



TECHNISCHE  
UNIVERSITÄT  
DARMSTADT

Fachbereich Mathematik

Masterthesis

# On Resolvent Estimates in $L^p$ for the Stokes Operator in Lipschitz Domains

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15.04.2018

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# Contents

<b>Introduction</b>	<b>3</b>
<b>1 Fundamentals</b>	<b>4</b>
1.1 Lipschitz-Domains . . . . .	4
1.2 The Stokes Operator . . . . .	9
<b>2 Estimating Fundamental Solutions</b>	<b>14</b>
2.1 Hankel Functions and the Helmholtz equation . . . . .	15
2.2 The Stokes Resolvent Problem . . . . .	22
<b>3 Single and Double Layer Potentials</b>	<b>30</b>
<b>4 Rellich Estimates</b>	<b>45</b>
<b>5 Solving the <math>L^2</math>-Dirichlet Problem</b>	<b>56</b>
<b>6 Derivation of Resolvent Estimates</b>	<b>69</b>
<b>Appendix</b>	<b>75</b>
<b>Bibliography</b>	<b>79</b>
<b>Erklärung zur Abschlussarbeit</b>	<b>81</b>

# Introduction

# Chapter 1

## Fundamentals

The purpose of this chapter is to collect basic definitions that will be used throughout the subsequent chapters. Furthermore we want to formulate the main problem regarding the resolvent estimates of the Stokes operator. Throughout this chapter we let  $d$  always denote a natural number greater or equal to 2.

### 1.1 Lipschitz-Domains

In this first section we will establish the fundamental notions regarding bounded Lipschitz domains.

defn:lipschitzDomain

**Definition 1.1.** Let  $\Omega \subset \mathbb{R}^d$  be a bounded open connected set. We call  $\Omega$  a *bounded Lipschitz domain* if there exist  $r_0, M > 0$  such that for all  $x \in \partial\Omega$  there exists a function  $\eta_x: \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  which is Lipschitz continuous and fulfills  $\eta_x(0) = 0$  and  $\|\nabla \eta_x\|_{L^\infty(\mathbb{R}^{d-1})} \leq M$ , and a rotation  $R_x: \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that for all  $0 < r \leq r_0$

$$\begin{aligned} R_x[\Omega - \{x\}] \cap D(r) &= D_{\eta_x}(r) \\ R_x[\partial\Omega - \{x\}] \cap D(r) &= I_{\eta_x}(r), \end{aligned}$$

where

$$\begin{aligned} D(r) &:= \{(x', x_d): |x'| < r, |x_d| < 10d(M+1)r\} \\ D_{\eta_x}(r) &:= \{(x', x_d): |x'| < r, \eta_x(x') < x_d < 10d(M+1)r\} \\ I_{\eta_x}(r) &:= \{(x', x_d): |x'| < r, \eta_x(x') = x_d\}. \end{aligned}$$

It is common to refer to sets of the form  $D_{\eta_x}$  as *Lipschitz cylinders*. If the point  $x$  in the definition of Lipschitz cylinders is not of particular importance we will denote the Lipschitz cylinder by  $D_\eta(r)$ .

If  $\Omega$  is a bounded Lipschitz domain,  $x \in \partial\Omega$  and  $0 < r \leq r_0$ , then we may define  $U_{x,r} := \{x\} + R_x^{-1}D(r)$ , where  $R_x$  is the rotation corresponding to  $x$  from Definition 1.1. This is all we need to define the Lipschitz character of a bounded Lipschitz domain  $\Omega$  as suggested by Pipher and Verchota in [18, Sec. 5].

**Definition 1.2.** Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ , be a bounded Lipschitz domain and  $x_1, \dots, x_n \in \partial\Omega$  be such that  $\{U_{x_i, r_0}\}_{i=1}^n$  covers  $\partial\Omega$ . Furthermore let  $M$  be the constant from Definition 1.1. Then a constant  $C > 0$  is said to depend on the *Lipschitz character* of  $\Omega$  if it depends on  $M$  and  $n$ .

That the Lipschitz character is indeed a fruitful concept will be emphasized by the following theorem. This result is a crucial ingredient in the proof of the Rellich estimates in Chapter 4 as it provides a useful approximating property of Lipschitz domains. In short it enables us to approximate a bounded Lipschitz domain  $\Omega$  by a sequence  $(\Omega_j)$  of  $C^\infty$  domains in such a way that estimates on  $\Omega_j$  with bounding constants that only depend on the Lipschitz characters may be extended to  $\Omega$  when taking the limit. The original proof of this Theorem goes back to Nečas [17] and Verchota [27]. The presented version of this theorem appeared in Brown [1].

thm:smoothApproximation

**Theorem 1.3** (Nečas, Verchota). *Let  $\Omega$  be a Lipschitz domain. Then there exists a sequence of  $C^\infty$ -domains  $(\Omega_k)$  with uniform Lipschitz characters, corresponding homeomorphisms  $\Lambda_k: \partial\Omega \rightarrow \partial\Omega_k$ , functions  $\vartheta_k: \partial\Omega \rightarrow \mathbb{R}^+$  and a smooth compactly supported vector field  $h: \mathbb{R}^d \rightarrow \mathbb{R}^d$  which satisfy the following properties:*

- i) *There exists a covering of  $\partial\Omega$  by coordinate cylinders which also serve as coordinate cylinders for  $\partial\Omega_k$ .*
- ii) *The homeomorphisms  $\Lambda_k: \partial\Omega \rightarrow \partial\Omega_k$  satisfy*

$$\sup_{Q \in \partial\Omega} |Q - \Lambda_k(Q)| \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

*and  $\Lambda_k(P)$  approaches  $P$  nontangentially meaning that for all  $k \in \mathbb{N}$*

$$|P - \Lambda_k(P)| < (1 + \beta) \text{dist}(\Lambda_k(P), \partial\Omega)$$

*for some constant  $\beta$  depending only on  $d$  and the Lipschitz character of  $\Omega$ .*

- iii) *The normals  $\nu_k$  of  $\partial\Omega_k$  satisfy  $\lim_{k \rightarrow \infty} \nu_k(\Lambda_k(P)) = \nu(P)$  a. e. for all  $P \in \partial\Omega$*

iv) The functions  $\vartheta_k$  satisfy  $\delta \leq \vartheta_k \leq \delta^{-1}$  for some  $\delta > 0$ ,  $\vartheta^k \rightarrow 1$  pointwise a. e. and

$$\int_E \vartheta_k(Q) d\sigma(Q) = \int_{\Lambda_k(E)} d\sigma_k(Q),$$

where  $E \subset \partial\Omega$  is measurable and  $\sigma_k$  denotes the surface measure on  $\Omega_k$ .

v) The vector field  $\alpha$  satisfies  $\langle \alpha, v_k \rangle \geq c > 0$  a.e. on each  $\partial\Omega_k$  where  $v_k$  denotes the unit inner normal to  $\partial\Omega_k$ .

The next concept we introduce will allow us to talk about boundary values of functions which are defined on  $\Omega$  by considering their nontangential behavior. The first step will be to define nontangential approach regions. Unfortunately, in the literature there exist at least two different concepts which will be introduced in the next definitions. In the following, by a cone we mean an open, circular, truncated cone with only one convex component.

defn:regularFamilyOfCones

**Definition 1.4** (Regular family of cones, Verchota). Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain. If  $q \in \partial\Omega$ , then  $\Gamma(q)$  will denote a cone with vertex  $q$  and one component in  $\Omega$ . Assigning to each  $q \in \partial\Omega$  one cone  $\Gamma(q)$  the family  $\{\Gamma(q) : q \in \partial\Omega\}$  will be called *regular* if there exist  $x_1, \dots, x_{n_0} \in \partial\Omega$ ,  $\tilde{r} > 0$  and rotations  $\tilde{R}_{x_1}, \dots, \tilde{R}_{x_{n_0}}$  such that

$$\partial\Omega \subset \bigcup_{i=1}^{n_0} \{x_i\} + \tilde{R}_{x_i}^{-1} D(4\tilde{r}/5),$$

and such that there exist Lipschitz continuous functions  $\tilde{\eta}_{x_i} : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  such that for all  $\tilde{r} \leq r \leq \nu\tilde{r}$  with

$$\nu := 1 + [1 + [10d(M+1)]^2]^{1/2}$$

we have

$$\begin{aligned} \tilde{R}_{x_i}[\Omega - \{x_i\}] \cap D(r) &= D_{\tilde{\eta}_{x_i}}(r) \\ \tilde{R}_{x_i}[\partial\Omega - \{x_i\}] \cap D(r) &= I_{\tilde{\eta}_{x_i}}(r). \end{aligned}$$

In addition for all  $i$  there exist cones  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$  with vertex at the origin and axis along the  $x_d$ -axis such that

$$\alpha_i \subset \overline{\beta_i} \setminus \{0\} \subset \gamma_i$$

and such that for all  $q \in [\{x_i\} + \tilde{R}_{x_i}^{-1} D(4\tilde{r}/5)] \cap \partial\Omega$ , we have

$$\begin{aligned} \tilde{R}_{x_i}^{-1} \alpha_i + \{q\} &\subset \Gamma(q) \subset \overline{\Gamma(q)} \setminus \{q\} \subset \tilde{R}_{x_i}^{-1} \beta_i + \{q\}, \\ \tilde{R}_{x_i}^{-1} \gamma_i + \{q\} &\subset [\{x_i\} + \tilde{R}_{x_i}^{-1} D(\tilde{r})] \cap \Omega. \end{aligned}$$

We will sometimes denote a regular cone as above by  $\Gamma_V(q)$ .

For the existence of such families of cones see the Appendix of Verchota [27].

In Verchota cones  $\Gamma_V(q)$  we have the properties that for all  $\Omega$  there exists a constant  $C > 0$  depending only on the Lipschitz character such that for all  $q, p \in \partial\Omega$  and any  $x \in \Gamma_V(p)$  we have that

$$|x - q| \geq C|x - p| \quad \text{eq:verCone1} \quad (1.1)$$

$$|x - q| \geq C|p - q|. \quad \text{eq:verCone2} \quad (1.2)$$

For a proof see Verchota [27, p. 9f.]

defn:nontangentialApproachRegion

**Definition 1.5** (Nontangential approach region, Shen). For  $\alpha > 1$  and  $q \in \partial\Omega$  we define

$$\Gamma_\alpha(q) := \{x \in \Omega \setminus \partial\Omega : |x - q| < \alpha \text{dist}(x, \partial\Omega)\}$$

If  $\alpha$  is chosen sufficiently large (see Shen [21]) we call  $\{\Gamma_\alpha(q) : q \in \partial\Omega\}$  a *family of nontangential approach regions*.

Note that in Shen cones  $\Gamma_\alpha(q)$ , we have that for  $q, y \in \partial\Omega$  and  $x \in \Gamma_\alpha(q)$

$$\begin{aligned} |q - y| &\leq |q - x| + |x - y| \leq \alpha \text{dist}(x, \partial\Omega) + |x - y| \\ &\leq (\alpha + 1)|x - y| \end{aligned} \quad \text{eq:shenConeEstimate} \quad (1.3)$$

where  $\alpha$  is the constant from Definition 1.5.

Depending on the type of cones used one may introduce similar concepts of nontangential convergence and nontangential maximal functions.

**Definition 1.6.** For a function  $u$  in  $\Omega$  and a fixed family of nontangential approach regions  $\{\Gamma_\alpha\}$ , we define the nontangential maximal function  $(u)_\alpha^*$  by

$$(u)_\alpha^*(q) = \sup \{|u(x)| : x \in \Gamma_\alpha(q)\} \quad \text{eq: defnNontangMaxFunction} \quad (1.4)$$

for  $q \in \partial\Omega$ . For a fixed regular family of cones  $\{\Gamma_V(q)\}$  we define the nontangential maximal function  $N(u)(q)$  via

$$N(u)(q) = \sup \{|u(x)| : x \in \Gamma_V(q)\}.$$

Note that Tolksdorf [25] and Shen [21] show that the choice of  $\alpha$  for the nontangential maximal function as in 1.4 does not affect their  $p$ -norms in an unpredictable way. In fact their  $p$ -norms for different  $\alpha_1$  and  $\alpha_2$  stay comparable with a constant only depending on  $d$ ,  $\alpha_1$ ,  $\alpha_2$  and the Lipschitz character. We will therefore for a given bounded Lipschitz domain always assume that  $\alpha > 1$  has been chosen big enough such that on the one hand

condition (ii) from Theorem 1.3 is fulfilled and that on the other hand  $\alpha$  is large enough such that  $\{\Gamma_\alpha(q) : q \in \partial\Omega\}$  is a family of nontangential approach regions. In the following we will thus ignore the parameter  $\alpha$  in cones and nontangential maximal functions and tacitly assume that it was chosen appropriately. We further note that the functions  $(u)^*$  and  $N(u)$  will not be comparable in general, see the discussion in Tolksdorf.

The above mentioned constructions of cones are not limited to cones that lay in the interior of the domain  $\Omega$ . In fact the same construction can be carried out for the exterior domain  $\mathbb{R}^d \setminus \overline{\Omega}$  yielding cones that lay outside of  $\Omega$ . While Verchota's cones from Definition 1.4 can be mirrored along the  $x_d = 0$  plane in a suitable local coordinate system, Shen's cones from Definition 1.5 have to be modified in a natural way to give cones lying inside of  $\mathbb{R}^d \setminus \overline{\Omega}$ , namely

$$\Gamma_\alpha^{\text{ext}}(q) := \{x \in \mathbb{R}^d \setminus \overline{\Omega} : |x - q| < \alpha \text{dist}(x, \partial\Omega)\}.$$

As the name *nontangential approach region* suggests, for functions  $u$  living on  $\Omega$  or  $\mathbb{R}^d \setminus \overline{\Omega}$  there will be a notion of convergence of function values  $u(x)$  as  $x$  goes to a point on  $p \in \partial\Omega$ . The idea is to restrict the set of directions from which one can approach  $p$  by only allowing sequences of points lying in cones  $\Gamma(q)$ .

**Definition 1.7** (Nontangential convergence). Let  $\Omega$  be a bounded Lipschitz domain and  $\{\Gamma(q) : q \in \partial\Omega\}$  be a family of nontangential approach regions with its exterior counterpart  $\{\Gamma^{\text{ext}}(q) : q \in \partial\Omega\}$ . Let furthermore  $u$  be a function on  $\mathbb{R}^d \setminus \partial\Omega$  and  $f$  a function on  $\partial\Omega$ . We say that  $u = f$  in the sense of nontangential convergence from the inside if

$$\lim_{\substack{x \rightarrow q \\ x \in \Gamma(q)}} u(x) = f(q), \quad \text{for a. e. } q \in \partial\Omega$$

and we say that  $u = f$  in the sense of nontangential convergence from the outside if

$$\lim_{\substack{x \rightarrow q \\ x \in \Gamma^{\text{ext}}(q)}} u(x) = f(q), \quad \text{for a. e. } q \in \partial\Omega.$$

If both of the above limits exist and coincide we say that  $u = f$  in the sense of nontangential convergence.

Usually the nontangential limits taken from inside and outside the domain will differ. For functions  $u$  on  $\mathbb{R}^d \setminus \partial\Omega$  we will therefore often use the notation  $u_+$  to denote the *inner* nontangential limit and  $u_-$  for the respective *outer* nontangential limit.



## 1.2 The Stokes Operator

sec:stokesOperator

In this section, we will introduce the Stokes operator on  $L^2(\Omega; \mathbb{C}^d)$  and  $L^p(\Omega; \mathbb{C}^d)$  for general  $p$  and establish a relation to the *Dirichlet problem for the Stokes resolvent system*

$$\begin{aligned} -\Delta u + \nabla \phi + \lambda u &= f & \text{in } \Omega, \\ \operatorname{div} u &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \quad \text{eq:stokesResolventSystem (1.5)}$$

where  $\lambda \in \Sigma_\theta := \{z \in \mathbb{C} : \lambda \neq 0 \text{ and } |\arg(z)| < \pi - \theta\}$  and  $\theta \in (0, \pi/2)$ .

We beginn by defining the relevant function spaces. Let  $\Omega \subseteq \mathbb{R}^d$  be a bounded Lipschitz domain and  $1 < p < \infty$ . We define

$$C_{c,\sigma}^\infty(\Omega) := \{\varphi \in C_c^\infty(\Omega; \mathbb{C}^d) : \operatorname{div}(\varphi) = 0\},$$

which can serve as a suitable space of test functions. We can now close this space in  $L^p(\Omega; \mathbb{C}^d)$  and the Sobolev space  $W^{1,p}(\Omega; \mathbb{C}^d)$  which gives

$$L_\sigma^p(\Omega) := \overline{C_{c,\sigma}^\infty(\Omega)}^{L^p}$$

and

$$W_{0,\sigma}^{1,p}(\Omega) := \overline{C_{c,\sigma}^\infty(\Omega)}^{W^{1,p}},$$

respectively. If  $p = 2$ , we will use the symbol  $H_{0,\sigma}^1(\Omega)$  to denote  $W_{0,\sigma}^{1,2}(\Omega)$  in order to emphasize that this space is a Hilbert space.

In order to define the Stokes operator, we introduce the following sesquilinear form

$$a : H_{0,\sigma}^1(\Omega) \times H_{0,\sigma}^1(\Omega) \rightarrow \mathbb{C}, \quad (u, v) \mapsto \int_\Omega \nabla u \cdot \overline{\nabla v} \, dx.$$

Note that for  $u \in H_{0,\sigma}^1(\Omega)$  the gradient  $\nabla u$  is a matrix and an element of the space  $L^2(\Omega; \mathbb{C}^{d \times d})$ .

defn:stokes

**Definition 1.8.** The *Stokes operator*  $A_2$  on  $L_\sigma^2(\Omega)$  is given via

$$\begin{aligned} \mathcal{D}(A_2) &:= \left\{ u \in H_{0,\sigma}^1(\Omega) : \exists! f \in L_\sigma^2(\Omega) \text{ s.t. } \forall v \in H_{0,\sigma}^1(\Omega) : a(u, v) = \int_\Omega f \cdot \bar{v} \, dx \right\} \\ A_2 u &:= f, \end{aligned}$$

where  $u \in \mathcal{D}(A_2)$  and  $f$  comes from the definition of the domain.

The following theorem from Mitrea and Monniaux [15, Thm 4.7] shows that our definition of the Stokes operator and the one used in Shen's paper [20] coincide. Another advantage of this characterization is the immediate link of the Stokes operator to the Stokes system.

thm:stokesOperatorL2

**Theorem 1.9.** *If  $\Omega \subseteq \mathbb{R}^d$ ,  $d \geq 2$ , is a bounded Lipschitz domain and  $A_2$  is the Stokes operator on  $L_\sigma^2(\Omega)$  then*

$$\mathcal{D}(A_2) = \{u \in H_{0,\sigma}^1(\Omega) : \exists \pi \in L^2(\Omega) \text{ s.t. } -\Delta u + \nabla \pi \in L_\sigma^2(\Omega)\},$$

where the expression  $\Delta u + \nabla \pi \in L_\sigma^2(\Omega)$  needs to be understood in the distributional sense. For  $u \in \mathcal{D}(A_2)$  and the corresponding pressure  $\pi$  we have

$$A_2 u = -\Delta u + \nabla \pi.$$

The following proposition summarizes some facts about the Stokes operator on  $L_\sigma^2(\Omega)$ . A proof can be found in Tolksdorf [25].

prop:stokesOperatorL2

**Proposition 1.10.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain and  $A_2$  the Stokes operator as in Definition 1.8. Then we have*

a)  $A_2$  is closed with dense domain. Furthermore  $0 \in \rho(A_2)$ .

b)  $\sigma(A) \subset [0, \infty)$  and for all  $\theta \in (0, \pi]$  there exists  $C > 0$  such that

$$\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(L_\sigma^2(\Omega))} \leq C, \quad \text{for all } \lambda \in \mathbb{C} \setminus \Sigma_\theta. \quad \text{eq:resolventEstimateL2} \quad (1.6)$$

In particular  $-A_2$  generates a bounded analytic semigroup on  $L_\sigma^2(\Omega)$ .

With these results at hand we can now give a quick recap of the solution theory to (1.5). Let  $f \in L_\sigma^2(\Omega)$  and  $\lambda \in \Sigma_\theta$ ,  $\theta \in (0, \pi/2)$ . By the previous theorem and proposition we know that there exists a unique  $u \in \mathcal{D}(A_2) \subseteq H_{0,\sigma}^1(\Omega)$  and some  $\pi \in L^2(\Omega)$  such that

$$-\Delta u + \nabla \pi + \lambda u = A_2 u + \lambda u = f.$$

For general  $f = L^2(\Omega; \mathbb{C}^d)$  we use the *Helmholtz projection*  $\mathbb{P}_2$  to get

$$\Delta u + \nabla \pi + \lambda u + (I - \mathbb{P}_2)f = \mathbb{P}_2 f + (I - \mathbb{P}_2)f = f,$$

where  $u$  and  $\pi$  now correspond to  $\mathbb{P}_2 f \in L_\sigma^2(\Omega)$ . On bounded Lipschitz domains the orthogonal complement to  $\mathbb{P}_2[L^2(\Omega; \mathbb{C}^d)] = L_\sigma^2(\Omega)$  is characterized via

$$L_\sigma^2(\Omega)^\perp = \{f \in L^2(\Omega; \mathbb{C}^d) : f = \nabla \phi, \text{ for some } \phi \in L^2(\Omega)\}.$$

A proof of this fact can be found in the book of Sohr [22, Lemma 2.5.3]. Using this result we find  $g \in L^2(\Omega)$  such that  $\nabla g = (I - \mathbb{P}_2)f$  in the sense of distributions and we see that

$$-\Delta u + \nabla(\pi + g) + \lambda u = f.$$

Consequently, we see that solving the resolvent equation for the Stokes operator and solving the Stokes resolvent system (1.5) are two sides of the same coin. Furthermore, we may deduce from the resolvent estimate (1.6) that the solution  $u$  which apparently is not affected by the additional part  $(I - \mathbb{P}_2)f$  fulfills the inequality

$$|\lambda|^{-1} \|u\|_{L^2(\Omega; \mathbb{C}^d)} = |\lambda|^{-1} \|(A_2 + \lambda)^{-1} \mathbb{P}_2 f\|_{L^2(\Omega; \mathbb{C}^d)} \leq C \|f\|_{L^2(\Omega; \mathbb{C}^d)},$$

where  $C$  depends only on  $\theta$ . By the calculations above it is understandable why this estimate on  $u$  instead of (1.6) is sometimes called *resolvent estimate*.

In order to develop an  $L^p$ -theory for system (1.5), one way is to study the Stokes operator on subspaces of  $L^p(\Omega; \mathbb{C}^d)$ . More precisely, we are interested in estimating solutions  $u \in H_0^1(\Omega; \mathbb{C}^d)$ , in  $L^p(\Omega; \mathbb{C}^d)$  provided that the right hand side of the Stokes resolvent system (1.5) is an element of the space  $L^2(\Omega; \mathbb{C}^d) \cap L^p(\Omega; \mathbb{C}^d)$ . This is once again just one side of the aforementioned coin. The other side just asks for a resolvent estimate on the Stokes operator, hoping that in analogy to Proposition 1.10 this leads to an analytic semigroup.

defn:stokeslp

**Definition 1.11.** Let  $\Omega \subseteq \mathbb{R}^d$ ,  $d \geq 2$  be a bounded Lipschitz domain. If  $p > 2$  we define the Stokes operator  $A_p$  via its *part of*  $A_2$  in  $L_\sigma^p(\Omega)$ .

$$\begin{aligned} \mathcal{D}(A_p) &:= \{u \in \mathcal{D}(A_2) \cap L_\sigma^p(\Omega) : A_2 u \in L_\sigma^p(\Omega)\} \\ A_p u &:= A_2 u, \quad u \in \mathcal{D}(A_p). \end{aligned}$$

For  $p > 2$  there exists an analog to Theorem 1.9. The peculiar range of  $p$  for which this theorem holds is due to the fact that the boundedness of the Helmholtz projection on  $L^p(\Omega; \mathbb{C}^d)$  is a crucial ingredient to the proof and a fundamental pillar of the  $L^p$ -theory of the Stokes equations. More details about the mechanics of the Helmholtz projection can be found in Tolksdorf [25, Sec. 5.1].

thm:domainStokesOperatorLp

**Theorem 1.12** (Thm. 5.2.11, [25]). *Let  $\Omega \subseteq \mathbb{R}^d$ ,  $d \geq 2$ , is a bounded Lipschitz domain. Then there exists  $\varepsilon > 0$  such that for all*

$$2 < p < \frac{2d}{d-1} + \varepsilon$$

the domain of the Stokes operator  $A_p$  is characterized as

$$\mathcal{D}(A_p) = \{u \in W_{0,\sigma}^{2,p}(\Omega) : \exists \pi \in L^p(\Omega) \text{ s.t. } -\Delta u + \nabla \pi \in L_\sigma^p(\Omega)\},$$

where the expression  $\Delta u + \nabla \pi \in L_\sigma^p(\Omega)$  needs to be understood in the distributional sense.

For  $u \in \mathcal{D}(A_p)$  and the corresponding pressure  $\pi$  we have

$$A_p u = -\Delta u + \nabla \pi.$$

For  $p < 2$  there exist various ways to define the Stokes operator. One adequate way is to dualize the operator  $A_{p'}$ , where  $p' = p/(1 + p)$  is the Hölder conjugate exponent. In order to carry out this construction we need the spaces  $L_\sigma^p(\Omega)$  to exhibit the same behavior regarding dualization as the spaces  $L^p(\Omega; \mathbb{C}^d)$ .

lem:duality

**Lemma 1.13** (Lem 5.2.13, [25]). *Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ , be a bounded Lipschitz domain. Then there exists  $\varepsilon > 0$  such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

*the spaces  $L_\sigma^p(\Omega)$  and  $(L_\sigma^{p'}(\Omega))^*$  are isomorphic, where  $(L_\sigma^{p'}(\Omega))^*$  denotes the antidual space and  $p' = p/(p-1)$  is the Hölder conjugate exponent of  $p$ . The isomorphism  $\Psi$  is given by*

$$[\Psi f](g) = \int_\Omega f \cdot \bar{g} \, dx, \quad g \in L_\sigma^{p'}(\Omega).$$

Now we define the Stokes operator for  $p < 2$  as announced.

**Definition 1.14.** Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ , be a bounded Lipschitz domain and let  $\varepsilon > 0$  be as in Lemma 1.13. Let furthermore

$$\frac{2d}{d+1} - \varepsilon < p < 2$$

and  $\Psi$  be the isomorphism from Lemma 1.13. Then the Stokes operator on  $L_\sigma^p(\Omega)$  is defined via

$$\begin{aligned} \mathcal{D}(A_p) &:= \{u \in L_\sigma^p(\Omega) : \Psi u \in \mathcal{D}(A_{p'}^*)\} \\ A_p &:= \Psi^{-1} A_{p'}^* \Psi u, \end{aligned}$$

where  $p' = p/(1 - p)$  denotes the Hölder conjugate exponent of  $p$  and  $A_{p'}^*$ , the adjoint operator to  $A_{p'}$ .

Without investing too much additional work, it is now possible to prove the following Theorem.

thm:stokesOperatorLp

**Theorem 1.15** (Thm. 5.2.9 and Thm. 5.2.17, [25]). *Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ , be a bounded Lipschitz domain. Then there exists  $\varepsilon > 0$  such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

*the operator  $A_p$  is closed with dense domain. Furthermore  $0 \in \rho(A_p)$ .*

The natural question arises when comparing Theorem 1.15 with the Hilbert space counterpart Theorem 1.10: Does the Stokes operator generate a bounded analytic semigroup in  $L^p_\sigma$ ? An affirmative answer was given by Shen in 2012 with his seminal paper [20] by proving the necessary resolvent estimates for  $d \geq 3$ .

thm:main

**Theorem 1.16** (Shen). *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^d$ ,  $d \geq 3$ . There exists  $\varepsilon > 0$ , such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

*there exists a constant  $C > 0$  such that for every  $f \in L^p_\sigma(\Omega)$  and all  $\lambda \in \Sigma_\theta$  the inequality*

$$|\lambda| \|(\lambda + A_p)^{-1}\|_{L^p(\Omega; \mathbb{C}^d)} \leq C \|f\|_{L^p(\Omega; \mathbb{C}^d)}$$

*holds. In particular  $-A_p$  is the generator of a bounded analytic semigroup on  $L^p_\sigma(\Omega)$ .*

This Theorem gave an affirmative answer to Taylor's conjecture in [23]. Curiously this positive result is limited to  $d \geq 3$  even though Shen states that the approach he developed should also work in the case  $d = 2$ . This sets the starting point for the present thesis. In the subsequent chapters we will not only present Shen's approach to the problem of the resolvent estimates, we will furthermore extend his results whenever possible to the two dimensional case.

# Chapter 2

## Estimating Fundamental Solutions

chap:2

The purpose of this section is to study fundamental solutions of the Stokes resolvent problem and to deduce related estimates which will be crucial for the next chapters. Before working on the Stokes resolvent problem we will take a look at the atoms of the fundamental solution of this problem: the Hankel functions.

