p} \\ \leq C\Big\{\Big(\frac{1}{s^d}\int_{\Omega\cap\alpha_1\mathrm{B}(y,s)} |f|^2\,\mathrm{d}x\Big)^{1/2} + \sup_{B'\cap \mathrm{B}(y,s)}\Big(\frac{1}{|B'|}\int_{\Omega\cap B'}|g|^2\,\mathrm{d}x\Big)^{1/2}\Big\}$$

holds for all balls $\mathrm{B}(y,s)$ with $\mathrm{B}(y,\alpha_2 s) \subset \mathrm{B}(x_0,\alpha_2 r)$ and which are either centered on $\partial\Omega$ or satisfy $\alpha_2\mathrm{B}(y,s) \subset \Omega$, then for each $\alpha \in (1,\alpha_2)$ there exists a constant C' such that

$$\Big(\frac{1}{r^d}\int_{\Omega\cap \mathrm{B}(x_0,r)} |f|^p \,\mathrm{d}x\Big)^{1/p} \\ \leq C'\Big\{\Big(\frac{1}{s^d}\int_{\Omega\cap\alpha_1\mathrm{B}(y,s)} |f|^2\,\mathrm{d}x\Big)^{1/2} + \sup_{B'\cap \mathrm{B}(y,s)}\Big(\frac{1}{|B'|}\int_{\Omega\cap B'}|g|^2\,\mathrm{d}x\Big)^{1/2}\Big\}.$$

As of now, our toolbox comprises enough tools to prove that solutions to the Stokes resolvent system satisfy a weak reverse Hölder inequality.

lem:reverseHoelder

Lemma 6.3. *Let $x_0 \in \overline{\Omega}$ and $0 < 2r < r_0$ and set $\alpha_1 = \sqrt{d^2 10^2 (1+M)^2 + 4}$ and $\alpha_2 = \alpha_1 + 1$. Let $u \in \mathrm{H}^1(\mathrm{B}(x_0,\alpha_2 r)\cap\Omega;\mathbb{C}^d)$ and $\phi \in \mathrm{L}^2(\mathrm{B}(x_0,\alpha_2 r)\cap\Omega)$ satisfy the Stokes resolvent system in $\mathrm{B}(x_0,\alpha_2 r)\cap\Omega$. If $\mathrm{B}(x_0,\alpha_2 r)\cap\partial\Omega \neq \emptyset$, we additionally assume $u = 0$ on $\mathrm{B}(x_0,\alpha_2 r)\cap\partial\Omega$. Then*

$$\Big(\fint_{\mathrm{B}(x_0,r)\cap\Omega} |u|^p\Big)^{1/p} \leq C\Big(\fint_{\mathrm{B}(x_0,2r)\cap\Omega}|u|^2\Big)^{1/2} \quad \text{eq:reverseHoelder} \tag{6.2}$$

holds, where $p = p_d$. Here, $C > 0$ only depends on d, θ and the Lipschitz character of Ω.

Proof. Due to Lemma 6.2 it suffices to consider only two cases: (1) $x_0 \in \Omega$ with $\alpha_2\mathrm{B}(x_0,r) \subset \Omega$ and (2) $x_0 \in \partial\Omega$.

Let $x_0 \in \Omega$ with $\alpha_2\mathrm{B}(x_0,r) \subset \Omega$. We may deploy the interior estimate (5.21) to derive that for all $x \in \mathrm{B}(x_0,r)$

$$|u(x)|^p \leq C\Big(\fint_{\mathrm{B}(x,r)}|u(y)|^2\,\mathrm{d}y\Big)^{p/2}$$

which after integrating x over $\mathrm{B}(x_0,r)$ yields

$$\fint_{\mathrm{B}(x_0,r)}|u(x)|^p\,\mathrm{d}x \leq C\Big(\alpha_1^d\fint_{\mathrm{B}(x_0,\alpha_1 r)}|u(z)|^2\,\mathrm{d}z\Big)^{p/2},$$

where we also used the fact that $\alpha_1 > 2$.

If $x_0 \in \partial\Omega$, then by Lemma 6.1

$$\begin{aligned}\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,r)\cap\Omega}|u|^p\Big)^{1/p} &\le \Big(\frac{1}{r^d}\int_{D_{\eta_{x_0}}(r)}|u|^p\Big)^{1/p}\\ &\le C\Big(\frac{1}{r^d}\int_{D_{\eta_{x_0}}(2r)}|u|^p\Big)^{1/p}\\ &\le C\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\alpha_1 r)\cap\Omega}|u|^2\Big)^{1/2}.\end{aligned}$$

Now the claim follows readily from an application of Lemma 6.2 with $\alpha = 2 \in (1,\alpha_2)$.

