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On Resolvent Estimates in L^p for the Stokes Operator in Lipschitz Domains

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Introduction

In the solution theory for nonlinear partial differential equations, an integral part of the solution process is often to develop a semigroup theory for the linearization of the equation. In the case of the famous *Navier-Stokes equations* which for a given domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, describe the behavior of a Newtonian fluid over time, the linearization is given by the *Stokes equation*s

$$
\begin{aligned}
\partial_t u - \Delta u + \nabla \pi &= 0 \quad \text{in } \Omega,\ t > 0, \\
\operatorname{div}(u) &= 0 \quad \text{in } \Omega,\ t > 0, \\
u(0) &= a \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,\ t > 0,
\end{aligned}
$$

where $u\colon \mathbb{R}^+ \times \Omega \to \mathbb{R}^d$ stands for the velocity field and $\pi\colon \mathbb{R}^+ \times \Omega \to \mathbb{R}$ represents the pressure of the fluid. The so-called *Stokes semigroup* $(\mathrm{e}^{-tA})_{t\geq 0}$ describes the evolution of the velocity u and the *Stokes operator* A corresponds to the term "$-\Delta u + \nabla \pi$" in the Stokes equations.

Having a semigroup makes it possible to look for *mild solutions* to the Navier-Stokes equations using a variation of constants formula to construct an iteration method. This approach was introduced by Fujita and Kato [9, 21] and builds mainly on resolvent estimates for the Stokes operator A and the analyticity property of the Stokes semigroup. Fujita and Kato applied their methods to find solutions in L^2 spaces, given that the initial values a were to be found in domains of fractional powers of the Stokes operator. Fractional power domains are in general difficult to describe which is why Fujita and Kato suggested that an L^p theory for the Navier-Stokes equations could overcome this problem, see [9, p. 313]. Resolvent estimates in L^p for the Stokes operator and the analyticity of the Stokes semigroup are crucial ingredients of this classical functional analytic approach to the solution theory of the Navier-Stokes equations.

On smooth domains, the first milestone in the direction of an L^p theory was laid by Giga [12] and Solonnikov [35] who proved that on bounded smooth domains the Stokes operator generates a bounded analytic semigroup in $\mathrm{L}^p(\Omega)$ for $1 < p < \infty$. This result was then used by Giga and Miyakawa [14] to prove the existence of mild solutions given L^3 initial data. Kato [20] then extended this theory to the whole space and later Giga [13] adjusted the approach for bounded smooth domains.

A natural assumption in real world applications is that the boundary of the domain Ω is not smooth but merely Lipschitz continuous. On Lipschitz domains, the analysis of partial differential equations is more complicated as classical localization techniques fail due to the lack of smoothness of the boundary. In order to overcome these problems, new techniques had to be developed which gave rise to the work of Fabes, Kenig and Verchota [7] who constructed solutions to the L^2 Dirichlet problem of the Stokes equations by the method of layer potentials.

As on smooth domains, the further expansion of the L^p theory by means of the classical iteration method was reliant on the analyticity of the Stokes semigroup. Due to the boundedness properties of the Helmholtz projection, Taylor conjectured in his paper [37] from 2000 that the analyticity of the Stokes semigroup would only hold in a neighborhood of $p = 2$, namely for $p \in (3/2 - \varepsilon, 3 + \varepsilon)$ in the three dimensional case. His conjecture was supported by the fact that, not much later, Deuring showed in [5] the existence of a three dimensional Lipschitz domain such that the needed resolvent estimates would fail for sufficiently large p.

Twelve years passed until a positive result could be given to Taylor's conjecture: Shen showed in his seminal paper [32] that in bounded Lipschitz domains $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$, for all $p \in (2d/(d+1) - \varepsilon, 2d/(d-1) + \varepsilon)$ the Stokes operator generates a bounded analytic semigroup on $\mathrm{L}^p_\sigma(\Omega)$. Shen's result on the analyticity of the semigroup thus made the Fujita-Kato approach available for the study of the Navier-Stokes equations on Lipschitz domains which, together with other methods, was used by Tolksdorf to prove the existence of mild solutions in L^3 to the Navier-Stokes equations [39].

In the present thesis, we will study Shen's approach to the analyticity problem of the Stokes semigroup in L^p for bounded Lipschitz domains in $\mathbb{R}^d$. While Shen's result was only formulated for $d \geq 3$, we will show that his approach can be extended to the two dimensional case by proving suitable estimates on the fundamental solutions of the Stokes resolvent problem.

Except for the first chapter, in which we will gather the fundamentals and formulate the problem under consideration, the rest of this thesis will closely follow the structure of Shen's work [32].

In Chapter 2, we will derive the central estimates of the fundamental solutions to the Stokes resolvent problem on $\mathbb{R}^d$ by taking advantage of their explicit representation formula.

Chapter 3 will introduce the method of boundary layer potentials for the solution of the Stokes resolvent problem on bounded Lipschitz domains. We will discover the central relation between boundary values that are attained nontangentially and the singular integral operators that provide a representation formula for solutions to the Stokes resolvent problem.

In Chapter 4, we take a more general look on solutions to the Stokes resolvent problem and derive Rellich-type estimates for these solutions.

Based on the results of Chapter 3 and 4, in Chapter 5, we will study the L^2 Dirichlet problem of the Stokes resolvent system. In this chapter, we will see that for given boundary values in $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$, there exists a unique solution to the L^2 Dirichlet problem which is given via a double layer potential.

Finally, in Chapter 6, we will tackle the problem of analyticity of the Stokes operator. In this chapter, we will derive the necessary resolvent estimates on L^p by making use of Shen's extrapolation theorem which can be considered a refined version of the Calderón-Zygmund Lemma.

This last paragraph is reserved to mention some of the persons that were involved in the course of this thesis. I thank Prof. Dr. Robert Haller-Dintelmann for taking the responsibility of supervising this thesis. Furthermore, I want to express my gratitude to Dr. Patrick Tolksdorf. This thesis would not have been possible without his generous support in innumerable occasions. I also want to thank Sebastian Bechtel for proofreading parts of this thesis. Last but not least, I want to thank my family for supporting me during my studies.

Chapter 1

Fundamentals

The purpose of this chapter is to collect basic definitions that will be used throughout the subsequent chapters. Furthermore, we want to formulate the main problem regarding the resolvent estimates of the Stokes operator. Throughout this chapter, we let d always denote a natural number greater or equal to 2.

1.1 Lipschitz Domains

In this first section, we will establish the fundamental notions regarding bounded Lipschitz domains.

Definition 1.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded open connected set. We call Ω a *bounded Lipschitz domain* if there exist $r_0, M > 0$ such that for all $x \in \partial\Omega$ there exists a function $\eta_x \colon \mathbb{R}^{d-1} \to \mathbb{R}$ which is Lipschitz continuous and additionally fulfills $\eta_x(0) = 0$ and $\|\nabla \eta_x\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} \leq M$, and a rotation $R_x \colon \mathbb{R}^d \to \mathbb{R}^d$ such that for all $0 < r \leq r_0$

$$\begin{aligned} R_x[\Omega - \{x\}] \cap D(r) &= D_{\eta_x}(r) \quad \text{and} \\ R_x[\partial\Omega - \{x\}] \cap D(r) &= I_{\eta_x}(r), \end{aligned}$$

where

$$\begin{aligned} D(r) &:= \big\{(x', x_d) \colon |x'| < r, |x_d| < 10d\,(M + 1)\,r\big\}, \\ D_{\eta_x}(r) &:= \big\{(x', x_d) \colon |x'| < r, \eta_x(x') < x_d < 10d\,(M + 1)\,r\big\}, \\ I_{\eta_x}(r) &:= \big\{(x', x_d) \colon |x'| < r, \eta_x(x') = x_d\big\}. \end{aligned}$$

It is common to refer to sets of the form $D_{\eta_x}(r)$ as *Lipschitz cylinders*. If the point x in the definition of Lipschitz cylinders is not of particular importance, we will denote the Lipschitz cylinder by $D_\eta(r)$.

If Ω is a bounded Lipschitz domain, $x \in \partial\Omega$ and $0 < r \leq r_0$, then we may define $U_{x,r} := \{x\} + R_x^{-1} D(r)$, where R_x is the rotation corresponding to x from Definition 1.1. This is all we need to define the Lipschitz character of a bounded Lipschitz domain Ω as suggested by Pipher and Verchota in [30, Sec. 5].

Definition 1.2. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and $x_1, \dots, x_n \in \partial\Omega$ be such that $\{U_{x_i,r_0}\}_{i=1}^n$ covers $\partial\Omega$. Furthermore, let M be the constant from Definition 1.1. Then a constant $C > 0$ is said to depend on the *Lipschitz character of* Ω if it depends on M and n.

That the Lipschitz character is indeed a fruitful concept will be emphasized by the following theorem. This result is a crucial ingredient in the proof of the Rellich estimates in Chapter 4 as it provides a useful approximating property of Lipschitz domains. In short it enables us to approximate a bounded Lipschitz domain Ω by a sequence $(\Omega_j)_{j\in\mathbb{N}}$ of C^∞ domains in such a way that estimates on Ω_j with bounding constants that only depend on the Lipschitz characters may be extended to Ω when taking the limit. The original proof of this theorem goes back to Nečas [29] and Verchota [40]. The presented version of this theorem appeared in Brown [2].

Theorem 1.3 (Nečas, Verchota)**.** *Let Ω be a bounded Lipschitz domain. Then, there exists a sequence of C^∞ domains $(\Omega_k)_{k\in\mathbb{N}}$ with uniform Lipschitz characters, corresponding homeomorphisms $\Lambda_k \colon \partial\Omega \to \partial\Omega_k$, functions $\vartheta_k \colon \partial\Omega \to \mathbb{R}^+$ and a smooth compactly supported vector field $h \colon \mathbb{R}^d \to \mathbb{R}^d$ which satisfy the following properties:*

i) There exists a covering of $\partial\Omega$ by coordinate cylinders which also serve as coordinate cylinders for $\partial\Omega_k$.

ii) The homeomorhpisms $\Lambda_k \colon \partial\Omega \to \partial\Omega_k$ satisfy

$$\sup_{q\in\partial\Omega} |q - \Lambda_k(q)| \to 0 \quad \text{as } k \to \infty$$

and $\Lambda_k(q)$ approaches q nontangentially *meaning that for all $k \in \mathbb{N}$*

$$|q - \Lambda_k(q)| < (1 + \beta)\,\mathrm{dist}(\Lambda_k(q), \partial\Omega)$$

for some constant β depending only on d and the Lipschitz character of Ω.

iii) The normals n_k *of* $\partial\Omega_k$ *satisfy* $\lim_{k\to\infty} n_k(\Lambda_k(q)) = n(q)$ *a.e. for all* $q \in \partial\Omega$.

iv) The functions ϑ_k *satisfy* $\delta \leq \vartheta_k \leq \delta^{-1}$ *for some* $\delta > 0$, $\vartheta^k \to 1$ *pointwise a.e. and*

$$\int_E \vartheta_k(q)\,\mathrm{d}\sigma(q) = \int_{\Lambda_k(E)} \mathrm{d}\sigma_k(q),$$

where $E \subset \partial\Omega$ *is measurable and* σ_k *denotes the surface measure on* $\partial\Omega_k$. *Furthermore, if* $u\colon \partial\Omega_k \to \mathbb{C}$ *is an integrable function on* $\partial\Omega_k$*, the following "transformation rule" holds:*

$$\int_{\partial\Omega} u(\Lambda_k(q))\vartheta_k(q)\,\mathrm{d}\sigma(q) = \int_{\partial\Omega_k} u(q)\,\mathrm{d}\sigma_k(q)\,.$$

v) The vector field h *satisfies* $h \cdot n_k \geq c > 0$ *a.e. on each* $\partial\Omega_k$ *where* n_k *denotes the unit outer normal to* $\partial\Omega_k$.

The next concept which we introduce will allow us to talk about boundary values of functions which are defined on Ω by considering their nontangential behavior. The first step will be to define nontangential approach regions. Unfortunately, in the literature there exist at least two different concepts which will be introduced in the next definitions. In the following, by a cone we mean an open, circular, truncated cone with only one convex component.

Definition 1.4 (Regular family of cones, Verchota)**.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. If $q \in \partial\Omega$, then $\Gamma(q)$ will denote a cone with vertex q and one component in Ω. Assigning to each $q \in \partial\Omega$ one cone $\Gamma(q)$, the family $\{\Gamma(q)\colon q \in \partial\Omega\}$ will be called *regular* if there exist $x_1, \ldots, x_{n_0} \in \partial\Omega$, $\tilde{r} > 0$ and rotations $\tilde{R}_{x_1}, \ldots, \tilde{R}_{x_{n_0}}$ such that

$$\partial\Omega \subset \bigcup_{i=1}^{n_0} \{x_i\} + \tilde{R}_{x_i}^{-1} D(4\tilde{r}/5),$$

and such that there exist Lipschitz continuous functions $\tilde{\eta}_{x_i}\colon \mathbb{R}^{d-1} \to \mathbb{R}$ such that for all $\tilde{r} \leq r \leq \nu\tilde{r}$ with

$$\nu := 1 + [1 + [10d(M+1)]^2]^{1/2}$$

we have

$$\tilde{R}_{x_i}[\Omega - \{x_i\}] \cap D(r) = D_{\tilde{\eta}_{x_i}}(r) \quad \text{and}$$
$$\tilde{R}_{x_i}[\partial\Omega - \{x_i\}] \cap D(r) = I_{\tilde{\eta}_{x_i}}(r)\,.$$

In addition, for all $i = 1, \dots, n_0$ there exist cones α_i, β_i and γ_i with vertex at the origin and axis along the x_d-axis such that

$$\alpha_i \subset \overline{\beta_i} \setminus \{0\} \subset \gamma_i$$

and such that for all $q \in [\{x_i\} + \tilde{R}_{x_i}^{-1} D(4\tilde{r}/5)] \cap \partial\Omega$, we have

$$\tilde{R}_{x_i}^{-1}\alpha_i + \{q\} \subset \Gamma(q) \subset \overline{\Gamma(q)} \setminus \{q\} \subset \tilde{R}_{x_i}^{-1}\beta_i + \{q\},$$

$$\tilde{R}_{x_i}^{-1}\gamma_i + \{q\} \subset [\{x_i\} + \tilde{R}_{x_i}^{-1} D(\tilde{r})] \cap \Omega\,.$$

We will sometimes denote a regular cone as above by $\Gamma_{\mathrm{V}}(q)$.

For the existence of such families of cones, see the appendix of Verchota [40] and [41].

In Verchota cones $\Gamma_{\mathrm{V}}(q)$, we have the properties that for all bounded Lipschitz domains Ω there exists a constant $C > 0$ depending only on the Lipschitz character of Ω such that for all $q, p \in \partial\Omega$ and any $x \in \Gamma_V(p)$ we have that

$$|x - q| \geq C\,|x - p| \tag{1.1}$$

$$|x - q| \geq C\,|p - q|. \tag{1.2}$$

For a proof, see Verchota [40, p. 9f]. The following lemma and definition are used by Shen, see [33, p. 174].

Lemma and Definition 1.5 (Nontangential approach region, Shen)**.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded Lipschitz domain. There exists $\alpha > 1$ depending only on d and the Lipschitz character such that*

$$\Gamma_\alpha(q) := \big\{x \in \Omega \colon |x - q| < \alpha \operatorname{dist}(x, \partial\Omega)\big\}$$

contains a cone of fixed height and aperture with vertex at q. In this case, we call $\big\{\Gamma_\alpha(q) \colon q \in \partial\Omega\big\}$ a family of nontangential approach regions.

Note that in Shen cones $\Gamma_\alpha(q)$, we have that for $p, q \in \partial\Omega$ and $x \in \Gamma_\alpha(p)$

$$\begin{aligned} |p - q| &\leq |p - x| + |x - q| \leq \alpha \operatorname{dist}(x, \partial\Omega) + |x - q| \\ &\leq (\alpha + 1)|x - q|, \end{aligned} \tag{1.3}$$

where α is the constant from Definition 1.5.

Depending on the type of cones used, one may introduce similar concepts of nontangential convergence and nontangential maximal functions.

Definition 1.6. For a function u in Ω and a fixed family of nontangential approach regions $\{\Gamma_\alpha\}$, we define the nontangential maximal function $(u)^*_\alpha$ by

$$(u)^*_\alpha(q) = \sup\left\{|u(x)| \colon x \in \Gamma_\alpha(q)\right\} \tag{1.4}$$

for $q \in \partial\Omega$. For a fixed regular family of cones $\{\Gamma_{\mathrm{V}}(q)\}$, we define the nontangential maximal function $N(u)(q)$ via

$$N(u)(q) = \sup\left\{|u(x)| \colon x \in \Gamma_{\mathrm{V}}(q)\right\}.$$

Note that Tolksdorf [39, Prop. 4.1.11] and Shen [33, Prop. 7.1.2] show that the choice of α for the nontangential maximal function as in (1.4) does not affect their p-norms in an unpredictable way. In fact, their p-norms for different α_1 and α_2 stay comparable with a constant only depending on d, α_1, α_2 and the Lipschitz character. For a given bounded Lipschitz domain, we will therefore always assume that $\alpha > 1$ has been chosen big enough such that on the one hand condition (ii) from Theorem 1.3 is fulfilled and that on the other hand α is large enough such that $\left\{\Gamma_\alpha(q) \colon q \in \partial\Omega\right\}$ is a family of nontangential approach regions. In the following, we will thus ignore the parameter α in cones and nontangential maximal functions and tacitly assume that it was chosen appropriately. We further note that the functions $(u)^*$ and $N(u)$ will not be comparable in general, see the discussion in Tolksdorf [39, p. 91].

The abovementioned constructions of cones are not limited to cones that lay in the interior of the domain Ω. In fact, the same construction can be carried out for the exterior domain $\mathbb{R}^d \setminus \overline{\Omega}$ yielding cones that lay outside of Ω. While Verchota's cones from Definition 1.4 can be mirrored along the $x_d = 0$ plane in a suitable local coordinate system, Shen's cones from Definition 1.5 have to be modified in a natural way to give cones lying inside of $\mathbb{R}^d \setminus \overline{\Omega}$, namely

$$\Gamma^{\mathrm{ext}}_\alpha(q) \coloneqq \left\{x \in \mathbb{R}^d \setminus \overline{\Omega} \colon |x - q| < \alpha \operatorname{dist}(x, \partial\Omega)\right\}.$$

As the name *nontangential approach region* suggests, for functions u living on Ω or $\mathbb{R}^d \setminus \overline{\Omega}$ there will be a notion of convergence of function values $u(x)$ as x approaches a point on $p \in \partial\Omega$. The idea is to restrict the set of directions from which one can approach p by only allowing sequences of points lying in cones $\Gamma(q)$.

Definition 1.7 (Nontangential convergence)**.** Let Ω be a bounded Lipschitz domain and $\left\{\Gamma(q) \colon q \in \partial\Omega\right\}$ be a family of nontangential approach regions with its exterior counterpart $\left\{\Gamma^{\mathrm{ext}}(q) \colon q \in \partial\Omega\right\}$. Let furthermore u be a function on $\mathbb{R}^d \setminus \partial\Omega$ and f a function on $\partial\Omega$.

We say that u converges to f in $q \in \partial\Omega$ *in the sense of nontangential convergence from the inside* if

$$\lim_{\substack{x \to q \\ x \in \Gamma(q)}} u(x) = f(q)$$

and we say that u converges to f in $q \in \partial\Omega$ *in the sense of nontangential convergence from the outside* if

$$\lim_{\substack{x \to q \\ x \in \Gamma^{\text{ext}}(q)}} u(x) = f(q).$$

If both of the above limits exist and coincide we say that u converges to f in $q \in \partial\Omega$ *in the sense of nontangential convergence*. We will also use this terminology for functions u living only on Ω which have a nontangential limit from the inside.

Usually, the nontangential limits taken from the inside of the domain will differ from the ones taken from the outside of the domain. For functions u on $\mathbb{R}^d \setminus \partial\Omega$ we will therefore often use the notation u_+ to denote the *inner* nontangential limit and u_- for the respective *outer* nontangential limit.

To put our new vocabulary to use, we formulate and prove the divergence theorem for functions on bounded Lipschitz domains that do not have a trace but a nontangential limit and integrable nontangential maximal functions. A similar statement was proven by Shen in [33, Thm. 7.1.6]. Note that in the following proposition and its proof we use Einstein's summation convention.

Proposition 1.8. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, a bounded Lipschitz domain and $f\colon \Omega \to \mathbb{C}^d$ smooth and $g\colon \partial\Omega \to \mathbb{C}$ measurable. Suppose that the nontangential limit f_+ exists almost everywhere and that the nontangential maximal function $(g)^*$ is integrable on $\partial\Omega$ and $|f_+| \leq (g)^*$ a.e. Furthermore, let* $\operatorname{div}(f)$ *be integrable on Ω. Then, Green's formula*

$$\int_{\partial\Omega} f_k(s) n_k(s)\, \mathrm{d}\sigma(s) = \int_{\Omega} \operatorname{div}(f)(x)\, \mathrm{d}x \tag{1.5}$$

holds, where n denotes the outer unit normal vector of $\partial\Omega$.

Proof. The proof rests heavily on the powerful Theorem 1.3 and uses its full capacity to uncover a very useful approximation argument.

Let's start by approximating Ω by a sequence $(\Omega_l)_{l\in\mathbb{N}}$ of C^∞ domains with uniform Lipschitz characters as described in Theorem 1.3. Remember that by Theorem 1.3 iv), the homeomorphisms $\Lambda_l\colon \partial\Omega \to \partial\Omega_l$ give rise to a transformation rule of the form

$$\int_{\partial\Omega_l} f_k(s) n_k^{(l)}(s)\, \mathrm{d}\sigma_l(s) = \int_{\partial\Omega} \vartheta_l(x)\, f_k(\Lambda_l(x))\, n_k^{(l)}(\Lambda_l(x))\, \mathrm{d}\sigma(x). \tag{1.6}$$

The idea of the proof is based on the approximation argument performed in Brown [2, Prop. 2.4]. Additionally, we have $\lim_{l\to\infty} \vartheta_l(x) = 1$ and $\lim_{l\to\infty} \Lambda_l(x) = x$ almost everywhere, where $\Lambda_l(x) \in \Gamma(x)$ for all $l \in \mathbb{N}$ thanks to Theorem 1.3 ii). Furthermore, we know that $\lim_{l\to\infty} n_k^{(l)}(\Lambda_l(x)) = n_k(x)$ almost everywhere by Theorem 1.3 and that f has a nontangential limit almost everywhere. This gives us that

$$\lim_{l\to\infty} \vartheta_l(x)\, f_k(\Lambda_l(x))\, n_k^{(l)}(\Lambda_l(x)) = f_k(x) n_k(x), \quad \text{a.e. } x \in \partial\Omega.$$

This sequence of integrands is dominated by $\delta (g)^*$, where $(g)^* \in \mathrm{L}^1(\partial\Omega)$ holds by assumption and δ is the uniform bound to ϑ_l due to Theorem 1.3 iv). Hence, the dominated convergence theorem is applicable and yields

$$\lim_{l\to\infty} \int_{\partial\Omega} \vartheta_l(s)\, f_k(\Lambda_l(s))\, n_k^{(l)}(\Lambda_l(s))\, \mathrm{d}\sigma(s) = \int_{\partial\Omega} f_k(s) n_k(s)\, \mathrm{d}\sigma(s). \tag{1.7}$$

Now consider the left-hand side of identity (1.6). By Green's formula [6, p. 711f.], we know that

$$\int_{\partial\Omega_l} f_k(s) n_k^{(l)}(s)\, \mathrm{d}\sigma_l(s) = \int_{\Omega_l} \operatorname{div}(f(x))\, \mathrm{d}x, \quad \text{for all } l \in \mathbb{N}.$$

As $\Omega_l \subseteq \Omega$ for all $l \in \mathbb{N}$, the dominated convergence theorem leaves us with

$$\lim_{l\to\infty} \int_{\Omega_l} \operatorname{div}(f(x))\, \mathrm{d}x = \int_{\Omega} \operatorname{div}(f(x))\, \mathrm{d}x. \tag{1.8}$$

Gluing together equations (1.7) and (1.8) gives the claim. □

1.2 The Stokes Operator

In this section, we will introduce the Stokes operator on the solenoidal counterparts $\mathrm{L}^2(\Omega;\mathbb{C}^d)$ and $\mathrm{L}^p(\Omega;\mathbb{C}^d)$ for $1 < p\infty$ and establish a relation to the *Dirichlet problem for the Stokes resolvent system*

$$\begin{aligned} -\Delta u + \nabla\phi + \lambda u &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.9}$$

where $\lambda \in \Sigma_\theta := \{z \in \mathbb{C} \colon \lambda \neq 0 \text{ and } |\arg(z)| < \pi - \theta\}$ and $\theta \in (0,\pi)$.

We begin by defining the relevant function spaces. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $1 < p < \infty$. We define

$$\mathrm{C}^\infty_{c,\sigma}(\Omega) := \{\varphi \in \mathrm{C}^\infty_c(\Omega;\mathbb{C}^d) \colon \operatorname{div}(\varphi) = 0\},$$

which can serve as a suitable space of test functions. We now close this space in $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ and the Sobolev space $\mathrm{W}^{1,p}(\Omega; \mathbb{C}^d)$ which gives

$$\mathrm{L}^p_\sigma(\Omega) := \overline{\mathrm{C}^\infty_{c,\sigma}(\Omega)}^{\mathrm{L}^p(\Omega)}$$

and

$$\mathrm{W}^{1,p}_{0,\sigma}(\Omega) := \overline{\mathrm{C}^\infty_{c,\sigma}(\Omega)}^{\mathrm{W}^{1,p}(\Omega)},$$

respectively. If $p = 2$, we will use the symbol $\mathrm{H}^1_{0,\sigma}(\Omega)$ to denote $\mathrm{W}^{1,2}_{0,\sigma}(\Omega)$ in order to emphasize that this space is a Hilbert space.

In order to define the Stokes operator, we introduce the sesquilinear form

$$\mathfrak{a}\colon \mathrm{H}^1_{0,\sigma}(\Omega) \times \mathrm{H}^1_{0,\sigma}(\Omega) \to \mathbb{C}, \quad (u, v) \mapsto \int_\Omega \nabla u \cdot \overline{\nabla v} \, \mathrm{d}x =: \int_\Omega \partial_k u_i \, \overline{\partial_k u_i} \, \mathrm{d}x.$$

Note that for $u \in \mathrm{H}^1_{0,\sigma}(\Omega)$ the gradient ∇u is a matrix and an element of the space $\mathrm{L}^2(\Omega; \mathbb{C}^{d\times d})$.

Definition 1.9. The *Stokes operator* A_2 on $\mathrm{L}^2_\sigma(\Omega)$ is given via

$$\begin{aligned} \mathcal{D}(A_2) &:= \Big\{ u \in \mathrm{H}^1_{0,\sigma}(\Omega) \colon \exists! f \in \mathrm{L}^2_\sigma(\Omega) \text{ s.t. } \forall v \in \mathrm{H}^1_{0,\sigma}(\Omega) \colon \mathfrak{a}(u, v) = \int_\Omega f \cdot \overline{v} \, \mathrm{d}x \Big\} \\ A_2 u &:= f, \end{aligned}$$

where $u \in \mathcal{D}(A_2)$ and f comes from the definition of the domain.

The following theorem by Mitrea and Monniaux [27, Thm. 4] shows that our definition of the Stokes operator and the one used in Shen's paper [32] coincide. Another advantage of this characterization is the immediate link of the Stokes operator to the Stokes system.

Theorem 1.10. *If $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is a bounded Lipschitz domain and A_2 is the Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$ then*

$$\mathcal{D}(A_2) = \big\{ u \in \mathrm{H}^1_{0,\sigma}(\Omega) \colon \exists \pi \in \mathrm{L}^2(\Omega) \textit{ s.t. } -\Delta u + \nabla \pi \in \mathrm{L}^2_\sigma(\Omega) \big\},$$

where the expression $-\Delta u + \nabla \pi \in \mathrm{L}^2_\sigma(\Omega)$ needs to be understood in the sense of distributions. For $u \in \mathcal{D}(A_2)$ and the corresponding pressure π we have

$$A_2 u = -\Delta u + \nabla \pi.$$

The following proposition summarizes some facts about the Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$. A proof can be found in Tolksdorf [39, Prop. 5.2.5]. We will always denote the norm of bounded linear operators between two Banach spaces X and Y or on a Banach space X by $\| \cdot \|_{\mathfrak{L}(X,Y)}$ or $\| \cdot \|_{\mathfrak{L}(X)}$, respectively.

Proposition 1.11. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and A_2 the Stokes operator as in Definition 1.9. Then we have*

a) A_2 is closed with dense domain. Furthermore, $0 \in \rho(A_2)$.

b) $\sigma(A) \subset (0, \infty)$ and for all $\theta \in (0, \pi]$ there exists $C > 0$ such that

$$\|\lambda(\lambda + A)^{-1}\|_{\mathfrak{L}(\mathrm{L}^2_\sigma(\Omega))} \leq C, \quad \text{for all } \lambda \in \mathbb{C} \setminus \overline{\Sigma}_\theta. \tag{1.10}$$

In particular, $-A_2$ generates a bounded analytic semigroup on $\mathrm{L}^2_\sigma(\Omega)$.

With these results at hand we can now give a quick recap of the solution theory to (1.9). Let $f \in \mathrm{L}^2_\sigma(\Omega)$ and $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi/2)$. By the previous theorem and proposition, we know that there exists a unique $u \in \mathcal{D}(A_2) \subseteq \mathrm{H}^1_{0,\sigma}(\Omega)$ and some $\pi \in \mathrm{L}^2(\Omega)$ such that

$$-\Delta u + \nabla \pi + \lambda u = A_2 u + \lambda u = f.$$

For general $f = \mathrm{L}^2(\Omega; \mathbb{C}^d)$ we use the *Helmholtz projection* $\mathbb{P}_2$ to get

$$\Delta u + \nabla \pi + \lambda u + (I - \mathbb{P}_2) f = \mathbb{P}_2 f + (I - \mathbb{P}_2) f = f,$$

where u and π now correspond to $\mathbb{P}_2 f \in \mathrm{L}^2_\sigma(\Omega)$. On bounded Lipschitz domains the orthogonal complement to $\mathbb{P}_2[\mathrm{L}^2(\Omega; \mathbb{C}^d)] = \mathrm{L}^2_\sigma(\Omega)$ is characterized via

$$\mathrm{L}^2_\sigma(\Omega)^\perp = \big\{ f \in \mathrm{L}^2(\Omega; \mathbb{C}^d) \colon f = \nabla \phi, \text{ for some } \phi \in \mathrm{L}^2(\Omega) \big\}.$$

A proof of this fact can be found in the book of Sohr [34, Lem. 2.5.3]. Using this result, we find $g \in \mathrm{L}^2(\Omega)$ such that $\nabla g = (I - \mathbb{P}_2) f$ in the sense of distributions and we see that

$$-\Delta u + \nabla(\pi + g) + \lambda u = f.$$

Consequently, we see that solving the resolvent equation for the Stokes operator and solving the Stokes resolvent system (1.9) are two sides of the same coin. Furthermore, we may deduce from the resolvent estimate (1.10) that the solution u which apparently is not affected by the additional part $(I - \mathbb{P}_2) f$ fulfills the inequality

$$|\lambda| \|u\|_{\mathrm{L}^2(\Omega; \mathbb{C}^d)} = |\lambda| \| (A_2 + \lambda)^{-1} \mathbb{P}_2 f \|_{\mathrm{L}^2(\Omega; \mathbb{C}^d)} \leq C \|f\|_{\mathrm{L}^2(\Omega; \mathbb{C}^d)},$$

where C depends only on θ. By the calculations above it is understandable why this estimate on u instead of (1.10) is sometimes called *resolvent estimate*.

In order to develop an L^p theory for system (1.9), one way is to study the Stokes operator on subspaces of $\mathrm{L}^p(\Omega; \mathbb{C}^d)$. More precisely, we are interested in estimating solutions $u \in \mathrm{H}^1_0(\Omega; \mathbb{C}^d)$, in $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ provided that the right hand side of the Stokes resolvent system (1.9) is an element of the space $\mathrm{L}^2(\Omega; \mathbb{C}^d) \cap \mathrm{L}^p(\Omega; \mathbb{C}^d)$. This is once again just one side of the aforementioned coin. The other side just asks for a resolvent estimate on the Stokes operator, hoping that, in analogy to Proposition 1.11, this leads to an analytic semigroup.

Definition 1.12. Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain: If $p > 2$, we define the Stokes operator A_p via its *part of* A_2 *in* $\mathrm{L}^p_\sigma(\Omega)$:

$$\begin{aligned} \mathcal{D}(A_p) &:= \Big\{u \in \mathcal{D}(A_2) \cap \mathrm{L}^p_\sigma(\Omega) \colon A_2 u \in \mathrm{L}^p_\sigma(\Omega)\Big\}, \\ A_p u &:= A_2 u, \quad u \in \mathcal{D}(A_p). \end{aligned}$$

For $p > 2$, there exists an analog to Theorem 1.10. The peculiar range of p for which this theorem holds is due to the fact that the boundedness of the Helmholtz projection on $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ is a crucial ingredient to the proof and a fundamental pillar of the L^p theory of the Stokes equations. More details about the mechanics of the Helmholtz projection can be found in Tolksdorf [39, Sec. 5.1].

Theorem 1.13 (Thm. 5.2.11, [39])**.** *Let* $\Omega \subseteq \mathbb{R}^d$*,* $d \geq 2$*, be a bounded Lipschitz domain. Then, there exists* $\varepsilon > 0$ *such that for all*

$$2 < p < \frac{2d}{d-1} + \varepsilon$$

the domain of the Stokes operator A_p *is characterized as*

$$\mathcal{D}(A_p) = \big\{u \in \mathrm{W}^{2,p}_{0,\sigma}(\Omega) \colon \exists \pi \in \mathrm{L}^p(\Omega) \text{ s.t. } -\Delta u + \nabla \pi \in \mathrm{L}^p_\sigma(\Omega)\big\},$$

where the expression $\Delta u + \nabla \pi \in \mathrm{L}^p_\sigma(\Omega)$ *needs to be understood in the sense of distributions. For* $u \in \mathcal{D}(A_p)$ *and the corresponding pressure* π *we have*

$$A_p u = -\Delta u + \nabla \pi.$$

For $p < 2$, there exist various ways to define the Stokes operator. One adequate way is to dualize the operator $A_{p'}$, where $p' = p/(1+p)$ is the Hölder conjugate exponent. In order to carry out this construction, we need the spaces $\mathrm{L}^p_\sigma(\Omega)$ to exhibit the same behavior regarding dualization as the spaces $\mathrm{L}^p(\Omega; \mathbb{C}^d)$.

Lemma 1.14 (Lem. 5.2.13, [39])**.** *Let* $\Omega \subset \mathbb{R}^d$*,* $d \geq 2$*, be a bounded Lipschitz domain. Then, there exists* $\varepsilon > 0$ *such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

the spaces $\mathrm{L}^p_\sigma(\Omega)$ *and* $(\mathrm{L}^{p'}_\sigma(\Omega))^*$ *are isomorphic, where* $(\mathrm{L}^{p'}_\sigma(\Omega))^*$ *denotes the antidual space and* $p' = p/(p-1)$ *is the Hölder conjugate exponent of* p. *The isomorphism* Ψ *is given by*

$$[\Psi f](g) = \int_\Omega f \cdot \overline{g}\, \mathrm{d}x, \quad g \in \mathrm{L}^{p'}_\sigma(\Omega).$$

Now we define the Stokes operator for $p < 2$ as announced.

Definition 1.15. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and let $\varepsilon > 0$ be as in Lemma 1.14. Let furthermore

$$\frac{2d}{d+1} - \varepsilon < p < 2$$

and Ψ be the isomorphism from Lemma 1.14. Then, the *Stokes operator on* $\mathrm{L}^p_\sigma(\Omega)$ is defined via

$$\begin{aligned} \mathcal{D}(A_p) &:= \{u \in \mathrm{L}^p_\sigma(\Omega) \colon \Psi u \in \mathcal{D}(A^*_{p'})\} \\ A_p &:= \Psi^{-1} A^*_{p'} \Psi u, \end{aligned}$$

where $p' = p/(1-p)$ denotes the Hölder conjugate exponent of p and $A^*_{p'}$ the adjoint operator to $A_{p'}$.

Without investing too much additional work, it is now possible to prove the following theorem.

Theorem 1.16 (Thm. 5.2.9 and Thm. 5.2.17, [39])**.** *Let* $\Omega \subset \mathbb{R}^d$, $d \geq 2$, *be a bounded Lipschitz domain. Then, there exists* $\varepsilon > 0$ *such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

the operator A_p *is closed with dense domain. Furthermore* $0 \in \rho(A_p)$.

The natural question arises when comparing Theorem 1.16 with the Hilbert space counterpart Theorem 1.11: Does the Stokes operator generate a bounded analytic semigroup in $\mathrm{L}^p_\sigma(\Omega)$?