As a basis for the subsequent sections and chapters let us fix recurring quantities regarding sectors in the complex plane  $\mathbb{C}$ .

Let  $\theta \in (0, \pi/2)$  and  $\lambda \in \Sigma_\theta$  as in Section 1.2. The polar form of  $\lambda$  is given as  $\lambda = r e^{i\tau}$  with  $0 < r < \infty$  and  $-\pi + \theta < \tau < \pi - \theta$ . Now set

$$k = \sqrt{r} e^{i(\pi+\tau)/2}.$$

Then

$$k^2 = -\lambda \quad \text{and} \quad \frac{\theta}{2} < \arg(k) < \pi - \frac{\theta}{2}$$

as

$$\arg(k) = \frac{\pi + \tau}{2} > \frac{\pi}{2} + \frac{-\pi + \theta}{2} = \frac{\theta}{2}$$

on the one hand and

$$< \frac{\pi}{2} + \frac{\pi - \theta}{2} = \pi - \frac{\theta}{2}$$

on the other hand. This gives rise to the following estimate

$$\operatorname{Im}(k) > \sqrt{|\lambda|} \sin(\theta/2) > 0. \quad \text{eq:imaginaryPartEstimate} \quad (2.1)$$

Indeed, we have

$$\operatorname{Im}(k) = \sqrt{r} \sin\left(\frac{\pi + \tau}{2}\right) = \sqrt{|\lambda|} \sin\left(\frac{\pi + \tau}{2}\right) \quad \text{and} \quad \frac{\theta}{2} < \frac{\pi + \tau}{2} < \pi - \frac{\theta}{2}$$

which gives for  $\tau$  with  $\frac{\pi+\tau}{2} \leq \frac{\pi}{2}$  that  $\sin(\frac{\pi+\tau}{2}) \geq \sin(\frac{\theta}{2})$  and for  $\tau$  with  $\frac{\pi+\tau}{2} > \frac{\pi}{2}$  that  $\sin(\frac{\pi+\tau}{2}) > \sin(\pi - \frac{\theta}{2}) = \sin(\frac{\theta}{2})$ .

## 2.1 Hankel Functions and the Helmholtz equation

Before diving into fundamental solutions of the Stokes resolvent problem, we will first consider a fundamental solution for the (scalar) Helmholtz equation in  $\mathbb{R}^d$

$$-\Delta u + \lambda u = 0.$$

One fundamental solution with pole at the origin is given by

$$G(x; \lambda) = \frac{i}{4(2\pi)^{\frac{d}{2}-1}} \cdot \frac{1}{|x|^{d-2}} \cdot \frac{1}{(k|x|)^{\frac{d}{2}-1}} H_{\frac{d}{2}-1}^{(1)}(k|x|), \quad \text{eq:definitionFundamentalHelmholtz} \quad (2.2)$$

see McLean [14, Eq. (9.14)], where  $H_\nu^{(1)}(z)$  is the Hankel function of the first kind which according to Lebedev [13, Sec. 5.11] can be also be written as

$$H_\nu^{(1)}(z) = \frac{2^{\nu+1} e^{i(z-\nu\pi)} z^\nu}{i\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{2zts} s^{\nu-\frac{1}{2}} (1+s)^{\nu-\frac{1}{2}} ds. \quad \text{eq:integralRepresentationHankel} \quad (2.3)$$

This formula holds for  $\nu > -\frac{1}{2}$  and  $0 < \arg(z) < \pi$ . We will usually set

$$\nu = \nu_d = \frac{d}{2} - 1 \quad \text{and} \quad z = k|x|.$$

Note that by (2.1) we will always have  $\text{Im}(z) > 0$ . Since  $\nu_d < \nu_{d+1}$  for all  $d \geq 2$  and  $\nu_2 = 0$ , formula (2.3) will hold for all dimensions  $d \geq 2$  and all  $x \in \mathbb{R}^d$ .

In the case  $d = 2$  formula (2.2) simplifies to

$$G(x; \lambda) = \frac{i}{4} H_0^{(1)}(k|x|). \quad \text{eq:2dDefinitionFundamentalHelmholtz} \quad (2.4)$$

In the case  $d = 3$  one has an even easier formula, namely

$$G(x; \lambda) = \frac{i}{4(2\pi)^{1/2}} \cdot \frac{1}{|x|} \cdot \frac{1}{(k|x|)^{1/2}} H_{1/2}^{(1)}(k|x|) = \frac{i}{4\pi|x|}, \quad \text{eq:3dDefinitionFundamentalHelmholtz} \quad (2.5)$$

which is due to an easy formula for  $H_{1/2}^{(1)}(z)$  in Lebedev, see [13, Eq. (5.8.4)] and [14, Eq. (9.15)].

Our first estimate is concerned with estimates on the fundamental solution and its derivatives for the (scalar) Helmholtz equation. The main concern of this lemma is with the asymptotic behavior of  $G(\cdot, \lambda)$  for large values of  $|x|$ .

lem:estimateHelmholtzDerivatives

**Lemma 2.1.** *Let  $\lambda \in \Sigma_\theta$ . Then*

$$|\nabla_x^l G(x; \lambda)| \leq \frac{C_l e^{-c\sqrt{|\lambda||x|}}}{|x|^{d-2+l}} \quad \text{eq:estimateHelmholtzDerivatives} \quad (2.6)$$

for any integer  $l \geq 0$  if  $d \geq 3$  and for  $l \geq 1$  if  $d = 2$ . Here,  $c > 0$  depends only on  $\theta$  and  $C_l$  depends only on  $d, l$  and  $\theta$ .

Let  $d = 2$ . Then  $|G(x; \lambda)| = o(1)$  as  $|x| \rightarrow \infty$ .

*Proof.* We start with the case  $l = 0$  and  $d \geq 3$ . Let  $\text{Im}(z) > 0$  and  $\nu - \frac{1}{2} \geq 0$ . Then (2.3) gives

$$|H_\nu^{(1)}(z)| \leq C_d e^{-\text{Im}(z)} |z|^\nu \int_0^\infty e^{-2s \text{Im}(z)} s^{\nu-\frac{1}{2}} (1+s)^{\nu-\frac{1}{2}} ds, \quad \text{eq: firstEstimate} \quad (2.7)$$

where  $C_d > 0$  depends only on  $d$ . We apply the substitution rule with  $t = s - (1/2)$  and calculate

$$\begin{aligned} e^{\frac{-\text{Im}(z)}{2}} \int_0^\infty e^{-s \text{Im}(z)} s^{\nu-\frac{1}{2}} (1+s)^{\nu-\frac{1}{2}} ds &= \int_0^\infty e^{-(s+\frac{1}{2}) \text{Im}(z)} s^{\nu-\frac{1}{2}} (1+s)^{\nu-\frac{1}{2}} ds \\ &= \int_{\frac{1}{2}}^\infty e^{-t \text{Im}(z)} \left(t^2 - \frac{1}{4}\right)^{\nu-\frac{1}{2}} dt \\ &\leq \int_0^\infty e^{-t \text{Im}(z)} t^{2\nu-1} dt \\ &= \int_0^\infty e^{-u} u^{2\nu-1} \text{Im}(z)^{1-2\nu} \text{Im}(z)^{-1} du \\ &= \text{Im}(z)^{-2\nu} \int_0^\infty e^{-u} u^{2\nu-1} du \\ &= C_\nu \text{Im}(z)^{-2\nu}, \end{aligned}$$

where we also used the substitution rule with  $u = t \text{Im}(z)$ . Now we multiply (2.7) by  $|z|^\nu$  and reuse the previous estimate to arrive at

$$|z|^\nu |H_\nu^{(1)}(z)| \leq C_d C_\nu |z|^{2\nu} |\text{Im}(z)|^{-2\nu} e^{-\frac{\text{Im}(z)}{2}},$$

which for  $z = k|x|$  gives

$$|kx|^\nu |H_\nu^{(1)}(k|x|)| \leq C \sin(\theta/2)^{-2\nu} e^{-\frac{1}{2} \sin(\theta/2) \sqrt{|\lambda|} |x|}, \quad \text{eq: zHEstimate} \quad (2.8)$$

where  $C > 0$  depends only on  $d$  and we used (2.1) to estimate

$$(|kx|)^{2\nu} \cdot |\text{Im}(k|x|)|^{-2\nu} = |\lambda|^\nu \cdot |\text{Im}(k)|^{-2\nu} \leq \sin(\theta/2)^{-2\nu}.$$

Using (2.2), we estimate for  $d \geq 3$  setting  $\nu = \frac{d}{2} - 1$

$$|G(x; \lambda)| \leq C |x|^{2-d} e^{-c \sqrt{|\lambda|} |x|}$$

and it is clear that the generic constants depends on  $d$  and  $\theta$ . This gives the estimate for  $l = 0$  and  $d \geq 3$ .

Using the relation for the derivatives of Hankel functions which one finds in the book of Lebedev [13, Eq. (5.6.3)],

$$\frac{d}{dz} \left\{ z^{-\nu} H_\nu^{(1)}(z) \right\} = -z^{-\nu} H_{\nu+1}^{(1)}(z),$$



we inductively establish the estimate (2.6) for  $l \geq 1$  and  $d \geq 2$ : For  $1 \leq j \leq d$ , we calculate using the product and chain rule

$$\begin{aligned} |\nabla_x G(x; \lambda)| &\leq C \cdot \left\{ |x|^{1-d} \cdot (k|x|)^{\frac{d}{2}-1} H_{\frac{d}{2}-1}^{(1)}(k|x|) - |x|^{2-d} (k|x|)^{\frac{d}{2}-1} H_{\frac{d}{2}}^{(1)}(k|x|) \cdot k \right\} \\ &\leq C \cdot |x|^{1-d} \left\{ (k|x|)^{\frac{d}{2}-1} H_{\frac{d}{2}-1}^{(1)}(k|x|) - (k|x|)^{\frac{d}{2}} H_{\frac{d}{2}}^{(1)}(k|x|) \right\}, \end{aligned} \quad \text{eq:derivativeOfG} \quad (2.9)$$

where  $C > 0$  is a generic constant that depends on  $d$ . Note that the first summand in (2.9) does not arise in the case  $d = 2$  as is easily seen from equation (2.4). The terms in the bracket can now be estimated individually by (2.8). The extension of this proof to orders of differentiation  $l \geq 2$  is straightforward using the Leibniz product rule for higher derivatives.

Now for the last part of the proof, let us verify the claim regarding the asymptotic behavior of  $|G(x; \lambda)|$  if  $d = 2$ . Based on the integral representation (2.3), Lebedev derived an asymptotic expansion for the Hankel function [13, Sec. 5.11, Eq. (5.11.3)]. For  $\nu = (d/2) - 1 = 0$  and  $z = k|x|$  this expansion reads

$$H_0^{(1)}(z) = \left( \frac{2}{\pi z} \right)^{1/2} e^{i(z-(1/4)\pi)} [1 + O(|z|^{-1})].$$

As  $\text{Im}(z) > 0$  we see that  $|H_0^{(1)}(k|x|)| = O(|x|^{-1/2})$ . Due to the simple structure of  $G(x; \lambda)$  for  $d = 2$  as shown in equation (2.4), the claim follows easily.  $\square$

In the derivation of the next estimates we will use the following useful interior estimate for solutions of Poisson's equation which we write down for further use.

lem:interiorEstimatePoisson

**Lemma 2.2.** *Let  $r > 0$  and  $x \in \mathbb{R}^d$ ,  $d \geq 2$ . If  $w \in C^k(B(x, r)) \cap C^0(\overline{B(x, r)})$  is a solution to  $\Delta w = f$  in  $B(x, r)$ , then*

$$|\nabla^l w(x)| \leq Cr^{-l} \sup_{B(x, r)} |w| + C \max_{0 \leq j \leq l-1} \sup_{B(x, r)} r^{j-l+2} |\nabla^j f|, \quad l \leq k, \quad \text{eq:interiorEstimatePoisson} \quad (2.10)$$

where  $C > 0$  only depends on  $d$  and  $l$ .

*Proof.* If  $l = 1$ , then estimate (2.10) is a consequence of the *comparison principle* and a proof of this fact can be found in the book of Gilbarg and Trudinger [9, Sec. 3.4, Eq. (3.16)]. We can now use this estimate to inductively deduce the estimates for higher derivatives. Note that by translating from  $x$  to 0 and rescaling like

$$u_r(x) := u(rx) \quad \text{and} \quad f_r(x) := r^2 f(rx)$$

we may assume that  $\Delta w = f$  in  $B(0, 1)$  and that it suffices to prove

$$|\nabla^l w(0)| \leq C \sup_{B(0,1)} |w| + C \max_{0 \leq j \leq l-1} \sup_{B(0,1)} |\nabla^j f|. \quad \text{eq:interiorEstimatePoissonSimple} \quad (2.11)$$

for  $l > 1$ . By the Schwartz theorem we have that if  $w$  solves Poisson's equation with right hand side  $f$  and  $w$  and  $f$  are sufficiently regular, then  $\nabla^l w$  solves Poisson's equation with right hand side  $\nabla^l f$ . We can thus estimate inductively

$$\begin{aligned} |\nabla^l w(0)| &\leq C_l \sup_{B(0,1/2^{l-1})} |\nabla^{l-1} w| + C_l \sup_{B(0,1/2^{l-1})} |\nabla^{l-1} f| \\ &\leq C_l \sup_{B(0,1/2^{l-2})} |\nabla^{l-2} w| + C_l \left\{ \sup_{B(0,1)} |\nabla^{l-2} f| + \sup_{B(0,1)} |\nabla^{l-1} f| \right\} \\ &\leq \dots \\ &\leq C_l \sup_{B(0,1)} |w| + C_l \sum_{j=0}^{l-1} \sup_{B(0,1)} |\nabla^j f| \end{aligned}$$

which readily yields the desired estimate.  $\square$

We will need the following asymptotic expansions for the function  $z^\nu H_\nu^{(1)}(z)$  in  $\mathbb{C} \setminus (-\infty, 0]$ . The derivation of these asymptotic expansions is based on asymptotic expansions of the *Bessel functions of the first and the second kind* and can be found in Tolksdorf [25, Sec. 4.2]:

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{i\pi} + \frac{i}{\pi} z^2 \log(z) + \omega z^2 + O(|z|^4 |\log(z)|) \quad \text{if } d = 4, \quad \text{eq:asymptoticd4} \quad (2.12)$$

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{i\pi} + \frac{2^\nu \Gamma(\nu-1)}{4\pi i} z^2 + \omega z^3 + O(|z|^4) \quad \text{if } d = 5, \quad \text{eq:asymptoticd5} \quad (2.13)$$

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{i\pi} + \frac{2^\nu \Gamma(\nu-1)}{4\pi i} z^2 + O(|z|^4 |\log z|) \quad \text{if } d = 6, \quad \text{eq:asymptoticd6} \quad (2.14)$$

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{i\pi} + \frac{2^\nu \Gamma(\nu-1)}{4\pi i} z^2 + O(|z|^4) \quad \text{if } d \geq 7. \quad \text{eq:asymptoticd7} \quad (2.15)$$

The next Lemma will be concerned with estimating the difference  $G(x; \lambda) - G(x; 0)$  (and derivatives of this difference) of the fundamental solution to the scalar Helmholtz equation and the fundamental solution for  $-\Delta = 0$  in  $\mathbb{R}^d$  which is given by

$$G(x; 0) := \begin{cases} -\frac{1}{2\pi} \log(|x|), & \text{for } d = 2, \\ c_d \frac{1}{|x|^{d-2}}, & \text{for } d > 2, \end{cases} \quad \text{eq:laplace} \quad (2.16)$$

where the coefficient  $c_d$  is given as as a multiple of the surface measure of the  $(d-1)$ -dimensional sphere  $\mathbb{S}^{d-1}$

$$c_d = \frac{1}{(d-2)\omega_d}, \quad \text{with} \quad \omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = |\mathbb{S}^{d-1}|. \quad \text{eq:wd} \quad (2.17)$$

Note that  $c_2 = (2\pi)^{-1}$ . By rearranging terms and using the functional equation of the Gamma function

$$\begin{aligned}\Gamma(z+1) &= z\Gamma(z), \quad z \in \mathbb{C}, \operatorname{Re}(z) > 0 \\ \Gamma(1) &= 1,\end{aligned}\tag{2.18} \quad \text{eq:functionalGamma}$$

we get

$$(d-2)\omega_d = 2\left(\frac{d}{2}-1\right)\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = 2\left(\frac{d}{2}-1\right)\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}-1)(\frac{d}{2}-1)} = \frac{4\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}-1)},$$

and thus we will also sometimes use the equivalent definition

$$c_d := \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}.\tag{2.19} \quad \text{eq:dfncd}$$

Furthermore the leading coefficient of the asymptotic expansions of the Hankel functions (2.12)-(2.15) for  $d \geq 3$  will be denoted as

$$a_d := \frac{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2}-1)}{i\pi}.\tag{2.20} \quad \text{eq:Defnad}$$

The coefficients  $a_d$  and  $c_d$  are related in the following way

$$c_d = \frac{i}{4(2\pi)^{\frac{d}{2}-1}} a_d.$$

This allows us to write for  $d \geq 3$

$$G(x; \lambda) - G(x; 0) = \frac{i}{4(2\pi)^{\frac{d}{2}-1}} \cdot \frac{1}{|x|^{d-2}} \left\{ (k|x|)^{\frac{d}{2}-1} H_{\frac{d}{2}-1}^{(1)}(k|x|) - a_d \right\}.\tag{2.21} \quad \text{eq:HelmholtzLaplaceDifference}$$

The following lemma will help us to estimate derivatives of (2.21) and the 2-dimensional counterpart.

lem:HelmholtzLaplaceDifference

**Lemma 2.3.** *Let  $\lambda \in \Sigma_\theta$ . Then*

$$\left| \nabla_x^l \{ G(x; \lambda) - G(x; 0) \} \right| \leq C |\lambda| |x|^{-\frac{d}{2}-l}.\tag{2.22} \quad \text{eq:HelmholtzLaplaceDifferenceEstimate}$$

if  $d \geq 5$  and  $l \geq 0$ , where  $C$  depends only on  $d$ ,  $l$  and  $\theta$ . If  $d = 3$  or  $4$ , estimate (2.22) holds for  $l \geq 1$  and if  $d = 2$ , the estimate holds for  $l \geq 3$ .

*Proof.* (a) In this part we will show that the desired estimate (2.22) holds if we assume that  $|\lambda||x|^2 > (1/2)$ . In this case, Lemma 2.1 gives

$$\left| \nabla_x^l \{ G(x; \lambda) - G(x; 0) \} \right| \leq C \left\{ \frac{e^{-c\sqrt{|\lambda||x|}}}{|x|^{d-2+l}} + \frac{1}{|x|^{d-2+l}} \right\} \leq C \frac{|\lambda|}{|x|^{d-4+l}},$$

where  $C$  depends only on  $d$ ,  $l$  and  $\theta$ . Therefore, for the remaining proof we will suppose  $|\lambda||x|^2 \leq (1/2)$ .

- (b) In this step, we show that we can restrict ourselves to proving (2.22) in three cases:  
 (1)  $d \geq 5$  and  $l = 0$ ; (2)  $d = 3$  or  $4$  and  $l = 1$ ; (3)  $d = 2$  and  $l = 3$ .

Suppose (2.22) holds in case (1) and let  $l > 1$ . If we set  $w(x) = G(x; \lambda) - G(x; 0)$ , we have  $\Delta_x w = \lambda G(x; \lambda)$  in  $\mathbb{R}^d \setminus \{0\}$ . For  $f = \lambda G(x; \lambda)$ , estimate (2.10) now gives

$$\begin{aligned} |\nabla^l w(x)| &\leq Cr^{-l} \sup_{B(x,r)} |w| + C \max_{0 \leq j \leq l-1} \sup_{B(x,r)} r^{j-l+2} |\nabla^j f| \\ &\leq Cr^{-l} \sup_{y \in B(x,r)} |\lambda| |y|^{4-d} + C \sum_{j=0}^{l-1} \sup_{y \in B(x,r)} r^{j-l+2} |\lambda| |y|^{2-d-j} \\ &= Cr^{-l} |\lambda| \left| x - r \frac{x}{|x|} \right|^{4-d} + C \sum_{j=0}^{l-1} r^{j-l+2} |\lambda| \left| x - r \frac{x}{|x|} \right|^{2-d-j}, \end{aligned}$$

for all  $0 < r < |x|$ , where we used (2.22) with  $l = 1$  for the first summand and (2.6) to estimate the second summand. We choose  $r = \frac{|x|}{2}$  and receive

$$\begin{aligned} |\nabla^l w(x)| &\leq C |\lambda| |x|^{-l} |x|^{4-d} + C \sum_{j=0}^{l-1} |x|^{j-l+2} |\lambda| |x|^{2-d-j} \\ &\leq C |\lambda| |x|^{4-d-l}. \end{aligned}$$

The proof for case (2) is completely analogous if one sets

$$w(x) = \nabla_x (G(x; \lambda) - G(x; 0)) \quad \text{and} \quad f(x) = \lambda \nabla_x G(x; \lambda).$$

Also case (3) is proven in a similar fashion.

- (c) In this step we prove (2.22) for  $d \geq 5$  and  $l = 0$ . First, note that for the functions

$$\begin{aligned} g(x) &:= (k|x|)^{\frac{d}{2}-1} H_{\frac{d}{2}-1}^{(1)}(k|x|), \quad g(0) = a_d, \\ h(z) &:= z^{\frac{d}{2}-1} H_{\frac{d}{2}-1}^{(1)}(z), \quad h(0) = a_d, \end{aligned}$$

the mean value theorem yields the estimate

$$|g(x) - g(0)| \leq |x| \sup_{y \in B(0, |x|)} |\nabla g(y)| \leq |x| |k| \sup_{y \in B(0, |x|)} \left| \left( \frac{d}{dz} h \right) (k|y|) \right|.$$

Using representation (2.21), we estimate

$$\begin{aligned} |G(x; \lambda) - G(x; 0)| &\leq C |x|^{2-d} \cdot |k| |x| \max_{\substack{|z| \leq |k||x| \\ \operatorname{Im}(z) > 0}} \left| \frac{d}{dz} \left\{ z^{\frac{d}{2}-1} H_{\frac{d}{2}-1}(z) \right\} \right| \\ &= C |x|^{2-d} \cdot |k| |x| \max_{\substack{|z| \leq |k||x| \\ \operatorname{Im}(z) > 0}} \left| z^{\frac{d}{2}-2} H_{\frac{d}{2}-2}(z) \right|, \end{aligned} \quad (2.23)$$

where for the last equality we used another useful relation that can be found in Lebedev [13, Eq. (5.6.3)],

$$\frac{d}{dz} \left\{ z^\nu H_\nu^{(1)}(z) \right\} = z^\nu H_{\nu-1}^{(1)}(z). \quad (2.24)$$

Since the asymptotic expansions yield that  $|z^\nu H_\nu^{(1)}(z)| \leq C_\nu$  for  $\nu > 0$  and  $|z| \leq 1$  with  $\text{Im}(z) > 0$ , it follows from (2.23) that

$$|G(x; \lambda) - G(x; 0)| \leq C|x|^{2-d} \cdot |k||x| \cdot |k||x| \max_{\substack{|z| \leq |k||x| \\ \text{Im}(z) > 0}} \left| z^{\frac{d}{2}-2} H_{\frac{d}{2}-2}^{(1)}(z) \right| \leq C|\lambda||x|^{4-d}.$$

(d) Now we consider the case  $d = 4$  and  $l = 1$ . The asymptotic expansion (2.12) gives that

$$\left| \frac{d}{dz} \left\{ \frac{zH_1^{(1)}(z) - a_4}{z^2} \right\} \right| \leq C|z|^{-1} \quad \text{eq: mwt4d (2.25)}$$

for all  $|z| \leq \frac{1}{2}$  with  $\text{Im}(z) > 0$ . Since

$$\frac{G(x; \lambda) - G(x; 0)}{\lambda} = -\frac{C(zH_1^{(1)}(z) - a_4)}{z^2},$$

where  $z = k|x|$ . With (2.25) we conclude that

$$\left| \frac{\nabla_x \{G(x; \lambda) - G(x; 0)\}}{\lambda} \right| \leq C|k| \left| \frac{d}{dz} \left\{ \frac{zH_1^{(1)}(z) - a_4}{z^2} \right\} \right|_{z=k|x|} \leq C|k||k|^{-1}|x|^{-1},$$

which after rearrangement of the involved terms gives the claim.

(e) For the case  $d = 3$  and  $l = 1$ , we get from equation (2.19) and a well known fact of the Gamma function,  $\Gamma(1/2) = \sqrt{\pi}$ , the following identity:

$$G(x; \lambda) - G(x; 0) = \frac{e^{ik|x|}}{4\pi|x|} - \frac{c_3}{|x|} = \frac{e^{ik|x|} - 1}{4\pi|x|}.$$

Now we calculate

$$\begin{aligned} \frac{\partial}{\partial x_j} \left\{ \frac{e^{ik|x|} - 1}{|x|} \right\} &= \frac{\partial}{\partial x_j} \left\{ \frac{e^{ik|x|} - 1 - ik|x|}{|x|} \right\} = \frac{\partial}{\partial x_j} \left\{ \sum_{n=2}^{\infty} \frac{(ik|x|)^n}{n!} \cdot \frac{1}{|x|} \right\} \\ &= \sum_{n=2}^{\infty} \frac{(ik)^n}{n!} (n-1) \cdot \frac{x_j}{|x|} |x|^{n-2} \end{aligned}$$

which in turn implies

$$\left| \frac{\partial}{\partial x_j} \left\{ \frac{e^{ik|x|} - 1}{|x|} \right\} \right| \leq |\lambda| \sum_{n=2}^{\infty} \frac{n-1}{n!} |k|^{n-2} |x|^{n-2} \leq C|\lambda|$$

since  $|\lambda||x| \leq (1/2)$ .

- (f) For the last case  $d = 2$  and  $l = 3$ , we will directly calculate the estimate using the asymptotic expansion of  $H_0^{(1)}(z)$  with  $z = k|x|$ . The calculations are omitted from this chapter. Instead, they can be found in the appendix of this thesis.  $\square$

rem:HelmholtzLaplaceDifference

**Remark 2.4.** In the situation of Lemma 2.3, one can show for  $|\lambda||x|^2 \leq (1/2)$  by considering the asymptotic expansions that

$$|G(x; \lambda) - G(x; 0)| \leq \begin{cases} C\sqrt{|\lambda|} & \text{if } d = 3, \\ C|\lambda|(|\log(|\lambda||x|^2)| + 1) & \text{if } d = 4. \end{cases}$$

Also using the asymptotic expansions it can be shown that if  $d = 2$ , then

$$|\nabla_x^l \{G(x; \lambda) - G(x; 0)\}| \leq C|\lambda||x|^{2-l}(|\log(|\lambda||x|^2)| + 1),$$

for  $l \in \{1, 2\}$ .

## 2.2 The Stokes Resolvent Problem

We will now analyze fundamental solutions to the *Stokes resolvent problem*

$$\begin{aligned} -\Delta u + \nabla \phi + \lambda u &= 0 & \text{eq:stokesResolventProblem} \\ \operatorname{div} u &= 0 \end{aligned} \tag{2.26}$$

in  $\mathbb{R}^d$  with  $\lambda \in \Sigma_\theta$  with the goal to deduce helpful estimates for the following chapters. The fundamental solutions to the (scalar) Helmholtz equation and the Laplace equation will form the main ingredients for the following matrix of fundamental solutions to the Stokes resolvent problem with pole at the origin:

$$\Gamma_{\alpha\beta}(x; \lambda) = G(x; \lambda)\delta_{\alpha\beta} - \frac{1}{\lambda} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \{G(x; \lambda) - G(x; 0)\}, \quad \alpha, \beta = 1, \dots, d. \tag{2.27}$$

eq:fundamentalMatrixStokes

As the matrix of fundamental solutions  $\Gamma(x; \lambda) = (\Gamma_{\alpha\beta}(x; \lambda))_{d \times d}$  carries two arguments it cannot be confused with the Gamma function. Having formula (2.27) at sight, the following observations are obvious:

$$\Gamma_{\alpha\beta}(x; \lambda) = \Gamma_{\beta\alpha}(x; \lambda), \quad \overline{\Gamma_{\alpha\beta}(x; \lambda)} = \Gamma_{\alpha\beta}(x; \bar{\lambda}) \quad \text{and} \quad \Gamma_{\alpha\beta}(x; \lambda) = \Gamma_{\alpha\beta}(-x; \lambda).$$

For the pressure, we define the vector of fundamental solutions

$$\Phi_\beta(x) = -\frac{\partial}{\partial x_\beta} \{G(x; 0)\} = \frac{x_\beta}{\omega_d |x|^d}, \quad \beta = 1, \dots, d. \tag{2.28}$$

eq:fundamentalVectorPressure

We note that  $\Phi_\beta(x) = \Phi_\beta(-x)$ .