□

We note that estimate (6.2) is a weak reverse Hölder inequality and thus possesses a self-improving property, see Giaquinta and Martinazzi [**?**] or Giaquinta and Modica [**?**].

prop:giaquinta

Proposition 6.4 (Giaquinta, Modica)**.** *Let $\Omega \subset \mathbb{R}^d$ be open, $f \in \mathrm{L}^1_{\mathrm{loc}}$, $q > 1$, be a non-negative function. If there exist constants $b > 0, R_0 > 0$ such that*

$$\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,r)} f^q\,\mathrm{d}x\Big)^{1/q} \le \frac{b}{r^d}\int_{\mathrm{B}(x_0,2r)} f\,\mathrm{d}x$$

for all $x_0 \in \Omega$ and $0 < r < \min\{R_0, \mathrm{dist}(x_0,\partial\Omega)/2\}$. Then $f \in \mathrm{L}^{q+\varepsilon}_{\mathrm{loc}}(\Omega)$ for some $\varepsilon > 0$, depending only on d, q, and b and there is contant $\tilde{C}$ depending only on d, q, ε and b such that

$$\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,r)} f^{q+\varepsilon}\,\mathrm{d}x\Big)^{1/(q+\varepsilon)} \le \frac{b}{r^d}\int_{\mathrm{B}(x_0,2r)} f^q\,\mathrm{d}x$$

for all $x_0 \in \Omega$ and $0 < r < \min\{R_0, \mathrm{dist}(x_0,\partial\Omega)/2\}$.

Remark 6.5. The self-improving property of reverse Hölder estimates can now be used to make the result of Lemma 6.3 a little bit better. We are aiming to apply Proposition 6.4 for $x_0 \in \overline{\Omega}$ on the open set $\Omega \cap \mathrm{B}(x_0,r)$. Let u be as in Lemma 6.3. Then for $f = |u|^2\chi_{\mathrm{B}(x_0,\alpha_2 r)\cap\Omega}$ which can be considered as a partial extension by 0 of u to $\mathbb{R}^d$ and $q = p/2$, inequality (6.2) reads

$$\begin{aligned}\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,r)} f^q\,\mathrm{d}x\Big)^{1/q} &= \Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,r)\cap\Omega}|u|^p\,\mathrm{d}x\Big)^{2/p}\\ &\le C^2\frac{1}{r^d}\int_{\mathrm{B}(x_0,2r)\cap\Omega}|u|^2\,\mathrm{d}x\end{aligned}$$

The following extrapolation theorem will be necessary in order to derive L^p-bounds on the solution of the Stokes resolvent system, [**?**, Thm. 3.3]. Note that a more recent result from Tolksdorf [**?**] generalizes this result to operators which are defined on spaces of Banach space valued functions.

thm:extrapolation

Theorem 6.6. *Let T be a bounded sublinear operator on $\mathrm{L}^2(\Omega;\mathbb{C}^d)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and $\|T\|_{\mathfrak{L}(\mathrm{L}^2(\Omega;\mathbb{C}^d))} \leq C_0$. Let $p > 2$. Suppose that there exist constants $R_0 > 0$, $N > 1$ and $\alpha_2 > \alpha_1 > 1$ such that for any bounded measurable function f with $\operatorname{supp}(f) \subseteq \Omega \setminus \alpha_2 B$,*

$$\Big\{\frac{1}{r^d}\int_{\Omega\cap B} |Tf|^p \,\mathrm{d}x\Big\}^{1/p} \leq N\Big\{\Big(\frac{1}{r^d}\int_{\Omega\cap\alpha_1 B} |Tf|^2 \,\mathrm{d}x\Big)^{1/2} + \sup_{B'\supset B}\Big(\frac{1}{|B'|}\int_{B'} |f|^p \,\mathrm{d}x\Big)^{1/p}\Big\},$$

where $B = B(x_0,r)$ is a ball with $0 < r < R_0$ and either $x_0 \in \partial\Omega$ or $B(x_0,\alpha_2 r) \subset \Omega$. Then T is bounded on $\mathrm{L}^q(\Omega;\mathbb{C}^d)$ for any $2 < q < p$. Moreover $\|T\|_{\mathfrak{L}(\mathrm{L}^q(\Omega;\mathbb{C}^d))}$ is bounded by a constant depending at most on d, N, C_0, p, q and the Lipschitz character of Ω.