An affirmative answer was given by Shen in 2012 with his seminal paper [32] in which he provided the necessary resolvent estimates for $d \geq 3$ and thus solved Taylor's conjecture from [37]. Curiously, this positive result was limited to the case $d \geq 3$ even though Shen stated that the approach he developed should also work for $d = 2$, see [32, p. 399]. This sets the starting point for the present thesis. In the subsequent chapters, we will not only present Shen's approach to the problem of the resolvent estimates but we will furthermore extend his results whenever possible to the two dimensional case. We will work towards a proof of the following theorem:

Theorem 1.17. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. There exists $\varepsilon > 0$, such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

there exists a constant $C > 0$ such that for every $f \in \mathrm{L}^p_\sigma(\Omega)$ and all $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi/2)$, the inequality

$$|\lambda| \left\| (\lambda + A_p)^{-1} f \right\|_{\mathrm{L}^p(\Omega;\, \mathbb{C}^d)} \leq C \, \|f\|_{\mathrm{L}^p(\Omega;\, \mathbb{C}^d)}$$

holds. In particular, $-A_p$ is the generator of a bounded analytic semigroup on $\mathrm{L}^p_\sigma(\Omega)$.

Chapter 2

Estimating Fundamental Solutions

The purpose of this section is to study fundamental solutions of the Stokes resolvent problem and to deduce related estimates which will be crucial for the next chapters. Before working on the Stokes resolvent problem, we will take a look at the atoms of the fundamental solution of this problem: the Hankel functions.

As a basis for the subsequent sections and chapters, let us fix recurring quantities regarding sectors in the complex plane $\mathbb{C}$.

Let $\theta \in (0, \pi/2)$ and $\lambda \in \Sigma_\theta$ as in Section 1.2. The polar form of λ is given as $\lambda = r\mathrm{e}^{\mathrm{i}\tau}$ with $0 < r < \infty$ and $-\pi + \theta < \tau < \pi - \theta$. Now, set

$$k := \sqrt{r}\,\mathrm{e}^{\mathrm{i}(\pi+\tau)/2}.$$

Then, we have

$$k^2 = -\lambda \quad \text{and} \quad \frac{\theta}{2} < \arg(k) < \pi - \frac{\theta}{2}$$

as it holds

$$\arg(k) = \frac{\pi + \tau}{2} > \frac{\pi}{2} + \frac{-\pi + \theta}{2} = \frac{\theta}{2}$$

on the one hand and

$$\arg(k) = \frac{\pi + \tau}{2} < \frac{\pi}{2} + \frac{\pi - \theta}{2} = \pi - \frac{\theta}{2}$$

on the other hand. The preceding calculation gives rise to the following estimate:

$$\operatorname{Im}(k) > \sqrt{|\lambda|}\sin(\theta/2) > 0. \tag{2.1}$$

Indeed, we have

$$\operatorname{Im}(k) = \sqrt{r}\sin\Big(\frac{\pi + \tau}{2}\Big) = \sqrt{|\lambda|}\sin\Big(\frac{\pi + \tau}{2}\Big) \quad \text{and} \quad \frac{\theta}{2} < \frac{\pi + \tau}{2} < \pi - \frac{\theta}{2}$$

which gives for τ with $\frac{\pi+\tau}{2} \leq \frac{\pi}{2}$ that $\sin(\frac{\pi+\tau}{2}) \geq \sin(\theta/2)$ and for τ with $\frac{\pi+\tau}{2} > \frac{\pi}{2}$ that $\sin(\frac{\pi+\tau}{2}) > \sin(\pi - \theta/2) = \sin(\theta/2)$.

2.1 Hankel Functions and the Helmholtz Equation

Before diving into fundamental solutions of the Stokes resolvent problem, we will first consider a fundamental solution for the (scalar) Helmholtz equation in $\mathbb{R}^d$

$$-\Delta u + \lambda u = 0.$$

One fundamental solution with pole at the origin is given by

$$G(x;\lambda) = \frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}} \cdot \frac{1}{|x|^{d-2}} \cdot (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|), \tag{2.2}$$

see McLean [24, Eq. 9.14)], where $H^{(1)}_\nu(z)$ is the Hankel function of the first kind which according to Lebedev [23, Sec. 5.11] can be also be written as

$$H^{(1)}_\nu(z) = \frac{2^{\nu+1}\,\mathrm{e}^{\mathrm{i}(z-\nu\pi)}\,z^\nu}{\mathrm{i}\,\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^\infty \mathrm{e}^{2z\mathrm{i}s} s^{\nu-\frac{1}{2}}(1+s)^{\nu-\frac{1}{2}}\,\mathrm{d}s. \tag{2.3}$$

This formula holds for $\nu > -\frac{1}{2}$ and $0 < \arg(z) < \pi$. We will usually set

$$\nu = \nu_d = \frac{d}{2} - 1 \quad \text{and} \quad z = k|x|.$$

Note that by (2.1) we will always have $\operatorname{Im}(z) > 0$. Since $\nu_d < \nu_{d+1}$ for all $d \geq 2$ and $\nu_2 = 0$, formula (2.3) will hold for all dimensions $d \geq 2$ and all $x \in \mathbb{R}^d$.

In the case $d = 2$, formula (2.2) simplifies to

$$G(x;\lambda) = \frac{\mathrm{i}}{4} H^{(1)}_0(k|x|). \tag{2.4}$$

In the case $d = 3$, one has an even easier formula, namely

$$G(x;\lambda) = \frac{\mathrm{i}}{4\,(2\pi)^{1/2}} \cdot \frac{1}{|x|} \cdot (k|x|)^{1/2} H^{(1)}_{1/2}(k|x|) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\,\pi\,|x|}, \tag{2.5}$$

which is due to this easy formula for $H^{(1)}_{1/2}(z)$:

$$H^{(1)}_{1/2}(z) = -\mathrm{i}\left(\frac{2}{\pi z}\right)^{1/2} \mathrm{e}^{\mathrm{i}z}, \tag{2.6}$$

see Lebedev [23, Eq. (5.8.4)] or McLean [24, Eq. (9.15)].

Our first estimate is concerned with estimates on the fundamental solution $G(\,\cdot\,;\lambda)$ and its derivatives. The main concern of this lemma is with the asymptotic behavior of $G(\,\cdot\,;\lambda)$ for large values of $|x|$.

Lemma 2.1. *Let $\lambda \in \Sigma_\theta$. Then,*

$$|\nabla_x^l G(x;\lambda)| \leq \frac{C_l\, \mathrm{e}^{-c\sqrt{|\lambda|}|x|}}{|x|^{d-2+l}} \tag{2.7}$$

for any integer $l \geq 0$ if $d \geq 3$ and for $l \geq 1$ if $d = 2$. Here, $c > 0$ depends only on θ and C_l depends only on d, l and θ.

Let $d = 2$. Then, $|G(x;\lambda)| = o(1)$ as $|x| \to \infty$.

Proof. We start with the case $l = 0$ and $d \geq 3$. Let $\mathrm{Im}(z) > 0$ and $\nu = \nu_d = (d/2) - 1$. In particular, we have $\nu - \frac{1}{2} \geq 0$. Then, (2.3) gives

$$|H_\nu^{(1)}(z)| \leq C_d\, \mathrm{e}^{-\mathrm{Im}(z)}|z|^\nu \int_0^\infty \mathrm{e}^{-2s\,\mathrm{Im}(z)} s^{\nu-\frac{1}{2}}(1+s)^{\nu-\frac{1}{2}}\,\mathrm{d}s, \tag{2.8}$$

where $C_d > 0$ depends only on d. We apply the substitution rule with $t = s + (1/2)$ and calculate

$$\begin{aligned}
\mathrm{e}^{\frac{-\mathrm{Im}(z)}{2}} \int_0^\infty \mathrm{e}^{-2s\,\mathrm{Im}(z)} s^{\nu-\frac{1}{2}}(1+s)^{\nu-\frac{1}{2}}\,\mathrm{d}s &\leq \int_0^\infty \mathrm{e}^{-(s+\frac{1}{2})\mathrm{Im}(z)} s^{\nu-\frac{1}{2}}(1+s)^{\nu-\frac{1}{2}}\,\mathrm{d}s \\
&= \int_{\frac{1}{2}}^\infty \mathrm{e}^{-t\,\mathrm{Im}(z)}\Big(t^2 - \frac{1}{4}\Big)^{\nu-\frac{1}{2}}\,\mathrm{d}t \\
&\leq \int_0^\infty e^{-t\,\mathrm{Im}(z)} t^{2\nu-1}\,\mathrm{d}t \\
&= \int_0^\infty e^{-u} u^{2\nu-1}\,\mathrm{Im}(z)^{1-2\nu}\,\mathrm{Im}(z)^{-1}\,\mathrm{d}u \\
&= \mathrm{Im}(z)^{-2\nu}\int_0^\infty \mathrm{e}^{-u}u^{2\nu-1}\,\mathrm{d}u \\
&= C_\nu\,\mathrm{Im}(z)^{-2\nu},
\end{aligned}$$

where we also used the substitution rule with $u = t\,\mathrm{Im}(z)$. Now, we multiply (2.8) by $|z|^\nu$ and reuse the previous estimate to arrive at

$$|z|^\nu \left|H_\nu^{(1)}(z)\right| \leq C_d\, C_\nu\, |z|^{2\nu}\ \mathrm{Im}(z)^{-2\nu}\mathrm{e}^{-\frac{\mathrm{Im}(z)}{2}},$$

which for $z = k|x|$ gives

$$|kx|^\nu \left|H_\nu^{(1)}(k|x|)\right| \leq C\sin(\theta/2)^{-2\nu}\,\mathrm{e}^{-\frac{1}{2}\sin(\theta/2)\sqrt{|\lambda|}|x|}, \tag{2.9}$$

where $C > 0$ depends only on d and we used (2.1) to estimate

$$(|kx|)^{2\nu}\cdot \mathrm{Im}(k|x|)^{-2\nu} = |\lambda|^\nu \cdot \mathrm{Im}(k)^{-2\nu} \leq \sin(\theta/2)^{-2\nu}.$$

Note that per constructionem k has positive imaginary part. Using (2.2), we estimate for $d \geq 3$

$$|G(x;\lambda)| \leq C\,|x|^{2-d} \mathrm{e}^{-c\sqrt{|\lambda|}|x|}$$

and it is clear that the generic constant $C > 0$ depends on d and θ while $c > 0$ depends only on θ. This gives the estimate for $l = 0$ and $d \geq 3$.

Using the relation for the derivatives of Hankel functions which one finds in the book of Lebedev [23, Eq. (5.6.3)],

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{z^{-\nu} H_\nu^{(1)}(z)\right\} = -z^{-\nu} H_{\nu+1}^{(1)}(z),$$

we inductively establish the estimate (2.7) for $l \geq 1$ and $d \geq 2$: For $1 \leq j \leq d$, we calculate using the product and chain rule

$$\begin{aligned}\nabla_x G(x;\lambda) &= C\left\{ |x|^{1-d} \cdot (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|) - |x|^{2-d} \cdot (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}}(k|x|) \cdot k \right\} \\ &= C\,|x|^{1-d}\left\{ (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|) - (k|x|)^{\frac{d}{2}} H^{(1)}_{\frac{d}{2}}(k|x|) \right\}, \end{aligned} \tag{2.10}$$

where $C > 0$ is a generic constant that depends on d. Note that the first summand in (2.10) does not arise in the case $d = 2$ as is easily seen from equation (2.4). The terms in the bracket can now be estimated individually by (2.9). The extension of this proof to orders of differentiation $l \geq 2$ is straightforward using the Leibniz product rule for higher derivatives.

Now, for the last part of the proof, let us verify the claim regarding the asymptotic behavior of $|G(x;\lambda)|$ if $d = 2$. Based on the integral representation (2.3), Lebedev derived an asymptotic expansion for the Hankel function [23, Sec. 5.11, Eq. (5.11.3)]. For $\nu = (d/2) - 1 = 0$ and $z = k|x|$ this expansion reads

$$H_0^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \mathrm{e}^{\mathrm{i}(z-(1/4)\pi)}\Big(1 + O(|z|^{-1})\Big).$$

As $\mathrm{Im}(z) > 0$, we see that $\left|H_0^{(1)}(k|x|)\right| = O(|x|^{-1/2})$. Due to the simple structure of $G(x;\lambda)$ for $d = 2$, as shown in equation (2.4), the claim follows easily. □

In the derivation of the next estimates, we will use the following useful interior estimate for solutions to Poisson's equation which we state her for further use.

Lemma 2.2. *Let $r > 0$ and $x \in \mathbb{R}^d$, $d \geq 2$. If $w \in \mathrm{C}^k(\mathrm{B}(x,r)) \cap \mathrm{C}^0(\overline{\mathrm{B}(x,r)})$ is a solution to $\Delta w = f$ in $\mathrm{B}(x,r)$ for $f \in \mathrm{C}^{k-1}(\mathrm{B}(x,r))$, then,*

$$\left|\nabla^l w(x)\right| \leq C r^{-l} \sup_{\mathrm{B}(x,r)} |w| + C \max_{0\leq j\leq l-1} \sup_{\mathrm{B}(x,r)} r^{j-l+2}\left|\nabla^j f\right|, \quad l \leq k, \tag{2.11}$$

where $C > 0$ only depends on d and l.

Proof. If $l = 1$, then estimate (2.11) is a consequence of the *comparison principle* and a proof of this fact can be found in the book of Gilbarg and Trudinger [15, Sec. 3.4, Eq. (3.16)]. We will now use this estimate to inductively deduce the estimates for higher derivatives: Note that by translating from x to 0 and rescaling like

$$u_r(x) := u(rx) \quad \text{and} \quad f_r(x) := r^2 f(rx)$$

we may assume that $\Delta w = f$ in $\mathrm{B}(0,1)$ and that it suffices to prove

$$\left|\nabla^l w(0)\right| \leq C \sup_{\mathrm{B}(0,1)} |w| + C \max_{0\leq j\leq l-1} \sup_{\mathrm{B}(0,1)} \left|\nabla^j f\right| \tag{2.12}$$

for $l > 1$. By the Schwarz theorem we have that if w solves Poisson's equation with right-hand side f and w and f are sufficiently regular, then $\nabla^l w$ solves Poisson's equation with right-hand side $\nabla^l f$. We thus estimate inductively

$$\begin{aligned}
\left|\nabla^l w(0)\right| &\leq C_l \sup_{\mathrm{B}(0,(1/2)^{l-1})} \left|\nabla^{l-1} w\right| + C_l \sup_{\mathrm{B}(0,(1/2)^{l-1})} \left|\nabla^{l-1} f\right| \\
&\leq C_l \sup_{\mathrm{B}(0,(1/2)^{l-2})} \left|\nabla^{l-2} w\right| + C_l \left(\sup_{\mathrm{B}(0,1)} \left|\nabla^{l-2} f\right| + \sup_{\mathrm{B}(0,1)} \left|\nabla^{l-1} f\right| \right) \\
&\leq C_l \sup_{\mathrm{B}(0,1)} |w| + C_l \sum_{j=0}^{l-1} \sup_{B(0,1)} |\nabla^j f|
\end{aligned}$$

which readily yields the desired estimate. □

We will need the following asymptotic expansions for the function $z^\nu H_\nu^{(1)}(z)$ in $\mathbb{C} \setminus (-\infty, 0]$. The derivation of these asymptotic expansions is based on asymptotic expansions of the *Bessel functions of the first* and *the second kind* and can be found in Tolksdorf [39, Sec. 4.2]:

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{\mathrm{i}}{\pi} z^2 \log(z) + \omega z^2 + O(|z|^4 \left|\log(z)\right|) \quad \text{if } d = 4, \tag{2.13}$$

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{2^\nu \Gamma(\nu-1)}{4\pi\mathrm{i}} z^2 + \omega z^3 + O(|z|^4) \quad \text{if } d = 5, \tag{2.14}$$

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{2^\nu \Gamma(\nu-1)}{4\pi\mathrm{i}} z^2 + O(|z|^4 \left|\log(z)\right|) \quad \text{if } d = 6, \tag{2.15}$$

$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{2^\nu \Gamma(\nu-1)}{4\pi\mathrm{i}} z^2 + O(|z|^4) \quad \text{if } d \geq 7, \tag{2.16}$$

where $\omega \in \mathbb{C}$ is a constant that needs to be chosen depending on d, see [39, Eq. (4.19)] and [39, Eq. (4.20)] for $d = 4$ and $d = 5$, respectively.

The next lemma will be concerned with estimating the difference $G(x;\lambda) - G(x;0)$ (and derivatives of this difference) of the fundamental solution to the scalar Helmholtz equation and the fundamental solution to $-\Delta u = 0$ in $\mathbb{R}^d$ which is given by

$$G(x;0) := \begin{cases} -\frac{1}{2\pi}\log(|x|), & \text{for } d = 2, \\ c_d \, \frac{1}{|x|^{d-2}}, & \text{for } d > 2, \end{cases} \tag{2.17}$$

where inverse of the coefficient c_d is given as as a multiple of the surface measure of the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$:

$$c_d = \frac{1}{(d-2)\,\omega_d}, \quad \text{with} \quad \omega_d = \frac{2\,\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = |\mathbb{S}^{d-1}|. \tag{2.18}$$

By rearranging terms and using the functional equation of the Gamma function

$$\begin{aligned} \Gamma(z+1) &= z\,\Gamma(z), \quad z \in \mathbb{C}, \mathrm{Re}(z) > 0 \\ \Gamma(1) &= 1, \end{aligned} \tag{2.19}$$

we get

$$(d-2)\,\omega_d = 2\left(\frac{d}{2} - 1\right)\frac{2\,\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = 2\left(\frac{d}{2} - 1\right)\frac{2\,\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}-1)\,(\frac{d}{2}-1)} = \frac{4\,\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}-1)},$$

and thus, we will also sometimes use the equivalent definition

$$c_d := \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}. \tag{2.20}$$

Furthermore, the leading coefficient of the asymptotic expansions of the Hankel functions (2.13)-(2.16) for $d \geq 4$ will be denoted as

$$a_d := \frac{2^{\frac{d}{2}-1}\,\Gamma(\frac{d}{2}-1)}{\mathrm{i}\pi}. \tag{2.21}$$

Note that the series expansion of the Hankel function in the case $d = 3$, see (2.6), gives that the above definition of a_d also extends to $d = 3$ since we have

$$-\mathrm{i}\left(\frac{2}{\pi}\right)^{1/2} = \frac{2\,\Gamma(\frac{1}{2})}{\mathrm{i}\pi}.$$

The coefficients a_d and c_d are related in the following way:

$$c_d = \frac{\mathrm{i}}{4\,(2\pi)^{\frac{d}{2}-1}}\,a_d\,.$$

This allows us to write for $d \geq 3$

$$G(x;\lambda) - G(x;0) = \frac{\mathrm{i}}{4\,(2\pi)^{\frac{d}{2}-1}} \cdot \frac{1}{|x|^{d-2}}\left\{(k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|) - a_d\right\}. \tag{2.22}$$

The following lemma will help us to estimate the expression (2.22) together with its derivatives and their two dimensional counterparts.

Lemma 2.3. *Let $\lambda \in \Sigma_\theta$. Then*

$$\left| \nabla_x^l \Big\{ G(x;\lambda) - G(x;0) \Big\} \right| \le C\, |\lambda||x|^{4-d-l} \tag{2.23}$$

if $d \ge 5$ and $l \ge 0$, where $C > 0$ depends only on d, l and θ. If $d = 3$ or 4, estimate (2.23) *holds for $l \ge 1$ and if $d = 2$, the estimate holds for $l \ge 3$.*

Proof. (a) In this part, we will show that the desired estimate (2.23) holds if we assume that $|\lambda||x|^2 > (1/2)$. In this case, Lemma 2.1 gives

$$\left|\nabla_x^l \Big\{ G(x;\lambda) - G(x;0) \Big\}\right| \le C \left\{ \frac{e^{-c\sqrt{|\lambda|}|x|}}{|x|^{d-2+l}} + \frac{1}{|x|^{d-2+l}} \right\} \le C\, \frac{|\lambda|}{|x|^{d-4+l}}\,,$$

where $C > 0$ depends only on d, l and θ. Therefore, for the remaining proof we will suppose $|\lambda||x|^2 \le (1/2)$.

(b) In this step, we show that we can restrict ourselves to proving (2.23) in three cases: (1) $d \ge 5$ and $l = 0$; (2) $d = 3$ or 4 and $l = 1$; (3) $d = 2$ and $l = 3$.

Suppose (2.23) holds in case (1) and let $l \ge 1$. If we set $w(x) = G(x;\lambda) - G(x;0)$, we have $\Delta_x w = \lambda G(x;\lambda)$ in $\mathbb{R}^d \setminus \{0\}$. For $f = \lambda G(x;\lambda)$, estimate (2.11) now gives

$$\begin{aligned}
\left|\nabla^l w(x)\right| &\le C r^{-l} \sup_{B(x,r)} |w| + C \max_{0 \le j \le l-1} \sup_{B(x,r)} r^{j-l+2} \left|\nabla^j f\right| \\
&\le C r^{-l} \sup_{y \in B(x,r)} |\lambda||y|^{4-d} + C \sum_{j=0}^{l-1} \sup_{y \in B(x,r)} r^{j-l+2} |\lambda||y|^{2-d-j} \\
&= C r^{-l} |\lambda| \left| x - r\frac{x}{|x|} \right|^{4-d} + C \sum_{j=0}^{l-1} r^{j-l+2} |\lambda| \left| x - r\frac{x}{|x|} \right|^{2-d-j},
\end{aligned}$$

for all $0 < r < |x|$, where we used (2.23) with $l = 0$ for the first summand and (2.7) to estimate the second summand. We choose $r = \frac{|x|}{2}$ and receive

$$\begin{aligned}
\left|\nabla^l w(x)\right| &\le C\, |\lambda| |x|^{-l} |x|^{4-d} + C \sum_{j=0}^{l-1} |x|^{j-l+2} |\lambda||x|^{2-d-j} \\
&\le C\, |\lambda||x|^{4-d-l}.
\end{aligned}$$

The proof for case (2) is completely analogous if one sets

$$w(x) = \nabla_x \Big\{ G(x;\lambda) - G(x;0) \Big\} \quad \text{and} \quad f(x) = \lambda\, \nabla_x G(x;\lambda).$$

Also case (3) is proven in a similar fashion.

(c) In this step we prove (2.23) for $d \geq 5$ and $l = 0$. First, note that for the functions

$$g(x) \coloneqq (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|), \quad g(0) = a_d\,,$$
$$h(z) \coloneqq z^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(z), \quad h(0) = a_d\,,$$

the mean value theorem yields the estimate

$$|g(x) - g(0)| \leq |x| \sup_{y \in \mathrm{B}(0,|x|)} |\nabla g(y)| \leq |x||k| \sup_{y \in \mathrm{B}(0,|x|)} \left| \frac{\mathrm{d}}{\mathrm{d}z} h(k|y|) \right|.$$

Using representation (2.22), we estimate

$$\begin{aligned} |G(x;\lambda) - G(x;0)| &\leq C|x|^{2-d} \cdot |k||x| \max_{\substack{|z| \leq |k||x| \\ \mathrm{Im}(z)>0}} \left| \frac{\mathrm{d}}{\mathrm{d}z} \left\{ z^{\frac{d}{2}-1} H_{\frac{d}{2}-1}(z) \right\} \right| \\ &= C|x|^{2-d} \cdot |k||x| \max_{\substack{|z| \leq |k||x| \\ \mathrm{Im}(z)>0}} \left| z^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-2}(z) \right|, \end{aligned} \tag{2.24}$$

where for the last equality we used another useful relation that can be found in the book of Lebedev [23, Eq. (5.6.3)],

$$\frac{\mathrm{d}}{\mathrm{d}z} \left\{ z^{\nu} H^{(1)}_{\nu}(z) \right\} = z^{\nu} H^{(1)}_{\nu-1}(z). \tag{2.25}$$

Since the asymptotic expansions yield that $|z^{\nu} H^{(1)}_{\nu}(z)| \leq C_{\nu}$ for $\nu > 0$ and $|z| \leq 1$ with $\mathrm{Im}(z) > 0$, it follows from (2.24) that

$$|G(x;\lambda) - G(x;0)| \leq C|x|^{2-d} \cdot |k||x| \cdot |k||x| \max_{\substack{|z| \leq |k||x| \\ \mathrm{Im}(z)>0}} \left| z^{\frac{d}{2}-2} H^{(1)}_{\frac{d}{2}-2}(z) \right| \leq C\,|\lambda||x|^{4-d}.$$

(d) Now we consider the case $d = 4$ and $l = 1$. The asymptotic expansion (2.13) gives that

$$\left| \frac{\mathrm{d}}{\mathrm{d}z} \left\{ \frac{z H^{(1)}_1(z) - a_4}{z^2} \right\} \right| \leq C\,|z|^{-1} \tag{2.26}$$

for all $|z| \leq \frac{1}{2}$ with $\mathrm{Im}(z) > 0$. Since by identity (2.22) we have that

$$\frac{G(x;\lambda) - G(x;0)}{\lambda} = -\frac{C\left(z H^{(1)}_1(z) - a_4\right)}{z^2},$$

where $z = k|x|$. With (2.26) we conclude that

$$\left| \frac{\nabla_x \{G(x;\lambda) - G(x;0)\}}{\lambda} \right| \leq C\,|k| \left| \frac{\mathrm{d}}{\mathrm{d}z} \left\{ \frac{z H^{(1)}_1(z) - a_4}{z^2} \right\} \Big|_{z=k|x|} \right| \leq C\,|k||k|^{-1}|x|^{-1},$$

which after rearrangement of the involved terms gives the claim.

(e) Next, we consider thee case $d = 3$ and $l = 1$. We start with the compact definition of $G(x;\lambda)$, see (2.5), which makes this dimension stand out. From equation (2.20) and a well known fact of the Gamma function, $\Gamma(1/2) = \sqrt{\pi}$, we then derive the following identity:

$$G(x;\lambda) - G(x;0) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\pi|x|} - \frac{c_3}{|x|} = \frac{\mathrm{e}^{\mathrm{i}k|x|} - 1}{4\pi|x|}.$$

Now we calculate

$$\begin{aligned}\frac{\partial}{\partial x_j}\left\{\frac{\mathrm{e}^{\mathrm{i}k|x|} - 1}{|x|}\right\} = \frac{\partial}{\partial x_j}\left\{\frac{\mathrm{e}^{\mathrm{i}k|x|} - 1 - \mathrm{i}k|x|}{|x|}\right\} &= \frac{\partial}{\partial x_j}\left\{\sum_{n=2}^{\infty}\frac{(\mathrm{i}k|x|)^n}{n!}\cdot\frac{1}{|x|}\right\}\\ &= \sum_{n=2}^{\infty}\frac{(\mathrm{i}k)^n}{n!}(n-1)\cdot\frac{x_j}{|x|}|x|^{n-2}\end{aligned}$$

which in turn implies

$$\left|\frac{\partial}{\partial x_j}\left\{\frac{\mathrm{e}^{\mathrm{i}k|x|} - 1}{|x|}\right\}\right| \le |\lambda|\sum_{n=2}^{\infty}\frac{n-1}{n!}|k|^{n-2}|x|^{n-2} \le C\,|\lambda|$$

since $|\lambda||x| \le (1/2)$.

(f) For the last case $d = 2$ and $l = 3$, we will directly calculate the estimate using the asymptotic expansion of $H_0^{(1)}(z)$ with $z = k|x|$. The calculations are omitted from this chapter. Instead, they can be found in the appendix of this thesis, see A.1. □

Remark 2.4. In the situation of Lemma 2.3, one can show for $|\lambda||x|^2 \le (1/2)$ by considering the asymptotic expansions that

$$|G(x;\lambda) - G(x;0)| \le \begin{cases} C\sqrt{|\lambda|} & \text{if } d = 3,\\ C\,|\lambda|\left\{\left|\log(|\lambda||x|^2)\right| + 1\right\} & \text{if } d = 4.\end{cases}$$

Also using the asymptotic expansions, it can be shown that if $d = 2$, then

$$\left|\nabla_x^l\{G(x;\lambda) - G(x;0)\}\right| \le C\,|\lambda||x|^{2-l}\left\{\left|\log(|\lambda||x|^2)\right| + 1\right\},$$

for $l \in \{1,2\}$.

2.2 The Stokes Resolvent Problem

We will now analyze fundamental solutions to the *Stokes resolvent problem*

$$\begin{aligned} -\Delta u + \nabla \phi + \lambda u &= 0 \\ \operatorname{div} u &= 0 \end{aligned} \tag{2.27}$$

in $\mathbb{R}^d$ with $\lambda \in \Sigma_\theta$ with the goal to deduce helpful estimates for the following chapters. The fundamental solutions to the (scalar) Helmholtz equation and the Laplace equation will form the main ingredients for the following matrix of fundamental solutions to the Stokes resolvent problem with pole at the origin:

$$\Gamma_{\alpha\beta}(x;\lambda) = G(x;\lambda)\delta_{\alpha\beta} - \frac{1}{\lambda}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\Big\{G(x;\lambda) - G(x;0)\Big\}, \quad \alpha,\beta = 1,\dots,d. \tag{2.28}$$

As the matrix of fundamental solutions $\Gamma(x;\lambda) = (\Gamma_{\alpha\beta}(x;\lambda))_{d\times d}$ carries two arguments it cannot be confused with the Gamma function. Having formula (2.28) at sight, the following observations are obvious:

$$\Gamma_{\alpha\beta}(x;\lambda) = \Gamma_{\beta\alpha}(x;\lambda), \quad \overline{\Gamma_{\alpha\beta}(x;\lambda)} = \Gamma_{\alpha\beta}(x;\bar{\lambda}) \quad \text{and} \quad \Gamma_{\alpha\beta}(x;\lambda) = \Gamma_{\alpha\beta}(-x;\lambda).$$

For the pressure, we define the vector of fundamental solutions

$$\Phi_\beta(x) = -\frac{\partial}{\partial x_\beta}\Big\{G(x;0)\Big\} = \frac{x_\beta}{\omega_d |x|^d}, \quad \beta = 1,\dots,d. \tag{2.29}$$

We note that $\Phi_\beta(x) = -\Phi_\beta(-x)$.

Using the fact that $\Delta_x G(x;\lambda) = \lambda G(x;\lambda)$ in $\mathbb{R}^d \setminus \{0\}$, one can see that on $\mathbb{R}^d \setminus \{0\}$ and for all $1 \le \beta \le d$

$$\begin{aligned} (-\Delta_x + \lambda)\,\Gamma_{\alpha\beta}(x;\lambda) + \frac{\partial}{\partial x_\alpha}\Big\{\Phi_\beta(x)\Big\} &= 0, \\ \frac{\partial}{\partial x_\alpha}\Big\{\Gamma_{\alpha\beta}(x;\lambda)\Big\} &= 0, \quad \text{for } 1 \le \alpha \le d. \end{aligned} \tag{2.30}$$

Note that in the last equation the summation convention was used.

We now keep up to the spirit of this exhausting chapter by proving further estimates, this time for the fundamental solutions to the Stokes resolvent problem (2.27).

Theorem 2.5. *Let $\lambda \in \Sigma_\theta$. Then, for any $d \ge 3$ and $l \ge 0$,*

$$\left|\nabla_x^l \Gamma(x;\lambda)\right| \le \frac{C}{(1+|\lambda||x|^2)|x|^{d-2+l}} \tag{2.31}$$

where $C > 0$ depends only on d, l and θ. For $d = 2$ and $l \ge 1$, the same estimate holds.

Proof. Let $|\lambda||x|^2 > (1/2)$. Then, there exist constants $C_a, C_b, C_c > 0$ such that

$$\mathrm{e}^{-c\sqrt{|\lambda|}|x|}(1+|\lambda||x|^2) \le C_a\,,$$

$$1 \le \frac{C_b|\lambda||x|^2}{1+|\lambda||x|^2}\,,$$

$$\mathrm{e}^{-c\sqrt{|\lambda|}|x|} \le \frac{C_c|\lambda||x|^2}{1+|\lambda||x|^2}\,,$$

where $c > 0$ is the constant from Lemma 2.1. Using these estimates and Lemma 2.1 gives

$$\begin{aligned}
\left|\nabla_x^l \Gamma(x;\lambda)\right| &\le \left|\nabla_x^l G(x;\lambda)\right| + \frac{1}{|\lambda|}\cdot\left|\nabla_x^{l+2}G(x;\lambda)\right| + \frac{1}{|\lambda|}\cdot\left|\nabla_x^{l+2}G(x;0)\right| \\
&\le \frac{C_l\,\mathrm{e}^{-c\sqrt{|\lambda|}|x|}}{|x|^{d-2+l}} + \frac{1}{|\lambda|}\cdot\frac{C_{l+2}\,\mathrm{e}^{-c\sqrt{|\lambda|}|x|}}{|x|^2|x|^{d-2+l}} + \frac{1}{|\lambda|}\cdot\frac{C}{|x|^2|x|^{d-2+l}} \\
&\le \frac{C}{1+|\lambda||x|^2}\cdot\frac{1}{|x|^{d-2+l}}.
\end{aligned}$$

Now let $|\lambda||x|^2 \le (1/2)$. Then, by Lemma 2.1 and Lemma 2.3 we get

$$\begin{aligned}
\left|\nabla_x^l \Gamma(x;\lambda)\right| &\le \left|\nabla_x^l G(x;\lambda)\right| + \frac{1}{|\lambda|}\cdot\left|\,\nabla_x^{l+2}\Big\{G(x;\lambda) - G(x;0)\Big\}\,\right| \\
&\le \frac{C}{|x|^{d-2+l}} + \frac{1}{|\lambda|}\cdot C\,|\lambda||x|^{4-d-(l+2)} \\
&\le \frac{C}{|x|^{d-2+l}}\cdot\frac{(1+|\lambda||x|^2)}{(1+|\lambda||x|^2)} \\
&\le \frac{C}{(1+|\lambda||x|^2)|x|^{d-2+l}}
\end{aligned}$$

which gives the claim. □

If $\lambda = 0$, the Stokes resolvent problem becomes just the Stokes problem in $\mathbb{R}^d$

$$\begin{aligned}
-\Delta u + \nabla\phi &= 0\,, \\
\operatorname{div} u &= 0\,.
\end{aligned} \tag{2.32}$$

Whereas the fundamental solution for the pressure is maintained, the matrix of fundamental solutions to the Stokes problem in $\mathbb{R}^d$ with pole at the origin is given by $\Gamma(x;0) = (\Gamma_{\alpha\beta}(x;0))_{d\times d}$, where

$$\Gamma_{\alpha\beta}(x;0) := \frac{1}{2\omega_d}\left\{\frac{\delta_{\alpha\beta}}{(d-2)\,|x|^{d-2}} + \frac{x_\alpha x_\beta}{|x|^d}\right\} \tag{2.33}$$

if $d \ge 3$ and

$$\Gamma_{\alpha\beta}(x;0) := \frac{1}{2\omega_2}\left\{-\,\delta_{\alpha\beta}\log(|x|) + \frac{x_\alpha x_\beta}{|x|^2}\right\} \tag{2.34}$$

for $d = 2$. Note that the given fundamental solution for the case $d = 2$ differs from the one given by Mitrea and Wright [28, Sec. 4.2] by having summands with alternating signs. Considering the structure of the fundamental solution for $d \geq 3$, our choice seems more natural with regard to the structure of the fundamental solutions to the Laplace equation (2.17). The alternating sign is necessary for $\Gamma_{\alpha\beta}$ to be divergence free. The ordering of the signs is also crucial as we will see in later calculations.

One important technique in the following chapter will be to reduce problems formulated for $\Gamma(x;\lambda)$ to problems formulated in $\Gamma(x;0)$ perturbed by the difference $\Gamma(x;\lambda) - \Gamma(x;0)$. Under this aspect it seems reasonable to study estimates of the difference of fundamental solutions. To this end, it is helpful to rewrite parts of the fundamental solution. Using the fact that for $d \geq 5$ or $d = 3$, we have

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left\{\frac{1}{|x|^{d-4}}\right\} = -(d-4)\frac{\partial}{\partial x_\alpha}\left\{\frac{x_\beta}{|x|^{d-2}}\right\} = -(d-4)\frac{\delta_{\alpha\beta}}{|x|^{d-2}} + \frac{(d-4)(d-2)\,x_\alpha x_\beta}{|x|^d}.$$

This allows us to write

$$\frac{x_\alpha x_\beta}{|x|^d} = \frac{\delta_{\alpha\beta}}{(d-2)\,|x|^{d-2}} + \frac{1}{(d-4)(d-2)}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left\{\frac{1}{|x|^{d-4}}\right\},$$

which, considering definition (2.33), gives

$$\Gamma_{\alpha\beta}(x;0) = G(x;0)\delta_{\alpha\beta} + \frac{1}{2\omega_d\,(d-4)(d-2)}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left\{\frac{1}{|x|^{d-4}}\right\}. \tag{2.35}$$

A similar trick works for $d = 4$: Since $\omega_4 = 2\pi^2$, we have

$$\begin{aligned}\Gamma_{\alpha\beta}(x;0) &= \frac{1}{2\omega_4}\frac{1}{|x|^2}\delta_{\alpha\beta} - \frac{1}{8\pi^2}\left(\frac{\delta_{\alpha\beta}}{|x|^2} - \frac{2x_\alpha x_\beta}{|x|^4}\right)\\ &= G(x;0)\delta_{\alpha\beta} - \frac{1}{8\pi^2}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left\{\log(|x|)\right\}\\ &= G(x;0)\delta_{\alpha\beta} - \frac{1}{4\omega_4}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left\{\log(|x|)\right\}. \end{aligned}\tag{2.36}$$

In the case $d = 2$, we use

$$\frac{1}{8\pi}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left\{|x|^2\log(|x|)\right\} = \frac{\delta_{\alpha\beta}}{4\pi}\log(|x|) + \frac{1}{4\pi}\frac{x_\alpha x_\beta}{|x|^2} + \frac{\delta_{\alpha\beta}}{8\pi}$$

to find the identity

$$\Gamma_{\alpha\beta}(x;0) = G(x;0)\delta_{\alpha\beta} - \frac{\delta_{\alpha\beta}}{8\pi} - \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left\{|x|^2\log(|x|)\right\}. \tag{2.37}$$

This ends the preparatory step and brings us to the next theorem.