Using the fact that  $\Delta_x G(x; \lambda) = \lambda G(x; \lambda)$  in  $\mathbb{R}^d \setminus \{0\}$ , one can see that on  $\mathbb{R}^d \setminus \{0\}$  and for all  $1 \leq \beta \leq d$

$$\begin{aligned} (-\Delta_x + \lambda)\Gamma_{\alpha\beta}(x; \lambda) + \frac{\partial}{\partial x_\alpha} \{\Phi_\beta(x)\} &= 0, \\ \frac{\partial}{\partial x_\alpha} \{\Gamma_{\alpha\beta}(x; \lambda)\} &= 0, \quad \text{for } 1 \leq \alpha \leq d. \end{aligned} \quad \begin{array}{l} \text{eq:solutionStokesSystem} \\ (2.29) \end{array}$$

Note that in the last equation the summation convention was used.

We now keep up to the spirit of this exhausting chapter by proving further estimates, this time for the fundamental solutions to the Stokes resolvent problem (2.26).

thm:fundamentalMatrixEstimate

**Theorem 2.5.** *Let  $\lambda \in \Sigma_\theta$ . Then for any  $d \geq 3$  and  $l \geq 0$*

$$|\nabla_x^l \Gamma(x; \lambda)| \leq \frac{C}{(1 + |\lambda||x|^2)|x|^{d-2+l}} \quad \begin{array}{l} \text{eq:fundamentalMatrixEstimate} \\ (2.30) \end{array}$$

where  $C$  depends only on  $d, l$  and  $\theta$ . For  $d = 2$  and  $l \geq 1$  the same estimate holds.

*Proof.* Let  $|\lambda||x|^2 > (1/2)$ . Then there exist constants  $C_a, C_b, C_c$  such that

$$\begin{aligned} e^{-c\sqrt{|\lambda||x|}}(1 + |\lambda||x|^2) &\leq C_a, \\ 1 &\leq \frac{C_b|\lambda||x|^2}{1 + |\lambda||x|^2}, \\ e^{-c\sqrt{|\lambda||x|}} &\leq \frac{C_c|\lambda||x|^2}{1 + |\lambda||x|^2}, \end{aligned}$$

where  $c$  is the constant from Lemma 2.1. Using these estimates and Lemma 2.1 gives

$$\begin{aligned} |\nabla_x^l \Gamma(x; \lambda)| &\leq |\nabla_x^l G(x; \lambda)| + \frac{1}{|\lambda|} |\nabla_x^{l+2} G(x; \lambda)| + \frac{1}{|\lambda|} |\nabla_x^{l+2} G(x; 0)| \\ &\leq \frac{C_l e^{-c\sqrt{|\lambda||x|}}}{|x|^{d-2+l}} + \frac{1}{|\lambda|} \frac{C_{l+2} e^{-c\sqrt{|\lambda||x|}}}{|x|^2 |x|^{d-2+l}} + \frac{1}{|\lambda|} \frac{C}{|x|^2 |x|^{d-2+l}} \\ &\leq \frac{C}{1 + |\lambda||x|^2} \frac{1}{|x|^{d-2+l}}. \end{aligned}$$

Now let  $|\lambda||x|^2 \leq (1/2)$ . Then by Lemma 2.1 and Lemma 2.3 we get

$$\begin{aligned} |\nabla_x^l \Gamma(x; \lambda)| &\leq |\nabla_x^l G(x; \lambda)| + \frac{1}{|\lambda|} \cdot |\nabla_x^{l+2}(G(x; \lambda) - G(x; 0))| \\ &\leq \frac{C}{|x|^{d-2+l}} + \frac{1}{|\lambda|} \cdot C|\lambda||x|^{4-d-(l+2)} \\ &\leq \frac{C}{|x|^{d-2+l}} \frac{(1 + |\lambda||x|^2)}{(1 + |\lambda||x|^2)} \\ &\leq \frac{C}{(1 + |\lambda||x|^2)|x|^{d-2+l}} \end{aligned}$$

which gives the claim.  $\square$

If  $\lambda = 0$ , the Stokes resolvent problem becomes just the Stokes problem in  $\mathbb{R}^d$

$$\begin{aligned} -\Delta u + \nabla \phi + \lambda u &= 0, \\ \operatorname{div} u &= 0. \end{aligned} \quad \text{eq:stokesProblem} \quad (2.31)$$

Whereas the fundamental solution for the pressure is maintained, the matrix of fundamental solutions to the Stokes problem in  $\mathbb{R}^d$  with pole at the origin is given by  $\Gamma(x; 0) = (\Gamma_{\alpha\beta}(x; 0))_{d \times d}$ , where

$$\Gamma_{\alpha\beta}(x; 0) := \frac{1}{2\omega_d} \left\{ \frac{\delta_{\alpha\beta}}{(d-2)|x|^{d-2}} + \frac{1}{|x|^d} \right\} \quad \text{eq:fundamentalSolutionStokes} \quad (2.32)$$

if  $d \geq 3$  and

$$\Gamma_{\alpha\beta}(x; 0) := \frac{1}{2\omega_2} \left\{ -\delta_{\alpha\beta} \log(|x|) + \frac{1}{|x|^2} \right\} \quad \text{eq:fundamentalSolutionStokes2d} \quad (2.33)$$

for  $d = 2$ . Note that the given fundamental solution for the case  $d = 2$  differs from the one given by Mitrea and Wright [16, Sec. 4.2] by having summands with alternating signs. Considering the structure of the fundamental solution for  $d \geq 3$ , our choice seems more natural with regard to the structure of the fundamental solutions to the Laplace equation (2.16). The alternating sign is necessary for  $\Gamma_{\alpha\beta}$  to be divergence free.

One important technique in the following chapter will be to reduce problems formulated for  $\Gamma(x; \lambda)$  to problems formulated in  $\Gamma(x; 0)$  perturbed by the difference  $\Gamma(x; \lambda) - \Gamma(x; 0)$ . Under this aspect it seems reasonable to study estimates of the difference of fundamental solutions. To this end it is helpful to rewrite parts of the fundamental solution. Using the fact that for  $d \geq 5$  or  $d = 3$ , we have

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left( \frac{1}{|x|^{d-4}} \right) = -(d-4) \frac{\partial}{\partial x_\alpha} \frac{x_\beta}{|x|^{d-2}} = -(d-4) \frac{\delta_{\alpha\beta}}{|x|^{d-2}} + \frac{(d-4)(d-2)x_\alpha x_\beta}{|x|^d}.$$

This allows us to write

$$\frac{x_\alpha x_\beta}{|x|^d} = \frac{\delta_{\alpha\beta}}{(d-2)|x|^{d-2}} + \frac{1}{(d-4)(d-2)} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left( \frac{1}{|x|^{d-4}} \right),$$

which, considering definition (2.32), gives

$$\Gamma_{\alpha\beta}(x; 0) = G(x; 0) \delta_{\alpha\beta} + \frac{1}{2\omega_d(d-4)(d-2)} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left( \frac{1}{|x|^{d-4}} \right). \quad \text{eq:fundamentalSdlutionStokes35} \quad (2.34)$$

A similar trick works for  $d = 4$ : Since  $\omega_4 = 2\pi^2$ , we have

$$\Gamma_{\alpha\beta}(x; 0) = \frac{1}{2\omega_4} \frac{1}{|x|^2} \delta_{\alpha\beta} - \frac{1}{8\pi^2} \left( \frac{\delta_{\alpha\beta}}{|x|^2} - \frac{2x_\alpha x_\beta}{|x|^4} \right)$$



$$\begin{aligned}
&= G(x; 0)\delta_{\alpha\beta} - \frac{1}{8\pi^2} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} (\log(|x|)) \\
&= G(x; 0)\delta_{\alpha\beta} - \frac{1}{4\omega_4} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} (\log(|x|)) \quad \text{eq: fundamentalSolutionStokes4} \quad (2.35)
\end{aligned}$$

In the case  $d = 2$ , we use

$$\frac{1}{8\pi} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} (|x|^2 \log(|x|)) = \frac{\delta_{\alpha\beta}}{4\pi} \log(|x|) + \frac{1}{4\pi} \frac{x_\alpha x_\beta}{|x|^2} + \frac{\delta_{\alpha\beta}}{8\pi}$$

to find the identity

$$\Gamma_{\alpha\beta}(x; 0) = G(x; 0)\delta_{\alpha\beta} - \frac{\delta_{\alpha\beta}}{8\pi} - \frac{\partial^2}{\partial x_\alpha \partial x_\beta} (|x|^2 \log(|x|)). \quad \text{eq: fundamentalSolutionStokes2} \quad (2.36)$$

This ends the preparatory step and brings us to the next theorem.

thm: differenceFundamentalSolutionStokes

**Theorem 2.6.** *Let  $\lambda \in \Sigma_\theta$ . Suppose that  $|\lambda||x|^2 \leq (1/2)$ . Then*

$$|\nabla_x \{\Gamma(x; \lambda) - \Gamma(x; 0)\}| \leq \begin{cases} C|\lambda||x|^{3-d} & \text{if } d \geq 7 \text{ or } d = 5, \\ C|\lambda||x|^{3-d} |\log(|\lambda||x|^2)| & \text{if } d = 4 \text{ or } 6, \\ C\sqrt{|\lambda|}|x|^{-1} & \text{if } d = 3, \\ C|\lambda||x| |\log(|\lambda||x|^2)| & \text{if } d = 2, \end{cases} \quad (2.37)$$

where  $C$  depends only on  $d$  and  $\theta$ .

*Proof.* We will split the proof in several parts. According to the preparatory step, for  $d \geq 2$  and all  $\alpha, \beta = 1, \dots, d$ , the difference  $\partial_\gamma \{\Gamma_{\alpha\beta}(x; \lambda) - \Gamma_{\alpha\beta}(x; 0)\}$ ,  $\gamma = 1, \dots, d$ , is always of the form

$$\begin{aligned}
&\frac{\partial}{\partial x_\gamma} \{\Gamma_{\alpha\beta}(x; \lambda) - \Gamma_{\alpha\beta}(x; 0)\} \\
&= \frac{\partial}{\partial x_\gamma} \{G(x; \lambda) - G(x; 0)\} \delta_{\alpha\beta} - \frac{1}{\lambda} \frac{\partial^3}{\partial x_\gamma \partial x_\alpha \partial x_\beta} \left\{ G(x; \lambda) - G(x; 0) + [\dots] \right\}.
\end{aligned}$$

But the first term on the right hand side of the above expression is already under control thanks to Lemma 2.3. It thus suffices to estimate the second term on the right hand side.

We start by considering the cases  $d = 3$  and  $d \geq 5$ . Taking into account identity (2.34), we have for all  $\alpha, \beta, \gamma = 1, \dots, d$ :

$$G(x; \lambda) - G(x; 0) + [\dots] = \frac{1}{\lambda} \left\{ G(x; \lambda) - G(x; 0) + \frac{\lambda}{2\omega_d(d-4)(d-2)|x|^{d-4}} \right\}.$$

If  $d = 3$ , a direct calculation will then yield the desired result: We start by noting that  $\omega_3 = 4\pi$  gives

$$\begin{aligned} G(x; \lambda) - G(x; 0) - \frac{\lambda}{2\omega_3|x|^{-1}} &= \frac{e^{ik|x|}}{4\pi|x|} - \frac{1}{4\pi|x|} - \frac{(ik)^2}{2\omega_3|x|^{-1}} \\ &= \frac{1}{4\pi|x|} \left( e^{ik|x|} - 1 - \frac{(ik)^2|x|^2}{2} \right) \\ &= \frac{1}{4\pi|x|} \left( ik|x| + \sum_{n=3}^{\infty} \frac{(ik|x|)^n}{n!} \right) \\ &= \frac{1}{4\pi} \left( ik + \sum_{n=3}^{\infty} \frac{(ik)^n|x|^{n-1}}{n!} \right). \end{aligned}$$

Taking the first derivative of this expression we get

$$\frac{\partial}{\partial x_\beta} \{ \dots \} = \frac{x_\beta}{4\pi} \sum_{n=3}^{\infty} \frac{(ik)^n(n-1)}{n!} |x|^{n-3}$$

and differentiating with respect to  $x_\alpha$  yields

$$\frac{\partial}{\partial x_\alpha} \{ \dots \} = \frac{\delta_{\alpha\beta}}{4\pi} \sum_{n=3}^{\infty} \frac{(ik)^n(n-1)}{n!} |x|^{n-3} + \frac{x_\beta x_\alpha}{4\pi} \sum_{n=4}^{\infty} \frac{(ik)^n(n-1)(n-3)}{n!} |x|^{n-5}.$$

As we are interested in estimating the *gradient* of the difference of  $\Gamma(x; \lambda)$  and  $\Gamma(x; 0)$ , we have to consider one additional derivative. This leaves us with

$$\begin{aligned} \frac{\partial}{\partial x_\gamma} \{ \dots \} &= \frac{\delta_{\alpha\beta} x_\gamma + \delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta}{4\pi} \sum_{n=4}^{\infty} \frac{(ik)^n(n-1)(n-3)}{n!} |x|^{n-5} \\ &\quad + \frac{x_\beta x_\alpha x_\gamma}{4\pi} \sum_{n=4}^{\infty} \frac{(ik)^n(n-1)(n-3)(n-5)}{n!} |x|^{n-7}. \end{aligned}$$

We can now prove the desired estimate via

$$\begin{aligned} \left| \frac{1}{\lambda} \frac{\partial^3}{\partial x_\gamma \partial x_\alpha \partial x_\beta} \{ \dots \} \right| &\leq \frac{1}{|k|^2 \pi} \sum_{n=4}^{\infty} \frac{|k|^n(n-1)(n-3)(1+(n-5))}{n!} |x|^{n-4} \\ &\leq \frac{1}{|k|^2 |x| \pi} |k|^3 \sum_{k=4}^{\infty} \frac{(n-1)(n-3)(1+(n-5))}{n!} |k|^{n-3} |x|^{n-3} \\ &\leq C \frac{1}{|k||x|}. \end{aligned}$$

This gives the claim for  $d = 3$ . If  $d \geq 5$ , equation (2.21) gives

$$\begin{aligned} G(x; \lambda) - G(x; 0) + \frac{\lambda}{2\omega_2(d-4)(d-2)|x|^{d-4}} &\quad \text{eq: secondTerm} \\ &= \frac{i}{4(2\pi)^{\frac{d-1}{2}} |x|^{\frac{d-2}{2}}} \left\{ z^{\frac{d}{2}-1} H_{\frac{d}{2}-1}^{(1)}(z) - a_d - b_d z^2 \right\}, \end{aligned} \quad (2.38)$$

where  $z = k|x|$ ,  $a_d$  was calculated in (2.20) and  $b_d$  is given by

$$b_d = -\frac{2i(2\pi)^{\frac{d}{2}-1}}{\omega_d(d-4)(d-2)}.$$

Using relation (2.17) and the functional equation of the Gamma function (2.18) twice, we see that

$$\begin{aligned} b_d &= -\frac{2i(2\pi)^{\frac{d}{2}-1}\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}(d-2)(d-4)} = \frac{2^{\frac{d}{2}-1}}{\pi i(d-4)} \frac{\Gamma(\frac{d}{2})}{(d-2)} = \frac{2^{\frac{d}{2}-1}}{2\pi i} \frac{\Gamma(\frac{d}{2}-1)}{(d-4)} \\ &= \frac{2^{\frac{d}{2}-1}}{4\pi i} \frac{\Gamma(\frac{d}{2}-1)}{(\frac{d}{2}-1-1)} = \frac{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2}-1-1)}{4\pi i} = \frac{2^{\nu_d}\Gamma(\nu_d-1)}{4\pi i}. \end{aligned}$$

This shows that for  $d \geq 5$ ,  $b_d$  is the second coefficient of the asymptotic expansions (2.13)-(2.15), respectively. Now we split the proof for  $d \geq 5$  into (1)  $d \geq 7$ , (2)  $d = 6$  and (3)  $d = 5$ . If  $d \geq 7$ , we use the asymptotic expansions (2.15) to estimate the part of (2.38) which involves the Hankel function as

$$\left| \frac{d^l}{dz^l} \left\{ z^{\frac{d}{2}-1} H_{\frac{d}{2}-1}^{(1)}(z) - a_d - b_d z^2 \right\} \right| \leq C|z|^{\frac{d}{2}-l-1} \quad (2.39)$$

for  $0 \leq l \leq 3$  and  $z \in \mathbb{C} \setminus (-\infty, 0]$ . For better readability we define the function

$$g(z) := z^{\frac{d}{2}-1} H_{\frac{d}{2}-1}^{(1)}(z) - a_d - b_d z^2$$

and consider the function  $f(x) := g(k|x|)$  on  $\mathbb{R}^d \setminus \{0\}$ . The derivatives of  $f$  read

$$\begin{aligned} \frac{\partial}{\partial x_\beta} f(x) &= \left( \frac{d}{dz} g \right)(k|x|) \cdot \frac{kx_\beta}{|x|}, \\ \frac{\partial^2}{\partial x_\alpha \partial x_\beta} f(x) &= \left( \frac{d^2}{dz^2} g \right)(k|x|) \cdot \frac{k^2 x_\alpha x_\beta}{|x|^2} + \left( \frac{d}{dz} g \right)(k|x|) \cdot k \left\{ \frac{\delta_{\alpha\beta}}{|x|} - \frac{x_\beta x_\alpha}{|x|^3} \right\}, \\ \frac{\partial^3}{\partial x_\gamma \partial x_\alpha \partial x_\beta} f(x) &= \left( \frac{d^3}{dz^3} g \right)(k|x|) \cdot \frac{k^3 x_\alpha x_\beta x_\gamma}{|x|^3} \\ &\quad + \left( \frac{d^2}{dz^2} g \right)(k|x|) \cdot k^2 \left\{ \frac{x_\alpha \delta_{\beta\gamma} + x_\beta \delta_{\alpha\gamma} + x_\gamma \delta_{\alpha\beta}}{|x|^2} - \frac{x_\alpha x_\beta x_\gamma}{|x|^4} \right\} \\ &\quad + \left( \frac{d}{dz} g \right)(k|x|) \cdot k \left\{ -\frac{x_\gamma \delta_{\alpha\beta} + x_\alpha \delta_{\beta\gamma} + x_\beta \delta_{\alpha\gamma}}{|x|^3} + \frac{3x_\alpha x_\beta x_\gamma}{|x|^5} \right\}. \end{aligned}$$

If we now look for estimates on the absolute value of the derivatives, we see that by (2.39)

$$|\nabla^l f(x)| \leq C|k|^4|x|^{4-l}, \quad 1 \leq l \leq 3,$$

where  $C$  only depends on  $l$ . We can now finally uncover the desired estimate via

$$\left| \frac{1}{\lambda} \nabla_x^3 \left\{ \frac{i}{4(2\pi)^{\frac{d}{2}-1}} \frac{1}{|x|^{d-2}} \left\{ z^{\frac{d}{2}-1} H_{\frac{d}{2}-1}^{(1)}(z) - a_d - b_d z^2 \right\} \right\} \right|$$

$$\leq C \frac{1}{|k|^2} \sum_{l=0}^3 \left| \nabla^{3-l} \left( \frac{1}{|x|^{d-2}} \right) \right| |\nabla^l f(x)| \leq C \sum_{l=0}^3 |x|^{-d+2-3+l} |k|^2 |x|^{4-l} = C |\lambda| |x|^{3-d},$$

where  $C$  is a constant only depending on  $d$ .

If  $d = 6$ , the asymptotic expansion (2.14) gives us in analogy to (2.39) the estimate

$$\left| \frac{d^l}{dz^l} \left\{ z^{\frac{d}{2}-1} H_{\frac{d}{2}-1}^{(1)}(z) - a_d - b_d z^2 \right\} \right| \leq C |z|^{4-l} |\log(z)|, \quad (2.40)$$

for  $0 \leq l \leq 3$  and  $z \in \mathbb{C} \setminus (-\infty, 0]$ . Using as before the expressions for the derivatives of  $f$ , we estimate their absolute values as

$$|\nabla^l f(x)| \leq C |k|^4 |x|^{4-l} |\log(|\lambda| |x|^2)|,$$

which, by a calculation analogous to the case  $d \geq 7$ , yields the claim.

For  $d = 5$  we differentiate (2.38) twice and use relation (2.34) for the fundamental solution of the Stokes problem to write

$$\begin{aligned} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left\{ G(x; \lambda) - G(x; 0) + \frac{\lambda}{6 \omega_5 |x|^{d-4}} \right\} \\ = \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left\{ \frac{i}{4(2\pi)^{\frac{3}{2}}} \cdot \frac{1}{|x|^3} \left[ z^{\frac{3}{2}} H_{\frac{3}{2}}^{(1)}(z) - a_5 - b_5 z^2 - w z^3 \right] \right\}, \end{aligned}$$

where  $w \in \mathbb{C}$  can be an arbitrary constant if we set  $z = k|x|$ . Now, for the appropriate choice of  $w \in \mathbb{C}$  the asymptotic expansion (2.13) gives the same estimate as (2.39) which, like for  $d \geq 7$ , proves the claim for  $d = 5$ .

In the case  $d = 4$  we use the respective relation for the fundamental solution (2.35) in order to simplify the difference

$$\begin{aligned} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left\{ G(x; \lambda) - G(x; 0) - \frac{\lambda \log(|x|)}{4 \omega_4} \right\} \\ = \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left\{ \frac{i}{8\pi |x|^2} \left[ z H_1^{(1)}(z) - a_4 - w z^2 - b_4 z^2 \log(z) \right] \right\}, \end{aligned}$$

where  $z = k|x|$ ,  $b_4 = (i/\pi)$  and  $w \in \mathbb{C}$  is an arbitrary constant. Using the asymptotic expansion (2.12) and the appropriate constant  $w \in \mathbb{C}$  we get the estimate

$$\left| \frac{d^l}{dz^l} \left\{ z H_1^{(1)}(z) - a_4 - w z^2 - b_4 z^2 \log(z) \right\} \right| \leq C |z|^{4-l} |\log(z)|.$$

The estimate has the same right hand side as (2.40) and the proof can be carried out just as in the previous cases.

For  $d = 2$  the claimed estimate follows from a direct calculation which is postponed until the appendix of this thesis.  $\square$

We can now use the assumption  $|\lambda||x|^2 \leq (1/2)$  to unify the structure of the estimates from Theorem 2.6.

cor: differenceFundamentalSolutionStokes

**Corollary 2.7.** *Let  $\lambda \in \Sigma_\theta$ . Suppose that  $|\lambda||x|^2 \leq (1/2)$ . Then for all  $d \geq 2$*

$$|\nabla_x \{\Gamma(x; \lambda) - \Gamma(x; 0)\}| \leq C\sqrt{|\lambda|}|x|^{2-d},$$

where  $C$  depends only on  $d$  and  $\theta$ .

*Proof.* We just extend the estimates given in Theorem 2.6. Let  $d \geq 7$  or  $d = 5$ . Since  $\sqrt{|\lambda|} \leq C|x|^{-1}$ , we have

$$C|\lambda||x|^{3-d} \leq C\sqrt{|\lambda|}|x|^{2-d}.$$

For  $d = 2, 4, 6$ , we have

$$|\lambda||x|^{3-d} |\log(|\lambda||x|^2)| = C\sqrt{|\lambda|}|x|^{2-d} \cdot \sqrt{|\lambda|}|x| |\log(|\lambda||x|^2)| \leq C\sqrt{|\lambda|}|x|^{2-d},$$

since  $\sqrt{|\lambda|}|x| |\log(|\lambda||x|^2)|$  is bounded for  $|\lambda||x|^2 \leq (1/2)$ . □

# Chapter 3

## Single and Double Layer Potentials

chap:3

In this chapter, we will deal with *single* and *double layer potentials*. Both will serve as “representation formulas” for solutions to the Stokes resolvent problem. We will study their properties as they will serve as the crucial ingredient to solving the  $L^2$  Dirichlet problem associated to the Stokes resolvent problem on bounded Lipschitz domains  $\Omega \subset \mathbb{R}^d$ : For  $\lambda \in \mathbb{C} \setminus (-\infty, 0)$  and

$$g \in L^2_\nu(\partial\Omega) := \left\{ g \in L^2(\partial\Omega; \mathbb{C}^d) : \int_{\partial\Omega} g \cdot \nu \, d\sigma = 0 \right\}$$

we are looking for smooth functions  $u$  and  $\phi$  that satisfy

$$(\text{Dir}_\lambda) \left\{ \begin{array}{ll} -\Delta u + \nabla \phi + \lambda u = 0 & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = g & \text{nontangentially on } \partial\Omega, \\ (u)^* \in L^2(\partial\Omega). \end{array} \right.$$

In this chapter we will thus always assume that  $\Omega$  is a bounded Lipschitz domain in  $\mathbb{R}^d$  with  $d \geq 2$  and  $1 < p < \infty$ . We will also tacitly use the summation convention.

We note that due to the new two dimensional estimates on fundamental solutions in Chapter 2, namely the continuation of Theorem 2.5 and Theorem 2.6 for the case  $d = 2$ , we could extend all results from Chapter 3 of Shen’s seminal paper [20] that are relevant to the analysis of the  $L^2$  Dirichlet problem in a straightforward way.

Let  $\lambda \in \Sigma_\theta$ . For  $f \in L^p(\partial\Omega; \mathbb{C}^d)$ , the single layer potential  $u = \mathcal{S}_\lambda(f)$  is defined by

$$u_j(x) = \int_{\partial\Omega} \Gamma_{jk}(x - y; \lambda) f_k(y) \, d\sigma(y), \quad \text{eq: defSingleLayer} \quad (3.1)$$

where  $\Gamma_{jk}$  is the fundamental solution to the Stokes resolvent problem given by (2.27). For the pressure, respectively, we define the single layer potential  $\phi = \mathcal{S}_\Phi(f)$  by

$$\phi(x) = \int_{\partial\Omega} \Phi_k(x-y) f_k(y) d\sigma(y), \quad \text{eq: defSingleLayerPressure} \quad (3.2)$$

where  $\Phi_k$  is given by (2.28). As we have already shown,  $(u, \phi)$  defines a solution to the Stokes resolvent problem (2.26).

We define two further integral operators

$$T_\lambda^*(f)(P) = \sup_{t>0} \left| \int_{\substack{y \in \partial\Omega \\ |y-P|>t}} \nabla_x \Gamma(P-y; \lambda) f(y) d\sigma(y) \right| \quad \text{eq: supTOperator} \quad (3.3)$$

$$T_\lambda(f)(P) = \text{p. v.} \int_{\partial\Omega} \nabla_x \Gamma(P-y; \lambda) f(y) d\sigma(y) \quad \text{eq: pvTOperator} \quad (3.4)$$

for  $P \in \partial\Omega$  which will be used to prove boundedness of maximal operators related to  $u$ .

The following lemma will be a good companion for the forthcoming calculation of estimates.

lem:compareBoundaryWithBall

**Lemma 3.1.** *Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ , be a bounded Lipschitz domain with corresponding numbers  $r_0$  and  $M$ . Then there exists a constant  $C > 0$  depending only on  $d$  and  $M$  such that*

$$\sigma(B(q, r) \cap \partial\Omega) \leq Cr^{d-1}$$

for all  $r > 0$  and  $q \in \partial\Omega$ . Furthermore, there exists a constant  $C > 0$ , depending only on  $d$  and the Lipschitz character of  $\Omega$ , such that

$$\sigma(\partial\Omega) \leq Cr_0^{d-1}.$$

Another cornerstone in the theory of the single and double layer potentials is the following lemma, see Tolksdorf [25, Lem. 4.3.2]

lem:central

**Lemma 3.2.** *Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ , be a bounded Lipschitz domain with corresponding numbers  $r_0$  and  $M$ . Let  $x \in \mathbb{R}^d$ ,  $0 < \varepsilon \leq r_0/4$ , and  $l \in \mathbb{N}_0$  with  $l < d-1$ . Then there exists a constant  $C > 0$  depending only on  $d$ ,  $l$  and  $M$  such that*

$$\int_{\partial\Omega \cap B(x, \varepsilon)} \frac{1}{|x-y|^l} d\sigma(y) \leq C\varepsilon^{d-l-1}.$$

We are now in the position to prove our first Lemma on the way to establish the Single layer potential as a benevolent operator for tackling boundary value problems on bounded Lipschitz domains.

lem:lpBoundednessT

**Lemma 3.3.** *Let  $1 < p < \infty$  and  $T_\lambda(f), T_\lambda^*(f)$  be defined by (3.3) and (3.4). Then  $T_\lambda(f)(P)$  exists for almost everywhere  $P \in \partial\Omega$  and*

$$\|T_\lambda(f)\|_{L^p(\partial\Omega)} \leq \|T_\lambda^*(f)\|_{L^p(\partial\Omega)} \leq C_p \|f\|_{L^p(\partial\Omega)}, \quad \text{eq:lpBoundednessT} \quad (3.5)$$

where  $C_p$  depends only on  $d, \theta, p$ , and the Lipschitz character of  $\Omega$ .

