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Masterthesis

On Resolvent Estimates in L^p for the Stokes Operator in Lipschitz Domains

Fabian Gabel

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Gutachter:	Prof. Dr. Robert Haller-Dintelmann
Betreut durch:	Dr. Patrick Tolksdorf

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Introduction

Chapter 1

Fundamentals

The purpose of this chapter is to collect basic definitions that will be used throughout the subsequent chapters. Furthermore we want to formulate the main problem regarding the resolvent estimates of the Stokes operator. Throughout this chapter we let d always denote a natural number greater or equal to 2.

1.1 Lipschitz-Domains

In this first section we will establish the fundamental notions regarding bounded Lipschitz domains.

defn:lipschitzDomain

Definition 1.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded open connected set. We call Ω a *bounded Lipschitz domain* if there exist $r_0, M > 0$ such that for all $x \in \partial\Omega$ there exists a function $\eta_x \colon \mathbb{R}^{d-1} \to \mathbb{R}$ which is Lipschitz continuous and fulfills $\eta_x(0) = 0$ and $\|\nabla \eta_x\|_{\mathrm{L}^\infty(\mathbb{R}^{d-1})} \leq M$, and a rotation $R_x \colon \mathbb{R}^d \to \mathbb{R}^d$ such that for all $0 < r \leq r_0$

$$R_x[\Omega - \{x\}] \cap D(r) = D_{\eta_x}(r)$$
$$R_x[\partial\Omega - \{x\}] \cap D(r) = I_{\eta_x}(r),$$

where

$$D(r) \coloneqq \{(x', x_d) \colon |x'| < r, |x_d| < 10d(M+1)r\}$$
$$D_{\eta_x}(r) \coloneqq \{(x', x_d) \colon |x'| < r, \eta_x(x') < x_d < 10d(M+1)r\}$$
$$I_{\eta_x}(r) \coloneqq \{(x', x_d) \colon |x'| < r, \eta_x(x') = x_d\}.$$

It is common to refer to sets of the form D_{η_x} as *Lipschitz cylinders*. If the point x in the definition of Lipschitz cylinders is not of particular importance we will denote the Lipschitz cylinder by $D_\eta(r)$.

If Ω is a bounded Lipschitz domain, $x \in \partial\Omega$ and $0 < r \leq r_0$, then we may define $U_{x,r} := \{x\} + R_x^{-1}D(r)$, where R_x is the rotation corresponding to x from Definition 1.1. This is all we need to define the Lipschitz character of a bounded Lipschitz domain Ω as suggested by Pipher and Verchota in [15, Sec. 5].

Definition 1.2. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and $x_1, \dots, x_n \in \partial\Omega$ be such that $\{U_{x_i,r_0}\}_{i=1}^n$ covers $\partial\Omega$. Furthermore let M be the constant from Definition 1.1. Then a constant $C > 0$ is said to depend on the *Lipschitz character of* Ω if it depends on M and n.

That the Lipschitz character is indeed a fruitful concept will be emphasized by the following theorem. This result is a crucial ingredient in the proof of the Rellich estimates in Chapter 4 as it provides a useful approximating property of Lipschitz domains. In short it enables us to approximate a bounded Lipschitz domain Ω by a sequence (Ω_j) of C^∞ domains in such a way that estimates on Ω_j with bounding constants that only depend on the Lipschitz characters may be extended to Ω when taking the limit. The original proof of this Theorem goes back to Nečas [14] and Verchota [23] . The presented version of this theorem appeared in Brown [1].

thm:smoothApproximation

Theorem 1.3 (Nečas, Verchota)**.** *Let Ω be a Lipschitz domain. Then there exists a sequence of C^∞-domains (Ω_k) with uniform Lipschitz characters, corresponding homeomorphisms $\Lambda_k\colon \partial\Omega \to \partial\Omega_k$, functions $\vartheta_k\colon \partial\Omega \to \mathbb{R}^+$ and a smooth compactly supported vector field $h\colon \mathbb{R}^d \to \mathbb{R}^d$ which satisfy the following properties:*

i) There exists a covering of $\partial\Omega$ by coordinate cylinders which also serve as coordinate cylinders for $\partial\Omega_k$.

ii) The homeomorhpisms $\Lambda_k\colon \partial\Omega \to \partial\Omega_k$ satisfy

$$\sup_{Q \in \partial\Omega} |Q - \Lambda_k(Q)| \to 0 \quad \text{as } k \to \infty$$

and $\Lambda_k(P)$ approaches P nontangentially meaning that for all $k \in \mathbb{N}$

$$|P - \Lambda_k(P)| < (1 + \beta)\operatorname{dist}(\Lambda_k(P), \partial\Omega)$$

for some constant β depending only on d and the Lipschitz character of Ω.

iii) The normals ν_k of $\partial\Omega_k$ satisfy $\lim_{k\to\infty} \nu_k(\Lambda_k(P)) = \nu(P)$ a. e. for all $P \in \partial\Omega$

iv) The functions ϑ_k *satisfy* $\delta \leq \vartheta_k \leq \delta^{-1}$ *for some* $\delta > 0$, $\vartheta^k \to 1$ *pointwise a. e. and*

$$\int_E \vartheta_k(Q)\,\mathrm{d}\sigma(Q) = \int_{\Lambda_k(E)} \mathrm{d}\sigma_k(Q),$$

where $E \subset \partial\Omega$ *is measurable and* σ_k *denotes the surface measure on* Ω_k.

v) The vector field h *satisfies* $\langle h, \nu_k \rangle \geq c > 0$ *a.e. on each* $\partial\Omega_k$ *where* ν_k *denotes the unit inner normal to* $\partial\Omega_k$.

The next concept we introduce will allow us to talk about boundary values of functions which are defined on Ω by considering their nontangential behavior. The first step will be to define nontangential approach regions. Unfortunately, in the literature there exist at least two different concepts which will be introduced in the next definitions. In the following, by a cone we mean an open, circular, truncated cone with only one convex component.

defn:regularFamilyOfCones

Definition 1.4 (Regular family of cones, Verchota)**.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. If $q \in \partial\Omega$, then $\Gamma(q)$ will denote a cone with vertex q and one component in Ω. Assigning to each $q \in \partial\Omega$ one cone $\Gamma(q)$ the family $\{\Gamma(q) \colon q \in \partial\Omega\}$ will be called *regular* if there exist $x_1, \ldots, x_{n_0} \in \partial\Omega$, $\tilde{r} > 0$ and rotations $\tilde{R}_{x_1}, \ldots, \tilde{R}_{x_{n_0}}$ such that

$$\partial\Omega \subset \bigcup_{i=1}^{n_0} \{x_i\} + \tilde{R}_{x_i}^{-1} D(4\tilde{r}/5),$$

and such that there exist Lipschitz continuous functions $\tilde{\eta}_{x_i} \colon \mathbb{R}^{d-1} \to \mathbb{R}$ such that for all $\tilde{r} \leq r \leq \nu\tilde{r}$ with

$$\nu := 1 + [1 + [10d(M+1)]^2]^{1/2}$$

we have

$$\begin{aligned} \tilde{R}_{x_i}[\Omega - \{x_i\}] \cap D(r) &= D_{\tilde{\eta}_{x_i}}(r) \\ \tilde{R}_{x_i}[\partial\Omega - \{x_i\}] \cap D(r) &= I_{\tilde{\eta}_{x_i}}(r). \end{aligned}$$

In addition for all i there exist cones α_i, β_i and γ_i with vertex at the origin and axis along the x_d-axis such that

$$\alpha_i \subset \overline{\beta_i} \setminus \{0\} \subset \gamma_i$$

and such that for all $q \in [\{x_i\} + \tilde{R}_{x_i}^{-1} D(4\tilde{r}/5)] \cap \partial\Omega$, we have

$$\begin{aligned} \tilde{R}_{x_i}^{-1}\alpha_i + \{q\} \subset \Gamma(q) \subset \overline{\Gamma(q)} \setminus \{q\} \subset \tilde{R}_{x_i}^{-1}\beta_i + \{q\}, \\ \tilde{R}_{x_i}^{-1}\gamma_i + \{q\} \subset [\{x_i\} + \tilde{R}_{x_i}^{-1} D(\tilde{r})] \cap \Omega. \end{aligned}$$

We will sometimes denote a regular cone as above by $\Gamma_{\mathrm{V}}(q)$.

For the existence of such families of cones see the Appendix of Verchota [23].

In Verchota cones $\Gamma_{\mathrm{V}}(q)$ we have the properties that for all Ω there exists a constant $C > 0$ depending only on the Lipschitz character such that for all $q, p \in \partial\Omega$ and any $x \in \Gamma_V(p)$ we have that

eq:verCone1

$$|x - q| \geq C|x - p| \tag{1.1}$$

eq:verCone2

$$|x - q| \geq C|p - q|. \tag{1.2}$$

For a proof see Verchota [23, p. 9f.]

defn:nontangentialApproachRegion

Definition 1.5 (Nontangential approach region, Shen)**.** For $\alpha > 1$ and $q \in \partial\Omega$ we define

$$\Gamma_\alpha(q) := \left\{x \in \Omega \setminus \partial\Omega \colon |x - q| < \alpha \operatorname{dist}(x, \partial\Omega)\right\}$$

If α is chosen sufficiently large (see Shen [**?**]) we call $\{\Gamma_\alpha(q) \colon q \in \partial\Omega\}$ a *family of nontangential approach regions*.

Note that in Shen cones $\Gamma_\alpha(q)$, we have that for $q, y \in \partial\Omega$ and $x \in \Gamma_\alpha(q)$

eq:shenConeEstimate

$$\begin{aligned}|q - y| &\leq |q - x| + |x - y| \leq \alpha \operatorname{dist}(x, \partial\Omega) + |x - y|\\ &\leq (\alpha + 1)|x - y|\end{aligned} \tag{1.3}$$

where α is the constant from Definition 1.5.

Depending on the type of cones used one may introduce similar concepts of nontangential convergence and nontangential maximal functions.

Definition 1.6. For a function u in Ω and a fixed family of nontangential approach regions $\{\Gamma_\alpha\}$, we define the nontangential maximal function $(u)^*_\alpha$ by

eq:defnNontangMaxFunction

$$(u)^*_\alpha(q) = \sup\left\{|u(x)| \colon x \in \Gamma_\alpha(q)\right\} \tag{1.4}$$

for $q \in \partial\Omega$. For a fixed regular family of cones $\{\Gamma_{\mathrm{V}}(q)\}$ we define the nontangential maximal function $N(u)(q)$ via

$$N(u)(q) = \sup\left\{|u(x)| \colon x \in \Gamma_{\mathrm{V}}(q)\right\}.$$

Note that Tolksdorf [22] and Shen [**?**] show that the choice of α for the nontangential maximal function as in 1.4 does not affect their p-norms in an unpredictable way. In fact their p-norms for different α_1 and α_2 stay comparable with a constant only depending on d, α_1, α_2 and the Lipschitz character. We will therefore for a given bounded Lipschitz domain always assume that $\alpha > 1$ has been choosen big enough such that on the one hand condition (ii) from Theorem 1.3 is fulfilled and that on the other hand α is large enough such that $\{\Gamma_\alpha(q)\colon q \in \partial\Omega\}$ is a family of nontangential aproach regions. In the following we will thus ignore the parameter α in cones and nontangential maximal functions and tacitly assume that it was chosen appropriately. We further note that the functions $(u)^*$ and $N(u)$ will not be comparable in general, see the discussion in Tolksdorf.

The above mentioned constructions of cones are not limited to cones that lay in the interior of the domain Ω. In fact the same construction can be carried out for the exterior domain $\mathbb{R}^d \setminus \overline{\Omega}$ yielding cones that lay outside of Ω. While Verchota's cones from Definition 1.4 can be mirrored along the $x_d = 0$ plane in a suitable local coordinate system, Shen's cones from Definition 1.5 have to be modified in a natural way to give cones lying inside of $\mathbb{R}^d \setminus \overline{\Omega}$, namely

$$\Gamma_\alpha^{\text{ext}}(q) \coloneqq \left\{x \in \mathbb{R}^d \setminus \overline{\Omega}\colon |x - q| < \alpha \operatorname{dist}(x, \partial\Omega)\right\}.$$

As the name *nontangential approach region* suggests, for functions u living on Ω or $\mathbb{R}^d \setminus \overline{\Omega}$ there will be a notion of convergence of function values $u(x)$ as x goes to a point on $p \in \partial\Omega$. The idea is to restrict the set of directions from which one can approach p by only allowing sequences of points lying in cones $\Gamma(q)$.

Definition 1.7 (Nontangential convergence)**.** Let Ω be a bounded Lipschitz domain and $\{\Gamma(q)\colon q \in \partial\Omega\}$ be a family of nontangential approach regions with its exterior counterpart $\{\Gamma^{\text{ext}}(q)\colon q \in \partial\Omega\}$. Let furthermore u be a function on $\mathbb{R}^d \setminus \partial\Omega$ and f a function on $\partial\Omega$. We say that $u = f$ *in the sense of nontangential convergence from the inside* if

$$\lim_{\substack{x \to q \\ x \in \Gamma(q)}} u(x) = f(q), \quad \text{for a. e. } q \in \partial\Omega$$

and we say that $u = f$ *in the sense of nontangential convergence from the outside* if

$$\lim_{\substack{x \to q \\ x \in \Gamma^{\text{ext}}(q)}} u(x) = f(q), \quad \text{for a. e. } q \in \partial\Omega.$$

If both of the above limits exist and coincide we say that $u = f$ *in the sense of nontangential convergence*.

Usually the nontangential limits taken from inside and outside the domain will differ. For functions u on $\mathbb{R}^d \setminus \partial\Omega$ we will therefore often use the notation u_+ to denote the *inner* nontangential limit and u_- for the respective *outer* nontangential limit.

To put our new vocabulary to use, we will formulate an prove the divergence theorem for functions on bounded Lipschitz domains that do not have a trace but nontangential limits and integrable nontangential maximal functions. A similar statement was proven by Shen in [**?**, Thm. 7.1.6].

prop:approximationArgument

Proposition 1.8. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, a bounded Lipschitz domain and $f\colon \Omega \to \mathbb{C}^d$ smooth and $g\colon \partial\Omega \to \mathbb{C}$ measurable. Suppose that the nontangential limit f_+ exists almost everywhere and that the nontangential maximal function $(g)^*$ is integrable on $\partial\Omega$ and $|f_+| \leq (g)^*$ a.e.. Then Green's formula*

$$\int_{\partial\Omega} f_k(s) n_k(s)\,\mathrm{d}\sigma(s) = \int_{\Omega} \operatorname{div}(f)(x)\,\mathrm{d}x \tag{1.5}$$

holds, where n denotes the outer unit normal vector of $\partial\Omega$.

Proof. The proof rests heavily on the powerful Theorem 1.3 and uses its full capacity to uncover a very useful approximation argument.

Let's start by approximating Ω by a sequence $(\Omega)_l$ of C^∞ domains with uniform Lipschitz characters as described in Theorem 1.3. Remember that by Theorem 1.3 iv), the homeomorphisms $\Lambda_l\colon \partial\Omega \to \partial\Omega_l$ give rise to a tranformation rule of the form

eq:transformation

$$\int_{\partial\Omega_l} f_k(s) n_k^{(l)}(s)\,\mathrm{d}\sigma_l(s) = \int_{\partial\Omega} \vartheta_l(x)\, f_k(\Lambda_l(x))\, n_k^{(l)}(\Lambda_l(x))\,\mathrm{d}\sigma(x). \tag{1.6}$$

The idea of the proof is based on the approximation argument performed in Brown [1, Prop. 2.4]. Additionally we have $\lim_{l\to\infty} \vartheta_l(x) = 1$ and $\lim_{l\to\infty} \Lambda_l(x) = x$ almost everywhere, where $\Lambda_l(x) \in \Gamma(x)$ for all $l \in \mathbb{N}$ thanks to Theorem 1.3 ii). Furthermore, we know that $\lim_{l\to\infty} n_k^{(l)}(\Lambda_l(x)) = n_k(x)$ almost everywhere by Theorem 1.3 and that f has a nontangential limit almost everywhere. This gives us that

$$\lim_{l\to\infty} \vartheta_l(x)\, f_k(\Lambda_l(x))\, n_k^{(l)}(\Lambda_l(x)) = f_k(x) n_k(x), \quad \text{a. e. } x \in \partial\Omega.$$

As this sequence of integrands is dominated by $\delta(g)^*$ with $(g)^* \in \mathrm{L}^1(\partial\Omega)$ by assumption and δ the uniform bound to ϑ_l due to Theorem 1.3 iv), the dominated convergence theorem is applicable and yields

eq:leftGreen

$$\lim_{l\to\infty} \int_{\partial\Omega} \vartheta_l(s)\, f_k(\Lambda_l(s))\, n_k^{(l)}(\Lambda_l(s))\,\mathrm{d}\sigma(s) = \int_{\partial\Omega} f_k(s) n_k(s)\,\mathrm{d}\sigma(s). \tag{1.7}$$

Now consider the left hand side of identity (1.6). By Green's formula [**?**, p. 711f.] we know that

$$\int_{\partial\Omega_l} f_k(s) n_k^{(l)}(s)\,\mathrm{d}\sigma_l(s) = \int_{\Omega_l} \operatorname{div}(f(x))\,\mathrm{d}x, \quad \text{for all } l \in \mathbb{N}.$$

As $\Omega_l \subseteq \Omega$ for all $l \in \mathbb{N}$, the monotone convergence Theorem leaves us with

eq:rightGreen

$$\lim_{l\to\infty} \int_{\Omega_l} \operatorname{div}(f(x))\,\mathrm{d}x = \int_{\Omega} \operatorname{div}(f(x)\,\mathrm{d}x. \tag{1.8}$$

Gluing together equations (1.7) and (1.8) gives the claim. □

1.2 The Stokes Operator

sec:stokesOperator

In this section, we will introduce the Stokes operator on $\mathrm{L}^2(\Omega; \mathbb{C}^d)$ and $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ for general p and establish a relation to the *Dirichlet problem for the Stokes resolvent system*

$$\begin{aligned} -\Delta u + \nabla \phi + \lambda u &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.9}$$

eq:stokesResolventSystem

where $\lambda \in \Sigma_\theta := \{z \in \mathbb{C} \colon \lambda \neq 0 \text{ and } |\arg(z)| < \pi - \theta\}$ and $\theta \in (0, \pi/2)$.

We beginn by defining the relevant function spaces. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $1 < p < \infty$. We define

$$\mathrm{C}^\infty_{c,\sigma}(\Omega) := \{\varphi \in \mathrm{C}^\infty_c(\Omega; \mathbb{C}^d) \colon \operatorname{div}(\varphi) = 0\},$$

which can serve as a suitable space of test functions. We can now close this space in $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ and the Sobolev space $\mathrm{W}^{1,p}(\Omega; \mathbb{C}^d)$ which gives

$$\mathrm{L}^p_\sigma(\Omega) := \overline{\mathrm{C}^\infty_{c,\sigma}(\Omega)}^{\mathrm{L}^p}$$

and

$$\mathrm{W}^{1,p}_{0,\sigma}(\Omega) := \overline{\mathrm{C}^\infty_{c,\sigma}(\Omega)}^{\mathrm{W}^{1,p}},$$

respectively. If $p = 2$, we will use the symbol $\mathrm{H}^1_{0,\sigma}(\Omega)$ to denote $\mathrm{W}^{1,2}_{0,\sigma}(\Omega)$ in order to emphasize that this space is a Hilbert space.

In order to define the Stokes operator, we introduce the following sesquilinear form

$$\mathfrak{a} \colon \mathrm{H}^1_{0,\sigma}(\Omega) \times \mathrm{H}^1_{0,\sigma}(\Omega) \to \mathbb{C}, \quad (u, v) \mapsto \int_\Omega \nabla u \cdot \overline{\nabla v} \, \mathrm{d}x.$$

Note that for $u \in \mathrm{H}^1_{0,\sigma}(\Omega)$ the gradient ∇u is a matrix and an element of the space $\mathrm{L}^2(\Omega; \mathbb{C}^{d \times d})$.

defn:stokes

Definition 1.9. The *Stokes operator* A_2 on $\mathrm{L}^2_\sigma(\Omega)$ is given via

$$\begin{aligned} \mathcal{D}(A_2) &:= \Big\{u \in \mathrm{H}^1_{0,\sigma}(\Omega) \colon \exists! f \in \mathrm{L}^2_\sigma(\Omega) \text{ s.t. } \forall v \in \mathrm{H}^1_{0,\sigma}(\Omega) \colon \mathfrak{a}(u, v) = \int_\Omega f \cdot \overline{v} \, \mathrm{d}x\Big\} \\ A_2 u &:= f, \end{aligned}$$

where $u \in \mathcal{D}(A_2)$ and f comes from the definition of the domain.

The following theorem from Mitrea and Monniaux [12, Thm 4.7] shows that our definition of the Stokes operator and the one used in Shen's paper [17] coincide. Another advantage of this characterization is the immediate link of the Stokes operator to the Stokes system.

thm:stokesOperatorL2

Theorem 1.10. *If $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is a bounded Lipschitz domain and A_2 is the Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$ then*

$$\mathcal{D}(A_2) = \big\{u \in H^1_{0,\sigma}(\Omega)\colon \exists \pi \in \mathrm{L}^2(\Omega) \text{ s.t. } -\Delta u + \nabla\pi \in \mathrm{L}^2_\sigma(\Omega)\big\},$$

where the expression $\Delta u + \nabla\pi \in \mathrm{L}^2_\sigma(\Omega)$ needs to be understood in the distributional sense. For $u \in \mathcal{D}(A_2)$ and the corresponding pressure π we have

$$A_2 u = -\Delta u + \nabla\pi.$$

The following proposition summarizes some facts about the Stokes operator on $\mathrm{L}^2_\sigma(\Omega)$. A proof can be found in Tolksdorf [22].

prop:stokesOperatorL2

Proposition 1.11. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and A_2 the Stokes operator as in Definition 1.9. Then we have*

a) A_2 is closed with dense domain. Furthermore $0 \in \rho(A_2)$.

b) $\sigma(A) \subset [0,\infty)$ and for all $\theta \in (0,\pi]$ there exists $C > 0$ such that

eq:resolventEstimateL2

$$\|\lambda(\lambda + A)^{-1}\|_{\mathfrak{L}(\mathrm{L}^2_\sigma(\Omega))} \leq C, \quad \text{for all } \lambda \in \mathbb{C} \setminus \Sigma_\theta. \tag{1.10}$$

In particular $-A_2$ generates a bounded analytic semigroup on $\mathrm{L}^2_\sigma(\Omega)$.

With these results at hand we can now give a quick recap of the solution theory to (1.9). Let $f \in \mathrm{L}^2_\sigma(\Omega)$ and $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi/2)$. By the previous theorem and proposition we know that there exists a unique $u \in \mathcal{D}(A_2) \subseteq \mathrm{H}^1_{0,\sigma}(\Omega)$ and some $\pi \in \mathrm{L}^2(\Omega)$ such that

$$-\Delta u + \nabla\pi + \lambda u = A_2 u + \lambda u = f.$$

For general $f = \mathrm{L}^2(\Omega;\mathbb{C}^d)$ we use the *Helmholtz projection* $\mathbb{P}_2$ to get

$$\Delta u + \nabla\pi + \lambda u + (I - \mathbb{P}_2)f = \mathbb{P}_2 f + (I - \mathbb{P}_2)f = f,$$

where u and π now correspond to $\mathbb{P}_2 f \in \mathrm{L}^2_\sigma(\Omega)$. On bounded Lipschitz domains the orthogonal complement to $\mathbb{P}_2[\mathrm{L}^2(\Omega;\mathbb{C}^d)] = \mathrm{L}^2_\sigma(\Omega)$ is characterized via

$$\mathrm{L}^2_\sigma(\Omega)^\perp = \big\{f \in \mathrm{L}^2(\Omega;\mathbb{C}^d)\colon f = \nabla\phi, \text{ for some } \phi \in \mathrm{L}^2(\Omega)\big\}.$$

A proof of this fact can be found in the book of Sohr [19, Lemma 2.5.3]. Using this result we find $g \in \mathrm{L}^2(\Omega)$ such that $\nabla g = (I - \mathbb{P}_2)f$ in the sense of distributions and we see that

$$-\Delta u + \nabla(\pi + g) + \lambda u = f.$$

Consequently, we see that solving the resolvent equation for the Stokes operator and solving the Stokes resolvent system (1.9) are two sides of the saime coin. Furthermore, we may deduce from the resolvent estimate (1.10) that the solution u which apparently is not affected by the additional part $(I - \mathbb{P}_2)f$ fulfills the inequality

$$|\lambda|^{-1}\|u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} = |\lambda|^{-1}\|(A_2 + \lambda)^{-1}\mathbb{P}_2 f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \leq C\|f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)},$$

where C depends only on θ. By the calculations above it is understandable why this estimate on u instead of (1.10) is sometimes called *resolvent estimate*.

In order to develop an L^p-theory for system (1.9), one way is to study the Stokes operator on subspaces of $\mathrm{L}^p(\Omega; \mathbb{C}^d)$. More precisely, we are interested in estimating solutions $u \in \mathrm{H}^1_0(\Omega; \mathbb{C}^d)$, in $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ provided that the right hand side of the Stokes resolvent system (1.9) is an element of the space $\mathrm{L}^2(\Omega; \mathbb{C}^d) \cap \mathrm{L}^p(\Omega; \mathbb{C}^d)$. This is once again just one side of the aforementioned coin. The other side just asks for a resolvent estimate on the Stokes operator, hoping that in analogy to Proposition 1.11 this leads to an analytic semigroup.

`defn:stokeslp`

Definition 1.12. Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ be a bounded Lipschit domain. If $p > 2$ we define the Stokes operator A_p via its *part of* A_2 *in* $\mathrm{L}^p_\sigma(\Omega)$.

$$\begin{aligned} \mathcal{D}(A_p) &:= \Big\{u \in \mathcal{D}(A_2) \cap \mathrm{L}^p_\sigma(\Omega) \colon A_2 u \in \mathrm{L}^p_\sigma(\Omega)\Big\} \\ A_p u &:= A_2 u, \quad u \in \mathcal{D}(A_p). \end{aligned}$$

For $p > 2$ there exists an analog to Theorem 1.10. The peculiar range of p for which this theorem holds is due to the fact that the boundedness of the Helmholtz projection on $\mathrm{L}^p(\Omega; \mathbb{C}^d)$ is a crucial ingredient to the proof and a fundamental pillar of the L^p-theory of the Stokes equations. More details about the mechanics of the Helmholtz projection can be found in Tolksdorf [22, Sec. 5.1].

`thm:domainStokesOperatorLp`

Theorem 1.13 (Thm. 5.2.11, [22]). *Let* $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, *is a bounded Lipschitz domain. Then there exists* $\varepsilon > 0$ *such that for all*

$$2 < p < \frac{2d}{d-1} + \varepsilon$$

the domain of the Stokes operator A_p is characterized as

$$\mathcal{D}(A_p) = \big\{u \in \mathrm{W}^{2,p}_{0,\sigma}(\Omega) \colon \exists \pi \in \mathrm{L}^p(\Omega) \ s.t. \ -\Delta u + \nabla\pi \in \mathrm{L}^p_\sigma(\Omega)\big\},$$

where the expression $\Delta u + \nabla\pi \in \mathrm{L}^p_\sigma(\Omega)$ needs to be understood in the distributional sense. For $u \in \mathcal{D}(A_p)$ and the corresponding pressure π we have

$$A_p u = -\Delta u + \nabla\pi.$$

For $p < 2$ there exist various ways to define the Stokes operator. One adequate way is to dualize the operator $A_{p'}$, where $p' = p/(1+p)$ is the Hölder conjugate exponent. In order to carry out this construction we need the spaces $\mathrm{L}^p_\sigma(\Omega)$ to exhibit the same behavior regarding dualization as the spaces $\mathrm{L}^p(\Omega; C^d)$.

lem:duality

Lemma 1.14 (Lem 5.2.13, [22])**.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

the spaces $\mathrm{L}^p_\sigma(\Omega)$ and $(\mathrm{L}^{p'}_\sigma(\Omega))^*$ are isomorphic, where $(\mathrm{L}^{p'}_\sigma(\Omega))^*$ denotes the antidual space and $p' = p/(p-1)$ is the Hölder conjugate exponent of p. The isomorphism Ψ is given by

$$[\Psi f](g) = \int_\Omega f \cdot \overline{g} \, \mathrm{d}x, \quad g \in \mathrm{L}^{p'}_\sigma(\Omega).$$

Now we define the Stokes operator for $p < 2$ as announced.

Definition 1.15. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and let $\varepsilon > 0$ be as in Lemma 1.14. Let furthermore

$$\frac{2d}{d+1} - \varepsilon < p < 2$$

and Ψ be the isomorphism from Lemma 1.14. Then the *Stokes operator on* $\mathrm{L}^p_\sigma(\Omega)$ is defined via

$$\begin{aligned} \mathcal{D}(A_p) &:= \big\{u \in \mathrm{L}^p_\sigma(\Omega) \colon \Psi u \in \mathcal{D}(A^*_{p'})\big\} \\ A_p &= := \Psi^{-1} A^*_{p'} \Psi u, \end{aligned}$$

where $p' = p/(1-p)$ denotes the Hölder conjugate exponent of p and $A^*_{p'}$ the adjoint operator to $A_{p'}$.

Without investing too much additional work, it is now possible to prove the following Theorem.

thm:stokesOperatorLp

Theorem 1.16 (Thm. 5.2.9 and Thm. 5.2.17, [22])**.** *Let* $\Omega \subset \mathbb{R}^d$*,* $d \geq 2$*, be a bounded Lipschitz domain. Then there exists* $\varepsilon > 0$ *such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

the operator A_p *is closed with dense domain. Furthermore* $0 \in \rho(A_p)$*.*

The natural question arises when comparing Theorem 1.16 with the Hilbert space counterpart Theorem 1.11: Does the Stokes operator generate a bounded analytic semigroup in L^p_σ? An affirmative answer was given by Shen in 2012 with his seminal paper [17] by proving the necessary resolvent estimates for $d \geq 3$.

thm:main

Theorem 1.17 (Shen)**.** *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^d$*,* $d \geq 3$*. There exists* $\varepsilon > 0$*, such that for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

there exists a constant $C > 0$ *such that for every* $f \in \mathrm{L}^p_\sigma(\Omega)$ *and all* $\lambda \in \Sigma_\theta$ *the inequality*

$$|\lambda| \|(\lambda + A_p)^{-1}\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)} \leq C \|f\|_{\mathrm{L}^p(\Omega;\mathbb{C}^d)}$$

holds. In particular $-A_p$ *is the generator of a bounded analytic semigroup on* $\mathrm{L}^p_\sigma(\Omega)$*.*

This Theorem gave an affirmative answer to Taylor's conjecture in [20]. Curiously this positive result is limited to $d \geq 3$ even though Shen states that the approach he developed should also work in the case $d = 2$. This sets the starting point for the present thesis. In the subsequent chapters we will not only present Shen's approach to the problem of the resolvent estimates, we will furthermore extend his results whenever possible to the two dimensional case.