Theorem 2.6. *Let $\lambda \in \Sigma_\theta$. Suppose that $|\lambda||x|^2 \leq (1/2)$. Then,*

$$\Big|\nabla_x\Big\{\Gamma(x;\lambda)-\Gamma(x;0)\Big\}\Big| \leq \begin{cases} C\,|\lambda||x|^{3-d} & \text{if } d=3,\,5 \text{ or } d\geq 7\,, \\ C\,|\lambda||x|^{3-d}\,\big|\log(|\lambda||x|^2)\big| & \text{if } d=2,\,4 \text{ or } 6\,, \end{cases} \tag{2.38}$$

where $C>0$ depends only on d and θ.

Proof. We will split the proof in several parts. According to the preparatory step, for $d\geq 2$ and all $\alpha,\beta=1,\ldots,d$, the difference $\partial_\gamma\big\{\Gamma_{\alpha\beta}(x;\lambda)-\Gamma_{\alpha\beta}(x;0)\big\}, \gamma=1,\ldots,d$, is always of the form

$$\begin{aligned} &\frac{\partial}{\partial x_\gamma}\Big\{\Gamma_{\alpha\beta}(x;\lambda)-\Gamma_{\alpha\beta}(x;0)\Big\} \\ &\qquad = \frac{\partial}{\partial x_\gamma}\Big\{G(x;\lambda)-G(x;0)\Big\}\delta_{\alpha\beta} \;-\frac{1}{\lambda}\,\frac{\partial^3}{\partial x_\gamma\partial x_\alpha\partial x_\beta}\Big\{G(x;\lambda)-G(x;0)+\big[\,\ldots\,\big]\Big\}. \end{aligned}$$

But the first term on the right-hand side of the above expression is already under control thanks to Lemma 2.3 and Remark 2.4. It thus suffices to estimate the second term on the right-hand side.

We start by considering the cases $d=3$ and $d\geq 5$. Taking into account identity (2.35), we have for all $\alpha,\beta,\gamma=1,\ldots,d$:

$$G(x;\lambda)-G(x;0)+\big[\,\ldots\,\big] = \frac{1}{\lambda}\left\{G(x;\lambda)-G(x;0)+\frac{\lambda}{2\,\omega_d\,(d-4)\,(d-2)\,|x|^{d-4}}\right\}.$$

If $d=3$, a direct calculation will then yield the desired result: We start by noting that $\omega_3=4\pi$ gives

$$\begin{aligned} G(x;\lambda)-G(x;0)-\frac{\lambda}{2\omega_3|x|^{-1}} &= \frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\pi|x|}-\frac{1}{4\pi|x|}-\frac{(\mathrm{i}k)^2}{2\omega_3|x|^{-1}} \\ &= \frac{1}{4\pi|x|}\left(\mathrm{e}^{\mathrm{i}k|x|}-1-\frac{(\mathrm{i}k)^2|x|^2}{2}\right) \\ &= \frac{1}{4\pi|x|}\left(\mathrm{i}k|x|+\sum_{n=3}^{\infty}\frac{(\mathrm{i}k|x|)^n}{n!}\right) \\ &= \frac{1}{4\pi}\left(\mathrm{i}k+\sum_{n=3}^{\infty}\frac{(\mathrm{i}k)^n|x|^{n-1}}{n!}\right) =: I. \end{aligned}$$

Taking the first derivative of this expression we get

$$\frac{\partial}{\partial x_\beta}\big\{\,I\,\big\} = \frac{x_\beta}{4\pi}\sum_{n=3}^{\infty}\frac{(ik)^n(n-1)}{n!}|x|^{n-3}$$

and differentiating with respect to x_α yields

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\{I\} = \frac{\delta_{\alpha\beta}}{4\pi}\sum_{n=3}^{\infty}\frac{(ik)^n(n-1)}{n!}|x|^{n-3} + \frac{x_\beta x_\alpha}{4\pi}\sum_{n=4}^{\infty}\frac{(\mathrm{i}k)^n(n-1)(n-3)}{n!}|x|^{n-5}.$$

As we are interested in estimating the *gradient* of the difference of $\Gamma(x;\lambda)$ and $\Gamma(x;0)$, we have to consider one additional derivative. This leaves us with

$$\begin{aligned}\frac{\partial^3}{\partial x_\gamma \partial x_\alpha \partial x_\beta}\{I\} = {}& \frac{\delta_{\alpha\beta}x_\gamma + \delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta}{4\pi}\sum_{n=4}^{\infty}\frac{(\mathrm{i}k)^n(n-1)(n-3)}{n!}|x|^{n-5} \\ &+ \frac{x_\beta x_\alpha x_\gamma}{4\pi}\sum_{n=4}^{\infty}\frac{(\mathrm{i}k)^n(n-1)(n-3)(n-5)}{n!}|x|^{n-7}.\end{aligned}$$

The desired estimate follows now via

$$\begin{aligned}\left|\frac{1}{\lambda}\frac{\partial^3}{\partial x_\gamma \partial x_\alpha \partial x_\beta}\{I\}\right| &\le \frac{1}{|k|^2\pi}\sum_{n=4}^{\infty}\frac{|k|^n(n-1)(n-3)(1+|n-5|)}{n!}|x|^{n-4} \\ &\le \frac{1}{|k|^2\pi}|k|^4\sum_{n=4}^{\infty}\frac{(n-1)(n-3)(1+|n-5|)}{n!}|k|^{n-4}|x|^{n-4} \\ &\le C\,|k|^2.\end{aligned}$$

This gives the claim for $d = 3$. If $d \ge 5$, equation (2.22) gives

$$\begin{aligned}G(x;\lambda) - G(x;0) + {}&\frac{\lambda}{2\omega_d(d-4)(d-2)|x|^{d-4}} \\ &= \frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}}\frac{1}{|x|^{d-2}}\left\{z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2\right\},\end{aligned} \tag{2.39}$$

where $z = k|x|$, a_d was calculated in (2.21) and b_d is given by

$$b_d = -\frac{2\mathrm{i}\,(2\pi)^{\frac{d}{2}-1}}{\omega_d(d-4)(d-2)}.$$

Using relation (2.18) and the functional equation of the Gamma function (2.19) twice, we see that

$$\begin{aligned}b_d &= -\frac{2\mathrm{i}\,(2\pi)^{\frac{d}{2}-1}\,\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}(d-2)(d-4)} = \frac{2^{\frac{d}{2}-1}}{\pi\mathrm{i}(d-4)}\,\frac{\Gamma(\frac{d}{2})}{(d-2)} = \frac{2^{\frac{d}{2}-1}}{2\pi\mathrm{i}}\,\frac{\Gamma(\frac{d}{2}-1)}{(d-4)} \\ &= \frac{2^{\frac{d}{2}-1}}{4\pi\mathrm{i}}\,\frac{\Gamma(\frac{d}{2}-1)}{(\frac{d}{2}-1-1)} = \frac{2^{\frac{d}{2}-1}\,\Gamma(\frac{d}{2}-1-1)}{4\pi\mathrm{i}} = \frac{2^{\nu_d}\,\Gamma(\nu_d-1)}{4\pi\mathrm{i}}.\end{aligned}$$

This shows that for $d \ge 5$, b_d is the second coefficient of the asymptotic expansions (2.14)-(2.16), respectively. Now we split the proof for $d \ge 5$ into (1) $d \ge 7$, (2) $d = 6$ and (3)

$d = 5$. If $d \geq 7$, we use the asymptotic expansion (2.16) to estimate the part of (2.39) which involves the Hankel function as

$$\left| \frac{\mathrm{d}^l}{\mathrm{d}z^l} \Big\{ z^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2 \Big\} \right| \leq C|z|^{4-l} \tag{2.40}$$

for $0 \leq l \leq 3$ and $z \in \mathbb{C} \setminus (-\infty, 0]$. For better readability, we define the function

$$g(z) := z^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2$$

and consider the function $f(x) := g(k|x|)$ on $\mathbb{R}^d \setminus \{0\}$. The derivatives of f read

$$\begin{aligned}
\frac{\partial}{\partial x_\beta} f(x) &= \Big(\frac{\mathrm{d}}{\mathrm{d}z} g\Big)(k|x|) \cdot k\, \frac{x_\beta}{|x|}, \\
\frac{\partial^2}{\partial x_\alpha \partial x_\beta} f(x) &= \Big(\frac{\mathrm{d}^2}{\mathrm{d}z^2} g\Big)(k|x|) \cdot k^2\, \frac{x_\alpha x_\beta}{|x|^2} + \Big(\frac{\mathrm{d}}{\mathrm{d}z} g\Big)(k|x|) \cdot k \left(\frac{\delta_{\alpha\beta}}{|x|} - \frac{x_\beta x_\alpha}{|x|^3} \right), \\
\frac{\partial^3}{\partial x_\gamma \partial x_\alpha \partial x_\beta} f(x) &= \Big(\frac{\mathrm{d}^3}{\mathrm{d}z^3} g\Big)(k|x|) \cdot k^3\, \frac{x_\alpha x_\beta x_\gamma}{|x|^3} \\
&\quad + \Big(\frac{\mathrm{d}^2}{\mathrm{d}z^2} g\Big)(k|x|) \cdot k^2 \left(\frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^2} - \frac{x_\alpha x_\beta x_\gamma}{|x|^4} \right) \\
&\quad + \Big(\frac{\mathrm{d}}{\mathrm{d}z} g\Big)(k|x|) \cdot k \left(- \frac{\delta_{\alpha\beta} x_\gamma + \delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta}{|x|^3} + \frac{3 x_\alpha x_\beta x_\gamma}{|x|^5} \right).
\end{aligned}$$

If we now look for estimates on the absolute value of the derivatives, we see that by (2.40)

$$\left|\nabla^l f(x)\right| \leq C\, |k|^4 |x|^{4-l}, \quad 1 \leq l \leq 3,$$

where $C > 0$ only depends on l. We finally uncover the desired estimate via

$$\begin{aligned}
&\left| \frac{1}{\lambda} \nabla_x^3 \left\{ \frac{\mathrm{i}}{4\,(2\pi)^{\frac{d}{2}-1}} \frac{1}{|x|^{d-2}} \Big\{ z^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2 \Big\} \right\} \right| \\
&\leq C\, \frac{1}{|k|^2} \sum_{l=0}^{3} \left| \nabla^{3-l} \left\{ \frac{1}{|x|^{d-2}} \right\} \right| \left|\nabla^l f(x)\right| \\
&\leq C \sum_{l=0}^{3} |x|^{-d+2-3+l} |k|^2 |x|^{4-l} = C\, |\lambda| |x|^{3-d},
\end{aligned}$$

where $C > 0$ is a constant only depending on d.

If $d = 6$, the asymptotic expansion (2.15) gives us in analogy to (2.40) the estimate

$$\left| \frac{\mathrm{d}^l}{\mathrm{d}z^l} \Big\{ z^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2 \Big\} \right| \leq C\, |z|^{4-l} \left|\log(z)\right|, \tag{2.41}$$

for $0 \le l \le 3$ as the absolute values of z are bounded by assumption.Using as before the expressions for the derivatives of f, we estimate their absolute values as

$$\left|\nabla^l f(x)\right| \le C\,|k|^4|x|^{4-l}\,\left|\log(|\lambda||x|^2)\right|,$$

which, by a calculation analogous to the case $d \ge 7$, yields the claim.

For $d = 5$, we differentiate (2.39) twice and use relation (2.35) for the fundamental solution of the Stokes problem to write

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left\{G(x;\lambda) - G(x;0) + \frac{\lambda}{6\,\omega_5\,|x|^{d-4}}\right\}$$
$$= \frac{\partial^2}{\partial x_\alpha x_\beta}\left\{\frac{\mathrm{i}}{4\,(2\pi)^{\frac{3}{2}}}\cdot\frac{1}{|x|^3}\left\{z^{\frac{3}{2}}H^{(1)}_{\frac{3}{2}}(z) - a_5 - b_5 z^2 - w z^3\right\}\right\},$$

where $w \in \mathbb{C}$ can be an arbitrary constant if we set $z = k|x|$. Now, for the appropriate choice of $w \in \mathbb{C}$ the asymptotic expansion (2.14) gives the same estimate as (2.40) which, like for $d \ge 7$, proves the claim for $d = 5$.

In the case $d = 4$, we use the respective relation for the fundamental solution (2.36) in order to simplify the difference

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left\{G(x;\lambda) - G(x;0) - \frac{\lambda\log(|x|)}{4\,\omega_4}\right\}$$
$$= \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left\{\frac{\mathrm{i}}{8\,\pi\,|x|^2}\left\{zH^{(1)}_1(z) - a_4 - wz^2 - b_4 z^2\log(z)\right\}\right\},$$

where $z = k|x|$, $b_4 = (\mathrm{i}/\pi)$ and $w \in \mathbb{C}$ is an arbitrary constant. Using the asymptotic expansion (2.13) and the appropriate constant $w \in \mathbb{C}$, we get the estimate

$$\left|\frac{\mathrm{d}^l}{\mathrm{d}z^l}\left\{zH^{(1)}_1(z) - a_4 - wz^2 - b_4 z^2\log(z)\right\}\right| \le C|z|^{4-l}\left|\log(z)\right|.$$

The estimate has the same right-hand side as (2.41) and the proof can be carried out just as in the previous cases.

For $d = 2$, the claimed estimate follows from a direct calculation which is postponed until appendix A.2. □

We can now use the assumption $|\lambda||x|^2 \le (1/2)$ to unify the structure of the estimates from Theorem 2.6.

Corollary 2.7. *Let $\lambda \in \Sigma_\theta$. Suppose that $|\lambda||x|^2 \le (1/2)$. Then, for all $d \ge 2$*

$$\left|\nabla_x\left\{\Gamma(x;\lambda) - \Gamma(x;0)\right\}\right| \le C\sqrt{|\lambda|}|x|^{2-d},$$

where C depends only on d and θ.

Proof. We just extend the estimates given in Theorem 2.6. Let $d \geq 7$ or $d = 5$. Since $\sqrt{|\lambda|} \leq C|x|^{-1}$, we have

$$C|\lambda||x|^{3-d} \leq C\sqrt{|\lambda|}|x|^{2-d}.$$

For $d = 2, 4, 6$, we have

$$|\lambda||x|^{3-d}\left|\log(|\lambda||x|^2)\right| = C\sqrt{|\lambda|}|x|^{2-d} \cdot \sqrt{|\lambda|}|x|\left|\log(|\lambda||x|^2)\right| \leq C\sqrt{|\lambda|}|x|^{2-d},$$

since $\sqrt{|\lambda|}|x|\left|\log(|\lambda||x|^2)\right|$ is bounded for $|\lambda||x|^2 \leq (1/2)$. $\square$

Chapter 3

Single and Double Layer Potentials

In this chapter, we will deal with *single* and *double layer potentials*. Both will serve as "representation formulas" for solutions to the Stokes resolvent problem. We will study their properties as they will serve as the crucial ingredient to solving the L^2 Dirichlet problem associated to the Stokes resolvent problem on bounded Lipschitz domains $\Omega \subseteq \mathbb{R}^d$: For $\lambda \in \mathbb{C} \setminus (-\infty, 0)$ and

$$g \in \mathrm{L}^2_n(\partial\Omega) := \left\{ g \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d) \colon \int_{\partial\Omega} g \cdot n \, \mathrm{d}\sigma = 0 \right\}$$

we are looking for smooth functions u and ϕ that satisfy

$$(\mathrm{Dir}_\lambda) \begin{cases} -\Delta u + \nabla \phi + \lambda u = 0 & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = g & \text{nontangentially on } \partial\Omega, \\ (u)^* \in \mathrm{L}^2(\partial\Omega). \end{cases}$$

In this chapter we will thus always assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$ with $d \geq 2$ and $1 < p < \infty$. We will also tacitly use the summation convention.

We note that due to the new two dimensional estimates on fundamental solutions in Chapter 2, namely the continuation of Theorem 2.5and Theorem 2.6 for the case $d = 2$, we could extend all results from Chapter 3 of Shen's seminal paper [32] that are relevant to the analysis of the L^2 Dirichlet problem in a straightforward way.

Let $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi/2)$. Furthermore, let $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$. The single layer potential $u = \mathcal{S}_\lambda(f)$ is defined by

$$(\mathcal{S}_\lambda(f))_j(x) := \int_{\partial\Omega} \Gamma_{jk}(x - y; \lambda) f_k(y) \, \mathrm{d}\sigma(y), \tag{3.1}$$

where Γ_{jk} is the fundamental solution to the Stokes resolvent problem given by (2.28). For the pressure, respectively, we define the single layer potential $\phi = \mathcal{S}_\Phi(f)$ by

$$\mathcal{S}_\Phi(f)(x) := \int_{\partial\Omega} \Phi_k(x-y) f_k(y)\, \mathrm{d}\sigma(y), \tag{3.2}$$

where Φ_k is given by (2.29). The pair (u, ϕ) defines a solution to the Stokes resolvent problem (2.27) in $\mathbb{R}^d \setminus \partial\Omega$ by the properties of the fundamental solution together with the dominated convergence theorem..

We define two further integral operators that map to functions living on $\partial\Omega$:

$$T_\lambda^*(f)(q) = \sup_{t>0} \left| \int_{\substack{y\in\partial\Omega \\ |y-q|>t}} \nabla_x \Gamma(q-y;\lambda) f(y)\, \mathrm{d}\sigma(y) \right| \tag{3.3}$$

$$T_\lambda(f)(q) = \mathrm{p.\,v.} \int_{\partial\Omega} \nabla_x \Gamma(q-y;\lambda) f(y)\, \mathrm{d}\sigma(y) \tag{3.4}$$

for $q \in \partial\Omega$ which will be used to prove boundedness of maximal operators related to u and its gradient.

The following lemma will be a good companion for the forthcoming calculation of estimates.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain with corresponding numbers r_0 and M. Then, there exist $c, C > 0$ depending only on d and M such that*

$$cr^{d-1} \leq \sigma(\mathrm{B}(q,r) \cap \partial\Omega) \leq Cr^{d-1}$$

for all $r > 0$ and $q \in \partial\Omega$. Furthermore, there exist constants $\tilde{c}, \tilde{C} > 0$, depending only on d and the Lipschitz character of Ω, such that

$$\tilde{c} r_0^{d-1} \leq \sigma(\partial\Omega) \leq C r_0^{d-1}.$$

Another cornerstone in the theory of the single and double layer potentials is the following lemma, see Tolksdorf [39, Lem. 4.3.2], as it will allow us to bring into play the estimates from Section 2.2.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain with corresponding numbers r_0 and M. Let $x \in \mathbb{R}^d$, $0 < \varepsilon \leq (r_0/4)$, and $l \in \mathbb{N}_0$ with $l < d-1$. Then there exists a constant $C > 0$ depending only on d, l and M such that*

$$\int_{\partial\Omega \cap \mathrm{B}(x,\varepsilon)} \frac{1}{|x-y|^l}\, \mathrm{d}\sigma(y) \leq C\varepsilon^{d-l-1}.$$

We are now in the position to prove our first lemma on the way to establish the single layer potential as a benevolent operator for tackling boundary value problems on bounded Lipschitz domains. The lemma deals with mapping properties of the aforementioned integral operators T_λ and T_λ^*. The main idea will be to deduce pointwise estimates that bound the operator T_λ^* by the *Hardy-Littlewood maximal operator* $M_{\partial\Omega}$ which is defined for functions $f \in \mathrm{L}^1_{\mathrm{loc}}(\partial\Omega)$ via

$$M_{\partial\Omega}(f)(q) \coloneqq \sup_{\varepsilon>0} \frac{1}{\sigma(\partial\Omega \cap \mathrm{B}(q,\varepsilon))} \int_{\partial\Omega\cap\mathrm{B}(q,\varepsilon)} |f(y)| \, \mathrm{d}\sigma(y), \quad q \in \partial\Omega.$$

Lemma 3.3. *Let* $1 < p < \infty$ *and* $T_\lambda(f), T_\lambda^*(f)$ *be defined by* (3.3) *and* (3.4). *Then* $T_\lambda(f)(P)$ *exists for almost every* $P \in \partial\Omega$ *and*

$$\|T_\lambda(f)\|_{\mathrm{L}^p(\partial\Omega;\,\mathbb{C}^{d\times d})} \leq \|T_\lambda^*(f)\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p \, \|f\|_{\mathrm{L}^p(\partial\Omega;\,\mathbb{C}^d)}, \tag{3.5}$$

where C_p *depends only on* d, θ, p, *and the Lipschitz character of* Ω.

Proof. If $\lambda = 0$, the lemma is a consequence of the seminal result of Coifman, McIntosh and Meyer [3, Thm.9]. One idea of the proof in the case $\lambda \in \Sigma_\theta$ will thus be to nourish from this result and to consider the difference $\Gamma(x-y;\lambda) - \Gamma(x-y;0)$.

We start with the second inequality of (3.5). To this end, let $t > 0$ and additionally assume that $t^2|\lambda| > (1/2)$. In this case, Theorem 2.5 gives us the estimate

$$\left| \int_{|y-q|>t} \nabla_x \Gamma(q-y;\lambda) f(y) \, \mathrm{d}\sigma(y) \right| \leq C \int_{|q-y|>t} \frac{|f(y)|}{|\lambda||q-y|^{d+1}} \, \mathrm{d}\sigma(y),$$

where C depends on d and θ. Choose now $N \in \mathbb{N}$ such that $2^N t \leq \mathrm{diam}(\Omega) < 2^{N+1} t$. We now exhaust the domain of integration by suitable annuli and use the inner radii to simplify the integrand and the outer radii to amplify the domain of integration:

$$\begin{aligned} &\sum_{k=0}^{N} \int_{2^k t < |q-y| < 2^{k+1} t} \frac{1}{|\lambda||q-y|^{d+1}} |f(y)| \, \mathrm{d}\sigma(y) \\ &\qquad \leq \sum_{k=0}^{N} \int_{2^k t < |q-y| < 2^{k+1} t} \frac{1}{|\lambda| 2^{k(d+1)} t^{d+1}} |f(y)| \, \mathrm{d}\sigma(y) \\ &\qquad \leq \frac{1}{|\lambda| t^2} \frac{1}{2^{1-d}} \sum_{k=0}^{N} \frac{1}{2^{2k}} \frac{1}{(2^{k+1}t)^{d-1}} \int_{\mathrm{B}(q,\,2^{k+1}t)\cap\partial\Omega} |f(y)| \, \mathrm{d}\sigma(y). \end{aligned} \tag{3.6}$$

Note that due to Lemma 3.1 we have that

$$\frac{1}{(2^{k+1}t)^{d-1}} \int_{\mathrm{B}(q,\,2^{k+1}t)\cap\partial\Omega} |f(y)| \, \mathrm{d}\sigma(y) \leq C M_{\partial\Omega}(f)(q), \quad k = 0, \dots, N, \tag{3.7}$$

with a constant $C > 0$ that depends on d and the Lipschitz character of Ω. Now, we glue together (3.6) and (3.7), take $N \to \infty$ noting the geometric series and get the estimate

$$\left| \int_{|y-q|>t} \nabla_x \Gamma(q-y;\lambda) f(y)\, \mathrm{d}\sigma(y) \right| \leq C M_{\partial\Omega}(f)(q). \tag{3.8}$$

Now let $t^2|\lambda| \leq (1/2)$. We then split the integral as follows:

$$\begin{aligned} \left| \int_{|q-y|>t} \nabla_x \Gamma(q-y;\lambda) f(y)\, \mathrm{d}\sigma(y) \right| &\leq \left| \int_{|q-y|\geq (2|\lambda|)^{-1/2}} \nabla_x \Gamma(q-y;\lambda) f(y)\, \mathrm{d}\sigma(y) \right| \\ &\quad + \left| \int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x \Gamma(q-y;\lambda) f(y)\, \mathrm{d}\sigma(y) \right|. \end{aligned}$$

For the first summand, note that estimate (3.8) holds for all $t > 0$ and thus in particular for $t = (2|\lambda|)^{-1/2}$. For the second term, we add a special zero and use the triangle inequality to estimate

$$\begin{aligned} &\left| \int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x \Gamma(q-y;\lambda) f(y)\, \mathrm{d}\sigma(y) \right| \\ &\qquad \leq \int_{t<|q-y|<(2|\lambda|)^{-1/2}} \left| \nabla_x \Gamma(q-y;\lambda) - \nabla_x \Gamma(q-y;0) \right| |f(y)|\, \mathrm{d}\sigma(y) \\ &\qquad\quad + \left| \int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x \Gamma(q-y;0) f(y)\, \mathrm{d}\sigma(y) \right|. \end{aligned}$$

We don't need to worry about the second summand here since the corresponding estimate is already covered by the $\lambda = 0$ case:

$$\begin{aligned} &\left| \int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x \Gamma(q-y;0)\, f(y)\, \mathrm{d}\sigma \right| \\ &\qquad \leq \left| \int_{|q-y|>t} \nabla_x \Gamma(q-y;0)\, f(y)\, \mathrm{d}\sigma \right| \leq T_0^*(f)(q). \end{aligned} \tag{3.9}$$

For the first summand we make use of Theorem 2.6 and more precisely of Corollary 2.7 which unifies all estimates: We start by estimating

$$\begin{aligned} &\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \left| \nabla_x \Gamma(q-y;\lambda) - \nabla_x \Gamma(q-y;0) \right| |f(y)|\, \mathrm{d}\sigma(y) \\ &\qquad \leq C \int_{t<|q-y|<(2|\lambda|)^{-1/2}} \sqrt{|\lambda|}\, |q-y|^{2-d}\, |f(y)|\, \mathrm{d}\sigma(y), \end{aligned} \tag{3.10}$$

where C depends on d and θ. Now we choose $N \in \mathbb{N}$ such that

$$2^{N+1} t > (2|\lambda|)^{-1/2} \geq 2^N t \tag{3.11}$$

holds. Once again we integrate over annuli, use the inner radii to loose the term $|q-y|^{2-d}$ and use the outer radii to expand the domain of integration to balls with this radius:

$$\begin{aligned}
&\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \frac{1}{|q-y|^{d-2}} |f(y)| \, \mathrm{d}\sigma(y) \\
&\qquad \leq \sum_{k=0}^{N} \int_{2^k t \leq |q-y| < 2^{k+1} t} \frac{1}{|q-y|^{d-2}} |f(y)| \, \mathrm{d}\sigma(y) \\
&\qquad \leq \sum_{k=0}^{N} \frac{1}{(2^k t)^{d-2}} \int_{\mathrm{B}(q,\, 2^{k+1} t) \cap \partial\Omega} |f(y)| \, \mathrm{d}\sigma(y) \\
&\qquad = 2^{d-1} \sum_{k=0}^{N} 2^k t \, \frac{1}{(2^{k+1} t)^{d-1}} \int_{\mathrm{B}(q,\, 2^{k+1} t) \cap \partial\Omega} |f(y)| \, \mathrm{d}\sigma(y).
\end{aligned}$$

As before we use Lemma 3.1 to bring the Hardy-Littlewood maximal operator into the game like for inequality (3.7). This time we cannot take $N \longrightarrow \infty$ as the resulting geometric series wouldn't converge. But N was chosen wisely, see (3.11), and thus

$$\sum_{k=0}^{N} 2^k t \leq 2^{N+1} t \leq 2^{1/2} |\lambda|^{-1/2}$$

which yields the inequality

$$\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \frac{1}{|q-y|^{d-2}} |f(y)| \, \mathrm{d}\sigma \leq C |\lambda|^{-1/2} M_{\partial\Omega}(f)(q),$$

where $C > 0$ depends on d and the Lipschitz character of Ω. Taking into account the foregoing calculations together with estimate (3.10) and (3.9) we derive

$$\left| \int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x \Gamma(q-y;\lambda) f(y) \, \mathrm{d}\sigma \right| \leq C \left\{ T_0^*(f)(q) + M_{\partial\Omega}(f)(q) \right\}, \tag{3.12}$$

with $C > 0$ depending only on d, θ and the Lipschitz character of Ω.

It is now the time to take the supremum over all $t > 0$, and, considering estimates (3.8) and (3.12), we finally see that

$$T_\lambda^*(f)(q) \leq C \left\{ T_0^*(f)(q) + M_{\partial\Omega}(f)(q) \right\},$$

for all $q \in \partial\Omega$. Once again using the result for $\lambda = 0$ and the L^p boundedness of the Hardy-Littlewood maximal operator, we conclude the first part of the claimed inequality

$$\|T_\lambda^*(f)\|_{\mathrm{L}^p(\partial\Omega)} \leq C \, \|f\|_{\mathrm{L}^p(\partial\Omega;\, \mathbb{C}^d)}.$$

To conclude the left inequality in (3.5), we want to use a standard result from argument harmonic analysis. To this end, we define the operators

$$T_\lambda^{(t)}(f)(q) := \int_{\substack{y\in\partial\Omega \\ |y-q|>t}} \nabla_x \Gamma(q-y;\lambda) f(y) \, \mathrm{d}\sigma(y), \quad t > 0.$$

Suppose we can show that

$$T_\lambda(f)(q) = \lim_{t\to 0} T_\lambda^{(t)}(f)(q) \tag{3.13}$$

exists for almost every $q \in \partial\Omega$ and all $f \in \mathrm{C}(\partial\Omega; \mathbb{C}^d)$. Now, note that $\mathrm{C}(\partial\Omega; \mathbb{C}^d)$ is dense in $\mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ and that $T_\lambda^*(f)$ is bounded on $\mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ as we showed earlier. Then, Grafakos [17, Thm. 2.1.14] gives that T_λ is bounded from $\mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ to $\mathrm{L}^p(\partial\Omega; \mathbb{C}^{d\times d})$.

In order to prove the existence of the pointwise limit (3.13), we split the operator T_λ as follows:

$$T_\lambda(f)(q) = T_0(f)(q) + \lim_{t\to 0} \int_{\substack{y\in\partial\Omega \\ |y-q|>t}} \nabla_x\big\{\Gamma(q-y;\lambda) - \Gamma(q-y;0)\big\} f(y)\,\mathrm{d}\sigma(y).$$

The right summand is well defined for $f \in \mathrm{C}(\partial\Omega; \mathbb{C}^d)$, once we prove integrability of the integral kernel $\big|\nabla_x\big\{\Gamma(q-y;\lambda) - \Gamma(q-y;0)\big\}\big|$ on $\partial\Omega$. To this end, we first note that it suffices to consider the integral

$$\int_{|q-y|\le\varepsilon} \big|\nabla_x\big\{\Gamma(q-y;\lambda) - \Gamma(q-y;0)\big\}\big|\,\mathrm{d}\sigma(y),$$

for $\varepsilon \le \min(2|\lambda|^{-1/2}, r_0/4)$ as the integrand is bounded on $\partial\Omega \setminus \mathrm{B}(q,\varepsilon)$ and the domain of integration $\partial\Omega$ is bounded. Now, Corollary 2.7 and Lemma 3.2 give that the integrand can be estimated by

$$C\int_{|q-y|\le\varepsilon} \sqrt{|\lambda|}|q-y|^{2-d}\,\mathrm{d}\sigma(y) \le C\sqrt{|\lambda|}\varepsilon \le C,$$

where C is a constant depending on d, θ and the Lipschitz character of Ω. Based on the preceding calculation we conclude that for all $f \in \mathrm{C}(\partial\Omega; \mathbb{C}^d)$ the limit $T_\lambda(f)(q)$ exists whenever $T_0(f)(q)$ exists. $T_0(f)(q)$ exists for almost every $q \in \partial\Omega$ because of Mitrea and Wright [28]. As furthermore, $T_\lambda^*(f)(q)$ is bounded on $\mathrm{L}^p(\partial\Omega)$ we may now apply Theorem 2.1.14 from Grafakos [17] to conclude that $T_\lambda(f)(q)$ exists now for all $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ and almost every $q \in \partial\Omega$. The desired L^p estimate for $T_\lambda(f)$ now follows from the observation that $|T_\lambda(f)(q)| \le T_\lambda^*(f)(q)$ for almost every $q \in \partial\Omega$. □

For further use, we state a very useful lemma which can be considered a *Young*-type inequality for L^p spaces on boundaries of Lipschitz domains. A proof can be found in Tolksdorf [39, Prop. 1.1.4].

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^d$, μ a σ-finite measure on Ω, and $1 \le p < \infty$. Let $g\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a function such that the function $\Omega\times\Omega \ni (x,y) \mapsto g(x-y)$ is measurable with respect to the product measure $\mu\times\mu$ and such that*

$$A + B := \sup_{x\in\Omega} \|g(x-\cdot\,)\|_{\mathrm{L}^1(\Omega,\mu)} + \sup_{y\in\Omega} \|g(\,\cdot - y)\|_{\mathrm{L}^1(\Omega,\mu)} < \infty.$$

If $f \in \mathrm{L}^p(\Omega, \mu)$*, then* $x \mapsto \int_\Omega g(x-y)f(y)\,\mathrm{d}\mu(y) \in \mathrm{L}^p(\Omega, \mu)$ *and*

$$\left\| \int_\Omega g(\cdot - y)f(y)\,\mathrm{d}\mu(y) \right\|_{\mathrm{L}^p(\Omega,\,\mu)} \leq A^{1-1/p}B^{1/p}\|f\|_{\mathrm{L}^p(\Omega,\,\mu)}.$$

For us, Lemma 3.4 will be applied often to integral kernels g that result from an application of the theorems in Chapter 2. The following lemma shows that these integral kernels fulfill the requirements from Lemma 3.4.

Lemma 3.5. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain. Then,*

$$\sup_{q \in \partial\Omega} \int_{\partial\Omega} \frac{1}{|q-y|^{d-2}} \leq Cr_0,$$

where C *is a constant depending only on* d *and the Lipschitz character of* Ω*.*

Proof. Let r_0 be the radius from the definition of Lipschitz cylinders and $q \in \partial\Omega$. Splitting the domain of integration and applying Lemma 3.2, we get

$$\begin{aligned}
&\int_{\partial\Omega} \frac{1}{|q-y|^{d-2}}\,\mathrm{d}\sigma(y) \\
&\quad \leq \int_{\partial\Omega\cap \mathrm{B}(q,\,r_0/4)} \frac{1}{|q-y|^{d-2}}\,\mathrm{d}\sigma(y) + \int_{\partial\Omega\setminus \mathrm{B}(q,\,r_0/4)} \frac{1}{|q-y|^{d-2}}\,\mathrm{d}\sigma(y) \\
&\quad \leq Cr_0 + r_0^{2-d}4^{d-2}\sigma(\partial\Omega) \leq C\,(r_0 + r_0^{2-d}r_0^{d-1}),
\end{aligned}$$

where $C > 0$ depends only on d and the Lipschitz character of Ω. This proves the claim. □

Now we prove the boundedness of certain nontangential maximal operators.

Lemma 3.6. *Let* $1 < p < \infty$ *and* (u, ϕ) *be given by* (3.1) *and* (3.2)*. Then,*

$$\|(\nabla u)^*\|_{\mathrm{L}^p(\partial\Omega)} + \|(\phi)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p\|f\|_{\mathrm{L}^p(\partial\Omega;\,\mathbb{C}^d)}, \tag{3.14}$$

where $C_p > 0$ *depends only on* d*,* θ*,* p *and the Lipschitz character of* Ω*. Let furthermore* $d \geq 3$*. Then,*

$$\|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p\|f\|_{\mathrm{L}^p(\partial\Omega;\,\mathbb{C}^d)}, \tag{3.15}$$

where $C_p > 0$ *depends only on* d*,* θ*,* p *and the Lipschitz character of* Ω*.*

Proof. A proof of the estimate $\|(\phi)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p\|f\|_{\mathrm{L}^p(\partial\Omega;\,\mathbb{C}^d)}$ can be found in Verchota's dissertation [40, Lem. 1.3]. The proof for $\|(\nabla u)^*\|_{\mathrm{L}^p(\partial\Omega)}$ works in the same way. We will provide a proof for the sake of completeness. To imitate the proof of Verchota, we will work with the corresponding type of cones. Therefore, the results for ∇u and ϕ will at first only be established for the type of maximal operators defined by Verchota. The transferability to Shen's maximal operators is given by Tolksdorf [39, p. 90ff.] as the solution (u, ϕ) has a representation as a single layer potential.