*Proof.* If  $\lambda = 0$ , the Lemma is known [5] as a consequence of the seminal result of Coifman et al. [2]. One idea of the proof in the case  $\lambda \in \Sigma_\theta$  will thus be to nourish from this result and to consider the difference  $\Gamma(x - y; \lambda) - \Gamma(x - y; 0)$  as a well-disposed integral kernel.

We start with the second inequality of 3.5. To this end, let  $t > 0$  and additionally assume that  $t^2|\lambda| \geq \frac{1}{2}$ . Theorem 2.5 gives

$$\left| \int_{|y-P|>t} \nabla \Gamma(P - y; \lambda) f(y) d\sigma(y) \right| \leq C \int_{|P-y|>t} \frac{|f(y)|}{|\lambda| |P - y|^{d+1}} d\sigma(y)$$

Choose now  $N \in \mathbb{N}$  such that  $2^N t \leq \text{diam}(\Omega) < 2^{N+1} t$ . We now exhaust the domain of integration by suitable annuli and calculate

$$\begin{aligned} & \sum_{k=0}^N \int_{B(P, 2^{k+1}t) \cap \partial\Omega} \frac{1}{|\lambda| 2^{k(d+1)} t^{d+1}} |f(y)| d\sigma(y) \\ & \leq \frac{1}{|\lambda| t^2} \frac{1}{2^{1-d}} \sum_{k=0}^N \frac{1}{2^{2k}} \frac{1}{(2^{k+1}t)^{d-1}} \int_{B(P, 2^{k+1}t \cap \partial\Omega)} |f(y)| d\sigma(y) \\ & \leq C \sum_{k=0}^N \frac{1}{2^{2k}} M_{\partial\Omega}(f)(P) \\ & \leq C M_{\partial\Omega}(f)(P) \end{aligned}$$

where for the second inequality we used Lemma 3.1 to estimate

$$\frac{1}{(2^{k+1}t)^{d-1}} \leq C(\sigma(B(P, 2^{k+1}t) \cap \partial\Omega))^{-1},$$

which gives the claimed estimate with a constant  $C$  that depends on  $d, \theta$  and the Lipschitz character of  $\Omega$ . Now let  $t^2|\lambda| < \frac{1}{2}$ . We then split the integral as follows

$$\begin{aligned} & \left| \int_{|y-P|>t} \nabla \Gamma(P - y; \lambda) f(y) d\sigma(y) \right| \\ & \leq \left| \int_{|y-P| \geq (2|\lambda|)^{-1/2}} \nabla \Gamma(P - y; \lambda) f(y) d\sigma \right| + \left| \int_{t < |y-P| < (2|\lambda|)^{-1/2}} \nabla \Gamma(P - y; \lambda) f(y) d\sigma \right|. \end{aligned}$$



The first summand can be estimated like in the step before, if we substitute  $t$  by  $(2|\lambda|)^{-1/2}$ . For the second term we use the principle of the nutrient zero and estimate

$$\begin{aligned} & \left| \int_{t < |y-P| < (2|\lambda|)^{-1/2}} \nabla \Gamma(P-y; \lambda) f(y) \, d\sigma \right| \\ & \leq \int_{t < |y-P| < (2|\lambda|)^{-1/2}} |\nabla \Gamma(P-y; \lambda) - \nabla \Gamma(P-y; 0)| f(y) \, d\sigma \\ & \quad + \left| \int_{t < |y-P| < (2|\lambda|)^{-1/2}} \nabla \Gamma(P-y; 0) f(y) \, d\sigma \right|. \end{aligned}$$

We don't need to worry about the second summand here since the corresponding estimate is already covered by the case of  $\lambda = 0$  and therefore

$$\begin{aligned} & \left| \int_{t < |y-P| < (2|\lambda|)^{-1/2}} \nabla \Gamma(P-y; 0) f(y) \, d\sigma \right| \\ & \leq \left| \int_{|y-P| > t} \nabla \Gamma(P-y; 0) f(y) \, d\sigma \right| + \left| \int_{|y-P| > (2|\lambda|)^{-1/2}} \nabla \Gamma(P-y; 0) f(y) \, d\sigma \right| \\ & \leq 2T_0^*(f)(P). \end{aligned}$$

For the first summand we make use of Theorem 2.6 and more precisely of Corollary 2.7 which unifies all estimates. We then calculate

$$\begin{aligned} & \int_{t < |y-P| < (2|\lambda|)^{-1/2}} |\nabla \Gamma(P-y; \lambda) - \nabla \Gamma(P-y; 0)| f(y) \, d\sigma \\ & \leq \int_{t < |y-P| < (2|\lambda|)^{-1/2}} |\lambda|^{\frac{1}{2}} |y-P|^{2-d} |f(y)| \, d\sigma, \end{aligned}$$

and as before we choose adequate  $N$  such that  $2^{N+1}t > (2|\lambda|)^{-1/2} \geq 2^N t$  which leads to

$$\begin{aligned} & \leq |\lambda|^{\frac{1}{2}} \sum_{k=0}^N \int_{2^k t < |y-P| < 2^{k+1} t} |y-P|^{2-d} |f(y)| \, d\sigma \\ & \leq |\lambda|^{\frac{1}{2}} t^{2-d} \sum_{k=0}^N 2^{k(2-d)} \int_{B(P, 2^{k+1} t)} |f(y)| \, d\sigma \\ & \leq 2^d |\lambda|^{\frac{1}{2}} t \sum_{k=0}^N 2^{k-1} 2^{(k+1)(1-d)} t^{1-d} \int_{B(P, 2^{k+1} t)} |f(y)| \, d\sigma \\ & \leq C |\lambda|^{\frac{1}{2}} t \frac{2^N - 1}{1} M_{\partial\Omega}(f)(P) \\ & \leq C |\lambda|^{\frac{1}{2}} (2|\lambda|)^{-\frac{1}{2}} M_{\partial\Omega}(f)(P). \end{aligned}$$

Taking now the supremum over all  $t > 0$  we see that

$$T_\lambda^*(f)(P) \leq C(M_{\partial\Omega}(f)(P) + T_0^*(f)(P)),$$

for all  $P \in \partial\Omega$ . Once again using the result for  $\lambda = 0$  and the  $L^p$ -boundedness of the Hardy-Littlewood maximal operator we see that

$$\|T_\lambda^*(f)\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega)}$$

To conclude the first inequality in (3.5), we want to use a standard result from harmonic analysis which can be found in the book of Grafakos [11, 1.1.14]. First we will show that the integral operator

$$T_\lambda(f)(P) = \lim_{t \rightarrow 0} \int_{\substack{y \in \partial\Omega \\ |y-P| > t}} \nabla_x \Gamma(P-y; \lambda) f(y) d\sigma(y)$$

exists for almost every  $P \in \partial\Omega$  and all  $f \in C(\partial\Omega; \mathbb{C}^d)$ . In a first step, we can split this operator formally in

$$T_\lambda(f)(P) = T_0(f)(P) + \lim_{t \rightarrow 0} \int_{\substack{y \in \partial\Omega \\ |y-P| > t}} \nabla_x \{\Gamma(P-y; \lambda) - \Gamma(P-y; 0)\} f(y) d\sigma(y)$$

The right expression is well defined for  $f \in C_0^\infty$ , once we prove integrability of

$$|\nabla\{\Gamma(P-y; \lambda) - \Gamma(P-y; 0)\}|$$

on  $\partial\Omega$ . To this end we first note that it suffices to consider the integral

$$\int_{|P-y| \leq \varepsilon} |\nabla\{\Gamma(P-y; \lambda) - \Gamma(P-y; 0)\}| d\sigma(y),$$

for  $\varepsilon \leq \min(2|\lambda|^{-1/2}, r_0/4)$  as the integrand is smooth away from 0 and the domain of integration is bounded. Now Corollary 2.7 and Lemma 3.2 give that this can be estimated by

$$\int_{|P-y| \leq \varepsilon} |\lambda|^{1/2} |P-y|^{2-d} d\sigma(y) \leq C|\lambda|^{1/2} \varepsilon \leq C.$$

Based on the preceding calculation we conclude that for all  $f \in C(\partial\Omega, \mathbb{C}^d)$  the operator  $T_\lambda(f)(P)$  exists whenever  $T_0(f)(P)$  exists.  $T_0(f)(P)$  exists for almost everywhere  $P \in \partial\Omega$  because of Fabes, Kenig and Verchota [5]. As furthermore  $T_\lambda^*(f)(P)$  is bounded on  $L^p(\partial\Omega)$  we may now apply Theorem 2.1.14 from Grafakos [11] to conclude that  $T_\lambda(f)(P)$  exists now for all  $f \in L^p(\partial\Omega; \mathbb{C}^d)$  and almost everywhere  $P \in \partial\Omega$ . The desired  $L^p$  estimate on  $T_\lambda(f)$  now follows from the observation that

$$|T_\lambda(f)(P)| \leq T_\lambda^*(f)(P)$$

for almost everywhere  $P \in \partial\Omega$ . □

For further use we state a very useful lemma which can be considered a *Young* type inequality for  $L^p$  spaces on boundaries of Lipschitz domains. A proof can be found in Tolksdorf [25, Prop 1.1.4].

lem:young

**Lemma 3.4.** *Let  $\Omega \subset \mathbb{R}^d$ ,  $\mu$  a  $\sigma$ -finite measure on  $\Omega$ , and  $1 \leq p < \infty$ . Let  $g: \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{C}$  be a function such that the function  $\Omega \times \Omega \ni (x, y) \mapsto g(x - y)$  is measurable with respect to the product measure  $\mu \times \mu$  and such that*

$$A + B := \sup_{x \in \Omega} \|g(x - \cdot)\|_{L^1(\Omega, \mu)} + \sup_{y \in \Omega} \|g(\cdot - y)\|_{L^1(\Omega, \mu)} < \infty.$$

*If  $f \in L^p(\Omega, \mu)$ , then  $x \mapsto \int_{\Omega} g(x - y)f(y) d\mu(y) \in L^p(\Omega, \mu)$  and*

$$\left\| \int_{\Omega} g(\cdot - y)f(y) d\mu(y) \right\|_{L^p(\Omega, \mu)} \leq A^{1-1/p} B^{1/p} \|f\|_{L^p(\Omega, \mu)}.$$

We can now prove the boundedness of certain nontangential maximal operators.

lem:nontangentialMaximalFunctions

**Lemma 3.5.** *Let  $1 < p < \infty$  and  $(u, \phi)$  be given by (3.1) and (3.2). Then*

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} + \|(\phi)^*\|_{L^p(\partial\Omega)} \leq C_p \|f\|_{L^p(\partial\Omega)}, \quad (3.6)$$

where  $C_p$  depends only on  $d$ ,  $\theta$ ,  $p$  and the Lipschitz character of  $\Omega$ . Let furthermore  $d \geq 3$ . Then

$$\|(u)^*\|_{L^p(\partial\Omega)} \leq C_p \|f\|_{L^p(\partial\Omega)}, \quad \text{eq:estimateUstar} \quad (3.7)$$

where  $C_p$  depends only on  $d$ ,  $\theta$ ,  $p$  and the Lipschitz character of  $\Omega$ .

*Proof.* A proof of the estimate  $\|(\phi)^*\|_{L^p(\partial\Omega)} \leq C_p \|f\|_{L^p(\partial\Omega)}$  can be found in Verchota [27]. The proof for  $\|(\nabla u)^*\|_{L^p(\partial\Omega)}$  works in the same way. We will provide a proof for the sake of completeness. To immitate the proof of Verchota, we will work with the corresponding type of cones. Therefore the results for  $\nabla u$  and  $\phi$  will at first only be established for the type of maximal operators defined by Verchota. The transferability to Shen's maximal operators is given by Tolksdorf [25, p. 90ff.] as the solution  $(u, \phi)$  has a representation as a single layer potential.

Let  $x \in \Gamma_V(P)$  and set  $t = |x - P|$ . Then,

$$\begin{aligned} |(\nabla u)(x)| &= \left| \int_{\partial\Omega} \nabla \Gamma_{jk}(x - y; \lambda) f_k d\sigma(y) \right| \\ &\leq \left| \int_{|y-P|>t} \nabla \Gamma_{jk}(x - y; \lambda) f_k d\sigma(y) \right| + \left| \int_{|y-P|\leq t} \nabla \Gamma_{jk}(x - y; \lambda) f_k d\sigma(y) \right| \\ &= I_1 + I_2. \end{aligned}$$

We will now estimate  $I_1$  and  $I_2$  separately. Note that in Verchota cones we have that for all  $Q \in \partial\Omega$  we have  $|x - Q| > C|x - P|$ , where  $C$  is a constant only depending on  $d$  and the Lipschitz character of  $\Omega$ . By Theorem 2.5 we know that

$$\begin{aligned} I_2 &\leq C \int_{|y-P| \leq t} \frac{1}{|x-y|^{d-1}} |f(y)| d\sigma(y) \\ &\leq \frac{C}{t^{n-1}} \int_{|y-P| \leq t} |f(y)| d\sigma(y) \leq CM_{\partial\Omega}(f)(P). \end{aligned}$$

For  $I_1$ , we calculate

$$\begin{aligned} &\left| \int_{|y-P| > t} \nabla\Gamma_{jk}(x-y; \lambda) f_k(y) - \nabla\Gamma_{jk}(x-y; 0) f_k(y) + \nabla\Gamma_{jk}(x-y; 0) f_k(y) d\sigma(y) \right| \\ &\leq \left| \int_{|y-P| > t} \nabla\Gamma_{jk}(x-y; \lambda) f_k(y) - \nabla\Gamma_{jk}(P-y; \lambda) f_k(y) d\sigma(y) \right| \\ &\quad + \left| \int_{|y-P| > t} \nabla\Gamma_{jk}(P-y; \lambda) f_k(y) d\sigma(y) \right|. \end{aligned}$$

The second summand can directly be estimated by  $T_\lambda^*(f)(P)$ . For the second one we apply the mean value theorem and derive using once again Theorem 2.5

$$\begin{aligned} &\int_{|y-P| > t} |\nabla\Gamma_{jk}(x-y; \lambda) - \nabla\Gamma_{jk}(P-y; \lambda)| |f(y)| d\sigma(y) \\ &\leq \int_{|y-P| > t} |\nabla^2\Gamma_{jk}(s-y; \lambda)| |x-P| |f(y)| d\sigma(y) \\ &\leq Ct \int_{|y-P| > t} \frac{1}{|s-y|^d} |f(y)| d\sigma(y) \\ &\leq Ct \int_{|y-P| > t} \frac{1}{|y-P|^d} |f(y)| d\sigma(y) \\ &\leq C \int_{\partial\Omega} \frac{t}{(t+|y-P|)^d} |f(y)| d\sigma(y). \end{aligned}$$

where  $s$  is an element on the line connecting  $x$  and  $P$  and we used the property of Verchota-cones that  $|s-y| \geq C|y-P|$ . Note that Verchota cones are convex. As in Verchota [27, Lem 1.3] the integral may now be bounded by the Hardy-Littlewood maximal operator due to an application of a suitable result from Grafakos [11, Thm. 2.1.10] as the kernel  $\frac{t}{(t+|y-P|)^d}$  is uniformly integrable and radially decreasing on  $\partial\Omega$ . Summing up we have shown that

$$|(\nabla u)(x)| \leq C\{M_{\partial\Omega}f(P) + T_\lambda^*(f)(P)\},$$

where  $C$  only depends on  $d$ ,  $\theta$  and the Lipschitz character of  $\Omega$ . We thus may take the supremum over all  $x \in \Gamma_V$  and conclude the desired estimate by the well known

mapping properties of the Hardy-Littlewood maximal operator and the respective results from Lemma 3.3.

We will now work on the proof of the estimate for  $(u)^*$  for  $d \geq 3$ . In order to derive  $L^p$  estimates on this maximal operator we will work directly with the Definition of the single layer potential (3.1). For  $P \in \partial\Omega$ , estimate (2.30) together with the estimate for Shen cones (1.3) gives that for all  $x \in \Gamma(P)$

$$|u^*(x)| \leq C \int_{\partial\Omega} \frac{1}{|x-y|^{d-2}} |f(y)| d\sigma(y) \leq C \int_{\partial\Omega} \frac{1}{|P-y|^{d-2}} |f(y)| d\sigma(y),$$

where  $C$  only depends on  $d$ ,  $\theta$  and the Lipschitz character of  $\Omega$ . Passing to the maximal operator yields the inequality

$$u^*(P) \leq C \int_{\partial\Omega} \frac{1}{|P-y|^{d-2}} |f(y)| d\sigma(y),$$

We are now left with the task to estimate the integral

$$\int_{\partial\Omega} \frac{1}{|P-y|^{d-2}} d\sigma(y)$$

uniformly for all  $P \in \partial\Omega$ , as the rest can be handled using the Young inequality from Lemma 3.4. Let  $r_0$  be the radius from the definition of Lipschitz cylinders. Then

$$\begin{aligned} & \int_{\partial\Omega} \frac{1}{|P-y|^{d-2}} d\sigma(y) \\ & \leq \int_{\partial\Omega \cap B(P; r_0/4)} \frac{1}{|P-y|^{d-2}} d\sigma(y) + \int_{\partial\Omega \setminus B(P; r_0/4)} \frac{1}{|P-y|^{d-2}} d\sigma(y). \\ & \leq Cr_0/4 + \sigma(\partial\Omega) r_0^{2-d} 4^{d-2}. \end{aligned}$$

where  $C$  only depends on  $d$  and the Lipschitz character of  $\Omega$ . □

**Remark 3.6.** We note that in addition to the consideration of  $d = 2$ , Lemma 3.5 differs in the form of estimate (3.7) from the original statement in Shen's work [20, Lem. 3.2]. There, the author derives an estimate of the form  $|\lambda|^{1/2} \|(u)^*\|_{L^p(\partial\Omega)} \leq C_p \|f\|_{L^p(\partial\Omega; \mathbb{C}^d)}$  which is based on Shen's version of Lemma 2.1, namely [20, Lem. 2.1]. As we could not follow the proof in [20], we provided a similar estimate and since the estimate won't be needed in the course of this thesis, we will not pursue the verification of Shen's estimate further.

The next lemma deals with *trace formulas* for  $\nabla u$  and  $\phi$ . We can now finally talk about boundary values as the existence of nontangential limits guarantees that there exists something on  $\partial\Omega$  that is related to the function inside  $\Omega$  or inside  $\mathbb{R}^d \setminus \overline{\Omega}$ , respectively.

lem:traceFormulas

**Lemma 3.7.** *Let  $(u, \phi)$  be given by (3.1) and (3.2) with  $f \in L^p(\partial\Omega; \mathbb{C}^d)$  and  $1 < p < \infty$ . Then*

$$\begin{aligned} \left(\frac{\partial u_i}{\partial x_j}\right)_\pm(x) &= \pm \frac{1}{2} \{n_j(x)f_i(x) - n_i(x)n_j(x)n_k(x)f_k(x)\} \\ &\quad + \text{p. v.} \int_{\partial\Omega} \frac{\partial}{\partial x_j} \{\Gamma_{ik}(x-y; \lambda)\} f_k(y) d\sigma(y), \\ \phi_\pm(x) &= \mp \frac{1}{2} n_k(x)f_k(x) + \text{p. v.} \int_{\partial\Omega} \Phi_k(x-y)f_k(y) d\sigma(y) \end{aligned} \quad \text{eq:traceFormula (3.8)}$$

for almost everywhere  $x \in \partial\Omega$ . The subscripts  $+$  and  $-$  indicate nontangential limits taken inside  $\Omega$  and outside  $\bar{\Omega}$ , respectively.

*Proof.* The correctness of the trace formulas (3.8) is known for the case  $\lambda = 0$  since Fabes, Kenig and Verchota [5]. This fact will now be reused for  $\lambda \in \Sigma_\theta$ . We insert a 0 to the nontangential limit as

$$(\nabla u_j)_\pm(x) = (\nabla v_j)_\pm(x) + (\nabla u_j - \nabla v_j)_\pm(x),$$

where  $v_j(x) = \int_{\partial\Omega} \Gamma_{jk}(x-y; 0)f_k(y) d\sigma(y)$ . Because of [5] we know that the first nontangential limit exists and is given by (3.8) with  $\lambda = 0$ . It therefore remains to show that

$$(\nabla u_j - \nabla v_j)_\pm(x) = \int_{\partial\Omega} \nabla \{\Gamma_{jk}(x-y; \lambda) - \Gamma_{jk}(x-y; 0)\} f_k(y) d\sigma(y)$$

for all  $x \in \partial\Omega$ . To this end let  $(x_l)_{l \in \mathbb{N}}$  a sequence in  $\Gamma(x)$  with  $\lim_{l \rightarrow \infty} x_l = x$ . Furthermore let us note that for almost everywhere  $x \in \partial\Omega$  we have that

$$\int_{\partial\Omega} \frac{1}{|x-y|^{d-2}} |f(y)| d\sigma(y) < \infty.$$

This is a consequence of the fact that

$$\sup_{x \in \partial\Omega} \left| \int_{\partial\Omega} \frac{1}{|x-y|^{d-2}} d\sigma(y) \right| < \infty$$

and an application of Lemma 3.4 regarding Young's inequality for convolutions: Let  $x \in \partial\Omega$ . Then

$$\begin{aligned} &\int_{\partial\Omega} \frac{1}{|x-y|^{d-2}} d\sigma(y) \\ &\leq \int_{\partial\Omega \cap B(x, r_0/4)} \frac{1}{|x-y|^{d-2}} d\sigma(y) + \int_{\partial\Omega \setminus B(x, r_0/4)} \frac{1}{|x-y|^{d-2}} d\sigma(y) \end{aligned}$$

$$\leq Cr_0 + r^{2-d}4^{d-2}\sigma(\partial\Omega)$$

by Lemma 3.1. Now Young's inequality gives us the desired result.

In the next step we will show that

$$\frac{1}{|x-y|^{d-2}}|f(y)|$$

gives a suitable function for dominated convergence. Set  $\varepsilon = (4|\lambda|^2)^{-1}$  and without loss of generality assume that  $\text{supp } f \subseteq B(x, \varepsilon)$ . Furthermore assume that  $|x_l - x| < \varepsilon$  for all  $l \in \mathbb{N}$ . Then  $|x_l - y| \leq (2|\lambda|^2)^{-1}$  and Corollary 2.7 give

$$\begin{aligned} & (\nabla u_j - \nabla v_j)(x_l) \int_{\partial\Omega} \nabla\{\Gamma_{jk}(x_l - y; \lambda) - \Gamma_{jk}(x_l - y; 0)\} f_k(y) \, d\sigma(y) \\ & \leq \int_{\partial\Omega} \frac{1}{\sqrt{|\lambda|}|x_l - y|^{d-2}} |f(y)| \, d\sigma(y) \\ & \leq \frac{C}{\sqrt{|\lambda|}} \int_{\partial\Omega} \frac{1}{|x - y|^{d-2}} |f(y)| \, d\sigma(y) < \infty. \end{aligned}$$

Now dominated convergence gives the claim for  $x_l \rightarrow x$ . Note that it does not affect the proof if the sequence  $x_l$  lays inside  $\Omega$  or outside  $\overline{\Omega}$ .  $\square$

The previous lemma enables us to talk about boundary values of partial derivatives. The next theorem will now give a similar result but for conormal derivatives which are defined by

$$\frac{\partial u}{\partial \nu} := \frac{\partial u}{\partial n} - \phi n. \quad \text{eq:conormalDerivative} \quad (3.9)$$

We will also be working with tangential derivatives which are defined via

$$\nabla_{\tan} u := \nabla u - \langle \nabla u, n \rangle n. \quad \text{eq:tangentialGradient} \quad (3.10)$$

thm:jumpConditions

**Theorem 3.8.** *Let  $\lambda \in \Sigma_\theta$  and  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^d$ ,  $d \geq 2$ . Let  $(u, \phi)$  be given by (3.1) and (3.2) with  $f \in L^p(\partial\Omega; \mathbb{C}^d)$  and  $1 < p < \infty$ . Then  $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$  and*

$$\left(\frac{\partial u}{\partial \nu}\right)_\pm = \left(\pm \frac{1}{2} I + \mathcal{K}_\lambda\right) f \quad \text{eq:nontangentialConormalDerivative} \quad (3.11)$$

on  $\partial\Omega$ , with  $\mathcal{K}_\lambda$  a bounded operator on  $L^p(\partial\Omega; \mathbb{C}^d)$  satisfying

$$\|\mathcal{K}_\lambda f\|_{L^p(\partial\Omega)} \leq C_p \|f\|_{L^p(\partial\Omega)},$$

where  $C_p$  depends only on  $d$ ,  $\theta$ ,  $p$  and the Lipschitz character of  $\Omega$ .

*Proof.* For the the  $j$ th component of the tangential derivative of  $u_i$ ,  $1 \leq i, j \leq d$ , we calculate using the results from Lemma 3.7

$$\begin{aligned}
((\nabla_{\tan} u_i)_+)_j &= \left(\frac{\partial u_i}{\partial x_j}\right)_+ - \langle (\nabla u_i)_+, n \rangle n_j \\
&= \left(\frac{\partial u_i}{\partial x_j}\right)_+ - \left(\frac{\partial u_i}{\partial x_k}\right)_+ n_k n_j \\
&= \frac{1}{2} \{n_j f_i - n_i n_j n_k f_k\} - \frac{1}{2} \{n_k f_i - n_i n_k n_l f_l\} n_k n_j \\
&\quad + \text{p. v.} \int_{\partial\Omega} \frac{\partial}{\partial x_j} \{\Gamma_{ik}(x-y; \lambda)\} f_k(y) d\sigma(y) \\
&\quad + \text{p. v.} \int_{\partial\Omega} \frac{\partial}{\partial x_k} \{\Gamma_{il}(x-y; \lambda)\} f_l(y) d\sigma(y) n_k n_j.
\end{aligned}$$

As the first two summands add up to zero, the entire expression does not depend on the direction of the nontangential limit. This gives

$$(\nabla_{\tan} u)_+ = (\nabla_{\tan} u)_-$$

We calculate for the  $j$ th component of the nontangential limit of the conormal derivative of  $u$  at  $x \in \partial\Omega$  using the results from Lemma 3.7

$$\begin{aligned}
\left(\frac{\partial u_j}{\partial x_i}\right)_+(x) n_i - \phi_+(x) n_j &= \frac{1}{2} \{n_i f_j(x) - n_j n_i n_k f_k(x)\} n_i + \text{p. v.} \int_{\partial\Omega} \frac{\partial}{\partial x_i} \{\Gamma_{jk}(x-y; \lambda)\} f_k(y) d\sigma(y) n_i \\
&\quad + \frac{1}{2} n_k f_k(x) n_j - \text{p. v.} \int_{\partial\Omega} \Phi_k(x-y) f_k(y) d\sigma(y) n_j \\
&= \frac{1}{2} f_j(x) + (\mathcal{K}_\lambda f)_j(x),
\end{aligned}$$

where  $n$  denotes the normal vector at  $x$  and

$$(\mathcal{K}_\lambda f)(x) = \text{p. v.} \int_{\partial\Omega} \nabla_x \Gamma(x-y; \lambda) f(y) d\sigma(y) n - \text{p. v.} \int_{\partial\Omega} \Phi_k(x-y) f_k(y) d\sigma(y) n. \quad \text{eq: defnKlambda (3.12)}$$

We note that  $\mathcal{K}_\lambda$  essentially consists of two boundary layer potentials. The  $L^p$ -boundedness of the first one was proven in Lemma 3.3. The  $L^p$ -boundedness of the second boundary layer potential follows in an analogous way using the fact that the operators

$$A^*(f)(P) = \sup_{t>0} \left| \int_{\substack{y \in \partial\Omega \\ |y-P|>t}} \frac{P-y}{|P-y|^n} f(y) d\sigma(y) \right|, \quad P \in \partial\Omega$$

are bounded by the corresponding result from Verchota [27, Lem. 1.2]. □



Similar to  $\mathcal{K}_\lambda$ , for  $\lambda = 0$  we have

$$(\mathcal{K}_0 f)(x) = \text{p. v.} \int_{\partial\Omega} \nabla_x \Gamma(x - y; 0) f(y) d\sigma(y) n - \text{p. v.} \int_{\partial\Omega} \Phi_k(x - y) f_k(y) d\sigma(y) n, \quad \text{eq: defnK0} \quad (3.13)$$

as was shown by Fabes, Kenig and Verchota [5, (0.12)].