We are now in the position to prove the main theorem of this thesis. For this the weak reverse Hölder inequality derived in Lemma 6.3 will be the crucial ingredient as it enables us to apply the extrapolation theorem 6.6.

Theorem 6.7 (Shen)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$. There exists $\varepsilon > 0$, depending only on d, θ and the Lipschitz character of Ω, such that if $f \in \mathrm{L}^2(\Omega;\mathbb{C}^d) \cap \mathrm{L}^p(\Omega;\mathbb{C}^d)$ and*

$$\Big|\frac{1}{p} - \frac{1}{2}\Big| < \frac{1}{2d} + \varepsilon,$$

then the unique solution u to (1.5) in $\mathrm{H}^1_0(\Omega;\mathbb{C}^d)$ satisfies the estimate

$$\|u\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \leq \frac{C_p}{|\lambda| + r_0^{-2}} \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)},$$

where $r_0 = \operatorname{diam}(\Omega)$ and C_p depends only on d, p, θ and the Lipschitz character of Ω.

Proof. Consider a family of scaled solution operators to the Stokes resolvent system (1.5), more precisely consider the family

$$T_\lambda \colon \mathrm{L}^2(\Omega;\mathbb{C}^d) \to \mathrm{L}^2(\Omega;\mathbb{C}^d), \quad f \mapsto (|\lambda|+1)(A_2+\lambda)^{-1}\mathbb{P}_2 f,$$

where $\lambda \in \Sigma_\theta$, $\theta \in (0,\pi/2)$. Let us first verify that $u := (|\lambda|+1)^{-1}T_\lambda(f)$ does indeed solve (1.5). First note that since $\mathbb{P}_2 f \in \mathrm{L}^2_\sigma(\Omega)$ we know that by the mapping properties of the Stokes resolvent we have $u \in \mathrm{H}^1_{0,\sigma}(\Omega)$ and

$$A_2 u + \lambda u = \mathbb{P}_2 f.$$

Therefore u is a weak solution to

$$-\Delta u + \lambda u = \mathbb{P}_2 f.$$

By the usual arguments (c.f. Chapter 1), there exists a pressure $\pi \in \mathrm{L}^2(\Omega)$ such that

$$-\Delta u + \nabla \pi + \lambda u = f.$$

Furthermore by testing (**??**) with u we derive the estimate

$$\|T_\lambda(f)\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d} = (|\lambda|+1)\|u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \leq C_0\|f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)},$$

where C_0 only depends on d, θ and the Lipschitz character of Ω. Accordingly the family T_λ is bounded on $\mathrm{L}^2(\Omega;C^d)$ and C_0 is a uniform bound on the operator norms $\|T_\lambda\|_{\mathfrak{L}(\mathrm{L}^2(\Omega;\mathbb{C}^d))}$.

We will now show that the operators T_λ fulfill estimate (**??**) in Shen's extrapolation theorem, in order to deduce their L^p-boundedness. To this end let $x_0 \in \overline{\Omega}$ and $0 < 2r < r_0$ such that $\alpha_2\mathrm{B}(x_0,r) \subseteq \Omega$ or $\mathrm{B}(x_0,r)$ is centered on $\partial\Omega$. Furthermore let $f \in \mathrm{L}^\infty(\Omega;\mathbb{C}^d)$ with support in $\Omega \setminus \alpha_2\mathrm{B}(x_0,r)$. By construction (u,π) does not only solve (**??**) in Ω, the pair also solves the dirichlet problem

$$\begin{aligned} -\Delta u + \nabla\phi + \lambda u &= 0 \\ \operatorname{div}(u) &= 0 \end{aligned}$$

in $\Omega \cap \alpha_2\mathrm{B}(x_0,r)$ where $u = 0$ on $\partial\Omega \cap \alpha_2\mathrm{B}(x_0,r)$. Therefore Lemma 6.3 gives that

$$\Big(\fint_{\Omega\cap\mathrm{B}(x_0,r)} |u|^p\Big)^{1/p} \leq C\Big(\fint_{\Omega\cap\alpha_1\mathrm{B}(x_0,r)} |u|^2\Big)^{1/2},$$