Let $q \in \partial\Omega$, $x \in \Gamma_{\mathrm{V}}(q)$, and set $t = |x - q|$. Then,

$$
\begin{aligned}
&|(\nabla u_j)(x)| \\
&\quad = \left| \int_{\partial\Omega} \nabla_x \Gamma_{jk}(x-y;\lambda) f_k(y)\,\mathrm{d}\sigma(y) \right| \\
&\quad \leq \left| \int_{|y-q|>t} \nabla_x \Gamma_{jk}(x-y;\lambda) f_k(y)\,\mathrm{d}\sigma(y) \right| + \left| \int_{|y-q|\leq t} \nabla_x \Gamma_{jk}(x-y;\lambda) f_k(y)\,\mathrm{d}\sigma(y) \right| \\
&\quad =: I_1 + I_2.
\end{aligned}
$$

We will now estimate I_1 and I_2 separately. Note that in Verchota cones $\Gamma_{\mathrm{V}}(q)$ it holds that for all $s \in \partial\Omega$ we have $|x - s| \geq C\,|x - q|$, where $C > 0$ is a constant only depending on d and the Lipschitz character of Ω, see inequality (1.1). By Theorem 2.5 we know that

$$
\begin{aligned}
I_2 &\leq C \int_{|y-q|\leq t} \frac{1}{|x-y|^{d-1}} |f(y)|\,\mathrm{d}\sigma(y) \\
&\leq \frac{C}{t^{d-1}} \int_{|y-q|\leq t} |f(y)|\,\mathrm{d}\sigma(y) \leq C\, M_{\partial\Omega}(f)(q),
\end{aligned}
$$

where we used also Lemma 3.1 to bring the Hardy-Littlewood maximal operator into play. Here, $C > 0$ depends on d, θ and the Lipschitz character of Ω. For I_1, we calculate

$$
\begin{aligned}
&\left| \int_{|x-q|>t} \nabla_x \Gamma_{jk}(x-y;\lambda) f_k(y) - \nabla_x \Gamma_{jk}(q-y;\lambda) f_k(y) + \nabla_x \Gamma_{jk}(q-y;\lambda) f_k(y)\,\mathrm{d}\sigma(y) \right| \\
&\quad \leq \left| \int_{|y-q|>t} \nabla_x \Big\{ \Gamma_{jk}(x-y;\lambda) - \Gamma_{jk}(q-y;\lambda) \Big\} f_k(y)\,\mathrm{d}\sigma(y) \right| \\
&\qquad + \left| \int_{|y-q|>t} \nabla_x \Gamma_{jk}(q-y;\lambda) f_k(y)\,\mathrm{d}\sigma(y) \right|.
\end{aligned}
$$

The second summand can directly be estimated by $T^*_\lambda(f)(q)$. For the first one we apply the mean value theorem and use Theorem 2.5 to derive the following chain of estimates:

$$
\begin{aligned}
&\int_{|y-q|>t} \left| \nabla_x \Gamma_{jk}(x-y;\lambda) - \nabla_x \Gamma_{jk}(q-y;\lambda) \right| |f(y)|\,\mathrm{d}\sigma(y) \\
&\qquad \leq \int_{|y-q|>t} \left| \nabla^2 \Gamma_{jk}(s-y;\lambda) \right| |x-q|\,|f(y)|\,\mathrm{d}\sigma(y)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C \int_{|y-q|>t} \frac{t}{|s-y|^d} |f(y)| \, \mathrm{d}\sigma(y) \\
&\leq C \int_{|y-q|>t} \frac{t}{|y-q|^d} |f(y)| \, \mathrm{d}\sigma(y) \\
&\leq C \int_{\partial\Omega} \frac{t}{(t+|y-q|)^d} |f(y)| \, \mathrm{d}\sigma(y),
\end{aligned}
$$

where s is an element on the line connecting x and q and we used the property of Verchota cones that $|s-y| \geq C\,|y-q|$, see inequality (1.2). Note that Verchota cones are convex. As in Verchota [40, Lem. 1.3], the integral may now be bounded by the Hardy-Littlewood maximal operator due to an application of a suitable result from Grafakos [17, Thm. 2.1.10] as the kernel $t(t+|y-q|)^{-d}$ is uniformly integrable and radially decreasing.Summing up we have shown that

$$
|(\nabla u)(x)| \leq C \Big\{ M_{\partial\Omega} f(P) + T_\lambda^*(f)(P) \Big\},
$$

where $C > 0$ only depends on d, θ and the Lipschitz character of Ω. We thus may take the supremum over all $x \in \Gamma_{\mathrm{V}}(q)$ and conclude the desired estimate by the well known mapping properties of the Hardy-Littlewood maximal operator and the respective results from Lemma 3.3.

We will now work on the proof of the estimate for $(u)^*$ for $d \geq 3$. In order to derive L^p bounds on this maximal operator, we will work directly with the Definition of the single layer potential (3.1). For $q \in \partial\Omega$, estimate (2.31) together with the estimate for Shen cones (1.3) gives that for all $x \in \Gamma(q)$

$$
|u(x)| \leq C \int_{\partial\Omega} \frac{1}{|x-y|^{d-2}} |f(y)| \, \mathrm{d}\sigma(y) \leq C \int_{\partial\Omega} \frac{1}{|q-y|^{d-2}} |f(y)| \, \mathrm{d}\sigma(y),
$$

where $C > 0$ only depends on d, θ and the Lipschitz character of Ω. Passing to the maximal operator yields the inequality

$$
(u)^*(q) \leq C \int_{\partial\Omega} \frac{1}{|q-y|^{d-2}} |f(y)| \, \mathrm{d}\sigma(y).
$$

Estimating the kernel via Lemma 3.5 and applying the Young inequality for convolutions from Lemma 3.4 the claim follows. □

Remark 3.7. We note that in addition to the consideration of $d = 2$, Lemma 3.6 differs in the form of estimate (3.15) from the original statement in Shen's work [32, Lem. 3.2]. There, the author derives an estimate of the form $|\lambda|^{1/2} \|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p \|f\|_{\mathrm{L}^p(\partial\Omega;\,\mathbb{C}^d)}$ which is based on Shen's version of Lemma 2.1, namely [32, Lem. 2.1]. As we could not follow the proof in [32], we provided a similar estimate and since the estimate won't be needed in the course of this thesis, we will not pursue the verification of Shen's estimate further. Another approach to the integrability of $(u)^*$ for the L^2 case will be given in Chapter 4.

The next lemma deals with *trace formulas* for ∇u and ϕ. We will then finally be able to talk about boundary values since the existence of nontangential limits guarantees that there exists something on $\partial\Omega$ that is related to the function inside Ω or inside $\mathbb{R}^d \setminus \overline{\Omega}$, respectively.

Lemma 3.8. *Let* (u, ϕ) *be given by* (3.1) *and* (3.2) *with* $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ *and* $1 < p < \infty$. *Then,*

$$\begin{aligned}\left(\frac{\partial u_i}{\partial x_j}\right)_{\pm}(x) &= \pm\frac{1}{2}\big\{n_j(x)f_i(x) - n_i(x)n_j(x)n_k(x)f_k(x)\big\} \\ &\quad + \mathrm{p.\,v.}\int_{\partial\Omega}\frac{\partial}{\partial x_j}\Big\{\Gamma_{ik}(x-y;\lambda)\Big\}f_k(y)\,\mathrm{d}\sigma(y), \qquad (3.16)\\ \phi_{\pm}(x) &= \mp\frac{1}{2}n_k(x)f_k(x) + \mathrm{p.\,v.}\int_{\partial\Omega}\Phi_k(x-y)f_k(y)\,\mathrm{d}\sigma(y)\end{aligned}$$

for almost every $x \in \partial\Omega$. *The subscripts + and − indicate nontangential limits taken inside* Ω *and outside* $\overline{\Omega}$, *respectively.*

Proof. The correctness of the trace formulas (3.16) is known for the case $\lambda = 0$ due to Mitrea and Wright [28, Prop. 4.4]. This fact will now be reused for $\lambda \in \Sigma_\theta$. We insert a zero to the nontangential limit such that

$$(\nabla u_j)_{\pm}(x) = (\nabla v_j)_{\pm}(x) + (\nabla u_j - \nabla v_j)_{\pm}(x),$$

where $v_j(x) = \int_{\partial\Omega}\Gamma_{jk}(x-y;0)f_k(y)\,\mathrm{d}\sigma(y)$. Because of [28] we know that the first nontangential limit exists and is given by (3.16) with $\lambda = 0$. It therefore remains to show the identity

$$(\nabla u_j - \nabla v_j)_{\pm}(x) = \int_{\partial\Omega}\nabla_x\Big\{\Gamma_{jk}(x-y;\lambda) - \Gamma_{jk}(x-y;0)\Big\}f_k(y)\,\mathrm{d}\sigma(y)$$

for all $x \in \partial\Omega$. To this end, let $(x_l)_{l\in\mathbb{N}}$ be a sequence in $\Gamma(x)$ with $\lim_{l\to\infty} x_l = x$. Furthermore, let us note that for almost every $x \in \partial\Omega$ we have that

$$\int_{\partial\Omega}\frac{1}{|x-y|^{d-2}}|f(y)|\,\mathrm{d}\sigma(y) < \infty.$$

This is a consequence of Lemma 3.5 and Young's inequality from Lemma 3.4. Now, we will show that for the members of the sequence x_l the function

$$\frac{1}{|x-y|^{d-2}}|f(y)|$$

serves as a suitable function for dominated convergence. Set $\varepsilon = (4|\lambda|^2)^{-1}$ and without loss of generality assume that $\operatorname{supp} f \subseteq \mathrm{B}(x, \varepsilon)$. Furthermore assume that $|x_l - x| < \varepsilon$ for all $l \in \mathbb{N}$. Then $|x_l - y| \leq (2|\lambda|^2)^{-1}$ and Corollary 2.7 give

$$\begin{aligned}
&\left| \int_{\partial\Omega} \nabla_x \Big\{ \Gamma_{jk}(x_l - y; \lambda) - \Gamma_{jk}(x_l - y; 0) \Big\} f_k(y)\, \mathrm{d}\sigma(y) \right| \\
&\qquad \leq C \int_{\partial\Omega} \sqrt{|\lambda|} \frac{1}{|x_l - y|^{d-2}} \, |f(y)| \, \mathrm{d}\sigma(y) \\
&\qquad \leq C \sqrt{|\lambda|} \int_{\partial\Omega} \frac{1}{|x - y|^{d-2}} \, |f(y)| \, \mathrm{d}\sigma(y) < \infty,
\end{aligned}$$

where for the last step we applied inequality (1.3) Now dominated convergence gives the claim for $x_l \longrightarrow x$. Note that it does not affect the proof if the sequence x_l lays inside Ω or outside $\overline{\Omega}$ and thus the same proof holds for a sequence $(x_l)_{l\in\mathbb{N}}$ in $\Gamma^{\mathrm{ext}}(x)$. □

The previous lemma enables us to talk about boundary values of partial derivatives. The next theorem will now give a similar result but for *conormal derivatives*, which are defined for solutions (u, ϕ) to the Stokes (resolvent) system via

$$\frac{\partial u}{\partial \nu} := \frac{\partial u}{\partial n} - \phi n, \tag{3.17}$$

see Mitrea and Wright [28, Eq. (1.2)], where n denotes the outer unit normal vector. We will also be working with the tangential gradient which is defined via

$$\nabla_{\tan} u_j := \nabla u_j - \big(\nabla u_j \cdot n\big) n, \tag{3.18}$$

see Mitrea and Wright [28, p. 17].

Theorem 3.9. *Let $\lambda \in \Sigma_\theta$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. Let (u, ϕ) be given by* (3.1) *and* (3.2) *with $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ and $1 < p < \infty$. Then $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$ and*

$$\left(\frac{\partial u}{\partial \nu}\right)_{\pm} = \Big(\pm \frac{1}{2} I + \mathcal{K}_\lambda \Big) f \tag{3.19}$$

on $\partial\Omega$, with $\mathcal{K}_\lambda$ a bounded operator on $\mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ satisfying

$$\|\mathcal{K}_\lambda f\|_{\mathrm{L}^p(\partial\Omega;\, \mathbb{C}^d)} \leq C_p \, \|f\|_{\mathrm{L}^p(\partial\Omega;\, \mathbb{C}^d)},$$

where $C_p > 0$ depends only on d, θ, p and the Lipschitz character of Ω.

Proof. For the jth component of the tangential derivative of u_i, $1 \le i, j \le d$, we calculate using the results from Lemma 3.8

$$\begin{aligned}
((\nabla_{\tan} u_i)_+)_j &= \left(\frac{\partial u_i}{\partial x_j}\right)_+ - (\nabla u_i)_+ \cdot n\, n_j \\
&= \left(\frac{\partial u_i}{\partial x_j}\right)_+ - \left(\frac{\partial u_i}{\partial x_k}\right)_+ n_k n_j \\
&= \frac{1}{2}\{n_j f_i - n_i n_j n_k f_k\} - \frac{1}{2}\{n_k f_i - n_i n_k n_l f_l\} n_k n_j \\
&\quad + \mathrm{p.\,v.} \int_{\partial\Omega} \frac{\partial}{\partial x_j}\Big\{\Gamma_{ik}(\,\cdot - y; \lambda)\Big\} f_k(y)\, \mathrm{d}\sigma(y) \\
&\quad + \mathrm{p.\,v.} \int_{\partial\Omega} \frac{\partial}{\partial x_k}\Big\{\Gamma_{il}(\,\cdot - y; \lambda)\Big\} f_l(y)\, \mathrm{d}\sigma(y)\, n_k n_j,
\end{aligned}$$

for almost every $x \in \partial\Omega$. As the first two summands add up to zero, the entire expression does not depend on the direction of the nontangential limit. This gives

$$(\nabla_{\tan} u)_+ = (\nabla_{\tan} u)_-\,,$$

for almost every $x \in \partial\Omega$. We calculate for the jth component of the nontangential limit of the conormal derivative of u on $\partial\Omega$ using the results from Lemma 3.8

$$\begin{aligned}
&\left(\frac{\partial u_j}{\partial x_i}\right)_+ n_i - \phi_+ n_j \\
&\quad = \frac{1}{2}\{n_i f_j - n_j n_i n_k f_k\} n_i + \mathrm{p.\,v.} \int_{\partial\Omega} \frac{\partial}{\partial x_i}\Big\{\Gamma_{jk}(\,\cdot - y; \lambda)\Big\} f_k(y)\, \mathrm{d}\sigma(y) n_i \\
&\quad\quad + \frac{1}{2} n_k f_k n_j - \mathrm{p.\,v.} \int_{\partial\Omega} \Phi_k(\,\cdot - y) f_k(y)\, \mathrm{d}\sigma(y) n_j \\
&\quad = \frac{1}{2} f_j + (\mathcal{K}_\lambda f)_j
\end{aligned}$$

almost every and where $\mathcal{K}_\lambda$ is a singular integral operator defined via

$$\begin{aligned}
(\mathcal{K}_\lambda f)_j(x) \coloneqq \mathrm{p.\,v.} &\int_{\partial\Omega} \frac{\partial}{\partial x_i}\Big\{\Gamma_{jk}(x - y; \lambda)\Big\} f_k(y)\, \mathrm{d}\sigma(y)\, n_i(x) \\
&- \mathrm{p.\,v.} \int_{\partial\Omega} \Phi_k(x - y) f_k(y)\, \mathrm{d}\sigma(y)\, n_j(x).
\end{aligned} \tag{3.20}$$

We note that $\mathcal{K}_\lambda$ essentially consists of two boundary layer potentials. The L^p boundedness of the first one was proven in Lemma 3.3. The L^p boundedness of the second boundary layer potential follows in an analogous way using the fact that the operators

$$A^*(f)(q) \coloneqq \sup_{t>0} \left| \int_{\substack{y \in \partial\Omega \\ |y-q|>t}} \frac{q-y}{|q-y|^d} f(y)\, \mathrm{d}\sigma(y) \right|, \quad q \in \partial\Omega,$$

are bounded by the corresponding result from Verchota [40, Lem. 1.2]. □

Similar to $\mathcal{K}_\lambda$, for $\lambda = 0$, we have

$$\begin{aligned}(\mathcal{K}_0 f)_j(x) = \mathrm{p.\,v.} &\int_{\partial\Omega} \frac{\partial}{\partial x_i}\Big\{\Gamma_{jk}(x-y;0)\Big\} f_k(y)\,\mathrm{d}\sigma(y)\, n_i(x) \\ &- \mathrm{p.\,v.}\int_{\partial\Omega} \Phi_k(x-y) f_k(y)\,\mathrm{d}\sigma(y)\, n_j(x),\end{aligned} \tag{3.21}$$

as was shown by Mitrea and Wright [28, Prop. 4.4]. If one compares (3.20) with (3.21), then the only difference lies within the boundary integral involving the fundamental solutions $\Gamma_{jk}(\,\cdot\,;0)$ instead of $\Gamma_{jk}(\,\cdot\,;\lambda)$.

The next result will be crucial for solving the L^2 Dirichlet problem in Chapter 5 and will fortify the hopes of translating results for $\lambda = 0$ to $\lambda \in \Sigma_\theta$.

Lemma 3.10. *Let $\lambda \in \Sigma_\theta$ and $d \geq 2$ and let $\mathcal{K}_\lambda$ and $\mathcal{K}_0$ be defined by* (3.20) *and* (3.21)*, respectively. Then the operator $\mathcal{K}_\lambda - \mathcal{K}_0$ on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ is compact.*

Proof. The idea of this proof is similar to the one in Tolksdorf [39, Lem. 4.3.5]. Let $f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ and let us denote $\mathcal{K} := \mathcal{K}_\lambda - \mathcal{K}_0$. We will now try to approximate $\mathcal{K}$ by compact operators in the operator norm. To this end, we define for all $\varepsilon > 0$

$$(\mathcal{K}^{(\varepsilon)} f)(x) := \int_{\partial\Omega\setminus \mathrm{B}(x,\varepsilon)} \nabla_x \Big\{\Gamma(x-y;\lambda) - \Gamma(x-y;0)\Big\} f(y)\,\mathrm{d}\sigma(y)\, n, \qquad x \in \partial\Omega.$$

Now we estimate using Young's inequality, see Lemma 3.4:

$$\begin{aligned}&\Big\|\mathcal{K} f - \mathcal{K}^{(\varepsilon)} f\Big\|_{\mathrm{L}^2(\partial\Omega;\,\mathbb{C}^d)} \\ &\qquad \leq \sup_{p\in\partial\Omega} \Big\|\nabla_x \Big\{\Gamma(p-\cdot;\lambda) - \Gamma(p-\cdot;0)\Big\} 1_{\mathrm{B}(p,\varepsilon)}\Big\|_{\mathrm{L}^1(\partial\Omega;\,\mathbb{C}^{d\times d})} \|f\|_{\mathrm{L}^2(\partial\Omega;\,\mathbb{C}^d)}.\end{aligned}$$

Our goal is to show that

$$\sup_{p\in\partial\Omega} \Big\|\nabla_x \Big\{\Gamma(p-\cdot;\lambda) - \Gamma(p-\cdot;0)\Big\} 1_{\mathrm{B}(p,\varepsilon)}\Big\|_{\mathrm{L}^1(\partial\Omega;\,\mathbb{C}^{d\times d})} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

To this end, let ε be small enough such that we can apply the estimates from Corollary 2.7 to calculate for some $p \in \partial\Omega$

$$\begin{aligned}&\Big\|\nabla_x \Big\{\Gamma(p-\cdot;\lambda) - \Gamma(p-\cdot;0)\Big\} 1_{\mathrm{B}(p,\varepsilon)}\Big\|_{\mathrm{L}^1(\partial\Omega;\,\mathbb{C}^{d\times d})} \\ &\qquad \leq C \int_{\partial\Omega\cap \mathrm{B}(p,\varepsilon)} \sqrt{|\lambda|}\, |p-y|^{2-d}\,\mathrm{d}\sigma(y) \leq C\sqrt{|\lambda|}\,\varepsilon\end{aligned}$$

where for the last step we applied Lemma 3.1. For $\varepsilon \to 0$ this gives us $\mathcal{K}^{(\varepsilon)} \to \mathcal{K}$ in the operator norm.

The last step is to verify the compactness of $\mathcal{K}^{(\varepsilon)}$. We note that the integral kernel of $\mathcal{K}^{(\varepsilon)}$ is bounded which gives us that in particular the kernel is an element of the space $\mathrm{L}^2(\partial\Omega \times \partial\Omega; \mathbb{C}^{d\times d})$. The compactness of $\mathcal{K}^{(\varepsilon)}$ now follows from Weidmann [43, Thm. 6.11].

As a consequence, $\mathcal{K}$ is compact since the limit of compact operators with respect to the operator norm gives again a compact operator. □

Our next step is to introduce the *double layer potential* $u(x) = \mathcal{D}_\lambda(f)(x)$ for the Stokes resolvent problem via

$$(\mathcal{D}_\lambda(f))_j(x) := \int_{\partial\Omega} \Big\{ \frac{\partial}{\partial y_i} \{\Gamma_{jk}(y-x;\lambda)\} n_i(y) - \Phi_j(y-x) n_k(y) \Big\} f_k(y)\,\mathrm{d}\sigma(y). \tag{3.22}$$

The corresponding pressure $\phi(x) = \mathcal{D}_\Phi(f)(x)$ is defined as

$$\begin{aligned}\mathcal{D}_\Phi(f)(x) := \frac{\partial^2}{\partial x_i \partial x_k} &\int_{\partial\Omega} G(y-x;0)\, n_i(y) f_k(y)\,\mathrm{d}\sigma(y) \\ &+ \lambda \int_{\partial\Omega} G(y-x;0)\, n_k(y) f_k(y)\,\mathrm{d}\sigma(y).\end{aligned} \tag{3.23}$$

Using (2.29) and (2.30) one can show that (u, ϕ) defines again a solution to the Stokes resolvent problem in $\mathbb{R}^d \setminus \partial\Omega$.

The following theorem will give us a suitable operator which maps a given function $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ to boundary values of $u = \mathcal{D}_\lambda(f)$ in the form of nontangential limits. It will then be the task of the following chapters to prove the invertibility of this operator.

Theorem 3.11. *Let $\lambda \in \Sigma_\theta$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. Let u be given by* (3.22) *for $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$, $1 < p < \infty$. Then,*

$$\|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p \|f\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)}, \tag{3.24}$$

where $C_p > 0$ depends only on d, p, θ and the Lipschitz character of Ω. Furthermore,

$$u_\pm = \Big(\mp \frac{1}{2} I + \mathcal{K}^*_{\bar\lambda} \Big) f, \tag{3.25}$$

where $\mathcal{K}^*_{\bar\lambda}$ is the adjoint of the operator $\mathcal{K}_{\bar\lambda}$ in (3.19).

Proof. The estimate for $(u)^*$ is a direct consequence of Lemma 3.6 and the estimates on the nontangential maximal functions for the single layer potentials $(\nabla\mathcal{S}_\lambda(f))^*$ and $(\mathcal{S}_\Phi(f))^*$: We have on the one hand

$$\begin{aligned}&\int_{\partial\Omega} \frac{\partial}{\partial y_i} \Big\{ \Gamma_{jk}(y-x;\lambda) \Big\}\, n_i(y) f_k(y)\,\mathrm{d}\sigma(y) \\ &\qquad = -\int_{\partial\Omega} \frac{\partial}{\partial x_i} \Big\{ \Gamma_{jk}(x-y;\lambda) \Big\}\, n_i(y) f_k(y)\,\mathrm{d}\sigma(y) = -\frac{\partial}{\partial x_i} \mathcal{S}_\lambda(n_i f)_j(x)\end{aligned} \tag{3.26}$$

and on the other hand

$$
\begin{aligned}
&-\int_{\partial\Omega} \Phi_j(y-x) n_k(y) f_k(y)\,\mathrm{d}\sigma(y) \\
&\qquad = \int_{\partial\Omega} \Phi_l(x-y)\,\delta_{lj} n_k(y) f_k(y)\,\mathrm{d}\sigma(y) = \mathcal{S}_\Phi(\tilde{f}^j)(x),
\end{aligned}
\tag{3.27}
$$

where $\tilde{f}_l^j = \delta_{lj} n_k f_k$. This shows that

$$
u_j(x) = (\mathcal{D}_\lambda(f))_j(x) = -\frac{\partial}{\partial x_i}\mathcal{S}_\lambda(n_i f)_j(x) + \mathcal{S}_\Phi(\tilde{f}^j)(x).
$$

Therefore, we have for $x \in \Gamma(q)$, $q \in \partial\Omega$,

$$
|u(x)| \leq C\left\{|\nabla_x \mathcal{S}_\lambda(n_i f)(x)| + |\mathcal{S}_\Phi(\tilde{f}^j)|\right\},
$$

with $C > 0$ depending only on d. Hence, by Lemma 3.6, we derive the followign chain of estimates:

$$
\|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C\left\{\sum_{i=1}^{d}\|n_i f\|_{\mathrm{L}^p(\partial\Omega;\,\mathbb{C}^d)} + \sum_{j=1}^{d}\|\mathcal{S}_\Phi(\tilde{f}^j)\|_{\mathrm{L}^p(\partial\Omega)}\right\} \leq C\,\|f\|_{\mathrm{L}^p(\partial\Omega;\,\mathbb{C}^d)},
$$

where $C > 0$ depends on d, p, θ and the Lipschitz character of Ω.

For the proof of (3.25), we begin by determining the adjoint of the operator $\mathcal{K}_{\overline{\lambda}}$. To this end, we will first work with truncated operators $\mathcal{K}_\lambda^{(\varepsilon)}\colon \mathrm{L}^2(\partial\Omega;\mathbb{C}^d) \to \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ which are defined via

$$
\begin{aligned}
(\mathcal{K}_\lambda^{(\varepsilon)} f)_j(x) := &\int_{\partial\Omega} 1_{\mathrm{E}(x,\,\varepsilon)}(y)\frac{\partial}{\partial x_i}\Gamma_{jk}(x-y;\lambda) f_k(y)\,\mathrm{d}\sigma(y)\, n_i(x) \\
&- \int_{\partial\Omega} 1_{\mathrm{E}(x,\,\varepsilon)}(y)\,\Phi_k(x-y) f_k(y)\,\mathrm{d}\sigma(y)\, n_j(x),
\end{aligned}
$$

for $x \in \partial\Omega$ and $\mathrm{E}(x,\varepsilon) := \mathbb{R}^d \setminus \mathrm{B}(x,\varepsilon)$. Now, for $f \in \mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ and $g \in \mathrm{L}^q(\partial\Omega;\mathbb{C}^d)$ with $1/p + 1/q = 1$ we calculate

$$
\begin{aligned}
\left\langle \mathcal{K}_{\overline{\lambda}}^{(\varepsilon)} f, g\right\rangle &= \int_{\partial\Omega} (\mathcal{K}_{\overline{\lambda}}^{(\varepsilon)} f)_j(x)\,\overline{g_j(x)}\,\mathrm{d}\sigma(x) \\
&= \int_{\partial\Omega}\int_{\partial\Omega} \frac{\partial}{\partial x_i}\left\{\Gamma_{jk}(x-y;\overline{\lambda})\right\} f_k(y) 1_{\mathrm{E}(x,\,\varepsilon)}(y)\,\mathrm{d}\sigma(y)\, n_i(x)\,\overline{g_j(x)}\,\mathrm{d}\sigma(x) \\
&\quad + \int_{\partial\Omega}\int_{\partial\Omega} \Phi_k(x-y) f_k(y) 1_{\mathrm{E}(x,\,\varepsilon)}(y)\,\mathrm{d}\sigma(y)\, n_j(x)\,\overline{g_j(x)}\,\mathrm{d}\sigma(x),
\end{aligned}
$$

where the dual pairing is denoted by the brackets $\langle\,\cdot\,,\cdot\,\rangle$. Note that $1_{\mathrm{E}(x,\,\varepsilon)}(y) = 1_{\mathrm{E}(y,\,\varepsilon)}(x)$ for all $x, y \in \partial\Omega$. Now, an application of Fubini's theorem and factoring out $f_k(y)$ gives us the identity

$$\left\langle \mathcal{K}_{\bar{\lambda}}^{(\varepsilon)} f, g \right\rangle = \int_{\partial\Omega} f_k(y) \int_{\partial\Omega} \left\{ \frac{\partial}{\partial x_i} \{\Gamma_{jk}(x-y;\overline{\lambda})\}\, n_i(x) \right.$$
$$\left. - \Phi_k(x-y) n_j(x) \right\} 1_{\mathrm{E}(y,\varepsilon)}(x)\, \overline{g_j(x)}\, \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(y).$$

Therefore, we see that the adjoint of the truncated operator $\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)}$ is given by

$$\left((\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)})^* g\right)_k(y) = \int_{\partial\Omega} \left\{ \frac{\partial}{\partial x_i} \{\Gamma_{jk}(x-y;\lambda)\} n_i(x) \right.$$
$$\left. - \Phi_k(x-y) n_j(x) \right\} 1_{\mathrm{E}(y,\varepsilon)}(x)\, g_j(x)\, \mathrm{d}\sigma(x),$$

for $y \in \partial\Omega$ since $\overline{\Gamma_{jk}(x-y;\lambda)} = \Gamma_{jk}(x-y;\overline{\lambda})$.

In the next step, we will go from truncated operators to principal value operators through the dominated convergence theorem. For this to work we will look for suitable majorants. For $x \in \partial\Omega$, we estimate

$$\left|(\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)} f)_j(x)\right| = \left| \int_{|x-y|>\varepsilon} \frac{\partial}{\partial x_i} \left\{\Gamma_{jk}(x-y;\lambda)\right\} f_k(y)\, \mathrm{d}\sigma(y)\, n_i(x) \right.$$
$$\left. - \int_{|x-y|>\varepsilon} \Phi_k(x-y) f_k(y)\, n_j(x)\, \mathrm{d}\sigma(y) \right|$$
$$\leq T_{\bar{\lambda}}^*(f)(x) + A^*(f n_j)(x). \tag{3.28}$$

We know from Lemma 3.3 and the respective result for A^* that the right-hand side of inequality (3.28) is p-integrable and hence we get from dominated convergence

$$\lim_{\varepsilon\to 0} \left\langle \mathcal{K}_{\bar{\lambda}}^{(\varepsilon)} f, g \right\rangle = \left\langle \mathcal{K}_{\bar{\lambda}} f, g \right\rangle.$$

With a similar argument we get

$$\left|\left((\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)})^* g\right)_k(y)\right| = \left| \int_{|x-y|>\varepsilon} \frac{\partial}{\partial x_i} \left\{\Gamma_{jk}(x-y;\overline{\lambda})\right\} n_i(x) g_j(x)\, \mathrm{d}\sigma(x) \right.$$
$$\left. - \int_{|x-y|>\varepsilon} \Phi_k(x-y) n_j(x) g_j(x)\, \mathrm{d}\sigma(x) \right|$$
$$\leq \sum_{i=1}^{d} T_{\lambda}^*(n_i g)(y) + \sum_{i=1}^{d} A^*(\tilde{g}^j)(y),$$

where $\tilde{g}_l^j = \delta_{lj} n_k f_k$ and therefore the dominated convergence theorem yields

$$\lim_{\varepsilon\to 0} \left\langle f, \left(\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)}\right)^* g \right\rangle = \left\langle f, \mathcal{K}_{\bar{\lambda}}^{(*)} g \right\rangle,$$

where the limit operator $\mathcal{K}_{\bar\lambda}^{(*)}$ is defined via

$$\big((\mathcal{K}_{\bar\lambda}^{(*)} g)\big)_k(y) := \text{p. v.} \int_{\partial\Omega} \Big\{ \frac{\partial}{\partial x_i} \big\{\Gamma_{kj}(x-y;\lambda)\big\} n_i(x) - \Phi_k(x-y) n_j(x) \Big\} g_j(x)\, \mathrm{d}\sigma(x).$$

Of course by the uniqueness of the adjoint operator, the identity $\langle \mathcal{K}_{\bar\lambda} f, g\rangle = \langle f, \mathcal{K}_{\bar\lambda}^{(*)} g\rangle$ shows that $\mathcal{K}_{\bar\lambda}^{*} = \mathcal{K}_{\bar\lambda}^{(*)}$. Note that we have used the symmetry of the matrix $(\Gamma_{jk})_{1\le j,k\le d}$.

In the last part of this proof we will show that the equality (3.25) holds. Note that by (3.26) and (3.27) we have made Lemma 3.8 accessible. For $x \in \partial\Omega$ we can now calculate

$$\begin{aligned}
&\Big(\int_{\partial\Omega} \frac{\partial}{\partial y_i} \Big\{ \Gamma_{jk}(y-\cdot\,;\lambda) \Big\} n_i(y) f_k(y)\, \mathrm{d}\sigma(y) \Big)_{\pm}(x) \\
&\qquad = -\Big(\frac{\partial}{\partial x_i} \mathcal{S}_\lambda (n_i f)_j \Big)_{\pm}(x) \\
&\qquad = \mp \frac{1}{2} \big\{ n_i(x) n_i(x) f_j(x) - n_j(x) n_i(x) n_k(x) n_i(x) f_k(x) \big\} \\
&\qquad\quad - \text{p. v.} \int_{\partial\Omega} \frac{\partial}{\partial x_i} \Big\{ \Gamma_{jk}(x-y;\lambda) \Big\}\, n_i(y) f_k(y)\, \mathrm{d}\sigma(y) \\
&\qquad = \mp \frac{1}{2} \big\{ f_j(x) - n_j(x) n_k(x) f_k(x) \big\} \\
&\qquad\quad + \text{p. v.} \int_{\partial\Omega} \frac{\partial}{\partial y_i} \Big\{ \Gamma_{jk}(y-x;\lambda) \Big\}\, n_i(y) f_k(y)\, \mathrm{d}\sigma(y),
\end{aligned}$$

where we used trace formula (3.16). A similar procedure for the second integral part of the double layer potential gives

$$\begin{aligned}
&-\Big(\int_{\partial\Omega} \Phi_j(y-\cdot\,)\, n_k(y) f_k(y)\, \mathrm{d}\sigma(y) \Big)_{\pm}(x) \\
&\qquad = \big(\mathcal{S}_\Phi(\tilde f^j) \big)_{\pm}(x) \\
&\qquad = \mp \frac{1}{2} n_k(x) \tilde f^j_k(x) - \text{p. v.} \int_{\partial\Omega} \Phi_k(x-y) \tilde f^j_k(x)\, \mathrm{d}\sigma(y) \\
&\qquad = \mp \frac{1}{2} n_j(x) n_k(x) f_k(x) - \text{p. v.} \int_{\partial\Omega} \Phi_j(x-y)\, n_k(x) f_k(x)\, \mathrm{d}\sigma(y).
\end{aligned}$$

Putting everything together we get

$$(u_j)_\pm(x) = \mp \frac{1}{2} f_j(x) + (\mathcal{K}_{\bar\lambda}^{*} f)_j(x)$$

and the proof is finished. □

Chapter 4

Rellich Estimates

In this section, we will establish Rellich-type estimates for solutions of the Stokes resolvent problem (1.9) which will be used in the following chapter to prove the invertibility of the operators $\pm(1/2)I + \mathcal{K}_\lambda$ and their adjoints from Theorems 3.9 and 3.11.

We will for this entire section always assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with connected boundary. Furthermore, we will use the shorthand notation

$$\|\cdot\|_\partial := \|\cdot\|_{L^2(\partial\Omega;\, \mathbb{C}^k)}, \quad k \in \mathbb{N},$$

and we will tacitly use the summation convention whenever it is applicable.

The following theorem formulates the aforementioned Rellich estimates and is the central result of this chapter:

Theorem 4.1. *Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0,1)$. Let (u,ϕ) be a smooth solution to the Stokes resolvent problem* (2.27) *in Ω and suppose that $(\nabla u)^* \in L^2(\partial\Omega)$ and $(\phi)^* \in L^2(\partial\Omega)$.*

Furthermore, assume that ∇u, ϕ have nontangential limits almost everywhere on $\partial\Omega$. Then

$$\begin{aligned}\|\nabla u\|_\partial &+ \left\| \phi - \left\{ \frac{1}{{r_0}^{d-1}} \int_{\partial\Omega} \phi \,\mathrm{d}\sigma \right\} \right\|_\partial \\ &\leq C \left\{ \left\| \nabla_{\tan} u \right\|_\partial + |\lambda|^{1/2} \|u\|_\partial + |\lambda|\, \|u \cdot n\|_{H^{-1}(\partial\Omega)} \right\}\end{aligned} \tag{4.1}$$

and

$$\|\nabla u\|_\partial + |\lambda|^{1/2}\|u\|_\partial + |\lambda|\, \|u \cdot n\|_{H^{-1}(\partial\Omega)} + \|\phi\|_\partial \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_\partial, \tag{4.2}$$

where $\frac{\partial}{\partial \nu}$ denotes the conormal derivative, and $C > 0$ depends only on d, τ, θ and the Lipschitz character of Ω.