Chapter 2

Estimating Fundamental Solutions

chap:2

The purpose of this section is to study fundamental solutions of the Stokes resolvent problem and to deduce related estimates which will be crucial for the next chapters. Before working on the Stokes resolvent problem we will take a look at the atoms of the fundamental solution of this problem: the Hankel functions.

As a basis for the subsequent sections and chapters let us fix recurring quantities regarding sectors in the complex plane $\mathbb{C}$.

Let $\theta \in (0, \pi/2)$ and $\lambda \in \Sigma_\theta$ as in Section 1.2. The polar form of λ is given as $\lambda = r\mathrm{e}^{\mathrm{i}\tau}$ with $0 < r < \infty$ and $-\pi + \theta < \tau < \pi - \theta$. Now set

$$k = \sqrt{r}\,\mathrm{e}^{\mathrm{i}(\pi+\tau)/2}.$$

Then we have

$$k^2 = -\lambda \quad \text{and} \quad \frac{\theta}{2} < \arg(k) < \pi - \frac{\theta}{2}$$

as it holds

$$\arg(k) = \frac{\pi + \tau}{2} > \frac{\pi}{2} + \frac{-\pi + \theta}{2} = \frac{\theta}{2}$$

on the one hand and

$$< \frac{\pi}{2} + \frac{\pi - \theta}{2} = \pi - \frac{\theta}{2}$$

on the other hand. The preceding calculation gives rise to the following estimate:

eq:imaginaryPartEstimate

$$\operatorname{Im}(k) > \sqrt{|\lambda|}\sin(\theta/2) > 0. \tag{2.1}$$

Indeed, we have

$$\operatorname{Im}(k) = \sqrt{r}\sin\left(\frac{\pi+\tau}{2}\right) = \sqrt{|\lambda|}\sin\left(\frac{\pi+\tau}{2}\right) \quad \text{and} \quad \frac{\theta}{2} < \frac{\pi+\tau}{2} < \pi - \frac{\theta}{2}$$

which gives for τ with $\frac{\pi+\tau}{2} \leq \frac{\pi}{2}$ that $\sin(\frac{\pi+\tau}{2}) \geq \sin(\frac{\theta}{2})$ and for τ with $\frac{\pi+\tau}{2} > \frac{\pi}{2}$ that $\sin(\frac{\pi+\tau}{2}) > \sin(\pi - \frac{\theta}{2}) = \sin(\frac{\theta}{2})$.

2.1 Hankel Functions and the Helmholtz equation

Before diving into fundamental solutions of the Stokes resolvent problem, we will first consider a fundamental solution for the (scalar) Helmholtz equation in $\mathbb{R}^d$

$$-\Delta u + \lambda u = 0.$$

One fundamental solution with pole at the origin is given by

eq:definitionFundamentalHelmholtz

$$G(x;\lambda) = \frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}} \cdot \frac{1}{|x|^{d-2}} \cdot (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|), \tag{2.2}$$

see McLean [?, Eq. (9.14)], where $H^{(1)}_\nu(z)$ is the Hankel function of the first kind which according to Lebedev [11, Sec. 5.11] can be also be written as

eq:integralRepresentationHankel

$$H^{(1)}_\nu(z) = \frac{2^{\nu+1}\mathrm{e}^{\mathrm{i}(z-\nu\pi)} z^\nu}{\mathrm{i}\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^\infty \mathrm{e}^{2z\mathrm{i}s} s^{\nu-\frac{1}{2}}(1+s)^{\nu-\frac{1}{2}}\,\mathrm{d}s. \tag{2.3}$$

This formula holds for $\nu > -\frac{1}{2}$ and $0 < \arg(z) < \pi$. We will usually set

$$\nu = \nu_d = \frac{d}{2} - 1 \quad \text{and} \quad z = k|x|.$$

Note that by (2.1) we will always have $\operatorname{Im}(z) > 0$. Since $\nu_d < \nu_{d+1}$ for all $d \geq 2$ and $\nu_2 = 0$, formula (2.3) will hold for all dimensions $d \geq 2$ and all $x \in \mathbb{R}^d$.

In the case $d = 2$, formula (2.2) simplifies to

eq:2dDefinitionFundamentalHelmholtz

$$G(x;\lambda) = \frac{\mathrm{i}}{4} H^{(1)}_0(k|x|). \tag{2.4}$$

In the case $d = 3$, one has an even easier formula, namely

eq:3dDefinitionFundamentalHelmholtz

$$G(x;\lambda) = \frac{\mathrm{i}}{4(2\pi)^{1/2}} \cdot \frac{1}{|x|} \cdot (k|x|)^{1/2} H^{(1)}_{1/2}(k|x|) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\pi|x|}, \tag{2.5}$$

which is due to an easy formula for $H^{(1)}_{1/2}(z)$ in Lebedev, see [11, Eq. (5.8.4)] and [?, Eq. (9.15)].

Our first estimate is concerned with estimates on the fundamental solution and its derivatives for the (scalar) Helmholtz equation. The main concern of this lemma is with the asymptotic behavior of $G(\,\cdot\,,\lambda)$ for large values of $|x|$.

lem:estimateHelmholtzDerivatives

Lemma 2.1. *Let $\lambda \in \Sigma_\theta$. Then*

eq:estimateHelmholtzDerivatives

$$|\nabla^l_x G(x;\lambda)| \leq \frac{C_l \mathrm{e}^{-c\sqrt{|\lambda|}|x|}}{|x|^{d-2+l}} \tag{2.6}$$

for any integer $l \geq 0$ if $d \geq 3$ and for $l \geq 1$ if $d = 2$. Here, $c > 0$ depends only on θ and C_l depends only on d, l and θ.

Let $d = 2$. Then $|G(x;\lambda)| = o(1)$ as $|x| \to \infty$.

Proof. We start with the case $l = 0$ and $d \geq 3$. Let $\operatorname{Im}(z) > 0$ and $\nu - \frac{1}{2} \geq 0$. Then (2.3) gives

$$|H_\nu^{(1)}(z)| \leq C_d\, \mathrm{e}^{-\operatorname{Im}(z)} |z|^\nu \int_0^\infty \mathrm{e}^{-2s\operatorname{Im}(z)} s^{\nu-\frac{1}{2}} (1+s)^{\nu-\frac{1}{2}}\, \mathrm{d}s, \qquad \text{eq:firstEstimate} \tag{2.7}$$

where $C_d > 0$ depends only on d. We apply the substitution rule with $t = s - (1/2)$ and calculate

$$\begin{aligned}
\mathrm{e}^{\frac{-\operatorname{Im}(z)}{2}} \int_0^\infty \mathrm{e}^{-s\operatorname{Im}(z)} s^{\nu-\frac{1}{2}} (1+s)^{\nu-\frac{1}{2}}\, \mathrm{d}s &= \int_0^\infty \mathrm{e}^{-(s+\frac{1}{2})\operatorname{Im}(z)} s^{\nu-\frac{1}{2}} (1+s)^{\nu-\frac{1}{2}}\, \mathrm{d}s \\
&= \int_{\frac{1}{2}}^\infty \mathrm{e}^{-t\operatorname{Im}(z)} \Big(t^2 - \frac{1}{4}\Big)^{\nu-\frac{1}{2}}\, \mathrm{d}t \\
&\leq \int_0^\infty e^{-t\operatorname{Im}(z)} t^{2\nu-1}\, \mathrm{d}t \\
&= \int_0^\infty e^{-u} u^{2\nu-1} \operatorname{Im}(z)^{1-2\nu} \operatorname{Im}(z)^{-1}\, \mathrm{d}u \\
&= \operatorname{Im}(z)^{-2\nu} \int_0^\infty \mathrm{e}^{-u} u^{2\nu-1}\, \mathrm{d}u \\
&= C_\nu \operatorname{Im}(z)^{-2\nu},
\end{aligned}$$

where we also used the substitution rule with $u = t\operatorname{Im}(z)$. Now we multiply (2.7) by $|z|^\nu$ and reuse the previous estimate to arrive at

$$|z|^\nu |H_\nu^{(1)}(z)| \leq C_d\, C_\nu\, |z|^{2\nu}\, |\operatorname{Im}(z)|^{-2\nu} \mathrm{e}^{-\frac{\operatorname{Im}(z)}{2}},$$

which for $z = k|x|$ gives

$$|kx|^\nu |H_\nu^{(1)}(k|x|)| \leq C \sin(\theta/2)^{-2\nu}\, \mathrm{e}^{-\frac{1}{2}\sin(\theta/2)\sqrt{|\lambda|}|x|}, \qquad \text{eq:zHEstimate} \tag{2.8}$$

where $C > 0$ depends only on d and we used (2.1) to estimate

$$(|kx|)^{2\nu} \cdot |\operatorname{Im}(k|x|)|^{-2\nu} = |\lambda|^\nu \cdot |\operatorname{Im}(k)|^{-2\nu} \leq \sin(\theta/2)^{-2\nu}.$$

Using (2.2), we estimate for $d \geq 3$ setting $\nu = \frac{d}{2} - 1$

$$|G(x;\lambda)| \leq C\, |x|^{2-d} e^{-c\sqrt{|\lambda|}|x|}$$

and it is clear that the generic constants depends on d and θ. This gives the estimate for $l = 0$ and $d \geq 3$.

Using the relation for the derivatives of Hankel functions which one finds in the book of Lebedev [11, Eq. (5.6.3)],

$$\frac{\mathrm{d}}{\mathrm{d}z}\Big\{z^{-\nu} H_\nu^{(1)}(z)\Big\} = -z^{-\nu} H_{\nu+1}^{(1)}(z),$$

we inductively establish the estimate (2.6) for $l \geq 1$ and $d \geq 2$: For $1 \leq j \leq d$, we calculate using the product and chain rule

eq:derivativeOfG

$$\begin{aligned} |\nabla_x G(x;\lambda)| &\leq C \cdot \left\{ |x|^{1-d} \cdot (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|) - |x|^{2-d} (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}}(k|x|) \cdot k \right\} \\ &\leq C \cdot |x|^{1-d} \left\{ (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|) - (k|x|)^{\frac{d}{2}} H^{(1)}_{\frac{d}{2}}(k|x|) \right\}, \end{aligned} \tag{2.9}$$

where $C > 0$ is a generic constant that depends on d. Note that the first summand in (2.9) does not arise in the case $d = 2$ as is easily seen from equation (2.4). The terms in the bracket can now be estimated individually by (2.8). The extension of this proof to orders of differentiation $l \geq 2$ is straightforward using the Leibniz product rule for higher derivatives.

Now for the last part of the proof, let us verify the claim regarding the asymptotic behavior of $|G(x;\lambda)|$ if $d = 2$. Based on the integral representation (2.3), Lebedev derived an asymptotic expansion for the Hankel function [11, Sec. 5.11, Eq. (5.11.3)]. For $\nu = (d/2) - 1 = 0$ and $z = k|x|$ this expansion reads

$$H_0^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \mathrm{e}^{\mathrm{i}(z-(1/4)\pi)} \left[1 + O(|z|^{-1})\right].$$

As $\mathrm{Im}(z) > 0$ we see that $|H_0^{(1)}(k|x|)| = O(|x|^{-1/2})$. Due to the simple structure of $G(x;\lambda)$ for $d = 2$ as shown in equation (2.4), the claim follows easily. $\square$

In the derivation of the next estimates we will use the following useful interior estimate for solutions of Poisson's equation which we write down for further use.

lem:interiorEstimatePoisson

Lemma 2.2. *Let $r > 0$ and $x \in \mathbb{R}^d$, $d \geq 2$. If $w \in \mathrm{C}^k(\mathrm{B}(x,r)) \cap \mathrm{C}^0(\overline{\mathrm{B}(x,r)})$ is a solution to $\Delta w = f$ in $\mathrm{B}(x,r)$, then*

eq:interiorEstimatePoisson

$$|\nabla^l w(x)| \leq C r^{-l} \sup_{\mathrm{B}(x,r)} |w| + C \max_{0 \leq j \leq l-1} \sup_{\mathrm{B}(x,r)} r^{j-l+2} |\nabla^j f|, \quad l \leq k, . \tag{2.10}$$

where $C > 0$ only depends on d and l.

Proof. If $l = 1$, then estimate (2.10) is a consequence of the *comparison principle* and a proof of this fact can be found in the book of Gilbarg and Trudinger [7, Sec. 3.4, Eq. (3.16)]. We can now use this estimate to inductively deduce the estimates for higher derivatives. Note that by translating from x to 0 and rescaling like

$$u_r(x) := u(rx) \quad \text{and} \quad f_r(x) := r^2 f(rx)$$

we may assume that $\Delta w = f$ in $\mathrm{B}(0,1)$ and that it suffices to prove

eq:interiorEstimatePoissonSimple
$$|\nabla^l w(0)| \leq C \sup_{\mathrm{B}(0,1)} |w| + C \max_{0\leq j\leq l-1} \sup_{\mathrm{B}(0,1)} |\nabla^j f|. \tag{2.11}$$

for $l > 1$. By the Schwartz theorem we have that if w solves Poisson's equation with right hand side f and w and f are sufficiently regular, then $\nabla^l w$ solves Poisson's equation with right hand side $\nabla^l f$. We can thus estimate inductively

$$\begin{aligned}
|\nabla^l w(0)| &\leq C_l \sup_{\mathrm{B}(0,1/2^{l-1})} |\nabla^{l-1} w| + C_l \sup_{\mathrm{B}(0,1/2^{l-1})} |\nabla^{l-1} f| \\
&\leq C_l \sup_{\mathrm{B}(0,1/2^{l-2})} |\nabla^{l-2} w| + C_l \left\{ \sup_{\mathrm{B}(0,1)} |\nabla^{l-2} f| + \sup_{\mathrm{B}(0,1)} |\nabla^{l-1} f| \right\} \\
&\leq \dots \\
&\leq C_l \sup_{\mathrm{B}(0,1)} |w| + C_l \sum_{j=0}^{l-1} \sup_{B(0,1)} |\nabla^j f|
\end{aligned}$$

which readily yields the desired estimate. □

We will need the following asymptotic expansions for the function $z^\nu H_\nu^{(1)}(z)$ in $\mathbb{C} \setminus (-\infty, 0]$. The derivation of these asymptotic expansions is based on asymptotic expansions of the *Bessel functions of the first* and *the second kind* and can be found in Tolksdorf [22, Sec. 4.2]:

eq:asymptoticd4
$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{\mathrm{i}}{\pi} z^2 \log(z) + \omega z^2 + O(|z|^4 |\log(z)) \quad \text{if } d = 4, \tag{2.12}$$

eq:asymptoticd5
$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{2^\nu \Gamma(\nu - 1)}{4\pi\mathrm{i}} z^2 + \omega z^3 + O(|z|^4) \quad \text{if } d = 5, \tag{2.13}$$

eq:asymptoticd6
$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{2^\nu \Gamma(\nu - 1)}{4\pi\mathrm{i}} z^2 + O(|z|^4 |\log z|) \quad \text{if } d = 6, \tag{2.14}$$

eq:asymptoticd7
$$z^\nu H_\nu^{(1)}(z) = \frac{2^\nu \Gamma(\nu)}{\mathrm{i}\pi} + \frac{2^\nu \Gamma(\nu - 1)}{4\pi\mathrm{i}} z^2 + O(|z|^4) \quad \text{if } d \geq 7. \tag{2.15}$$

The next Lemma will be concerned with estimating the difference $G(x;\lambda) - G(x;0)$ (and derivatives of this difference) of the fundamental solution to the scalar Helmholtz equation and the fundamental solution for $-\Delta = 0$ in $\mathbb{R}^d$ which is given by

eq:laplace
$$G(x;0) := \begin{cases} -\frac{1}{2\pi} \log(|x|), & \text{for } d = 2, \\ c_d \frac{1}{|x|^{d-2}}, & \text{for } d > 2, \end{cases} \tag{2.16}$$

where the coefficient c_d is given as as a multiple of the surface measure of the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$

eq:wd
$$c_d = \frac{1}{(d-2)\omega_d}, \quad \text{with} \quad \omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = |\mathbb{S}^{d-1}|. \tag{2.17}$$

Note that $c_2 = (2\pi)^{-1}$. By rearranging terms and using the functional equation of the Gamma function

eq:functionalGamma

$$\begin{aligned} \Gamma(z+1) &= z\,\Gamma(z), \quad z \in \mathbb{C}, \mathrm{Re}(z) > 0 \\ \Gamma(1) &= 1, \end{aligned} \tag{2.18}$$

we get

$$(d-2)\omega_d = 2\left(\frac{d}{2}-1\right)\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} = 2\left(\frac{d}{2}-1\right)\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}-1)(\frac{d}{2}-1)} = \frac{4\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}-1)},$$

and thus we will also sometimes use the equivalent definition

eq:defncd

$$c_d := \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}. \tag{2.19}$$

Furthermore, the leading coefficient of the asymptotic expansions of the Hankel functions (2.12)-(2.15) for $d \geq 3$ will be denoted as

eq:Defnad

$$a_d := \frac{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2}-1)}{\mathrm{i}\pi}. \tag{2.20}$$

The coefficients a_d and c_d are related in the following way:

$$c_d = \frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}}\, a_d.$$

This allows us to write for $d \geq 3$

eq:HelmholtzLaplaceDifference

$$G(x;\lambda) - G(x;0) = \frac{i}{4(2\pi)^{\frac{d}{2}-1}} \cdot \frac{1}{|x|^{d-2}}\left\{(k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|) - a_d\right\}. \tag{2.21}$$

The following lemma will help us to estimate derivatives of (2.21) and the 2-dimensional counterpart.

lem:HelmholtzLaplaceDifference

Lemma 2.3. *Let $\lambda \in \Sigma_\theta$. Then*

eq:HelmholtzLaplaceDifferenceEstimate

$$\left|\nabla^l_x\left\{G(x;\lambda) - G(x;0)\right\}\right| \leq C|\lambda||x|^{4-d-l} \tag{2.22}$$

if $d \geq 5$ and $l \geq 0$, where C depends only on d, l and θ. If $d = 3$ or 4, estimate (2.22) *holds for $l \geq 1$ and if $d = 2$, the estimate holds for $l \geq 3$.*

Proof. (a) In this part we will show that the desired estimate (2.22) holds if we assume that $|\lambda||x|^2 > (1/2)$. In this case, Lemma 2.1 gives

$$\left|\nabla^l_x\left\{G(x;\lambda) - G(x;0)\right\}\right| \leq C\left\{\frac{\mathrm{e}^{-c\sqrt{|\lambda|}|x|}}{|x|^{d-2+l}} + \frac{1}{|x|^{d-2+l}}\right\} \leq C\frac{|\lambda|}{|x|^{d-4+l}},$$

where C depends only on d, l and θ. Therefore, for the remaining proof we will suppose $|\lambda||x|^2 \leq (1/2)$.

(b) In this step we show that we can restrict ourselves to proving (2.22) in three cases: (1) $d \geq 5$ and $l = 0$; (2) $d = 3$ or 4 and $l = 1$; (3) $d = 2$ and $l = 3$.

Suppose (2.22) holds in case (1) and let $l > 1$. If we set $w(x) = G(x;\lambda) - G(x;0)$, we have $\Delta_x w = \lambda G(x;\lambda)$ in $\mathbb{R}^d \setminus \{0\}$. For $f = \lambda G(x;\lambda)$, estimate (2.10) now gives

$$\begin{aligned}
|\nabla^l w(x)| &\leq Cr^{-l} \sup_{\mathrm{B}(x,r)} |w| + C \max_{0\leq j\leq l-1} \sup_{\mathrm{B}(x,r)} r^{j-l+2} |\nabla^j f| \\
&\leq Cr^{-l} \sup_{y\in\mathrm{B}(x,r)} |\lambda||y|^{4-d} + C \sum_{j=0}^{l-1} \sup_{y\in\mathrm{B}(x,r)} r^{j-l+2}|\lambda||y|^{2-d-j} \\
&= Cr^{-l}|\lambda| \left| x - r\frac{x}{|x|} \right|^{4-d} + C \sum_{j=0}^{l-1} r^{j-l+2} |\lambda| \left| x - r\frac{x}{|x|}\right|^{2-d-j},
\end{aligned}$$

for all $0 < r < |x|$, where we used (2.22) with $l = 1$ for the first summand and (2.6) to estimate the second summand. We choose $r = \frac{|x|}{2}$ and receive

$$\begin{aligned}
|\nabla^l w(x)| &\leq C|\lambda||x|^{-l}|x|^{4-d} + C \sum_{j=0}^{l-1} |x|^{j-l+2}|\lambda||x|^{2-d-j} \\
&\leq C|\lambda||x|^{4-d-l}.
\end{aligned}$$

The proof for case (2) is completely analogous if one sets

$$w(x) = \nabla_x (G(x;\lambda) - G(x;0)) \quad \text{and} \quad f(x) = \lambda \nabla_x G(x;\lambda).$$

Also case (3) is proven in a similar fashion.

(c) In this step we prove (2.22) for $d \geq 5$ and $l = 0$. First, note that for the functions

$$\begin{aligned}
g(x) &:= (k|x|)^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(k|x|), \quad g(0) = a_d, \\
h(z) &:= z^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-1}(z), \quad h(0) = a_d,
\end{aligned}$$

the mean value theorem yields the estimate

$$|g(x) - g(0)| \leq |x| \sup_{y\in\mathrm{B}(0,|x|)} |\nabla g(y)| \leq |x||k| \sup_{y\in\mathrm{B}(0,|x|)} \left| \Big(\frac{\mathrm{d}}{\mathrm{d}z} h\Big)(k|y|) \right|.$$

Using representation (2.21), we estimate

eq:HelmholtzLaplaceDifferenceGt5

$$\begin{aligned}
|G(x;\lambda) - G(x;0)| &\leq C|x|^{2-d} \cdot |k||x| \max_{\substack{|z|\leq|k||x| \\ \mathrm{Im}(z)>0}} \left| \frac{\mathrm{d}}{\mathrm{d}z} \left\{ z^{\frac{d}{2}-1} H_{\frac{d}{2}-1}(z) \right\} \right| \\
&= C|x|^{2-d} \cdot |k||x| \max_{\substack{|z|\leq|k||x| \\ \mathrm{Im}(z)>0}} \left| z^{\frac{d}{2}-1} H^{(1)}_{\frac{d}{2}-2}(z) \right|, \qquad (2.23)
\end{aligned}$$

where for the last equality we used another useful relation that can be found in Lebedev [11, Eq. (5.6.3)],

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{z^\nu H_\nu^{(1)}(z)\right\} = z^\nu H_{\nu-1}^{(1)}(z). \tag{2.24}$$

Since the asymptotic expansions yield that $|z^\nu H_\nu^{(1)}(z)| \le C_\nu$ for $\nu > 0$ and $|z| \le 1$ with $\mathrm{Im}(z) > 0$, it follows from (2.23) that

$$|G(x;\lambda) - G(x;0)| \le C|x|^{2-d} \cdot |k||x| \cdot |k||x| \max_{\substack{|z|\le|k||x|\\ \mathrm{Im}(z)>0}} \left|z^{\frac{d}{2}-2} H^{(1)}_{\frac{d}{2}-2}(z)\right| \le C|\lambda||x|^{4-d}.$$

(d) Now we consider the case $d = 4$ and $l = 1$. The asymptotic expansion (2.12) gives that

eq:mwt4d

$$\left|\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{zH_1^{(1)}(z) - a_4}{z^2}\right\}\right| \le C|z|^{-1} \tag{2.25}$$

for all $|z| \le \frac{1}{2}$ with $\mathrm{Im}(z) > 0$. Since

$$\frac{G(x;\lambda) - G(x;0)}{\lambda} = -\frac{C(zH_1^{(1)}(z) - a_4)}{z^2},$$

where $z = k|x|$. With (2.25) we conclude that

$$\left|\frac{\nabla_x\{G(x;\lambda) - G(x;0)\}}{\lambda}\right| \le C|k| \left|\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{zH_1^{(1)}(z) - a_4}{z^2}\right\}\Big|_{z=k|x|}\right| \le C|k||k|^{-1}|x|^{-1},$$

which after rearrangement of the involved terms gives the claim.

(e) For the case $d = 3$ and $l = 1$, we get from equation (2.19) and a well known fact of the Gamma function, $\Gamma(1/2) = \sqrt{\pi}$, the following identity:

$$G(x;\lambda) - G(x;0) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\pi|x|} - \frac{c_3}{|x|} = \frac{\mathrm{e}^{\mathrm{i}k|x|} - 1}{4\pi|x|}.$$

Now we calculate

$$\begin{aligned}\frac{\partial}{\partial x_j}\left\{\frac{\mathrm{e}^{\mathrm{i}k|x|} - 1}{|x|}\right\} &= \frac{\partial}{\partial x_j}\left\{\frac{\mathrm{e}^{\mathrm{i}k|x|} - 1 - \mathrm{i}k|x|}{|x|}\right\} = \frac{\partial}{\partial x_j}\left\{\sum_{n=2}^\infty \frac{(\mathrm{i}k|x|)^n}{n!} \cdot \frac{1}{|x|}\right\}\\ &= \sum_{n=2}^\infty \frac{(\mathrm{i}k)^n}{n!}(n-1)\cdot\frac{x_j}{|x|}|x|^{n-2}\end{aligned}$$

which in turn implies

$$\left|\frac{\partial}{\partial x_j}\left\{\frac{\mathrm{e}^{\mathrm{i}k|x|} - 1}{|x|}\right\}\right| \le |\lambda| \sum_{n=2}^\infty \frac{n-1}{n!}|k|^{n-2}|x|^{n-2} \le C\,|\lambda|$$

since $|\lambda||x| \le (1/2)$.

(f) For the last case $d = 2$ and $l = 3$, we will directly calculate the estimate using the asymptotic expansion of $H_0^{(1)}(z)$ with $z = k|x|$. The calculations are omitted from this chapter. Instead, they can be found in the appendix of this thesis. □

rem:HelmholtzLaplaceDifference

Remark 2.4. In the situation of Lemma 2.3, one can show for $|\lambda||x|^2 \leq (1/2)$ by considering the asymptotic expansions that

$$|G(x;\lambda) - G(x;0)| \leq \begin{cases} C\sqrt{|\lambda|} & \text{if } d = 3, \\ C|\lambda|\{|\log(|\lambda||x|^2)| + 1\} & \text{if } d = 4. \end{cases}$$

Also using the asymptotic expansions it can be shown that if $d = 2$, then

$$|\nabla_x^l\{G(x;\lambda) - G(x;0)\}| \leq C|\lambda||x|^{2-l}\{|\log(|\lambda||x|^2)| + 1\},$$

for $l \in \{1, 2\}$.

2.2 The Stokes Resolvent Problem

sec:2.2

We will now analyze fundamental solutions to the *Stokes resolvent problem*

$$\begin{aligned} -\Delta u + \nabla\phi + \lambda u &= 0 \\ \operatorname{div} u &= 0 \end{aligned} \qquad \text{eq:stokesResolventProblem} \tag{2.26}$$

in $\mathbb{R}^d$ with $\lambda \in \Sigma_\theta$ with the goal to deduce helpful estimates for the following chapters. The fundamental solutions to the (scalar) Helmholtz equation and the Laplace equation will form the main ingredients for the following matrix of fundamental solutions to the Stokes resolvent problem with pole at the origin:

$$\Gamma_{\alpha\beta}(x;\lambda) = G(x;\lambda)\delta_{\alpha\beta} - \frac{1}{\lambda}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\Big\{G(x;\lambda) - G(x;0)\Big\}, \quad \alpha,\beta = 1,\ldots,d. \qquad \text{eq:fundamentalMatrixStokes} \tag{2.27}$$

As the matrix of fundamental solutions $\Gamma(x;\lambda) = (\Gamma_{\alpha\beta}(x;\lambda))_{d\times d}$ carries two arguments it cannot be confused with the Gamma function. Having formula (2.27) at sight, the following observations are obvious:

$$\Gamma_{\alpha\beta}(x;\lambda) = \Gamma_{\beta\alpha}(x;\lambda), \quad \overline{\Gamma_{\alpha\beta}(x;\lambda)} = \Gamma_{\alpha\beta}(x;\bar\lambda) \quad \text{and} \quad \Gamma_{\alpha\beta}(x;\lambda) = \Gamma_{\alpha\beta}(-x;\lambda).$$

For the pressure, we define the vector of fundamental solutions

$$\Phi_\beta(x) = -\frac{\partial}{\partial x_\beta}\Big\{G(x;0)\Big\} = \frac{x_\beta}{\omega_d|x|^d}, \quad \beta = 1,\ldots,d. \qquad \text{eq:fundamentalVectorPressure} \tag{2.28}$$

We note that $\Phi_\beta(x) = \Phi_\beta(-x)$.

Using the fact that $\Delta_x G(x;\lambda) = \lambda G(x;\lambda)$ in $\mathbb{R}^d \setminus \{0\}$, one can see that on $\mathbb{R}^d \setminus \{0\}$ and for all $1 \leq \beta \leq d$

eq:solutionStokesSystem

$$\begin{aligned} (-\Delta_x + \lambda)\Gamma_{\alpha\beta}(x;\lambda) + \frac{\partial}{\partial x_\alpha}\Big\{\Phi_\beta(x)\Big\} &= 0, \\ \frac{\partial}{\partial x_\alpha}\Big\{\Gamma_{\alpha\beta}(x;\lambda)\Big\} &= 0, \quad \text{for } 1 \leq \alpha \leq d. \end{aligned} \tag{2.29}$$

Note that in the last equation the summation convention was used.

We now keep up to the spirit of this exhausting chapter by proving further estimates, this time for the fundamental solutions to the Stokes resolvent problem (2.26).

thm:fundamentalMatrixEstimate

Theorem 2.5. *Let $\lambda \in \Sigma_\theta$. Then for any $d \geq 3$ and $l \geq 0$*

eq:fundamentalMatrixEstimate

$$\left|\nabla_x^l \Gamma(x;\lambda)\right| \leq \frac{C}{(1+|\lambda||x|^2)|x|^{d-2+l}} \tag{2.30}$$

where C depends only on d, l and θ. For $d = 2$ and $l \geq 1$ the same estimate holds.