We now note a fact that will be crucial for solving the  $L^2$  Dirichlet problem in Chapter 5 and will fortify the hopes of translating results for  $\lambda = 0$  to  $\lambda \in \Sigma_\theta$ .

lem: compactness

**Lemma 3.9.** *Let  $\lambda \in \Sigma_\theta$  and  $d \geq 2$ . Then the operator  $\mathcal{K}_\lambda - \mathcal{K}_0$  on  $L^2(\partial\Omega; \mathbb{C}^d)$  is compact.*

*Proof.* The idea of this proof is similar to the one in Tolksdorf [25, Lemma 4.3.5]. Let  $f \in L^2(\partial\Omega; \mathbb{C}^d)$ . Let's denote  $\mathcal{K} := \mathcal{K}_\lambda - \mathcal{K}_0$ . We will now try to approximate  $\mathcal{K}$  by compact operators in the operator norm. To this end we define for all  $\varepsilon > 0$

$$(\mathcal{K}^{(\varepsilon)} f)(x) := \int_{\partial\Omega \setminus B(x, \varepsilon)} \nabla \{\Gamma(x - y; \lambda) - \Gamma(x - y; 0)\} f(y) d\sigma(y), \quad x \in \partial\Omega.$$

We can now estimate by Young's inequality 3.4

$$\|(\mathcal{K}^{(\varepsilon)} f)(x)\|_{L^2(\partial\Omega)} \leq \sup_{p \in \partial\Omega} \|\nabla \{\Gamma(p - \cdot; \lambda) - \Gamma(p - \cdot; 0)\} 1_{B(p, \varepsilon)}\|_{L^1(\partial\Omega)} \|f\|_{L^2(\partial\Omega)}.$$

Our goal is to show that

$$\sup_{p \in \partial\Omega} \|\nabla \{\Gamma(p - \cdot; \lambda) - \Gamma(p - \cdot; 0)\} 1_{B(p, \varepsilon)}\|_{L^1(\partial\Omega)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

To this end, let  $\varepsilon$  be small enough such that we can apply the estimates from Corollary 2.7 to calculate for some  $p \in \partial\Omega$

$$\begin{aligned} & \|\nabla \{\Gamma(p - \cdot; \lambda) - \Gamma(p - \cdot; 0)\} 1_{B(p, \varepsilon)}\|_{L^1(\partial\Omega)} \\ & \leq C \int_{\partial\Omega \cap B(p, \varepsilon)} \sqrt{|\lambda|} |p - y|^{2-d} d\sigma(y) \leq C \sqrt{|\lambda|} \varepsilon \end{aligned}$$

where for the last step we applied Lemma 3.1. For  $\varepsilon \rightarrow 0$  this gives us  $\mathcal{K}^{(\varepsilon)} \rightarrow \mathcal{K}$  in the operator norm.

The last step is to verify the compactness of  $\mathcal{K}^{(\varepsilon)}$ . We note that the integral kernel of  $\mathcal{K}^{(\varepsilon)}$  is bounded which gives us that in particular the kernel is an element of  $L^2(\partial\Omega \times \partial\Omega; \mathbb{C}^{d \times d})$ . The compactness of  $\mathcal{K}^{(\varepsilon)}$  now follows from Weidmann [29, Thm. 6.11].

As a consequence,  $\mathcal{K}$  is compact since the limit of compact operators with respect to the operator norm gives again a compact operator.  $\square$

Our next step is to introduce the *double layer potential*  $u(x) = \mathcal{D}_\lambda(f)(x)$  for the Stokes resolvent problem, where

$$u_j(x) = \int_{\partial\Omega} \left\{ \frac{\partial}{\partial y_i} \{ \Gamma_{jk}(y-x; \lambda) \} n_i(y) - \Phi_j(y-x) n_k(y) \right\} f_k(y) d\sigma(y). \quad \text{eq: defDoubleLayer} \quad (3.14)$$

The corresponding pressure  $\phi(x) = \mathcal{D}_\phi(f)(x)$  is defined via

$$\begin{aligned} \phi(x) = & \frac{\partial^2}{\partial x_i \partial x_k} \int_{\partial\Omega} G(y-x; 0) n_i(y) f_k(y) d\sigma(y) \\ & + \lambda \int_{\partial\Omega} G(y-x; 0) n_k(y) f_k(y) d\sigma(y). \end{aligned} \quad \text{eq: defDoubleLayerPressure} \quad (3.15)$$

Using (2.28) and (2.29) one can show that  $(u, \phi)$  defines again a solution to the Stokes resolvent problem in  $\mathbb{R}^d \setminus \partial\Omega$ .

The next theorem will give us a suitable operator which maps a given function  $f \in L^p(\partial\Omega; \mathbb{C}^d)$  to boundary values of  $u = \mathcal{D}_\lambda(f)$  in the form of nontangential limits.

thm: nontangentialLimitDoubleLayer

**Theorem 3.10.** *Let  $\lambda \in \Sigma_\theta$  and  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^d$ ,  $d \geq 2$ . Let  $u$  be given by (3.14) for  $f \in L^p(\partial\Omega; \mathbb{C}^d)$ ,  $1 < p < \infty$ . Then*

$$\|(u)^*\|_{L^p(\partial\Omega)} \leq C_p \|f\|_{L^p(\partial\Omega)} \quad \text{eq: lpBoundednessUNontangentialMax} \quad (3.16)$$

where  $C_p$  depends only on  $d, p, \theta$  and the Lipschitz character of  $\Omega$ . Furthermore

$$u_\pm = \left( \mp \frac{1}{2} I + \mathcal{K}_\lambda^* \right) f, \quad \text{eq: nontangentialLimitDoubleLayer} \quad (3.17)$$

where  $K_\lambda^*$  is the adjoint of the operator  $K_\lambda$  in (3.11)

*Proof.* The estimate for  $(u)^*$  is a direct consequence of Lemma 3.5, in particular of the estimates on  $(\nabla u)^*$  and  $(\phi)^*$ .

For the proof of (3.17), we begin by determining the adjoint of the operator  $\mathcal{K}_\lambda$ . To this end we first work with truncated operators  $\mathcal{K}_\lambda^{(\varepsilon)}$  which are defined as

$$(\mathcal{K}_\lambda^{(\varepsilon)} f)(x) = \int_{\partial\Omega} 1_{E(x, \varepsilon)} \nabla_x \Gamma(x-y; \lambda) f(y) d\sigma(y) n - \int_{\partial\Omega} 1_{E(x, \varepsilon)} \Phi_k(x-y) f_k(y) d\sigma(y) n,$$

for  $x \in \partial\Omega$  and  $E(x, \varepsilon) := \mathbb{R}^d \setminus B(x, \varepsilon)$ . Now for  $f \in L^p(\partial\Omega; \mathbb{C}^d)$  and  $g \in L^q(\partial\Omega; \mathbb{C}^d)$  with  $1/p + 1/q = 1$  we calculate

$$\begin{aligned} \langle \mathcal{K}_\lambda^{(\varepsilon)} f, g \rangle &= \int_{\partial\Omega} (\mathcal{K}_\lambda^{(\varepsilon)} f_j)(x) \overline{g_j(x)} d\sigma(x) \\ &= \int_{\partial\Omega} \int_{\partial\Omega} \frac{\partial}{\partial x_i} \{ \Gamma_{jk}(x-y; \bar{\lambda}) \} f_k(y) 1_{E(x, \varepsilon)}(y) d\sigma(y) n_i(x) \overline{g_j(x)} d\sigma(x) \end{aligned}$$

$$+ \int_{\partial\Omega} \int_{\partial\Omega} \Phi_k(x-y) f_k(y) 1_{E(x,\varepsilon)}(y) d\sigma(y) n_j(x) \overline{g_j(x)} d\sigma(x).$$

Note that  $1_{E(x,\varepsilon)}(y) = 1_{E(y,\varepsilon)}(x)$ . Now an application of Fubini and factoring out  $f_k(y)$  gives that the lengthy expression is equal to

$$\int_{\partial\Omega} f_k(y) \int_{\partial\Omega} \left\{ \frac{\partial}{\partial x_i} \{ \Gamma_{jk}(x-y; \bar{\lambda}) \} n_i(x) - \Phi_k(x-y) n_j(x) \right\} 1_{E(y,\varepsilon)}(x) \overline{g_j(x)} d\sigma(x) d\sigma(y).$$

Therefore we see that the adjoint of the truncated operator  $\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)}$  is given by

$$((K_{\bar{\lambda}}^{(\varepsilon)})^* g)_k(y) = \int_{\partial\Omega} \left\{ \frac{\partial}{\partial x_i} \{ \Gamma_{jk}(x-y; \lambda) \} n_i(x) - \Phi_k(x-y) n_j(x) \right\} 1_{E(y,\varepsilon)}(x) g_j(x) d\sigma(x),$$

for  $y \in \partial\Omega$  since  $\overline{\Gamma_{jk}(x-y; \bar{\lambda})} = \Gamma_{jk}(x-y; \bar{\lambda})$ .

In the next step we will go from truncated operators to principal value operators. For this to work we will look for suitable majorants. If  $x \in \partial\Omega$  we estimate

$$\begin{aligned} |(\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)} f)_j(x)| &= \left| \int_{|x-y|>\varepsilon} \frac{\partial}{\partial x_i} \{ \Gamma_{jk}(x-y; \lambda) \} f_k(y) d\sigma(y) n_i(x) \right. \\ &\quad \left. - \int_{|x-y|>\varepsilon} \Phi_k(x-y) f_k(y) n_j(x) d\sigma(y) \right| \\ &\leq T_{\lambda}^*(f)(x) + A^*(f)(x). \end{aligned}$$

Now dominated convergence gives

$$\lim_{\varepsilon \rightarrow 0} \langle K_{\bar{\lambda}}^{(\varepsilon)} f, g \rangle = \langle K_{\bar{\lambda}} f, g \rangle.$$

A similar argument gives

$$\lim_{\varepsilon \rightarrow 0} \langle f, K_{\bar{\lambda}}^{(\varepsilon)*} g \rangle = \langle f, K_{\bar{\lambda}}^* g \rangle,$$

where

$$((K_{\bar{\lambda}}^* g)_k(y) = \text{p. v.} \int_{\partial\Omega} \left\{ \frac{\partial}{\partial x_i} \{ \Gamma_{kj}(x-y; \lambda) \} n_i(x) - \Phi_k(x-y) n_j(x) \right\} g_j(x) d\sigma(x).$$

Note that we have used the symmetry of  $(\Gamma_{\alpha\beta})$ .

The last part now consists of proving that the equality (3.17) holds. To simplify the calculations and make Lemma 3.7 more accessible note that on the one hand

$$\begin{aligned} \int_{\partial\Omega} \frac{\partial}{\partial y_i} \{ \Gamma_{jk}(y-x; \lambda) \} n_i(y) f_k(y) d\sigma(y) &= - \int_{\partial\Omega} \frac{\partial}{\partial x_i} \{ \Gamma_{jk}(x-y; \lambda) \} n_i(y) f_k(y) d\sigma(y) \\ &= - \frac{\partial}{\partial x_i} \mathcal{S}(n_i f)_j(x) \end{aligned}$$

and on the other hand

$$- \int_{\partial\Omega} \Phi_j(y-x) n_k(y) f_k(y) d\sigma(y) = \int_{\partial\Omega} \Phi_l(x-y) \delta_{lj} n_k(y) f_k(y) d\sigma(y) = \mathcal{S}_\Phi(\tilde{f}^j)(x),$$

where  $\tilde{f}_l^j = \delta_{lj} n_k f_k$ . For  $x \in \partial\Omega$  we can now calculate

$$\begin{aligned} & \left( \int_{\partial\Omega} \frac{\partial}{\partial y_i} \{ \Gamma_{jk}(y-\cdot; \lambda) \} n_i(y) f_k(y) d\sigma(y) \right)_\pm(x) \\ &= - \left( \frac{\partial}{\partial x_i} \mathcal{S}_\lambda(n_i f)_j \right)_\pm(x) \\ &= \mp \frac{1}{2} \{ n_i(x) n_i(x) f_j(x) - n_j(x) n_i(x) n_k(x) n_i(x) f_k(x) \} \\ &\quad - \text{p. v.} \int_{\partial\Omega} \frac{\partial}{\partial x_i} \{ \Gamma_{jk}(x-y; \lambda) \} n_i(y) f_k(y) d\sigma(y) \\ &= \mp \frac{1}{2} \{ f_j(x) - n_j(x) n_k(x) f_k(x) \} \\ &\quad + \text{p. v.} \int_{\partial\Omega} \frac{\partial}{\partial y_i} \{ \Gamma_{jk}(x-y; \lambda) \} n_i(y) f_k(y) d\sigma(y), \end{aligned}$$

where we used trace formula (3.8). A similar procedure for the second integral part of the double layer potential gives

$$\begin{aligned} & - \left( \int_{\partial\Omega} \Phi_j(y-\cdot) n_k(y) f_k(y) d\sigma(y) \right)_\pm(x) \\ &= (\mathcal{S}_\Phi(\tilde{f}^j))_\pm(x) \\ &= \mp \frac{1}{2} n_k(x) \tilde{f}_k^j(x) - \text{p. v.} \int_{\partial\Omega} \Phi_k(x-y) \tilde{f}_k^j(x) d\sigma(y) \\ &= \mp \frac{1}{2} n_j(x) n_k(x) f_k(x) - \text{p. v.} \int_{\partial\Omega} \Phi_j(x-y) n_k(x) f_k(x) d\sigma(y) \end{aligned}$$

Putting everything together we get

$$(u_j)_\pm(x) = \mp \frac{1}{2} f_j(x) + (K_\lambda^* f)_j(x)$$

which proves the claim. □

# Chapter 4

## Rellich Estimates

chap:4

In this section we will establish Rellich type estimates for the Stokes resolvent problem. We will for this entire section always assume that  $\Omega$  is a bounded Lipschitz domain in  $\mathbb{R}^d$ ,  $d \geq 2$  with connected boundary and  $|\partial\Omega| = 1$ . Furthermore we will use the shorthand notation

$$\|\cdot\|_{\partial} := \|\cdot\|_{L^2(\partial\Omega)}.$$

The goal of this section is to derive an Rellich type inequality which will be used to prove the invertibility of the operators  $\pm(1/2)I + \mathcal{K}_{\lambda}$ . This inequality is part of the following theorem.

thm:rellich

**Theorem 4.1.** *Let  $\lambda \in \Sigma_{\theta}$  and  $|\lambda| \geq \tau$ , where  $\tau \in (0, 1)$ . Let  $(u, \phi)$  be a solution to the Stokes resolvent problem in  $\Omega$  and suppose that  $(\nabla u)^* \in L^2(\partial\Omega)$  and  $(\phi)^* \in L^2(\partial\Omega)$ . Furthermore, assume that  $\nabla u, \phi$  have nontangential limits almost everywhere on  $\partial\Omega$ . Then*

$$\|\nabla u\|_{\partial} + \|\phi - \int_{\partial\Omega} \phi\|_{\partial} \leq C\{\|\nabla_{\tan} u\|_{\partial} + |\lambda|^{1/2}\|u\|_{\partial} + |\lambda|\|u \cdot n\|_{H^{-1}(\partial\Omega)}\} \quad \text{eq:rellich1} \quad (4.1)$$

and

$$\|\nabla u\|_{\partial} + |\lambda|^{1/2}\|u\|_{\partial} + |\lambda|\|u \cdot n\|_{H^{-1}(\partial\Omega)} + \|\phi\|_{\partial} \leq C\left\|\frac{\partial u}{\partial \nu}\right\|_{\partial}, \quad \text{eq:rellich2} \quad (4.2)$$

where  $\frac{\partial}{\partial \nu}$  denotes the conormal derivative, and  $C$  depends only on  $d, \tau, \theta$  and the Lipschitz character of  $\Omega$ .

We will now prepare the proof of this theorem by proving several helpful lemmata.

lem:rellichIdentity

**Lemma 4.2.** *Under the same conditions on  $(u, \phi)$  as in Theorem 4.1, we have*

$$\begin{aligned} \int_{\partial\Omega} h_k n_k |\nabla u|^2 d\sigma &= 2 \operatorname{Re} \int_{\partial\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k} \left( \frac{\partial u}{\partial \nu} \right)_i d\sigma + \int_{\Omega} \operatorname{div}(h) |\nabla u|^2 dx \\ &\quad - 2 \operatorname{Re} \int_{\Omega} \frac{\partial h_k}{\partial x_j} \cdot \frac{\partial u_i}{\partial x_k} \cdot \frac{\partial \bar{u}_i}{\partial x_j} dx + 2 \operatorname{Re} \int_{\Omega} \frac{\partial h_k}{\partial x_i} \cdot \frac{\partial u_i}{\partial x_k} \bar{\phi} dx \\ &\quad - 2 \operatorname{Re} \int_{\Omega} h_k \frac{\partial u_i}{\partial x_k} \cdot \bar{\lambda} \bar{u}_i dx \end{aligned} \quad \text{eq:rellichIdentity (4.3)}$$

and

$$\begin{aligned} \int_{\partial\Omega} h_k n_k |\nabla u|^2 d\sigma &= 2 \operatorname{Re} \int_{\partial\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_j} \left\{ n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k} \right\} d\sigma \\ &\quad + 2 \operatorname{Re} \int_{\partial\Omega} h_k \bar{\phi} \left\{ n_i \frac{\partial u_i}{\partial x_k} - n_k \frac{\partial u_i}{\partial x_i} \right\} d\sigma - \int_{\Omega} \operatorname{div}(h) |\nabla u|^2 dx \\ &\quad + 2 \operatorname{Re} \int_{\Omega} \frac{\partial h_k}{\partial x_j} \cdot \frac{\partial u_i}{\partial x_k} \cdot \frac{\partial \bar{u}_i}{\partial x_j} dx - 2 \operatorname{Re} \int_{\Omega} \frac{\partial h_k}{\partial x_i} \cdot \frac{\partial u_i}{\partial x_k} \bar{\phi} dx \\ &\quad + 2 \operatorname{Re} \int_{\Omega} h_k \frac{\partial u_i}{\partial x_k} \cdot \bar{\lambda} \bar{u}_i dx, \end{aligned} \quad \text{eq:rellichIdentity2 (4.4)}$$

where  $h = (h_1, \dots, h_d) \in C_0^1(\mathbb{R}^d, \mathbb{R}^d)$ .

*Proof.* The proof of the stated identities reduces to several integrations by part once we establish its applicability. Approximating  $\partial\Omega$  by a sequence of  $C^\infty$  domains with uniform Lipschitz characters as described in Theorem 1.3 and the facts that on the one hand  $(\nabla u)^*, (\phi)^* \in L^2(\partial\Omega)$  and on the other hand  $\nabla u$  and  $\phi$  have nontantential limits almost everywhere, the integration by parts is justified. Details on how the approximation argument works can be found in Brown [1, Prop. 2.4]. For the sake of completeness we show the approximation argument once for a solid integral and once for a boundary integral.

Let  $(\Omega_l)_{l \in \mathbb{N}}$  denote the approximating sequence of  $C^\infty$  domains with outer normal  $n^{(l)}$ . Then

$$\int_{\partial\Omega_l} h_k n_k^{(l)} |\nabla u|^2 d\sigma_l = \int_{\partial\Omega} w_l(x) h_k(\Lambda_l(x)) n_k^{(l)}(\Lambda_l(x)) |\nabla u|^2(\Lambda_l(x)) d\sigma$$

Now we know that  $\lim_{l \rightarrow \infty} w_l(x) = 1$  and  $\lim_{l \rightarrow \infty} \Lambda_l(x) = x$  almost everywhere and  $\Lambda_l(x) \in \Gamma(x)$  for all  $l \in \mathbb{N}$ . Furthermore we know that  $\lim_{l \rightarrow \infty} n_k^{(l)} = n_k$  almost everywhere and that  $\nabla u$  has a nontangential limit almost everywhere. This gives us that the integrand converges almost everywhere to  $h_k(x) n_k(x) |\nabla u|^2(x)$ . Now furthermore we have that the integrand is dominated by  $\delta \|h\|_\infty ((\nabla u)^*)^2$ , where  $\delta$  is a uniform bound to  $w_l$ . Since by assumption  $(\nabla u)^* \in L^2(\partial\Omega)$ , the dominated convergence theorem gives us

$$\lim_{l \rightarrow \infty} \int_{\partial\Omega} w_l(x) h_k(\Lambda_l(x)) n_k^{(l)}(\Lambda_l(x)) |\nabla u|^2(\Lambda_l(x)) d\sigma = \int_{\partial\Omega} h_k n_k |\nabla u|^2 d\sigma.$$

It is easy to check, that the only differences in the approximation argument when applied to the other boundary integrals lie in the choice of the majorant. Now consider for instance the solid integral

We now start a formal calculation on  $\Omega$  and  $\partial\Omega$  keeping in mind that the stated equalities sometimes hold only after the application of the former approximation argument.

Let's expand the first summand in (4.3) using the definition of conormal derivatives

$$\begin{aligned} 2 \operatorname{Re} \int_{\partial\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k} \left( \frac{\partial u}{\partial \nu} \right)_i d\sigma &= 2 \operatorname{Re} \int_{\partial\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k} \left( \frac{\partial u_i}{\partial x_j} \right) n_j d\sigma - 2 \operatorname{Re} \int_{\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k} \phi n_i dx \\ &=: I_1 - I_2. \end{aligned}$$

For  $I_1$  we find using the divergence theorem

$$\begin{aligned} I_1 &= 2 \operatorname{Re} \int_{\Omega} \frac{\partial}{\partial x_j} \left( h_k \frac{\partial}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_k} \frac{\partial u_i}{\partial x_j} \right) dx \\ &= 2 \operatorname{Re} \int_{\Omega} \frac{\partial h_k}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_k} \frac{\partial u_i}{\partial x_j} + h_k \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_k} \frac{\partial u_i}{\partial x_j} + h_k \frac{\partial \bar{u}_i}{\partial x_k} \frac{\partial^2 u_i}{\partial x_j^2} dx \\ &=: I_3 + I_4 + I_5. \end{aligned}$$

For  $I_5$  we use the fact that  $u$  solves the Stokes resolvent problem which gives

$$\begin{aligned} I_5 &= 2 \operatorname{Re} \int_{\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k} \frac{\partial \phi}{\partial x_i} dx + 2 \operatorname{Re} \int_{\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k} \lambda u_i dx \\ &=: I_6 + I_7. \end{aligned}$$

Another application of the divergence theorem gives

$$\begin{aligned} I_2 &= 2 \operatorname{Re} \int_{\Omega} \frac{\partial}{\partial x_i} \left( h_k \frac{\partial \bar{u}_i}{\partial x_k} \phi \right) dx \\ &= 2 \operatorname{Re} \int_{\Omega} \frac{\partial h_k}{\partial x_i} \frac{\partial \bar{u}_i}{\partial x_k} \phi + h_k \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_k} \phi + h_k \frac{\partial \bar{u}_i}{\partial x_k} \frac{\partial \phi}{\partial x_i} dx \\ &=: I_8 + I_9 + I_{10}. \end{aligned}$$

One term that hasn't come up so far, the second summand of the right side in (4.3), will now be expanded

$$\begin{aligned} \operatorname{div}(h) |\nabla u|^2 dx &= \int_{\Omega} \operatorname{div}(h |\nabla u|^2) dx - \int_{\Omega} h_k \frac{\partial}{\partial x_i} |\nabla u|^2 dx \\ &=: I_{10} - I_{11}. \end{aligned}$$

Expanding this further gives us

$$I_{11} = \int_{\Omega} h_i \frac{\partial}{\partial x_i} \left( \frac{\partial u_k}{\partial x_j} \frac{\partial \bar{u}_k}{\partial x_j} \right) dx = \int_{\Omega} h_i \frac{\partial^2 u_k}{\partial x_i \partial x_j} \frac{\partial \bar{u}_k}{\partial x_j} + \frac{\partial u_k}{\partial x_j} \frac{\partial^2 \bar{u}_k}{\partial x_i \partial x_j} = I_4.$$

If we now put everything together, the right side of (4.3) reads

$$\begin{aligned} & (I_1 - I_2) + (I_{10} - I_{11}) - I_3 + I_8 - I_7 \\ &= (I_3 + I_4 + I_6 + I_7) - (I_8 + I_9 + I_6) + I_{10} - I_{11} - I_3 + I_8 - I_7 = I_{10}. \end{aligned}$$

Noting that by the divergence theorem we have

$$I_{10} = \int_{\partial\Omega} h_k n_k |\nabla u|^2 d\sigma,$$

the first identity is proven.

In order to prove identity (4.4), we show that the expression we get from considering ((4.3) + (4.4)) holds, i.e. we show the identity

$$\begin{aligned} 2 \int_{\partial\Omega} h_k n_k |\nabla u|^2 d\sigma &= 2 \operatorname{Re} \int_{\partial\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k} \left( \frac{\partial u}{\partial \nu} \right)_i \\ &\quad + 2 \operatorname{Re} \int_{\partial\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_j} \left\{ n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k} \right\} d\sigma \\ &\quad + 2 \operatorname{Re} \int_{\partial\Omega} h_k \bar{\phi} \left\{ n_i \frac{\partial u_i}{\partial x_i} - n_k \frac{\partial u_i}{\partial x_i} \right\} d\sigma. \end{aligned}$$

To this end, note that the left side of the identity equals  $2I_{10}$ , whereas the right side can be written as

$$I_1 - I_2 + 2I_{10} - I_1 + I_2 = 0,$$

where we used the fact that  $\operatorname{div} u = 0$ . □

We note that the operators

$$\left\{ n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k} \right\} \quad \text{and} \quad \left\{ n_i \frac{\partial u_i}{\partial x_k} - n_k \frac{\partial u_i}{\partial x_i} \right\}$$

are the *first-order tangential derivative operators* which can be found in Mitrea and Wrigth [16].

We make a quick detour that gives us the following lemma.

lem:lambdaIneq

**Lemma 4.3.** *Let  $\theta \in (0, \pi/2)$ . Then there exists  $\alpha$  depending only on  $\theta$  such that*

$$\operatorname{Re}(\lambda) + \alpha |\operatorname{Im}(\lambda)| \geq |\lambda|$$

for all  $\lambda \in \Sigma_\theta$ .



*Proof.* For  $|\lambda| = 1$  we have  $\operatorname{Re}(\lambda) = \cos(\varphi)$  and  $\operatorname{Im}(\lambda) = \sin(\varphi)$  for some  $\varphi \in (0, \pi - \theta)$ . Set

$$\alpha = \frac{1 - \cos(\pi - \theta)}{\sin(\pi - \theta)} \geq \frac{1 - \cos(\varphi)}{\sin(\varphi)}.$$

Then we have

$$\operatorname{Re}(\lambda) + \alpha |\operatorname{Im}(\lambda)| = \cos(\varphi) + \alpha \sin(\varphi) \geq 1.$$

For arbitrary  $\lambda$  the claim follows by considering the normalized value  $\lambda/|\lambda|$ .  $\square$

The next lemma enables us to handle the solid integrals in (4.3) and (4.4).

lem:laxMilgramIneq

**Lemma 4.4.** *Under the same assumptions on  $(u, \phi)$  and  $\lambda$  as in Theorem 4.1, we have*

$$\int_{\Omega} |\nabla u|^2 \, dx + |\lambda| \int_{\Omega} |u|^2 \, dx \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial} \|u\|_{\partial}, \quad \text{eq:laxMilgramIneq (4.5)}$$

where  $C$  depends only on  $\theta$ .

*Proof.* Testing the Stokes resolvent problem against the solution  $u$  gives us

$$\int_{\Omega} -\Delta u \cdot \bar{u} \, dx + \lambda \int_{\Omega} u \cdot \bar{u} \, dx = \int_{\Omega} -\nabla \phi \cdot \bar{u} \, dx$$

Using integration by parts which may, as in the proof of the previous lemma, be justified by an approximation argument, we get

$$\int_{\Omega} |\nabla u|^2 \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot \bar{u} \, d\sigma + \lambda \int_{\Omega} |u|^2 \, dx = \int_{\partial\Omega} \Phi n \cdot \bar{u} \, d\sigma$$

or with the definition of conormal derivatives

$$\int_{\Omega} |\nabla u|^2 \, dx + \lambda \int_{\Omega} |u|^2 \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \bar{u} \, d\sigma. \quad \text{eq:testedStokes (4.6)}$$

If we now take the real and imaginary part of (4.6) and sum them up with a prefactor  $\alpha > 0$ , we get

$$\int_{\Omega} |\nabla u|^2 \, dx + \{\operatorname{Re}(\lambda) + \alpha |\operatorname{Im}(\lambda)|\} \int_{\Omega} |u|^2 \, dx \leq (1 + \alpha) \left| \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \bar{u} \, d\sigma \right|.$$

Lemma 4.3 now gives

$$\int_{\Omega} |\nabla u|^2 \, dx + |\lambda| \int_{\Omega} |u|^2 \, dx \leq C \left| \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \bar{u} \, d\sigma \right|,$$

from which we get estimate (4.5) after applying the Cauchy-Schwartz inequality.  $\square$

The next lemma combines Rellich identities (4.3) and (4.4) with estimate (4.5).

**Lemma 4.5.** *Under the same assumptions on  $(u, \phi)$  and  $\lambda$  as in Theorem 4.1, we have*

$$\|\nabla u\|_{\partial} \leq C_{\varepsilon} \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial} + \varepsilon \left\{ \|\nabla u\|_{\partial} + \|\phi\|_{\partial} + \|\lambda\|^{1/2} \|u\|_{\partial} \right\} \quad \text{eq:gradEstimateRellich} \quad (4.7)$$

and

$$\|\nabla u\|_{\partial} \leq C_{\varepsilon} \left\{ \|\nabla_{\tan} u\|_{\partial} + \|\lambda\|^{1/2} \|u\|_{\partial} \right\} + \varepsilon \left\{ \|\nabla u\|_{\partial} + \|\phi\|_{\partial} \right\} \quad \text{eq:gradEstimateRellich2} \quad (4.8)$$

for all  $\varepsilon \in (0, 1)$ , where  $C_{\varepsilon}$  depends only on  $d, \theta, \tau, \varepsilon$  and the Lipschitz character of  $\Omega$ .