Remark 4.2. The assumptions on u in Theorem 4.1 are sufficient for u to have a nontangential limit and a square integrable maximal function $(u)^*$. Indeed for $d = 2$ we have $(u)^* \in \mathrm{L}^\infty(\partial\Omega)$, for $d = 3$ we have $(u)^* \in \mathrm{L}^p(\partial\Omega), p \in (1, \infty)$, and for $d > 3$ we have $(u)^* \in \mathrm{L}^p(\partial\Omega)$, $p \in \big(1, 2(d-1)/(d-3)\big)$. A proof of these facts can be found in Shen's notes [33, Prop. 7.1.3 and Rem. 7.1.4].

We will now prepare the proof of Theorem 4.1 by proving several helpful lemmata. The first lemma deals with so-called *Rellich identities* for solutions of the Stokes resolvent system (2.27).

Lemma 4.3. *Under the same conditions on (u, ϕ) as in Theorem 4.1, we have*

$$\begin{aligned}\int_{\partial\Omega} h_k n_k |\nabla u|^2 \,\mathrm{d}\sigma = 2\,\mathrm{Re} \int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k} \Big(\frac{\partial u}{\partial \nu}\Big)_i \,\mathrm{d}\sigma + \int_\Omega \mathrm{div}\,(h)\, |\nabla u|^2 \,\mathrm{d}x \\ - 2\,\mathrm{Re} \int_\Omega \frac{\partial h_k}{\partial x_j} \cdot \frac{\partial u_i}{\partial x_k} \cdot \frac{\partial \overline{u}_i}{\partial x_j} \,\mathrm{d}x + 2\,\mathrm{Re} \int_\Omega \frac{\partial h_k}{\partial x_i} \cdot \frac{\partial u_i}{\partial x_k}\, \overline{\phi} \,\mathrm{d}x \\ - 2\,\mathrm{Re} \int_\Omega h_k \frac{\partial u_i}{\partial x_k} \cdot \overline{\lambda u_i} \,\mathrm{d}x \end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\int_{\partial\Omega} h_k n_k |\nabla u|^2 \,\mathrm{d}\sigma = 2\,\mathrm{Re} \int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_j} \Big\{ n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k} \Big\} \,\mathrm{d}\sigma \\ + 2\,\mathrm{Re} \int_{\partial\Omega} h_k\, \overline{\phi} \Big\{ n_i \frac{\partial u_i}{\partial x_k} - n_k \frac{\partial u_i}{\partial x_i} \Big\} \,\mathrm{d}\sigma - \int_\Omega \mathrm{div}\,(h)\, |\nabla u|^2 \,\mathrm{d}x \\ + 2\,\mathrm{Re} \int_\Omega \frac{\partial h_k}{\partial x_j} \cdot \frac{\partial u_i}{\partial x_k} \cdot \frac{\partial \overline{u}_i}{\partial x_j} \,\mathrm{d}x - 2\,\mathrm{Re} \int_\Omega \frac{\partial h_k}{\partial x_i} \cdot \frac{\partial u_i}{\partial x_k}\, \overline{\phi} \,\mathrm{d}x \\ + 2\,\mathrm{Re} \int_\Omega h_k \frac{\partial u_i}{\partial x_k} \cdot \overline{\lambda u_i} \,\mathrm{d}x, \end{aligned} \tag{4.4}$$

where $h = (h_1, \ldots, h_d) \in \mathrm{C}_0^1(\mathbb{R}^d; \mathbb{R}^d)$.

Proof. The proof of the stated identities reduces to several applications of the divergence theorem once we establish its applicability. To this end, we want to make Proposition 1.8 available. We note that the assumptions given in Theorem 4.1 are sufficient for this purpose and we will verify them, once they are used.

Let us expand the first summand in (4.3) using the definition of the conormal derivative

$$\begin{aligned} 2\,\mathrm{Re} \int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k} \Big(\frac{\partial u}{\partial \nu}\Big)_i \,\mathrm{d}\sigma &= 2\,\mathrm{Re} \int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k} \cdot \frac{\partial u_i}{\partial x_j} n_j \,\mathrm{d}\sigma - 2\,\mathrm{Re} \int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k}\, \phi\, n_i \,\mathrm{d}x \\ &=: I_1 - I_2. \end{aligned}$$

The divergence theorem, see Proposition 1.8, is applicable for I_1 as h is bounded and defined everywhere and the integrand has nontangential limits that can be dominated by $|(\nabla u)^*|^2 \in \mathrm{L}^1(\partial\Omega)$. Therefore, we find using the divergence theorem and the product rule:

$$\begin{aligned} I_1 &= 2\,\mathrm{Re}\int_\Omega \frac{\partial}{\partial x_j}\left\{h_k \frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial u_i}{\partial x_j}\right\}\mathrm{d}x \\ &= 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_j}\cdot\frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial u_i}{\partial x_j}\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_k}\cdot\frac{\partial u_i}{\partial x_j}\,\mathrm{d}x \\ &\qquad + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial^2 u_i}{\partial x_j^2}\,\mathrm{d}x \\ &=: I_3 + I_4 + I_5. \end{aligned}$$

For I_5, we use the fact that u solves the Stokes resolvent problem which gives

$$\begin{aligned} I_5 &= 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial \phi}{\partial x_i}\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k}\lambda u_i\,\mathrm{d}x \\ &=: I_6 + I_7. \end{aligned}$$

Now, we want to apply the divergence theorem, i.e. Proposition 1.8, to integral I_2. This is possible since h is defined everywhere and bounded, $(\partial_k u_i)\cdot\phi$ has a nontangential limit and can be bounded by $\big(|(\nabla u)^*||(\phi)^*|\big)$ which is integrable due to Hölder's inequality as $(\nabla u)^*$ and $(\phi)^*$ are square integrable by assumption. Thus, the divergence theorem is applicable and yields together with the product rule:

$$\begin{aligned} I_2 &= 2\,\mathrm{Re}\int_\Omega \frac{\partial}{\partial x_i}\left\{h_k \frac{\partial \overline{u}_i}{\partial x_k}\,\phi\right\}\mathrm{d}x \\ &= 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_i}\cdot\frac{\partial \overline{u}_i}{\partial x_k}\,\phi\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial^2 \overline{u}_i}{\partial x_i \partial x_k}\,\phi\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial \phi}{\partial x_i}\,\mathrm{d}x \\ &=: I_8 + I_9 + I_6. \end{aligned}$$

Note that since $\operatorname{div}(u) = 0$ it follows that $I_9 = 0$. One term that has not come up so far, the second summand of the right-hand side in (4.3), will now be expanded:

$$\begin{aligned} \int_\Omega \operatorname{div}(h)\,|\nabla u|^2\,\mathrm{d}x &= \int_\Omega \operatorname{div}(h\,|\nabla u|^2)\,\mathrm{d}x - \int_\Omega h_k \frac{\partial}{\partial x_i}\left\{|\nabla u|^2\right\}\mathrm{d}x \\ &=: I_{10} - I_{11}. \end{aligned}$$

Expanding the Integral I_{11} gives us the identity

$$I_{11} = \int_\Omega h_i \frac{\partial}{\partial x_i}\left\{\frac{\partial u_k}{\partial x_j}\cdot\frac{\partial \overline{u}_k}{\partial x_j}\right\}\mathrm{d}x = \int_\Omega h_i\left\{\frac{\partial^2 u_k}{\partial x_i \partial x_j}\cdot\frac{\partial \overline{u}_k}{\partial x_j} + \frac{\partial u_k}{\partial x_j}\cdot\frac{\partial^2 \overline{u}_k}{\partial x_i \partial x_j}\right\}\mathrm{d}x = I_4.$$

If we now put everything together, the right-hand side of (4.3) reads

$$\begin{aligned}&(I_1 - I_2) + (I_{10} - I_{11}) - I_3 + I_8 - I_7\\ &\quad = (I_3 + I_4 + I_6 + I_7) - (I_8 + I_9 + I_6) + I_{10} - I_{11} - I_3 + I_8 - I_7 = I_{10}.\end{aligned}$$

Noting that by the divergence theorem, which is applicable with the same justification as for the integral I_1, we have

$$I_{10} = \int_{\partial\Omega} h_k n_k |\nabla u|^2 \, \mathrm{d}\sigma.$$

Thus, the first identity is proven.

In order to prove identity (4.4), we show that the expression we get from considering ((4.3) + (4.4)) holds, i.e. we show the identity

$$\begin{aligned}2\int_{\partial\Omega} h_k n_k |\nabla u|^2 \, \mathrm{d}\sigma = 2\,\mathrm{Re}\int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k}\left(\frac{\partial u}{\partial \nu}\right)_i \mathrm{d}\sigma&\\ + 2\,\mathrm{Re}\int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_j}\left\{n_k\frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k}\right\}\mathrm{d}\sigma&\\ + 2\,\mathrm{Re}\int_{\partial\Omega} h_k\,\overline{\phi}\left\{n_i\frac{\partial u_i}{\partial x_k} - n_k\frac{\partial u_i}{\partial x_i}\right\}\mathrm{d}\sigma.&\end{aligned}$$

To this end, note that the left side of the identity equals $2\,I_{10}$, whereas the right-hand side can be written as

$$(I_1 - I_2) + (2\,I_{10} - I_1) + (I_2 - 0),$$

where we also used the fact that $\operatorname{div} u = \partial_i u_i = 0$. □

Consider the operators $\partial_{\tau_{jk}}$ which act on compactly supported continuously differentiable functions ψ in the neighborhood of $\partial\Omega$ by

$$\partial_{\tau_{jk}}\psi := n_j \frac{\partial\psi}{\partial x_k}\bigg|_{\partial\Omega} - n_k\frac{\partial\psi}{\partial x_j}\bigg|_{\partial\Omega}, \qquad j,k = 1,\dots,d. \tag{4.5}$$

These operators show up in identity (4.4) as

$$\partial_{\tau_{kj}} u_i = \left\{n_k\frac{\partial u_i}{\partial x_j} - n_j\frac{\partial u_i}{\partial x_k}\right\} \quad \text{and} \quad \partial_{\tau_{ik}} u_i = \left\{n_i\frac{\partial u_i}{\partial x_k} - n_k\frac{\partial u_i}{\partial x_i}\right\}$$

and are introduced for example in the work of Mitrea and Wright [28, p. 16]. These operators are called *first-order tangential derivative operators* and relate to the tangential gradient, which has been introduced in (3.18), in the following way:

$$\left(\nabla_{\tan}\psi\right)_j = \frac{\partial\psi}{\partial x_j} - n_k n_j \frac{\partial\psi}{\partial x_k} = n_k\,\partial_{\tau_{kj}}\psi. \tag{4.6}$$

We can use relation (4.6) to establish the converse relation

$$\begin{aligned} \partial_{\tau_{jk}}\psi &= n_j \frac{\partial \psi}{\partial x_k} - n_k \frac{\partial \psi}{\partial x_j} \\ &= n_j \big(\nabla_{\tan}\psi\big)_k + n_j n_l n_k \frac{\partial \psi}{\partial x_l} - n_k \big(\nabla_{\tan}\psi\big)_j - n_k n_l n_j \frac{\partial \psi}{\partial x_l} \\ &= n_j \big(\nabla_{\tan}\psi\big)_k - n_k \big(\nabla_{\tan}\psi\big)_j. \end{aligned} \tag{4.7}$$

The tangential derivative operators come with a helpful “integration by parts” rule and can be used to define Sobolev spaces on the boundary $\partial\Omega$: For $f \in \mathrm{L}^1_{\mathrm{loc}}(\partial\Omega)$, we start by defining antilinear functionals on $\mathrm{C}_0^\infty(\mathbb{R}^d)$ by setting

$$\partial_{\tau_{kj}} f \colon \mathrm{C}_0^\infty(\mathbb{R}^d) \ni \psi \mapsto \int_{\partial\Omega} f\, \overline{\partial_{\tau_{jk}}\psi}\, \mathrm{d}\sigma.$$

Now the weak tangential derivatives of f are given by those functionals which are regular, i.e. elements of $\mathrm{L}^1_{\mathrm{loc}}(\partial\Omega)$. In this case, the following integration by parts formula holds:

$$\int_{\partial\Omega} f\, \overline{(\partial_{\tau_{jk}}\psi)}\, \mathrm{d}\sigma = \int_{\partial\Omega} (\partial_{\tau_{kj}} f)\, \overline{\psi}\, \mathrm{d}\sigma. \tag{4.8}$$

Note that the minus sign that usually comes with the integration by parts is hidden in the new order of indices. For $p \in (1, \infty)$, we define the corresponding Sobolev space via

$$\mathrm{W}^{1,p}(\partial\Omega) = \Big\{ f \in \mathrm{L}^p(\partial\Omega) \colon \partial_{\tau_{jk}} f \in \mathrm{L}^p(\partial\Omega),\ j, k = 1, \ldots, d \Big\}$$

with the norm defined via

$$\|f\|_{\mathrm{W}^{1,p}(\partial\Omega)} \coloneqq \|f\|_{\partial} + \|\nabla_{\tan} f\|_{\partial}.$$

As for the case of Sobolev spaces on Ω, we will denote the spaces $\mathrm{W}^{1,2}(\partial\Omega)$ by $\mathrm{H}^1(\partial\Omega)$.

We extend our detour by the following basic lemma on a reverse triangle inequality for elements of the sector Σ_θ. A powerful generalization of this lemma can be found in Tolksdorf [39, Lem. 5.2.4].

Lemma 4.4. *Let $\theta \in (0, \pi/2)$. Then, there exists α depending only on θ such that for all $\lambda \in \Sigma_\theta$ the following inequality holds:*

$$\mathrm{Re}(\lambda) + \alpha \big|\mathrm{Im}(\lambda)\big| \geq |\lambda|.$$

Proof. For the time being, suppose $|\lambda| = 1$. Then, we have $\mathrm{Re}(\lambda) = \cos(\varphi)$ and $\mathrm{Im}(\lambda) = \sin(\varphi)$ with $|\varphi| \in [0, \pi - \theta)$. Set

$$\alpha \coloneqq \frac{1 - \cos(\pi - \theta)}{\sin(\pi - \theta)} \geq \frac{1 - \cos(|\varphi|)}{\sin(|\varphi|)}, \tag{4.9}$$

where the former estimate a consequence of the fact that tan is strictly increasing on $(0, \pi/2)$ and the identity

$$\frac{1-\cos(x)}{\sin(x)} = \frac{1-\cos^2(x/2)+\sin^2(x/2)}{2\cos(x/2)\sin(x/2)} = \tan(x/2)$$

which is derived using well-known trigonometric identities for $\sin(2x)$ and $\cos(2x)$.

If $\varphi = |\varphi|$, we use (4.9) and derive the inequality

$$\mathrm{Re}(\lambda) + \alpha\,|\,\mathrm{Im}(\lambda)\,| = \cos(\varphi) + \alpha\sin(\varphi) \geq 1.$$

Conversely, if $\varphi = -|\varphi|$, then we have by the symmetry properties of sin and cos that

$$\mathrm{Re}(\lambda) + \alpha\,|\,\mathrm{Im}(\lambda)\,| = \cos(-\varphi) + \alpha\sin(-\varphi) \geq 1.$$

For arbitrary λ, the claim follows by considering the normalized value $(\lambda/|\lambda|)$. □

The next lemma enables us to handle the solid integrals in (4.3) and (4.4).

Lemma 4.5. *Under the same assumptions on (u, ϕ) and λ as in Theorem 4.1, we have*

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x + |\lambda| \int_\Omega |u|^2 \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_\partial \|u\|_\partial, \tag{4.10}$$

where $C > 0$ depends only on θ.

Proof. In a first step, we show that by approximating Ω through a sequence of smooth domains Ω_j, $j \in \mathbb{N}$, with uniformly bounded Lipschitz characters it suffices to prove inequality (4.10) for elements of this sequence. To this end, suppose that we have shown

$$\int_{\Omega_j} |\nabla u|^2\,\mathrm{d}x + |\lambda| \int_{\Omega_j} |u|^2 \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_{\mathrm{L}^2(\partial\Omega_j)} \|u\|_{\mathrm{L}^2(\partial\Omega_j)},$$

with a constant $C > 0$ which only depends on the Lipschitz character of Ω. By the monotone convergence theorem, it is clear that the left-hand side of the inequality above converges to the left-hand side of (4.10) for $j \to \infty$. The integrals on the right-hand side converge to their counterparts on $\partial\Omega$ through an application of the dominated convergence theorem: The conormal derivative may, modulo a constant that depends on the dimension d, be dominated by the nontangential maximal function $(\nabla u)^*$ and $(\phi)^*$ which are both square integrable by assumption. Similarly, u may be dominated by its nontangential maximal function which is square integrable thanks to Remark 4.2.

Thus, for the reminder of this proof, we assume that (u, ϕ) is a smooth solution to the Stokes resolvent problem on $\overline{\Omega}$ with smooth boundary $\partial\Omega$. Multiplying the Stokes equation by $\overline{u}$ and integrating over Ω gives

$$\int_\Omega -\Delta u \cdot \overline{u}\,\mathrm{d}x + \lambda \int_\Omega u \cdot \overline{u}\,\mathrm{d}x = -\int_\Omega \nabla\phi \cdot \overline{u}\,\mathrm{d}x. \tag{4.11}$$

Rewriting the first term of equation (4.11) and using the product rule leads to

$$-\int_\Omega \frac{\partial^2 u_j}{\partial x_i \partial x_i}\,\overline{u}_j\,\mathrm{d}x = -\int_\Omega \frac{\partial}{\partial x_i}\left\{\overline{u}_j \frac{\partial u_j}{\partial x_i}\right\}\mathrm{d}x + \int_\Omega \frac{\partial u_j}{\partial x_i}\cdot\frac{\partial \overline{u}_j}{\partial x_i}\,\mathrm{d}x.$$

Note that since u is solenoidal, we have for the third term of equation (4.11)

$$-\int_\Omega \frac{\partial \phi}{\partial x_i}\,\overline{u}_i\,\mathrm{d}x = -\int_\Omega \frac{\partial}{\partial x_i}\left\{\phi\,\overline{u}_i\right\}\mathrm{d}x.$$

Now we want to transform the first and third of the above solid integrals into boundary integrals through an application of the standard divergence theorem which is applicable as $\partial\Omega$ is smooth and all involved functions are smooth on $\overline{\Omega}$. This allows to transform equation (4.11) into

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial n}\cdot\overline{u}\,\mathrm{d}\sigma + \lambda\int_\Omega |u|^2\,\mathrm{d}x = -\int_{\partial\Omega} \phi\, n\cdot\overline{u}\;\mathrm{d}\sigma.$$

We rearrange the terms of this identity and use the definition of the conormal derivative, see equation (3.17), to derive

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x + \lambda\int_\Omega |u|^2\,\mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\cdot\overline{u}\;\mathrm{d}\sigma. \tag{4.12}$$

If we now take the real and imaginary part of (4.12) and sum them up with the prefactor $\alpha(\theta) > 0$ from Lemma 4.4, we get

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x + \left\{\mathrm{Re}(\lambda) + \alpha\left|\mathrm{Im}(\lambda)\right|\right\}\int_\Omega |u|^2\,\mathrm{d}x \le (1+\alpha)\left|\int_{\partial\Omega}\frac{\partial u}{\partial \nu}\cdot\bar{u}\,\mathrm{d}\sigma\right|.$$

Lemma 4.4 now gives

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x + |\lambda|\int_\Omega |u|^2\,\mathrm{d}x \le C\left|\int_{\partial\Omega}\frac{\partial u}{\partial \nu}\cdot\bar{u}\,\mathrm{d}\sigma\right|,$$

with $C = (1+\alpha)$ from which we readily derive estimate (4.10) after applying the Cauchy-Schwarz inequality. □

One useful tool in this chapter and the next one, will be the following lemma which enables us to control a function on Ω by its nontangential maximal function. A proof of this Lemma was carried out by Wei and Zhang [42, Lem. 3.3] and can also be found in Shen's paper [32, p. 418f.] although it was not formulated as a separate result. Note furthermore that this result, despite the fact that it was originally proven for dimensions $d \ge 3$, holds also in the case $d = 2$.

Lemma 4.6. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and $q > 0$. For all functions $\varphi \colon \Omega \to \mathbb{C}$ the estimate*

$$\left(\int_\Omega |\varphi|^{qd/(d-1)} \, \mathrm{d}x \right)^{(d-1)/d} \leq C \int_{\partial\Omega} |(\varphi)^*|^q \, \mathrm{d}x \tag{4.13}$$

holds for a constant $C > 0$ depending only on d and the Lipschitz constant of Ω.

For the next lemma and also later parts of this chapter, we state the following theorem about the Dirichlet problem, the regularity problem and the Neumann problem for the Laplacian on bounded Lipschitz domains:

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. Then the following statements hold:*

a) Given $f \in \mathrm{L}^2(\partial\Omega)$, there exists a unique harmonic function ψ with $(\psi)^* \in \mathrm{L}^2(\partial\Omega)$ and ψ converges nontangentially to f a.e. Furthermore, the estimate

$$\|(\psi)^*\|_\partial \leq C \, \|f\|_\partial.$$

holds with a constant only depending on the Lipschitz character of Ω.

b) Given $f \in \mathrm{H}^1(\partial\Omega)$, there exists a unique harmonic function ψ with $(\nabla\psi)^* \in \mathrm{L}^2(\partial\Omega)$ and ψ converges nontangentially to f a.e. Furthermore, the estimate

$$\|(\nabla\psi)^*\|_\partial \leq C \, \|f\|_{\mathrm{H}^1(\partial\Omega)}$$

holds with a constant only depending on the Lipschitz character of Ω.

c) Given $f \in \mathrm{L}^2(\partial\Omega)$ with $\int_{\partial\Omega} f \, \mathrm{d}\sigma = 0$, there exists a harmonic function ψ on Ω with $\frac{\partial\psi}{\partial n} = f$ a.e. Furthermore, the estimate

$$\|\psi\|_{\mathrm{H}^1(\partial\Omega)} \leq C \, \|f\|_\partial$$

holds with $C > 0$ only depending on the Lipschitz character of Ω.

Proof. According to Kenig, for the differential operator Δ, the *Dirichlet problem*, $(D)_2$ is solvable for data in $\mathrm{L}^2(\partial\Omega)$, see [22, Thm. 2.1.5]. Checking the definition of $(D)_2$, see [22, Defn. 1.7.4], one sees that the desired statement a) is a consequence of Theorem 1.7.7 in [22]. We also refer to the work of Dahlberg [4] on the Dirichlet problem.

The *regularity problem*, $(R)_2$, is solvable for data $f \in \mathrm{H}^1(\partial\Omega)$, see [22, Thm. 2.1.10]. The definition of $(R)_2$, see [22, Defn. 1.7.4], reveals that the claimed statement b) is a consequence of Theorem 1.8.2 in [22]. We refer also to the work of Jerison and Kenig on the L^2 regularity problem [18].

The *Neumann problem*, $(N)_2$, is solvable for data $f \in \mathrm{L}^2(\partial\Omega)$ due to Theorem 2.1.10 in [22]. Definition 1.7.9 in [22] shows that the stated properties in c) follow from Theorem 1.8.3 in [22]. See also Jerison's and Kenig's work on the Neumann problem [19]. □

Our next lemma combines Rellich identities (4.3) and (4.4) with estimate (4.10).

Lemma 4.8. *Under the same assumptions on* (u, ϕ) *and* λ *as in Theorem 4.1, we have*

$$\|\nabla u\|_\partial \le C_\varepsilon \left\| \frac{\partial u}{\partial \nu} \right\|_\partial + \varepsilon \left\{ \|\nabla u\|_\partial + \|\phi\|_\partial + |\lambda|^{1/2} \|u\|_\partial \right\} \tag{4.14}$$

and

$$\|\nabla u\|_\partial \le C_\varepsilon \left\{ \left\|\nabla_{\tan} u\right\|_\partial + |\lambda|^{1/2} \|u\|_\partial \right\} + \varepsilon \left\{ \|\nabla u\|_\partial + \|\phi\|_\partial \right\} \tag{4.15}$$

for all $\varepsilon \in (0, 1)$*, where* $C_\varepsilon > 0$ *depends only on* d*,* θ*,* τ*,* ε *and the Lipschitz character of* Ω*.*

Proof. Let $h = (h_1, \dots, h_d) \in \mathrm{C}^1_0(\mathbb{R}^d; \mathbb{R}^d)$ with $h_k n_k \ge c > 0$ on $\partial\Omega$ as given by Theorem 1.3 v). The idea of the proof of the desired estimates (4.14) and (4.15) is to first use the Rellich identities from Lemma 4.3 with this particular h to estimate $\|\nabla u\|_\partial$ and then to bound the resulting right-hand side by providing individual estimates.

Before we start, note that we have $\Delta\phi = 0$ on the one hand and for the nontangential maximal function $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$ on the other hand. Now we apply Hölder's inequality and Lemma 4.6 to bound the L^2 norm of ϕ via

$$\int_\Omega |\phi|^2 \,\mathrm{d}x \le C \left(\int_\Omega |\phi|^{2d/(d-1)} \,\mathrm{d}x \right)^{(d-1)/d} \le C \,\|(\phi)^*\|_\partial^2,$$

where $C > 0$ depends only d and the Lipschitz constant of Ω. Then, we use Theorem 4.7 on the L^2 Dirichlet problem of the Laplacian to bound $\|(\phi)^*\|_\partial$ by $\|\phi\|_\partial$ which gives

$$\int_\Omega |\phi|^2 \,\mathrm{d}x \le C \,\|\phi\|_\partial^2. \tag{4.16}$$

with $C > 0$ only depending on d and the Lipschitz character of Ω.

We will now prove the first estimate (4.14). In view of identity (4.3), we have

$$\begin{aligned} \|\nabla u\|_\partial^2 \le C \Bigg\{ & \|\nabla u\|_\partial \left\|\frac{\partial u}{\partial \nu}\right\|_\partial + \int_\Omega |\nabla u|^2 \,\mathrm{d}x \\ & + \int_\Omega |\nabla u|\, |\phi| \,\mathrm{d}x + |\lambda| \int_\Omega |\nabla u|\, |u| \,\mathrm{d}x \Bigg\}, \end{aligned} \tag{4.17}$$

where the first term follows from the Cauchy-Schwarz inequality and $C > 0$ only depends on d and the Lipschitz character of Ω.

For now, we keep the first term of (4.17) as it is, the second term can be handled via Lemma 4.5. The goal for the remaining two integrals will be to bound each of them by a product of norms $\|\cdot\|_\partial$. To this end, for the third integral we calculate

$$\int_\Omega |\nabla u||\phi|\,\mathrm{d}x \le \left(\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |\phi|^2\,\mathrm{d}x\right)^{1/2} \le C\left\|\frac{\partial u}{\partial \nu}\right\|_\partial^{1/2}\|u\|_\partial^{1/2}\,\|\phi\|_\partial, \tag{4.18}$$

where the first step is due to the Cauchy-Schwarz inequality and the second step combines estimate (4.10) with estimate (4.16).

The last integral of (4.17) can be estimated as follows:

$$|\lambda|\int_\Omega |\nabla u|\,|u|\,\mathrm{d}x \le \frac{|\lambda|^{3/2}}{2}\int_\Omega |u|^2\,\mathrm{d}x + \frac{|\lambda|^{1/2}}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x \le C\left\|\frac{\partial u}{\partial \nu}\right\|_\partial |\lambda|^{1/2}\|u\|_\partial, \tag{4.19}$$

where in the first step we used the weighted Young inequality and in the second step we applied estimate (4.10). Putting everything together, we calculate

$$\|\nabla u\|_\partial^2 \le C\left\{\|\nabla u\|_\partial\left\|\frac{\partial u}{\partial \nu}\right\|_\partial + \left\|\frac{\partial u}{\partial \nu}\right\|_\partial\|u\|_\partial + \left\|\frac{\partial u}{\partial \nu}\right\|_\partial^{1/2}\|u\|_\partial^{1/2}\|\phi\|_\partial + \left\|\frac{\partial u}{\partial \nu}\right\|_\partial|\lambda|^{1/2}\|u\|_\partial\right\}.$$

If we now use the assumption $|\lambda| \ge \tau$ which allows us to bound $\|u\|_\partial$ via

$$\|u\|_\partial \le \frac{|\lambda|^{1/2}}{\tau^{1/2}}\|u\|_\partial = C\,|\lambda|^{1/2}\|u\|_\partial,$$

the desired estimate (4.14) now follows applying Young's weighted inequality with an ε and the norm equivalence on finite dimensional vector spaces. Note that for the product of three norms from inequality (4.18) we need to apply the Young inequality twice:

$$\begin{aligned}\left\|\frac{\partial u}{\partial \nu}\right\|_\partial^{1/2}\|u\|_\partial^{1/2}\|\phi\|_\partial &\le \left\{\frac{1}{4\varepsilon}\left\|\frac{\partial u}{\partial \nu}\right\|_\partial + \varepsilon\,\|u\|_\partial\right\}\|\phi\|_\partial \\ &\le \frac{1}{32\,\varepsilon^3}\left\|\frac{\partial u}{\partial \nu}\right\|_\partial^2 + \frac{\varepsilon}{2}\|\phi\|_\partial^2 + \frac{\varepsilon}{2}\|u\|_\partial^2 + \frac{\varepsilon}{2}\|\phi\|_\partial^2 \\ &\le C_\varepsilon\left\|\frac{\partial u}{\partial \nu}\right\|_\partial^2 + \varepsilon\left\{\|u\|_\partial^2 + \|\phi\|_\partial^2\right\}.\end{aligned}$$

For inequality (4.15), we use the Rellich identity (4.4) and the relation (4.7) to obtain the estimate

$$\begin{aligned}\|\nabla u\|_\partial^2 \le C\Big\{&\left\|\nabla_{\tan} u\right\|_\partial\left\{\|\nabla u\|_\partial + \|\phi\|_\partial\right\} \\ &+ \int_\Omega |\nabla u|^2\,\mathrm{d}x + \int_\Omega |\nabla u||\phi|\,\mathrm{d}x + |\lambda|\int_\Omega |\nabla u||u|\,\mathrm{d}x\Big\},\end{aligned} \tag{4.20}$$

where $C > 0$ only depends on d and the Lipschitz character of Ω. As before, we estimate the three terms on the right side of (4.20) using (4.10), (4.18) and (4.19), respectively, and obtain the estimate

$$\|\nabla u\|_\partial^2 \leq C \Big\{ \big\|\nabla_{\tan} u\big\|_\partial \Big\{ \|\nabla u\|_\partial + \|\phi\|_\partial \Big\} + \Big\| \frac{\partial u}{\partial \nu} \Big\|_\partial \|u\|_\partial + \Big\| \frac{\partial u}{\partial \nu} \Big\|_\partial^{1/2} \|u\|_\partial^{1/2} \|\phi\|_\partial + \Big\| \frac{\partial u}{\partial \nu} \Big\|_\partial |\lambda|^{1/2} \big\|u\big\|_\partial \Big\}.$$

If we now use the Young inequality with an ε, we get

$$\|\nabla u\|_\partial^2 \leq C_\varepsilon \Big\{ \|\nabla_{\tan} u\|_\partial^2 + |\lambda| \, \|u\|_\partial^2 \Big\} + \varepsilon \Big\{ \|\nabla u\|_\partial^2 + \|\phi\|_\partial^2 + \frac{1}{4} \Big\| \frac{\partial u}{\partial \nu} \Big\|_\partial^2 \Big\}.$$

The claim now follows if we use the definition of the conormal derivative and the norm equivalence on finite dimensional vector spaces. □

We prove one last lemma before we tackle the central theorem of this chapter. The following lemma will not depend on the lemmata which were proven in the preceding part of this chapter, as the approach to derive the desired boundary estimates will be different: We will not rely upon the Rellich identities from Lemma 4.3 but directly part from a variational formulation of the Stokes resolvent problem on the boundary. Furthermore, we will rely on properties of solutions to the regularity problem and the Neumann problem for the Laplacian on bounded Lipschitz domains, see Theorem 4.7.

Lemma 4.9. *Assume that (u, ϕ) satisfies the same conditions as in Theorem 4.1. Then,*

$$\Big\| \phi - \fint_{\partial\Omega} \phi \, \mathrm{d}\sigma \Big\|_\partial \leq C \Big\{ \|\nabla u\|_\partial + |\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \Big\} \tag{4.21}$$

and

$$|\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C \Big\{ \|\phi\|_\partial + \|\nabla u\|_\partial \Big\}, \tag{4.22}$$

where $C > 0$ depends only on d and the Lipschitz character of Ω.

Proof. Our first goal will be to show that without loss of generality we may assume that $\Delta u = \nabla \phi + \lambda u$ on $\partial\Omega$. The central player will once again be Theorem 1.3. To this end, let (Ω_j) be a sequence of approximating C^∞ domains as in Theorem 1.3 and suppose that inequality (4.21) holds for all Ω_j, i.e., we have proven the inequality

$$\Big\| \phi - \fint_{\partial\Omega} \phi \, \mathrm{d}\sigma \Big\|_{\mathrm{L}^2(\partial\Omega_j)} \leq C \Big\{ \|\nabla u\|_{\mathrm{L}^2(\partial\Omega_j;\, \mathbb{C}^{d\times d})} + |\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega_j)} \Big\}.$$

Analogously, suppose that an inequality similar to (4.22) holds. It is now crucial that the constants in these inequalities only depend on the Lipschitz character of Ω and not on other geometric properties of the domain. Thus, C does not depend on j and we may take the limit $j \to \infty$ using dominated convergence for the terms that do not involve the operator norms $\|\cdot\|_{\mathrm{H}^{-1}(\partial\Omega_j)}$. For these terms, we deploy another strategy. Let $f \in \mathrm{H}^1(\partial\Omega_j)$ and let ψ solve the regularity problem for the Laplacian on $\tilde{\Omega} := \mathrm{B}_R \setminus \overline{\Omega_j}$ with boundary data $\tilde{f}$, where $\mathrm{B}_R \supseteq \overline{\Omega}$ is a sufficiently large ball and $\tilde{f}$ is defined via

$$\tilde{f} \colon \partial\mathrm{B}_R \cup \partial\Omega_j, \quad x \mapsto \tilde{f}(x) := \begin{cases} f(x), & \text{if } x \in \partial\Omega_j, \\ 0, & \text{if } x \in \partial\mathrm{B}_R. \end{cases}$$

See Theorem 4.7 b) for the results on the regularity problem. Suppose that j is chosen large enough such that every $q \in \partial\Omega$ is part of the outer cone $\Gamma_{\mathrm{ext}}(\Lambda_j(q))$, where the nontangential approach regions are defined with respect to the domain $\tilde{\Omega}$. We calculate using the divergence theorem

$$\begin{aligned}\Big| \int_{\partial\Omega_j} u \cdot (-n_j)\overline{f} \, \mathrm{d}\sigma \Big| &= \Big| \int_{\Omega\setminus\Omega_j} \operatorname{div}(u \cdot \overline{\psi}) \, \mathrm{d}y - \int_{\partial\Omega} u \cdot n \overline{\psi} \, \mathrm{d}\sigma \Big| \\ &\leq C\, \|u\|_{\mathrm{L}^2(\Omega\setminus\Omega_j)} \|f\|_{\mathrm{H}^1(\partial\Omega_j)} + \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \|\psi\|_{\mathrm{H}^1(\partial\Omega)}, \end{aligned} \tag{4.23}$$

where $C > 0$ depends on d. Now we have to bound the norm $\|\psi\|_{\mathrm{H}^1(\partial\Omega)}$ by $\|f\|_{\mathrm{H}^1(\partial\Omega_j)}$. To this end, we estimate

$$\begin{aligned}\int_{\partial\Omega} |\psi(p)|^2 \, \mathrm{d}\sigma(p) &\leq \int_{\partial\Omega} |(\psi(\Lambda_j(p)))^*|^2 \, \mathrm{d}\sigma(p) \\ &\leq \delta^{-1} \int_{\partial\Omega} |(\psi)^*(\Lambda_j(p))|^2 \vartheta(p) \, \mathrm{d}\sigma(p) \\ &= \delta^{-1} \int_{\partial\Omega_j} |(\psi)^*(p)|^2 \, \mathrm{d}\sigma(p),\end{aligned}$$

where $\delta > 0$ is the bound from Theorem 1.3 iv). A similar calculation works for the tangential derivatives of ψ. This shows that

$$\begin{aligned}&\|\psi\|_{\mathrm{L}^2(\partial\Omega)} + \|\nabla_{\tan}\psi\|_{\mathrm{L}^2(\partial\Omega)} \\ &\quad \leq C \Big\{ \|(\psi)^*\|_{\mathrm{L}^2(\partial\Omega_j)} + \|(\nabla_{\tan}\psi)^*\|_{\mathrm{L}^2(\partial\Omega_j)} \Big\} \leq C\, \|f\|_{\mathrm{H}^1(\partial\Omega_j)},\end{aligned} \tag{4.24}$$

where for the last inequality we used further results for the regularity problem of the Dirichlet Laplacian, see Mitrea [25, Thm. 5.1] and Mitrea [26, p. 91] for the two dimensional case. Note that the cited results hold for bounded Lipschitz domains with arbitrary topology. Inequality (4.24) is now used to transform inequality (4.23) into the estimate

$$\|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega_j)} \leq C \Big\{ \|u\|_{\mathrm{L}^2(\Omega\setminus\Omega_j)} + \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \Big\},$$

with a constant that only depends on d, $\operatorname{diam}(\Omega)$ and the Lipschitz character of Ω. This shows that

$$\limsup_{j\to\infty} \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega_j)} \leq C\, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}.$$

In a similar fashion one may prove that the estimate

$$\|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C \liminf_{j\to\infty} \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega_j)}$$

holds.