Proof. Let $|\lambda||x|^2 > (1/2)$. Then there exist constants C_a, C_b, C_c such that

$$\begin{aligned} \mathrm{e}^{-c\sqrt{|\lambda|}|x|}(1+|\lambda||x|^2) &\leq C_a, \\ 1 &\leq \frac{C_b|\lambda||x|^2}{1+|\lambda||x|^2}, \\ \mathrm{e}^{-c\sqrt{|\lambda|}|x|} &\leq \frac{C_c|\lambda||x|^2}{1+|\lambda||x|^2}, \end{aligned}$$

where c is the constant from Lemma 2.1. Using these estimates and Lemma 2.1 gives

$$\begin{aligned} \left|\nabla_x^l \Gamma(x;\lambda)\right| &\leq \left|\nabla_x^l G(x;\lambda)\right| + \frac{1}{|\lambda|}\left|\nabla_x^{l+2} G(x;\lambda)\right| + \frac{1}{|\lambda|}\left|\nabla_x^{l+2} G(x;0)\right| \\ &\leq \frac{C_l\,\mathrm{e}^{-c\sqrt{|\lambda|}|x|}}{|x|^{d-2+l}} + \frac{1}{|\lambda|}\frac{C_{l+2}\,\mathrm{e}^{-c\sqrt{|\lambda|}|x|}}{|x|^2|x|^{d-2+l}} + \frac{1}{|\lambda|}\frac{C}{|x|^2|x|^{d-2+l}} \\ &\leq \frac{C}{1+|\lambda||x|^2}\,\frac{1}{|x|^{d-2+l}}. \end{aligned}$$

Now let $|\lambda||x|^2 \leq (1/2)$. Then by Lemma 2.1 and Lemma 2.3 we get

$$\begin{aligned} \left|\nabla_x^l \Gamma(x;\lambda)\right| &\leq \left|\nabla_x^l G(x;\lambda)\right| + \frac{1}{|\lambda|}\cdot\left|\nabla_x^{l+2}(G(x;\lambda) - G(x;0))\right| \\ &\leq \frac{C}{|x|^{d-2+l}} + \frac{1}{|\lambda|}\cdot C|\lambda||x|^{4-d-(l+2)} \\ &\leq \frac{C}{|x|^{d-2+l}}\frac{(1+|\lambda||x|^2)}{(1+|\lambda||x|^2)} \\ &\leq \frac{C}{(1+|\lambda||x|^2)|x|^{d-2+l}} \end{aligned}$$

which gives the claim. $\square$

If $\lambda = 0$, the Stokes resolvent problem becomes just the Stokes problem in $\mathbb{R}^d$

eq:stokesProblem

$$\begin{aligned} -\Delta u + \nabla\phi + \lambda u &= 0, \\ \operatorname{div} u &= 0. \end{aligned} \tag{2.31}$$

Whereas the fundamental solution for the pressure is maintained, the matrix of fundamental solutions to the Stokes problem in $\mathbb{R}^d$ with pole at the origin is given by $\Gamma(x;0) = (\Gamma_{\alpha\beta}(x;0))_{d\times d}$, where

eq:fundamentalSolutionStokes

$$\Gamma_{\alpha\beta}(x;0) := \frac{1}{2\omega_d}\left\{\frac{\delta_{\alpha\beta}}{(d-2)|x|^{d-2}} + \frac{x_\alpha x_\beta}{|x|^d}\right\} \tag{2.32}$$

if $d \geq 3$ and

eq:fundamentalSolutionStokes2d

$$\Gamma_{\alpha\beta}(x;0) := \frac{1}{2\omega_2}\left\{ - \delta_{\alpha\beta}\log(|x|) + \frac{x_\alpha x_\beta}{|x|^2}\right\} \tag{2.33}$$

for $d = 2$. Note that the given fundamental solution for the case $d = 2$ differs from the one given by Mitrea and Wright [13, Sec. 4.2] by having summands with alternating signs. Considering the structure of the fundamental solution for $d \geq 3$, our choice seems more natural with regard to the structure of the fundamental solutions to the Laplace equation (2.16). The alternating sign is necessary for $\Gamma_{\alpha\beta}$ to be divergence free.

One important technique in the following chapter will be to reduce problems formulated for $\Gamma(x;\lambda)$ to problems formulated in $\Gamma(x;0)$ perturbed by the difference $\Gamma(x;\lambda) - \Gamma(x;0)$. Under this aspect it seems reasonable to study estimates of the difference of fundamental solutions. To this end it is helpful to rewrite parts of the fundamental solution. Using the fact that for $d \geq 5$ or $d = 3$, we have

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{1}{|x|^{d-4}}\right) = -(d-4)\frac{\partial}{\partial x_\alpha}\frac{x_\beta}{|x|^{d-2}} = -(d-4)\frac{\delta_{\alpha\beta}}{|x|^{d-2}} + \frac{(d-4)(d-2)x_\alpha x_\beta}{|x|^d}.$$

This allows us to write

$$\frac{x_\alpha x_\beta}{|x|^d} = \frac{\delta_{\alpha\beta}}{(d-2)|x|^{d-2}} + \frac{1}{(d-4)(d-2)}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{1}{|x|^{d-4}}\right),$$

which, considering definition (2.32), gives

eq:fundamentalSolutionStokes35

$$\Gamma_{\alpha\beta}(x;0) = G(x;0)\delta_{\alpha\beta} + \frac{1}{2\omega_d(d-4)(d-2)}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{1}{|x|^{d-4}}\right). \tag{2.34}$$

A similar trick works for $d = 4$: Since $\omega_4 = 2\pi^2$, we have

$$\Gamma_{\alpha\beta}(x;0) = \frac{1}{2\omega_4}\frac{1}{|x|^2}\delta_{\alpha\beta} - \frac{1}{8\pi^2}\left(\frac{\delta_{\alpha\beta}}{|x|^2} - \frac{2x_\alpha x_\beta}{|x|^4}\right)$$

$$= G(x;0)\delta_{\alpha\beta} - \frac{1}{8\pi^2}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}(\log(|x|))$$

eq:fundamentalSolutionStokes4

$$= G(x;0)\delta_{\alpha\beta} - \frac{1}{4\omega_4}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}(\log(|x|)) \tag{2.35}$$

In the case $d = 2$, we use

$$\frac{1}{8\pi}\frac{\partial^2}{\partial_\alpha \partial_\beta}(|x|^2\log(|x|)) = \frac{\delta_{\alpha\beta}}{4\pi}\log(|x|) + \frac{1}{4\pi}\frac{x_\alpha x_\beta}{|x|^2} + \frac{\delta_{\alpha\beta}}{8\pi}$$

to find the identity

eq:fundamentalSolutionStokes2

$$\Gamma_{\alpha\beta}(x;0) = G(x;0)\delta_{\alpha\beta} - \frac{\delta_{\alpha\beta}}{8\pi} - \frac{\partial^2}{\partial x_\alpha \partial x_\beta}(|x|^2\log(|x|)). \tag{2.36}$$

This ends the preparatory step and brings us to the next theorem.

thm:differenceFundamentalSolutionStokes

Theorem 2.6. *Let $\lambda \in \Sigma_\theta$. Suppose that $|\lambda||x|^2 \le (1/2)$. Then*

$$\left|\nabla_x\{\Gamma(x;\lambda) - \Gamma(x;0)\}\right| \le \begin{cases} C|\lambda||x|^{3-d} & \text{if } d \ge 7 \text{ or } d = 5, \\ C|\lambda||x|^{3-d}\left|\log(|\lambda||x|^2)\right| & \text{if } d = 4 \text{ or } 6, \\ C\sqrt{|\lambda|}|x|^{-1} & \text{if } d = 3, \\ C|\lambda||x|\left|\log(|\lambda||x|^2)\right| & \text{if } d = 2, \end{cases} \tag{2.37}$$

where C depends only on d and θ.

Proof. We will split the proof in several parts. According to the preparatory step, for $d \ge 2$ and all $\alpha, \beta = 1, \ldots, d$, the difference $\partial_\gamma\{\Gamma_{\alpha\beta}(x;\lambda) - \Gamma_{\alpha\beta}(x;0)\}, \gamma = 1, \ldots, d$, is always of the form

$$\frac{\partial}{\partial x_\gamma}\left\{\Gamma_{\alpha\beta}(x;\lambda) - \Gamma_{\alpha\beta}(x;0)\right\}$$
$$= \frac{\partial}{\partial x_\gamma}\left\{G(x;\lambda) - G(x;0)\right\}\delta_{\alpha\beta} - \frac{1}{\lambda}\frac{\partial^3}{\partial x_\gamma \partial x_\alpha \partial x_\beta}\left\{G(x;\lambda) - G(x;0) + \left[\ \ldots\ \right]\right\}.$$

But the first term on the right hand side of the above expression is already under control thanks to Lemma 2.3. It thus suffices to estimate the second term on the right hand side.

We start by considering the cases $d = 3$ and $d \ge 5$. Taking into account identity (2.34), we have for all $\alpha, \beta, \gamma = 1, \ldots, d$:

$$G(x;\lambda) - G(x;0) + \left[\ \ldots\ \right] = \frac{1}{\lambda}\left\{G(x;\lambda) - G(x;0) + \frac{\lambda}{2\omega_d(d-4)(d-2)|x|^{d-4}}\right\}.$$

If $d = 3$, a direct calculation will then yield the desired result: We start by noting that $\omega_3 = 4\pi$ gives

$$\begin{aligned}
G(x;\lambda) - G(x;0) - \frac{\lambda}{2\omega_3|x|^{-1}} &= \frac{e^{ik|x|}}{4\pi|x|} - \frac{1}{4\pi|x|} - \frac{(ik)^2}{2\omega_3|x|^{-1}} \\
&= \frac{1}{4\pi|x|}\Big(e^{ik|x|} - 1 - \frac{(ik)^2|x|^2}{2}\Big) \\
&= \frac{1}{4\pi|x|}\Big(ik|x| + \sum_{n=3}^{\infty}\frac{(ik|x|)^n}{n!}\Big) \\
&= \frac{1}{4\pi}\Big(ik + \sum_{n=3}^{\infty}\frac{(ik)^n|x|^{n-1}}{n!}\Big).
\end{aligned}$$

Taking the first derivative of this expression we get

$$\frac{\partial}{\partial x_\beta}\Big\{\dots\Big\} = \frac{x_\beta}{4\pi}\sum_{n=3}^{\infty}\frac{(ik)^n(n-1)}{n!}|x|^{n-3}$$

and differentiating with respect to x_α yields

$$\frac{\partial}{\partial x_\alpha}\Big\{\dots\Big\} = \frac{\delta_{\alpha\beta}}{4\pi}\sum_{n=3}^{\infty}\frac{(ik)^n(n-1)}{n!}|x|^{n-3} + \frac{x_\beta x_\alpha}{4\pi}\sum_{n=4}^{\infty}\frac{(ik)^n(n-1)(n-3)}{n!}|x|^{n-5}.$$

As we are interested in estimating the *gradient* of the difference of $\Gamma(x;\lambda)$ and $\Gamma(x;0)$, we have to consider one additional derivative. This leaves us with

$$\begin{aligned}
\frac{\partial}{\partial x_\gamma}\Big\{\dots\Big\} &= \frac{\delta_{\alpha\beta}x_\gamma + \delta_{\beta\gamma}x_\alpha + \delta_{\alpha\gamma}x_\beta}{4\pi}\sum_{n=4}^{\infty}\frac{(ik)^n(n-1)(n-3)}{n!}|x|^{n-5} \\
&\quad + \frac{x_\beta x_\alpha x_\gamma}{4\pi}\sum_{n=4}^{\infty}\frac{(ik)^n(n-1)(n-3)(n-5)}{n!}|x|^{n-7}.
\end{aligned}$$

We can now prove the desired estimate via

$$\begin{aligned}
\left|\frac{1}{\lambda}\frac{\partial^3}{\partial x_\gamma\partial x_\alpha\partial x_\beta}\Big\{\dots\Big\}\right| &\le \frac{1}{|k|^2\pi}\sum_{n=4}^{\infty}\frac{|k|^n(n-1)(n-3)(1+(n-5))}{n!}|x|^{n-4} \\
&\le \frac{1}{|k|^2|x|\pi}|k|^3\sum_{k=4}^{\infty}\frac{(n-1)(n-3)(1+(n-5))}{n!}|k|^{n-3}|x|^{n-3} \\
&\le C\frac{1}{|k||x|}.
\end{aligned}$$

This gives the claim for $d = 3$. If $d \ge 5$, equation (2.21) gives

$$\begin{aligned}
G(x;\lambda) - G(x;0) + \frac{\lambda}{2\omega_2(d-4)(d-2)|x|^{d-4}} \\
= \frac{i}{4(2\pi)^{\frac{d}{2}-1}}\frac{1}{|x|^{d-2}}\Big\{z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2\Big\},
\end{aligned} \tag{2.38}$$

eq:secondTerm

where $z = k|x|$, a_d was calculated in (2.20) and b_d is given by

$$b_d = -\frac{2\mathrm{i}(2\pi)^{\frac{d}{2}-1}}{\omega_d(d-4)(d-2)}.$$

Using relation (2.17) and the functional equation of the Gamma function (2.18) twice, we see that

$$\begin{aligned} b_d &= -\frac{2\mathrm{i}(2\pi)^{\frac{d}{2}-1}\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}(d-2)(d-4)} = \frac{2^{\frac{d}{2}-1}}{\pi\mathrm{i}(d-4)}\frac{\Gamma(\frac{d}{2})}{(d-2)} = \frac{2^{\frac{d}{2}-1}}{2\pi\mathrm{i}}\frac{\Gamma(\frac{d}{2}-1)}{(d-4)} \\ &= \frac{2^{\frac{d}{2}-1}}{4\pi\mathrm{i}}\frac{\Gamma(\frac{d}{2}-1)}{(\frac{d}{2}-1-1)} = \frac{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2}-1-1)}{4\pi\mathrm{i}} = \frac{2^{\nu_d}\Gamma(\nu_d-1)}{4\pi\mathrm{i}}. \end{aligned}$$

This shows that for $d \geq 5$, b_d is the second coefficient of the asymptotic expansions (2.13)-(2.15), respectively. Now we split the proof for $d \geq 5$ into (1) $d \geq 7$, (2) $d = 6$ and (3) $d = 5$. If $d \geq 7$, we use the asymptotic expansions (2.15) to estimate the part of (2.38) which involves the Hankel function as

eq:estimateDerivativesd7

$$\left|\frac{\mathrm{d}^l}{\mathrm{d}z^l}\left\{z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2\right\}\right| \leq C|z|^{4-l} \tag{2.39}$$

for $0 \leq l \leq 3$ and $z \in \mathbb{C} \setminus (-\infty, 0]$. For better readability we define the function

$$g(z) := z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2$$

and consider the function $f(x) := g(k|x|)$ on $\mathbb{R}^d \setminus \{0\}$. The derivatives of f read

$$\begin{aligned} \frac{\partial}{\partial x_\beta} f(x) &= \Big(\frac{\mathrm{d}}{\mathrm{d}z} g\Big)(k|x|) \cdot \frac{k x_\beta}{|x|}, \\ \frac{\partial^2}{\partial x_\alpha \partial x_\beta} f(x) &= \Big(\frac{\mathrm{d}^2}{\mathrm{d}z^2} g\Big)(k|x|) \cdot \frac{k^2 x_\alpha x_\beta}{|x|^2} + \Big(\frac{\mathrm{d}}{\mathrm{d}z} g\Big)(k|x|) \cdot k\left\{\frac{\delta_{\alpha\beta}}{|x|} - \frac{x_\beta x_\alpha}{|x|^3}\right\}, \\ \frac{\partial^3}{\partial x_\gamma \partial x_\alpha \partial x_\beta} f(x) &= \Big(\frac{\mathrm{d}^3}{\mathrm{d}z^3} g\Big)(k|x|) \cdot \frac{k^3 x_\alpha x_\beta x_\gamma}{|x|^3} \\ &\quad + \Big(\frac{\mathrm{d}^2}{\mathrm{d}z^2} g\Big)(k|x|) \cdot k^2\left\{\frac{x_\alpha\delta_{\beta\gamma} + x_\beta\delta_{\alpha\gamma} + x_\gamma\delta_{\alpha\beta}}{|x|^2} - \frac{x_\alpha x_\beta x_\gamma}{|x|^4}\right\} \\ &\quad + \Big(\frac{\mathrm{d}}{\mathrm{d}z} g\Big)(k|x|) \cdot k\left\{-\frac{x_\gamma\delta_{\alpha\beta} + x_\alpha\delta_{\beta\gamma} + x_\beta\delta_{\alpha\gamma}}{|x|^3} + \frac{3x_\alpha x_\beta x_\gamma}{|x|^5}\right\}. \end{aligned}$$

If we now look for estimates on the absolute value of the derivatives, we see that by (2.39)

$$\left|\nabla^l f(x)\right| \leq C|k|^4|x|^{4-l}, \quad 1 \leq l \leq 3,$$

where C only depends on l. We can now finally uncover the desired estimate via

$$\left|\frac{1}{\lambda}\nabla_x^3\left\{\frac{\mathrm{i}}{4(2\pi)^{\frac{d}{2}-1}}\frac{1}{|x|^{d-2}}\left\{z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2\right\}\right\}\right|$$

$$\leq C\frac{1}{|k|^2}\sum_{l=0}^{3}\left|\nabla^{3-l}\Big(\frac{1}{|x|^{d-2}}\Big)\right|\left|\nabla^l f(x)\right| \leq C\sum_{l=0}^{3}|x|^{-d+2-3+l}|k|^2|x|^{4-l} = C|\lambda||x|^{3-d},$$

where C is a constant only depending on d.

If $d = 6$, the asymptotic expansion (2.14) gives us in analogy to (2.39) the estimate

eq:estimateDerivativesd6

$$\left|\frac{\mathrm{d}^l}{\mathrm{d}z^l}\Big\{z^{\frac{d}{2}-1}H^{(1)}_{\frac{d}{2}-1}(z) - a_d - b_d z^2\Big\}\right| \leq C|z|^{4-l}\left|\log(z)\right|, \tag{2.40}$$

for $0 \leq l \leq 3$ and $z \in \mathbb{C}\setminus(-\infty, 0]$. Using as before the expressions for the derivatives of f, we estimate their absolute values as

$$\left|\nabla^l f(x)\right| \leq C|k|^4|x|^{4-l}\left|\log(|\lambda||x|^2)\right|,$$

which, by a calculation analogous to the case $d \geq 7$, yields the claim.

For $d = 5$, we differentiate (2.38) twice and use relation (2.34) for the fundamental solution of the Stokes problem to write

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\Big\{G(x;\lambda) - G(x;0) + \frac{\lambda}{6\,\omega_5|x|^{d-4}}\Big\}$$
$$= \frac{\partial^2}{\partial x_\alpha x_\beta}\Big\{\frac{\mathrm{i}}{4(2\pi)^{\frac{3}{2}}}\cdot\frac{1}{|x|^3}\Big[z^{\frac{3}{2}}H^{(1)}_{\frac{3}{2}}(z) - a_5 - b_5 z^2 - wz^3\Big]\Big\},$$

where $w \in \mathbb{C}$ can be an arbitrary constant if we set $z = k|x|$. Now, for the appropriate choice of $w \in \mathbb{C}$ the asymptotic expansion (2.13) gives the same estimate as (2.39) which, like for $d \geq 7$, proves the claim for $d = 5$.

In the case $d = 4$ we use the respective relation for the fundamental solution (2.35) in order to simplify the difference

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\Big\{G(x;\lambda) - G(x;0) - \frac{\lambda\log(|x|)}{4\,\omega_4}\Big\}$$
$$= \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\Big\{\frac{\mathrm{i}}{8\pi|x|^2}\Big[zH^{(1)}_1(z) - a_4 - wz^2 - b_4 z^2\log(z)\Big]\Big\},$$

where $z = k|x|$, $b_4 = (\mathrm{i}/\pi)$ and $w \in \mathbb{C}$ is an arbitrary constant. Using the asymptotic expansion (2.12) and the appropriate constant $w \in \mathbb{C}$ we get the estimate

$$\left|\frac{\mathrm{d}^l}{\mathrm{d}z^l}\Big\{zH^{(1)}_1(z) - a_4 - wz^2 - b_4 z^2\log(z)\Big\}\right| \leq C|z|^{4-l}\left|\log(z)\right|.$$

The estimate has the same right hand side as (2.40) and the proof can be carried out just as in the previous cases.

For $d = 2$, the claimed estimate follows from a direct calculation which is postponed until the appendix of this thesis. □

We can now use the assumption $|\lambda||x|^2 \leq (1/2)$ to unify the structure of the estimates from Theorem 2.6.

cor:differenceFundamentalSolutionStokes

Corollary 2.7. *Let* $\lambda \in \Sigma_\theta$. *Suppose that* $|\lambda||x|^2 \leq (1/2)$. *Then for all* $d \geq 2$

$$\left|\nabla_x\{\Gamma(x;\lambda) - \Gamma(x;0)\}\right| \leq C\sqrt{|\lambda|}|x|^{2-d},$$

where C *depends only on* d *and* θ.

Proof. We just extend the estimates given in Theorem 2.6. Let $d \geq 7$ or $d = 5$. Since $\sqrt{|\lambda|} \leq C|x|^{-1}$, we have

$$C|\lambda||x|^{3-d} \leq C\sqrt{|\lambda|}|x|^{2-d}.$$

For $d = 2, 4, 6$, we have

$$|\lambda||x|^{3-d}\left|\log(|\lambda||x|^2)\right| = C\sqrt{|\lambda|}|x|^{2-d} \cdot \sqrt{|\lambda|}|x|\left|\log(|\lambda||x|^2)\right| \leq C\sqrt{|\lambda|}|x|^{2-d},$$

since $\sqrt{|\lambda|}|x|\left|\log(|\lambda||x|^2)\right|$ is bounded for $|\lambda||x|^2 \leq (1/2)$. □

Chapter 3

Single and Double Layer Potentials

chap:3

In this chapter, we will deal with *single* and *double layer potentials*. Both will serve as "representation formulas" for solutions to the Stokes resolvent problem. We will study their properties as they will serve as the crucial ingredient to solving the L^2 Dirichlet problem associated to the Stokes resolvent problem on bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$: For $\lambda \in \mathbb{C} \setminus (-\infty, 0)$ and

$$g \in \mathrm{L}^2_\nu(\partial\Omega) := \left\{ g \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d) \colon \int_{\partial\Omega} g \cdot \nu \, \mathrm{d}\sigma = 0 \right\}$$

we are looking for smooth functions u and ϕ that satisfy

$$(\mathrm{Dir}_\lambda) \begin{cases} -\Delta u + \nabla \phi + \lambda u = 0 & \in \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = g & \text{nontangentially on } \partial\Omega, \\ (u)^* \in \mathrm{L}^2(\partial\Omega). \end{cases}$$

In this chapter we will thus always assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$ with $d \geq 2$ and $1 < p < \infty$. We will also tacitly use the summation convention.

We note that due to the new two dimensional estimates on fundamental solutions in Chapter 2, namely the continuation of Theorem 2.5and Theorem 2.6 for the case $d = 2$, we could extend all results from Chapter 3 of Shen's seminal paper [17] that are relevant to the analysis of the L^2 Dirichlet problem in a straightforward way.

Let $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi/2)$. Furthermore, let $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$. The single layer potential $u = \mathcal{S}_\lambda(f)$ is defined by

eq:defSingleLayer

$$(\mathcal{S}_\lambda(f))_j(x) := \int_{\partial\Omega} \Gamma_{jk}(x - y; \lambda) f_k(y) \, \mathrm{d}\sigma(y), \tag{3.1}$$

where Γ_{jk} is the fundamental solution to the Stokes resolvent problem given by (2.27). For the pressure, respectively, we define the single layer potential $\phi = \mathcal{S}_\Phi(f)$ by

eq:defSingleLayerPressure

$$\mathcal{S}_\Phi(f)(x) := \int_{\partial\Omega} \Phi_k(x-y) f_k(y) \, \mathrm{d}\sigma(y), \tag{3.2}$$

where Φ_k is given by (2.28). As we have already shown, (u, ϕ) defines a solution to the Stokes resolvent problem (2.26) in $\mathbb{R}^d \setminus \partial\Omega$.

We define two further integral operators that map to functions living on $\partial\Omega$:

eq:supTOperator

$$T_\lambda^*(f)(q) = \sup_{t>0} \left| \int_{\substack{y \in \partial\Omega \\ |y-q|>t}} \nabla_x \Gamma(q-y;\lambda) f(y) \, \mathrm{d}\sigma(y) \right| \tag{3.3}$$

eq:pvTOperator

$$T_\lambda(f)(q) = \mathrm{p.\,v.} \int_{\partial\Omega} \nabla_x \Gamma(q-y;\lambda) f(y) \, \mathrm{d}\sigma(y) \tag{3.4}$$

for $q \in \partial\Omega$ which will be used to prove boundedness of maximal operators related to u and its gradient.

The following lemma will be a good companion for the forthcoming calculation of estimates.

lem:compareBoundaryWithBall

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain with corresponding numbers r_0 and M. Then there exists $C > 0$ depending only on d and M such that*

$$\sigma(\mathrm{B}(q,r) \cap \partial\Omega) \leq C r^{d-1}$$

for all $r > 0$ and $q \in \partial\Omega$. Furthermore, there exists a constant $C > 0$, depending only on d and the Lipschitz character of Ω, such that

$$\sigma(\partial\Omega) \leq C r_0^{d-1}.$$

Another cornerstone in the theory of the single and double layer potentials is the following lemma, see Tolksdorf [22, Lem. 4.3.2], as it will allow us to bring into play the estimates from Section 2.2.

lem:central

Lemma 3.2. *Let $\Omega \subset R^d$, $d \geq 2$, be a bounded Lipschitz domain with corresponding numbers r_0 and M. Let $x \in \mathbb{R}^d$, $0 < \varepsilon \leq (r_0/4)$, and $l \in \mathbb{N}_0$ with $l < d-1$. Then there exists a constant $C > 0$ depending only on d, l and M such that*

$$\int_{\partial\Omega \cap \mathrm{B}(x,\varepsilon)} \frac{1}{|x-y|^l} \, \mathrm{d}\sigma(y) \leq C \varepsilon^{d-l-1}.$$

We are now in the position to prove our first lemma on the way to establish the single layer potential as a benevolent operator for tackling boundary value problems on bounded Lipschitz domains. The lemma deals with mapping properties of the aforementioned integral operators T_λ and T_λ^*.

lem:lpBoundednessT

Lemma 3.3. *Let* $1 < p < \infty$ *and* $T_\lambda(f), T^*_\lambda(f)$ *be defined by* (3.3) *and* (3.4). *Then* $T_\lambda(f)(P)$ *exists for almost everywhere* $P \in \partial\Omega$ *and*

eq:lpBoundednessT

$$\|T_\lambda(f)\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^{d\times d})} \leq \|T^*_\lambda(f)\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p \, \|f\|_{\mathrm{L}^p(\partial\Omega);\mathbb{C}^d}, \tag{3.5}$$

where C_p *depends only on* d, θ, p, *and the Lipschitz character of* Ω.

Proof. If $\lambda = 0$, the lemma is known due to Fabes, Kenig and Verchota [3] as a consequence of the seminal result of Coifman, McIntosh and Meyer [**?**]. One idea of the proof in the case $\lambda \in \Sigma_\theta$ will thus be to nourish from this result and to consider the difference $\Gamma(x-y;\lambda) - \Gamma(x-y;0)$.