where $p = p_d$

$\square$

Appendix A

For $d=2$ we have that $G(x;\lambda)=\frac{\mathrm{i}}{4}H_0^{(1)}(k|x|)$. Furthermore we set $z=k|x|$. Then applications of chain rule and product rule of differentiation give

$$
\begin{aligned}
\partial_\gamma G(x;\lambda) &= \frac{\mathrm{i}}{4}k\,\frac{x_\gamma}{|x|}\frac{\mathrm{d}}{\mathrm{d}z}H_0^{(1)}(z)\\
\partial_\alpha\partial_\gamma G(x;\lambda) &= \frac{\mathrm{i}}{4}k^2\,\frac{x_\alpha x_\gamma}{|x|^2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}H_0^{(1)}(z)+\frac{\mathrm{i}}{4}k\Big(\frac{\delta_{\alpha\gamma}}{|x|}-\frac{x_\alpha x_\gamma}{|x|^3}\Big)\frac{\mathrm{d}}{\mathrm{d}z}H_0^{(1)}(z)\\
\partial_\beta\partial_\alpha\partial_\gamma G(x;\lambda) &= \frac{\mathrm{i}}{4}k^3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^3}\frac{\mathrm{d}^3}{\mathrm{d}z^3}H_0^{(1)}(z)\\
&\quad+\frac{\mathrm{i}}{4}k^2\Big(\frac{\delta_{\beta\gamma}x_\alpha+\delta_{\alpha\gamma}x_\beta+\delta_{\alpha\beta}x_\gamma}{|x|^2}-3\frac{x_\alpha x_\beta x_\gamma}{|x|^4}\Big)\frac{\mathrm{d}^2}{\mathrm{d}z^2}H_0^{(1)}(z)\\
&\quad+\frac{\mathrm{i}}{4}k\Big(3\frac{x_\alpha x_\beta x_\gamma}{|x|^5}-\frac{\delta_{\beta\gamma}x_\alpha+\delta_{\alpha\gamma}x_\beta+\delta_{\alpha\beta}x_\gamma}{|x|^3}\Big)\frac{\mathrm{d}}{\mathrm{d}z}H_0^{(1)}(z).
\end{aligned}
$$

The series expansions for the Hankel function $H_0^{(1)}(z)$ read according to Lebedev [?]

$$
\begin{aligned}
H_0^{(1)}(z) &= J_0(z)+\mathrm{i}Y_0(z)\\
&= \frac{2\mathrm{i}}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{(l!)^2\,4^l}\,z^{2l}\Big(-\frac{\mathrm{i}\pi}{2}-\log(2)-\psi(l+1)\Big)+\frac{2\mathrm{i}}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{(l!)^2\,4^l}\,z^{2l}\log(z)\\
&= \frac{2\mathrm{i}}{\pi}\sum_{l=0}^{\infty}a_l\,z^{2l}\,C_l+\frac{2\mathrm{i}}{\pi}\sum_{l=0}^{\infty}a_l z^{2l}\log(z)\\
\frac{\mathrm{d}}{\mathrm{d}z}H_0^{(1)}(z) &= \frac{2\mathrm{i}}{\pi}\sum_{l=1}^{\infty}a_l\,(2l)\,z^{2l-1}\,C_l+\frac{2\mathrm{i}}{\pi}\sum_{l=1}^{\infty}a_l\,(2l)\,z^{2l-1}\log(z)\;+\frac{2\mathrm{i}}{\pi}\sum_{l=0}^{\infty}a_l z^{2l-1}\\
&= \frac{2\mathrm{i}}{\pi}\sum_{l=1}^{\infty}b_l\,z^{2l-1}\,C_l+\frac{2\mathrm{i}}{\pi}\sum_{l=1}^{\infty}b_l\,z^{2l-1}\log(z)\;+\frac{2\mathrm{i}}{\pi}\sum_{l=0}^{\infty}a_l\,z^{2l-1}\\
\frac{\mathrm{d}^2}{\mathrm{d}z^2}H_0^{(1)}(z) &= \frac{2\mathrm{i}}{\pi}\sum_{l=1}^{\infty}b_l\,(2l-1)\,z^{2l-2}\,C_l+\frac{2\mathrm{i}}{\pi}\sum_{l=1}^{\infty}b_l\,(2l-1)\,z^{2l-2}\log(z)+\frac{2\mathrm{i}}{\pi}\sum_{l=1}^{\infty}b_l\,z^{2l-2}\\
&\quad+\frac{2\mathrm{i}}{\pi}\sum_{l=0}^{\infty}a_l\,(2l-1)\,z^{2l-2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2\mathrm{i}}{\pi}\sum_{l=1}^{\infty} c_l\, z^{2l-2} C_l + \frac{2\mathrm{i}}{\pi}\sum_{l=1}^{\infty} c_l\, z^{2l-2}\log(z) + \frac{2\mathrm{i}}{\pi}\sum_{l=1}^{\infty} b_l\, z^{2l-2} \\
&\quad + \frac{2\mathrm{i}}{\pi}\sum_{l=0}^{\infty} a_l (2l-1)\, z^{2l-2} \\
\frac{\mathrm{d}^3}{\mathrm{d}z^3} H_0^{(1)}(z) &= \frac{2\mathrm{i}}{\pi}\sum_{l=2}^{\infty} c_l\,(2l-2)\, z^{2l-3} C_l + \frac{2\mathrm{i}}{\pi}\sum_{l=2}^{\infty} c_l\,(2l-2)\, z^{2l-3}\log(z) + \frac{2\mathrm{i}}{\pi}\sum_{l=1}^{\infty} c_l\, z^{2l-3} \\
&\quad + \frac{2\mathrm{i}}{\pi}\sum_{l=2}^{\infty} b_l (2l-2)\, z^{2l-3} + \frac{2\mathrm{i}}{\pi}\sum_{l=0}^{\infty} a_l (2l-1)(2l-2)\, z^{2l-3} \\
&= \frac{2\mathrm{i}}{\pi}\sum_{l=2}^{\infty} d_l\, z^{2l-3} C_l + \frac{2\mathrm{i}}{\pi}\sum_{l=2}^{\infty} d_l\, z^{2l-3}\log(z) + \frac{2\mathrm{i}}{\pi}\sum_{l=1}^{\infty} c_l\, z^{2l-3} \\
&\quad + \frac{2\mathrm{i}}{\pi}\sum_{l=2}^{\infty} b_l (2l-2)\, z^{2l-3} + \frac{2\mathrm{i}}{\pi}\sum_{l=0}^{\infty} a_l (2l-1)(2l-2)\, z^{2l-3},
\end{aligned}
$$