By virtue of the previous discussion, we see that it is sufficient to prove inequalities (4.21) and (4.22) for smooth subdomains. For the rest of the proof we will thus assume that (u, ϕ) satisfies the Stokes resolvent problem in a smooth domain Ω' for some $\overline{\Omega} \subseteq \Omega'$. In particular we have $\Delta u = \nabla\phi + \lambda u$ on $\partial\Omega$. Multiplying this identity on $\partial\Omega$ with the outer normal vector n and using the triangle inequality gives the following set of estimates:

$$\begin{aligned} \|\nabla\phi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} &\leq \|\Delta u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + |\lambda|\, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}, \\ |\lambda|\, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} &\leq \|\Delta u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\nabla\phi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}. \end{aligned} \tag{4.25}$$

This looks almost like the desired pair of inequalities. We will now show that on the one hand the estimate

$$\|\Delta u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C\, \|\nabla u\|_{\partial} \tag{4.26}$$

and on the other hand the estimate

$$c \left\| \phi - \fint_{\partial\Omega} \phi \,\mathrm{d}\sigma \right\|_{\partial} \leq \|\nabla\phi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C\, \|\phi\|_{\partial} \tag{4.27}$$

holds for constants $c, C > 0$ that only depend on d and the Lipschitz character of Ω. Using these two estimates applied to the respective terms of (4.25), we can directly verify (4.21) and (4.22).

In order to prove (4.26), we note the following identity of differential operators

$$\Delta u \cdot n = n_i \frac{\partial^2 u_i}{\partial x_j^2} = \Big\{ n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i} \Big\} \frac{\partial u_i}{\partial x_j},$$

where we used the fact that $\operatorname{div} u = 0$ in $\overline{\Omega}$. The expression between the brackets is the first-order tangential derivative $\partial_{\tau_{ij}}$, see (4.5). We can thus calculate that for $f \in \mathrm{H}^1(\partial\Omega)$ the estimate

$$\Big| \Big\langle \Delta u \cdot n, f \Big\rangle \Big| = \Big| \Big\langle \partial_{\tau_{ij}} \Big\{ \frac{\partial u}{\partial x_j} \Big\}, f \Big\rangle \Big| = \Big| \Big\langle \frac{\partial u}{\partial x_j}, \partial_{\tau_{ji}} f \Big\rangle \Big| \leq C\, \|\nabla u\|_{\partial} \|\nabla_{\tan} f\|_{\partial}$$

holds, where we applied the Cauchy-Schwarz inequality and used relation (4.7) to compare the first-order tangential derivative to the tangential gradient. Identifying $\Delta u \cdot n$ with an element of $\mathrm{H}^{-1}(\partial\Omega)$, the estimate (4.26) follows.

For the proof of estimate (4.27), we will use L^2 estimates for the Neumann and regularity problems for the Laplace equation in Lipschitz domains. Jerison and Kenig showed that for $g \in \mathrm{L}^2(\partial\Omega)$ with mean value zero the *Neumann problem* for Laplace's equation on the Lipschitz domain Ω has a unique solution ψ with $(\nabla\psi)^* \in \mathrm{L}^2(\partial\Omega)$, $\frac{\partial\psi}{\partial n} = g$ a.e. on $\partial\Omega$ and the solution fulfills the estimate $\|\psi\|_{\mathrm{H}^1(\partial\Omega)} \le C\|g\|_\partial$, see Theorem 4.7. By the divergence theorem, which is applicable since $\partial\Omega$ is assumed to be smooth, we see that

$$\int_{\partial\Omega} \phi\, \frac{\partial\psi}{\partial x_i} n_i \,\mathrm{d}\sigma = \int_\Omega \frac{\partial}{\partial x_i}\left\{\phi \frac{\partial\psi}{\partial x_i}\right\} dx = \int_\Omega \frac{\partial\phi}{\partial x_i} \cdot \frac{\partial\psi}{\partial x_i}\,\mathrm{d}x = \int_{\partial\Omega} \psi\, \frac{\partial\phi}{\partial x_i} n_i \,\mathrm{d}\sigma.$$

We can then use this identity and the estimate of ψ against the data g to derive

$$\begin{aligned} \left| \int_{\partial\Omega} \phi\, g \,\mathrm{d}\sigma \right| = \left| \int_{\partial\Omega} \frac{\partial\phi}{\partial n}\, \psi \,\mathrm{d}\sigma \right| &\le \left\| \frac{\partial\phi}{\partial n} \right\|_{\mathrm{H}^{-1}(\partial\Omega)} \|\psi\|_{\mathrm{H}^1(\partial\Omega)} \\ &\le C \left\| \frac{\partial\phi}{\partial n} \right\|_{\mathrm{H}^{-1}(\partial\Omega)} \|g\|_\partial . \end{aligned} \tag{4.28}$$

Now, if we set $\overline{g} = \phi - \tilde{\phi}$, with $\tilde{\phi} := \fint_{\partial\Omega} \phi \,\mathrm{d}\sigma$, we arrive at the following estimate:

$$\|\phi - \tilde{\phi}\|_\partial^2 = \int_{\partial\Omega} (\phi - \tilde{\phi})\, \overline{(\phi - \tilde{\phi})} \,\mathrm{d}\sigma = \int_{\partial\Omega} \phi\, \overline{(\phi - \tilde{\phi})} \,\mathrm{d}\sigma \le C \left\| \frac{\partial\phi}{\partial n} \right\|_{\mathrm{H}^{-1}(\partial\Omega)} \|\phi - \tilde{\phi}\|_\partial ,$$

where in the last step we used (4.28). This together with Lemma 3.1 proves the left side of inequality (4.27).

For the right side of inequality (4.27), we work in a similar way. We will use results for the *regularity problem* of Laplace's equation by Jerison and Kenig, see Theorem 4.7: Given $f \in \mathrm{H}^1(\partial\Omega)$, there exists a harmonic function ψ in Ω such that $(\nabla\psi)^* \in \mathrm{L}^2(\partial\Omega)$ and $\psi = f$ on $\partial\Omega$ nontangentially. Furthermore, the estimate $\|\nabla\psi\|_\partial \le C\,\|f\|_{\mathrm{H}^1(\partial\Omega)}$ holds. As for (4.28), we calculate

$$\left| \int_{\partial\Omega} \frac{\partial\phi}{\partial n} f \,\mathrm{d}\sigma \right| = \left| \int_{\partial\Omega} \phi\, \frac{\partial\psi}{\partial n} \,\mathrm{d}\sigma \right| \le \|\phi\|_\partial \|\nabla\psi\|_\partial \le C\,\|\phi\|_\partial \|f\|_{\mathrm{H}^1(\partial\Omega)} ,$$

Interpreting the function $\frac{\partial\phi}{\partial n} \in \mathrm{L}^2(\partial\Omega)$ as a functional on $\mathrm{H}^1(\partial\Omega)$, we obtain

$$\left\| \frac{\partial\phi}{\partial n} \right\|_{\mathrm{H}^{-1}(\partial\Omega)} \le C\,\|\phi\|_\partial .$$

This finishes the proof of inequality (4.27). □

Remark 4.10. Throughout the proof of Lemma 4.9, we derived several estimates under the assumptions of Theorem 4.1. Some estimates prove to hold under more general conditions if we study them closely. In particular, we will not have to assume that (u, ϕ) solves the Stokes resolvent system (2.27).

A careful look at the proof of inequality (4.26) reveals that the estimate

$$\|\Delta u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C\,\|\nabla u\|_{\partial}$$

holds for *all* smooth divergence free vector fields u such that $(\nabla u)^* \in \mathrm{L}^2(\partial\Omega)$ and ∇u has a nontangential limit.

Additionally, examining the proof of inequality (4.27) shows us that the estimate

$$c\,\|\phi\|_{\partial} \leq \|\nabla\phi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)},$$

holds for *all* harmonic functions ϕ that fulfill $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$ and have a nontangential limit with vanishing mean on $\partial\Omega$.

After all this preparation, we have acquainted enough tools and are now able to prove Theorem 4.1 about the Rellich estimates on solutions of the Stokes resolvent problem (2.27).

Proof of Theorem 4.1. For the proof of estimate (4.1), we can assume without loss of generality that $\int_{\partial\Omega} \phi \,\mathrm{d}\sigma = 0$.

We start by proving estimate (4.1). Using (4.21) to bound the second summand in (4.1) and then (4.15) for ∇u, we get

$$\begin{aligned}\|\nabla u\|_{\partial} + \|\phi\|_{\partial} &\leq C\left\{\|\nabla u\|_{\partial} + |\lambda|\,\|u \cdot n\|_{\mathrm{H}^1(\partial\Omega)}\right\}\\ &\leq C_\varepsilon\left\{\left\|\nabla_{\tan} u\right\|_{\partial} + |\lambda|^{1/2}\|u\|_{\partial} + |\lambda|\,\|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}\right\} + C\,\varepsilon\left\{\|\nabla u\|_{\partial} + \|\phi\|_{\partial}\right\}\end{aligned}$$

for all $\varepsilon \in (0,1)$. Choosing ε such that $C\,\varepsilon < (1/2)$, we can rearrange the above inequality and obtain estimate (4.1).

Estimate (4.2) will need more effort to be proven. In order to obtain the desired estimate, we will divide the left-hand side of inequality (4.2) into two groups as the following line suggests and then bound both groups separately

$$\left\{\|\nabla u\|_{\partial} + |\lambda|\,\|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\phi\|_{\partial}\right\} + |\lambda|^{1/2}\|u\|_{\partial} =: G_1 + G_2.$$

We start with inequality (4.22) and derive

$$G_1 \leq C\left\{\|\nabla u\|_{\partial} + \|\phi\|_{\partial}\right\} \leq C\left\{\left\|\frac{\partial u}{\|\nabla u\|_{\partial} + \partial\nu}\right\|_{\partial}\right\},$$

where in the last step we used the definition of the conormal derivative to bound the pressure term. If we now apply (4.14), we get

$$G_1 \le C_\varepsilon \left\| \frac{\partial u}{\partial \nu} \right\|_\partial + \varepsilon \left\{ \|\nabla u\|_\partial + \|\phi\|_\partial + |\lambda|^{1/2}\|u\|_\partial \right\}$$

for all $\varepsilon \in (0,1)$. Choosing ε appropriately yields

$$\|\nabla u\|_\partial + |\lambda|\, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\phi\|_\partial \le C \left\{ \left\| \frac{\partial u}{\partial \nu} \right\|_\partial + |\lambda|^{1/2}\|u\|_\partial \right\}, \tag{4.29}$$

and the first group has been successfully bounded.

Now we need to estimate G_2. For this, we will work with the next identity which is a consequence of the divergence theorem:

$$\begin{aligned} \int_{\partial\Omega} h_k n_k |u|^2 \, \mathrm{d}\sigma &= \int_\Omega \frac{\partial h_k}{\partial x_k} |u|^2 \, \mathrm{d}x + \int_\Omega h_k \frac{\partial |u|^2}{\partial x_k} \, \mathrm{d}x \\ &= \int_\Omega \operatorname{div}(h)\, |u|^2 \, \mathrm{d}x + 2 \operatorname{Re} \int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k} u_i \, \mathrm{d}x, \end{aligned} \tag{4.30}$$

where $h \in \mathrm{C}_0^1(\mathbb{R}^d; \mathbb{R}^d)$. This calculation is valid since Remark 4.2 assures the existence of nontangential limits of u and furthermore gives $(u)^* \in \mathrm{L}^2(\partial\Omega)$. Thus, $(h_k n_k |u|^2)$ can be dominated by the integrable function $(\|h\|_\infty |(u)^*|^2)$ and the claim follows from Proposition 1.8.

Next, for identity (4.30), we choose $h \in \mathrm{C}_0^1(\mathbb{R}^d; \mathbb{R}^d)$ with $h_k n_k \ge c > 0$ on $\partial\Omega$, which is possible due to Theorem 1.3. Then we take absolute values and estimate

$$\|u\|_\partial^2 \le C \left\{ \int_\Omega |u|^2 \, \mathrm{d}x + \int_\Omega |u||\nabla u| \, \mathrm{d}x \right\}. \tag{4.31}$$

Multiplying (4.31) with $|\lambda|$ and using (4.10) three times, once directly and twice after an application of the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |\lambda|\, \|u\|_\partial^2 &\le C \left\{ |\lambda| \int_\Omega |u|^2 \, \mathrm{d}x + |\lambda|^{1/2} \int_\Omega \left(|\lambda|^{1/2}|u| \right) |\nabla u| \, \mathrm{d}x \right\} \\ &\le C \left\{ \left\| \frac{\partial u}{\partial \nu} \right\|_\partial \|u\|_\partial + |\lambda|^{1/2} \left(|\lambda| \int_\Omega |u|^2 \right)^{1/2} \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x \right)^{1/2} \right\} \\ &\le C \left\| \frac{\partial u}{\partial \nu} \right\|_\partial |\lambda|^{1/2} \|u\|_\partial . \end{aligned}$$

Note that for the last estimate we also used the fact that $|\lambda| \ge \tau$ helps us to bound $\|u\|_\partial$ by $C\,|\lambda|^{1/2}\|u\|_\partial$. Rearranging terms in the last estimate, we now derive

$$G_2 = |\lambda|^{1/2}\|u\|_\partial \le C \left\| \frac{\partial u}{\partial \nu} \right\|_\partial . \tag{4.32}$$

This bounds the second group and concludes our proof. □

Shen proved that, under reasonable assumptions, a theorem similar to Theorem 4.1 holds for exterior domains, see [32, Thm. 4.6].

Theorem 4.11. *Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0,1)$. Let (u, ϕ) be a solution of the Stokes resolvent problem in $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$. Suppose additionally that $(\nabla u)^*$, $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$ and that ∇u, ϕ have nontangential limits almost everywhere on $\partial\Omega$. Furthermore, let for $|x| \to \infty$*

$$|\phi(x)| + |\nabla u(x)| = O(|x|^{1-d}) \quad \textit{and} \quad u(x) = \begin{cases} O(|x|^{2-d}) & \textit{if } d \geq 3\,, \\ o(1) & \textit{if } d = 2\,. \end{cases}$$

Then, the estimates

$$\|\nabla u\|_\partial + \|\phi\|_\partial \leq C \left\{ \|\nabla_{\tan} u\|_\partial + |\lambda|^{1/2} \|u\|_\partial + |\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \right\} \tag{4.33}$$

and

$$\|\nabla u\|_\partial + |\lambda|^{1/2} \|u\|_\partial + |\lambda| \, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\phi\|_\partial \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_\partial \tag{4.34}$$

hold, where $C > 0$ depends only on d, τ, θ and the Lipschitz character of Ω.

Chapter 5

Solving the L^2 Dirichlet Problem

This section is all about the application of the method of layer potentials to solve the L^2 Dirichlet problem (Dir_λ) for the Stokes resolvent system. The results from Chapter 3 on the jump relations and the corresponding operators, see Theorem 3.9 and Theorem 3.11, together with the Rellich estimates from Chapter 4 will be the essential ingredients.

In the first part of this chapter, we will show that the operators $\pm(1/2)I + \mathcal{K}_\lambda$ and their adjoints are isomorphisms on suitable Hilbert spaces. Then, in the second part of this chapter, we will build on this information to construct solutions to (Dir_λ) via double layer potentials.

For the remainder of this chapter, let Ω always denote a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with connected boundary. We will use $\mathrm{L}^2_n(\partial\Omega)$ to denote the function space

$$\mathrm{L}^2_n(\partial\Omega) := \Big\{ f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d) \colon \int_{\partial\Omega} f \cdot n \,\mathrm{d}\sigma = 0 \Big\},$$

and $\mathrm{L}^2_0(\partial\Omega; \mathbb{C}^d)$ to denote the function space of L^2 functions with mean value zero. As before, $\|\cdot\|_\partial$ stands for the norms of the spaces $\mathrm{L}^2(\partial\Omega; \mathbb{C}^k)$, $k \in \mathbb{N}$.

The following results will build on the application of Theorem 4.11. Therefore, the next lemma shows that solutions to the Stokes resolvent problem (2.27) that are given by single layer potentials fulfill the requirements of the theorem.

Lemma 5.1. *Let $\lambda \in \Sigma_\theta$ and (u, ϕ) be given by* (3.1) *and* (3.2)*, respectively. Then the following holds for $|x| \to \infty$:*

$$|\phi(x)| + |\nabla u(x)| = O(|x|^{1-d}) \quad \textit{and} \quad u(x) = \begin{cases} O(|x|^{2-d}) & \textit{if } d \geq 3\,, \\ o(1) & \textit{if } d = 2\,. \end{cases}$$

Proof. If $d \geq 2$, then an application of the dominated convergence theorem gives that $|\phi(x)| + |\nabla u(x)| = O(|x|^{1-d})$ as $|\phi(x)| = O(|\Phi(x)|) = O(|x|^{1-d})$ by (2.29). Furthermore, we have $|\nabla u(x)| = O(|x|^{1-d})$ by estimate (2.31).

If $d \geq 3$, then the first part of Lemma 2.1 and an application of the dominated convergence theorem give $u(x) = O(|x|^{2-d})$. If $d = 2$, consider the definition of the fundamental matrix (2.28). According to the first part of Lemma 2.1 and the asymptotic behavior of the second derivative of $\log(|x|)$, we only have to worry about the first summand in (2.28). But the asymptotic behavior of the fundamental solution to the scalar Helmholtz equation is already available thanks to the second part of Lemma 2.1. Through dominated convergence, the same asymptotic behavior holds for $u(x)$. □

As announced in the introduction to this chapter, we will study the invertibility of the operators $\pm(1/2)I + \mathcal{K}_\lambda$ from Chapter 3, starting with the one corresponding to $+$. We will furthermore be concerned about bounds on the inverse of the operator $(1/2)I + \mathcal{K}_\lambda$.

Lemma 5.2. *Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0,1)$. Suppose that $|\partial\Omega| = 1$. Then $(1/2)I + \mathcal{K}_\lambda$ is an isomorphism on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ and*

$$\|f\|_\partial \leq C \left\| \left((1/2)I + \mathcal{K}_\lambda\right) f \right\|_\partial \quad \textit{for any } f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d), \tag{5.1}$$

where $C > 0$ depends only on d, θ, τ and the Lipschitz character of Ω.

Proof. We start with $f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ and the corresponding single layer potentials $u = \mathcal{S}_\lambda(f)$ and $\phi = \mathcal{S}_\Phi(f)$ given by (3.1) and (3.2). We saw in Chapter 3 that (u, ϕ) solves the Stokes resolvent problem in $\mathbb{R}^d \setminus \partial\Omega$ and got from Lemma 3.6 with $p = 2$ for the nontangential maximal functions that $(\nabla u)^*$, $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$. We furthermore saw in Lemma 3.8 that ∇u and ϕ have nontangential limits almost everywhere on $\partial\Omega$. Finally, in Theorem 3.9 we saw that $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$ and derived the jump relation $\left(\frac{\partial u}{\partial \nu}\right)_\pm = (\pm(1/2)I + \mathcal{K}_\lambda) f$.

Let us assume for a moment that the estimate

$$\|\nabla u_-\|_\partial + \|\phi_-\|_\partial \leq C \left\| \left(\frac{\partial u}{\partial \nu}\right)_+ \right\|_\partial. \tag{5.2}$$

holds for a constant $C > 0$ depending only on d, θ, τ and the Lipschitz character of Ω. Using (5.2), we can prove (5.1): Note that the jump relation (3.19) gives us $f = \left(\frac{\partial u}{\partial \nu}\right)_+ - \left(\frac{\partial u}{\partial \nu}\right)_-$. With the definition of the conormal derivative and estimate (5.2), we calculate that

$$\begin{aligned}
\|f\|_\partial &\leq \left\| \left(\frac{\partial u}{\partial \nu}\right)_+ \right\|_\partial + \left\| \left(\frac{\partial u}{\partial \nu}\right)_- \right\|_\partial \\
&\leq \left\| \left(\frac{\partial u}{\partial \nu}\right)_+ \right\|_\partial + \left\| \left(\frac{\partial u}{\partial n}\right)_- \right\|_\partial + \|\phi_-\|_\partial
\end{aligned}$$

$$\leq C\left\|\left(\frac{\partial u}{\partial \nu}\right)_+\right\|_\partial = C\left\|\left((1/2)I + \mathcal{K}_\lambda\right)f\right\|_\partial.$$

In order to prove (5.2), note that due to Lemma 5.1 we can use Theorem 4.11 to derive the following inequality for the outer nontangential limit:

$$\begin{aligned}\|\nabla u_-\|_\partial + \|\phi_-\|_\partial &\leq C\left\{\left\|\nabla_{\tan} u_-\right\|_\partial + |\lambda|^{1/2}\|u_-\|_\partial + |\lambda|\,\|n\cdot u_-\|_{\mathrm{H}^{-1}(\partial\Omega)}\right\}\\ &= C\left\{\left\|\nabla_{\tan} u_+\right\|_\partial + |\lambda|^{1/2}\|u_+\|_\partial + |\lambda|\,\|n\cdot u_+\|_{\mathrm{H}^{-1}(\partial\Omega)}\right\}, \qquad (5.3)\end{aligned}$$

where we used the fact that $u_+ = u_-$ and $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$ on $\partial\Omega$. The former fact is a consequence of the continuity of the single layer potential across $\partial\Omega$, see Mitrea and Wright [28, Prop. 4.7]. Inequality (4.2) of Theorem 4.1 now allows us to estimate the right-hand side of (5.3) by $C\,\|\left(\frac{\partial u}{\partial \nu}\right)_+\|_\partial$ and thus the desired estimate (5.2) follows.

Let us now work on the invertibility of $(1/2)I + \mathcal{K}_\lambda$. In the case $\lambda = 0$, Mitrea and Wright showed in [28, Eq. (5.166)] that $(1/2)I + \mathcal{K}_0$ as an operator on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ has a one dimensional null space and as range the space $\mathrm{L}^2_0(\partial\Omega; \mathbb{C}^d)$. In other words, $(1/2)I + \mathcal{K}_0$ has Fredholm index 0 as the orthogonal complement of $\mathrm{L}^2_0(\partial\Omega; \mathbb{C}^d)$ is the span of the normal vector n which is one dimensional. Since the operator $\mathcal{K}_\lambda - \mathcal{K}_0$ is compact on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ by Lemma 3.10, we deduce that for all $\lambda \in \Sigma_\theta$ the operator

$$(1/2)I + \mathcal{K}_\lambda = (1/2)I + \mathcal{K}_0 + (\mathcal{K}_\lambda - \mathcal{K}_0)$$

has the Fredholm index zero as well. Now inequality (5.1) gives that $(1/2)I + \mathcal{K}_\lambda$ is injective and thus the Fredholm index of zero implies that it is also surjective and hence an isomorphism. □

The next lemma is the counterpart to Lemma 5.2 and proves a similar result for the operator $-(1/2)I + \mathcal{K}_\lambda$ on the slightly smaller space $\mathrm{L}^2_n(\partial\Omega)$.

Lemma 5.3. *Let $\lambda \in \Sigma_\theta$. Then $-(1/2)I + \mathcal{K}_\lambda$ is a Fredholm operator on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ with index zero and*

$$\|f\|_\partial \leq C\left\|\left(-(1/2)I + \mathcal{K}_\lambda\right)f\right\|_\partial \quad \textit{for all } f \in \mathrm{L}^2_n(\partial\Omega), \qquad (5.4)$$

where $C > 0$ depends only on d, θ, the Lipschitz character of Ω and $\mathrm{diam}(\Omega)$.

Proof. Let us assume without loss of generality that $\sigma(\partial\Omega) = 1$. In the case $\lambda = 0$, Mitrea and Wright showed in [28, Eq. (5.166)] that the Fredholm index of the operator $-(1/2)I + \mathcal{K}_0$ on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ is zero and estimate (5.4) holds. Since $\mathcal{K}_\lambda - \mathcal{K}_0$ is compact on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ and the Fredholm index remains unchanged under compact perturbations, we know that the Fredholm index of $-(1/2)I + \mathcal{K}_\lambda$ on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ is zero for all $\lambda \in \Sigma_\theta$. This proves the first claim of the lemma.

Now let $\tau < (2\,\mathrm{diam}(\Omega)^2 + 1)^{-1}$ and $|\lambda| < \tau$. We claim that

$$\big\| (\mathcal{K}_\lambda - \mathcal{K}_0) f \big\|_\partial \leq C\, |\lambda|^{1/2} \|f\|_\partial.$$

In order to prove this inequality, we want to apply Young's inequality from Lemma 3.4. To this end, we start by estimating

$$\big\| (\mathcal{K}_\lambda - \mathcal{K}_0) f \big\|_\partial \leq \sup_{\substack{p \in \partial\Omega \\ i,j=1,\dots,d}} \Big\| \nabla_x \Big\{ \Gamma_{ij}(p - \cdot\,; \lambda) - \Gamma_{ij}(p - \cdot\,; 0) \Big\} \Big\|_{\mathrm{L}^1(\partial\Omega;\, \mathbb{C}^d)} \|f\|_\partial.$$

In the next step, we prove that for $p \in \partial\Omega$ the integral over the gradients of Γ can be estimated independently of p and of course i and j. This is straightforward using Lemma 3.2 as Corollary 2.7 gives us

$$\begin{aligned}
&\int_{\partial\Omega} \Big| \nabla_x \big\{ \Gamma_{ij}(p - y; \lambda) - \Gamma_{ij}(p - y; 0) \big\} \Big| \, \mathrm{d}\sigma(y) \\
&\quad \leq C\, |\lambda|^{1/2} \int_{\partial\Omega} \frac{1}{|p - y|^{d-2}} \, \mathrm{d}\sigma(y) \\
&\quad = C\, |\lambda|^{1/2} \int_{\partial\Omega \cap \mathrm{B}(p,\, r_0/4)} \frac{1}{|p - y|^{d-2}} \, \mathrm{d}\sigma(y) + C|\lambda|^{1/2} \int_{\partial\Omega \setminus \mathrm{B}(p,\, r_0/4)} \frac{1}{|p - y|^{d-2}} \, \mathrm{d}\sigma(y) \\
&\quad \leq C\, |\lambda|^{1/2} \big(r_0/4 + 4^{2-d} r_0^{d-2} \sigma(\partial\Omega) \big),
\end{aligned}$$

where r_0 is the radius from the definition of Lipschitz domains. But as $\sigma(\partial\Omega) = 1$ and r_0 can be related to $|\partial\Omega|$ by Lemma 3.1, we get

$$\int_{\partial\Omega} \Big| \nabla_x \big\{ \Gamma_{ij}(p - y; \lambda) - \Gamma_{ij}(p - y; 0) \big\} \Big| \, \mathrm{d}\sigma(y) \leq C\, |\lambda|^{1/2}$$

with a constant $C > 0$ that only depends on d, θ and the Lipschitz character of Ω. Note that by the choice of τ the estimate from Corollary 2.7 applies on the whole domain of integration.

For $f \in \mathrm{L}^2_n(\partial\Omega)$, we now estimate

$$\begin{aligned}
\|f\|_\partial &\leq C \big\| (-(1/2)I + \mathcal{K}_0) f \big\|_\partial \\
&\leq C \big\| (-(1/2)I + \mathcal{K}_\lambda) f \big\|_\partial + \big\| (\mathcal{K}_\lambda - \mathcal{K}_0) f \big\|_\partial \\
&\leq C \big\| (-(1/2)I + \mathcal{K}_\lambda) f \big\|_\partial + C\, |\lambda|^{1/2} \|f\|_\partial,
\end{aligned}$$

with a constant $C > 0$ which depends only on d, θ and the Lipschitz character of $\partial\Omega$. Choosing τ smaller than $(2C)^{-2}$ allows us to rearrange the terms in the estimate above such that estimate (5.4) holds for $\lambda \in \Sigma_\theta$ and $|\lambda| < \tau$, with τ depending on d, θ, the Lipschitz character of Ω and diam(Ω) by the first choice of τ.

Now leave τ fixed and consider the case $|\lambda| \geq \tau$. This case will be handled using the Rellich estimates from Section 4. We use the facts that for $\nabla_{\tan} u$ and u the inner and outer nontangential limits coincide and apply Theorems 4.1 and 4.11 to conclude that

$$
\begin{aligned}
&\|\nabla u_+\|_\partial + \Big\| \phi_+ - \frac{1}{r_0^{d-1}} \int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma \Big\|_\partial \\
&\quad \leq C \Big\{ \big\|(\nabla_{\tan} u)_+\big\|_\partial + |\lambda|^{1/2} \|u_+\|_\partial + |\lambda| \, \|u_+ \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \Big\} \\
&\quad = C \Big\{ \big\|(\nabla_{\tan} u)_-\big\|_\partial + |\lambda|^{1/2} \|u_-\|_\partial + |\lambda| \, \|u_- \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \Big\} \leq C \Big\| \Big(\frac{\partial u}{\partial \nu}\Big)_- \Big\|_\partial .
\end{aligned}
$$

We can now use this inequality to estimate $\|\big(\frac{\partial u}{\partial \nu}\big)_+\|_\partial$ via

$$
\begin{aligned}
\Big\| \Big(\frac{\partial u}{\partial \nu}\Big)_+ \Big\|_\partial &\leq \Big\| \Big(\frac{\partial u}{\partial n}\Big)_+ \Big\|_\partial + C \, \|\phi_+\|_\partial \\
&\leq C \Big\{ \|(\nabla u)_+\|_\partial + \Big\| \phi_+ - \frac{1}{r_0^{d-1}} \int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma \Big\|_\partial + \Big| \frac{1}{r_0^{d-1}} \int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma \Big| \Big\} \\
&\leq C \Big\{ \Big\| \Big(\frac{\partial u}{\partial \nu}\Big)_- \Big\|_\partial + \Big| \int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma \Big| \Big\}.
\end{aligned}
$$

Furthermore, considering the jump relation (3.19) and the previous estimate, we get that

$$
\begin{aligned}
\|f\|_\partial &\leq \Big\| \Big(\frac{\partial u}{\partial \nu}\Big)_+ \Big\|_\partial + \Big\| \Big(\frac{\partial u}{\partial \nu}\Big)_- \Big\|_\partial \\
&\leq C \Big\{ \Big\| \Big(\frac{\partial u}{\partial \nu}\Big)_- \Big\|_\partial + \Big| \int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma \Big| \Big\} \\
&\leq C \Big\{ \big\| (-(1/2)I + \mathcal{K}_\lambda) f \big\|_\partial + \Big| \int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma \Big| \Big\}.
\end{aligned} \tag{5.5}
$$

Now we are left with the term $\int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma$ that needs to be estimated. To this end, note that multiplying the conormal derivatives of u by n gives

$$
\Big(\frac{\partial u}{\partial \nu}\Big)_+ \cdot n = \Big(\frac{\partial u_i}{\partial x_j}\Big)_+ n_i n_j - \phi_+ = n_j \Big(n_i \frac{\partial u_i}{\partial x_j} - n_j \Big(\frac{\partial u_i}{\partial x_i}\Big)\Big)_+ - \phi_+ ,
$$

where for the second equality we used that $\operatorname{div}(u) = 0$ in Ω and thus this also holds for the nontangential limit. Note that the expression on the right-hand side involves a first-order tangential derivative operator, see (4.5), and thus we can also write

$$
\Big(\frac{\partial u}{\partial \nu}\Big)_+ \cdot n = n_j \big(\partial_{\tau_{ij}} u_i\big)_+ - \phi_+ .
$$

Using the above identity and relation (4.7) to bring the tangential gradient into the game, it follows that

$$\begin{aligned}\Big|\int_{\partial\Omega}\phi_+\,\mathrm{d}\sigma\Big| &\le \Big|\int_{\partial\Omega}\Big(\frac{\partial u}{\partial\nu}\Big)_+\cdot n\,\mathrm{d}\sigma\Big| + C\,\big\|(\nabla_{\tan}u)_+\big\|_\partial\\ &\le \Big|\int_{\partial\Omega}\Big(\frac{\partial u}{\partial\nu}\Big)_-\cdot n\,\mathrm{d}\sigma\Big| + C\,\big\|(\nabla_{\tan}u)_-\big\|_\partial\\ &\le C\,\Big\|\Big(\frac{\partial u}{\partial\nu}\Big)_-\Big\|_\partial,\end{aligned}\tag{5.6}$$

where in the second step, we used the jump relation to exchange $\big(\frac{\partial u}{\partial\nu}\big)_+\cdot n$ by $\big(\frac{\partial u}{\partial\nu}\big)_- + f\cdot n$ and then used the fact $f\in\mathrm{L}^2_n(\partial\Omega)$. The third step follows from Theorem 4.11 considering that $\big\|(\nabla_{\tan}u)_-\big\|_\partial \le C\,\|(\nabla u)_-\|_\partial$ with a constant that only depends on d. Now, extending estimate (5.5) by (5.6) gives

$$\|f\|_\partial \le C\,\big\|\big(-(1/2)I+\mathcal{K}_\lambda\big)f\big\|_\partial + C\,\Big\|\Big(\frac{\partial u}{\partial\nu}\Big)_-\Big\|_\partial \le C\,\big\|\big(-(1/2)I+\mathcal{K}_\lambda\big)f\big\|_\partial\,,$$

where we used the jump relation (3.19) again. This proves estimate (5.4) in the case $|\lambda|\ge\tau$ and thus concludes the proof. □

With the following lemma that looks like a reverse trace theorem, we will later show the uniqueness of solutions to the L^2 Dirichlet problem (Dir_λ) for the Stokes resolvent system.

Lemma 5.4. *Let $\lambda\in\Sigma_\theta$ and (u,ϕ) be a solution to the Stokes resolvent problem* (2.27) *in Ω. Furthermore, suppose that the nontangential limit of u exists almost everywhere on $\partial\Omega$ and that $(u)^*\in\mathrm{L}^2(\partial\Omega)$. Then,*

$$\int_\Omega |u|^2\,\mathrm{d}x \le C\int_{\partial\Omega}|u|^2\,\mathrm{d}\sigma,\tag{5.7}$$

where $C>0$ depends only on d, θ and the Lipschitz character of Ω.

Proof. We use the approximation theorem, Theorem 1.3, and approximate Ω by a sequence of smooth domains with uniform Lipschitz characters from inside. It suffices to prove (5.7) for elements of this sequence of domains as $(u)^*\in\mathrm{L}^2(\partial\Omega)$. As a consequence, we will assume for the rest of the proof that Ω is smooth and that u, ϕ are smooth in $\overline{\Omega}$. Let $(w,\psi)\in\mathrm{H}^1_0(\Omega;\mathbb{C}^d)\times\mathrm{H}^1(\Omega)$ be a solution to the inhomogeneous system

$$\begin{aligned}-\Delta w+\lambda w+\nabla\psi &= \overline{u} \quad \text{in }\Omega\,,\\ \operatorname{div}(w) &= 0 \quad \text{in }\Omega\,.\end{aligned}\tag{5.8}$$

In fact, the regularity theory for the Stokes equation gives us that w and ψ are even smooth in Ω as $\overline{u}$ is smooth. It follows from testing (5.8) against u that

$$\int_\Omega |u|^2 \,\mathrm{d}x = \int_\Omega u \cdot \big\{ -\Delta w + \lambda w + \nabla\psi \big\} \,\mathrm{d}x. \tag{5.9}$$

The left-hand side of (5.9) gives the starting point for the proof of inequality (5.7).