We start with the second inequality of (3.5). To this end, let $t > 0$ and additionally assume that $t^2|\lambda| > (1/2)$. In this case, Theorem 2.5 gives us the estimate

$$\left| \int_{|y-q|>t} \nabla_x \Gamma(q-y;\lambda) f(y) \, \mathrm{d}\sigma(y) \right| \leq C \int_{|q-y|>t} \frac{|f(y)|}{|\lambda||q-y|^{d+1}} \, \mathrm{d}\sigma(y),$$

where C depends on d and θ. Choose now $N \in \mathbb{N}$ such that $2^N t \leq \operatorname{diam}(\Omega) < 2^{N+1} t$. We now exhaust the domain of integration by suitable annuli and use the inner radii to simplify the integrand and the outer radii to amplify the domain of integration:

eq:gtt

$$\begin{aligned}
&\sum_{k=0}^{N} \int_{2^k t < |q-y| < 2^{k+1}} \frac{1}{|\lambda||q-y|^{d+1}} |f(y)| \, \mathrm{d}\sigma(y) \\
&\qquad \leq \sum_{k=0}^{N} \int_{2^k t < |q-y| < 2^{k+1}} \frac{1}{|\lambda| 2^{k(d+1)} t^{d+1}} |f(y)| \, \mathrm{d}\sigma(y) \\
&\qquad \leq \frac{1}{|\lambda| t^2} \, \frac{1}{2^{1-d}} \sum_{k=0}^{N} \frac{1}{2^{2k}} \frac{1}{(2^{k+1}t)^{d-1}} \int_{\mathrm{B}(q,2^{k+1}t)\cap\partial\Omega} |f(y)| \, \mathrm{d}\sigma(y).
\end{aligned} \tag{3.6}$$

Note that due to Lemma 3.1 we have that

eq:applLem31

$$\frac{1}{(2^{k+1}t)^{d-1}} \int_{\mathrm{B}(q,2^{k+1}t)\cap\partial\Omega} |f(y)| \, \mathrm{d}\sigma(y) \leq C M_{\partial\Omega}(f)(q), \quad k = 0, \ldots, N, \tag{3.7}$$

with a constant C that depends on d and the Lipschitz character of Ω. Now we glue together (3.6) and (3.7), take $N \to \infty$ noting the geometric series and get the estimate

eq:finalgtt

$$\left| \int_{|y-q|>t} \nabla_x \Gamma(q-y;\lambda) f(y) \, \mathrm{d}\sigma(y) \right| \leq C M_{\partial\Omega}(f)(q). \tag{3.8}$$

Now let $t^2|\lambda| \leq (1/2)$. We then split the integral as follows:

$$\begin{aligned}\left|\int_{|q-y|>t} \nabla_x\Gamma(q-y;\lambda)f(y)\,\mathrm{d}\sigma(y)\right| \leq \left|\int_{|q-y|\geq(2|\lambda|)^{-1/2}} \nabla_x\Gamma(q-y;\lambda)f(y)\,\mathrm{d}\sigma(y)\right| \\ + \left|\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x\Gamma(q-y;\lambda)f(y)\,\mathrm{d}\sigma(y)\right|.\end{aligned}$$

For the first summand, note that estimate (3.8) holds for all $t > 0$ and thus in particular for $t = (2|\lambda|)^{-1/2}$. For the second term, we add a special zero and use the triangle inequality to estimate

$$\begin{aligned}&\left|\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x\Gamma(q-y;\lambda)f(y)\,\mathrm{d}\sigma(y)\right| \\ &\qquad\leq \int_{t<|q-y|<(2|\lambda|)^{-1/2}} \left|\nabla_x\Gamma(q-y;\lambda) - \nabla_x\Gamma(q-y;0)\right| |f(y)|\,\mathrm{d}\sigma(y) \\ &\qquad\quad + \left|\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x\Gamma(q-y;0)f(y)\,\mathrm{d}\sigma(y)\right|.\end{aligned}$$

We don't need to worry about the second summand here since the corresponding estimate is already covered by the $\lambda = 0$ case:

eq:lambda0case

$$\begin{aligned}&\left|\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x\Gamma(q-y;0)\,f(y)\,\mathrm{d}\sigma\right| \\ &\qquad\leq \left|\int_{|q-y|>t} \nabla_x\Gamma(q-y;0)\,f(y)\,\mathrm{d}\sigma\right| \leq T_0^*(f)(q).\end{aligned}\tag{3.9}$$

For the first summand we make use of Theorem 2.6 and more precisely of Corollary 2.7 which unifies all estimates: We start by estimating

eq:estDiff

$$\begin{aligned}&\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \left|\nabla_x\Gamma(q-y;\lambda) - \nabla_x\Gamma(q-y;0)\right| |f(y)|\,\mathrm{d}\sigma(y) \\ &\qquad\leq C\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \sqrt{|\lambda|}\,|q-y|^{2-d}\,|f(y)|\,\mathrm{d}\sigma(y),\end{aligned}\tag{3.10}$$

where C depends on d and θ. Now we choose N such that

eq:choiceOfN

$$2^{N+1}t > (2|\lambda|)^{-1/2} \geq 2^N t \tag{3.11}$$

holds. Once again we integrate over annuli, use the inner radii to loose the term $|q-y|^{2-d}$ and use the outer radii to expand the domain of integration to balls with this radius:

$$\begin{aligned}\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \frac{1}{|q-y|^{d-2}}\,|f(y)|\,\mathrm{d}\sigma &\leq \sum_{k=0}^{N}\int_{2^k t\leq|q-y|<2^{k+1}t} \frac{1}{|q-y|^{d-2}}\,|f(y)|\,\mathrm{d}\sigma \\ &\leq \sum_{k=0}^{N}\frac{1}{(2^k t)^{d-2}}\int_{\mathrm{B}(q,2^{k+1}t)\cap\partial\Omega} |f(y)|\,\mathrm{d}\sigma\end{aligned}$$

$$= 2^d \sum_{k=0}^{N} 2^k t \, \frac{1}{(2^{k+1}t)^{d-1}} \int_{\mathrm{B}(q,2^{k+1}t)\cap\partial\Omega} |f(y)| \, \mathrm{d}\sigma.$$

As before we use Lemma 3.1 to bring the Hardy-Littlewood maximal operator into the game like for inequality (3.7). This time we cannot take $N \longrightarrow \infty$ as the resulting geometric series wouldn't converge. But N was chosen wisely, see (3.11) and thus

$$\sum_{k=0}^{N} 2^k t \leq 2^{N+1} t \leq 2^{1/2} |\lambda|^{-1/2}.$$

which yields the inequality

$$\int_{t<|q-y|<(2|\lambda|)^{-1/2}} \frac{1}{|q-y|^{d-2}} \, |f(y)| \, \mathrm{d}\sigma \leq C|\lambda|^{-1/2} M_{\partial\Omega}(f)(q),$$

where C depends on d and the Lipschitz character of Ω. Taking into account the foregoing calculations together with estimate (3.10) and (3.9) we derive

$$\left| \int_{t<|q-y|<(2|\lambda|)^{-1/2}} \nabla_x \Gamma(q-y;0) f(y) \, \mathrm{d}\sigma \right| \leq C \Big\{ T_0^*(f)(q) + M_{\partial\Omega}(f)(q) \Big\}, \quad \text{eq:finalltt} \tag{3.12}$$

with C depending only on d, θ and the Lipschitz character of Ω.

It is now the time to take the supremum over all $t > 0$, and considering estimates (3.8) and (3.12) we finally see that

$$T_\lambda^*(f)(q) \leq C \Big\{ T_0^*(f)(q) + M_{\partial\Omega}(f)(q) \Big\},$$

for all $q \in \partial\Omega$. Once again using the result for $\lambda = 0$ and the L^p-boundedness of the Hardy-Littlewood maximal operator, we conclude the first part of the claimed inequality

$$\|T_\lambda^*(f)\|_{\mathrm{L}^p(\partial\Omega)} \leq C \, \|f\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)}$$

To conclude the left inequality in (3.5), we want to use a standard result argument harmonic analysis. To this end we define the operators

$$T_\lambda^{(t)}(f)(q) := \int_{\substack{y\in\partial\Omega \\ |y-q|>t}} \nabla_x \Gamma(q-y;\lambda) f(y) \, \mathrm{d}\sigma(y).$$

Suppose we can show that

$$T_\lambda(f)(q) = \lim_{t\to 0} T_\lambda^{(t)}(f)(q) \quad \text{eq:pointwiseLimit} \tag{3.13}$$

exists for almost every $q \in \partial\Omega$ and all $f \in \mathrm{C}(\partial\Omega;\mathbb{C}^d)$. Now, note that $\mathrm{C}(\partial\Omega;\mathbb{C}^d)$ is dense in $\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ and that $T_\lambda^*(f)$ is bounded on $\mathrm{L}^p(\partial\Omega;C^d)$ as we showed earlier. Then Grafakos [9, Thm. 2.1.14] gives that T_λ is bounded on $\mathrm{L}^p(\partial\Omega;\mathbb{C}^{d\times d})$.

In order to prove the existence of the pointwise limit (3.13), we split the operator T_λ as follows:

$$T_\lambda(f)(q) = T_0(f)(q) + \lim_{t\to 0} \int_{\substack{y\in\partial\Omega \\ |y-q|>t}} \nabla_x\big\{\Gamma(q-y;\lambda) - \Gamma(q-y;0)\big\} f(y)\,\mathrm{d}\sigma(y).$$

The right summand is well defined for $f \in \mathrm{C}(\partial\Omega;\mathbb{C}^d)$, once we prove integrability of the integral kernel $\big|\nabla_x\big\{\Gamma(q-y;\lambda) - \Gamma(q-y;0)\big\}\big|$ on $\partial\Omega$. To this end, we first note that it suffices to consider the integral

$$\int_{|q-y|\le\varepsilon} \big|\nabla_x\big\{\Gamma(q-y;\lambda) - \Gamma(P-y;0)\big\}\big|\,\mathrm{d}\sigma(y),$$

for $\varepsilon \le \min(2|\lambda|^{-1/2}, r_0/4)$ as the integrand is bounded away from 0 and the domain of integration is bounded. Now Corollary 2.7 and Lemma 3.2 give that the integrand can be estimated by

$$C\int_{|q-y|\le\varepsilon} \sqrt{|\lambda|}|q-y|^{2-d}\,\mathrm{d}\sigma(y) \le C\sqrt{|\lambda|}\varepsilon \le C,$$

where C is a constant depending on d, θ and the Lipschitz character of Ω. Based on the preceding calculation we conclude that for all $f \in \mathrm{C}(\partial\Omega, \mathbb{C}^d)$ the operator $T_\lambda(f)(q)$ exists whenever $T_0(f)(q)$ exists. $T_0(f)(q)$ exists for almost everywhere $q \in \partial\Omega$ because of Fabes, Kenig and Verchota [3]. As furthermore $T^*_\lambda(f)(q)$ is bounded on $\mathrm{L}^p(\partial\Omega)$ we may now apply Theorem 2.1.14 from Grafakos [9] to conclude that $T_\lambda(f)(q)$ exists now for all $f \in \mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ and almost everywhere $q \in \partial\Omega$. The desired L^p estimate for $T_\lambda(f)$ now follows from the observation that $|T_\lambda(f)(q)| \le T^*_\lambda(f)(q)$ for almost everywhere $q \in \partial\Omega$. □

For further use, we state a very useful lemma which can be considered a *Young*-type inequality for L^p spaces on boundaries of Lipschitz domains. A proof can be found in Tolksdorf [22, Prop 1.1.4].

lem:young

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^d$, μ a σ-finite measure on Ω, and $1 \le p < \infty$. Let $g\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a function such that the function $\Omega\times\Omega \ni (x,y) \mapsto g(x-y)$ is measurable with respect to the product measure $\mu\times\mu$ and such that*

$$A + B \coloneqq \sup_{x\in\Omega} \|g(x-\cdot)\|_{\mathrm{L}^1(\Omega,\mu)} + \sup_{y\in\Omega} \|g(\cdot - y)\|_{\mathrm{L}^1(\Omega,\mu)} < \infty.$$

If $f \in \mathrm{L}^p(\Omega,\mu)$, then $x \mapsto \int_\Omega g(x-y)f(y)\,\mathrm{d}\mu(y) \in \mathrm{L}^p(\Omega,\mu)$ and

$$\left\| \int_\Omega g(\cdot - y) f(y)\,\mathrm{d}\mu(y) \right\|_{\mathrm{L}^p(\Omega,\mu)} \le A^{1-1/p} B^{1/p} \|f\|_{\mathrm{L}^p(\Omega,\mu)}.$$

For us, Lemma 3.4 will be applied often to integral kernels g that result from an application of the Theorems in Chapter 2. The following Lemma shows that these integral kernels fulfill the requirements from Lemma 3.4.

lem:youngApp

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then*

$$\sup_{q\in\partial\Omega}\int_{\partial\Omega}\frac{1}{|q-y|^{d-2}} \le Cr_0,$$

where C is a constant depending only on d and the Lipschitz character of Ω.

Proof. Let r_0 be the radius from the definition of Lipschitz cylinders and $q \in \partial\Omega$. Splitting the domain of integration and applying Lemma 3.2, we get

$$\begin{aligned}
&\int_{\partial\Omega}\frac{1}{|q-y|^{d-2}}\,\mathrm{d}\sigma(y)\\
&\quad\le \int_{\partial\Omega\cap \mathrm{B}(q;r_0/4)}\frac{1}{|q-y|^{d-2}}\,\mathrm{d}\sigma(y) + \int_{\partial\Omega\setminus \mathrm{B}(q;r_0/4)}\frac{1}{|q-y|^{d-2}}\,\mathrm{d}\sigma(y)\\
&\quad\le Cr_0 + r_0^{2-d}4^{d-2}\sigma(\partial\Omega) \le C(r_0 + r_0^{2-d}r_0^{d-1}),
\end{aligned}$$

where C depends only on d and the Lipschitz character of Ω. This proves the claim. □

We can now prove the boundedness of certain nontangential maximal operators.

lem:nontangentialMaximalFunctions

Lemma 3.6. *Let $1 < p < \infty$ and (u, ϕ) be given by* (3.1) *and* (3.2). *Then*

$$\|(\nabla u)^*\|_{\mathrm{L}^p(\partial\Omega)} + \|(\phi)^*\|_{\mathrm{L}^p(\partial\Omega)} \le C_p\|f\|_{\mathrm{L}^p(\partial\Omega)}, \tag{3.14}$$

where C_p depends only on d, θ, p and the Lipschitz character of Ω. Let furthermore $d \ge 3$. Then

eq:estimateUstar

$$\|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \le C_p\|f\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)}, \tag{3.15}$$

where C_p depends only on d, θ, p and the Lipschitz character of Ω.

Proof. A proof of the estimate $\|(\phi)^*\|_{\mathrm{L}^p(\partial\Omega)} \le C_p\|f\|_{\mathrm{L}^p(\partial\Omega,\mathbb{C}^d)}$ can be found in Verchota's dissertation [23, Lem. 1.3]. The proof for $\|(\nabla u)^*\|_{\mathrm{L}^p(\partial\Omega)}$ works in the same way. We will provide a proof for the sake of completeness. To imitate the proof of Verchota, we will work with the corresponding type of cones. Therefore the results for ∇u and ϕ will at first only be established for the type of maximal operators defined by Verchota. The transferability to Shen's maximal operators is given by Tolksdorf [22, p. 90ff.] as the solution (u, ϕ) has a representation as a single layer potential.

Let $q \in \partial\Omega$, $x \in \Gamma_{\mathrm{V}}(q)$ and set $t = |x - q|$. Then,

$$
\begin{aligned}
|(\nabla u)(x)| &= \left| \int_{\partial\Omega} \nabla_x \Gamma_{jk}(x - y; \lambda) f_k \, \mathrm{d}\sigma(y) \right| \\
&\leq \left| \int_{|y-q|>t} \nabla_x \Gamma_{jk}(x - y; \lambda) f_k \, \mathrm{d}\sigma(y) \right| + \left| \int_{|y-q|\leq t} \nabla_x \Gamma_{jk}(x - y; \lambda) f_k \, \mathrm{d}\sigma(y) \right| \\
&=: I_1 + I_2.
\end{aligned}
$$

We will now estimate I_1 and I_2 separately. Note that in Verchota cones $\Gamma_{\mathrm{V}}(q)$ we have that for all $s \in \partial\Omega$ we have $|x - s| \geq C|x - q|$, where C is a constant only depending on d and the Lipschitz character of Ω, see inequality 1.1. By Theorem 2.5 we know that

$$
\begin{aligned}
I_2 &\leq C \int_{|y-q|\leq t} \frac{1}{|x - y|^{d-1}} |f(y)| \, \mathrm{d}\sigma(y) \\
&\leq \frac{C}{t^{d-1}} \int_{|y-q|\leq t} |f(y)| \, \mathrm{d}\sigma(y) \leq C M_{\partial\Omega}(f)(q),
\end{aligned}
$$

where we used also Lemma 3.1 to bring the Hardy-Littlewood maximal operator into play. Here, C depends on d, θ and the Lipschitz character of Ω. For I_1, we calculate

$$
\begin{aligned}
&\left| \int_{|y-q|>t} \nabla_x \Gamma_{jk}(q - y; \lambda) f_k(y) - \nabla_x \Gamma_{jk}(q - y; \lambda) f_k(y) + \nabla_x \Gamma_{jk}(q - y; \lambda) f_k(y) \, \mathrm{d}\sigma(y) \right| \\
&\quad \leq \left| \int_{|y-q|>t} \nabla_x \Big\{ \Gamma_{jk}(x - y; \lambda) - \Gamma_{jk}(q - y; \lambda) \Big\} f_k(y) \, \mathrm{d}\sigma(y) \right| \\
&\qquad + \left| \int_{|y-q|>t} \nabla_x \Gamma_{jk}(q - y; \lambda) f_k(y) \, \mathrm{d}\sigma(y) \right|.
\end{aligned}
$$

The second summand can directly be estimated by $T^*_\lambda(f)(q)$. For the first one we apply the mean value theorem and use Theorem 2.5 to derive the following estimation:

$$
\begin{aligned}
&\int_{|y-q|>t} \big|\nabla_x \Gamma_{jk}(x - y; \lambda) - \nabla_x \Gamma_{jk}(P - y; \lambda)\big| \, |f(y)| \, \mathrm{d}\sigma(y) \\
&\qquad \leq \int_{|y-q|>t} \big|\nabla^2 \Gamma_{jk}(s - y; \lambda)\big| |x - q| \, |f(y)| \, \mathrm{d}\sigma(y) \\
&\qquad \leq C \int_{|y-q|>t} \frac{t}{|s - y|^d} |f(y)| \, \mathrm{d}\sigma(y) \\
&\qquad \leq C \int_{|y-q|>t} \frac{t}{|y - q|^d} |f(y)| \, \mathrm{d}\sigma(y) \\
&\qquad \leq C \int_{\partial\Omega} \frac{t}{(t + |y - q|)^d} |f(y)| \, \mathrm{d}\sigma(y),
\end{aligned}
$$

where s is an element on the line connecting x and q and we used the property of Verchota cones that $|s - y| \geq C|y - q|$, see inequality (1.2). Note that Verchota cones are convex.

As in Verchota [23, Lem 1.3], the integral may now be bounded by the Hardy-Littlewood maximal operator due to an application of a suitable result from Grafakos [9, Thm. 2.1.10] as the kernel $t(t + |y - P|)^{-d}$ is uniformly integrable and radially decreasing on $\partial\Omega$. Summing up we have shown that

$$|(\nabla u)(x)| \leq C\Big\{M_{\partial\Omega} f(P) + T^*_\lambda(f)(P)\Big\},$$

where C only depends on d, θ and the Lipschitz character of Ω. We thus may take the supremum over all $x \in \Gamma_{\mathrm{V}}(q)$ and conclude the desired estimate by the well known mapping properties of the Hardy-Littlewood maximal operator and the respective results from Lemma 3.3.

We will now work on the proof of the estimate for $(u)^*$ for $d \geq 3$. In order to derive L^p-bounds on this maximal operator, we will work directly with the Definition of the single layer potential (3.1). For $q \in \partial\Omega$, estimate (2.30) together with the estimate for Shen cones (1.3) gives that for all $x \in \Gamma(q)$

$$|u^*(x)| \leq C \int_{\partial\Omega} \frac{1}{|x - y|^{d-2}} |f(y)| \, \mathrm{d}\sigma(y) \leq C \int_{\partial\Omega} \frac{1}{|q - y|^{d-2}} |f(y)| \, \mathrm{d}\sigma(y),$$

where C only depends on d, θ and the Lipschitz character of Ω. Passing to the maximal operator yields the inequality

$$u^*(q) \leq C \int_{\partial\Omega} \frac{1}{|q - y|^{d-2}} |f(y)| \, \mathrm{d}\sigma(y).$$

Estimating the kernel via Lemma 3.5 and applying the Young inequality for convolutions from Lemma 3.4 the claim follows. □

Remark 3.7. We note that in addition to the consideration of $d = 2$, Lemma 3.6 differs in the form of estimate (3.15) from the original statement in Shen's work [17, Lem. 3.2]. There, the author derives an estimate of the form $|\lambda|^{1/2} \|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p \|f\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)}$ which is based on Shen's version of Lemma 2.1, namely [17, Lem. 2.1]. As we could not follow the proof in [17], we provided a similar estimate and since the estimate won't be needed in the course of this thesis, we will not pursue the verification of Shen's estimate further.

The next lemma deals with *trace formulas* for ∇u and ϕ. We will then finally be able to talk about boundary values since the existence of nontangential limits guarantees that there exists something on $\partial\Omega$ that is related to the function inside Ω or inside $\mathbb{R}^d \setminus \overline{\Omega}$, respectively.

lem:traceFormulas

Lemma 3.8. *Let* (u, ϕ) *be given by* (3.1) *and* (3.2) *with* $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ *and* $1 < p < \infty$. *Then*

$$\begin{aligned}\Big(\frac{\partial u_i}{\partial x_j}\Big)_\pm(x) &= \pm\frac{1}{2}\big\{n_j(x)f_i(x) - n_i(x)n_j(x)n_k(x)f_k(x)\big\} \\ &\quad + \mathrm{p.\,v.}\int_{\partial\Omega}\frac{\partial}{\partial x_j}\Big\{\Gamma_{ik}(x-y;\lambda)\Big\}f_k(y)\,\mathrm{d}\sigma(y), \qquad \text{eq:traceFormula} \quad (3.16)\\ \phi_\pm(x) &= \mp\frac{1}{2}n_k(x)f_k(x) + \mathrm{p.\,v.}\int_{\partial\Omega}\Phi_k(x-y)f_k(y)\,\mathrm{d}\sigma(y)\end{aligned}$$

for almost everywhere $x \in \partial\Omega$. *The subscripts* + *and* − *indicate nontangential limits taken inside* Ω *and outside* $\overline{\Omega}$, *respectively.*

Proof. The correctness of the trace formulas (3.16) is known for the case $\lambda = 0$ due to Mitrea and Write [13, Prop 4.4]. This fact will now be reused for $\lambda \in \Sigma_\theta$. We insert a 0 to the nontangential limit such that

$$(\nabla u_j)_\pm(x) = (\nabla v_j)_\pm(x) + (\nabla u_j - \nabla v_j)_\pm(x),$$

where $v_j(x) = \int_{\partial\Omega}\Gamma_{jk}(x-y;0)f_k(y)\,\mathrm{d}\sigma(y)$. Because of [13] we know that the first nontangential limit exists and is given by (3.16) with $\lambda = 0$. It therefore remains to show the identity

$$(\nabla u_j - \nabla v_j)_\pm(x) = \int_{\partial\Omega}\nabla_x\Big\{\Gamma_{jk}(x-y;\lambda) - \Gamma_{jk}(x-y;0)\Big\}f_k(y)\,\mathrm{d}\sigma(y)$$

for all $x \in \partial\Omega$. To this end let $(x_l)_{l\in\mathbb{N}}$ a sequence in $\Gamma(x)$ with $\lim_{l\to\infty} x_l = x$. Furthermore let us note that for almost everywhere $x \in \partial\Omega$ we have that

$$\int_{\partial\Omega}\frac{1}{|x-y|^{d-2}}|f(y)|\,\mathrm{d}\sigma(y) < \infty.$$

This is a consequence of Lemma 3.5 and Young's inequality from Lemma 3.4. Now, we will show that

$$\frac{1}{|x-y|^{d-2}}|f(y)|$$

gives a suitable function for dominated convergence. Set $\varepsilon = (4|\lambda|^2)^{-1}$ and without loss of generality assume that $\operatorname{supp} f \subseteq \mathrm{B}(x,\varepsilon)$. Furthermore assume that $|x_l - x| < \varepsilon$ for all $l \in \mathbb{N}$. Then $|x_l - y| \le (2|\lambda|^2)^{-1}$ and Corollary 2.7 give

$$\Big|\int_{\partial\Omega}\nabla_x\Big\{\Gamma_{jk}(x_l-y;\lambda) - \Gamma_{jk}(x_l-y;0)\Big\}f_k(y)\,\mathrm{d}\sigma(y)\Big|$$

$$\begin{aligned} &\leq \int_{\partial\Omega} \sqrt{|\lambda|} \frac{1}{|x_l - y|^{d-2}} |f(y)| \, \mathrm{d}\sigma(y) \\ &\leq C\sqrt{|\lambda|} \int_{\partial\Omega} \frac{1}{|x_l - y|^{d-2}} |f(y)| \, \mathrm{d}\sigma(y) < \infty. \end{aligned}$$

Now dominated convergence gives the claim for $x_l \longrightarrow x$. Note that it does not affect the proof if the sequence x_l lays inside Ω or outside $\overline{\Omega}$ and thus the same proof holds for a sequence (x_l) in $\Gamma^{\mathrm{ext}}(x)$. □

The previous lemma enables us to talk about boundary values of partial derivatives. The next theorem will now give a similar result but for *conormal derivatives*, which are defined for solutions (u, ϕ) to the Stokes (resolvent) system via

eq:conormalDerivative

$$\frac{\partial u}{\partial \nu} := \frac{\partial u}{\partial n} - \phi n, \tag{3.17}$$

see Mitrea and Wright [13, Eq. (1.2)], where n denotes the outer unit normal vector. We will also be working with the tangential gradient which is defined via

eq:tangentialGradient

$$\nabla_{\tan} u := \nabla u - \langle \nabla u, n \rangle n, \tag{3.18}$$

see Mitrea and Wright [13, p. 17].

thm:jumpConditions

Theorem 3.9. *Let $\lambda \in \Sigma_\theta$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. Let (u, ϕ) be given by* (3.1) *and* (3.2) *with $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ and $1 < p < \infty$. Then $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$ and*

eq:nontangentialConormalDerivative

$$\left(\frac{\partial u}{\partial \nu}\right)_\pm = \left(\pm \frac{1}{2} I + \mathcal{K}_\lambda\right) f \tag{3.19}$$

on $\partial\Omega$, with $\mathcal{K}_\lambda$ a bounded operator on $\mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ satisfying

$$\|\mathcal{K}_\lambda f\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p \|f\|_{\mathrm{L}^p(\partial\Omega)},$$

where C_p depends only on d, θ, p and the Lipschitz character of Ω.

Proof. For the jth component of the tangential derivative of u_i, $1 \leq i, j \leq d$, we calculate using the results from Lemma 3.8

$$\begin{aligned} ((\nabla_{\tan} u_i)_+)_j &= \left(\frac{\partial u_i}{\partial x_j}\right)_+ - \langle (\nabla u_i)_+, n \rangle n_j \\ &= \left(\frac{\partial u_i}{\partial x_j}\right)_+ - \left(\frac{\partial u_i}{\partial x_k}\right)_+ n_k n_j \\ &= \frac{1}{2}\left\{n_j f_i - n_i n_j n_k f_k\right\} - \frac{1}{2}\left\{n_k f_i - n_i n_k n_l f_l\right\} n_k n_j \end{aligned}$$

$$+\,\mathrm{p.\,v.}\int_{\partial\Omega}\frac{\partial}{\partial x_j}\Big\{\Gamma_{ik}(\,\cdot - y;\lambda)\Big\}f_k(y)\,\mathrm{d}\sigma(y)$$
$$+\,\mathrm{p.\,v.}\int_{\partial\Omega}\frac{\partial}{\partial x_k}\Big\{\Gamma_{il}(\,\cdot - y;\lambda)\Big\}f_l(y)\,\mathrm{d}\sigma(y)\,n_k n_j,$$

for almost everywhere $x \in \partial\Omega$. As the first two summands add up to zero, the entire expression does not depend on the direction of the nontangential limit. This gives

$$(\nabla_{\tan}u)_+ = (\nabla_{\tan}u)_-,$$

for almost everywhere $x \in \partial\Omega$. We calculate for the jth component of the nontangential limit of the conormal derivative of u at $x \in \partial\Omega$ using the results from Lemma 3.8

$$\begin{aligned}
&\Big(\frac{\partial u_j}{\partial x_i}\Big)_+ n_i - \phi_+ n_j \\
&\quad = \frac{1}{2}\big\{n_i f_j - n_j n_i n_k f_k\big\}n_i + \mathrm{p.\,v.}\int_{\partial\Omega}\frac{\partial}{\partial x_i}\Big\{\Gamma_{jk}(\,\cdot - y;\lambda)\Big\}f_k(y)\,\mathrm{d}\sigma(y)n_i \\
&\qquad + \frac{1}{2}n_k f_k n_j - \mathrm{p.\,v.}\int_{\partial\Omega}\Phi_k(\,\cdot - y)f_k(y)\,\mathrm{d}\sigma(y)n_j \\
&\quad = \frac{1}{2}f_j + (\mathcal{K}_\lambda f)_j
\end{aligned}$$

almost everywhere and where $\mathcal{K}_\lambda$ is a singular integral operator defined via

$$\begin{aligned}
(\mathcal{K}_\lambda f)_j(x) \coloneqq \mathrm{p.\,v.}&\int_{\partial\Omega}\frac{\partial}{\partial x_i}\Big\{\Gamma_{jk}(x-y;\lambda)\Big\}f_k(y)\,\mathrm{d}\sigma(y)\,n_i(x) \\
&- \mathrm{p.\,v.}\int_{\partial\Omega}\Phi_k(x-y)f_k(y)\,\mathrm{d}\sigma(y)\,n_j(x).
\end{aligned} \quad \text{eq:defnKlambda} \tag{3.20}$$

We note that $\mathcal{K}_\lambda$ essentially consists of two boundary layer potentials. The L^p-boundedness of the first one was proven in Lemma 3.3. The L^p-boundedness of the second boundary layer potential follows in an analogous way using the fact that the operators

$$A^*(f)(q) = \sup_{t>0}\left|\int_{\substack{y\in\partial\Omega\\ |y-q|>t}}\frac{q-y}{|q-y|^d}f(y)\,\mathrm{d}\sigma(y)\right|, \qquad q \in \partial\Omega,$$

are bounded by the corresponding result from Verchota [23, Lem. 1.2]. □

Similar to $\mathcal{K}_\lambda$, for $\lambda = 0$ we have

$$\begin{aligned}
(\mathcal{K}_0 f)_j(x) = \mathrm{p.\,v.}&\int_{\partial\Omega}\frac{\partial}{\partial x_i}\Big\{\Gamma_{jk}(x-y;0)\Big\}f_k(y)\,\mathrm{d}\sigma(y)\,n_i(x) \\
&- \mathrm{p.\,v.}\int_{\partial\Omega}\Phi_k(x-y)f_k(y)\,\mathrm{d}\sigma(y)\,n_j(x),
\end{aligned} \quad \text{eq:defnK0} \tag{3.21}$$

as was shown by Mitrea and Wright [13, Prop. 4.4]. If one compares (3.20) with (3.21), then the only difference lies withing the boundary integral involving the fundamental solutions Γ_{jk}.

The next result will be crucial for solving the L^2 Dirichlet problem in Chapter 5 and will fortify the hopes of translating results for $\lambda = 0$ to $\lambda \in \Sigma_\theta$.

lem:compactness

Lemma 3.10. *Let $\lambda \in \Sigma_\theta$ and $d \geq 2$ and let $\mathcal{K}_\lambda$ and $\mathcal{K}_0$ be defined by* (3.20) *and* (3.21), *respectively. Then the operator $\mathcal{K}_\lambda - \mathcal{K}_0$ on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ is compact.*

Proof. The idea of this proof is similar to the one in Tolksdorf [22, Lemma 4.3.5]. Let $f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$and let's denote $\mathcal{K} := \mathcal{K}_\lambda - \mathcal{K}_0$. We will now try to approximate $\mathcal{K}$ by compact operators in the operator norm. To this end, we define for all $\varepsilon > 0$

$$(\mathcal{K}^{(\varepsilon)} f)(x) := \int_{\partial\Omega \setminus \mathrm{B}(x,\varepsilon)} \nabla_x \Big\{ \Gamma(x-y;\lambda) - \Gamma(x-y;0) \Big\} f(y) \,\mathrm{d}\sigma(y)\, n, \qquad x \in \partial\Omega.$$

We can now estimate by Young's inequality 3.4

$$\begin{aligned} &\Big\| \mathcal{K} f - \mathcal{K}^{(\varepsilon)} f \Big\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)} \\ &\quad \leq \sup_{p \in \partial\Omega} \Big\| \nabla_x \Big\{ \Gamma(p - \cdot\,;\lambda) - \Gamma(p - \cdot\,;0) \Big\} 1_{\mathrm{B}(p,\varepsilon)} \Big\|_{\mathrm{L}^1(\partial\Omega;\mathbb{C}^{d\times d})} \|f\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)}. \end{aligned}$$

Our goal is to show that

$$\sup_{p \in \partial\Omega} \Big\| \nabla_x \Big\{ \Gamma(p - \cdot\,;\lambda) - \Gamma(p - \cdot\,;0) \Big\} 1_{\mathrm{B}(p,\varepsilon)} \Big\|_{\mathrm{L}^1(\partial\Omega;\mathbb{C}^{d\times d})} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

To this end, let ε be small enough such that we can apply the estimates from Corollary 2.7 to calculate for some $p \in \partial\Omega$

$$\begin{aligned} &\Big\| \nabla_x \Big\{ \Gamma(p - \cdot\,;\lambda) - \Gamma(p - \cdot\,;0) \Big\} 1_{\mathrm{B}(p,\varepsilon)} \Big\|_{\mathrm{L}^1(\partial\Omega;\mathbb{C}^{d\times d})} \\ &\qquad \leq C \int_{\partial\Omega \cap \mathrm{B}(p,\varepsilon)} \sqrt{|\lambda|} |p-y|^{2-d} \,\mathrm{d}\sigma(y) \leq C\sqrt{|\lambda|}\varepsilon \end{aligned}$$

where for the last step we applied Lemma 3.1. For $\varepsilon \to 0$ this gives us $\mathcal{K}^{(\varepsilon)} \to \mathcal{K}$ in the operator norm.