*Proof.* Let  $h = (h_1, \dots, h_d) \in C_0^1(\mathbb{R}^d, \mathbb{R}^d)$  with  $h_k n_k \geq c > 0$  on  $\partial\Omega$ . The existence of this vector field follows from Verchota. Now in view of identity (??), we have

$$\|\nabla u\|_{\partial}^2 \leq C \left\{ \|\nabla u\|_{\partial} \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial} + \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla u| |\phi| dx + |\lambda| \int_{\Omega} |\nabla u| |u| dx \right\}, \quad \text{eq:normRellich} \quad (4.9)$$

where the first term follows from the Cauchy-Schwartz inequality. Since  $\Delta\phi = 0$  and the nontangential maximal function  $(\phi)^* \in L^2(\partial\Omega)$  a result from Dahlberg [3] gives

$$\int_{\Omega} |\phi|^2 dx \leq C \|(\phi)^*\|_{\partial}^2 \leq C \|\phi\|_{\partial}^2. \quad \text{eq:dahlbergEstimate} \quad (4.10)$$

The last summand of (4.9) can be estimated as follows

$$\begin{aligned} |\lambda| \int_{\Omega} |\nabla u| |u| dx &\leq |\lambda| \left\{ \frac{|\lambda|^{1/2}}{2} \int_{\Omega} |u|^2 dx + \frac{1}{2|\lambda|^{1/2}} \int_{\Omega} |\nabla u|^2 dx \right\} \\ &\leq C \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial} \|\lambda\|^{1/2} \|u\|_{\partial}, \end{aligned} \quad \text{eq:lambdaNablaU} \quad (4.11)$$

where in the first step we used the weighed Young inequality and in the second step we applied estimate (4.5). Similarly we calculate

$$\int_{\Omega} |\nabla u| |\phi| dx \leq \left( \int_{\Omega} |\nabla u|^2 dx \right)^{1/2} \left( \int_{\Omega} |\phi|^2 dx \right)^{1/2} \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial}^{1/2} \|u\|_{\partial}^{1/2} \|\phi\|_{\partial}, \quad \text{eq:nablaPhi} \quad (4.12)$$

where the first step is just the Cauchy-Schwartz inequality and the second step combines estimate (??) with estimate (4.10). Putting everything together, we calculate

$$\begin{aligned} \|\nabla u\|_{\partial}^2 &\leq C \|\nabla u\|_{\partial} \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial} + C \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial} \|u\|_{\partial} + C \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial}^{1/2} \|u\|_{\partial}^{1/2} \|\phi\|_{\partial} \\ &\quad + C \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial} \|\lambda\|^{1/2} \|u\|_{\partial}. \end{aligned}$$

Note that we used the fact that  $|\lambda| \geq \tau$  to bound  $\|u\|_{\partial}$  as

$$\|u\|_{\partial} \leq \frac{|\lambda|^{1/2}}{\tau^{1/2}} \|u\|_{\partial} = C |\lambda|^{1/2} \|u\|_{\partial}.$$

The desired estimate (4.7) now follows applying Young's weighted inequality and the norm equivalence on finite dimensional vector spaces.

For inequality (4.8) we use the identity (4.4) and obtain

$$\begin{aligned} \|\nabla u\|_{\partial}^2 &\leq C\|\nabla_{\tan} u\|_{\partial}\{\|\nabla u\|_{\partial} + \|\phi\|_{\partial}\} + C \int_{\Omega} |\nabla u|^2 dx \\ &\quad + C \int_{\Omega} |\nabla u| |\phi| dx + C|\lambda| \int_{\Omega} |\nabla u| |u| dx. \end{aligned}$$

Using estimates (4.5), (4.10), (4.11) and (4.12) together with the weighted Young inequality gives us

$$\|\nabla u\|_{\partial}^2 \leq C_{\varepsilon}\{\|\nabla_{\tan} u\|_{\partial}^2 + \|\lambda\|^{1/2} u\|_{\partial}^2\} + \varepsilon\{\|\nabla u\|_{\partial}^2 + \|\phi\|_{\partial}^2 + \frac{1}{4}\|\frac{\partial u}{\partial \nu}\|_{\partial}^2\}.$$

The claim now follows if we use the definition of the conormal derivative and the norm equivalence on finite dimensional vector spaces.  $\square$

We prove one last lemma before we tackle the central theorem of this chapter.

**Lemma 4.6.** *Assume that  $(u, \phi)$  satisfies the same conditions as in Theorem 4.1. Then*

$$\|\phi - \int_{\partial\Omega} \phi\|_{\partial} \leq C\{\|\nabla u\|_{\partial} + |\lambda|\|u \cdot n\|_{H^{-1}(\partial\Omega)}\} \quad \text{eq:phiDashintPhi} \quad (4.13)$$

and

$$|\lambda|\|u \cdot n\|_{H^{-1}(\partial\Omega)} \leq C\{\|\phi\|_{\partial} + \|\nabla u\|_{\partial}\}, \quad \text{eq:lambdaun} \quad (4.14)$$

where  $C$  depends only on  $d$  and the Lipschitz character of  $\Omega$ .

*Proof.* By Theorem 1.3 we may assume that  $\Delta u = \nabla \phi + \lambda u$  on  $\partial\Omega$ . Multiplying the Stokes resolvent equation on  $\partial\Omega$  with  $n$  and using the triangle inequality gives

$$\begin{aligned} \|\nabla \phi \cdot n\|_{H^{-1}(\partial\Omega)} &\leq \|\Delta u \cdot n\|_{H^{-1}(\partial\Omega)} + |\lambda|\|u \cdot n\|_{H^{-1}(\partial\Omega)}, \\ |\lambda|\|u \cdot n\|_{H^{-1}(\partial\Omega)} &\leq \|\Delta u \cdot n\|_{H^{-1}(\partial\Omega)} + \|\nabla \phi \cdot n\|_{H^{-1}(\partial\Omega)}. \end{aligned} \quad \text{eq:stokesEquationH1} \quad (4.15)$$

We will now show that

$$\|\Delta u \cdot n\|_{H^{-1}(\partial\Omega)} \leq C\|\nabla u\|_{\partial} \quad \text{eq:deltaun} \quad (4.16)$$

and

$$c\|\phi - \int_{\partial\Omega} \phi d\sigma\|_{\partial} \leq \|\nabla \phi \cdot n\|_{H^{-1}(\partial\Omega)} \leq C\|\phi\|_{\partial} \quad \text{eq:nablaPhiin} \quad (4.17)$$

Using these two estimates applied to (4.15), we can directly derive (4.13) and (4.14).

In order to prove (4.16), note that

$$\Delta u \cdot n = n_i \frac{\partial^2 u_i}{\partial x_j^2} = \left( n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j}$$

since  $\operatorname{div} u = 0$  in  $\overline{\Omega}$ . As the expression in between the brackets is a tangential derivative we derive estimate (4.16) from

$$|\langle \Delta u \cdot n, u \rangle| = |\langle \nabla u, \nabla_{\tan} u \rangle| \leq \|\nabla u\|_{\partial}^2$$

since this implies

$$\|\nabla u \cdot n\|_{H^{-1}(\partial\Omega)} \leq \|\nabla u\|_{\partial}.$$

Now for the proof of estimate (4.17) we will use  $L^2$ -estimates for the Neumann and regularity problems for the Laplace equation in Lipschitz domains. For  $g \in L^2(\partial\Omega)$  with mean value zero, by Jerison and Kenig [?] the Neumann problem for Laplace's equation on the Lipschitz domain  $\Omega$  has a solution  $\psi$  with  $(\nabla\psi)^* \in L^2(\partial\Omega)$  and  $\frac{\partial\psi}{\partial n} = g$  on  $\partial\Omega$ . Green's identity we have that since  $\phi$  and  $\psi$  are harmonic

$$\begin{aligned} \left| \int_{\partial\Omega} \phi g \, d\sigma \right| &= \left| \int_{\partial\Omega} \phi \frac{\partial\psi}{\partial n} \, d\sigma \right| = \left| \int_{\partial\Omega} \frac{\partial\phi}{\partial n} \psi \, d\sigma \right| \\ &\leq \left\| \frac{\partial\phi}{\partial n} \right\|_{H^{-1}(\partial\Omega)} \|\psi\|_{H^1(\partial\Omega)} \leq C \left\| \frac{\partial\phi}{\partial n} \right\|_{H^{-1}(\partial\Omega)} \|g\|_{\partial}, \end{aligned} \quad \text{eq:dualityPhi (4.18)}$$

where in the last step we used the estimate  $\|\psi\|_{H^1(\partial\Omega)} \leq C\|g\|_{\partial}$  for the  $L^2$  Neumann problem which can be found in Jerison and Kenig [?]. Now if we set  $\bar{g} = \phi - \tilde{\phi}$ , with  $\tilde{\phi} = \int_{\partial\Omega} \phi \, d\sigma$  and use that  $\int_{\partial\Omega} (\phi - \tilde{\phi})(\phi - \tilde{\phi}) \, d\sigma = \int_{\partial\Omega} \phi(\phi - \tilde{\phi}) \, d\sigma$ , we get from (4.18)

$$\|\phi - \tilde{\phi}\|_{\partial}^2 \leq C \left\| \frac{\partial\phi}{\partial n} \right\|_{H^{-1}(\partial\Omega)} \|\phi - \tilde{\phi}\|_{\partial}$$

or, after rearranging and expanding

$$\left\| \phi - \int_{\partial\Omega} \phi \, d\sigma \right\|_{\partial} \leq C \left\| \frac{\partial\phi}{\partial n} \right\|_{H^{-1}(\partial\Omega)}$$

We work in a similar way with results from the regularity problem of Laplace's equation by Jerison and Kenig [?]. Given  $f \in H^1(\partial\Omega)$ , there exists a harmonic function  $\psi$  in  $\Omega$  such that  $(\nabla\psi)^* \in L^2(\partial\Omega)$  and  $\psi = f$  on  $\partial\Omega$ . As for (4.18), we calculate

$$\begin{aligned} \left| \int_{\partial\Omega} \frac{\partial\phi}{f} \, d\sigma \right| &= \left| \int_{\partial\Omega} \frac{\partial\phi}{\psi} \, d\sigma \right| = \left| \int_{\partial\Omega} \phi \frac{\partial\psi}{\partial n} \, d\sigma \right| \\ &\leq \|\phi\|_{\partial} \|\nabla\psi\|_{\partial} \leq C \|\phi\|_{\partial} \|f\|_{H^1(\partial\Omega)}, \end{aligned}$$

where in the last step we used the estimate  $\|\nabla\psi\|_{\partial} \leq C\|f\|_{H^1(\partial\Omega)}$  for the  $L^2$  regularity problem. By duality this gives that

$$\left\|\frac{\partial\phi}{\partial n}\right\|_{H^{-1}(\partial\Omega)} \leq C\|\phi\|_{\partial}.$$

□

rem:harmonicEstimate

**Remark 4.7.** A careful look at the proof of inequality (4.17) reveals that the estimate

$$c\|\phi\|_{\partial} \leq \|\nabla\phi \cdot n\|_{H^{-1}(\partial\Omega)},$$

holds for all harmonic functions  $\phi$  with vanishing mean on  $\partial\Omega$ .

After all this preparation we are now able to prove Theorem 4.1.

*Proof of Theorem 4.1.* For the proof of estimate (4.1), without loss of generality we can assume that  $\int_{\partial\Omega} \phi \, d\sigma = 0$ .

Using (4.13) for the second summand in (4.1) and then (4.8) for the terms involving  $\nabla u$  we get

$$\begin{aligned} \|\nabla u\|_{\partial} + \|\phi\|_{\partial} &\leq C\{\|\nabla u\|_{\partial} + |\lambda|\|u \cdot n\|_{H^1(\partial\Omega)}\} \\ &\leq C_{\varepsilon}\{\|\nabla_{\tan} u\|_{\partial} + |\lambda|^{1/2}\|u\|_{\partial} + |\lambda|\|u \cdot n\|_{H^{-1}(\partial\Omega)}\} \\ &\quad + C\varepsilon\{\|\nabla u\|_{\partial} + \|\phi\|_{\partial}\} \end{aligned}$$

for all  $\varepsilon \in (0, 1)$ . Choosing  $\varepsilon$  such that  $C\varepsilon < (1/2)$  we can rearrange the above inequality and obtain estimate (4.1).

Estimate (4.2) will need more effort to be proven. We start with inequality (4.14) and derive

$$\|\nabla u\|_{\partial} + \|\phi\|_{\partial} + |\lambda|\|u \cdot n\|_{H^{-1}(\partial\Omega)} \leq C\{\|\nabla u\|_{\partial} + \|\phi\|_{\partial}\} \leq C\left\{\left\|\frac{\partial u}{\partial \nu}\right\|_{\partial} + \|\nabla u\|_{\partial}\right\},$$

where in the last step we used the definition of conormal derivatives. If we now apply (4.7) we get

$$\|\nabla u\|_{\partial} + \|\phi\|_{\partial} + |\lambda|\|u \cdot n\|_{H^{-1}(\partial\Omega)} \leq C_{\varepsilon}\left\|\frac{\partial u}{\partial \nu}\right\|_{\partial} + \varepsilon\{\|\nabla u\|_{\partial} + \|\phi\|_{\partial} + \||\lambda|^{1/2}u\|_{\partial}\}$$

for all  $\varepsilon \in (0, 1)$ . Choosing  $\varepsilon$  appropriately yields

$$\|\nabla u\|_{\partial} + \|\phi\|_{\partial} + |\lambda|\|u \cdot n\|_{H^{-1}(\partial\Omega)} \leq C\left\|\frac{\partial u}{\partial \nu}\right\|_{\partial} + C|\lambda|^{1/2}\|u\|_{\partial}. \quad (4.19)$$

Now we need to estimate  $|\lambda|^{1/2}\|u\|_{\partial}$ . Green's identity yields

$$\begin{aligned} \int_{\partial\Omega} h_k n_k |u|^2 d\sigma &= \int_{\Omega} \frac{\partial}{\partial x_k} (h_k |u|^2) dx = \int_{\Omega} \frac{\partial h_k}{\partial x_k} |u|^2 dx + \int_{\Omega} h_k \frac{\partial |u|^2}{\partial x_k} dx \\ &= \int_{\Omega} \operatorname{div}(h) |u|^2 dx + 2 \operatorname{Re} \int_{\Omega} h_k \frac{\partial \bar{u}_i}{\partial x_k} u_i dx. \end{aligned} \quad \text{eq:hknkgreen (4.20)}$$

We choose  $h \in C_0^1(\mathbb{R}^d, \mathbb{R}^d)$  with  $h_k n_k \geq c > 0$  on  $\partial\Omega$ . The existence of such a function  $h$  follows from Theorem 1.3. Using this, we can continue the estimate (4.20) as

$$\|u\|_{\partial}^2 \leq C \int_{\Omega} |u|^2 dx + C \int_{\Omega} |u| |\nabla u| dx. \quad \text{eq:estupartial (4.21)}$$

The next estimate uses (4.21) and (4.5) which gives

$$\begin{aligned} |\lambda| \|u\|_{\partial}^2 &\leq |\lambda| C \int_{\Omega} |u|^2 dx + |\lambda| C \int_{\Omega} |u| |\nabla u| dx \\ &\leq C \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial} \|u\|_{\partial} + |\lambda|^{1/2} C \int_{\Omega} (|\lambda|^{1/2} |u|) |\nabla u| dx \\ &\leq C \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial} \|u\|_{\partial} + |\lambda|^{1/2} C \left( \int_{\Omega} |\lambda| |u|^2 \right)^{1/2} \left( \int_{\Omega} |\nabla u|^2 dx \right)^{1/2} \\ &\leq C \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial} \| |\lambda|^{1/2} u \|_{\partial}. \end{aligned}$$

Note that for the last estimate we also used the fact that  $|\lambda| \geq \tau$  helps us to bound  $\|u\|_{\partial}$  by  $C|\lambda|^{1/2}\|u\|_{\partial}$ . Rearranging terms in the last estimate, we now derive

$$\| |\lambda|^{1/2} u \|_{\partial} \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_{\partial}. \quad \text{eq:lambda12u (4.22)}$$

Estimate (4.2) follows directly from (4.19) in combination with (4.22) and this concludes our proof.  $\square$

Shen proved that under reasonable assumptions a theorem similar to 4.1 also holds for exterior domains.

thm:rellichExterior

**Theorem 4.8.** *Let  $\lambda \in \Sigma_{\theta}$  and  $|\lambda| \geq \tau$ , where  $\tau \in (0, 1)$ . Let  $(u, \phi)$  be a solution of the Stokes resolvent Problem in  $\Omega_- = \mathbb{R}^d \setminus \bar{\Omega}$ . Suppose additionally that  $(\nabla u)^*, (\phi)^* \in L^2(\partial\Omega)$  and that  $\nabla u, \phi$  have nontangential limits almost everywhere on  $\partial\Omega$ . Furthermore let for  $|x| \rightarrow \infty$*

$$|\phi(x)| + |\nabla u(x)| = O(|x|^{1-d}) \quad \text{and} \quad u(x) = \begin{cases} O(|x|^{2-d}) & \text{if } d \geq 3 \\ o(1) & \text{if } d = 2. \end{cases}$$

Then

$$\|\nabla u\|_{\partial} + \|\phi\|_{\partial} \leq C\{\|\nabla_{\tan} u\|_{\partial} + |\lambda|^{1/2}\|u\|_{\partial} + |\lambda|\|u \cdot n\|_{H^{-1}(\partial\Omega)}\} \quad \text{eq:rellich1ext} \quad (4.23)$$

and

$$\|\nabla u\|_{\partial} + |\lambda|^{1/2}\|u\|_{\partial} + |\lambda|\|u \cdot n\|_{H^{-1}(\partial\Omega)} + \|\phi\|_{\partial} \leq C\left\|\frac{\partial u}{\partial \nu}\right\|_{\partial}, \quad \text{eq:rellich2ext} \quad (4.24)$$

where  $C$  depends only on  $d$ ,  $\tau$ ,  $\theta$  and the Lipschitz character of  $\Omega$ .

# Chapter 5

## Solving the $L^2$ -Dirichlet Problem

This section is all about the application of the method of layer potentials to solve the  $L^2$ -Dirichlet problem for the Stokes resolvent system. Furthermore we will establish a uniform  $L^p$ -estimate for the nontangential maximal function which will be important for the proof of our central theorem.

For the remainder of this chapter let  $\Omega$  always denote a bounded Lipschitz domain in  $\mathbb{R}^d$ ,  $d \geq 2$ , with connected boundary. We will use  $L_n^2(\partial\Omega)$  to denote the function space

$$L_n^2(\partial\Omega) := \left\{ f \in L^2(\partial\Omega; \mathbb{C}^d) : \int_{\partial\Omega} f \cdot n \, d\sigma = 0 \right\},$$

and  $L_0^2(\partial\Omega; \mathbb{C}^d)$  to denote the function space of  $L^2$  functions with mean value zero. As before,  $\|\cdot\|_{\partial}$  stands for the norm of  $L^2(\partial\Omega)$ .

The following results will build on the application of Theorem 4.8. Therefore the next Lemma shows that solutions to the Stokes resolvent problem that are given by single layer potentials fulfill the requirements of the theorem.

lem:requirements

**Lemma 5.1.** *Let  $\lambda \in \Sigma_\theta$  and  $(u, \phi)$  be given by (3.1) and (3.2), respectively. Then the following holds for  $|x| \rightarrow \infty$ :*

$$|\phi(x)| + |\nabla u(x)| = O(|x|^{1-d}) \quad \text{and} \quad u(x) = \begin{cases} O(|x|^{2-d}) & \text{if } d \geq 3 \\ o(1) & \text{if } d = 2. \end{cases}$$

*Proof.* If  $d \geq 2$  then an application of the dominated convergence theorem gives that  $|\phi(x)| + |\nabla u(x)| = O(|x|^{1-d})$  as  $|\phi(x)| = O(|\Phi_k(x)|) = O(|x|^{1-d})$  by (2.28) and  $|\nabla u(x)| = O(|x|^{1-d})$  by estimate 2.30.

If  $d \geq 3$  then Lemma 2.1 and an application of the dominated convergence Theorem gives  $u(x) = O(|x|^{2-d})$ . □



We will now derive bounds on the inverse operator of  $(1/2)I + \mathcal{K}_\lambda$  from Chapter 3.

**Lemma 5.2.** *Let  $\lambda \in \Sigma_\theta$  and  $|\lambda| \geq \tau$ , where  $\tau \in (0, 1)$ . Suppose that  $|\partial\Omega| = 1$ . Then  $(1/2)I + \mathcal{K}_\lambda$  is an isomorphism on  $L^2(\partial\Omega; \mathbb{C}^d)$  and*

$$\|f\|_\partial \leq C\|((1/2)I + \mathcal{K}_\lambda)f\|_\partial \quad \text{for any } f \in L^2(\partial\Omega; \mathbb{C}^d), \quad \text{eq:inverseEstimate} \quad (5.1)$$

where  $C$  depends only on  $d, \theta, \tau$  and the Lipschitz character of  $\Omega$ .

*Proof.* We start with  $f \in L^2(\partial\Omega; \mathbb{C}^d)$  and the corresponding single layer potentials  $u = \mathcal{S}_\lambda(f)$  and  $\phi = \mathcal{S}_\Phi(f)$  given by (3.1) and (3.2). We saw in Chapter 3 that  $(u, \phi)$  solves the Stokes resolvent problem in  $\mathbb{R}^d \setminus \partial\Omega$  and got from Lemma 3.5 with  $p = 2$  for the nontangential maximal functions that  $(\nabla u)^*, (\phi)^* \in L^2(\partial\Omega)$ . We furthermore saw in Lemma 3.7 that  $\nabla u$  and  $\phi$  have nontangential limits almost everywhere on  $\partial\Omega$ . Finally in Theorem 3.8 we saw that  $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$  and derived the jump condition  $(\frac{\partial u}{\partial \nu})_\pm = (\pm(1/2)I + \mathcal{K}_\lambda)f$ .

Our next step will be to show the estimate

$$\|\nabla u_-\|_\partial + \|\phi_-\|_\partial \leq C\|(\frac{\partial u}{\partial \nu})_-\|_\partial. \quad \text{eq:negNablaPhi} \quad (5.2)$$

Assuming that (5.2) holds we can prove (5.1): Set  $f = (\frac{\partial u}{\partial \nu})_+ - (\frac{\partial u}{\partial \nu})_-$ . Then this gives with the definition of the conormal derivative and estimate (5.2) that

$$\begin{aligned} \|f\|_\partial &\leq \|(\frac{\partial u}{\partial \nu})_+\|_\partial + \|(\frac{\partial u}{\partial \nu})_-\|_\partial \\ &\leq \|(\frac{\partial u}{\partial \nu})_+\|_\partial + \|(\frac{\partial u}{\partial n})_-\|_\partial + \|\phi_-\|_\partial \\ &\leq C\|(\frac{\partial u}{\partial \nu})_+\|_\partial = C\|(1/2)I + \mathcal{K}_\lambda)f\|_\partial. \end{aligned}$$

In order to prove (5.2), note that since  $|u(x)| + |\nabla u(x)| = O(|x|^{-N})$  for all  $N > 0$  and  $\phi(x) = O(|x|^{1-d})$  as  $|x| \rightarrow \infty$  we can use Theorem 4.8 to derive

$$\begin{aligned} \|\nabla u_-\|_\partial + \|\phi_-\|_\partial &\leq C\left\{\|\nabla_{\tan} u_-\|_\partial + |\lambda|^{1/2}\|u_-\|_\partial + |\lambda|\|n \cdot u_-\|_{H^{-1}(\partial\Omega)}\right\} \\ &= C\left\{\|\nabla_{\tan} u_+\|_\partial + |\lambda|^{1/2}\|u_+\|_\partial + |\lambda|\|n \cdot u_+\|_{H^{-1}(\partial\Omega)}\right\}, \quad \text{eq:nablauMinus} \quad (5.3) \end{aligned}$$

where we used the fact that  $u_+ = u_-$  and  $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$  on  $\partial\Omega$ . Inequality (4.2) of Theorem 4.1 now allows us to estimate the right hand side of (5.3) by  $C\|(\frac{\partial u}{\partial \nu})_+\|_\partial$  and thus the desired estimate (5.2) follows.

Let's now work on the invertibility of  $(1/2)I + \mathcal{K}_\lambda$ . In the case  $\lambda = 0$ , Fabes, Kenig and Verchota showed in [5] that  $(1/2)I + \mathcal{K}_0$  as an operator on  $L^2(\partial\Omega; \mathbb{R}^d)$  has a

one dimensional null space and as range the space  $L^2_0(\partial\Omega; \mathbb{R}^d)$ . Thus  $(1/2)I + \mathcal{K}_0$  has Fredholm index 0. This remains true if we replace  $L^2(\partial\Omega; \mathbb{R}^d)$  by  $L^2(\partial\Omega; \mathbb{C}^d)$  as this just corresponds to a complexification of the vector space and the operator. Since the operator  $\mathcal{K}_\lambda - \mathcal{K}_0$  is compact on  $L^2(\partial\Omega; \mathbb{C}^d)$  by Lemma 3.9, we deduce that the operator

$$(1/2)I + \mathcal{K}_\lambda = (1/2)I + \mathcal{K}_0 + (\mathcal{K}_\lambda - \mathcal{K}_0)$$

has the Fredholm index zero as well for all  $\lambda \in \Sigma_\theta$ . Now inequality (5.1) gives that  $(1/2)I + \mathcal{K}_\lambda$  is injective and thus the Fredholm index of zero implies that it is also surjective and hence an isomorphism.  $\square$

The next lemma works with the counterpart of  $(1/2)I + \mathcal{K}_\lambda$ .

lem:inverseEstimate

**Lemma 5.3.** *Let  $\lambda \in \Sigma_\theta$ . Then  $-(1/2)I + \mathcal{K}_\lambda$  is a Fredholm operator on  $L^2(\partial\Omega; \mathbb{C}^d)$  with index zero and*

$$\|f\|_\partial \leq C \|(-(1/2)I + \mathcal{K}_\lambda)f\|_\partial \quad \text{for all } f \in L^2_n(\partial\Omega), \quad (5.4)$$

eq:inverseEstimate2

where  $C$  depends only on  $d$ ,  $\theta$  and the Lipschitz character of  $\Omega$ .

*Proof.* In the case  $\lambda = 0$ , Fabes, Kenig and Verchota proved in [5] that the Fredholm index of the operator  $-(1/2)I + \mathcal{K}_0$  on  $L^2(\partial\Omega; \mathbb{R}^d)$  is zero and estimate (5.4) holds. As in the previous proof, this still remains true if we complexify the operator making it a Fredholm operator with index zero on  $L^2(\partial\Omega; \mathbb{C}^d)$ . Since  $\mathcal{K}_\lambda - \mathcal{K}_0$  is compact on  $L^2(\partial\Omega; \mathbb{C}^d)$  and the Fredholm index remains unchanged under compact perturbations, we know that the Fredholm index of  $-(1/2)I + \mathcal{K}_\lambda$  on  $L^2(\partial\Omega; \mathbb{C}^d)$  is zero for all  $\lambda \in \Sigma_\theta$ . This proves the first claim of the lemma.

Now let  $\tau < (2 \operatorname{diam}(\Omega)^2 + 1)^{-1}$  and  $|\lambda| < \tau$ . Then

$$\|(\mathcal{K}_\lambda - \mathcal{K}_0)f\|_\partial \leq C |\lambda|^{1/2} \|f\|_\partial.$$

In order to prove this inequality we once again apply Young's inequality from Lemma 3.4. To this end we start by estimating

$$\|(\mathcal{K}_\lambda - \mathcal{K}_0)f\|_\partial \leq \sup_{p \in \partial\Omega} \|\nabla_x \{\Gamma(p - \cdot; \lambda) - \Gamma(p - \cdot; 0)\}\|_{L^1(\partial\Omega)} \|f\|_{L^2(\partial\Omega)}.$$

In the next step we prove that for  $p \in \partial\Omega$  the integral over the gradients of  $\Gamma$  can be estimated independent of  $p$ . This is straightforward using Lemma 3.2 as Corollary 2.7 gives us

$$\int_{\partial\Omega} |\nabla_x \{\Gamma(p - y; \lambda) - \Gamma(p - y; 0)\}| d\sigma(y)$$

$$\begin{aligned}
&\leq C|\lambda|^{1/2} \int_{\partial\Omega} \frac{1}{|p-y|^{d-2}} d\sigma(y) \\
&= |\lambda|^{1/2} C \int_{\partial\Omega \cap B(p, r_0/4)} \frac{1}{|p-y|^{d-2}} d\sigma(y) + |\lambda|^{1/2} C \int_{\partial\Omega \setminus B(p, r_0/4)} \frac{1}{|p-y|^{d-2}} d\sigma(y) \\
&\leq C|\lambda|^{1/2} (r_0/4 + 4^{2-d} r_0^{d-2} |\partial\Omega|),
\end{aligned}$$

where  $r_0$  is the radius from the definition of Lipschitz domains. Note that by the choice of  $\tau$  the estimate from Corollary 2.7 applies on the whole domain of integration.