Using one of Green's identities, see [6, Thm. 3, App. C.2], on the first summand and the fact that u is the solution to the Stokes resolvent problem gives that

$$\begin{aligned}\int_\Omega -u \cdot \Delta w \,\mathrm{d}x &= \int_\Omega -w \cdot \Delta u \,\mathrm{d}x - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} \,\mathrm{d}\sigma,\\ &= \int_\Omega w \cdot (-\lambda u - \nabla\phi) \,\mathrm{d}x - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} \,\mathrm{d}\sigma\\ &= \int_\Omega -\lambda w \cdot u \,\mathrm{d}x - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} \,\mathrm{d}\sigma,\end{aligned}$$

where in the last step we used integration by parts and the fact that w vanishes on $\partial\Omega$ and is divergence free:

$$\int_\Omega w \cdot \nabla\phi \,\mathrm{d}x = -\int_\Omega \operatorname{div}(w)\,\phi \,\mathrm{d}x + \int_{\partial\Omega} \phi\, w \cdot n \,\mathrm{d}\sigma = 0.$$

For the third summand in (5.9), we do the same with the only difference that the second integral does not vanish:

$$\int_\Omega u \cdot \nabla\psi \,\mathrm{d}x = -\int_\Omega \operatorname{div}(u)\,\psi \,\mathrm{d}x + \int_{\partial\Omega} u \cdot n\,\psi \,\mathrm{d}\sigma = \int_{\partial\Omega} u \cdot n\,\psi \,\mathrm{d}\sigma.$$

Putting everything together gives

$$\int_\Omega |u|^2 \,\mathrm{d}x = \left| \int_{\partial\Omega} u \cdot \left\{ -\frac{\partial w}{\partial n} + n\psi \right\} \mathrm{d}\sigma \right| \le \|u\|_\partial \left\{ \|\nabla w\|_\partial + \|\psi\|_\partial \right\} \tag{5.10}$$

by the Cauchy-Schwarz inequality. As the pressure ψ is only specified modulo additive constants, we may as well assume that $\int_{\partial\Omega} \psi \,\mathrm{d}\sigma = 0$. Furthermore, by the Schwarz theorem, we see from (5.8) that $\Delta\psi = \operatorname{div}(\overline{u}) = 0$ in Ω. As stated in Remark 4.10, this allows us to use the results from the proof of (4.27) with $\phi = \psi$ to conclude that

$$\|\psi\|_\partial \le C\, \|\nabla\psi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}$$

and since w has vanishing trace on $\partial\Omega$ we can use that property together with the fact that (w, ψ) solves (5.8) to further estimate

$$\le C \left\{ \|\Delta w \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\overline{u} \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \right\}$$

$$\leq C\left\{\|\nabla w\|_\partial + \|u\|_\partial\right\}, \tag{5.11}$$

where for the last estimate we used (4.26) which is applicable since $\operatorname{div} w = 0$ on Ω, see Remark 4.10. If we combine inequalities (5.10) and (5.11), we get

$$\int_\Omega |u|^2 \,\mathrm{d}x \leq C\left\{\|u\|_\partial \|\nabla w\|_\partial + \|u\|_\partial^2\right\}. \tag{5.12}$$

We are left with the task to estimate the first term on the right-hand side of (5.12). To this end, it will suffice to show the following inequality

$$\int_{\partial\Omega} |\nabla w|^2 \,\mathrm{d}\sigma \leq C\left\{\int_\Omega |u|^2 \,\mathrm{d}x + \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma\right\} \tag{5.13}$$

with a constant $C > 0$ depending only on d, θ and the Lipschitz character of Ω since by the weighted Young inequality for real numbers this would make the estimate

$$C\,\|u\|_\partial \|\nabla w\|_\partial \leq \frac{1}{2}\int_\Omega |u|^2 \,\mathrm{d}x + C\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma$$

available which after rearranging terms in (5.12) yields (5.7).

In order to derive estimate (5.13), we will need the Rellich-type identity

$$\begin{aligned}\int_{\partial\Omega} h_k n_k |\nabla w|^2 \,\mathrm{d}\sigma = &-\int_\Omega \operatorname{div}(h)\,|\nabla w|^2 \,\mathrm{d}x + 2\operatorname{Re}\int_\Omega \frac{\partial h_k}{\partial x_j}\cdot\frac{\partial w_i}{\partial x_k}\cdot\frac{\partial \overline{w_i}}{\partial x_j}\,\mathrm{d}x \\ &- 2\operatorname{Re}\int_\Omega \frac{\partial h_k}{\partial x_i}\cdot\frac{\partial w_i}{\partial x_k}\,\overline{\psi}\,\mathrm{d}x + 2\operatorname{Re}\int_\Omega h_k\frac{\partial w_i}{\partial x_k}\cdot\overline{\lambda w_i}\,\mathrm{d}x \\ &+ 2\operatorname{Re}\int_\Omega h_k\frac{\partial \overline{w_i}}{\partial x_k}\,\overline{u_i}\,\mathrm{d}x\,, \end{aligned} \tag{5.14}$$

where $h = (h_1, \ldots, h_d) \in \mathrm{C}_0^1(\mathbb{R}^d; \mathbb{R}^d)$. Note that since all involved quantities are smooth up to the boundary, integration by parts is allowed and the proof of the stated Rellich identity boils down to a formal calculation. The proof is analogous to the proof of Rellich identity (4.4) and one can prove that for (w, ψ) an equality like (4.4) holds with one extra term, namely

$$-2\operatorname{Re}\int_\Omega h_k\frac{\partial \overline{w_i}}{\partial x_k}\,\overline{u_i}\,\mathrm{d}x.$$

This term will show up in the proof of Lemma 4.3 in the calculation of I_5 when one uses the fact that (w, ψ) solves problem (5.8). Finally, the first two terms on the right-hand side of (4.4) vanish due to the integration by parts rule for tangential derivatives, see (4.8), and the fact that w is equal to 0 on $\partial\Omega$.

Now, let $h \in \mathrm{C}_0^1(\mathbb{R}^d;\mathbb{R}^d)$ with $h_k n_k \geq c > 0$ on $\partial\Omega$ be the function from Theorem 1.3. We apply the triangle inequality to the Rellich-type identity (5.14) to obtain

$$\begin{aligned}\int_{\partial\Omega} |\nabla w|^2 \,\mathrm{d}\sigma \leq C \Big\{ &\int_\Omega |\nabla w|^2 \,\mathrm{d}x + \int_\Omega |\nabla w||\psi| \,\mathrm{d}x \\ &+ |\lambda| \int_\Omega |\nabla w||w| \,\mathrm{d}x + \int_\Omega |\nabla w||u| \,\mathrm{d}x \Big\}\end{aligned} \tag{5.15}$$

with a constant $C > 0$ that only depends on d and the Lipschitz character of Ω. The left-hand side of this estimate establishes the starting point for the proof of inequality (5.13).

The next step consists in deriving estimates which are compatible with the right-hand side of (5.15). Testing the first equation of (5.8) with $\overline{w}$, integration by parts and Lemma 4.4 give us as in the proof of Lemma 4.5 the estimate

$$\int_\Omega |\nabla w|^2 \,\mathrm{d}x + |\lambda| \int_\Omega |w|^2 \,\mathrm{d}x \leq C \int_\Omega |w||u| \,\mathrm{d}x,$$

where $C > 0$ depends only on θ. The next step consists in using the previous inequality and the Poincaré inequality to estimate

$$\begin{aligned}\int_\Omega |\nabla w|^2 \,\mathrm{d}x + (1 + |\lambda|) \int_\Omega |w|^2 \,\mathrm{d}x &\leq (1 + C) \int_\Omega |\nabla w|^2 \,\mathrm{d}x + |\lambda| \int_\Omega |w|^2 \,\mathrm{d}x \\ &\leq C \int_\Omega |w||u| \,\mathrm{d}x \\ &\leq C \Big(\int_\Omega |w|^2 \,\mathrm{d}x \Big)^{1/2} \Big(\int_\Omega |u|^2 \,\mathrm{d}x \Big)^{1/2}\end{aligned}$$

where for the last step we used the Cauchy-Schwarz inequality. The weighted Young inequality for real numbers allows us to further estimate

$$\begin{aligned}&\leq \frac{C}{4\varepsilon} \int_\Omega |u|^2 \,\mathrm{d}x + C\,\varepsilon \int_\Omega |w|^2 \,\mathrm{d}x \\ &= \frac{\tilde{C}}{1 + |\lambda|} \int_\Omega |u|^2 \,\mathrm{d}x + \frac{1}{2}(1 + |\lambda|) \int_\Omega |w|^2 \,\mathrm{d}x\end{aligned}$$

if we set $\varepsilon = \frac{(1+|\lambda|)}{2C}$. Rearranging terms, we can produce our next estimate

$$\int_\Omega |\nabla w|^2 \,\mathrm{d}x + (1 + |\lambda|) \int_\Omega |w|^2 \,\mathrm{d}x \leq \frac{C}{1 + |\lambda|} \int_\Omega |u|^2 \,\mathrm{d}x, \tag{5.16}$$

where $C > 0$ depends on d, θ, the Lipschitz character of Ω and $\operatorname{diam}(\Omega)$.

Now it's time to harvest: Using the weighted Young inequality, we see that we can simplify the right-hand side of (5.15) via the chain of estimates

$$\int_{\partial\Omega} |\nabla w|^2 \,\mathrm{d}\sigma \leq C_\varepsilon (1 + |\lambda|) \int_\Omega |\nabla w|^2 \,\mathrm{d}x + C\,|\lambda| \int_\Omega |w|^2 \,\mathrm{d}x$$

$$
\begin{aligned}
&+ C \int_\Omega |u|^2 \,\mathrm{d}x + \varepsilon \int_\Omega |\psi|^2 \,\mathrm{d}x \\
&\le C_\varepsilon \int_\Omega |u|^2 \,\mathrm{d}x + \varepsilon \int_\Omega |\psi|^2 \,\mathrm{d}x,
\end{aligned}
$$

where the second inequality is thanks to estimate (5.16). The first term on the right-hand side is already fine for (5.13). For the second one, we use the estimate $\|\psi\|_{\mathrm{L}^2(\Omega)} \le C\,\|\psi\|_\partial$ and inequality (5.11) and arrive at

$$
\varepsilon \int_\Omega |\psi|^2 \,\mathrm{d}x \le \varepsilon\, C \int_{\partial\Omega} |\nabla w|^2 \,\mathrm{d}\sigma + C_\varepsilon \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma.
$$

Choosing $\varepsilon = \frac{1}{2C}$ and rearranging finally gives the desired estimate (5.13). This concludes our proof. $\square$

The next theorem states the important fact that, in L^2, the Dirichlet Stokes resolvent problem has a unique solution.

Theorem 5.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \ge 2$, with connected boundary and let $\lambda \in \Sigma_\theta$. For all $g \in \mathrm{L}^2_n(\partial\Omega)$ there exists a unique vector field u and harmonic function ϕ which is unique up to constants such that (u, ϕ) satisfies* (2.27)*, $(u)^* \in \mathrm{L}^2(\partial\Omega)$ and $u = g$ on $\partial\Omega$ in the sense of nontangential convergence. Moreover, the estimate $\|(u)^*\|_\partial \le C\,\|g\|_\partial$ holds and u may be represented by the double layer potential $\mathcal{D}_\lambda(f)$ with $\|f\|_\partial \le C\,\|g\|_\partial$, where in both cases $C > 0$ depends only on d, θ, the Lipschitz character of* Ω *and* $\mathrm{diam}(\Omega)$.

Proof. By Lemma 5.4, we already know that the problem under consideration admits at most one solution. Therefore, we only have to worry about the existence of a solution. In Chapter 3, it was already established that a solution to the Stokes resolvent problem is given by the double layer potentials $u := \mathcal{D}_\lambda(f)$ and $\phi := \mathcal{D}_\Phi(f)$, $f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$. It was also shown that in this way one also solves the L^2 Dirichlet problem with boundary data $\mathcal{D}_\lambda(f)_+$. From Theorem 3.11 we know that we find this nontangential limit as $\big((-1/2)I + \mathcal{K}^*_{\bar\lambda}\big)f$. Thus, the central idea of this proof will be to invert the operator $(-1/2)I + \mathcal{K}^*_{\bar\lambda}$ in order to find the *right* f to plug into the double layer potentials in order to attain the given boundary data $g \in \mathrm{L}^2_n(\partial\Omega)$ as a nontangential limit.

We first note that due to Lemma 5.3 the operator

$$
T\colon \mathrm{L}^2(\partial\Omega; \mathbb{C}^d) \to \mathrm{L}^2(\partial\Omega; \mathbb{C}^d), \quad \mathrm{x} \mapsto -(1/2)x + \mathcal{K}_{\bar\lambda} x,
$$

is a Fredholm operator on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ with index 0 and thus the same is true for its adjoint

$$
T^*\colon \mathrm{L}^2(\partial\Omega; \mathbb{C}^d) \to \mathrm{L}^2(\partial\Omega; \mathbb{C}^d), \quad x \mapsto -(1/2)x + \mathcal{K}^*_{\bar\lambda} x.
$$

In the following paragraphs, we will show that T^* has a bounded inverse.

We know that for all $f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ we have $\operatorname{div}(\mathcal{D}_\lambda(f)) = 0$ and therefore

$$\int_{\partial\Omega} T^* f \cdot n \,\mathrm{d}\sigma = \int_{\partial\Omega} u_+ \cdot n \,\mathrm{d}\sigma = 0$$

holds, where for the first equality we applied Theorem 3.11. The second equality uses the fact that since $(u)^*$ is integrable, Proposition 1.8 and hence the divergence theorem are available. This gives $\mathrm{Im}(T^*) \subseteq \mathrm{L}^2_n(\partial\Omega)$. Now, on the one hand we have

$$\operatorname{span}(n) = \mathrm{L}^2_n(\partial\Omega)^\perp \subseteq \mathrm{Im}(T^*)^\perp = \ker(T)$$

and on the other hand, as T is injective on $\mathrm{L}^2_n(\partial\Omega)$ by (5.4), we have that

$$\operatorname{span}(n) \supseteq \ker(T).$$

This yields $\operatorname{span}(n) = \ker(T)$. We can use this equality and show that

$$\mathrm{L}^2_n(\partial\Omega) = \ker(T)^\perp = \overline{\mathrm{Im}(T^*)} = \mathrm{Im}(T^*),$$

where for the last equality we used the fact that the range of T^* is closed, as usual for Fredholm operators. With the same argument we can show for T that

$$\ker(T^*)^\perp = \overline{\mathrm{Im}(T)} = \mathrm{Im}(T).$$

Now we want to consider restrictions of the operators T and T^* and derive estimates on the operator norms of their inverses. To make the following proof more readable let

$$X = \mathrm{L}^2_n(\partial\Omega) \quad \text{and} \quad Y = \mathrm{Im}(T).$$

Both spaces are closed subspaces of the Hilbert space $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ and therefore again Hilbert spaces. Consequently, the operator

$$K'_{Y,X} \colon Y \to X, \quad x \mapsto T^* x$$

is invertible by the continuous inverse theorem. At this point we could already establish the solvability of the L^2 Dirichlet problem. But before we do this, in order to derive the additional estimates which were stated in the theorem, we want to bound the operator norm of $\left(K'_{Y,X}\right)^{-1}$ by a constant that does not depend on λ but on the sectoriality parameter θ. To this end, let us introduce the operator

$$K_{X,Y} \colon X \to Y, \quad x \mapsto Tx\,.$$

Now, for $x \in X$ and $y \in Y$ we have that

$$\left\langle x, K^*_{X,Y} y\right\rangle_X = \left\langle K_{X,Y} x, y\right\rangle_Y = \left\langle T x, y\right\rangle_Y = \left\langle x, T^* y\right\rangle_X = \left\langle x, K'_{Y,X} y\right\rangle_X$$

which shows that $K'_{Y,X} = K^*_{X,Y}$ on Y. With the above definitions at hand, Lemma 5.3 states that $K_{X,Y}$ is an invertible operator with operator norm of the inverse bounded from above by some $C > 0$ and C depends at most on d, θ, the Lipschitz character of Ω and $\operatorname{diam}(\Omega)$. As $\|K^*_{X,Y}\|_{\mathfrak{L}(Y,X)} = \|K_{X,Y}\|_{\mathfrak{L}(X,Y)}$ the same holds for the adjoint operator $K^*_{X,Y}$. In particular, we have that $\|(K^*_{X,Y})^{-1}\|_{\mathfrak{L}(X,Y)} = \|K^{-1}_{X,Y}\|_{\mathfrak{L}(Y,X)}$. Therefore, for all $f \in Y = \operatorname{Im}(T) = \operatorname{Im}\left((1/2)I + \mathcal{K}_{\bar{\lambda}}\right)$ we have

$$\begin{aligned}
\|f\|_\partial &= \left\|(K'_{Y,X})^{-1} K'_{Y,X} f\right\|_\partial \\
&\leq \left\|(K^*_{X,Y})^{-1}\right\|_{\mathfrak{L}(X,Y)} \left\|K^*_{X,Y} f\right\|_\partial \leq C \left\|(-(1/2)I + \mathcal{K}^*_{\bar{\lambda}}) f\right\|_\partial .
\end{aligned} \tag{5.17}$$

We are now in position to derive the missing estimates which were stated in the theorem. For $g \in \mathrm{L}^2_n(\partial\Omega)$, let $f \in \operatorname{Im}(T)$ with $T^* f = g$. Fix this f and let (u, ϕ) be the respective double layer potentials which were defined in equations (3.22) and (3.23). Then $u_+ = T^* f = g$ on $\partial\Omega$ by Theorem 3.11. Additionally, we have that

$$\|(u)^*\|_\partial \leq C \|f\|_\partial \leq C \|g\|_\partial$$

where we used inequality (3.24) and (5.17). In particular, this gives $(u)^* \in \mathrm{L}^2(\partial\Omega)$. Consequently, all claims of the theorem have been proven. □

The next theorem can in some sense be regarded as a reverse trace theorem and will play an important role for the proof of the needed reverse Hölder inequality in the forthcoming chapter.

Theorem 5.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with connected boundary. Let $u \in \mathrm{H}^1(\Omega; \mathbb{C}^d)$ and $\pi \in \mathrm{L}^2(\Omega)$ satisfy the Stokes resolvent problem in Ω for some $\lambda \in \Sigma_\theta$. Then*

$$\left(\int_\Omega |u|^{p_d} \,\mathrm{d}x\right)^{1/p_d} \leq C \left(\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma\right)^{1/2}, \tag{5.18}$$

where $p_d = \frac{2d}{d-1}$ and $C > 0$ depends only on d, θ the Lipschitz character of Ω and $\operatorname{diam}(\Omega)$.

Proof. We start our proof of (5.18) on the left-hand side by using inequality (4.13):

$$\left(\int_\Omega |u|^{p_d} \,\mathrm{d}x\right)^{1/p_d} \leq C \left(\int_{\partial\Omega} |(u)^*|^2 \,\mathrm{d}\sigma\right)^{1/2}, \tag{5.19}$$

where $C > 0$ only depends on d and the Lipschitz character of Ω.

Up to now, we do not know whether the right-hand side of inequality (5.19) equals infinity. We only know that (u, ϕ) solves the Stokes resolvent problem and that it implicitly solves a Dirichlet problem with boundary data given as $f = \mathrm{Tr}_{\partial\Omega}(u) \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$. The following part of this proof will show that u coincides with the solution $w := \mathcal{D}_\lambda(g)$, $g \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$, of the L^2 Dirichlet problem (Dir_λ) with boundary data f as given by Theorem 5.5. If this is the case, then the knowledge about $(w)^*$ that is contained in Theorem 5.5 will help us to complete estimate (5.19).

In order to show that $u = w$ on Ω, consider a sequence $(\Omega_j)_{j\in\mathbb{N}}$ of smooth domains that approximates Ω from inside as described by Theorem 1.3. Then, an application of Lemma 5.4 shows

$$\int_{\Omega_j} |u - w|^2 \, \mathrm{d}x \leq C \int_{\partial\Omega_j} |u - w|^2 \, \mathrm{d}\sigma_j, \tag{5.20}$$

where $C > 0$ does not depend on j but on the Lipschitz character of Ω and $\mathrm{diam}(\Omega)$. Furthermore, the trace theorem on bounded Lipschitz domains gives us for all Sobolev functions $h \in \mathrm{H}^1(\Omega; \mathbb{C}^d)$ that

$$\|h\|_\partial^2 \leq C \, \|h\|^2_{\mathrm{H}^1(\Omega;\, \mathbb{C}^d)}, \tag{5.21}$$

where $C > 0$ only depends on d and the Lipschitz character of Ω, see Wei and Zhang [42, Lem. 2.2]. Now, let $\varepsilon > 0$ be given. From the theory of Sobolev spaces it is known that there exists $\varphi_\varepsilon \in \mathrm{C}^\infty(\overline{\Omega}; \mathbb{C}^d)$ such that $\left\|\varphi_\varepsilon - u\right\|^2_{\mathrm{H}^1(\Omega;\, \mathbb{C}^d)} \leq \varepsilon/(6\tilde{C})$, see Adams and Fournier [1, Thm. 3.18]. Here, $\tilde{C} > 0$ denotes the fixed constant from (5.21) which is uniform for all Ω_j and Ω. Thanks to Theorem 1.3, we know that

$$\int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \, \mathrm{d}\sigma_j \longrightarrow \int_{\partial\Omega} |\varphi_\varepsilon - u|^2 \, \mathrm{d}\sigma, \qquad \text{as } j \to \infty,$$

since $w = f$ on $\partial\Omega$ in the sense of nontangential convergence. Therefore, we choose J large enough such that for all $j \geq J$ we have

$$\int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \, \mathrm{d}\sigma_j \leq \int_{\partial\Omega} |\varphi_\varepsilon - u|^2 \, \mathrm{d}\sigma + \frac{\varepsilon}{6}\,.$$

Plugging everything together, this gives us the chain of estimates

$$\begin{aligned}
&\int_{\partial\Omega_j} |u - w|^2 \, \mathrm{d}\sigma_j \\
&\quad \leq 2 \left\{ \int_{\partial\Omega_j} |u - \varphi_\varepsilon|^2 \, \mathrm{d}\sigma_j + \int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \, \mathrm{d}\sigma_j \right\} \leq 4\,\tilde{C} \left\|u - \varphi_\varepsilon\right\|^2_{\mathrm{H}^1(\Omega;\, \mathbb{C}^d)} + \frac{\varepsilon}{3} \leq \varepsilon.
\end{aligned}$$

Gluing together this inequality and (5.20), we see that $w = u$ holds a.e. in Ω once we take the limit $j \to \infty$ and remember that ε was chosen arbitrarily. In particular, we have $(u)^* = (w)^*$ a.e. on $\partial\Omega$. Now, Theorem 5.5 gives

$$\int_{\partial\Omega} |(u)^*|^2 \,\mathrm{d}\sigma = \int_{\partial\Omega} |(w)^*|^2 \,\mathrm{d}\sigma \leq C \int_{\partial\Omega} |f|^2 \,\mathrm{d}\sigma = C \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma$$

where $C > 0$ depends on d, θ the Lipschitz character of Ω and $\operatorname{diam}(\Omega)$. Using this inequality to continue estimate (4.13) concludes our proof. □

Remark 5.7. At this point, the choice $p = \frac{2d}{d-1}$ in Theorem 5.6 may seem arbitrary. Taking a closer look at the proof of inequality (5.19) which is part of Lemma 4.6 and appeared in Wei and Zhang [42, Lem. 3.3], one sees that the choice of p is due to two facts: (1) For the dual exponent we have $p' = \frac{2d}{d+1}$. (2) It holds $\frac{1}{p'} - \frac{1}{p} = \frac{1}{d}$ and thus the *Hardy-Littlewood-Sobolev theorem on fractional integration* may be applied to estimate the p-norm of the *Riesz potential* $I_1(f)$ of a function $f \in \mathrm{L}^{p'}(\mathbb{R}^d)$ by the p'-norm of f, see Grafakos [16, Thm. 6.1.3].

In the following remark and the forthcoming chapter, we will make use of an integration argument which can be considered an application of the following theorem on *integration along slices*. A proof of this result can be found in Federer [8, Thm. 3.2.12].

Theorem 5.8 (Co-area formula)**.** *If $f\colon \mathbb{R}^d \to \mathbb{R}$, $d \geq 2$, is Lipschitz continuous and* $\mathfrak{g}$ *the representative of a function $g \in \mathrm{L}^2(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} \mathfrak{g}(x) \Bigg[\sum_{i=1}^{d} \left| \frac{\partial f}{\partial x_i}(x) \right|^2 \Bigg]^{1/2} \mathrm{d}x = \int_{\mathbb{R}} \int_{f^{-1}(y)} \mathfrak{g}(x) \,\mathrm{d}m_{d-1}(x) \,\mathrm{d}y, \tag{5.22}$$

where m_{d-1} denotes the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$.

Note that the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$ is comparable to the surface measure σ of a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$. When using Theorem 5.8 in estimations, we will therefore always be working with the surface measure only.

Remark 5.9. Let (u, ϕ) be a solution of the Stokes resolvent system in the domain $\mathrm{B}(x_0, r) \subseteq \mathbb{R}^d$. Then the interior estimate

$$|\nabla^l u(x_0)| \leq \frac{C_l}{r^l} \left(\frac{1}{r^d} \int_{\mathrm{B}(x_0,\, r)} |u(x)|^2 \,\mathrm{d}x \right)^{1/2} \tag{5.23}$$

holds for all $l \in \mathbb{N}_0$, where $C_l > 0$ only depends on d, l and θ: Without loss of generality, we may rescale and translate and thus assume that $x_0 = 0$ and $r = 2$. Let $t \in (1, 2)$. By

Theorem 5.5, we know that a solution to the Stokes resolvent system on $\mathrm{B}(0,\ t) \subsetneq \mathrm{B}(0,\ 2)$ with boundary values $g_t \coloneqq \mathrm{Tr}_{\partial \mathrm{B}(0,\,t)}(u) \in \mathrm{L}^2(\partial \mathrm{B}(0,\ t); \mathbb{C}^d)$ is given by a boundary layer potential $\mathcal{D}_\lambda(f_t)$, $f_t \in \mathrm{L}^2(\partial \mathrm{B}(0,\ t); \mathbb{C}^d)$. We use this fact to derive the desired estimate via

$$\begin{aligned} |\nabla^l u(0)|^2 &\leq C \bigg(\int_{\partial \mathrm{B}(0,\,t)} \Big\{ \big|\nabla_x^{l+1} \Gamma(y; \lambda)\big| + \big|\nabla_x^l \Phi(y)\big| \Big\} \, |f_t(y)| \, \mathrm{d}\sigma(y) \bigg)^2 \\ &\leq C \bigg(\int_{\partial \mathrm{B}(0,\,t)} \frac{|f_t(y)|}{t^{d-1+l}} \, \mathrm{d}\sigma(y) \bigg)^2 \\ &\leq C \int_{\partial \mathrm{B}(0,\,t)} |f_t(y)|^2 \, \mathrm{d}\sigma(y) \\ &\leq C \int_{\partial \mathrm{B}(0,\,t)} |u(y)|^2 \, \mathrm{d}\sigma(y), \end{aligned}$$

where in the last step we used the estimate of f_t against the "data" from Theorem 5.5. Integrating this inequality in t over the interval $(1, 2)$ and using the co-area formula (5.22) with Lipschitz function $|\cdot|$ gives

$$|\nabla^l u(0)|^2 \leq C \int_{\mathrm{B}(0,\,2)} |u(x)|^2 \, \mathrm{d}x.$$

Note that for this argument to work it is crucial that $C > 0$ does only depend on d, the Lipschitz character and diameter of $\mathrm{B}(x_0, 2)$. All this quantities are comparable for the involved domains $\mathrm{B}(x, t)$, $1 \leq t \leq 2$. Now, the claim follows readily.

Chapter 6

Derivation of Resolvent Estimates

In this final chapter, we will prove that the Stokes semigroup is analytic on $\mathrm{L}^p_\sigma(\Omega)$ for bounded Lipschitz domains $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$.

The first step will be to establish a *weak reverse Hölder estimate* for local solutions of the Stokes resolvent problem (2.27). We start with a similar result on Lipschitz cylinders, see Definition 1.1. Recall the definition of the constant $M > 0$ related to Lipschitz domains and the sets

$$\begin{aligned} D(r) &:= \big\{(x', x_d)\colon |x'| < r, |x_d| < 10d\,(M+1)\,r\big\}, \\ D_\eta(r) &:= \big\{(x', x_d)\colon |x'| < r, \eta_x(x') < x_d < 10d\,(M+1)\,r\big\}, \\ I_\eta(r) &:= \big\{(x', x_d)\colon |x'| < r, \eta_x(x') = x_d\big\}, \end{aligned}$$

where $\eta_x\colon \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz character bounded by M.

Lemma 6.1. *Let $\eta\colon \mathbb{R}^{d-1} \to \mathbb{R}$ be a Lipschitz function. Furthermore, let the functions $u \in \mathrm{H}^1(D_\eta(r); \mathbb{C}^d)$ and $\phi \in \mathrm{L}^2(D_\eta(2r))$ solve the Stokes resolvent problem* (2.27) *in $D_\eta(2r)$ with $u = 0$ on $I_\eta(2r)$ for some $0 < r < \infty$ and $\lambda \in \Sigma_\theta$. Let $p_d = 2d/(d-1)$. Then,*

$$\left(\frac{1}{r^{d-1}} \int_{D_\eta(r)} |u|^{p_d}\,\mathrm{d}x\right)^{1/p_d} \leq C \left(\frac{1}{r^{d-1}} \int_{D_\eta(2r)} |u|^2\,\mathrm{d}x\right)^{1/2}, \tag{6.1}$$

where $C > 0$ only depends on d, M and θ.

Proof. Without loss of generality, we rescale and assume that $r = 1$. Let $t \in (1, 2)$. We note that thanks to a thorough investigation carried out by Tolksdorf, see [39, Lem. 1.3.25], a Lipschitz cylinder is itself a Lipschitz domain. It is therefore admissible to apply the trace estimate from Theorem 5.6 to u in $D_\eta(t)$ which yields

$$\left(\int_{D_\eta(t)} |u|^{p_d}\,\mathrm{d}x\right)^{2/p_d} \leq C \int_{\partial D_\eta(t)} |u|^2\,\mathrm{d}\sigma,$$

where $C > 0$ depends only on d, θ, the Lipschitz character of $D_\eta(t)$ and $\operatorname{diam}(D_\eta(t))$. In particular, C does not depend on t. Since u vanishes on $I(2)$, we have that

$$\begin{aligned}\left(\int_{D_\eta(1)} |u|^{p_d}\,\mathrm{d}x\right)^{2/p} &\le C\int_1^2 \left\{\int_{\partial D(t)} |u(y)|^2 \chi_{D_\eta(2)}(y)\,\mathrm{d}\sigma(y) + \int_{I_\eta(2)} |u(y)|^2\,\mathrm{d}\sigma(y)\right\}\mathrm{d}t\\ &= C\int_1^2\int_{\partial D(t)} |u(y)|^2\chi_{D_\eta(2)}(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}t.\end{aligned}$$

We define the function

$$f\colon \mathbb{R}^d \to \mathbb{R}, \quad x \mapsto t \quad \text{iff} \quad x \in \partial D(t).$$

For $x \in \partial D(t)$ and $y \in \partial D(s)$, $s, t > 0$, we have that

$$|f(x) - f(y)| = |t - s| \le \operatorname{dist}(\partial D(t), \partial D(s)) \le |x - y|$$

and, consequently, the function f is Lipschitz continuous with Lipschitz constant 1. Applying the co-area formula from Theorem 5.8 with function f to integrate both sides over the interval $(1, 2)$ gives

$$\left(\int_{D_\eta(1)} |u|^{p_d}\,\mathrm{d}x\right)^{2/p_d} \le C\int_{D_\eta(2)} |u|^2\,\mathrm{d}x.$$

Estimate (6.1) now follows readily. □

The next step consists in extending the previous result to arbitrary Lipschitz domains. The following Lemma, which appeared in Tolksdorf [38, Lem. 4.2], reduces the amount of work that needs to be done to a few special cases.

Lemma 6.2. *Let $\Omega \subseteq \mathbb{R}^d$ be Lebesgue-measurable, $f, g \in \mathrm{L}^2(\Omega)$, $\alpha_2 > \alpha_1 > 1$, $p > 2$, $r > 0$ and $x_0 \in \mathbb{R}^d$ be such that $\mathrm{B}(x_0, r) \cap \Omega \neq \emptyset$. If there exists $C > 0$ such that*

$$\begin{aligned}&\left(\frac{1}{s^d}\int_{\Omega\cap \mathrm{B}(y,s)} |f|^p\,\mathrm{d}x\right)^{1/p}\\ &\quad\le C\left\{\left(\frac{1}{s^d}\int_{\Omega\cap\mathrm{B}(y,\alpha_1 s)} |f|^2\,\mathrm{d}x\right)^{1/2} + \sup_{B'\supset \mathrm{B}(y,s)}\left(\frac{1}{|B'|}\int_{\Omega\cap B'} |g|^2\,\mathrm{d}x\right)^{1/2}\right\}\end{aligned}$$

holds for all balls $\mathrm{B}(y, s)$ with $\mathrm{B}(y, \alpha_2 s) \subseteq \mathrm{B}(x_0, \alpha_2 r)$ and which are either centered on $\partial\Omega$ or satisfy $\mathrm{B}(y, \alpha_2 s) \subseteq \Omega$, then for each $\alpha \in (1, \alpha_2)$ there exists a constant $C' > 0$ such that

$$\begin{aligned}&\left(\frac{1}{r^d}\int_{\Omega\cap \mathrm{B}(x_0,r)} |f|^p\,\mathrm{d}x\right)^{1/p}\\ &\quad\le C'\left\{\left(\frac{1}{r^d}\int_{\Omega\cap\mathrm{B}(x_0,\alpha r)} |f|^2\,\mathrm{d}x\right)^{1/2} + \sup_{B'\supset \mathrm{B}(x_0,r)}\left(\frac{1}{|B'|}\int_{\Omega\cap B'} |g|^2\,\mathrm{d}x\right)^{1/2}\right\}.\end{aligned}$$

This constant C' only depends on d, α, α_1, α_2, p and C.

As of now, our toolbox comprises enough tools to prove that solutions to the Stokes resolvent problem (2.27) satisfy a weak reverse Hölder inequality.

Lemma 6.3. *Let* $x_0 \in \overline{\Omega}$ *and* $0 < 2r < r_0$ *and set*

$$\alpha_1 \coloneqq \sqrt{d^2 10^2 (1+M)^2 + 4} \quad \text{and} \quad \alpha_2 \coloneqq \alpha_1 + 1.$$

Let $u \in \mathrm{H}^1(\mathrm{B}(x_0, \alpha_2 r) \cap \Omega; \mathbb{C}^d)$ *and* $\phi \in \mathrm{L}^2(\mathrm{B}(x_0, \alpha_2 r) \cap \Omega)$ *satisfy the Stokes resolvent problem* (2.27) *in the domain* $\mathrm{B}(x_0, \alpha_2 r) \cap \Omega$. *If* $\mathrm{B}(x_0, \alpha_2 r) \cap \partial\Omega \neq \emptyset$, *we additionally assume* $u = 0$ *on* $\mathrm{B}(x_0, \alpha_2 r) \cap \partial\Omega$. *Then,*

$$\Big(\frac{1}{r^d} \int_{\mathrm{B}(x_0,\, r) \cap \Omega} |u(x)|^{p_d} \, \mathrm{d}x \Big)^{1/p_d} \leq C \Big(\frac{1}{r^d} \int_{\mathrm{B}(x_0,\, 2r) \cap \Omega} |u(x)|^2 \, \mathrm{d}x \Big)^{1/2} \tag{6.2}$$

holds, where $p_d = 2d/(d-1)$. *Here,* $C > 0$ *only depends on* d, θ *and the Lipschitz character of* Ω.

Proof. Due to Lemma 6.2, it suffices to consider only two cases: (1) $x_0 \in \Omega$ with $\mathrm{B}(x_0, \alpha_2 r) \subseteq \Omega$ and (2) $x_0 \in \partial\Omega$.

In order to prove (1), let $x_0 \in \Omega$ with $\mathrm{B}(x_0, \alpha_2 r) \subseteq \Omega$. We deploy the interior estimate (5.23) to derive that for all $x \in \mathrm{B}(x_0, r)$ the estimate

$$|u(x)|^{p_d} \leq C \Big(\frac{1}{r^d} \int_{\mathrm{B}(x,\, r)} |u(y)|^2 \, \mathrm{d}y \Big)^{p_d/2}$$

holds which after integrating x over $\mathrm{B}(x_0, r)$ yields

$$\frac{1}{r^d} \int_{\mathrm{B}(x_0,\, r)} |u(x)|^{p_d} \, \mathrm{d}x \leq C \Big(\frac{1}{r^d} \int_{\mathrm{B}(x_0,\, \alpha_1 r)} |u(z)|^2 \, \mathrm{d}z \Big)^{p_d/2},$$

where we used the fact that $\alpha_1 > 2$. Here, $C > 0$ depends only on d and θ.

For (2), note that if $x_0 \in \partial\Omega$, then by Lemma 6.1 and Pythagoras' theorem we have

$$\begin{aligned} \Big(\frac{1}{r^d} \int_{\mathrm{B}(x_0,\, r) \cap \Omega} |u(x)|^{p_d} \, \mathrm{d}x \Big)^{1/p_d} &\leq \Big(\frac{1}{r^d} \int_{D_{\eta x_0}(r)} |u(x)|^{p_d} \, \mathrm{d}x \Big)^{1/p_d} \\ &\leq C \Big(\frac{1}{r^d} \int_{D_{\eta x_0}(2r)} |u(x)|^{p_d} \, \mathrm{d}x \Big)^{1/p_d} \\ &\leq C \Big(\frac{1}{r^d} \int_{\mathrm{B}(x_0,\, \alpha_1 r) \cap \Omega} |u(x)|^2 \, \mathrm{d}x \Big)^{1/2}, \end{aligned}$$

with a constant $C > 0$ that only depends on d, M and θ. Now, the claim follows readily from an application of Lemma 6.2 with the parameter $\alpha = 2 \in (1, \alpha_2)$. □

We note that estimate (6.2) is a weak reverse Hölder inequality and thus possesses a self-improving property, see Giaquinta and Martinazzi [10, Thm. 6.38] or Giaquinta and Modica [11, Prop. 5.1].