The last step is to verify the compactness of $\mathcal{K}^{(\varepsilon)}$. We note that the integral kernel of $\mathcal{K}^{(\varepsilon)}$ is bounded which gives us that in particular the kernel is an element of the space $\mathrm{L}^2(\partial\Omega \times \partial\Omega; \mathbb{C}^{d\times d})$. The compactness of $\mathcal{K}^{(\varepsilon)}$ now follows from Weidmann [25, Thm. 6.11].

As a consequence, $\mathcal{K}$ is compact since the limit of compact operators with respect to the operator norm gives again a compact operator. □

Our next step is to introduce the *double layer potential* $u(x) = \mathcal{D}_\lambda(f)(x)$ for the Stokes resolvent problem via

eq:defDoubleLayer

$$(\mathcal{D}_\lambda(f))_j(x) := \int_{\partial\Omega} \Big\{ \frac{\partial}{\partial y_i} \{\Gamma_{jk}(y-x;\lambda)\} n_i(y) - \Phi_j(y-x) n_k(y) \Big\} f_k(y) \,\mathrm{d}\sigma(y). \tag{3.22}$$

The corresponding pressure $\phi(x) = \mathcal{D}_\Phi(f)(x)$ is defined as

eq:defDoubleLayerPressure

$$\mathcal{D}_\Phi(f)(x) := \frac{\partial^2}{\partial x_i \partial x_k} \int_{\partial\Omega} G(y-x;0)\, n_i(y) f_k(y)\, \mathrm{d}\sigma(y) \\ + \lambda \int_{\partial\Omega} G(y-x;0)\, n_k(y) f_k(y)\, \mathrm{d}\sigma(y). \tag{3.23}$$

Using (2.28) and (2.29) one can show that (u, ϕ) defines again a solution to the Stokes resolvent problem in $\mathbb{R}^d \setminus \partial\Omega$.

The following theorem will give us a suitable operator which maps a given function $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$ to boundary values of $u = \mathcal{D}_\lambda(f)$ in the form of nontangential limits. It will then be the task of the following chapters to prove the invertibility of this operator.

thm:nontangentialLimitDoubleLayer

Theorem 3.11. *Let $\lambda \in \Sigma_\theta$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. Let u be given by* (3.22) *for $f \in \mathrm{L}^p(\partial\Omega; \mathbb{C}^d)$, $1 < p < \infty$. Then*

eq:lpBoundednessUNontangentialMax

$$\|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \leq C_p \|f\|_{\mathrm{L}^p(\partial\Omega)}, \tag{3.24}$$

where C_p depends only on d, p, θ and the Lipschitz character of Ω. Furthermore

eq:nontangentialLimitDoubleLayer

$$u_\pm = \Big(\mp \frac{1}{2} I + \mathcal{K}^*_{\bar\lambda} \Big) f, \tag{3.25}$$

where $\mathcal{K}^*_{\bar\lambda}$ is the adjoint of the operator $\mathcal{K}_{\bar\lambda}$ in (3.19)

Proof. The estimate for $(u)^*$ is a direct consequence of Lemma 3.6 and the estimates on the nontangential maximal functions for the single layer potentials $(\nabla \mathcal{S}_\lambda(f))^*$ and $(\mathcal{S}_\Phi(f))^*$: We have on the one hand

eq:rep1

$$\int_{\partial\Omega} \frac{\partial}{\partial y_i} \Big\{ \Gamma_{jk}(y-x;\lambda) \Big\}\, n_i(y) f_k(y)\, \mathrm{d}\sigma(y) \\ = - \int_{\partial\Omega} \frac{\partial}{\partial x_i} \Big\{ \Gamma_{jk}(x-y;\lambda) \Big\}\, n_i(y) f_k(y)\, \mathrm{d}\sigma(y) = - \frac{\partial}{\partial x_i} \mathcal{S}_\lambda(n_i f)_j(x) \tag{3.26}$$

and on the other hand

eq:rep2

$$- \int_{\partial\Omega} \Phi_j(y-x) n_k(y) f_k(y)\, \mathrm{d}\sigma(y) \\ = \int_{\partial\Omega} \Phi_l(x-y)\, \delta_{lj} n_k(y) f_k(y)\, \mathrm{d}\sigma(y) = \mathcal{S}_\Phi(\tilde{f}^j)(x), \tag{3.27}$$

where $\tilde{f}^j_l = \delta_{lj} n_k f_k$. This shows that

$$u_j(x) = (\mathcal{D}_\lambda(f))_j(x) = - \frac{\partial}{\partial x_i} \mathcal{S}_\lambda(n_i f)_j(x) + \mathcal{S}_\Phi(\tilde{f}^j)(x).$$

Therefore we have for $x \in \Gamma(q)$, $q \in \partial\Omega$,

$$|u(x)| \le C\Big\{|\nabla_x \mathcal{S}_\lambda(n_i f)(x)| + |\mathcal{S}_\Phi(\tilde{f}^j)|\Big\},$$

with C depending only on d. Hence by Lemma 3.6 we derive the estimation

$$\|(u)^*\|_{\mathrm{L}^p(\partial\Omega)} \le C\Big\{\sum_{i=1}^d \|n_i f\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)} + \sum_{i=1}^d \|\mathcal{S}_\Phi(\tilde{f}^j)\|_{\mathrm{L}^p(\partial\Omega)}\Big\} \le C\|f\|_{\mathrm{L}^p(\partial\Omega;\mathbb{C}^d)},$$

where C depends on d, p, θ and the Lipschitz character of Ω.

For the proof of (3.25), we begin by determining the adjoint of the operator $\mathcal{K}_{\bar{\lambda}}$. To this end we will first work with truncated operators $\mathcal{K}_\lambda^{(\varepsilon)} : \mathrm{L}^2(\partial\Omega;\mathbb{C}^d) \to \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ which are defined via

$$\begin{aligned}(\mathcal{K}_\lambda^{(\varepsilon)} f)_j(x) := &\int_{\partial\Omega} 1_{\mathrm{E}(x;\varepsilon)}(y)\frac{\partial}{\partial x_i}\Gamma_{jk}(x-y;\lambda) f_k(y)\,\mathrm{d}\sigma(y)\, n_i(x)\\ &- \int_{\partial\Omega} 1_{\mathrm{E}(x;\varepsilon)}(y)\,\Phi_k(x-y) f_k(y)\,\mathrm{d}\sigma(y)\, n_j(x),\end{aligned}$$

for $x \in \partial\Omega$ and $\mathrm{E}(x,\varepsilon) := \mathbb{R}^d \setminus \mathrm{B}(x;\varepsilon)$. Now for $f \in \mathrm{L}^p(\partial\Omega;\mathbb{C}^d)$ and $g \in \mathrm{L}^q(\partial\Omega;\mathbb{C}^d)$ with $1/p + 1/q = 1$ we calculate

$$\begin{aligned}\Big\langle \mathcal{K}_{\bar{\lambda}}^{(\varepsilon)} f, g\Big\rangle &= \int_{\partial\Omega} (\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)} f)_j(x)\,\overline{g_j(x)}\,\mathrm{d}\sigma(x)\\ &= \int_{\partial\Omega}\int_{\partial\Omega} \frac{\partial}{\partial x_i}\Big\{\Gamma_{jk}(x-y;\overline{\lambda})\Big\} f_k(y) 1_{\mathrm{E}(x;\varepsilon)}(y)\,\mathrm{d}\sigma(y)\, n_i(x)\,\overline{g_j(x)}\,\mathrm{d}\sigma(x)\\ &\quad+ \int_{\partial\Omega}\int_{\partial\Omega} \Phi_k(x-y) f_k(y) 1_{\mathrm{E}(x;\varepsilon)}(y)\,\mathrm{d}\sigma(y)\, n_j(x)\,\overline{g_j(x)}\,\mathrm{d}\sigma(x).\end{aligned}$$

Note that $1_{\mathrm{E}(x;\varepsilon)}(y) = 1_{\mathrm{E}(y;\varepsilon)}(x)$ for all $x, y \in \partial\Omega$. Now an application of Fubini's theorem and factoring out $f_k(y)$ gives that

$$\begin{aligned}\Big\langle \mathcal{K}_{\bar{\lambda}}^{(\varepsilon)} f, g\Big\rangle = \int_{\partial\Omega} f_k(y) \int_{\partial\Omega} \Big\{&\frac{\partial}{\partial x_i}\{\Gamma_{jk}(x-y;\overline{\lambda})\}\, n_i(x)\\ &- \Phi_k(x-y) n_j(x) \qquad \Big\}\, 1_{\mathrm{E}(y,\varepsilon)}(x)\,\overline{g_j(x)}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y).\end{aligned}$$

Therefore we see that the adjoint of the truncated operator $\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)}$ is given by

$$\begin{aligned}\big((\mathcal{K}_{\bar{\lambda}}^{(\varepsilon)})^* g\big)_k(y) = \int_{\partial\Omega} \Big\{&\frac{\partial}{\partial x_i}\{\Gamma_{jk}(x-y;\lambda)\} n_i(x)\\ &- \Phi_k(x-y) n_j(x) \qquad \Big\}\, 1_{\mathrm{E}(y,\varepsilon)}(x)\, g_j(x)\,\mathrm{d}\sigma(x),\end{aligned}$$

for $y \in \partial\Omega$ since $\overline{\Gamma_{jk}(x-y;\lambda)} = \Gamma_{jk}(x-y;\overline{\lambda})$.

In the next step we, will go from truncated operators to principal value operators through the dominated convergence theorem. For this to work we will look for suitable majorants. For $x \in \partial\Omega$, we estimate

$$\begin{aligned}\left|(\mathcal{K}_{\bar\lambda}^{(\varepsilon)} f)_j(x)\right| &= \Bigg| \int_{|x-y|>\varepsilon} \frac{\partial}{\partial x_i} \Big\{\Gamma_{jk}(x-y;\lambda)\Big\} f_k(y)\,\mathrm{d}\sigma(y)\, n_i(x) \\ &\qquad - \int_{|x-y|>\varepsilon} \Phi_k(x-y) f_k(y)\, n_j(x)\,\mathrm{d}\sigma(y) \Bigg| \\ &\le T_\lambda^*(f)(x) + A^*(f n_j)(x). \end{aligned} \tag{3.28}$$

eq:pIntegrableMajorant

We know from Lemma 3.3 and the respective result for A^* that the right hand side of inequality (3.28) is p-integrable and hence we get from dominated convergence

$$\lim_{\varepsilon\to 0} \left\langle \mathcal{K}_{\bar\lambda}^{(\varepsilon)} f, g \right\rangle = \left\langle \mathcal{K}_{\bar\lambda} f, g \right\rangle .$$

With a similar argument we get

$$\begin{aligned}\left|\big((\mathcal{K}_{\bar\lambda}^{(\varepsilon)})^* g\big)_k(y)\right| &= \Bigg| \int_{|x-y|>\varepsilon} \frac{\partial}{\partial x_i} \Big\{\Gamma_{jk}(x-y;\overline{\lambda})\Big\}\, n_i(x) g_j(x)\,\mathrm{d}\sigma(x) \\ &\qquad - \int_{|x-y|>\varepsilon} \Phi_k(x-y) n_j(x) g_j(x)\,\mathrm{d}\sigma(x) \Bigg| \\ &\le \sum_{i=1}^{d} T_\lambda^*(n_i g)(y) + \sum_{i=1}^{d} A^*(\tilde g^j)(y), \end{aligned}$$

where $\tilde g_l^j = \delta_{lj} n_k f_k$ and therefore the dominated convergence theorem yields

$$\lim_{\varepsilon\to 0} \left\langle f, \left(\mathcal{K}_{\bar\lambda}^{(\varepsilon)}\right)^* g \right\rangle = \left\langle f, \mathcal{K}_{\bar\lambda}^{(*)} g \right\rangle,$$

where the limit operator $\mathcal{K}_{\bar\lambda}^{(*)}$ is defined via

$$\big((\mathcal{K}_{\bar\lambda}^{(*)} g\big)_k(y) := \mathrm{p.\,v.} \int_{\partial\Omega} \Big\{ \frac{\partial}{\partial x_i} \big\{\Gamma_{kj}(x-y;\lambda)\big\} n_i(x) - \Phi_k(x-y) n_j(x) \Big\} g_j(x)\,\mathrm{d}\sigma(x).$$

Of course by the uniqueness of the adjoint operator, the identity $\langle \mathcal{K}_{\bar\lambda} f, g\rangle = \langle f, \mathcal{K}_{\bar\lambda}^{(*)} g\rangle$ shows that $\mathcal{K}_{\bar\lambda}^* = \mathcal{K}_{\bar\lambda}^{(*)}$. Note that we have used the symmetry of (Γ_{jk}).

In the last part of this proof we will show that the equality (3.25) holds. Note that by (3.26) and (3.27) we have made Lemma 3.8 accessible. For $x \in \partial\Omega$ we can now calculate

$$\begin{aligned}&\Big(\int_{\partial\Omega} \frac{\partial}{\partial y_i} \Big\{\Gamma_{jk}(y-\cdot\,;\lambda)\Big\} n_i(y) f_k(y)\,\mathrm{d}\sigma(y) \Big)_{\pm}(x) \\ &\qquad = -\big(\frac{\partial}{\partial x_i} \mathcal{S}_\lambda (n_i f)_j\big)_{\pm}(x)\end{aligned}$$

$$
\begin{aligned}
&= \mp\frac{1}{2}\big\{n_i(x)n_i(x)f_j(x) - n_j(x)n_i(x)n_k(x)n_i(x)f_k(x)\big\} \\
&\qquad - \text{p. v.} \int_{\partial\Omega} \frac{\partial}{\partial x_i}\Big\{\Gamma_{jk}(x-y;\lambda)\Big\}\, n_i(y)f_k(y)\,\mathrm{d}\sigma(y) \\
&= \mp\frac{1}{2}\big\{f_j(x) - n_j(x)n_k(x)f_k(x)\big\} \\
&\qquad + \text{p. v.} \int_{\partial\Omega} \frac{\partial}{\partial y_i}\Big\{\Gamma_{jk}(y-x;\lambda)\Big\}\, n_i(y)f_k(y)\,\mathrm{d}\sigma(y),
\end{aligned}
$$

where we used trace formula (3.16). A similar procedure for the second integral part of the double layer potential gives

$$
\begin{aligned}
&-\Big(\int_{\partial\Omega} \Phi_j(y-\cdot\,)\, n_k(y)f_k(y)\,\mathrm{d}\sigma(y)\Big)_{\pm}(x) \\
&\qquad = \big(\mathcal{S}_\Phi(\tilde{f}^j)\big)_{\pm}(x) \\
&\qquad = \mp\frac{1}{2}n_k(x)\tilde{f}^j_k(x) - \text{p. v.} \int_{\partial\Omega} \Phi_k(x-y)\tilde{f}^j_k(x)\,\mathrm{d}\sigma(y) \\
&\qquad = \mp\frac{1}{2}n_j(x)n_k(x)f_k(x) - \text{p. v.} \int_{\partial\Omega} \Phi_j(x-y)\, n_k(x)f_k(x)\,\mathrm{d}\sigma(y).
\end{aligned}
$$

Putting everything together we get

$$
(u_j)_\pm(x) = \mp\frac{1}{2}f_j(x) + (\mathcal{K}^*_{\bar{\lambda}}f)_j(x)
$$

and the proof is finished. □

Chapter 4

Rellich Estimates

chap:4

In this section we will establish Rellich type estimates for the Stokes resolvent problem which will be used to prove the invertibility of the operators $\pm(1/2)I + \mathcal{K}_\lambda$ and their adjoints from Theorems 3.9 and 3.11. We will for this entire section always assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with connected boundary. Furthermore we will use the shorthand notation

$$\|\cdot\|_\partial := \|\cdot\|_{\mathrm{L}^2(\partial\Omega;\mathbb{C}^k)}, \quad k \in \mathbb{N},$$

and we will tacitly use the summation convention whenever it is applicable.

The following Theorem is the central result of this chapter:

thm:rellich

Theorem 4.1. *Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0,1)$. Let (u, ϕ) be a smooth solution to the Stokes resolvent problem in Ω and suppose that $(\nabla u)^* \in \mathrm{L}^2(\partial\Omega)$ and $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$. Furthermore, assume that ∇u, ϕ have nontangential limits almost everywhere on $\partial\Omega$. Then*

$$\begin{aligned}\|\nabla u\|_\partial &+ \left\| \phi - \left\{ \frac{1}{{r_0}^{d-1}} \int_{\partial\Omega} \phi \,\mathrm{d}\sigma \right\} \right\|_\partial \\ &\leq C \left\{ \|\nabla_{\tan} u\|_\partial + |\lambda|^{1/2} \|u\|_\partial + |\lambda|\, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \right\}\end{aligned} \qquad \text{eq:rellich1} \quad (4.1)$$

and

$$\|\nabla u\|_\partial + |\lambda|^{1/2}\|u\|_\partial + |\lambda|\, \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\phi\|_\partial \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_\partial, \qquad \text{eq:rellich2} \quad (4.2)$$

where $\frac{\partial}{\partial\nu}$ denotes the conormal derivative, and C depends only on d, τ, θ and the Lipschitz character of Ω.

rem:shenNontangential

Remark 4.2. The assumptions on u in Theorem 4.1 are sufficient for u to have a nontangential limit and a square integrable maximal function $(u)^*$. Indeed for $d = 2$ we have $(u)^* \in \mathrm{L}^\infty(\partial\Omega)$, for $d = 3$ we have $(u)^* \in \mathrm{L}^p(\partial\Omega), p \in (1, \infty)$, and for $d \geq 3$ we have $(u)^* \in \mathrm{L}^p(\partial\Omega)$, $p \in \big(1, 2(d-1)/(d-3)\big)$. A proof of these facts can be found in Shen's notes [**?**, Prop. 7.1.3].

We will now prepare the proof of Theorem 4.1 by proving several helpful lemmata. The first lemma deals with so called *Rellich identities* for solutions to the Stokes resolvent system.

lem:rellichIdentity

Lemma 4.3. *Under the same conditions on (u, ϕ) as in Theorem 4.1, we have*

$$\begin{aligned}\int_{\partial\Omega} h_k n_k |\nabla u|^2 \,\mathrm{d}\sigma = {} & 2\,\mathrm{Re} \int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k} \Big(\frac{\partial u}{\partial \nu}\Big)_i \,\mathrm{d}\sigma + \int_\Omega \operatorname{div}(h) |\nabla u|^2 \,\mathrm{d}x \\ & - 2\,\mathrm{Re} \int_\Omega \frac{\partial h_k}{\partial x_j} \cdot \frac{\partial u_i}{\partial x_k} \cdot \frac{\partial \overline{u}_i}{\partial x_j} \,\mathrm{d}x + 2\,\mathrm{Re} \int_\Omega \frac{\partial h_k}{\partial x_i} \cdot \frac{\partial u_i}{\partial x_k} \overline{\phi} \,\mathrm{d}x \\ & - 2\,\mathrm{Re} \int_\Omega h_k \frac{\partial u_i}{\partial x_k} \cdot \overline{\lambda u_i} \,\mathrm{d}x \qquad \text{eq:rellichIdentity} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\int_{\partial\Omega} h_k n_k |\nabla u|^2 \,\mathrm{d}\sigma = {} & 2\,\mathrm{Re} \int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_j} \Big\{ n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k} \Big\} \,\mathrm{d}\sigma \\ & + 2\,\mathrm{Re} \int_{\partial\Omega} h_k \overline{\phi} \Big\{ n_i \frac{\partial u_i}{\partial x_k} - n_k \frac{\partial u_i}{\partial x_i} \Big\} \,\mathrm{d}\sigma - \int_\Omega \operatorname{div}(h) |\nabla u|^2 \,\mathrm{d}x \\ & + 2\,\mathrm{Re} \int_\Omega \frac{\partial h_k}{\partial x_j} \cdot \frac{\partial u_i}{\partial x_k} \cdot \frac{\partial \overline{u}_i}{\partial x_j} \,\mathrm{d}x - 2\,\mathrm{Re} \int_\Omega \frac{\partial h_k}{\partial x_i} \cdot \frac{\partial u_i}{\partial x_k} \overline{\phi} \,\mathrm{d}x \\ & + 2\,\mathrm{Re} \int_\Omega h_k \frac{\partial u_i}{\partial x_k} \cdot \overline{\lambda} \overline{u}_i \,\mathrm{d}x, \qquad \text{eq:rellichIdentity2} \end{aligned} \tag{4.4}$$

where $h = (h_1, \dots, h_d) \in \mathrm{C}_0^1(\mathbb{R}^d, \mathbb{R}^d)$.

Proof. The proof of the stated identities reduces to several applications of the divergence theorem once we establish its applicability. To this end, we want to make Proposition 1.8 available. We note that the assumptions given in Theorem 4.1 are sufficient for this purpose and we will verify them, once they are used.

Let's expand the first summand in (4.3) using the definition of conormal derivatives

$$2\,\mathrm{Re} \int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k} \Big(\frac{\partial u}{\partial \nu}\Big)_i \,\mathrm{d}\sigma = 2\,\mathrm{Re} \int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k} \cdot \frac{\partial u_i}{\partial x_j} n_j \,\mathrm{d}\sigma - 2\,\mathrm{Re} \int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k} \phi n_i \,\mathrm{d}x$$

$$=: I_1 - I_2.$$

The divergence theorem is applicable for I_1 as h is bounded and defined everywhere and the integrand has nontangential limits that can be dominated by $|(\nabla u)^*|^2 \in \mathrm{L}^2(\partial\Omega)$. Therefore, we find using the divergence theorem and the product rule:

$$\begin{aligned}
I_1 &= 2\,\mathrm{Re}\int_\Omega \frac{\partial}{\partial x_j}\left\{h_k \frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial u_i}{\partial x_j}\right\}\mathrm{d}x \\
&= 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_j}\cdot\frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial u_i}{\partial x_j}\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_k}\cdot\frac{\partial u_i}{\partial x_j}\,\mathrm{d}x \\
&\qquad\qquad\qquad\qquad\qquad + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial^2 u_i}{\partial x_j^2}\,\mathrm{d}x \\
&=: I_3 + I_4 + I_5.
\end{aligned}$$

For I_5 we use the fact that u solves the Stokes resolvent problem which gives

$$\begin{aligned}
I_5 &= 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial \phi}{\partial x_i}\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k}\lambda u_i\,\mathrm{d}x \\
&=: I_6 + I_7.
\end{aligned}$$

Now we want to apply the divergence theorem, i.e. Proposition 1.8 to integral I_2. This is possible since h is defined everywhere and bounded, $(\partial_k u_i)\cdot\phi$ has a nontangential limit and can be bounded by $\big(|(\nabla u)^*||(\phi)^*|\big)$ which is integrable due to Hölder's inequality as $(\nabla u)^*$ and $(\phi)^*$ are square integrable by assumption. Thus the divergence theorem is applicable and yields together with the product rule:

$$\begin{aligned}
I_2 &= 2\,\mathrm{Re}\int_\Omega \frac{\partial}{\partial x_i}\left\{h_k \frac{\partial \overline{u}_i}{\partial x_k}\phi\right\}\mathrm{d}x \\
&= 2\,\mathrm{Re}\int_\Omega \frac{\partial h_k}{\partial x_i}\cdot\frac{\partial \overline{u}_i}{\partial x_k}\,\phi\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_k}\,\phi\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial \overline{u}_i}{\partial x_k}\cdot\frac{\partial \phi}{\partial x_i}\,\mathrm{d}x \\
&=: I_8 + I_9 + I_6.
\end{aligned}$$

One term that hasn't come up so far, the second summand of the right hand side in (4.3), will now be expanded:

$$\begin{aligned}
\int_\Omega \mathrm{div}\,(h)\,|\nabla u|^2\,\mathrm{d}x &= \int_\Omega \mathrm{div}\,(h|\nabla u|^2)\,\mathrm{d}x - \int_\Omega h_k \frac{\partial}{\partial x_i}\left\{|\nabla u|^2\right\}\mathrm{d}x \\
&=: I_{10} - I_{11}.
\end{aligned}$$

Expanding the Integral I_{11} gives us the identity

$$I_{11} = \int_\Omega h_i \frac{\partial}{\partial x_i}\left\{\frac{\partial u_k}{\partial x_j}\cdot\frac{\partial \overline{u}_k}{\partial x_j}\right\}\mathrm{d}x = \int_\Omega h_i\left\{\frac{\partial^2 u_k}{\partial x_i \partial x_j}\cdot\frac{\partial \overline{u}_k}{\partial x_j} + \frac{\partial u_k}{\partial x_j}\cdot\frac{\partial^2 \overline{u}_k}{\partial x_i \partial x_j}\right\}\mathrm{d}x = I_4.$$

If we now put everything together, the right hand side of (4.3) reads

$$\begin{aligned}&(I_1 - I_2) + (I_{10} - I_{11}) - I_3 + I_8 - I_7\\&\quad = (I_3 + I_4 + I_6 + I_7) - (I_8 + I_9 + I_6) + I_{10} - I_{11} - I_3 + I_8 - I_7 = I_{10}.\end{aligned}$$

Noting that by the divergence theorem, which is applicable with the same justification as for the integral I_1, we have

$$I_{10} = \int_{\partial\Omega} h_k n_k |\nabla u|^2 \,\mathrm{d}\sigma.$$

Thus, the first identity is proven.

In order to prove identity (4.4), we show that the expression we get from considering ((4.3) + (4.4)) holds, i.e. we show the identity

$$\begin{aligned}2\int_{\partial\Omega} h_k n_k |\nabla u|^2 \,\mathrm{d}\sigma = 2\operatorname{Re}\int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_k}\left(\frac{\partial u}{\partial \nu}\right)_i \mathrm{d}\sigma&\\+ 2\operatorname{Re}\int_{\partial\Omega} h_k \frac{\partial \overline{u}_i}{\partial x_j}\left\{n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k}\right\}\mathrm{d}\sigma&\\+ 2\operatorname{Re}\int_{\partial\Omega} h_k\, \overline{\phi}\left\{n_i \frac{\partial u_i}{\partial x_i} - n_k \frac{\partial u_i}{\partial x_i}\right\}\mathrm{d}\sigma.&\end{aligned}$$

To this end, note that the left side of the identity equals $2I_{10}$, whereas the right hand side can be written as

$$(I_1 - I_2) + 2(I_{10} - I_1) + (I_2 - 0),$$

where we also used the fact that $\operatorname{div} u = \partial_i u_i = 0$. □

Consider the operators $\partial_{\tau_{jk}}$ which act on compactly supported continuously differentiable functions ψ in the neighborhood of $\partial\Omega$ by

$$\partial_{\tau_{jk}}\psi := n_j \frac{\partial\psi}{\partial x_k}\bigg|_{\partial\Omega} - n_k \frac{\partial\psi}{\partial x_j}\bigg|_{\partial\Omega}, \qquad j,k = 1,\dots,d. \tag{4.5}$$

eq:defnTangDerivative

They have been introduced by Mitrea and Wrigth [13, p. 16] and come with a helpful "integration by parts" rule that can be used to define Sobolev spaces on the boundary $\partial\Omega$. However for our purposes it will suffice to formulate this rule for the specific case

These operators show up in identity (4.4) as

$$\partial_{\tau_{kj}} u_i = \left\{n_k \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_i}{\partial x_k}\right\} \quad \text{and} \quad \partial_{\tau_{ik}} u_i = \left\{n_i \frac{\partial u_i}{\partial x_k} - n_k \frac{\partial u_i}{\partial x_i}\right\}$$

We make a quick detour that gives us the following basic lemma on elements of the sector Σ_θ. A powerful generalization of this Lemma can be found in Tolksdorf [22, Lem. 5.2.4].

lem:lambdaIneq

Lemma 4.4. *Let $\theta \in (0, \pi/2)$. Then there exists α depending only on θ such that for all $\lambda \in \Sigma_\theta$ the following inequality holds:*

$$\mathrm{Re}(\lambda) + \alpha \left|\mathrm{Im}(\lambda)\right| \geq |\lambda|.$$

Proof. For the moment being, suppose $|\lambda| = 1$ Then, we have $\mathrm{Re}(\lambda) = \cos(\varphi)$ and $\mathrm{Im}(\lambda) = \sin(\varphi)$ with $|\varphi| \in (0, \pi - \theta)$. Set

$$\alpha = \frac{1 - \cos(\pi - \theta)}{\sin(\pi - \theta)} \geq \frac{1 - \cos(|\varphi|)}{\sin(|\varphi|)}.$$

If $\varphi = |\varphi|$, this gives the inequality

$$\mathrm{Re}(\lambda) + \alpha \,|\, \mathrm{Im}(\lambda) \,|\, = \cos(\varphi) + \alpha \sin(\varphi) \geq 1.$$

Conversely, if $\varphi = -|\varphi|$, then we have by the symmetry properties of sin and cos that

$$\mathrm{Re}(\lambda) + \alpha \,|\, \mathrm{Im}(\lambda) \,|\, = \cos(-\varphi) + \alpha \sin(-\varphi) \geq 1.$$

For arbitrary λ the claim follows by considering the normalized value $(\lambda/|\lambda|)$. □

The next lemma enables us to handle the solid integrals in (4.3) and (4.4).

lem:laxMilgramIneq

Lemma 4.5. *Under the same assumptions on (u, ϕ) and λ as in Theorem 4.1, we have*

eq:laxMilgramIneq

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + |\lambda| \int_\Omega |u|^2 \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_\partial \|u\|_\partial, \tag{4.6}$$

where C depends only on θ.