For  $f \in L_n^2(\partial\Omega)$  we can now estimate

$$\begin{aligned}
\|f\|_{\partial} &\leq C\|(-(1/2)I + \mathcal{K}_0)f\|_{\partial} \\
&\leq C\|(-(1/2)I + \mathcal{K}_{\lambda})f\|_{\partial} + \|(\mathcal{K}_{\lambda} - \mathcal{K}_0)f\|_{\partial} \\
&\leq C\|(-(1/2)I + \mathcal{K}_{\lambda})f\|_{\partial} + C|\lambda|^{1/2}\|f\|_{\partial},
\end{aligned}$$

with a constant  $C$  which depends only on  $d, \theta$  and the Lipschitz character of  $\partial\Omega$ . Choosing  $\tau$  even smaller allows us to rearrange the terms in the above estimate such that estimate (5.4) holds for  $\lambda \in \Sigma_{\theta}$  and  $|\lambda| < \tau$ , with  $\tau$  depending on  $d, \theta$  and the Lipschitz character of  $\Omega$ .

Now leave  $\tau$  fixed and consider the case  $|\lambda| \geq \tau$ . This case will be handled using the Rellich estimates from Section 4. We use the facts that for  $\nabla_{\tan} u$  and  $u$  the inner and outer nontangential limits coincide and apply Theorems 4.1 and 4.8 to conclude that

$$\begin{aligned}
&\|\nabla u_+\|_{\partial} + \|\phi_+ - \int_{\partial\Omega} \phi_+ \| \\
&\leq C\{\|(\nabla_{\tan} u)_+\|_{\partial} + |\lambda|^{1/2}\|u_+\|_{\partial} + |\lambda|\|u_+ \cdot n\|_{H^{-1}(\partial\Omega)}\} \\
&= C\{\|(\nabla_{\tan} u)_-\|_{\partial} + |\lambda|^{1/2}\|u_-\|_{\partial} + |\lambda|\|u_- \cdot n\|_{H^{-1}(\partial\Omega)}\} \\
&\leq C\|(\frac{\partial u}{\partial \nu})_-\|_{\partial}.
\end{aligned}$$

We can now use this inequality to estimate  $\|(\frac{\partial u}{\partial \nu})_+\|_{\partial}$  since

$$\begin{aligned}
\|(\frac{\partial u}{\partial \nu})_+\|_{\partial} &\leq \|(\frac{\partial u}{\partial n})_+\|_{\partial} + C\|\phi_+\|_{\partial} \\
&\leq C\|(\nabla u)_+\|_{\partial} + C\|\phi_+ - \int_{\partial\Omega} \phi_+ d\sigma\|_{\partial} + C|\int_{\partial\Omega} \phi_+ d\sigma| \\
&\leq C\|(\frac{\partial u}{\partial \nu})_-\|_{\partial} + C|\int_{\partial\Omega} \phi_+ d\sigma|
\end{aligned}$$

Considering the jump relation (3.11) and the previous estimate we get that

$$\|f\|_{\partial} \leq \|(\frac{\partial u}{\partial \nu})_+\|_{\partial} + \|(\frac{\partial u}{\partial \nu})_-\|_{\partial}$$

$$\begin{aligned}
&\leq C \|(\frac{\partial u}{\partial \nu})_-\|_{\partial} + C \left| \int_{\partial\Omega} \phi_+ d\sigma \right| \\
&\leq C \|(-(1/2)I + \mathcal{K}_\lambda)f\|_{\partial} + C \left| \int_{\partial\Omega} \phi_+ d\sigma \right|. \quad \text{eq:estimatef} \tag{5.5}
\end{aligned}$$

We now are left with the term  $\int_{\partial\Omega} \phi_+ d\sigma$  that needs to be estimated. To this end, note that multiplying the conormal derivatives of  $u$  by  $n$  gives

$$(\frac{\partial u}{\partial \nu})_+ \cdot n = (\frac{\partial u_i}{\partial x_j})_+ n_i n_j - \phi_+ = n_j (n_i \frac{\partial}{\partial x_j} - n_j (\frac{\partial}{\partial x_i}) u_i)_+ - \phi_+,$$

where for the second equality we used that  $\operatorname{div}(u) = 0$  in  $\Omega$  and thus this also holds for the nontangential limit. This identity now implies

$$\begin{aligned}
\left| \int_{\partial\Omega} \phi_+ d\sigma \right| &\leq \left| \int_{\partial\Omega} (\frac{\partial u}{\partial \nu})_+ \cdot n d\sigma \right| + C \|\nabla_{\tan} u\|_{\partial} \\
&\leq \left| \int_{\partial\Omega} (\frac{\partial u}{\partial \nu})_- \cdot n d\sigma \right| + C \|\nabla_{\tan} u\|_{\partial} \\
&\leq C \|(\frac{\partial u}{\partial \nu})_-\|_{\partial}, \quad \text{eq:estimatephiplus} \tag{5.6}
\end{aligned}$$

where in the second step, we used the jump relation to exchange  $(\frac{\partial u}{\partial \nu})_+ \cdot n$  by  $(\frac{\partial u}{\partial \nu})_- + f \cdot n$  and then used the fact  $f \in L_n^2(\partial\Omega)$ . The third step follows from Theorem 4.8 considering that  $\|\nabla_{\tan} u\|_{\partial} \leq C \|\nabla u\|_{\partial}$ . Now extending estimate (5.5) by (5.6) gives

$$\|f\|_{\partial} \leq C \|(-(1/2)I + \mathcal{K}_\lambda)f\|_{\partial} + C \|(\frac{\partial u}{\partial \nu})_-\|_{\partial} \leq C \|(-(1/2)I + \mathcal{K}_\lambda)f\|_{\partial},$$

where we used the jump relation (3.11) again. This proves estimate (5.4) in the case  $|\lambda| \geq \tau$  and thus concludes the proof.  $\square$

In the following lemma we will show the uniqueness of solutions to the  $L^2$  Dirichlet problem to the Stokes resolvent system.

lem:l2unique

**Lemma 5.4.** *Let  $\lambda \in \Sigma_\theta$  and  $(u, \phi)$  be a solution to the Stokes resolvent problem in  $\Omega$ . Furthermore suppose that the nontangential limit of  $u$  exists almost everywhere on  $\partial\Omega$  and that  $(u)^* \in L^2(\partial\Omega)$ . Then*

$$\int_{\Omega} |u|^2 dx \leq C \int_{\partial\Omega} |u|^2 d\sigma, \quad \text{eq:OmegaBoundaryEstimate} \tag{5.7}$$

where  $C$  depends only on  $d$ ,  $\theta$  and the Lipschitz character of  $\Omega$ .

*Proof.* We use Verchota's approximation theorem 1.3 and approximate  $\Omega$  by a sequence of smooth domains with uniform Lipschitz characters from inside. As a consequence we

may assume that  $\Omega$  is smooth and  $u, \phi$  are smooth in  $\overline{\Omega}$ . Let  $(w, \psi) \in H_0^1(\Omega; \mathbb{C}^d) \times H^1(\Omega)$  be a solution to the inhomogenous system

$$\begin{aligned} -\Delta w + \lambda w + \nabla \psi &= \bar{u} \text{ in } \Omega, \\ \operatorname{div}(w) &= 0 \text{ in } \Omega. \end{aligned} \quad \text{eq:inhomogenousStokes} \quad (5.8)$$

In fact the regularity theory for the Stokes equation gives us that  $w$  and  $\psi$  are smooth. It follows from testing (5.8) against  $u$  that

$$\int_{\Omega} |u|^2 dx = \int_{\Omega} u \cdot \{-\Delta w + \lambda w + \nabla \psi\} dx. \quad \text{eq:testingInhomogenousStokes} \quad (5.9)$$

Using one of Green's identities on the first summand and the fact that  $u$  is the solution to the Stokes resolvent problem gives that

$$\begin{aligned} \int_{\Omega} -u \cdot \Delta w dx &= \int_{\Omega} -w \cdot \Delta u dx - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} d\sigma, \\ &= \int_{\Omega} w \cdot (-\lambda u - \nabla \phi) dx - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} d\sigma \\ &= \int_{\Omega} -\lambda w \cdot u dx - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} d\sigma, \end{aligned}$$

where in the last step we used partial integration and the fact that  $w$  vanishes on  $\partial\Omega$  and is divergence free:

$$\int_{\Omega} w \cdot \nabla \phi dx = - \int_{\Omega} \operatorname{div}(w) \phi dx + \int_{\partial\Omega} \phi w \cdot n d\sigma = 0.$$

For the third summand in (5.9) we do the same with the only difference that the second integral does not vanish. Putting everything together gives

$$\begin{aligned} \int_{\Omega} |u|^2 dx &= - \int_{\partial\Omega} u \cdot \left\{ \frac{\partial w}{\partial n} - \psi n \right\} d\sigma \\ &\leq \|u\|_{\partial} \{ \|\nabla w\|_{\partial} + \|\psi\|_{\partial} \} \end{aligned} \quad \text{eq:u2estimate} \quad (5.10)$$

by the Cauchy-Schwartz inequality. As the pressure  $\psi$  is only specified modulo additive constants, we may as well assume that  $\int_{\partial\Omega} \psi d\sigma = 0$ . Furthermore by the Schwartz theorem we see from (5.8) that  $\Delta \psi = \operatorname{div}(\bar{u}) = 0$  in  $\Omega$ . As stated in Remark 4.7, this allows us to use the results from the proof of (4.17) by setting  $\phi = \psi$  to conclude that

$$\|\psi\|_{\partial} \leq C \|\nabla \psi \cdot n\|_{H^{-1}(\partial\Omega)}$$

and since  $w$  has vanishing trace on  $\partial\Omega$  we can use that fact that  $(w, \psi)$  solves (5.8) to further estimate

$$\leq C \{ \|\Delta w \cdot n\|_{H^{-1}(\partial\Omega)} + \|u \cdot n\|_{H^{-1}(\partial\Omega)} \}$$

$$\leq C\{\|\nabla w\|_{\partial} + \|u\|_{\partial}\}, \quad \text{eq:psiEstimate} \quad (5.11)$$

where for the last estimate we used (4.16) which is applicable since  $\operatorname{div} w = 0$  on  $\Omega$ . If we combine inequalities (5.10) and (5.11), we get

$$\int_{\Omega} |u|^2 dx \leq C\|u\|_{\partial}\|\nabla w\|_{\partial} + C\|u\|_{\partial}^2. \quad \text{eq:u2estimate2} \quad (5.12)$$

We are left with estimating the first term in (5.12). In fact it will suffice to show the following inequality

$$\int_{\partial\Omega} |\nabla w|^2 d\sigma \leq C \int_{\Omega} |u|^2 dx + C \int_{\partial\Omega} |u|^2 d\sigma \quad \text{eq:w2estimate} \quad (5.13)$$

since by the weighted Young inequality for real numbers this would make the estimate

$$C\|u\|_{\partial}\|\nabla w\|_{\partial} \leq \frac{1}{2} \int_{\Omega} |u|^2 dx + C \int_{\partial\Omega} |u|^2 d\sigma$$

available which after rearranging terms yields (5.7).

To this end, we will first prove the Rellich type identity

$$\begin{aligned} \int_{\partial\Omega} h_k n_k |\nabla w|^2 d\sigma &= \int_{\Omega} \operatorname{div}(h) |\nabla w|^2 dx + 2 \operatorname{Re} \int_{\Omega} h_j \frac{\partial \psi}{\partial x_j} \frac{\bar{w}_j}{\partial x_k} dx \\ &\quad + 2 \operatorname{Re} \int_{\Omega} h_k \lambda w_j \frac{\partial \bar{w}_j}{\partial x_k} + 2 \operatorname{Re} \int_{\Omega} h_k \bar{u}_j \frac{\partial w_j}{\partial x_k} dx. \end{aligned} \quad \text{eq:rellichIdentity3} \quad (5.14)$$

Note that since all involved quantities are smooth up to the boundary, integration by parts is allowed and the proof the stated Rellich identity boils down to a formal calculation. Let  $h \in C_0^1(\mathbb{R}^d, \mathbb{R}^d)$  with  $h_k n_k \geq c > 0$  on  $\partial\Omega$  be the function from Theorem 1.3. Then the divergence theorem gives

$$\int_{\partial\Omega} h_k n_k |\nabla w|^2 d\sigma = \int_{\Omega} \operatorname{div}(h |\nabla w|^2) dx = \int_{\Omega} \operatorname{div}(h) |\nabla w|^2 dx + \int_{\Omega} h_k \frac{\partial}{\partial x_k} (|\nabla w|^2) dx$$

and we can rewrite the second summand as

$$\begin{aligned} \int_{\Omega} h_k \frac{\partial}{\partial x_k} (|\nabla w|^2) dx &= \int_{\Omega} h_k \frac{\partial}{\partial x_k} \left( \frac{\partial w_j}{\partial x_i} \frac{\partial \bar{w}_j}{\partial x_i} \right) dx \\ &= 2 \operatorname{Re} \int_{\Omega} h_k \frac{\partial^2 w_j}{\partial x_k \partial x_i} \frac{\partial \bar{w}_j}{\partial x_i} dx \\ &= -2 \operatorname{Re} \int_{\Omega} h_k \frac{\partial}{\partial x_k} \left( \frac{\partial^2}{\partial x_i^2} w_j \right) \bar{w}_j dx + \int_{\partial\Omega} h_k \left( \frac{\partial^2}{\partial x_k \partial x_i} w_j \right) \bar{w}_j d\sigma \\ &= 2 \operatorname{Re} \int_{\Omega} h_k (\Delta w_j) \frac{\partial \bar{w}_j}{\partial x_k} dx + 0 \\ &= 2 \operatorname{Re} \int_{\Omega} h_k \left( \frac{\partial \psi}{\partial x_j} + \lambda w_j - \bar{u}_j \right) \frac{\partial \bar{w}_j}{\partial x_k} dx, \end{aligned}$$

where in addition to partial integration we used (5.8) and the fact that  $w = 0$  on  $\partial\Omega$ . Now we can apply the triangle inequality to the Rellich type identity (5.14) to obtain

$$\begin{aligned} \int_{\partial\Omega} |\nabla w|^2 d\sigma &\leq C \left\{ \int_{\Omega} |\nabla w|^2 dx + \int_{\Omega} |\nabla w| |\psi| dx \right. \\ &\quad \left. + |\lambda| \int_{\Omega} |\nabla w| |w| dx + \int_{\Omega} |\nabla w| |u| dx \right\}. \end{aligned} \quad (5.15)$$

The next step consists in deriving estimates which are compatible with the right hand side of (5.15). Testing (5.8) with  $\bar{w}$ , itegration by parts gives us as in the proof of Lemma 4.4

$$\int_{\Omega} |\nabla w|^2 dx + |\lambda| \int_{\Omega} |w|^2 dx \leq C \int_{\Omega} |w| |u| dx.$$

The next step consists in using the previous inequality and the Poincaré inequality to estimate

$$\begin{aligned} \int_{\Omega} |\nabla w|^2 dx + (1 + |\lambda|) \int_{\Omega} |w|^2 dx &\leq (1 + C) \int_{\Omega} |\nabla w|^2 dx + |\lambda| \int_{\Omega} |w|^2 dx \\ &\leq C \int_{\Omega} |w| |u| dx \\ &\leq C \|w\|_{\partial} \|u\|_{\partial}, \end{aligned}$$

where for the last step we used the Cauchy-Schwartz inequality. The weighted Young inequality for real numbers allows us to further estimate

$$\begin{aligned} &\leq \frac{C}{4\varepsilon} \int_{\Omega} |u|^2 dx + C\varepsilon \int_{\Omega} |w|^2 dx \\ &= \frac{\tilde{C}}{1 + |\lambda|} \int_{\Omega} |u|^2 dx + \frac{1}{2}(1 + |\lambda|) \int_{\Omega} |w|^2 dx \end{aligned}$$

if we set  $\varepsilon = \frac{(1+|\lambda|)}{2C}$ . Rearranging terms, we can produce our next estimate

$$\int_{\Omega} |\nabla w|^2 dx + (1 + |\lambda|) \int_{\Omega} |w|^2 dx \leq \frac{C}{1 + |\lambda|} \int_{\Omega} |u|^2 dx. \quad (5.16)$$

Now it's time to harvest: Using estimate (5.15) together with (5.16) gives

$$\begin{aligned} \int_{\partial\Omega} |\nabla w|^2 d\sigma &\leq C \left\{ \int_{\Omega} |\nabla w|^2 dx + \int_{\Omega} |\nabla w| |\psi| dx + |\lambda| \int_{\Omega} |\nabla w| |w| dx + \int_{\Omega} |\nabla w| |u| dx \right\}. \end{aligned}$$

Using the weighted Young inequality, we see that we can simplify the right hand to

$$C_{\varepsilon}(1 + |\lambda|) \int_{\Omega} |\nabla w|^2 dx + C|\lambda| \int_{\Omega} |w|^2 dx + C \int_{\Omega} |u|^2 dx + \varepsilon \int_{\Omega} |\psi|^2 dx,$$

which with (5.16) can be bounded in this way

$$\leq C_\varepsilon \int_{\Omega} |u|^2 dx + \varepsilon \int_{\Omega} |\psi|^2 dx.$$

Using the estimate  $\|\psi\|_{L^2(\partial\Omega)} \leq C\|\psi\|_{\partial}$  and inequality (5.11) gives

$$\varepsilon \int_{\Omega} |\psi|^2 dx \leq \varepsilon C \int_{\partial\Omega} |\nabla w|^2 d\sigma + C_\varepsilon \int_{\partial\Omega} |u|^2 d\sigma.$$

Choosing  $\varepsilon = \frac{1}{2C}$  and rearranging gives the desired estimate (5.13). This concludes our proof.  $\square$

The next Theorem states the important fact that in  $L^2$  the Dirichlet Stokes resolvent problem has a unique solution.

thm:exAndUniqueSolution

**Theorem 5.5.** *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^d$ ,  $d \geq 3$  with connected boundary and let  $\lambda \in \Sigma_\theta$ . For all  $g \in L_n^2(\partial\Omega)$  there exists a unique  $u$  and harmonic function  $\phi$  which is unique up to constants such that  $(u, \phi)$  satisfies (2.26),  $(u)^* \in L^2(\partial\Omega)$  and  $u = g$  on  $\partial\Omega$  in the sense of nontangential convergence. Moreover the estimate  $\|(u)^*\|_{\partial} \leq C\|g\|_{\partial}$  holds and  $u$  may be represented by the double layer potential  $\mathcal{D}_\lambda(f)$  with  $\|f\|_{\partial} \leq C\|g\|_{\partial}$ , where  $C$  depends only on  $d$ ,  $\theta$  and the Lipschitz character of  $\Omega$ .*

*Proof.* By Lemma 5.4 we already now that the problem under consideration admits at most one solution. Therefore we only have to worry about the existence of a solution. To this end we want to apply Lemma 5.3.

We first note that since

$$T: L^2(\partial\Omega; \mathbb{C}^d) \rightarrow L^2(\partial\Omega; \mathbb{C}^d), \quad x \mapsto -(1/2)x + \mathcal{K}_{\bar{\lambda}}x$$

is a Fredholm operator on  $L^2(\partial\Omega; \mathbb{C}^d)$  with index 0 the same is true for its adjoint

$$T^*: L^2(\partial\Omega; \mathbb{C}^d) \rightarrow L^2(\partial\Omega; \mathbb{C}^d), \quad x \mapsto (-1/2)x + \mathcal{K}_{\bar{\lambda}}^*x.$$

We know that for all  $f \in L^2(\partial\Omega; \mathbb{C}^d)$  we have  $\operatorname{div}(\mathcal{D}_\lambda f) = 0$  and therefore

$$\int_{\partial\Omega} T^* f \cdot n d\sigma = \int_{\partial\Omega} u_+ \cdot n d\sigma = 0,$$

where for the first inequality we applied Theorem 3.10. The second equality uses Verchota's approximation scheme from Theorem 1.3 in order to apply the divergence theorem and the fact that  $(u)^*$  is integrable together with dominated convergence. This gives  $\operatorname{Im}(T^*) \subseteq L_n^2(\partial\Omega)$ . Now on the one hand we have

$$\operatorname{span}(n) = L_n^2(\partial\Omega)^\perp \subseteq \operatorname{Im}(T^*)^\perp = \ker(T)$$



and on the other hand, as  $T$  is injective on  $L_n^2(\partial\Omega)$  by (5.4), we have that  $\text{span}(n) \supseteq \ker(T)$ . This yields  $\text{span}(n) = \ker(T)$ . We can use this equality and show that

$$L_n^2(\partial\Omega) = \ker(T)^\perp = \overline{\text{Im}(T^*)} = \text{Im}(T^*),$$

where for the last equality we used the fact that the range of  $T^*$  is closed, as usual for Fredholm operators. With the same argument we can show for  $T$  that

$$\ker(T^*)^\perp = \overline{\text{Im}(T)} = \text{Im}(T).$$

Now let

$$X = L_n^2(\partial\Omega) \quad \text{and} \quad Y = \text{Im}(T).$$

Both spaces are closed subspaces of the Hilbert space  $L^2(\partial\Omega; \mathbb{C}^d)$  and therefore again Hilbertspaces. Consequently the operator

$$K'_{Y,X}: Y \rightarrow X, \quad x \mapsto T^*x$$

is invertible by the continuous inverse theorem. We now want to bound the operator norm of  $K_{X,Y}$  by a constant that does not depend on  $\lambda$  but on the sectorial parameter  $\theta$ . To this end let us introduce the operator

$$K_{X,Y}: X \rightarrow Y, \quad x \mapsto Tx$$

Now for  $f \in X$  and  $g \in Y$  we have that

$$\langle x, K_{X,Y}^* y \rangle_X = \langle K_{X,Y} x, y \rangle_Y = \langle x, T^* y \rangle_X = \langle x, K'_{Y,X} y \rangle_X$$

which shows that  $K'_{Y,X} = K_{X,Y}^*$  on  $Y$ . With the above definitions at hand, Lemma 5.3 states that  $K_{X,Y}$  is an invertible operator on  $Y$  with operator norm of the inverse bounded by some  $C > 0$  and  $C$  depends at most on  $d$ ,  $\theta$  and the Lipschitz character of  $\Omega$ . Therefore for all  $f \in Y = \text{Im}(T) = \text{Im}((1/2)I + \mathcal{K}_{\bar{\lambda}})$  we have

$$\|f\|_\partial = \|(K'_{Y,X})^{-1} K'_{Y,X} f\|_\partial \leq \|(K_{X,Y}^*)^{-1}\|_{X,Y} \|K_{X,Y}^* f\|_\partial \leq C \|(-(1/2)I + \mathcal{K}_{\bar{\lambda}}^*) f\|_\partial. \quad \text{eq:dualityArgument (5.17)}$$

We are now in position to derive the missing estimates which were stated in the theorem. For  $g \in L_n^2(\partial\Omega)$  let  $f \in \text{Im}(T)$  with  $T^* f = g$ . Furthermore let  $(u, \phi)$  be the double layer potential defined in equations (3.14) and (3.15). Then  $u_+ = T^* f = g$  on  $\partial\Omega$  by Theorem 3.10. Additionally we have that

$$\|(u)^*\|_\partial \leq C \|f\|_\partial \leq C \|g\|_\partial$$

where we used inequality (3.16) and (5.17). □

The next theorem can in some sense be regarded as a reverse trace theorem and will play an important role for the proof of the needed reverse Hölder inequality.

thm:reverseTrace

**Theorem 5.6.** *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^d$ ,  $d \geq 3$  with connected boundary. Let  $u \in H^1(\Omega; \mathbb{C}^d)$  and  $\pi \in L^2(\Omega)$  satisfy the Stokes resolvent problem in  $\Omega$  for some  $\lambda \in \Sigma_\theta$ . Then*

$$\left( \int_{\Omega} |u|^p dx \right)^{1/p} \leq C \left( \int_{\partial\Omega} |u|^2 d\sigma \right)^{1/2}, \quad \text{eq:reverseTrace} \quad (5.18)$$

where  $p = \frac{2d}{d-1}$  and  $C$  depends only on  $d$ ,  $\theta$  and the Lipschitz character of  $\Omega$ .

*Proof.* Let us denote the trace of  $u$  on  $\partial\Omega$  by  $f$  and let furthermore  $w = \mathcal{D}(g)$ ,  $g \in L^2(\partial\Omega; \mathbb{C}^d)$ , be the solution of the  $L^2$  Dirichlet problem with boundary data  $f$  as given by Theorem 5.5. For the sequence  $(\Omega_j)_{j \in \mathbb{N}}$  of smooth domains that approximates  $\Omega$  from inside as described by Theorem 1.3, an application of Lemma 5.4 shows

$$\int_{\Omega_j} |u - w|^2 dx \leq C \int_{\partial\Omega_j} |u - w|^2 d\sigma, \quad (5.19)$$

where  $C$  does not depend on  $j$  but on the Lipschitz character of  $\Omega$ . Now let  $\varepsilon > 0$  be given. Then there exists  $\varphi_\varepsilon \in C^\infty(\overline{\Omega})$  such that  $\|\varphi_\varepsilon - u\|_{H^1(\Omega)}^2 \leq \frac{\varepsilon}{3}$ . Due to Theorem 1.3 we know that

$$\int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 d\sigma \rightarrow \int_{\partial\Omega} |\varphi_\varepsilon - u|^2 d\sigma, \text{ as } j \rightarrow \infty$$

since  $w = f$  on  $\partial\Omega$  in the sense of nontangential convergence. Therefore we choose  $J$  large enough such that for all  $j \geq J$  we have

$$\int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 d\sigma \leq \frac{2}{3}\varepsilon.$$

This gives together with the trace theorem that

$$\begin{aligned} \int_{\partial\Omega_j} |u - w|^2 d\sigma &\leq \int_{\partial\Omega_j} |u - \varphi_\varepsilon|^2 d\sigma + \int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 d\sigma \\ &\leq C \|u - \varphi_\varepsilon\|_{H^1(\Omega)}^2 + \int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 d\sigma, \end{aligned}$$

where once again  $C$  only depends on the Lipschitz character of  $\Omega$  and is thus independent of  $j$ . Due to Verchota Theorem 1.3 we may take  $j$  large enough such that also the second summand of the previous inequality is smaller than  $\varepsilon/2$ . As a consequence we get that  $w = u$  in  $\Omega$ . Now Theorem 5.5 gives

$$\|(u)^*\|_\partial = \|(w)^*\|_\partial \leq C \|f\|_\partial = C \|u\|_\partial,$$

where  $C$  depends on  $d$ ,  $\theta$ , and the Lipschitz character of  $\Omega$ . The claimed inequality now follows from estimate

$$\left( \int_{\Omega} |u|^p dx \right)^{1/p} \leq C \left( \int_{\partial\Omega} |(u)^*|^2 d\sigma \right)^{1/2}, \quad \text{weizhangestimate (5.20)}$$

where  $C$  only depends on  $d$  and the Lipschitz constant of  $\Omega$ . The proof of (5.20) was carried out by Wei and Zhang in [28, Lem. 3.3].  $\square$

**Remark 5.7.** At this point the choice  $p = \frac{2d}{d-1}$  in Theorem 5.6 may seem arbitrary. Taking a closer look at the proof by Wei and Zhang in [28, Lem. 3.3] one sees that the choice of  $p$  is due to 2 facts: (1) For the dual exponent we have  $p' = \frac{2d}{d+1}$ . (2) It holds  $\frac{1}{p} - \frac{1}{p'} = \frac{1}{d}$  and thus the *Hardy-Littlewood-Sobolev theorem on fractional integration* may be applied to estimate the  $p'$ -norm of the Riesz potential of a function  $f \in L^p(\mathbb{R}^d)$  by the  $p$ -norm of  $f$ , see Grafakos [10, Thm 6.1.3].

In the following remark and the forthcoming chapter we will make use of an integration argument which can be considered as an application of the following Theorem on *integration along slices*. A proof of this result can be found in Federer [6, Thm. 3.2.12]. thm:coarea

**Theorem 5.8** (Co-area formula). *If  $f: \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \geq 2$ , is Lipschitzian and  $g$  the representative of a function  $g \in L^2(\mathbb{R}^d)$ , then*

$$\int_{\mathbb{R}^d} g(x) \left[ \sum_{i=1}^d |\partial_i f(x)|^2 \right]^{1/2} dx = \int_{\mathbb{R}} \int_{f^{-1}(y)} g(x) dm_{d-1}(x) dy, \quad \text{eq:coarea (5.21)}$$

where  $m_{d-1}$  denotes the  $(d-1)$ -dimensional Hausdorff measure on  $\mathbb{R}^d$ .