Proposition 6.4 (Giaquinta, Modica)**.** *Let* $\Omega \subseteq \mathbb{R}^d$ *be open,* $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ *a non-negative function and* $q > 1$. *If there exist constants* $b > 0, R_0 > 0$ *such that*

$$\left(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\,r)} f^q \,\mathrm{d}x\right)^{1/q} \le \frac{b}{r^d}\int_{\mathrm{B}(x_0,\,2r)} f \,\mathrm{d}x$$

for all $x_0 \in \Omega$ *and* $0 < r < \min\left\{R_0, \operatorname{dist}(x_0, \partial\Omega)/2\right\}$*, then* $f \in \mathrm{L}^{q+\varepsilon}_{\mathrm{loc}}(\Omega)$ *for some* $\varepsilon > 0$*, depending only on d, q and b and there is a constant* $\tilde{C} > 0$ *depending only on d, q,* ε *and b such that*

$$\left(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\,r)} f^{q+\varepsilon} \,\mathrm{d}x\right)^{1/(q+\varepsilon)} \le \tilde{C}\left(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\,2r)} f^q \,\mathrm{d}x\right)^{1/q}$$

for all $x_0 \in \Omega$ *and* $0 < r < \min\{R_0, \operatorname{dist}(x_0, \partial\Omega)/2\}$.

Remark 6.5. The self-improving property of reverse Hölder estimates can now be used to make the result of Lemma 6.3 a little bit better. Fix $0 < 2r < r_0$ and choose $x_0 \in \overline{\Omega}$. Let α_2 and $u \in \mathrm{H}^1(\mathrm{B}(x_0, \alpha_2 r) \cap \Omega; \mathbb{C}^d)$ be as in Lemma 6.3. We are aiming to apply Proposition 6.4 on the open set $\tilde{\Omega} := \mathrm{B}(x_0, r)$ for $f := |u|^2 \chi_{\Omega \cap \mathrm{B}(x_0, \alpha_2 r)}$. To this end, let $y \in \mathrm{B}(x_0, r)$ and choose

$$0 < s < \frac{\operatorname{dist}(y, \partial\tilde{\Omega})}{2} = \frac{r - |y - x_0|}{r}.$$

Then, the inclusion $\mathrm{B}(y, \alpha_2 s) \subseteq \mathrm{B}(x_0, \alpha_2 s)$ holds and u fulfills also the requirements of Lemma 6.3 on the subdomain $\mathrm{B}(y, \alpha_2 s) \cap \Omega$. Therefore, using Lemma 6.3, we derive the inequality

$$\begin{aligned}\left(\frac{1}{s^d}\int_{\mathrm{B}(y,\,s)} f^q \,\mathrm{d}x\right)^{1/q} &= \left(\frac{1}{s^d}\int_{\mathrm{B}(y,\,s)\cap\Omega} |u|^p \,\mathrm{d}x\right)^{2/p} \\ &\le C\frac{1}{s^d}\int_{\mathrm{B}(y,\,s)\cap\Omega} |u|^2 \,\mathrm{d}x = C\frac{1}{s^d}\int_{\mathrm{B}(y,\,s)} f \,\mathrm{d}x.\end{aligned}$$

As $y \in \tilde{\Omega}$ and s were chosen according to the requirements of Proposition 6.4, we see that there exists some $\varepsilon > 0$ which depends only on d, q and C and a constant $\tilde{C} > 0$ depending only on d, q, ε and C such that

$$\left(\frac{1}{s^d}\int_{\mathrm{B}(y,\,s)} f^{q+\varepsilon/2} \,\mathrm{d}x\right)^{1/(q+\varepsilon/2)} \le \tilde{C}\left(\frac{1}{s^d}\int_{\mathrm{B}(y,\,2s)} f^q \,\mathrm{d}x\right)^{1/q}.$$

In particular, this inequality holds for $y = x_0$ and $s = r/2$ by dominated convergence. In this case, the inequality reads

$$\left(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\,r/2)\cap\Omega}|u|^{p_d+\varepsilon}\,\mathrm{d}x\right)^{2/(p_d+\varepsilon)} \leq \tilde{C}\left(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\,r)\cap\Omega}|u|^p\,\mathrm{d}x\right)^{2/p}$$

As $r/2 < r_0$ by assumption, Lemma 6.3 helps us to further extend the estimate above in the following way:

$$\left(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\,r/2)\cap\Omega}|u|^{p_d+\varepsilon}\,\mathrm{d}x\right)^{2/(p_d+\varepsilon)} \leq \tilde{C}\,\frac{1}{r^d}\int_{\mathrm{B}(x_0,\,2r)\cap\Omega}|u|^2\,\mathrm{d}x$$

Summing up, we see that for $0 < 4r' < r_0$ we have that we improved inequality (6.2) in Lemma 6.3 to

$$\left(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\,r')\cap\Omega}|u|^{p_d+\varepsilon}\,\mathrm{d}x\right)^{2/(p_d+\varepsilon)} \leq \tilde{C}\,\frac{1}{r^d}\int_{\mathrm{B}(x_0,\,4r)\cap\Omega}|u|^2\,\mathrm{d}x \tag{6.3}$$

The following extrapolation theorem by Shen [31, Thm. 3.3] will be necessary in order to derive L^p bounds on the solution of the Stokes resolvent system. Essentially, this theorem states that if the non locality of an L^2 bounded operator T can be quantified via a reverse Hölder estimate, then this operator extends to an L^p bounded operator for certain values p. In this sense, this extrapolation theorem can also be considered a p-sensitive version of the famous Calderón-Zygmund Lemma. Note that a more recent result from Tolksdorf [38, Thm. 4.1] generalizes this result to operators which are defined on spaces of Banach space valued functions.

Theorem 6.6. *Let T be a bounded sublinear operator on $\mathrm{L}^2(\Omega;\mathbb{C}^d)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^d$ and $\|T\|_{\mathfrak{L}(\mathrm{L}^2(\Omega;\mathbb{C}^d))} \leq C_0$. Let $p > 2$. Suppose that there exist constants $R_0 > 0$, $N > 1$ and $\alpha_2 > \alpha_1 > 1$ such that for any bounded measurable function f with $\operatorname{supp}(f) \subseteq \Omega \setminus \alpha_2 B$,*

$$\left\{\frac{1}{r^d}\int_{\Omega\cap B}|Tf|^p\,\mathrm{d}x\right\}^{1/p} \leq N\left\{\left(\frac{1}{r^d}\int_{\Omega\cap\alpha_1 B}|Tf|^2\,\mathrm{d}x\right)^{1/2} + \sup_{B'\supset B}\left(\frac{1}{|B'|}\int_{B'}|f|^2\,\mathrm{d}x\right)^{1/2}\right\},$$

where $B = \mathrm{B}(x_0, r)$ is a ball with $0 < r < R_0$ and either $x_0 \in \partial\Omega$ or $B(x_0, \alpha_2 r) \subseteq \Omega$. Then, the restriction of T to $\mathrm{L}^q(\Omega;\mathbb{C}^d)$ yields a bounded operator on $\mathrm{L}^q(\Omega;\mathbb{C}^d)$ for any $2 < q < p$. Moreover, the operator norm $\|T\|_{\mathfrak{L}(\mathrm{L}^q(\Omega;\,\mathbb{C}^d))}$ is bounded by a constant depending at most on d, N, C_0, p, q and the Lipschitz character of Ω.

We are now in the position to prove Theorem 1.17, the main theorem of this thesis. For this, the improved weak reverse Hölder inequality derived in Remark 6.5 will serve as the crucial ingredient, enabling us to apply the extrapolation theorem, Theorem 6.6, to a suitable family of operators. As we want to prove resolvent estimates of the Stokes operator on L^p, the family of operators under consideration will basically consist of resolvent operators. To this end, note that from the last sentence of Theorem 6.6 we get uniform bounds on our operator family on L^p provided that we start with a uniform bound C_0 on L^2. This aspect regarding the uniformity of the estimates and the p-sensitivity of the extrapolation theorem are the distinguished properties of this theorem compared to classic results from the Calderón-Zygmund theory of convolution operators, see Grafakos [17, Sec. 5.3] or Stein [36, Ch. 2].

Proof of Theorem 1.17. Consider a family of scaled solution operators to the Stokes resolvent system (1.9) with right-hand side $f \in \mathrm{L}^2(\Omega; \mathbb{C}^d)$, more precisely consider the family

$$T_\lambda \colon \mathrm{L}^2(\Omega; \mathbb{C}^d) \to \mathrm{L}^2(\Omega; \mathbb{C}^d), \quad f \mapsto (|\lambda| + 1)(A_2 + \lambda)^{-1}\mathbb{P}_2 f,$$

where $\lambda \in \Sigma_\theta$, for fixed $\theta \in (0, \pi/2)$. Let us first verify that $u := (|\lambda| + 1)^{-1} T_\lambda(f)$ does indeed solve (1.9). To this end, note that since $\mathbb{P}_2 f \in \mathrm{L}^2_\sigma(\Omega)$ we know that by the mapping properties of the Stokes resolvent we have $u \in \mathrm{H}^1_{0,\sigma}(\Omega)$ and

$$A_2 u + \lambda u = \mathbb{P}_2 f.$$

By testing this equation with elements of the space $\mathrm{H}^1_{0,\sigma}$, we see that u is a weak solution of

$$-\Delta u + \lambda u = \mathbb{P}_2 f.$$

By the usual arguments (c.f. Section 1.2), there exists a pressure $\pi \in \mathrm{L}^2(\Omega)$ such that

$$-\Delta u + \nabla\pi + \lambda u = f$$

holds in the sense of distributions. Furthermore, by testing this identity with $\overline{u}$ as in the proof of Lemma 4.5 we derive the estimate

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}x + \Big\{ \mathrm{Re}(\lambda) + \alpha \big| \mathrm{Im}(\lambda) \big| \Big\} \int_\Omega |u|^2 \, \mathrm{d}x \leq (1 + \alpha) \bigg| \int_\Omega f \cdot \overline{u} \, \mathrm{d}x \bigg|. \tag{6.4}$$

Now, using Lemma 4.5 to apply the reverse triangle inequality, we arrive at the estimate

$$\|T_\lambda(f)\|_{\mathrm{L}^2(\Omega;\, \mathbb{C}^d)} = (|\lambda| + 1) \, \|u\|_{\mathrm{L}^2(\Omega;\, \mathbb{C}^d)} \leq C_0 \, \|f\|_{\mathrm{L}^2(\Omega;\, \mathbb{C}^d)},$$

where $C_0 > 0$ only depends on d and θ. Accordingly, the family T_λ is bounded on $\mathrm{L}^2(\Omega; \mathbb{C}^d)$ and C_0 is a uniform bound on the operator norms $\|T_\lambda\|_{\mathfrak{L}(\mathrm{L}^2(\Omega;\mathbb{C}^d))}$.

We will now show that the operators T_λ fulfill the estimate in Theorem 6.6, in order to deduce their L^p boundedness. To this end, for α_2 as in Lemma 6.3 let $x_0 \in \overline{\Omega}$ and $0 < 4r < r_0$ such that $\mathrm{B}(x_0, \alpha_2 r) \subseteq \Omega$ or $\mathrm{B}(x_0, r)$ is centered on $\partial\Omega$. Furthermore, let $f \in \mathrm{L}^\infty(\Omega; \mathbb{C}^d)$ with support in $\Omega \setminus \mathrm{B}(x_0, \alpha_2 r)$. As shown before, the solution (u, π) fulfills (1.9) with right-hand side f in Ω. Furthermore, the pair also solves the Dirichlet problem

$$\begin{aligned} -\Delta u + \nabla\pi + \lambda u &= 0 \\ \operatorname{div}(u) &= 0 \end{aligned}$$

in $\Omega \cap \mathrm{B}(x_0, \alpha_2 r)$ where $u = 0$ on $\partial\Omega \cap \mathrm{B}(x_0, \alpha_2 r)$. Therefore, Remark 6.5 and more precisely the improved reverse Hölder inequality (6.3) give that

$$\left(\frac{1}{r^d}\int_{\Omega\cap\mathrm{B}(x_0,\,r)} |u|^p \,\mathrm{d}x\right)^{1/p} \leq C\left(\frac{1}{r^d}\int_{\Omega\cap\mathrm{B}(x_0,\,4r)} |u|^2 \,\mathrm{d}x\right)^{1/2},$$

where $p = p_d + \varepsilon$. Multiplying this inequality on both sides with $(|\lambda| + 1)$ gives

$$\left(\frac{1}{r^d}\int_{\Omega\cap\mathrm{B}(x_0,\,r)} |T_\lambda(f)|^p \,\mathrm{d}x\right)^{1/p} \leq C\left(\frac{1}{r^d}\int_{\Omega\cap\mathrm{B}(x_0,\,4r)} |T_\lambda(f)|^2 \,\mathrm{d}x\right)^{1/2}, \tag{6.5}$$

where $C > 0$ depends only on d, θ and the Lipschitz character of Ω. Now, Shen's extrapolation theorem, Theorem 6.6, gives that, for all $2 < q < p_d + \varepsilon$, the restriction of T_λ to $\mathrm{L}^q(\Omega; \mathbb{C}^d)$ is bounded on $\mathrm{L}^q(\Omega; \mathbb{C}^d)$ and that the operator norms $\|T_\lambda\|_{\mathfrak{L}(\mathrm{L}^q(\Omega;\mathbb{C}^d))}$ are uniformly bounded by a constant $C_q > 0$ depending only on d, θ, q and the Lipschitz character of Ω.

In the next step of the proof, we study the relationship between the operator T_λ and the resolvent of the Stokes operator A_q on $\mathrm{L}^q_\sigma(\Omega)$ for $q \in (2, p_d + \varepsilon)$. We will later treat the operators $A_{q'}$, where q' denotes the dual exponent to q, by a dualization argument. If necessary, we chose ε smaller than the parameter of the same name from Theorem 1.16 such that we may assume A_q to be a closed operator. Now, let $f \in \mathrm{L}^q_\sigma(\Omega)$. We already know that

$$u = (1 + |\lambda|)^{-1} T_\lambda(f) = (A_2 + \lambda)^{-1}\mathbb{P}_2 f \in \mathrm{L}^q_\sigma(\Omega) \cap \mathcal{D}(A_2)$$

by the mapping properties of $T_\lambda(f)$. As $\mathrm{L}^q_\sigma(\Omega) \subseteq \mathrm{L}^2_\sigma(\Omega)$, we have furthermore that

$$\lambda u + A_2 u = f \in \mathrm{L}^q_\sigma(\Omega)$$

and thus $A_2 u \in \mathrm{L}^q_\sigma(\Omega)$. Appealing to Definition 1.12, we showed that $u \in \mathcal{D}(A_q)$ and that $A_2 u = A_q u$. Therefore, we have that

$$\lambda u + A_q u = f \in \mathrm{L}^q_\sigma(\Omega).$$

By the uniqueness of u, which follows from the L^2 theory of the Stokes resolvent problem, we have that $u = (\lambda + A_q)^{-1} f$. Hence, the L^q boundedness of T_λ gives

$$\|u\|_{\mathrm{L}^q(\Omega;\,\mathbb{C}^d)} = \big\| (\lambda + A_q)^{-1} f \big\|_{\mathrm{L}^q(\Omega;\,\mathbb{C}^d)} \leq \frac{C_q}{1 + |\lambda|} \|f\|_{\mathrm{L}^q(\Omega;\,\mathbb{C}^d)}$$

and thus A_q is sectorial on $\mathrm{L}^q_\sigma(\Omega)$. Now consider the dual exponent $q' \in ((p_d + \varepsilon)', 2)$, where $q' = q/(q-1)$ and $p_d' = 2d/(d+1)$. As the results in Chapter 1 show, the space $\mathrm{L}^q_\sigma(\Omega)$ is reflexive and $\mathrm{L}^q_\sigma(\Omega)^*$ is isometrically isomorphic to $\mathrm{L}^{q'}_\sigma(\Omega)$.Therefore, standard properties of the dual operator yield that

$$\big\| (A_{q'} + \lambda)^{-1} \big\|_{\mathfrak{L}(\mathrm{L}^{q'}_\sigma(\Omega))} = \big\| \big((A_q + \lambda)^{-1}\big)^* \big\|_{\mathfrak{L}(\mathrm{L}^q_\sigma(\Omega)^*)} = \big\| (A_q + \lambda)^{-1} \big\|_{\mathfrak{L}(\mathrm{L}^q_\sigma(\Omega))} \leq \frac{C_q}{1 + |\lambda|}.$$

Consequently, also the operators $A_{q'}$ are sectorial as Theorem 1.16 assures that $A_{q'}$ is closed. This completes the proof. □

Appendix

In this chapter, we will provide the missing calculations for the proofs of Lemma 2.3 and Theorem 2.6. Therefore, we will build on the notations which were already established at the beginning of Chapter 2.

We first collect expressions for the derivatives of the fundamental solutions to the scalar Helmholtz equation and the Laplace equation in $d = 2$. For the fundamental solution to the Laplace equation $G(x;0) = -\frac{1}{2\pi}\log(|x|)$, the partial derivatives read:

$$\begin{aligned}
\partial_\gamma G(x;0) &= -\frac{1}{2\pi}\,\frac{x_\gamma}{|x|^2},\\
\partial_\alpha\partial_\gamma G(x;0) &= -\frac{1}{2\pi}\,\frac{\delta_{\alpha\gamma}}{|x|^2} + \frac{1}{\pi}\,\frac{x_\alpha x_\gamma}{|x|^4},\\
\partial_\beta\partial_\alpha\partial_\gamma G(x;0) &= \frac{1}{\pi}\,\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^4} - \frac{4}{\pi}\,\frac{x_\alpha x_\beta x_\gamma}{|x|^6}.
\end{aligned}$$

The fundamental solution for the scalar Helmholtz equation is given via

$$G(x;\lambda) = \frac{\mathrm{i}}{4}\,H_0^{(1)}(k|x|),$$

where $H_0^{(1)}(z)$ is the Hankel function of the first kind, see Section 2.1. Complex derivatives will be denoted by $\frac{\mathrm{d}}{\mathrm{d}z}$. We calculate using the chain rule and the product rule:

$$\begin{aligned}
\partial_\gamma G(x;\lambda) &= \frac{\mathrm{i}}{4}\,k\,\frac{x_\gamma}{|x|}\,\frac{\mathrm{d}}{\mathrm{d}z}H_0^{(1)}(k|x|),\\
\partial_\alpha\partial_\gamma G(x;\lambda) &= \frac{\mathrm{i}}{4}\,k\left(\frac{\delta_{\alpha\gamma}}{|x|} - \frac{x_\alpha x_\gamma}{|x|^3}\right)\frac{\mathrm{d}}{\mathrm{d}z}H_0^{(1)}(k|x|) + \frac{\mathrm{i}}{4}\,k^2\,\frac{x_\alpha x_\gamma}{|x|^2}\,\frac{\mathrm{d}^2}{\mathrm{d}z^2}H_0^{(1)}(k|x|),\\
\partial_\beta\partial_\alpha\partial_\gamma G(x;\lambda) &= \frac{\mathrm{i}}{4}\,k^3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^3}\,\frac{\mathrm{d}^3}{\mathrm{d}z^3}H_0^{(1)}(k|x|)\\
&\quad + \frac{\mathrm{i}}{4}\,k^2\left(\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^2} - 3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^4}\right)\frac{\mathrm{d}^2}{\mathrm{d}z^2}H_0^{(1)}(k|x|)\\
&\quad + \frac{\mathrm{i}}{4}\,k\left(3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^5} - \frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^3}\right)\frac{\mathrm{d}}{\mathrm{d}z}H_0^{(1)}(k|x|)\,.
\end{aligned}$$

We will now present the derivatives of the Hankel function $H_0^{(1)}(z)$ which appeared in the previous calculations by calculating the complex derivatives of the series expansion of $H_0^{(1)}(z)$ which may be found in Lebedev [23, Sec. 5.6]. The expansions below are a consequence of the series expansions of the Bessel functions of the first and second kind J_0 and Y_0, respectively. In the following, ψ will denote the *Digamma function*. The aforementioned expansions read:

$$\begin{aligned}
\frac{\pi}{2\mathrm{i}} H_0^{(1)}(z) &= \frac{\pi}{2\mathrm{i}} J_0(z) + \frac{\pi}{2} Y_0(z) \\
&= \sum_{l=0}^{\infty} \frac{(-1)^l}{(l!)^2\, 4^l} z^{2l} \Big(-\frac{\mathrm{i}\pi}{2} - \log(2) - \psi(l+1) \Big) + \sum_{l=0}^{\infty} \frac{(-1)^l}{(l!)^2\, 4^l} z^{2l} \log(z) \\
&= \sum_{l=0}^{\infty} a_l\, z^{2l}\, C_l + \sum_{l=0}^{\infty} a_l\, z^{2l} \log(z),
\end{aligned}$$

$$\begin{aligned}
\frac{\pi}{2\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z} H_0^{(1)}(z) &= \sum_{l=1}^{\infty} a_l\, (2l)\, z^{2l-1}\, C_l + \sum_{l=1}^{\infty} a_l\, (2l)\, z^{2l-1} \log(z) + \sum_{l=0}^{\infty} a_l\, z^{2l-1} \\
&= \sum_{l=1}^{\infty} b_l\, z^{2l-1}\, C_l + \sum_{l=1}^{\infty} b_l\, z^{2l-1} \log(z) + \sum_{l=0}^{\infty} a_l\, z^{2l-1},
\end{aligned}$$

$$\begin{aligned}
\frac{\pi}{2\mathrm{i}} \frac{\mathrm{d}^2}{\mathrm{d}z^2} H_0^{(1)}(z) &= \sum_{l=1}^{\infty} b_l\, (2l-1)\, z^{2l-2}\, C_l + \sum_{l=1}^{\infty} b_l\, (2l-1)\, z^{2l-2} \log(z) + \sum_{l=1}^{\infty} b_l\, z^{2l-2} \\
&\quad + \sum_{l=0}^{\infty} a_l\, (2l-1)\, z^{2l-2} \\
&= \sum_{l=1}^{\infty} c_l\, z^{2l-2}\, C_l + \sum_{l=1}^{\infty} c_l\, z^{2l-2} \log(z) + \sum_{l=1}^{\infty} b_l\, z^{2l-2} + \sum_{l=0}^{\infty} a_l\, (2l-1)\, z^{2l-2},
\end{aligned}$$

$$\begin{aligned}
\frac{\pi}{2\mathrm{i}} \frac{\mathrm{d}^3}{\mathrm{d}z^3} H_0^{(1)}(z) &= \sum_{l=2}^{\infty} c_l\, (2l-2)\, z^{2l-3}\, C_l + \sum_{l=2}^{\infty} c_l\, (2l-2)\, z^{2l-3} \log(z) + \sum_{l=1}^{\infty} c_l\, z^{2l-3} \\
&\quad + \sum_{l=2}^{\infty} b_l\, (2l-2)\, z^{2l-3} + \sum_{l=0}^{\infty} a_l\, (2l-1)\, (2l-2)\, z^{2l-3} \\
&= \sum_{l=2}^{\infty} d_l\, z^{2l-3}\, C_l + \sum_{l=2}^{\infty} d_l\, z^{2l-3} \log(z) + \sum_{l=1}^{\infty} c_l\, z^{2l-3} \\
&\quad + \sum_{l=2}^{\infty} b_l\, (2l-2)\, z^{2l-3} + \sum_{l=0}^{\infty} a_l\, (2l-1)\, (2l-2)\, z^{2l-3},
\end{aligned}$$

where, in order to increase readability, we introduced the following coefficients:

$$\begin{aligned}
C_l &:= -\frac{\mathrm{i}\pi}{2} - \log(2) - \psi(l+1), \\
a_l &:= \frac{(-1)^l}{(l!)^2\, 4^l}, \quad b_l := a_l \cdot 2l, \quad c_l := b_l \cdot (2l-1) \quad \text{and} \quad d_l := c_l \cdot (2l-2).
\end{aligned} \tag{D1}$$

A.1 Proof of Lemma 2.3 for $d = 2$

For $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi/2)$, we need to show that for $|\lambda||x|^2 \le (1/2)$ the estimate

$$\Big| \partial_\beta \partial_\alpha \partial_\gamma \big\{ G(x;\lambda) - G(x;0) \big\} \Big| \le C\, |\lambda||x|^{-1}$$

holds with a constant $C > 0$ that only depends on θ. The strategy will be to first estimate all absolute values of the resulting terms individually and to extract those which cannot be estimated in this way. We will call the terms whose absolute value that cannot be bounded individually *problematic*. In the second step, we will show that all problematic terms cancel when added together which shows that the claimed estimate holds.

In order to make the next calculations better to digest, we decompose the third derivative of $G(\,\cdot\,;\lambda)$ as follows:

$$\partial_\beta \partial_\alpha \partial_\gamma G(x;\lambda) = A_3 + A_2 + A_1,$$

where each A_i, $i = 1, \ldots, 3$, corresponds to the term involving the ith derivative of $H_0^{(1)}$. Let us start with A_3. We have

$$\begin{aligned}
A_3 = -\frac{1}{2\pi} k^3 \frac{x_\alpha x_\beta x_\gamma}{|x|^3} \cdot \Bigg\{ &\sum_{l=2}^{\infty} d_l\, (k|x|)^{2l-3}\, C_l + \sum_{l=2}^{\infty} d_l\, (k|x|)^{2l-3} \log(k|x|) \\
&+ \sum_{l=1}^{\infty} c_l\, (k|x|)^{2l-3} + \sum_{l=2}^{\infty} b_l\, (2l-2)\, (k|x|)^{2l-3} \\
&+ \sum_{l=0}^{\infty} a_l\, (2l-1)\, (2l-2)\, (k|x|)^{2l-3} \Bigg\}.
\end{aligned}$$

First, note that

$$\left| \frac{1}{2\pi} k^3 \frac{x_\alpha x_\beta x_\gamma}{|x|^3} \cdot c_1\, (k|x|)^{2\cdot 1-3} \right| \le C\, |\lambda||x|^{-1},$$

with a constant $C > 0$. This shows that the first term of the third sum in A_3 is not problematic. For the rest of A_3 we use the fact $|\lambda||x|^2 \le (1/2)$ to trade one $|k| = \sqrt{|\lambda|}$ in the prefactor for a constant times $|x|^{-1}$ and show that the prefactor behaves as

$$\left| k^3 \frac{x_\alpha x_\beta x_\gamma}{|x|^3} \right| \le C\, |\lambda||x|^{-1},$$

with a constant $C > 0$. Therefore, the only problematic term in A_3 is the first element of the last sum

$$-\frac{1}{2\pi}k^3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^3}\;\cdot\;a_0\,(2\cdot 0-1)\,(2\cdot 0-2)\,(k|x|)^{2\cdot 0-3}. \tag{P1}$$

For A_2, we calculate

$$\begin{aligned} A_2 = -\frac{1}{2\pi}k^2 &\left(\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^2} - 3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^4}\right) \\ &\cdot\Bigg\{\sum_{l=1}^{\infty} c_l\,(k|x|)^{2l-2}\,C_l + \sum_{l=1}^{\infty} c_l\,(k|x|)^{2l-2}\log(k|x|) \\ &\quad + \sum_{l=1}^{\infty} b_l\,(k|x|)^{2l-2} + \sum_{l=0}^{\infty} a_l\,(2l-1)\,(k|x|)^{2l-2}\Bigg\}. \end{aligned}$$

As the prefactor already behaves like $|\lambda||x|^{-1}$, we identify the following to terms as being problematic:

$$\begin{aligned} &-\frac{1}{2\pi}k^2\left(\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^2} - 3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^4}\right)\cdot\;c_1\,(k|x|)^{2\cdot 1-2}\log(k|x|) \quad \text{and} \\ &-\frac{1}{2\pi}k^2\left(\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^2} - 3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^4}\right)\cdot\;a_0\,(2\cdot 0-1)\,(k|x|)^{2\cdot 0-2}. \end{aligned} \tag{P2}$$

For the last component, we have the following identity:

$$\begin{aligned} A_1 = -\frac{1}{2\pi}k &\left(3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^5} - \frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^3}\right) \\ &\cdot\Bigg\{\sum_{l=1}^{\infty} b_l\,(k|x|)^{2l-1}\,C_l + \sum_{l=1}^{\infty} b_l\,(k|x|)^{2l-1}\log(k|x|) + \sum_{l=0}^{\infty} a_l\,(k|x|)^{2l-1}\Bigg\}. \end{aligned}$$

In this case, the prefactor behaves like $\sqrt{|\lambda|}|x|^{-2}$. Therefore, problematic terms only arise in the last two sums

$$\begin{aligned} &-\frac{1}{2\pi}k\left(3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^5} - \frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^3}\right)\cdot\;b_1\,(k|x|)^{2\cdot 1-1}\log(k|x|) \quad \text{and} \\ &-\frac{1}{2\pi}k\left(3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^5} - \frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^3}\right)\cdot\;a_0\,(k|x|)^{2\cdot 0-1}. \end{aligned} \tag{P3}$$

Note that we get from definition (D1)

$$a_0 = 1, \qquad c_1 = b_1 = -\frac{1}{2}.$$

Therefore, we can already see that the logarithmic terms in the sum (P2) + (P3) cancel. Next, we observe that if we take the sum over the problematic terms (P1), (P2) and (P3) and subtract $\partial_\beta\partial_\alpha\partial_\gamma G(x;0)$, the result is 0 which is easily seen by grouping the terms having the same power of $|x|$:

$$\begin{aligned}
&(\mathrm{P1}) + (\mathrm{P2}) + (\mathrm{P3}) - \partial_\beta\partial_\alpha\partial_\gamma G(x;0)\\
&\qquad = -\frac{1}{\pi}\,\frac{x_\alpha x_\beta x_\gamma}{|x|^6}\\
&\qquad\quad + \frac{1}{2\pi}\left(\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^4} - 3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^6}\right)\\
&\qquad\quad - \frac{1}{2\pi}\left(3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^6} - \frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^4}\right)\\
&\qquad\quad - \frac{1}{\pi}\,\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^4} + \frac{4}{\pi}\,\frac{x_\alpha x_\beta x_\gamma}{|x|^6} \quad = \quad 0\,.
\end{aligned}$$

This completes the proof of Theorem 2.3. □

A.2 Proof of Theorem 2.6 for $d = 2$

In this section, we show that, for all $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi/2)$ and $x \in \mathbb{R}^2 \setminus \{0\}$ satisfying $|\lambda||x|^2 \le (1/2)$, we have

$$\Big|\nabla_x\big\{\Gamma(x;\lambda) - \Gamma(x;0)\big\}\Big| \le C\,|\lambda||x|\,\big|\log(|\lambda||x|^2)\big|,$$

where $C > 0$ depends only on θ. The means and the strategy to prove this estimate are similar to the procedure in the previous section. In addition to the derivatives that were calculated at the beginning of this appendix, we will furthermore need the first partial derivatives for the matrix of fundamental solutions of the Stokes problem:

$$\begin{aligned}
\Gamma_{\alpha\beta}(x;0) &= \frac{1}{4\pi}\left\{-\delta_{\alpha\beta}\log(|x|) + \frac{x_\alpha x_\beta}{|x|^2}\right\}\\
\partial_\gamma\Gamma_{\alpha\beta}(x;0) &= \frac{1}{4\pi}\left(\frac{\delta_{\alpha\gamma}x_\beta + \delta_{\beta\gamma}x_\alpha - \delta_{\alpha\beta}x_\gamma}{|x|^2}\right) - \frac{1}{2\pi}\,\frac{x_\alpha x_\beta x_\gamma}{|x|^4}.
\end{aligned}$$

Now consider the difference

$$\begin{aligned}
&\partial_\gamma\Gamma_{\alpha\beta}(x;\lambda) - \partial_\gamma\Gamma_{\alpha\beta}(x;0)\\
&\qquad = \partial_\gamma G(x;\lambda)\,\delta_{\alpha\beta} + \frac{1}{k^2}\,\partial_\beta\partial_\alpha\partial_\gamma\Big\{G(x;\lambda) - G(x;0)\Big\} - \partial_\gamma\Gamma_{\alpha\beta}(x;0) =: B_1 + B_2 + B_3\,,
\end{aligned}$$

where we introduced the variables B_i, $i = 1, \ldots, 3$, for the sake of readability. As in the previous section, we will study the terms B_i independently and extract the terms that do not exhibit the desired behavior. For B_1, we have

$$B_1 = -\frac{1}{2\pi} k \, \frac{\delta_{\alpha\beta} x_\gamma}{|x|} \cdot \left\{ \sum_{l=1}^{\infty} b_l \, (k|x|)^{2l-1} \, C_l + \sum_{l=1}^{\infty} b_l \, (k|x|)^{2l-1} \log(k|x|) + \sum_{l=0}^{\infty} a_l \, (k|x|)^{2l-1} \right\}.$$

In this expression, we only detect one problematic term, namely the term corresponding to $l = 0$ in the last sum

$$-\frac{1}{2\pi} k \, \frac{\delta_{\alpha\beta} x_\gamma}{|x|} \cdot a_0 \, (k|x|)^{2\cdot 0-1}. \tag{Q1}$$

For the expression B_2, we have

$$B_2 = k^{-2} \Big(A_3 + A_2 + A_1 - \partial_\beta \partial_\alpha \partial_\gamma G(x;0) \Big) =: A_3' + A_2' + A_1',$$

with the variables A_i, $i = 1, \ldots, 3$, which were introduced in the previous section. We will now list the problematic terms for A_i' where we will already take into account the cancellations from the previous section. This will result on the one hand in additional terms and on the other hand in subsequent terms in the same sums compared to (P1), (P2) and (P3).

For A_3', we see that the following term does not meet the desired behavior:

$$-\frac{1}{2\pi} k \, \frac{x_\alpha x_\beta x_\gamma}{|x|^3} \cdot c_1 \, (k|x|)^{2\cdot 1-3}. \tag{Q2}$$

For A_2', we see that, compared to A_2, every summand is problematic which after cancellation leads to:

$$\begin{aligned}
&-\frac{1}{2\pi} \left(\frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^2} - 3 \, \frac{x_\alpha x_\beta x_\gamma}{|x|^4} \right) \cdot c_1 \, (k|x|)^{2\cdot 1-2} \, C_1 \\
&-\frac{1}{2\pi} \left(\frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^2} - 3 \, \frac{x_\alpha x_\beta x_\gamma}{|x|^4} \right) \cdot b_1 \, (k|x|)^{2\cdot 1-2} \\
&-\frac{1}{2\pi} \left(\frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^2} - 3 \, \frac{x_\alpha x_\beta x_\gamma}{|x|^4} \right) \cdot a_1 (2 \cdot 1 - 1) \, (k|x|)^{2\cdot 1-2}.
\end{aligned} \tag{Q3}$$

The same holds for the expression A_1':

$$\begin{aligned}
&-\frac{1}{2\pi} \frac{1}{k} \left(3 \, \frac{x_\alpha x_\beta x_\gamma}{|x|^5} - \frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^3} \right) \cdot b_1 \, (k|x|)^{2\cdot 1-1} \, C_1 \\
&-\frac{1}{2\pi} \frac{1}{k} \left(3 \, \frac{x_\alpha x_\beta x_\gamma}{|x|^5} - \frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^3} \right) \cdot a_1 \, (k|x|)^{2\cdot 1-1}.
\end{aligned} \tag{Q4}$$

Now it is time to sum all problematic terms, expand the variables and add the term B_3. As before, all the terms will cancel which will then prove our initial claim. With

$$a_0 = 1, \quad a_1 = -\frac{1}{4} \quad \text{and} \quad c_1 = b_1 = -\frac{1}{2},$$

we see that already within the sum (Q3) + (Q4) a lot of terms cancel which leaves us with:

$$-\frac{1}{2\pi}\left(\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^2} - 3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^4}\right) \cdot b_1\,(k|x|)^{2\cdot 1-2}. \tag{Q3'}$$

Now let us consider the expression (Q1) + (Q2) + (Q3') + B_3:

$$\begin{aligned} &-\frac{1}{2\pi}\frac{\delta_{\alpha\beta}x_\gamma}{|x|^2} + \frac{1}{4\pi}\frac{x_\alpha x_\beta x_\gamma}{|x|^4} + \frac{1}{4\pi}\left(\frac{\delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta + \delta_{\alpha\beta}x_\gamma}{|x|^2} - 3\,\frac{x_\alpha x_\beta x_\gamma}{|x|^4}\right) \\ &-\frac{1}{4\pi}\left(\frac{\delta_{\alpha\gamma}x_\beta + \delta_{\gamma\beta}x_\alpha - \delta_{\alpha\beta}x_\gamma}{|x|^2}\right) + \frac{1}{2\pi}\frac{x_\alpha x_\beta x_\gamma}{|x|^4} \quad = \quad 0\,. \end{aligned}$$

As all problematic terms add up to zero, we have proven the initial claim. □

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