Proof. Inserting the solution u into the the variational problem of the Stokes resolvent problem gives us

eq:variationalStokes

$$\int_\Omega -\Delta u \cdot \overline{u} \,\mathrm{d}x + \lambda \int_\Omega u \cdot \overline{u} \,\mathrm{d}x = -\int_\Omega \nabla\phi \cdot \overline{u} \,\mathrm{d}x. \tag{4.7}$$

Rewriting the first term of equation (4.7) leads to

$$-\int_\Omega \frac{\partial^2 u_j}{\partial x_i \partial x_i} \overline{u}_j \,\mathrm{d}x = -\int_\Omega \frac{\partial}{\partial x_i}\left\{\overline{u}_j \frac{\partial u_j}{\partial x_i}\right\} \mathrm{d}x + \int_\Omega \frac{\partial u_j}{\partial x_i} \cdot \frac{\partial \overline{u}_j}{\partial x_i} \,\mathrm{d}x.$$

Note that since u is solenoidal, we have for the third term of equation (4.7)

$$-\int_\Omega \frac{\partial \phi}{\partial x_i} \overline{u}_i \,\mathrm{d}x = -\int_\Omega \frac{\partial}{\partial x_i}\left\{\phi\, \overline{u}_i\right\} \mathrm{d}x.$$

Now we want to transform the first and third of the above solid integrals into boundary integrals through Proposition 1.8. By the assumptions formulated in Theorem 4.1, ϕ and ∇u have a nontangential limit and for both nontangential maximal functions the inclusion $(\phi)^*, (\nabla u)^* \in \mathrm{L}^2(\partial\Omega)$ holds. Furthermore, according to Remark 4.2, also u has a nontangential limit and the nontangential maximal function satisfies $(u)^* \in \mathrm{L}^2(\partial\Omega)$. Therefore, the function $|\phi\, \overline{u}_i|$ may be dominated by $|(\phi)^*(u)^*| \in \mathrm{L}^2(\partial\Omega)$ and the function $|(\partial_j u_i)u_i|$ may be dominated by $|(\nabla u)^*(u)^*|$, respectively. Thus, the door to Proposition 1.8 has been opened which allows to transform equation (4.7) into

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot \overline{u} \,\mathrm{d}\sigma + \lambda \int_\Omega |u|^2 \,\mathrm{d}x = - \int_{\partial\Omega} \phi n \cdot \overline{u} \,\mathrm{d}\sigma.$$

We can rearrange the terms of this identity and use the definition of conormal derivatives, see equation (3.17), to derive

eq:testedStokes

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \lambda \int_\Omega |u|^2 \,\mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \overline{u} \,\mathrm{d}\sigma. \tag{4.8}$$

If we now take the real and imaginary part of (4.8) and sum them up with the prefactor $\alpha(\theta) > 0$ from Lemma 4.4, we get

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \Big\{ \mathrm{Re}(\lambda) + \alpha \left| \mathrm{Im}(\lambda) \right| \Big\} \int_\Omega |u|^2 \,\mathrm{d}x \leq (1+\alpha) \left| \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \bar{u} \,\mathrm{d}\sigma \right|.$$

Lemma 4.4 now gives

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x + |\lambda| \int_\Omega |u|^2 \,\mathrm{d}x \leq C \left| \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \bar{u} \,\mathrm{d}\sigma \right|,$$

with $C = (1+\alpha)$ from which we readily derive estimate (4.6) after applying the Cauchy-Schwartz inequality. □

The next lemma combines Rellich identities (4.3) and (4.4) with estimate (4.6).

Lemma 4.6. *Under the same assumptions on (u, ϕ) and λ as in Theorem 4.1, we have*

eq:gradEstimateRellich

$$\|\nabla u\|_\partial \leq C_\varepsilon \left\| \frac{\partial u}{\partial \nu} \right\|_\partial + \varepsilon \Big\{ \|\nabla u\|_\partial + \|\phi\|_\partial + \| \, |\lambda|^{1/2} u \|_\partial \Big\} \tag{4.9}$$

and

eq:gradEstimateRellich2

$$\|\nabla u\|_\partial \leq C_\varepsilon \Big\{ \|\nabla_{\tan} u\|_\partial + \| \, |\lambda|^{1/2} u \|_\partial \Big\} + \varepsilon \{ \|\nabla u\|_\partial + \|\phi\|_\partial \} \tag{4.10}$$

for all $\varepsilon \in (0,1)$, where C_ε depends only on d, θ, τ, ε and the Lipschitz character of Ω.

Proof. Let $h = (h_1, \dots, h_d) \in \mathrm{C}^1_0(\mathbb{R}^d, \mathbb{R}^d)$ with $h_k n_k \geq c > 0$ on $\partial\Omega$ as given by Therorem 1.3 v). The idea of the proof of the desired estimates (4.9) and (4.10) is to first use the Rellich identities from Lemma 4.3 with this particular h to estimate $\|\nabla u\|_\partial$ and then to bound the resulting right hand side by providing individual estimates.

Before we start, note that we have $\Delta\phi = 0$ on the one hand and for the nontangential maximal function $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$ on the other hand. According to Shen [17, p. 410], a result from Dahlberg [**?**] gives the estimation

$$\int_\Omega |\phi|^2 \,\mathrm{d}x \leq C\|(\phi)^*\|_\partial^2 \leq C\|\phi\|_\partial^2. \qquad \text{eq:dahlbergEstimate} \tag{4.11}$$

We will now prove the first estimate (4.9). In view of identity (4.3), we have

$$\begin{aligned}\|\nabla u\|_\partial^2 \leq C\bigg\{&\|\nabla u\|_\partial \left\|\frac{\partial u}{\partial \nu}\right\|_\partial + \int_\Omega |\nabla u|^2 \,\mathrm{d}x \\ &+ \int_\Omega |\nabla u|\,|\phi| \,\mathrm{d}x + |\lambda| \int_\Omega |\nabla u|\,|u| \,\mathrm{d}x\bigg\},\end{aligned} \qquad \text{eq:normRellich} \tag{4.12}$$

where the first term follows from the Cauchy-Schwartz inequality and C only depends on d and the Lipschitz character of Ω.

For now, we keep the first term of (4.12) as it is, the second term can be handled via Lemma 4.5. The goal for the remaining two integrals will be to bound each of them by a product of norms $\|\cdot\|_\partial$. To this end, for the third integral we calculate

$$\int_\Omega |\nabla u||\phi| \,\mathrm{d}x \leq \Big(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega |\phi|^2 \,\mathrm{d}x\Big)^{1/2} \leq C\left\|\frac{\partial u}{\partial \nu}\right\|_\partial^{1/2} \|u\|_\partial^{1/2}\,\|\phi\|_\partial, \qquad \text{eq:nablaPhi} \tag{4.13}$$

where the first step is due to the Cauchy-Schwartz inequality and the second step combines estimate (4.6) with estimate (4.11).

The last integral of (4.12) can be estimated as follows:

$$\begin{aligned}|\lambda| \int_\Omega |\nabla u|\,|u| \,\mathrm{d}x &\leq \frac{|\lambda|^{3/2}}{2}\int_\Omega |u|^2 \,\mathrm{d}x + \frac{|\lambda|^{1/2}}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x \\ &\leq C\left\|\frac{\partial u}{\partial \nu}\right\|_\partial \big\|\, |\lambda|^{1/2} u \,\big\|_\partial,\end{aligned} \qquad \text{eq:lambdaNablaU} \tag{4.14}$$

where in the first step we used the weighted Young inequality and in the second step we applied estimate (4.6). Putting everything together, we calculate

$$\|\nabla u\|_\partial^2 \leq C\bigg\{\|\nabla u\|_\partial \left\|\frac{\partial u}{\partial \nu}\right\|_\partial + \left\|\frac{\partial u}{\partial \nu}\right\|_\partial \|u\|_\partial + \left\|\frac{\partial u}{\partial \nu}\right\|_\partial^{1/2} \|u\|_\partial^{1/2}\|\phi\|_\partial + \left\|\frac{\partial u}{\partial \nu}\right\|_\partial \big\|\, |\lambda|^{1/2} u \,\big\|_\partial\bigg\}$$

If we now use the assumption $|\lambda| \geq \tau$ which allows us to bound $\|u\|_\partial$ via

$$\|u\|_\partial \leq \frac{|\lambda|^{1/2}}{\tau^{1/2}}\|u\|_\partial = C|\lambda|^{1/2}\|u\|_\partial,$$

the desired estimate (4.9) now follows applying Young's weighted inequality with an ε and the norm equivalence on finite dimensional vector spaces. Note that for the product of three norms from inequality (4.13) we need to apply the Young inequality twice:

$$\begin{aligned}
\left\|\frac{\partial u}{\partial \nu}\right\|_\partial^{1/2} \|u\|_\partial^{1/2} \|\phi\|_\partial &\leq \left\{ \frac{1}{4\varepsilon}\left\|\frac{\partial u}{\partial \nu}\right\|_\partial + \varepsilon \|u\|_\partial \right\} \|\phi\|_\partial \\
&\leq \frac{1}{32\,\varepsilon^3}\left\|\frac{\partial u}{\partial \nu}\right\|_\partial^2 + \frac{\varepsilon}{2}\|\phi\|_\partial^2 + \frac{1}{2}\|u\|_\partial^2 + \frac{\varepsilon^2}{2}\|\phi\|_\partial^2 \\
&\leq C_\varepsilon \left\{ \left\|\frac{\partial u}{\partial \nu}\right\|_\partial^2 + \|u\|_\partial^2 \right\} + \varepsilon \|\phi\|_\partial^2,
\end{aligned}$$

where for the last inequality we used the fact that $\varepsilon < 1$.

For inequality (4.10), we use the Rellich identity (4.4) to obtain the estimate

eq:onTheWay

$$\begin{aligned}
\|\nabla u\|_\partial^2 \leq C\Big\{ &\|\nabla_{\tan} u\|_\partial \Big\{ \|\nabla u\|_\partial + \|\phi\|_\partial \Big\} \\
&+ \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega |\nabla u||\phi| \,\mathrm{d}x + |\lambda| \int_\Omega |\nabla u||u| \,\mathrm{d}x \Big\}.
\end{aligned} \tag{4.15}$$

As before we estimate the three terms on the right side of (4.15) using (4.6), (4.13) and (4.14), respectively and obtain the estimate

$$\begin{aligned}
\|\nabla u\|_\partial^2 \leq C\Big\{ &\|\nabla_{\tan} u\|_\partial \Big\{ \|\nabla u\|_\partial + \|\phi\|_\partial \Big\} \\
&+ \left\|\frac{\partial u}{\partial \nu}\right\|_\partial \|u\|_\partial + \left\|\frac{\partial u}{\partial \nu}\right\|_\partial^{1/2} \|u\|_\partial^{1/2} \|\phi\|_\partial + \left\|\frac{\partial u}{\partial \nu}\right\|_\partial \left\| |\lambda|^{1/2} u \right\|_\partial \Big\}.
\end{aligned}$$

If we now use the Young inequality with an ε, we get

$$\|\nabla u\|_\partial^2 \leq C_\varepsilon \left\{ \|\nabla_{\tan} u\|_\partial^2 + \||\lambda|^{1/2} u\|_\partial^2 \right\} + \varepsilon \left\{ \|\nabla u\|_\partial^2 + \|\phi\|_\partial^2 + \frac{1}{4}\left\|\frac{\partial u}{\partial \nu}\right\|_\partial^2 \right\}.$$

The claim now follows if we use the definition of the conormal derivative and the norm equivalence on finite dimensional vector spaces. □

We prove one last lemma befor we tackle the central theorem of this chapter.

Lemma 4.7. *Assume that (u, ϕ) satisfies the same conditions as in Theorem 4.1. Then*

eq:phiDashintPhi

$$\|\phi - \fint_{\partial\Omega} \phi\|_\partial \leq C\{\|\nabla u\|_\partial + |\lambda|\|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega} \tag{4.16}$$

and

eq:lambdaun

$$|\lambda|\|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C\{\|\phi\|_\partial + \|\nabla u\|_\partial, \tag{4.17}$$

where C depends only on d and the Lipschitz character of Ω.

Proof. By Theorem 1.3 we may assume that $\Delta u = \nabla\phi + \lambda u$ on $\partial\Omega$. Multiplying the Stokes resolvent equation on $\partial\Omega$ with n and using the triangle inequality gives

eq:stokesEquationH1

$$\begin{aligned} \|\nabla\phi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} &\le \|\Delta u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + |\lambda|\|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}, \\ |\lambda|\|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} &\le \|\Delta u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\nabla\phi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}. \end{aligned} \tag{4.18}$$

We will now show that

eq:deltaun

$$\|\Delta u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \le C\|\nabla u\|_{\partial} \tag{4.19}$$

and

eq:nablaPhin

$$c\|\phi - ⨍_{\partial\Omega} \phi \,\mathrm{d}\sigma\|_{\partial} \le \|\nabla\phi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \le C\|\phi\|_{\partial} \tag{4.20}$$

Using these two estimates applied to (4.18), we can directly derive (4.16) and (4.17).

In order to prove (4.19), note that

$$\Delta u \cdot n = n_i \frac{\partial^2 u_i}{\partial x_j^2} = \Big(n_i \frac{\partial}{\partial x_j} - n_j \frac{\partial}{\partial x_i}\Big)\frac{\partial u_i}{\partial x_j}$$

since $\operatorname{div} u = 0$ in $\overline{\Omega}$. As the expression in between the brackets is a tangential derivative we derive estimate (4.19) from

$$|\langle \Delta u \cdot n, u\rangle| = |\langle \nabla u, \nabla_{\tan} u\rangle| \le \|\nabla u\|_{\partial}^2$$

since this implies

$$\|\nabla u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \le \|\nabla u\|_{\partial}.$$

Now for the proof of estimate (4.20) we will use L^2-estimates for the Neumann and regularity problems for the Laplace equation in Lipschitz domains. For $g \in \mathrm{L}^2(\partial\Omega)$ with mean value zero, by Jerison and Kenig [**?**] the Neumann problem for Laplace's equation on the Lipschitz domain Ω has a solution ψ with $(\nabla\psi)^* \in \mathrm{L}^2(\partial\Omega)$ and $\frac{\partial\psi}{\partial n} = g$ on $\partial\Omega$. Green's identity we have that since ϕ and ψ are harmonic

eq:dualityPhi

$$\begin{aligned} \Big|\int_{\partial\Omega} \phi g \,\mathrm{d}\sigma\Big| &= \Big|\int_{\partial\Omega} \phi \frac{\partial\psi}{\partial n} \,\mathrm{d}\sigma\Big| = \Big|\int_{\partial\Omega} \frac{\partial\phi}{\partial n}\psi \,\mathrm{d}\sigma\Big| \\ &\le \|\frac{\partial\phi}{\partial n}\|_{\mathrm{H}^{-1}(\partial\Omega)}\|\psi\|_{\mathrm{H}^{1}(\partial\Omega)} \le C\|\frac{\partial\phi}{\partial n}\|_{\mathrm{H}^{-1}(\partial\Omega)}\|g\|_{\partial}, \end{aligned} \tag{4.21}$$

where in the last step we used the estimate $\|\psi\|_{\mathrm{H}^1(\partial\Omega)} \le C\|g\|_{\partial}$ for the L^2 Neumann problem which can be found in Jerison and Kenig [**?**]. Now if we set $\bar{g} = \phi - \tilde{\phi}$, with $\tilde{\phi} = ⨍_{\partial\Omega} \phi \,\mathrm{d}\sigma$ and use that $\int_{\partial\Omega}(\phi - \tilde{\phi})\overline{(\phi - \tilde{\phi})}\,\mathrm{d}\sigma = \int_{\partial\Omega} \phi\overline{(\phi - \tilde{\phi})}\,\mathrm{d}\sigma$, we get from (4.21)

$$\|\phi - \tilde{\phi}\|_{\partial}^2 \le C\|\frac{\partial\phi}{\partial n}\|_{\mathrm{H}^{-1}(\partial\Omega)}\|\phi - \tilde{\phi}\|_{\partial}$$

or, after rearranging and expanding

$$\|\phi - \fint_{\partial\Omega} \phi \, \mathrm{d}\sigma\|_\partial \leq C \|\frac{\partial\phi}{\partial n}\|_{\mathrm{H}^{-1}(\partial\Omega)}$$

We work in a similar way with results from the regularity problem of Laplace's equation by Jerison and Kenig [?]. Given $f \in \mathrm{H}^1(\partial\Omega)$, there exists a harmonic function ψ in Ω such that $(\nabla\psi)^* \in \mathrm{L}^2(\partial\Omega)$ and $\psi = f$ on $\partial\Omega$. As for (4.21), we calculate

$$\begin{aligned} |\int_{\partial\Omega} \frac{\partial\phi}{f} \, \mathrm{d}\sigma| = |\int_{\partial\Omega} \frac{\partial\phi}{\psi} \, \mathrm{d}\sigma| &= |\int_{\partial\Omega} \phi \frac{\partial\psi}{\partial n} \, \mathrm{d}\sigma| \\ &\leq \|\phi\|_\partial \|\nabla\psi\|_\partial \leq C\|\phi\|_\partial \|f\|_{\mathrm{H}^1(\partial\Omega)}, \end{aligned}$$

where in the last step we used the estimate $\|\nabla\psi\|_\partial \leq C\|f\|_{\mathrm{H}^1(\partial\Omega)}$ for the L^2 regularity problem. By duality this gives that

$$\|\frac{\partial\phi}{\partial n}\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C\|\phi\|_\partial.$$

□

rem:harmonicEstimate

Remark 4.8. A careful look at the proof of inequality (4.20) reveals that the estimate

$$c\|\phi\|_\partial \leq \|\nabla\phi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)},$$

holds for all harmonic functions ϕ with vanishing mean on $\partial\Omega$.

After all this preparation we are now able to prove Theorem 4.1.

Proof of Theorem 4.1. For the proof of estimate (4.1), without loss of generality we can assume that $\int_{\partial\Omega} \phi \, \mathrm{d}\sigma = 0$.

Using (4.16) for the second summand in (4.1) and and then (4.10) for the terms involving ∇u we get

$$\begin{aligned} \|\nabla u\|_\partial + \|\phi\|_\partial &\leq C\{\|\nabla u\|_\partial + |\lambda|\|u \cdot n\|_{\mathrm{H}^1(\partial\Omega)}\} \\ &\leq C_\varepsilon\Big\{\|\nabla_{\tan} u\|_\partial + |\lambda|^{1/2}\|u\|_\partial + |\lambda|\|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}\Big\} \\ &\quad + C\varepsilon\{\|\nabla u\|_\partial + \|\phi\|_\partial\} \end{aligned}$$

for all $\varepsilon \in (0, 1)$. Chosing ε such that $C\varepsilon < (1/2)$ we can rearrange the above inequality and obtain estimate (4.1).

Estimate (4.2) will need more effort to be proven. We start with inequality (4.17) and derive

$$\|\nabla u\|_\partial + \|\phi\|_\partial + |\lambda|\|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \leq C\{\|\nabla u\|_\partial + \|\phi\|_\partial\} \leq C\Big\{\|\frac{\partial u}{\partial\nu}\|_\partial + \|\nabla u\|_\partial\Big\},$$

where in the last step we used the definition of conormal derivatives. If we now apply (4.9) we get

$$\|\nabla u\|_\partial + \|\phi\|_\partial + |\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \le C_\varepsilon \|\frac{\partial u}{\partial \nu}\|_\partial + \varepsilon\big\{\|\nabla u\|_\partial + \|\phi\|_\partial + \||\lambda|^{1/2}u\|_\partial\big\}$$

for all $\varepsilon \in (0,1)$. Choosing ε appropriately yields

eq:partOfRellich2

$$\|\nabla u\|_\partial + \|\phi\|_\partial + |\lambda|\|u\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} \le C\|\frac{\partial u}{\partial \nu}\|_\partial + C|\lambda|^{1/2}\|u\|_\partial. \tag{4.22}$$

Now we need to estimate $|\lambda|^{1/2}\|u\|_\partial$. Green's identity yields

$$\begin{aligned}\int_{\partial\Omega} h_k n_k |u|^2 \,\mathrm{d}\sigma &= \int_\Omega \frac{\partial}{\partial x_k}\big(h_k|u|^2\big)\,\mathrm{d}x = \int_\Omega \frac{\partial h_k}{\partial x_k}|u|^2\,\mathrm{d}x + \int_\Omega h_k \frac{\partial |u|^2}{\partial x_k}\,\mathrm{d}x \\ &= \int_\Omega \operatorname{div}(h)|u|^2\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega h_k \frac{\partial \bar{u}_i}{\partial x_k} u_i\,\mathrm{d}x.\end{aligned} \tag{4.23}$$

eq:hknkgreen

We choose $h \in \mathrm{C}^1_0(\mathbb{R}^d,\mathbb{R}^d)$ with $h_k n_k \ge c > 0$ on $\partial\Omega$. The existence of such a function h follows from Theorem 1.3. Using this, we can continue the estimate (4.23) as

eq:estupartial

$$\|u\|_\partial^2 \le C\int_\Omega |u|^2\,\mathrm{d}x + C\int_\Omega |u||\nabla u|\,\mathrm{d}x. \tag{4.24}$$

The next estimate uses (4.24) and (4.6) which gives

$$\begin{aligned}|\lambda|\|u\|_\partial^2 &\le |\lambda| C\int_\Omega |u|^2\,\mathrm{d}x + |\lambda| C\int_\Omega |u||\nabla u|\,\mathrm{d}x \\ &\le C\|\frac{\partial u}{\partial \nu}\|_\partial\|u\|_\partial + |\lambda|^{1/2}C\int_\Omega (|\lambda|^{1/2}|u|)|\nabla u|\,\mathrm{d}x \\ &\le C\|\frac{\partial u}{\partial \nu}\|_\partial\|u\|_\partial + |\lambda|^{1/2}C\big(\int_\Omega |\lambda||u|^2\big)^{1/2}\big(\int_\Omega (|\nabla u|^2\,\mathrm{d}x\big)^{1/2} \\ &\le C\|\frac{\partial u}{\partial \nu}\|_\partial\||\lambda|^{1/2}u\|_\partial.\end{aligned}$$

Note that for the last estimate we also used the fact that $|\lambda| \ge \tau$ helps us to bound $\|u\|_\partial$ by $C|\lambda|^{1/2}\|u\|_\partial$. Rearranging terms in the last estimate, we now derive

eq:lambda12u

$$\||\lambda|^{1/2}u\|_\partial \le C\|\frac{\partial u}{\partial \nu}\|_\partial. \tag{4.25}$$

Estimate (4.2) follows directly from (4.22) in combination with (4.25) and this concludes our proof. □

Shen proved that under reasonable assumptions a theorem similar to 4.1 also holds for exterior domains.

thm:rellichExterior

Theorem 4.9. *Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0,1)$. Let (u,ϕ) be a solution of the Stokes resolvent Problem in $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$. Suppose additionally that $(\nabla u)^*$, $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$ and that ∇u, ϕ have nontangential limits almost everywhere on $\partial\Omega$. Furthermore let for $|x| \to \infty$*

$$|\phi(x)| + |\nabla u(x)| = O(|x|^{1-d}) \quad \textit{and} \quad u(x) = \begin{cases} O(|x|^{2-d}) & \textit{if } d \geq 3 \\ o(1) & \textit{if } d = 2. \end{cases}$$

Then

eq:rellich1ext

$$\|\nabla u\|_\partial + \|\phi\|_\partial \leq C\big\{\|\nabla_{\tan} u\|_\partial + |\lambda|^{1/2}\|u\|_\partial + |\lambda|\|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}\big\} \tag{4.26}$$

and

eq:rellich2ext

$$\|\nabla u\|_\partial + |\lambda|^{1/2}\|u\|_\partial + |\lambda|\|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|\phi\|_\partial \leq C\|\frac{\partial u}{\partial \nu}\|_\partial, \tag{4.27}$$

where C depends only on d, τ, θ and the Lipschitz character of Ω.

Chapter 5

Solving the L^2-Dirichlet Problem

chap:5

This section is all about the application of the method of layer potentials to solve the L^2-Dirichlet problem for the Stokes resolvent system. Furthermore we will establish a uniform L^p-estimate for the nontangential maximal function which will be important for the proof of our central theorem.

For the remainder of this chapter let Ω always denote a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with connected boundary. We will use $\mathrm{L}^2_n(\partial\Omega)$ to denote the function space

$$\mathrm{L}^2_n(\partial\Omega) := \Big\{ f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d) \colon \int_{\partial\Omega} f \cdot n \, \mathrm{d}\sigma = 0 \Big\},$$

and $\mathrm{L}^2_0(\partial\Omega; \mathbb{C}^d)$ to denote the function space of L^2 functions with mean value zero. As before, $\| \cdot \|_\partial$ stands for the norm of $\mathrm{L}^2(\partial\Omega; \mathbb{C}^k)$, $k \in \mathbb{N}$.

The following results will build on the application of Theorem 4.9. Therefore the next lemma shows that solutions to the Stokes resolvent problem that are given by single layer potentials fulfill the requirements of the theorem.

lem:requirements

Lemma 5.1. *Let $\lambda \in \Sigma_\theta$ and (u, ϕ) be given by* (3.1) *and* (3.2)*, respectively. Then the following holds for $|x| \to \infty$:*

$$|\phi(x)| + |\nabla u(x)| = O(|x|^{1-d}) \quad \textit{and} \quad u(x) = \begin{cases} O(|x|^{2-d}) & \textit{if } d \geq 3 \\ o(1) & \textit{if } d = 2. \end{cases}$$

Proof. If $d \geq 2$ then an application of the dominated convergence theorem gives that $|\phi(x)| + |\nabla u(x)| = O(|x|^{1-d})$ as $|\phi(x)| = O(|\Phi_k(x)|) = O(|x|^{1-d})$ by (2.28). Furthermore, we have $|\nabla u(x)| = O(|x|^{1-d})$ by estimate 2.30.

If $d \geq 3$ then the first part of Lemma 2.1 and an application of the dominated convergence Theorem gives $u(x) = O(|x|^{2-d})$. If $d = 2$, then the asymptotic behavior of the fundamental solution to the scalar Helmholtz equation is already available. Through dominated convergence the same asymptotic behavior holds for $u(x)$. □

We will now derive bounds on the inverse operator of $(1/2)I + \mathcal{K}_\lambda$ from Chapter 3.

Lemma 5.2. *Let $\lambda \in \Sigma_\theta$ and $|\lambda| \geq \tau$, where $\tau \in (0,1)$. Suppose that $|\partial\Omega| = 1$. Then $(1/2)I + \mathcal{K}_\lambda$ is an isomorphism on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ and*

eq:inverseEstimate

$$\|f\|_\partial \leq C\|\big((1/2)I + \mathcal{K}_\lambda\big)f\|_\partial \quad \textit{for any } f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d), \tag{5.1}$$

where C depends only on d, θ, τ and the Lipschitz character of Ω.

Proof. We start with $f \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ and the corresponding single layer potentials $u = \mathcal{S}_\lambda(f)$ and $\phi = \mathcal{S}_\Phi(f)$ given by (3.1) and (3.2). We saw in Chapter 3 that (u, ϕ) solves the Stokes resolvent problem in $\mathbb{R}^d \setminus \partial\Omega$ and got from Lemma 3.6 with $p = 2$ for the nontangential maximal functions that $(\nabla u)^*$, $(\phi)^* \in \mathrm{L}^2(\partial\Omega)$. We furthermore saw in Lemma 3.8 that ∇u and ϕ have nontangential limits almost everywhere on $\partial\Omega$. Finally in Theorem 3.9 we saw that $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$ and derived the jump condition $\big(\frac{\partial u}{\partial \nu}\big)_\pm = (\pm(1/2)I + \mathcal{K}_\lambda)f$.

Our next step will be to show the estimate

eq:negNablaPhi

$$\|\nabla u_-\|_\partial + \|\phi_-\|_\partial \leq C\|\big(\frac{\partial u}{\partial \nu}\big)\|_\partial. \tag{5.2}$$

Assuming that (5.2) holds we can prove (5.1): Set $f = \big(\frac{\partial u}{\partial \nu}\big)_+ - \big(\frac{\partial u}{\partial \nu}\big)_-$. Then this gives with the definition of the conormal derivative and estimate (5.2) that

$$\begin{aligned} \|f\|_\partial &\leq \|\big(\frac{\partial u}{\partial \nu}\big)_+\|_\partial + \|\big(\frac{\partial u}{\partial \nu}\big)_-\|_\partial \\ &\leq \|\big(\frac{\partial u}{\partial \nu}\big)_+\|_\partial + \|\big(\frac{\partial u}{\partial n}\big)_-\|_\partial + \|\phi_- n\|_\partial \\ &\leq C\|\big(\frac{\partial u}{\partial \nu}\big)_+\|_\partial = C\|(1/2)I + K_\lambda)f\|_\partial. \end{aligned}$$

In order to prove (5.2), note that since $|u(x)| + |\nabla u(x)| = O(|x|^{-N})$ for all $N > 0$ and $\phi(x) = O(|x|^{1-d})$ as $|x| \to \infty$ we can use Theorem 4.9 to derive

eq:nablauminus

$$\begin{aligned} \|\nabla u_-\|_\partial + \|\phi_-\|_\partial &\leq C\Big\{\|\nabla_{\tan} u_-\|_\partial + |\lambda|^{1/2}\|u_-\|_\partial + |\lambda|\|n \cdot u_-\|_{\mathrm{H}^{-1}(\partial\Omega)}\Big\} \\ &= C\Big\{\|\nabla_{\tan} u_+\|_\partial + |\lambda|^{1/2}\|u_+\|_\partial + |\lambda|\|n \cdot u_+\|_{\mathrm{H}^{-1}(\partial\Omega)}\Big\}, \end{aligned} \tag{5.3}$$

where we used the fact that $u_+ = u_-$ and $\nabla_{\tan} u_+ = \nabla_{\tan} u_-$ on $\partial\Omega$. Inequality (4.2) of Theorem 4.1 now allows us to estimate the right hand side of (5.3) by $C\|(\frac{\partial u}{\partial \nu})_+\|_\partial$ and thus the desired estimate (5.2) follows.

Let's now work on the invertibility of $(1/2)I + \mathcal{K}_\lambda$. In the case $\lambda = 0$, Fabes, Kenig and Verchota showed in [3] that $(1/2)I + \mathcal{K}_0$ as an operator on $\mathrm{L}^2(\partial\Omega; \mathbb{R}^d)$ has a one dimensional null space and as range the space $\mathrm{L}^2_0(\partial\Omega; \mathbb{R}^d)$. Thus $(1/2)I + \mathcal{K}_0$ has Fredholm index 0. This remains true if we replace $\mathrm{L}^2(\partial\Omega; \mathbb{R}^d)$ by $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ as this just corresponds to a complexification of the vector space and the operator. Since the operator $\mathcal{K}_\lambda - \mathcal{K}_0$ is compact on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ by Lemma 3.10, we deduce that the operator

$$(1/2)I + \mathcal{K}_\lambda = (1/2)I + \mathcal{K}_0 + (\mathcal{K}_\lambda - \mathcal{K}_0)$$

has the Fredholm index zero as well for all $\lambda \in \Sigma_\theta$. Now inequality (5.1) gives that $(1/2)I + \mathcal{K}_\lambda$ is injective and thus the Fredholm index of zero implies that it is also surjective and hence an isomorphism. □

The next lemma works with the counterpart of $(1/2)I + \mathcal{K}_\lambda$.

lem:inverseEstimate

Lemma 5.3. *Let $\lambda \in \Sigma_\theta$. Then $-(1/2)I + \mathcal{K}_\lambda$ is a Fredholm operator on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ with index zero and*

eq:inverseEstimate2

$$\|f\|_\partial \leq C\|\big(-(1/2)I + \mathcal{K}_\lambda\big)f\|_\partial \quad \textit{for all } f \in \mathrm{L}^2_n(\partial\Omega), \tag{5.4}$$

where C depends only on d, θ and the Lipschitz character of Ω.