**Remark 5.9.** Let  $(u, \phi)$  be a solution of the Stokes resolvent system in the domain  $B(x_0, r) \subset \mathbb{R}^d$ .

$$|\nabla^l u(x_0)| \leq \frac{C_l}{r^l} \left( \int_{B(x_0, r)} |u(x)|^2 dx \right)^{1/2}, \quad \text{eq:interiorEstimateDoubleLayer (5.22)}$$

for all  $l \geq 0$ , where  $C_l$  only depends on  $d$ ,  $l$  and  $\theta$ . Without loss of generality we may rescale and translate and assume that  $x_0 = 0$  and  $r = 2$ . Let  $t \in (1, 2)$ . By Theorem 5.5 we know that a solution to the Stokes resolvent system on  $B(0, t) \subsetneq B(0, 2)$  with boundary values  $g_t := \text{Tr}_{\partial B(0, t)} u \in L^2(\partial B(0, t))$  is given by a boundary layer potential  $\mathcal{D}_\lambda(f_t)$ . We can use this fact to derive the desired estimate via

$$\begin{aligned} |\nabla^l u(0)|^2 &\leq C \left( \int_{\partial B(0, t)} \{ |\nabla_x^{l+1} \Gamma(y; \lambda)| + |\nabla_x^l \Phi(y)| \} |f_t(y)| d\sigma(y) \right)^2 \\ &\leq C \left( \int_{\partial B(0, t)} \frac{|f_t(y)|}{t^{d-1+l}} d\sigma(y) \right)^2 \end{aligned}$$

$$\begin{aligned}
&\leq C \int_{\partial B(0,t)} |f_t(y)|^2 d\sigma(y) \\
&\leq C \int_{\partial B(0,t)} |u(y)|^2 d\sigma(y),
\end{aligned}$$

where we also used the Cauchy inequality in the estimate of  $f_t$  against the “data” from Theorem 5.5. Integrating this inequality in  $t$  over the interval  $(1, 2)$  and using the co-area formula (5.21) gives

$$|\nabla^l u(0)|^2 \leq C \int_{B(0,2)} |u(x)|^2 dx.$$

Now the claim follows readily.

# Chapter 6

## Derivation of Resolvent Estimates

In this final chapter we will prove that the Stokes semigroup is analytic on  $L^p_\sigma(\Omega, \mathbb{C}^d)$  for bounded Lipschitz domains  $\Omega \subseteq \mathbb{R}^d$ ,  $d \geq 3$ .

The first step will be to establish a weak reverse Hölder estimate for local solutions of the Stokes resolvent problem. We start with a similar result on Lipschitz cylinders.

lem:reverseHoelderCylinder

**Lemma 6.1.** *Let  $\eta: \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  be a Lipschitz function. Furthermore, let  $u \in H^1(D_\eta(r); \mathbb{C}^d)$  and  $\phi \in L^2(D_\eta(2r))$  solve the Stokes resolvent problem in  $D_\eta(2r)$  with  $u = 0$  on  $I_\eta(2r)$  for some  $0 < r < \infty$  and  $\lambda \in \Sigma_\theta$ . Let  $p_d = \frac{2d}{d-1}$ . Then*

$$\left( \int_{D_\eta(r)} |u|^{p_d} dx \right)^{1/p_d} \leq C \left( \int_{D_\eta(2r)} |u|^2 dx \right)^{1/2}, \quad (6.1)$$

eq:reverseHoelderCylinder

where  $C$  only depends on  $d$ ,  $M$  and  $\theta$ .

*Proof.* Without loss of generality we rescale and assume that  $r = 1$ . Let  $t \in (1, 2)$ . We note that by [25, Lemma 1.3.25] a Lipschitz cylinder is itself a Lipschitz domain. It is therefore admissible to apply Theorem 5.18 to  $u$  in  $D_\eta(t)$  which yields

$$\left( \int_{D_\eta(t)} |u|^{p_d} dx \right)^{2/p_d} \leq C \int_{\partial D_\eta(t)} |u|^2 d\sigma,$$

where  $C$  depends only on  $d$ ,  $\theta$  and the Lipschitz character of  $\Omega$ . In particular  $C$  does not depend on  $t$ . Since  $u$  vanishes on  $I(2)$  we have that

$$\left( \int_{D_\eta(1)} |u|^p dx \right)^{2/p} \leq C \int_{\partial D_\eta(t) \setminus I(2)} |u|^2 d\sigma.$$

Applying the co-area formula to integrate both sides over the interval  $(1, 2)$  gives

$$\left( \int_{D_\eta(1)} |u|^p dx \right)^{2/p} \leq C \int_{D_\eta(2)} |u|^2 dx.$$

Estimate (6.1) now follows after dividing by  $|D_\eta(1)|$ . □

The next step is to extend the result to arbitrary Lipschitz domains. The following Lemma reduces the amount of work to a few special cases.

lem:ballsforballs

**Lemma 6.2** (Tolksdorf). *Let  $\Omega \subset \mathbb{R}^d$  be Lebesgue-measurable,  $f, g \in L^2(\Omega)$ ,  $\alpha_2 > \alpha_1 > 1$ ,  $p > 2$ ,  $r > 0$  and  $x_0 \in \mathbb{R}^d$  be such that  $B(x_0, r) \cap \Omega \neq \emptyset$ . If there exists  $C > 0$  such that*

$$\begin{aligned} & \left( \frac{1}{s^d} \int_{\Omega \cap B(y, s)} |f|^p dx \right)^{1/p} \\ & \leq C \left\{ \left( \frac{1}{s^d} \int_{\Omega \cap \alpha_1 B(y, s)} |f|^2 dx \right)^{1/2} + \sup_{B' \cap B(y, s)} \left( \frac{1}{|B'|} \int_{\Omega \cap B'} |g|^2 dx \right)^{1/2} \right\} \end{aligned}$$

*holds for all balls  $B(y, s)$  with  $B(y, \alpha_2 s) \subset B(x_0, \alpha_2 r)$  and which are either centered on  $\partial\Omega$  or satisfy  $\alpha_2 B(y, s) \subset \Omega$ , then for each  $\alpha \in (1, \alpha_2)$  there exists a constant  $C'$  such that*

$$\begin{aligned} & \left( \frac{1}{r^d} \int_{\Omega \cap B(x_0, r)} |f|^p dx \right)^{1/p} \\ & \leq C' \left\{ \left( \frac{1}{r^d} \int_{\Omega \cap \alpha B(x_0, r)} |f|^2 dx \right)^{1/2} + \sup_{B' \cap B(x_0, r)} \left( \frac{1}{|B'|} \int_{\Omega \cap B'} |g|^2 dx \right)^{1/2} \right\}. \end{aligned}$$

*Proof.* A proof of this lemma was given by Tolksdorf [24]. We will give the proof for the sake of completeness. □

As of now, our toolbox comprises enough tools to prove that solutions to the Stokes resolvent system satisfy a weak reverse Hölder inequality.

lem:reverseHoelder

**Lemma 6.3.** *Let  $x_0 \in \overline{\Omega}$  and  $0 < 2r < r_0$  and set  $\alpha_1 = \sqrt{d^2 10^2 (1 + M)^2 + 4}$  and  $\alpha_2 = \alpha_1 + 1$ . Let  $u \in H^1(B(x_0, \alpha_2 r) \cap \Omega; \mathbb{C}^d)$  and  $\phi \in L^2(B(x_0, \alpha_2 r) \cap \Omega)$  satisfy the Stokes resolvent system in  $B(x_0, \alpha_2 r) \cap \Omega$ . If  $B(x_0, \alpha_2 r) \cap \partial\Omega \neq \emptyset$ , we additionally assume  $u = 0$  on  $B(x_0, \alpha_2 r) \cap \partial\Omega$ . Then*

$$\left( \int_{B(x_0, r) \cap \Omega} |u|^p \right)^{1/p} \leq C \left( \int_{B(x_0, 2r) \cap \Omega} |u|^2 \right)^{1/2} \quad \text{eq:reverseHoelder (6.2)}$$

*holds, where  $p = p_d$ . Here,  $C > 0$  only depends on  $d$ ,  $\theta$  and the Lipschitz character of  $\Omega$ .*

*Proof.* Due to Lemma 6.2 it suffices to consider only two cases: (1)  $x_0 \in \Omega$  with  $\alpha_2 B(x_0, r) \subset \Omega$  and (2)  $x_0 \in \partial\Omega$ .

Let  $x_0 \in \Omega$  with  $\alpha_2 B(x_0, r) \subset \Omega$ . We may deploy the interior estimate (5.22) to derive that for all  $x \in B(x_0, r)$

$$|u(x)|^p \leq C \left( \int_{B(x, r)} |u(y)|^2 dy \right)^{p/2}$$

which after integrating  $x$  over  $B(x_0, r)$  yields

$$\int_{B(x_0, r)} |u(x)|^p dx \leq C \left( \alpha_1^d \int_{B(x_0, \alpha_1 r)} |u(z)|^2 dz \right)^{p/2},$$

where we also used the fact that  $\alpha_1 > 2$ .

If  $x_0 \in \partial\Omega$ , then by Lemma 6.1

$$\begin{aligned} \left( \frac{1}{r^d} \int_{B(x_0, r) \cap \Omega} |u|^p \right)^{1/p} &\leq \left( \frac{1}{r^d} \int_{D_{\eta x_0}(r)} |u|^p \right)^{1/p} \\ &\leq C \left( \frac{1}{r^d} \int_{D_{\eta x_0}(2r)} |u|^p \right)^{1/p} \\ &\leq C \left( \frac{1}{r^d} \int_{B(x_0, \alpha_1 r) \cap \Omega} |u|^2 \right)^{1/2}. \end{aligned}$$

Now the claim follows readily from an application of Lemma 6.2 with  $\alpha = 2 \in (1, \alpha_2)$ .  $\square$

We note that estimate (6.2) is a weak reverse Hölder inequality and thus possesses a self-improving property, see Giaquinta and Martinazzi [7, Thm. 6.38] or Giaquinta and Modica [8, Prop. 5.1].

prop:giaquinta

**Proposition 6.4** (Giaquinta, Modica). *Let  $\Omega \subset \mathbb{R}^d$  be open,  $f \in L^1_{\text{loc}}(\Omega)$ ,  $q > 1$ , be a non-negative function. If there exist constants  $b > 0, R_0 > 0$  such that*

$$\left( \frac{1}{r^d} \int_{B(x_0, r)} f^q dx \right)^{1/q} \leq \frac{b}{r^d} \int_{B(x_0, 2r)} f dx$$

*for all  $x_0 \in \Omega$  and  $0 < r < \min\{R_0, \text{dist}(x_0, \partial\Omega)/2\}$ . Then  $f \in L^{q+\varepsilon}_{\text{loc}}(\Omega)$  for some  $\varepsilon > 0$ , depending only on  $d, q$ , and  $b$  and there is a constant  $\tilde{C}$  depending only on  $d, q, \varepsilon$  and  $b$  such that*

$$\left( \frac{1}{r^d} \int_{B(x_0, r)} f^{q+\varepsilon} dx \right)^{1/(q+\varepsilon)} \leq \tilde{C} \left( \frac{1}{r^d} \int_{B(x_0, 2r)} f^q dx \right)^{1/q}$$

*for all  $x_0 \in \Omega$  and  $0 < r < \min\{R_0, \text{dist}(x_0, \partial\Omega)/2\}$ .*

rem:reverseHoelder

**Remark 6.5.** The self-improving property of reverse Hölder estimates can now be used to make the result of Lemma 6.3 a little bit better. Let  $0 < 2r < r_0$ . We are aiming to apply Proposition 6.4 for  $x_0 \in \overline{\Omega}$  on the open set  $\Omega \cap B(x_0, \alpha_2 r)$ , for  $\alpha_2$  as in Lemma 6.3. Let also  $u$  be as in Lemma 6.3 and set  $R_0 = r_0/2$ . Then for  $f = |u|^2 \chi_{B(x_0, \alpha_2 r) \cap \Omega}$  which can be considered as a partial extension of  $u$  by 0 to  $\mathbb{R}^d$  and  $q = p/2$ , inequality (6.2) reads

$$\left( \frac{1}{r^d} \int_{B(x_0, r)} f^q dx \right)^{1/q} = \left( \frac{1}{r^d} \int_{B(x_0, r) \cap \Omega} |u|^p dx \right)^{2/p}$$

$$\leq C^2 \frac{1}{r^d} \int_{B(x_0, 2r) \cap \Omega} |u|^2 dx = C^2 \frac{1}{r^d} \int_{B(x_0, 2r)} f dx.$$

Consequently Proposition 6.4 gives us that there exists some  $\varepsilon > 0$  which depends only on  $d, q$  and  $C^2$  and a constant  $\tilde{C}$  depending only on  $d, q, \varepsilon$  and  $C^2$  such that

$$\begin{aligned} \left( \frac{1}{r^d} \int_{B(x_0, r/2) \cap \Omega} |u|^{p+\varepsilon'} dx \right)^{2/(p+\varepsilon')} &= \left( \frac{1}{r^d} \int_{B(x_0, r/2)} f^{q+\varepsilon} dx \right)^{1/(q+\varepsilon)} \\ &\leq \tilde{C} \left( \frac{1}{r^d} \int_{B(x_0, r) \cap \Omega} |u|^p dx \right)^{1/p} \\ &\leq \tilde{C} C \left( \frac{1}{r^d} \int_{B(x_0, 2r) \cap \Omega} |u|^2 dx \right)^{1/2}. \end{aligned}$$

Another application of Lemma 6.2 gives us that for all  $r < r_0/4$  it holds that

$$\left( \frac{1}{r^d} \int_{B(x_0, r) \cap \Omega} |u|^{p+\varepsilon'} dx \right)^{2/(p+\varepsilon')} \leq C \left( \frac{1}{r^d} \int_{B(x_0, 2r) \cap \Omega} |u|^2 dx \right)^{1/2}. \quad \text{eq:improvedreverseHoelder (6.3)}$$

The following extrapolation theorem by Shen will be necessary in order to derive  $L^p$ -bounds on the solution of the Stokes resolvent system, [19, Thm. 3.3]. Note that a more recent result from Tolksdorf [24] generalizes this result to operators which are defined on spaces of Banach space valued functions.

thm:extrapolation

**Theorem 6.6.** *Let  $T$  be a bounded sublinear operator on  $L^2(\Omega; \mathbb{C}^d)$ , where  $\Omega$  is a bounded Lipschitz domain in  $\mathbb{R}^n$  and  $\|T\|_{\mathcal{L}(L^2(\Omega; \mathbb{C}^d))} \leq C_0$ . Let  $p > 2$ . Suppose that there exist constants  $R_0 > 0$ ,  $N > 1$  and  $\alpha_2 > \alpha_1 > 1$  such that for any bounded measurable function  $f$  with  $\text{supp}(f) \subseteq \Omega \setminus \alpha_2 B$ ,*

$$\left\{ \frac{1}{r^d} \int_{\Omega \cap B} |Tf|^p dx \right\}^{1/p} \leq N \left\{ \left( \frac{1}{r^d} \int_{\Omega \cap \alpha_1 B} |Tf|^2 dx \right)^{1/2} + \sup_{B' \supset B} \left( \frac{1}{|B'|} \int_{B'} |f|^p dx \right)^{1/p} \right\},$$

where  $B = B(x_0, r)$  is a ball with  $0 < r < R_0$  and either  $x_0 \in \partial\Omega$  or  $B(x_0, \alpha_2 r) \subset \Omega$ . Then  $T$  is bounded on  $L^q(\Omega; \mathbb{C}^d)$  for any  $2 < q < p$ . Moreover  $\|T\|_{\mathcal{L}(L^q(\Omega; \mathbb{C}^d))}$  is bounded by a constant depending at most on  $d, N, C_0, p, q$  and the Lipschitz character of  $\Omega$ .

We are now in the position to prove Theorem 1.16, the main theorem of this thesis. For this, the improved weak reverse Hölder inequality derived in Remark 6.5 will be the crucial ingredient as it enables us to apply the extrapolation theorem 6.6 to a suitable family of operators.

*Proof of Theorem 1.16.* Consider a family of scaled solution operators to the Stokes resolvent system (1.5), more precisely consider the family

$$T_\lambda : L^2(\Omega; \mathbb{C}^d) \rightarrow L^2(\Omega; \mathbb{C}^d), \quad f \mapsto (|\lambda| + 1)(A_2 + \lambda)^{-1} \mathbb{P}_2 f,$$



where  $\lambda \in \Sigma_\theta$ ,  $\theta \in (0, \pi/2)$ . Let us first verify that  $u := (|\lambda| + 1)^{-1}T_\lambda(f)$  does indeed solve (1.5). First note that since  $\mathbb{P}_2 f \in L^2_\sigma(\Omega)$  we know that by the mapping properties of the Stokes resolvent we have  $u \in H^1_{0,\sigma}(\Omega)$  and

$$A_2 u + \lambda u = \mathbb{P}_2 f.$$

Therefore  $u$  is a weak solution to

$$-\Delta u + \lambda u = \mathbb{P}_2 f.$$

By the usual arguments (c.f. Chapter 1), there exists a pressure  $\pi \in L^2(\Omega)$  such that

$$-\Delta u + \nabla \pi + \lambda u = f.$$

Furthermore by testing (??) with  $u$  we derive the estimate

$$\|T_\lambda(f)\|_{L^2(\Omega; \mathbb{C}^d)} = (|\lambda| + 1)\|u\|_{L^2(\Omega; \mathbb{C}^d)} \leq C_0 \|f\|_{L^2(\Omega; \mathbb{C}^d)},$$

where  $C_0$  only depends on  $d, \theta$  and the Lipschitz character of  $\Omega$ . Accordingly the family  $T_\lambda$  is bounded on  $L^2(\Omega; \mathbb{C}^d)$  and  $C_0$  is a uniform bound on the operator norms  $\|T_\lambda\|_{\mathcal{L}(L^2(\Omega; \mathbb{C}^d))}$ .

We will now show that the operators  $T_\lambda$  fulfill the estimate in Theorem 6.6, in order to deduce their  $L^p$ -boundedness. To this end let  $x_0 \in \overline{\Omega}$  and  $0 < 4r < r_0$  such that  $3B(x_0, r) \subseteq \Omega$  or  $B(x_0, r)$  is centered on  $\partial\Omega$ . Furthermore let  $f \in L^\infty(\Omega; \mathbb{C}^d)$  with support in  $\Omega \setminus 3B(x_0, r)$ . By construction  $(u, \pi)$  does not only solve (2.26) in  $\Omega$ , the pair also solves the dirichlet problem

$$\begin{aligned} -\Delta u + \nabla \phi + \lambda u &= 0 \\ \operatorname{div}(u) &= 0 \end{aligned}$$

in  $\Omega \cap 3B(x_0, r)$  where  $u = 0$  on  $\partial\Omega \cap 3B(x_0, r)$ . Therefore Remark 6.5 and more precisely inequality (6.3) give that

$$\left( \frac{1}{r^d} \int_{\Omega \cap B(x_0, r)} |u|^p \, dx \right)^{1/p} \leq C \left( \frac{1}{r^d} \int_{\Omega \cap 2B(x_0, r)} |u|^2 \, dx \right)^{1/2},$$

where  $p = p_d + \varepsilon$ . Multiplying this inequality on both sides with  $(|\lambda| + 1)$  gives

$$\left( \frac{1}{r^d} \int_{\Omega \cap B(x_0, r)} |T_\lambda(f)|^p \, dx \right)^{1/p} \leq C \left( \frac{1}{r^d} \int_{\Omega \cap 2B(x_0, r)} |T_\lambda(f)|^2 \, dx \right)^{1/2}, \quad (6.4)$$

where  $C$  depends only on  $d, \theta$  and the Lipschitz character of  $\Omega$ . Now Shen's extrapolation theorem 6.6 gives that  $T_\lambda$  is bounded on  $L^q(\Omega; \mathbb{C}^d)$  for all  $2 < q < p_d + \varepsilon$  and that the

operator norms  $\|T_\lambda\|_{\mathfrak{L}(L^q(\Omega; \mathbb{C}^d))}$  are uniformly bounded by a constant  $C_q$  depending only on  $d, \theta, q$  and the Lipschitz character of  $\Omega$ .

In the next step of the proof we now study the relationship between the operator  $T_\lambda$  and the resolvent of the Stokes operator  $A_q$  on  $L_\sigma^q(\Omega)$  for  $q \in (2, p_d + \varepsilon)$ . To this end let  $f \in L_\sigma^q(\Omega)$ . We already know that  $u = (1 + |\lambda|)^{-1} T_\lambda(f) = (A_2 + \lambda)^{-1} \mathbb{P}_2(f) \in L_\sigma^p(\Omega) \cap \mathcal{D}(A_2)$  by the mapping properties of  $T_\lambda(f)$ . As  $L_\sigma^q(\Omega) \subset L_\sigma^2(\Omega)$  we have furthermore that

$$\lambda u + A_2 u = f \in L_\sigma^q(\Omega)$$

and thus  $A_2 u \in L_\sigma^q(\Omega)$ . Appealing to Definition 1.11 we showed that  $u \in \mathcal{D}(A_p)$  and that  $A_2 u = A_q u$ . Therefore we have that

$$\lambda u + A_q u = f \in L_\sigma^p(\Omega)$$

By the uniqueness of  $u$ , which follows from the  $L^2$ -theory of the Stokes resolvent problem, we have that  $u = (\lambda + A_p)^{-1} f$ . Hence estimate 6.4 gives

$$\|u\|_{L^q(\Omega; \mathbb{C}^d)} = \|(\lambda + A_q)^{-1} f\|_{L^q(\Omega; \mathbb{C}^d)} \leq \frac{C_q}{1 + |\lambda|} \|f\|_{L^q(\Omega; \mathbb{C}^d)}.$$

And thus  $A_p$  is sectorial on  $L_\sigma^p(\Omega)$ . If necessary we take  $\varepsilon$  to be the minimum of the parameter  $\varepsilon$  used in the first part of this proof and the one from Theorem ???. In this case it can be shown, that the spaces  $L_\sigma^q(\Omega)$  are reflexive and that  $L_\sigma^q(\Omega)^* = L_\sigma^{q'}(\Omega)$  where  $q'$  denotes the dual exponent  $q' = q(q - 1)^{-1}$ . By abstract operator theory [12] we get that  $A_q$  is indeed densely defined and that  $A_q^* = A_{q'}'$ . Therefore

$$\|(A_q + \lambda)^{-1}\|_{\mathfrak{L}(L_\sigma^q(\Omega))} = \|(A_q + \lambda)^{-1} * \|_{\mathfrak{L}(L_\sigma^{q'}(\Omega))} = \|(A_{q'} + \lambda)^{-1}\|_{\mathfrak{L}(L_\sigma^{q'}(\Omega))}.$$

Consequently also the operators  $A_{q'}$  are sectorial, densely defined and closed. This completes the proof.  $\square$

# Appendix

chap : app

For  $d = 2$  we have that  $G(x; \lambda) = \frac{i}{4} H_0^{(1)}(k|x|)$ . Furthermore we set  $z = k|x|$ . Then applications of chain rule and product rule of differentiation give

$$\begin{aligned}\partial_\gamma G(x; \lambda) &= \frac{i}{4} k \frac{x_\gamma}{|x|} \frac{d}{dz} H_0^{(1)}(z) \\ \partial_\alpha \partial_\gamma G(x; \lambda) &= \frac{i}{4} k^2 \frac{x_\alpha x_\gamma}{|x|^2} \frac{d^2}{dz^2} H_0^{(1)}(z) + \frac{i}{4} k \left( \frac{\delta_{\alpha\gamma}}{|x|} - \frac{x_\alpha x_\gamma}{|x|^3} \right) \frac{d}{dz} H_0^{(1)}(z) \\ \partial_\beta \partial_\alpha \partial_\gamma G(x; \lambda) &= \frac{i}{4} k^3 \frac{x_\alpha x_\beta x_\gamma}{|x|^3} \frac{d^3}{dz^3} H_0^{(1)}(z) \\ &\quad + \frac{i}{4} k^2 \left( \frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^2} - 3 \frac{x_\alpha x_\beta x_\gamma}{|x|^4} \right) \frac{d^2}{dz^2} H_0^{(1)}(z) \\ &\quad + \frac{i}{4} k \left( 3 \frac{x_\alpha x_\beta x_\gamma}{|x|^5} - \frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^3} \right) \frac{d}{dz} H_0^{(1)}(z).\end{aligned}$$

The series expansions for the Hankel function  $H_0^{(1)}(z)$  read according to Lebedev [13]

$$\begin{aligned}H_0^{(1)}(z) &= J_0(z) + iY_0(z) \\ &= \frac{2i}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{(l!)^2 4^l} z^{2l} \left( -\frac{i\pi}{2} - \log(2) - \psi(l+1) \right) + \frac{2i}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{(l!)^2 4^l} z^{2l} \log(z) \\ &= \frac{2i}{\pi} \sum_{l=0}^{\infty} a_l z^{2l} C_l + \frac{2i}{\pi} \sum_{l=0}^{\infty} a_l z^{2l} \log(z) \\ \frac{d}{dz} H_0^{(1)}(z) &= \frac{2i}{\pi} \sum_{l=1}^{\infty} a_l (2l) z^{2l-1} C_l + \frac{2i}{\pi} \sum_{l=1}^{\infty} a_l (2l) z^{2l-1} \log(z) + \frac{2i}{\pi} \sum_{l=0}^{\infty} a_l z^{2l-1} \\ &= \frac{2i}{\pi} \sum_{l=1}^{\infty} b_l z^{2l-1} C_l + \frac{2i}{\pi} \sum_{l=1}^{\infty} b_l z^{2l-1} \log(z) + \frac{2i}{\pi} \sum_{l=0}^{\infty} a_l z^{2l-1} \\ \frac{d^2}{dz^2} H_0^{(1)}(z) &= \frac{2i}{\pi} \sum_{l=1}^{\infty} b_l (2l-1) z^{2l-2} C_l + \frac{2i}{\pi} \sum_{l=1}^{\infty} b_l (2l-1) z^{2l-2} \log(z) + \frac{2i}{\pi} \sum_{l=1}^{\infty} b_l z^{2l-2}\end{aligned}$$

$$\begin{aligned}
& + \frac{2i}{\pi} \sum_{l=0}^{\infty} a_l (2l-1) z^{2l-2} \\
& = \frac{2i}{\pi} \sum_{l=1}^{\infty} c_l z^{2l-2} C_l + \frac{2i}{\pi} \sum_{l=1}^{\infty} c_l z^{2l-2} \log(z) + \frac{2i}{\pi} \sum_{l=1}^{\infty} b_l z^{2l-2} \\
& \quad + \frac{2i}{\pi} \sum_{l=0}^{\infty} a_l (2l-1) z^{2l-2} \\
\frac{d^3}{dz^3} H_0^{(1)}(z) & = \frac{2i}{\pi} \sum_{l=2}^{\infty} c_l (2l-2) z^{2l-3} C_l + \frac{2i}{\pi} \sum_{l=2}^{\infty} c_l (2l-2) z^{2l-3} \log(z) + \frac{2i}{\pi} \sum_{l=1}^{\infty} c_l z^{2l-3} \\
& \quad + \frac{2i}{\pi} \sum_{l=2}^{\infty} b_l (2l-2) z^{2l-3} + \frac{2i}{\pi} \sum_{l=0}^{\infty} a_l (2l-1)(2l-2) z^{2l-3} \\
& = \frac{2i}{\pi} \sum_{l=2}^{\infty} d_l z^{2l-3} C_l + \frac{2i}{\pi} \sum_{l=2}^{\infty} d_l z^{2l-3} \log(z) + \frac{2i}{\pi} \sum_{l=1}^{\infty} c_l z^{2l-3} \\
& \quad + \frac{2i}{\pi} \sum_{l=2}^{\infty} b_l (2l-2) z^{2l-3} + \frac{2i}{\pi} \sum_{l=0}^{\infty} a_l (2l-1)(2l-2) z^{2l-3},
\end{aligned}$$

where

$$\begin{aligned}
C_l &:= -\frac{i\pi}{2} - \log(2) - \psi(l+1) \\
a_l &:= \frac{(-1)^l}{(l!)^2 4^l}, \quad b_l := a_l \cdot 2l, \quad c_l := b_l \cdot (2l-1), \quad d_l := c_l \cdot (2l-2).
\end{aligned}$$

For  $G(x; 0) = -\frac{1}{2\pi} \log(|x|)$ , we have

$$\begin{aligned}
\partial_\gamma G(x; 0) &= -\frac{1}{2\pi} \frac{x_\gamma}{|x|^2} \\
\partial_\alpha \partial_\gamma G(x; 0) &= -\frac{1}{2\pi} \delta_{\alpha\gamma} \cdot \frac{1}{|x|^2} + \frac{1}{\pi} \cdot \frac{x_\alpha x_\gamma}{|x|^4} \\
\partial_\beta \partial_\alpha \partial_\gamma G(x; 0) &= \frac{1}{\pi} \frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^4} - \frac{4}{\pi} \frac{x_\alpha x_\beta x_\gamma}{|x|^6}.
\end{aligned}$$

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