where

$$
\begin{aligned}
C_l &:= -\frac{\mathrm{i}\pi}{2} - \log(2) - \psi(l+1) \\
a_l &:= \frac{(-1)^l}{(l!)^2\, 4^l}, \quad b_l := a_l \cdot 2l, \quad c_l := b_l \cdot (2l-1), \quad d_l := c_l \cdot (2l-2).
\end{aligned}
$$

For $G(x;0) = -\frac{1}{2\pi}\log(|x|)$, we have

$$
\begin{aligned}
\partial_\gamma G(x;0) &= -\frac{1}{2\pi}\frac{x_\gamma}{|x|^2} \\
\partial_\alpha\partial_\gamma G(x;0) &= -\frac{1}{2\pi}\delta_{\alpha\gamma}\cdot\frac{1}{|x|^2} + \frac{1}{\pi}\cdot\frac{x_\alpha x_\gamma}{|x|^4} \\
\partial_\beta\partial_\alpha\partial_\gamma G(x;0) &= \frac{1}{\pi}\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^4} - \frac{4}{\pi}\frac{x_\alpha x_\beta x_\gamma}{|x|^6}.
\end{aligned}
$$

Erklärung zur Abschlussarbeit gemäß § 22 Abs. 7 und § 23 Abs. 7 APB TU Darmstadt

Hiermit versichere ich, Fabian Gabel, die vorliegende Master-Thesis gemäß § 22 Abs. 7 APB der TU Darmstadt ohne Hilfe Dritter und nur mit den angegebenen Quellen und Hilfsmitteln angefertigt zu haben. Alle Stellen, die Quellen entnommen wurden, sind als solche kenntlich gemacht worden. Diese Arbeit hat in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegen.

Mir ist bekannt, dass im Falle eines Plagiats (§ 38 Abs.2 APB) ein Täuschungsversuch vorliegt, der dazu führt, dass die Arbeit mit 5,0 bewertet und damit ein Prüfungsversuch verbraucht wird. Abschlussarbeiten dürfen nur einmal wiederholt werden.

Bei der abgegebenen Thesis stimmen die schriftliche und die zur Archivierung eingereichte elektronische Fassung gemäß § 23 Abs. 7 APB überein.

Darmstadt, Datum/Date

Unterschrift des Autors/Signature of author