Proof. In the case $\lambda = 0$, Fabes, Kenig and Verchota proved in [3] that the Fredholm index of the operator $-(1/2)I + \mathcal{K}_0$ on $\mathrm{L}^2(\partial\Omega; \mathbb{R}^d)$ is zero and estimate (5.4) holds. As in the previous proof, this still remains true if we complexify the operator making it a Fredholm operator with index zero on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$. Since $\mathcal{K}_\lambda - \mathcal{K}_0$ is compact on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ and the Fredholm index remains unchanged under compact perturbations, we know that the Fredholm index of $-(1/2)I + \mathcal{K}_\lambda$ on $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ is zero for all $\lambda \in \Sigma_\theta$. This proves the first claim of the lemma.

Now let $\tau < (2\,\mathrm{diam}(\Omega)^2 + 1)^{-1}$ and $|\lambda| < \tau$. Then

$$\|(\mathcal{K}_\lambda - \mathcal{K}_0 f\|_\partial \leq C|\lambda|^{1/2}\|f\|_\partial.$$

In order to prove this inequality we once again apply Young's inequality from Lemma 3.4. To this end we start by estimating

$$\|(\mathcal{K}_\lambda - \mathcal{K}_0 f\|_\partial \leq \sup_{p \in \partial\Omega} \|\nabla_x\{\Gamma(p - \cdot\,; \lambda) - \Gamma(p - \cdot\,; 0)\}\|_{\mathrm{L}^1(\partial\Omega)}\|f\|_{\mathrm{L}^2(\partial\Omega)}.$$

In the next step we prove that for $p \in \partial\Omega$ the integral over the gradients of Γ can be estimated independent of p. This is straightforward using Lemma 3.2 as Corollary 2.7

gives us

$$\begin{aligned}
&\int_{\partial\Omega} |\nabla_x\{\Gamma(p-y;\lambda)-\Gamma(p-y;0)\}|\,\mathrm{d}\sigma(y)\\
&\quad\le C|\lambda|^{1/2}\int_{\partial\Omega}\frac{1}{|p-y|^{d-2}}\,\mathrm{d}\sigma(y)\\
&\quad= |\lambda|^{1/2}C\int_{\partial\Omega\cap \mathrm{B}(p,r_0/4)}\frac{1}{|p-y|^{d-2}}\,\mathrm{d}\sigma(y)+|\lambda|^{1/2}C\int_{\partial\Omega\setminus \mathrm{B}(p,r_0/4)}\frac{1}{|p-y|^{d-2}}\,\mathrm{d}\sigma(y)\\
&\quad\le C|\lambda|^{1/2}(r_0/4+4^{2-d}r_0^{d-2}|\partial\Omega|),
\end{aligned}$$

where r_0 is the radius from the definition of Lipschitz domains. Note that by the choice of τ the estimate from Corollary 2.7 applies on the whole domain of integration.

For $f\in \mathrm{L}^2_n(\partial\Omega)$ we can now estimate

$$\begin{aligned}
\|f\|_\partial &\le C\|(-(1/2)I+\mathcal{K}_0)f\|_\partial\\
&\le C\|(-(1/2)I+\mathcal{K}_\lambda)f\|_\partial+\|(\mathcal{K}_\lambda-\mathcal{K}_0)f\|_\partial\\
&\le C\|(-(1/2)I+\mathcal{K}_\lambda)f\|_\partial+C|\lambda|^{1/2}\|f\|_\partial,
\end{aligned}$$

with a constant C which depends only on d, θ and the Lipschitz character of $\partial\Omega$. Choosing τ even smaller allows us to rearrange the terms in the above estimate such that estimate (5.4) holds for $\lambda\in\Sigma_\theta$ and $|\lambda|<\tau$, with τ depending on d, θ and the Lipschitz character of Ω.

Now leave τ fixed and consider the case $|\lambda|\ge\tau$. This case will be handled using the Rellich estimates from Section 4. We use the facts that for $\nabla_{\tan}u$ and u the inner and outer nontangential limits coincide and apply Theorems 4.1 and 4.9 to conclude that

$$\begin{aligned}
&\|\nabla u_+\|_\partial+\|\phi_+ - ⨍_{\partial\Omega}\phi_+\|\\
&\quad\le C\{\|(\nabla_{\tan}u)_+\|_\partial+|\lambda|^{1/2}\|u_+\|_\partial+|\lambda|\|u_+\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}\}\\
&\quad= C\{\|(\nabla_{\tan}u)_-\|_\partial+|\lambda|^{1/2}\|u_-\|_\partial+|\lambda|\|u_-\cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}\}\\
&\quad\le C\|\Big(\frac{\partial u}{\partial\nu}\Big)_-\|_\partial.
\end{aligned}$$

We can now use this inequality to estimate $\|\big(\frac{\partial u}{\partial\nu}\big)_+\|_\partial$ since

$$\begin{aligned}
\|\Big(\frac{\partial u}{\partial\nu}\Big)_+\|_\partial &\le \|\Big(\frac{\partial u}{\partial n}\Big)_+\|_\partial + C\|\phi_+\|_\partial\\
&\le C\|(\nabla u)_+\|_\partial + C\|\phi_+ - ⨍_{\partial\Omega}\phi_+\,\mathrm{d}\sigma\|_\partial + C\Big|⨍_{\partial\Omega}\phi_+\,\mathrm{d}\sigma\Big|\\
&\le C\|\Big(\frac{\partial u}{\partial\nu}\Big)_-\|_\partial + C\Big|\int_{\partial\Omega}\phi_+\,\mathrm{d}\sigma\Big|
\end{aligned}$$

Considering the jump relation (3.19) and the previous estimate we get that

$$\begin{aligned}\|f\|_\partial &\le \|(\frac{\partial u}{\partial \nu})_+\|_\partial + \|(\frac{\partial u}{\partial \nu})_-\|_\partial \\ &\le C\|(\frac{\partial u}{\partial \nu})_-\|_\partial + C\Big|\int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma\Big| \\ &\le C\|(-(1/2)I + \mathcal{K}_\lambda)f\|_\partial + C\Big|\int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma\Big|. \end{aligned} \quad \text{eq:estimatef} \tag{5.5}$$

We now are left with the term $\int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma$ that needs to be estimated. To this end, note that multiplying the conormal derivatives of u by n gives

$$(\frac{\partial u}{\partial \nu})_+ \cdot n = (\frac{\partial u_i}{\partial x_j})_+ n_i n_j - \phi_+ = n_j\big(n_i \frac{\partial}{\partial x_j} - n_j(\frac{\partial}{\partial x_i}) u_i\big)_+ - \phi_+,$$

where for the second equality we used that $\operatorname{div}(u) = 0$ in Ω and thus this also holds for the nontangential limit. This identity now implies

$$\begin{aligned}\Big|\int_{\partial\Omega} \phi_+ \,\mathrm{d}\sigma\Big| &\le \Big|\int_{\partial\Omega} (\frac{\partial u}{\partial \nu})_+ \cdot n \,\mathrm{d}\sigma\Big| + C\|\nabla_{\tan} u\|_\partial \\ &\le \Big|\int_{\partial\Omega} (\frac{\partial u}{\partial \nu})_- \cdot n \,\mathrm{d}\sigma\Big| + C\|\nabla_{\tan} u\|_\partial \\ &\le C\|(\frac{\partial u}{\partial \nu})_-\|_\partial, \end{aligned} \quad \text{eq:estimatephiplus} \tag{5.6}$$

where in the second step, we used the jump relation to exchange $\big(\frac{\partial u}{\partial \nu}\big)_+ \cdot n$ by $\big(\frac{\partial u}{\partial \nu}\big)_- + f \cdot n$ and then used the fact $f \in \mathrm{L}^2_n(\partial\Omega)$. The third step follows from Theorem 4.9 considering that $\|\nabla_{\tan} u\|_\partial \le C\|\nabla u\|_\partial$. Now extending estimate (5.5) by (5.6) gives

$$\|f\|_\partial \le C\|\big(-(1/2)I + \mathcal{K}_\lambda\big)f\|_\partial + C\|(\frac{\partial u}{\partial \nu})_-\|_\partial \le C\|\big(-(1/2)I + \mathcal{K}_\lambda\big)f\|_\partial,$$

where we used the jump relation (3.19) again. This proves estimate (5.4) in the case $|\lambda| \ge \tau$ and thus concludes the proof. □

In the following lemma we will show the uniqueness of solutions to the L^2 Dirichlet problem to the Stokes resolvent system.

lem:l2unique

Lemma 5.4. *Let $\lambda \in \Sigma_\theta$ and (u, ϕ) be a solution to the Stokes resolvent problem in Ω. Furthermore suppose that the nontangential limit of u exists almost everywhere on $\partial\Omega$ and that $(u)^* \in \mathrm{L}^2(\partial\Omega)$. Then*

$$\int_\Omega |u|^2 \,\mathrm{d}x \le C \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma, \quad \text{eq:OmegaBoundaryEstimate} \tag{5.7}$$

where C depends only on d, θ and the Lipschitz character of Ω.

Proof. We use Verchota's approximation theorem 1.3 and approximate Ω by a sequence of smooth domains with uniform Lipschitz characters from inside. As a consequence we may assume that Ω is smooth and u, ϕ are smooth in $\overline{\Omega}$. Let $(w,\psi) \in \mathrm{H}^1_0(\Omega;\mathbb{C}^d) \times \mathrm{H}^1(\Omega)$ be a solution to the inhomogenous system

$$\begin{aligned} -\Delta w + \lambda w + \nabla\psi &= \bar{u} \text{ in } \Omega, \\ \operatorname{div}(w) &= 0 \text{ in } \Omega. \end{aligned} \qquad \texttt{eq:inhomogenousStokes} \quad (5.8)$$

In fact the regularity theory for the Stokes equation gives us that w and ψ are smooth. It follows from testing (5.8) against u that

$$\int_\Omega |u|^2 \,\mathrm{d}x = \int_\Omega u \cdot \{-\Delta w + \lambda w + \nabla\psi\} \,\mathrm{d}x. \qquad \texttt{eq:testingInhomogenousStokes} \quad (5.9)$$

Using one of Green's identities on the first summand and the fact that u is the solution to the Stokes resolvent problem gives that

$$\begin{aligned} \int_\Omega -u \cdot \Delta w \,\mathrm{d}x &= \int_\Omega -w \cdot \Delta u \,\mathrm{d}x - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} \,\mathrm{d}\sigma, \\ &= \int_\Omega w \cdot (-\lambda u - \nabla\phi) \,\mathrm{d}x - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} \,\mathrm{d}\sigma \\ &= \int_\Omega -\lambda w \cdot u \,\mathrm{d}x - \int_{\partial\Omega} u \cdot \frac{\partial w}{\partial n} \,\mathrm{d}\sigma, \end{aligned}$$

where in the last step we used partial integration and the fact that w vanishes on $\partial\Omega$ and is divergence free:

$$\int_\Omega w \cdot \nabla\phi \,\mathrm{d}x = -\int_\Omega \operatorname{div}(w)\phi \,\mathrm{d}x + \int_{\partial\Omega} \phi w \cdot n \,\mathrm{d}\sigma = 0.$$

For the third summand in (5.9) we do the same with the only difference that the second integral does not vanish. Putting everything together gives

$$\begin{aligned} \int_\Omega |u|^2 \,\mathrm{d}x &= -\int_{\partial\Omega} u \cdot \Big\{\frac{\partial w}{\partial n} - \psi n\Big\} \,\mathrm{d}\sigma \\ &\leq \|u\|_\partial \{\|\nabla w\|_\partial + \|\psi\|_\partial\} \end{aligned} \qquad \texttt{eq:u2estimate} \quad (5.10)$$

by the Cauchy-Schwartz inequality. As the pressure ψ is only specified modulo additive constants, we may as well assume that $\int_{\partial\Omega} \psi \,\mathrm{d}\sigma = 0$. Furthermore by the Schwartz theorem we see from (5.8) that $\Delta\psi = \operatorname{div}(\bar{u}) = 0$ in Ω. As stated in Remark 4.8, this allows us to use the results from the proof of (4.20) by setting $\phi = \psi$ to conclude that

$$\|\psi\|_\partial \leq C\|\nabla\psi \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}$$

and since w has vanishing trace on $\partial\Omega$ we can use that fact that (w, ψ) solves (5.8) to further estimate

eq:psiEstimate

$$\begin{aligned} &\leq C\{\|\Delta w \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)} + \|u \cdot n\|_{\mathrm{H}^{-1}(\partial\Omega)}\} \\ &\leq C\{\|\nabla w\|_\partial + \|u\|_\partial\}, \end{aligned} \tag{5.11}$$

where for the last estimate we used (4.19) which is applicable since $\operatorname{div} w = 0$ on Ω. If we combine inequalities (5.10) and (5.11), we get

eq:u2estimate2

$$\int_\Omega |u|^2 \,\mathrm{d}x \leq C\|u\|_\partial \|\nabla w\|_\partial + C\|u\|_\partial^2. \tag{5.12}$$

We are left with estimating the first term in (5.12). In fact it will suffice to show the following inequality

eq:w2estimate

$$\int_{\partial\Omega} |\nabla w|^2 \,\mathrm{d}\sigma \leq C \int_\Omega |u|^2 \,\mathrm{d}x + C \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma \tag{5.13}$$

since by the weighted Young inequality for real numbers this would make the estimate

$$C\|u\|_\partial \|\nabla w\|_\partial \leq \frac{1}{2} \int_\Omega |u|^2 \,\mathrm{d}x + C \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma$$

available which after rearranging terms yields (5.7).

To this end, we will first prove the Rellich type identity

eq:rellichIdentity3

$$\begin{aligned} \int_{\partial\Omega} h_k n_k |\nabla w|^2 \,\mathrm{d}\sigma = &\int_\Omega \operatorname{div}(h) |\nabla w|^2 \,\mathrm{d}x + 2\operatorname{Re} \int_\Omega h_j \frac{\partial \psi}{\partial x_j} \frac{\bar{w}_j}{\partial x_k} \,\mathrm{d}x \\ &+ 2\operatorname{Re} \int_\Omega h_k \lambda w_j \frac{\partial \bar{w}_j}{\partial x_k} + 2\operatorname{Re} \int_\Omega h_k \bar{u}_j \frac{\partial \bar{w}_j}{\partial x_k} \,\mathrm{d}x. \end{aligned} \tag{5.14}$$

Note that since all involved quantities are smooth up to the boundary, integration by parts is allowed and the proof the stated Rellich identity boils down to a formal calculation. Let $h \in C_0^1(\mathbb{R}^d, \mathbb{R}^d)$ with $h_k n_k \geq c > 0$ on $\partial\Omega$ be the function from Theorem 1.3. Then the divergence theorem gives

$$\int_{\partial\Omega} h_k n_k |\nabla w|^2 \,\mathrm{d}\sigma = \int_\Omega \operatorname{div}(h|\nabla w|^2) \,\mathrm{d}x = \int_\Omega \operatorname{div}(h) |\nabla w|^2 \,\mathrm{d}x + \int_\Omega h_k \frac{\partial}{\partial x_k} \big(|\nabla w|^2\big) \,\mathrm{d}x$$

and we can rewrite the second summand as

$$\begin{aligned} \int_\Omega h_k \frac{\partial}{\partial x_k} (|\nabla w|^2) \,\mathrm{d}x &= \int_\Omega h_k \frac{\partial}{\partial x_k} \Big(\frac{\partial w_j}{\partial x_i} \frac{\partial \bar{w}_j}{\partial x_i}\Big) \,\mathrm{d}x \\ &= 2\operatorname{Re} \int_\Omega h_k \frac{\partial^2 w_j}{\partial x_k \partial x_i} \frac{\partial \bar{w}_j}{\partial x_i} \,\mathrm{d}x \\ &= -2\operatorname{Re} \int_\Omega h_k \frac{\partial}{\partial x_k} \Big(\frac{\partial^2}{\partial x_i^2} w_j\Big) \bar{w}_j \,\mathrm{d}x + \int_{\partial\Omega} h_k \Big(\frac{\partial^2}{\partial x_k \partial x_i} w_j\Big) \bar{w}_j \,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= 2\,\mathrm{Re} \int_\Omega h_k (\Delta w_j) \frac{\partial \bar{w}_j}{\partial x_k}\,\mathrm{d}x + 0 \\
&= 2\,\mathrm{Re} \int_\Omega h_k \big(\frac{\partial \psi}{\partial x_j} + \lambda w_j - \bar{u}_j \big) \frac{\partial \bar{w}_j}{\partial x_k}\,\mathrm{d}x,
\end{aligned}
$$

where in addition to partial integration we used (5.8) and the fact that $w = 0$ on $\partial\Omega$. Now we can apply the triangle inequality to the Rellich type identity (5.14) to obtain

eq:rellichInequality3

$$
\begin{aligned}
\int_{\partial\Omega} |\nabla w|^2\,\mathrm{d}\sigma \leq C \Big\{ &\int_\Omega |\nabla w|^2\,\mathrm{d}x + \int_\Omega |\nabla w||\psi|\,\mathrm{d}x \\
&+ |\lambda| \int_\Omega |\nabla w||w|\,\mathrm{d}x + \int_\Omega |\nabla w||u|\,\mathrm{d}x \Big\}.
\end{aligned} \tag{5.15}
$$

The next step consists in deriving estimates which are compatible with the right hand side of (5.15). Testing (5.8) with $\bar{w}$, itegration by parts gives us as in the proof of Lemma 4.5

$$
\int_\Omega |\nabla w|^2\,\mathrm{d}x + |\lambda| \int_\Omega |w|^2\,\mathrm{d}x \leq C \int_\Omega |w||u|\,\mathrm{d}x.
$$

The next step consists in using the previous inequality and the Poincaré inequality to estimate

$$
\begin{aligned}
\int_\Omega |\nabla w|^2\,\mathrm{d}x + (1 + |\lambda|) \int_\Omega |w|^2\,\mathrm{d}x &\leq (1 + C) \int_\Omega |\nabla w|^2\,\mathrm{d}x + |\lambda| \int_\Omega |w|^2\,\mathrm{d}x \\
&\leq C \int_\Omega |w||u|\,\mathrm{d}x \\
&\leq C \|w\|_\partial \|u\|_\partial,
\end{aligned}
$$

where for the last step we used the Cauchy-Schwartz inequality. The weighted Young inequality for real numbers allows us to further estimate

$$
\begin{aligned}
&\leq \frac{C}{4\varepsilon} \int_\Omega |u|^2\,\mathrm{d}x + C\varepsilon \int_\Omega |w|^2\,\mathrm{d}x \\
&= \frac{\tilde{C}}{1 + |\lambda|} \int_\Omega |u|^2\,\mathrm{d}x + \frac{1}{2}(1 + |\lambda|) \int_\Omega |w|^2\,\mathrm{d}x
\end{aligned}
$$

if we set $\varepsilon = \frac{(1+|\lambda|)}{2C}$. Rearranging terms, we can produce our next estimate

eq:w2lambdaEstimate

$$
\int_\Omega |\nabla w|^2\,\mathrm{d}x + (1 + |\lambda|) \int_\Omega |w|^2\,\mathrm{d}x \leq \frac{C}{1 + |\lambda|} \int_\Omega |u|^2\,\mathrm{d}x. \tag{5.16}
$$

Now it's time to harvest: Using estimate (5.15) together with (5.16) gives

$$
\begin{aligned}
&\int_{\partial\Omega} |\nabla w|^2\,\mathrm{d}\sigma \\
&\quad \leq C \Big\{ \int_\Omega |\nabla w|^2\,\mathrm{d}x + \int_\Omega |\nabla w||\psi|\,\mathrm{d}x + |\lambda| \int_\Omega |\nabla w||w|\,\mathrm{d}x + \int_\Omega |\nabla w||u|\,\mathrm{d}x \Big\}.
\end{aligned}
$$

Using the weighted Young inequality, we see that we can simplify the right hand to

$$C_\varepsilon(1+|\lambda|)\int_\Omega |\nabla w|^2\,\mathrm{d}x + C|\lambda|\int_\Omega |w|^2\,\mathrm{d}x + C\int_\Omega |u|^2\,\mathrm{d}x + \varepsilon\int_\Omega |\psi|^2\,\mathrm{d}x,$$

which with (5.16) can be bounded in this way

$$\leq C_\varepsilon\int_\Omega |u|^2\,\mathrm{d}x + \varepsilon\int_\Omega |\psi|^2\,\mathrm{d}x.$$

Using the estimate $\|\psi\|_{\mathrm{L}^2(\partial\Omega)} \leq C\|\psi\|_\partial$ and inequality (5.11) gives

$$\varepsilon\int_\Omega |\psi|^2\,\mathrm{d}x \leq \varepsilon C\int_{\partial\Omega} |\nabla w|^2\,\mathrm{d}\sigma + C_\varepsilon\int_{\partial\Omega} |u|^2\,\mathrm{d}\sigma.$$

Choosing $\varepsilon = \frac{1}{2C}$ and rearranging gives the desired estimate (5.13). This concludes our proof. □

The next Theorem states the important fact that in L^2 the Dirichlet Stokes resolvent problem has a unique solution.

thm:exAndUniqueSolution

Theorem 5.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$ with connected boundary and let $\lambda \in \Sigma_\theta$. For all $g \in \mathrm{L}^2_n(\partial\Omega)$ there exists a unique u and harmonic function ϕ which is unique up to constants such that (u,ϕ) satisfies* (2.26)*, $(u)^* \in \mathrm{L}^2(\partial\Omega)$ and $u = g$ on $\partial\Omega$ in the sense of nontangential convergence. Moreover the estimate $\|(u)^*\|_\partial \leq C\|g\|_\partial$ holds and u may be represented by the double layer potential $\mathcal{D}_\lambda(f)$ with $\|f\|_\partial \leq C\|g\|_\partial$, where C depends only on d, θ and the Lipschitz character of Ω.*

Proof. By Lemma 5.4 we already now that the problem under consideration admits at most one solution. Therefore we only have to worry about the existence of a solution. To this end we want to apply Lemma 5.3.

We first note that since

$$T\colon \mathrm{L}^2(\partial\Omega;\mathbb{C}^d) \to \mathrm{L}^2(\partial\Omega;C^d), \quad \mathrm{x} \mapsto -(1/2)x + \mathcal{K}_{\bar\lambda}x$$

is a Fredholm operator on $\mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ with index 0 the same is true for its adjoint

$$T^*\colon \mathrm{L}^2(\partial\Omega;\mathbb{C}^d) \to \mathrm{L}^2(\partial\Omega;C^d), \quad x \mapsto (-1/2)x + \mathcal{K}^*_{\bar\lambda}x.$$

We know that for all $f \in \mathrm{L}^2(\partial\Omega;\mathbb{C}^d)$ we have $\operatorname{div}(\mathcal{D}_\lambda f) = 0$ and therefore

$$\int_{\partial\Omega} T^*f\cdot n\,\mathrm{d}\sigma = \int_{\partial\Omega} u_+\cdot n\,\mathrm{d}\sigma = 0,$$

where for the first inequality we applied Theorem 3.11. The second equality uses Verchota's approximation scheme from Theorem 1.3 in order to apply the divergence theorem and the fact that $(u)^*$ is integrable together with dominated convergence. This gives $\mathrm{Im}(T^*) \subseteq \mathrm{L}^2_n(\partial\Omega)$. Now on the one hand we have

$$\mathrm{span}(n) = \mathrm{L}^2_n(\partial\Omega)^\perp \subseteq \mathrm{Im}(T^*)^\perp = \ker(T)$$

and on the other hand, as T is injective on $\mathrm{L}^2_n(\partial\Omega)$ by (5.4), we have that $\mathrm{span}(n) \supseteq \ker(T)$. This yields $\mathrm{span}(n) = \ker(T)$. We can use this equality and show that

$$\mathrm{L}^2_n(\partial\Omega) = \ker(T)^\perp = \overline{\mathrm{Im}(T^*)} = \mathrm{Im}(T^*),$$

where for the last equality we used the fact that the range of T^* is closed, as usual for Fredholm operators. With the same argument we can show for T that

$$\ker(T^*)^\perp = \overline{\mathrm{Im}(T)} = \mathrm{Im}(T).$$

Now let

$$X = \mathrm{L}^2_n(\partial\Omega) \quad \text{and} \quad Y = \mathrm{Im}(T).$$

Both spaces are closed subspaces of the Hilbert space $\mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$ and therefore again Hilbertspaces. Consequently the operator

$$K'_{Y,X} \colon Y \to X, \quad x \mapsto T^* x$$

is invertible by the continuous inverse theorem. We now want to bound the operator norm of $K_{X,Y}$ by a constant that does not depend on λ but on the sectorial parameter θ. To this end let us introduce the operator

$$K_{X,Y} \colon X \to Y, \quad x \mapsto Tx$$

Now for $f \in X$ and $g \in Y$ we have that

$$\langle x, K^*_{X,Y} y\rangle_X = \langle K_{X,Y} x, y\rangle_Y = \langle x, T^* y\rangle_X = \langle x, K'_{Y,X} y\rangle_X$$

which shows that $K'_{Y,X} = K_{X,Y}*$ on Y. With the above definitions at hand, Lemma 5.3 states that $K_{X,Y}$ is an invertible operator on Y with operator norm of the inverse bounded by some $C > 0$ and C depends at most on d, θ and the Lipschitz character of Ω. Therefore for all $f \in Y = \mathrm{Im}(T) = \mathrm{Im}((1/2)I + \mathcal{K}_{\bar{\lambda}})$ we have

eq:dualityArgument

$$\|f\|_\partial = \|(K'_{Y,X})^{-1} K'_{Y,X} f\|_\partial \leq \|(K^*_{X,Y})^{-1}\|_{X,Y} \|K^*_{X,Y} f\|_\partial \leq C\|(-(1/2)I + \mathcal{K}^*_{\bar{\lambda}})f\|_\partial. \quad (5.17)$$

We are now in position to derive the missing estimates which were stated in the theorem. For $g \in \mathrm{L}^2_n(\partial\Omega)$ let $f \in \mathrm{Im}(T)$ with $T^* f = g$. Furthermore let (u, ϕ) be the double layer potential defined in equations (3.22) and (3.23). Then $u_+ = T^* f = g$ on $\partial\Omega$ by Theorem 3.11. Additionally we have that

$$\|(u)^*\|_\partial \leq C\|f\|_\partial \leq C\|g\|_\partial$$

where we used inequality (3.24) and (5.17). □

The next theorem can in some sense be regarded as a reverse trace theorem and will play an important role for the proof of the needed reverse Hölder inequality.

thm:reverseTrace

Theorem 5.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$ with connected boundary. Let $u \in \mathrm{H}^1(\Omega; \mathbb{C}^d)$ and $\pi \in \mathrm{L}^2(\Omega)$ satisfy the Stokes resolvent problem in Ω for some $\lambda \in \Sigma_\theta$. Then*

$$\Big(\int_\Omega |u|^p \,\mathrm{d}x\Big)^{1/p} \leq C\Big(\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma\Big)^{1/2}, \qquad (5.18)$$

eq:reverseTrace

where $p = \frac{2d}{d-1}$ and C depends only on d, θ and the Lipschitz character of Ω.

Proof. Let us denote the trace of u on $\partial\Omega$ by f and let furthermore $w = \mathcal{D}(g)$, $g \in \mathrm{L}^2(\partial\Omega; \mathbb{C}^d)$, be the solution of the L^2 Dirichlet problem with boundary data f as given by Theorem 5.5. For the sequence $(\Omega_j)_{j\in\mathbb{N}}$ of smooth domains that approximates Ω from inside as described by Theorem 1.3, an application of Lemma 5.4 shows

$$\int_{\Omega_j} |u - w|^2 \,\mathrm{d}x \leq C \int_{\partial\Omega_j} |u - w|^2 \,\mathrm{d}\sigma, \qquad (5.19)$$

where C does not depend on j but on the Lipschitz character of Ω. Now let $\varepsilon > 0$ be given. Then there exists $\varphi_\varepsilon \in \mathrm{C}^\infty(\overline{\Omega})$ such that $\|\varphi_\varepsilon - u\|^2_{\mathrm{H}^1(\Omega)} \leq \frac{\varepsilon}{3}$. Due to Theorem 1.3 we know that

$$\int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \,\mathrm{d}\sigma \to \int_{\partial\Omega} |\varphi_\varepsilon - u|^2 \,\mathrm{d}\sigma, \text{ as } j \to \infty$$

since $w = f$ on $\partial\Omega$ in the sense of nontangential convergence. Therefore we choose J large enough such that for all $j \geq J$ we have

$$\int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \,\mathrm{d}\sigma \leq \frac{2}{3}\varepsilon.$$

This gives together with the trace theorem that

$$\int_{\partial\Omega_j} |u - w|^2 \,\mathrm{d}\sigma \leq \int_{\partial\Omega_j} |u - \varphi_\varepsilon|^2 \,\mathrm{d}\sigma + \int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \,\mathrm{d}\sigma$$

$$\leq C\|u-\varphi_\varepsilon\|^2_{\mathrm{H}^1(\Omega)} + \int_{\partial\Omega_j} |\varphi_\varepsilon - w|^2 \,\mathrm{d}\sigma,$$

where once again C only depends on the Lipschitz character of Ω and is thus independent of j. Due to Verchota Theorem 1.3 we may take j large enough such that also the second summand of the previous inequality is smaller than $\varepsilon/2$. As a consequence we get that $w = u$ in Ω. Now Theorem 5.5 gives

$$\|(u)^*\|_\partial = \|(w)^*\|_\partial \leq C\|f\|_\partial = C\|u\|_\partial,$$

where C depends on d, θ, and the Lipschitz character of Ω. The claimed inequality now follows from estimate

weizhangestimate

$$\Big(\int_\Omega |u|^p \,\mathrm{d}x\Big)^{1/p} \leq C\Big(\int_{\partial\Omega} |(u)^*|^2 \,\mathrm{d}\sigma\Big)^{1/2}, \tag{5.20}$$

where C only depends on d and the Lipschitz constant of Ω. The proof of (5.20) was carried out by Wei and Zhang in [24, Lem. 3.3]. □

Remark 5.7. At this point the choice $p = \frac{2d}{d-1}$ in Theorem 5.6 may seem arbitrary. Taking a closer look at the proof by Wei and Zhang in [24, Lem. 3.3] one sees that the choice of p is due to 2 facts: (1) For the dual exponent we have $p' = \frac{2d}{d+1}$. (2) It holds $\frac{1}{p} - \frac{1}{p'} = \frac{1}{d}$ and thus the *Hardy-Littlewood-Sobolev theorem on fractional integration* may be applied to estimate the p'-norm of the Riesz potential of a function $f \in \mathrm{L}^p(\mathbb{R}^d)$ by the p-norm of f, see Grafakos [8, Thm 6.1.3].

In the following remark and the forthcoming chapter we will make use of an integration argument which can be considered as an application of the following Theorem on *integration along slices*. A proof of this result can be found in Federer [4, Thm. 3.2.12].

thm:coarea

Theorem 5.8 (Co-area formula). *If $f\colon \mathbb{R}^d \to \mathbb{R}$, $d \geq 2$, is Lipschitzian and $\mathfrak{g}$ the representative of a function $g \in \mathrm{L}^2(\mathbb{R}^d)$, then*

eq:coarea

$$\int_{R^d} \mathfrak{g}(x)\Big[\sum_{i=1}^d |\partial_i f(x)|^2\Big]^{1/2} \mathrm{d}x = \int_{\mathbb{R}}\int_{f^{-1}(y)} \mathfrak{g}(x)\,\mathrm{d}m_{d-1}(x)\,\mathrm{d}y, \tag{5.21}$$

where m_{d-1} denotes the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$.

Remark 5.9. Let (u, ϕ) be a solution of the Stokes resolvent system in the domain $\mathrm{B}(x_0, r) \subset \mathbb{R}^d$.

eq:interiorEstimateDoubleLayer

$$|\nabla^l u(x_0)| \leq \frac{C_l}{r^l}\Big(\fint_{\mathrm{B}(x_0,r)} |u(x)|^2 \,\mathrm{d}x\Big)^{1/2} \tag{5.22}$$

for all $l \geq 0$, where C_l only depends on d, l and θ. Without loss of generality we may rescale and translate and assume that $x_0 = 0$ and $r = 2$. Let $t \in (1,2)$. By Theorem 5.5 we know that a solution to the Stokes resolvent system on $\mathrm{B}(0,t) \subsetneq \mathrm{B}(0,2)$ with boundary values $g_t \coloneqq \mathrm{Tr}_{\partial \mathrm{B}(0,t)}\, u \in \mathrm{L}^2(\partial \mathrm{B}(0,t))$ is given by a boundary layer potential $\mathcal{D}_\lambda(f_t)$. We can use this fact to derive the desired estimate via

$$
\begin{aligned}
|\nabla^l u(0)|^2 &\leq C\Big(\int_{\partial \mathrm{B}(0,t)} \big\{ |\nabla_x^{l+1} \Gamma(y;\lambda)| + |\nabla_x^l \Phi(y)| \big\} |f_t(y)| \,\mathrm{d}\sigma(y) \Big)^2 \\
&\leq C\Big(\int_{\partial \mathrm{B}(0,t)} \frac{|f_t(y)|}{t^{d-1+l}} \,\mathrm{d}\sigma(y) \Big)^2 \\
&\leq C \int_{\partial \mathrm{B}(0,t)} |f_t(y)|^2 \,\mathrm{d}\sigma(y) \\
&\leq C \int_{\partial \mathrm{B}(0,t)} |u(y)|^2 \,\mathrm{d}\sigma(y),
\end{aligned}
$$

where we also used the Cauchy inequality in the estimate of f_t against the "data" from Theorem 5.5. Integrating this inequality in t over the interval $(1,2)$ and using the co-area formula (5.21) gives

$$
|\nabla^l u(0)|^2 \leq C \int_{\mathrm{B}(0,2)} |u(x)|^2 \,\mathrm{d}x.
$$

Now the claim follows readily.

Chapter 6

Derivation of Resolvent Estimates

In this final chapter we will prove that the Stokes semigroup is analytic on $\mathrm{L}^p_\sigma(\Omega, \mathbb{C}^d)$ for bounded Lipschitz domains $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$.

The first step will be to establich a weak reverse Hölder estimate for local solutions of the Stokes resolvent problem. We start with a similar result on Lipschitz cylinders.

lem:reverseHoelderCylinder

Lemma 6.1. *Let* $\eta \colon \mathbb{R}^{d-1} \to \mathbb{R}$ *be a Lipschitz function. Furthermore, let* $u \in \mathrm{H}^1(D_\eta(r); \mathbb{C}^d)$ *and* $\phi \in \mathrm{L}^2(D_\eta(2r))$ *solve the Stokes resolvent problem in* $D_\eta(2r)$ *with* $u = 0$ *on* $I_\eta(2r)$ *for some* $0 < r < \infty$ *and* $\lambda \in \Sigma_\theta$*. Let* $p_d = \frac{2d}{d-1}$*. Then*

eq:reverseHoelderCylinder

$$\Big(\fint_{D_\eta(r)} |u|^{p_d} \, \mathrm{d}x \Big)^{1/p_d} \leq C \Big(\fint_{D_\eta(2r)} |u|^2 \, \mathrm{d}x \Big)^{1/2}, \tag{6.1}$$

where C only depends on d, M and θ.

Proof. Without loss of generality we rescale and assume that $r = 1$. Let $t \in (1, 2)$. We note that by [22, Lemma 1.3.25] a Lipschitz cylinder is itself a Lipschitz domain. It is therefore admissible to apply Theorem 5.18 to u in $D_\eta(t)$ which yields

$$\Big(\int_{D_\eta(t)} |u|^{p_d} \, \mathrm{d}x \Big)^{2/p_d} \leq C \int_{\partial D_\eta(t)} |u|^2 \, \mathrm{d}\sigma,$$

where C depends only on d, θ and the Lipschitz character of Ω. In particular C does not depend on t. Since u vanishes on $I(2)$ we have that

$$\Big(\int_{D_\eta(1)} |u|^p \, \mathrm{d}x \Big)^{2/p} \leq C \int_{\partial D_\eta(t) \setminus I(2)} |u|^2 \, \mathrm{d}\sigma.$$

Applying the co-area formula to integrate both sides over the interval $(1, 2)$ gives

$$\Big(\int_{D_\eta(1)} |u|^p \, \mathrm{d}x \Big)^{2/p} \leq C \int_{D_\eta(2)} |u|^2 \, \mathrm{d}x.$$

Estimate (6.1) now follows after dividing by $|D_\eta(1)|$. □

The next step is to extend the result to arbitrary Lipschitz domains. The following Lemma reduces the amount of work to a few special cases.

lem:ballsforballs

Lemma 6.2 (Tolksdorf)**.** *Let $\Omega \subset \mathbb{R}^d$ be Lebesgue-measurable, $f, g \in \mathrm{L}^2(\Omega)$, $\alpha_2 > \alpha_1 > 1$, $p > 2$, $r > 0$ and $x_0 \in \mathbb{R}^d$ be such that $\mathrm{B}(x_0, r) \cap \Omega \neq \emptyset$. If there exists $C > 0$ such that*

$$\Big(\frac{1}{s^d}\int_{\Omega\cap\mathrm{B}(y,s)} |f|^p \,\mathrm{d}x\Big)^{1/p} \leq C\Big\{\Big(\frac{1}{s^d}\int_{\Omega\cap\alpha_1\mathrm{B}(y,s)} |f|^2 \,\mathrm{d}x\Big)^{1/2} + \sup_{B'\cap\mathrm{B}(y,s)}\Big(\frac{1}{|B'|}\int_{\Omega\cap B'} |g|^2 \,\mathrm{d}x\Big)^{1/2}\Big\}$$

holds for all balls $\mathrm{B}(y, s)$ with $\mathrm{B}(y, \alpha_2 s) \subset \mathrm{B}(x_0, \alpha_2 r)$ and which are either centered on $\partial\Omega$ or satisfy $\alpha_2\mathrm{B}(y, s) \subset \Omega$, then for each $\alpha \in (1, \alpha_2)$ there exists a constant C' such that

$$\Big(\frac{1}{r^d}\int_{\Omega\cap\mathrm{B}(x_0,r)} |f|^p \,\mathrm{d}x\Big)^{1/p} \leq C'\Big\{\Big(\frac{1}{r^d}\int_{\Omega\cap\alpha\mathrm{B}(x_0,r)} |f|^2 \,\mathrm{d}x\Big)^{1/2} + \sup_{B'\cap\mathrm{B}(x_0,r)}\Big(\frac{1}{|B'|}\int_{\Omega\cap B'} |g|^2 \,\mathrm{d}x\Big)^{1/2}\Big\}.$$

Proof. A proof of this lemma was given by Tolksdorf [21]. We will give the proof for the sake of completeness. □

As of now, our toolbox comprises enough tools to prove that solutions to the Stokes resolvent system satisfy a weak reverse Hölder inequality.

lem:reverseHoelder

Lemma 6.3. *Let $x_0 \in \overline{\Omega}$ and $0 < 2r < r_0$ and set $\alpha_1 = \sqrt{d^2 10^2 (1 + M)^2 + 4}$ and $\alpha_2 = \alpha_1 + 1$. Let $u \in \mathrm{H}^1(\mathrm{B}(x_0, \alpha_2 r) \cap \Omega; \mathbb{C}^d)$ and $\phi \in \mathrm{L}^2(\mathrm{B}(x_0, \alpha_2 r) \cap \Omega)$ satisfy the Stokes resolvent system in $\mathrm{B}(x_0, \alpha_2 r) \cap \Omega$. If $\mathrm{B}(x_0, \alpha_2 r) \cap \partial\Omega \neq \emptyset$, we additionally assume $u = 0$ on $\mathrm{B}(x_0, \alpha_2 r) \cap \partial\Omega$. Then*

$$\Big(\fint_{\mathrm{B}(x_0,r)\cap\Omega} |u|^p\Big)^{1/p} \leq C\Big(\fint_{\mathrm{B}(x_0,2r)\cap\Omega} |u|^2\Big)^{1/2} \qquad \text{eq:reverseHoelder} \quad (6.2)$$

holds, where $p = p_d$. Here, $C > 0$ only depends on d, θ and the Lipschitz character of Ω.

Proof. Due to Lemma 6.2 it suffices to consider only two cases: (1) $x_0 \in \Omega$ with $\alpha_2\mathrm{B}(x_0, r) \subset \Omega$ and (2) $x_0 \in \partial\Omega$.

Let $x_0 \in \Omega$ with $\alpha_2\mathrm{B}(x_0, r) \subset \Omega$. We may deploy the interior estimate (5.22) to derive that for all $x \in \mathrm{B}(x_0, r)$

$$|u(x)|^p \leq C\Big(\fint_{\mathrm{B}(x,r)} |u(y)|^2 \,\mathrm{d}y\Big)^{p/2}$$

which after integrating x over $\mathrm{B}(x_0, r)$ yields

$$\fint_{\mathrm{B}(x_0,r)} |u(x)|^p \,\mathrm{d}x \le C\Big(\alpha_1^d \fint_{\mathrm{B}(x_0,\alpha_1 r)} |u(z)|^2 \,\mathrm{d}z\Big)^{p/2},$$

where we also used the fact that $\alpha_1 > 2$.

If $x_0 \in \partial\Omega$, then by Lemma 6.1

$$\begin{aligned}\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,r)\cap\Omega} |u|^p\Big)^{1/p} &\le \Big(\frac{1}{r^d}\int_{D_{\eta x_0}(r)} |u|^p\Big)^{1/p} \\ &\le C\Big(\frac{1}{r^d}\int_{D_{\eta x_0}(2r)} |u|^p\Big)^{1/p} \\ &\le C\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,\alpha_1 r)\cap\Omega} |u|^2\Big)^{1/2}.\end{aligned}$$

Now the claim follows readily from an application of Lemma 6.2 with $\alpha = 2 \in (1, \alpha_2)$. □

We note that estimate (6.2) is a weak reverse Hölder inequality and thus possesses a self-improving property, see Giaquinta and Martinazzi [5, Thm. 6.38] or Giaquinta and Modica [6, Prop. 5.1].

prop:giaquinta

Proposition 6.4 (Giaquinta, Modica)**.** *Let $\Omega \subset \mathbb{R}^d$ be open, $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, $q > 1$, be a non-negative function. If there exist constants $b > 0$, $R_0 > 0$ such that*

$$\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,r)} f^q \,\mathrm{d}x\Big)^{1/q} \le \frac{b}{r^d}\int_{\mathrm{B}(x_0,2r)} f \,\mathrm{d}x$$

for all $x_0 \in \Omega$ and $0 < r < \min\{R_0, \mathrm{dist}(x_0, \partial\Omega)/2\}$. Then $f \in \mathrm{L}^{q+\varepsilon}_{\mathrm{loc}}(\Omega)$ for some $\varepsilon > 0$, depending only on d, q, and b and there is a contant $\tilde{C}$ depending only on d, q, ε and b such that

$$\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,r)} f^{q+\varepsilon} \,\mathrm{d}x\Big)^{1/(q+\varepsilon)} \le \tilde{C}\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,2r)} f^q \,\mathrm{d}x\Big)^{1/q}$$

for all $x_0 \in \Omega$ and $0 < r < \min\{R_0, \mathrm{dist}(x_0, \partial\Omega)/2\}$.

rem:reverseHoelder

Remark 6.5. The self-improving property of reverse Hölder estimates can now be used to make the result of Lemma 6.3 a little bit better. Let $0 < 2r < r_0$. We are aiming to apply Proposition 6.4 for $x_0 \in \overline{\Omega}$ on the open set $\Omega \cap \mathrm{B}(x_0, \alpha_2 r)$, for α_2 as in Lemma 6.3. Let also u be as in Lemma 6.3 and set $R_0 = r_0/2$. Then for $f = |u|^2 \chi_{\mathrm{B}(x_0,\alpha_2 r)\cap\Omega}$ which can be considered as a partial extension of u by 0 to $\mathbb{R}^d$ and $q = p/2$, inequality (6.2) reads

$$\Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,r)} f^q \,\mathrm{d}x\Big)^{1/q} = \Big(\frac{1}{r^d}\int_{\mathrm{B}(x_0,r)\cap\Omega} |u|^p \,\mathrm{d}x\Big)^{2/p}$$

$$\leq C^2 \frac{1}{r^d} \int_{\mathrm{B}(x_0,2r)\cap\Omega} |u|^2 \,\mathrm{d}x = C^2 \frac{1}{r^d} \int_{\mathrm{B}(x_0,2r)} f \,\mathrm{d}x.$$

Consequently Proposition 6.4 gives us that there exists some $\varepsilon > 0$ which depends only on d, q and C^2 and a constant $\tilde{C}$ depending only on d, q, ε and C^2 such that

$$\begin{aligned}\Big(\frac{1}{r^d} \int_{\mathrm{B}(x_0,r/2)\cap\Omega} |u|^{p+\varepsilon'} \,\mathrm{d}x\Big)^{2/(p+\varepsilon')} &= \Big(\frac{1}{r^d} \int_{\mathrm{B}(x_0,r/2)} f^{q+\varepsilon} \,\mathrm{d}x\Big)^{1/(q+\varepsilon)} \\ &\leq \tilde{C}\Big(\frac{1}{r^d} \int_{\mathrm{B}(x_0,r)\cap\Omega} |u|^p \,\mathrm{d}x\Big)^{1/p} \\ &\leq \tilde{C}C\Big(\frac{1}{r^d} \int_{\mathrm{B}(x_0,2r)\cap\Omega} |u|^2 \,\mathrm{d}x\Big)^{1/2}.\end{aligned}$$

Another application of Lemma 6.2 gives us that for all $r < r_0/4$ it holds that

eq:improvedreverseHoelder

$$\Big(\frac{1}{r^d} \int_{\mathrm{B}(x_0,r)\cap\Omega} |u|^{p+\varepsilon'} \,\mathrm{d}x\Big)^{2/(p+\varepsilon')} \leq C\Big(\frac{1}{r^d} \int_{\mathrm{B}(x_0,2r)\cap\Omega} |u|^2 \,\mathrm{d}x.\Big)^{1/2}. \tag{6.3}$$

The following extrapolation theorem by Shen will be necessary in order to derive L^p-bounds on the solution of the Stokes resolvent system, [16, Thm. 3.3]. Note that a more recent result from Tolksdorf [21] generalizes this result to operators which are defined on spaces of Banach space valued functions.

thm:extrapolation

Theorem 6.6. *Let T be a bounded sublinear operator on $\mathrm{L}^2(\Omega;\mathbb{C}^d)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and $\|T\|_{\mathfrak{L}(\mathrm{L}^2(\Omega;\mathbb{C}^d))} \leq C_0$. Let $p > 2$. Suppose that there exist constants $R_0 > 0$, $N > 1$ and $\alpha_2 > \alpha_1 > 1$ such that for any bounded measurable function f with $\mathrm{supp}(f) \subseteq \Omega \setminus \alpha_2 B$,*

$$\Big\{\frac{1}{r^d} \int_{\Omega\cap B} |Tf|^p \,\mathrm{d}x\Big\}^{1/p} \leq N\Big\{\Big(\frac{1}{r^d} \int_{\Omega\cap\alpha_1 B} |Tf|^2 \,\mathrm{d}x\Big)^{1/2} + \sup_{B'\supset B}\Big(\frac{1}{|B'|} \int_{B'} |f|^p \,\mathrm{d}x\Big)^{1/p}\Big\},$$

where $B = B(x_0, r)$ is a ball with $0 < r < R_0$ and either $x_0 \in \partial\Omega$ or $B(x_0, \alpha_2 r) \subset \Omega$. Then T is bounded on $\mathrm{L}^q(\Omega;\mathbb{C}^d)$ for any $2 < q < p$. Moreover $\|T\|_{\mathfrak{L}(\mathrm{L}^q(\Omega;\mathbb{C}^d))}$ is bounded by a constant depending at most on d, N, C_0, p, q and the Lipschitz character of Ω.

We are now in the position to prove Theorem 1.17, the main theorem of this thesis. For this, the improved weak reverse Hölder inequality derived in Remark 6.5 will be the crucial ingredient as it enables us to apply the extrapolation theorem 6.6 to a suitable family of operators.

Proof of Theorem 1.17. Consider a family of scaled solution operators to the Stokes resolvent system (1.9), more precisely consider the family

$$T_\lambda \colon \mathrm{L}^2(\Omega;\mathbb{C}^d) \to \mathrm{L}^2(\Omega;\mathbb{C}^d), \quad f \mapsto (|\lambda| + 1)(A_2 + \lambda)^{-1}\mathbb{P}_2 f,$$

where $\lambda \in \Sigma_\theta$, $\theta \in (0, \pi/2)$. Let us first verify that $u := (|\lambda| + 1)^{-1} T_\lambda(f)$ does indeed solve (1.9). First note that since $\mathbb{P}_2 f \in \mathrm{L}^2_\sigma(\Omega)$ we know that by the mapping properties of the Stokes resolvent we have $u \in \mathrm{H}^1_{0,\sigma}(\Omega)$ and

$$A_2 u + \lambda u = \mathbb{P}_2 f.$$

Therefore u is a weak solution to

$$-\Delta u + \lambda u = \mathbb{P}_2 f.$$

By the usual arguments (c.f. Chapter 1), there exists a pressure $\pi \in \mathrm{L}^2(\Omega)$ such that

$$-\Delta u + \nabla \pi + \lambda u = f.$$

Furthermore by testing (**??**) with u we derive the estimate

$$\|T_\lambda(f)\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} = (|\lambda| + 1)\|u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)} \leq C_0 \|f\|_{\mathrm{L}^2(\Omega;\mathbb{C}^d)},$$

where C_0 only depends on d, θ and the Lipschitz character of Ω. Accordingly the family T_λ is bounded on $\mathrm{L}^2(\Omega; C^d)$ and C_0 is a uniform bound on the operator norms $\|T_\lambda\|_{\mathfrak{L}(\mathrm{L}^2(\Omega;\mathbb{C}^d))}$.

We will now show that the operators T_λ fulfill the estimate in Theorem 6.6, in order to deduce their L^p-boundedness. To this end let $x_0 \in \overline{\Omega}$ and $0 < 4r < r_0$ such that $3\mathrm{B}(x_0, r) \subseteq \Omega$ or $\mathrm{B}(x_0, r)$ is centered on $\partial\Omega$. Furthermore let $f \in \mathrm{L}^\infty(\Omega; \mathbb{C}^d)$ with support in $\Omega \setminus 3\mathrm{B}(x_0, r)$. By construction (u, π) does not only solve (2.26) in Ω, the pair also solves the dirichlet problem

$$\begin{aligned} -\Delta u + \nabla \phi + \lambda u &= 0 \\ \operatorname{div}(u) &= 0 \end{aligned}$$

in $\Omega \cap 3\mathrm{B}(x_0, r)$ where $u = 0$ on $\partial\Omega \cap 3\mathrm{B}(x_0, r)$. Therefore Remark 6.5 and more precisely inequality (6.3) give that

$$\Big(\frac{1}{r^d} \int_{\Omega \cap \mathrm{B}(x_0,r)} |u|^p \,\mathrm{d}x\Big)^{1/p} \leq C\Big(\frac{1}{r^d} \int_{\Omega \cap 2\mathrm{B}(x_0,r)} |u|^2 \,\mathrm{d}x\Big)^{1/2},$$

where $p = p_d + \varepsilon$. Multiplying this inequality on both sides with $(|\lambda| + 1)$ gives

eq:tlambdaEstimate

$$\Big(\frac{1}{r^d} \int_{\Omega \cap \mathrm{B}(x_0,r)} |T_\lambda(f)|^p \,\mathrm{d}x\Big)^{1/p} \leq C\Big(\frac{1}{r^d} \int_{\Omega \cap 2\mathrm{B}(x_0,r)} |T_\lambda(f)|^2 \,\mathrm{d}x\Big)^{1/2}, \tag{6.4}$$

where C depends only on d, θ and the Lipschitz character of Ω. Now Shen's extrapolation theorem 6.6 gives that T_λ is bounded on $\mathrm{L}^q(\Omega; \mathbb{C}^d)$ for all $2 < q < p_d + \varepsilon$ and that the operator norms $\|T_\lambda\|_{\mathfrak{L}(\mathrm{L}^q(\Omega;\mathbb{C}^d))}$ are uniformly bounded by a constant C_q depending only on d, θ, q and the Lipschitz character of Ω.

In the next step of the proof we now study the relationship between the operator T_λ and the resolvent of the Stokes operator A_q on $\mathrm{L}^q_\sigma(\Omega)$ for $q \in (2, p_d + \varepsilon)$. To this end let $f \in \mathrm{L}^q_\sigma(\Omega)$. We already know that $u = (1+|\lambda|)^{-1}T_\lambda(f) = (A_2+\lambda)^{-1}\mathbb{P}_2(f) \in \mathrm{L}^p_\sigma(\Omega) \cap \mathcal{D}(A_2)$ by the mapping properties of $T_\lambda(f)$. As $\mathrm{L}^q_\sigma(\Omega) \subset \mathrm{L}^2_\sigma(\Omega)$ we have furthermore that

$$\lambda u + A_2 u = f \in \mathrm{L}^q_\sigma(\Omega)$$

and thus $A_2 u \in \mathrm{L}^q_\sigma(\Omega)$. Appealing to Definition 1.12 we showed that $u \in \mathcal{D}(A_p)$ and that $A_2 u = A_q u$. Therefore we have that

$$\lambda u + A_q u = f \in \mathrm{L}^p_\sigma(\Omega)$$

By the uniqueness of u, which follows from the L^2-theory of the Stokes resolvent problem, we have that $u = (\lambda + A_p)^{-1} f$. Hence estimate 6.4 gives

$$\|u\|_{\mathrm{L}^q(\Omega;\mathbb{C}^d)} = \|(\lambda + A_q)^{-1} f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^d)} \leq \frac{C_q}{1+|\lambda|}\|f\|_{\mathrm{L}^q(\Omega;\mathbb{C}^d)}.$$

And thus A_p is sectorial on $\mathrm{L}^p_\sigma(\Omega)$. If necessary we take ε to be the minimum of the parameter ε used in the first part of this proof and the one from Theorem **??**. In this case it can be shown, that the spaces $\mathrm{L}^q_\sigma(\Omega)$ are reflexive and that $\mathrm{L}^q_\sigma(\Omega)^* = \mathrm{L}^{q'}_\sigma(\Omega)$ where q' denotes the dual exponent $q' = q(q-1)^{-1}$. By abstract operator theory [10] we get that A_q is indeed densely defined and that $A_q^* = A_q'$. Therefore

$$\|(A_q+\lambda)^{-1}\|_{\mathfrak{L}(\mathrm{L}^q_\sigma(\Omega))} = \|(A_q+\lambda)^{-1} * \|_{\mathfrak{L}(\mathrm{L}^{q'}_\sigma(\Omega))} = \|(A_{q'}+\lambda)^{-1}\|_{\mathfrak{L}(\mathrm{L}^{q'}_\sigma(\Omega))}.$$

Consequently also the operators $A_{q'}$ are sectorial, densely defined and closed. This completes the proof. $\square$

Appendix

chap:app

For $d = 2$ we have that $G(x;\lambda) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x|)$. Furthermore we set $z = k|x|$. Then applications of chain rule and product rule of differentiation give

$$
\begin{aligned}
\partial_\gamma G(x;\lambda) &= \frac{\mathrm{i}}{4} k \, \frac{x_\gamma}{|x|} \frac{\mathrm{d}}{\mathrm{d}z} H_0^{(1)}(z) \\
\partial_\alpha \partial_\gamma G(x;\lambda) &= \frac{\mathrm{i}}{4} k^2 \, \frac{x_\alpha x_\gamma}{|x|^2} \frac{\mathrm{d}^2}{\mathrm{d}z^2} H_0^{(1)}(z) + \frac{\mathrm{i}}{4} k \Big(\frac{\delta_{\alpha\gamma}}{|x|} - \frac{x_\alpha x_\gamma}{|x|^3} \Big) \frac{\mathrm{d}}{\mathrm{d}z} H_0^{(1)}(z) \\
\partial_\beta \partial_\alpha \partial_\gamma G(x;\lambda) &= \frac{\mathrm{i}}{4} k^3 \, \frac{x_\alpha x_\beta x_\gamma}{|x|^3} \frac{\mathrm{d}^3}{\mathrm{d}z^3} H_0^{(1)}(z) \\
&\quad + \frac{\mathrm{i}}{4} k^2 \Big(\frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^2} - 3 \frac{x_\alpha x_\beta x_\gamma}{|x|^4} \Big) \frac{\mathrm{d}^2}{\mathrm{d}z^2} H_0^{(1)}(z) \\
&\quad + \frac{\mathrm{i}}{4} k \Big(3 \frac{x_\alpha x_\beta x_\gamma}{|x|^5} - \frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^3} \Big) \frac{\mathrm{d}}{\mathrm{d}z} H_0^{(1)}(z).
\end{aligned}
$$

The series expansions for the Hankel function $H_0^{(1)}(z)$ read according to Lebedev [11]

$$
\begin{aligned}
H_0^{(1)}(z) &= J_0(z) + \mathrm{i} Y_0(z) \\
&= \frac{2\mathrm{i}}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{(l!)^2 \, 4^l} \, z^{2l} \Big(-\frac{\mathrm{i}\pi}{2} - \log(2) - \psi(l+1) \Big) + \frac{2\mathrm{i}}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{(l!)^2 \, 4^l} \, z^{2l} \log(z) \\
&= \frac{2\mathrm{i}}{\pi} \sum_{l=0}^{\infty} a_l \, z^{2l} \, C_l + \frac{2\mathrm{i}}{\pi} \sum_{l=0}^{\infty} a_l z^{2l} \log(z) \\
\frac{\mathrm{d}}{\mathrm{d}z} H_0^{(1)}(z) &= \frac{2\mathrm{i}}{\pi} \sum_{l=1}^{\infty} a_l \, (2l) \, z^{2l-1} \, C_l + \frac{2\mathrm{i}}{\pi} \sum_{l=1}^{\infty} a_l \, (2l) \, z^{2l-1} \log(z) \, + \frac{2\mathrm{i}}{\pi} \sum_{l=0}^{\infty} a_l z^{2l-1} \\
&= \frac{2\mathrm{i}}{\pi} \sum_{l=1}^{\infty} b_l \, z^{2l-1} \, C_l + \frac{2\mathrm{i}}{\pi} \sum_{l=1}^{\infty} b_l \, z^{2l-1} \log(z) \, + \frac{2\mathrm{i}}{\pi} \sum_{l=0}^{\infty} a_l \, z^{2l-1} \\
\frac{\mathrm{d}^2}{\mathrm{d}z^2} H_0^{(1)}(z) &= \frac{2\mathrm{i}}{\pi} \sum_{l=1}^{\infty} b_l \, (2l-1) \, z^{2l-2} \, C_l + \frac{2\mathrm{i}}{\pi} \sum_{l=1}^{\infty} b_l \, (2l-1) \, z^{2l-2} \log(z) + \frac{2\mathrm{i}}{\pi} \sum_{l=1}^{\infty} b_l \, z^{2l-2}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{2\mathrm{i}}{\pi} \sum_{l=0}^{\infty} a_l \, (2l-1) \, z^{2l-2} \\
&= \frac{2\mathrm{i}}{\pi} \sum_{l=1}^{\infty} c_l \, z^{2l-2} \, C_l + \frac{2\mathrm{i}}{\pi} \sum_{l=1}^{\infty} c_l \, z^{2l-2} \log(z) + \frac{2\mathrm{i}}{\pi} \sum_{l=1}^{\infty} b_l \, z^{2l-2} \\
&\quad + \frac{2\mathrm{i}}{\pi} \sum_{l=0}^{\infty} a_l (2l-1) \, z^{2l-2} \\
\frac{\mathrm{d}^3}{\mathrm{d}z^3} H_0^{(1)}(z) &= \frac{2\mathrm{i}}{\pi} \sum_{l=2}^{\infty} c_l \, (2l-2) \, z^{2l-3} \, C_l + \frac{2\mathrm{i}}{\pi} \sum_{l=2}^{\infty} c_l \, (2l-2) \, z^{2l-3} \log(z) + \frac{2\mathrm{i}}{\pi} \sum_{l=1}^{\infty} c_l \, z^{2l-3} \\
&\quad + \frac{2\mathrm{i}}{\pi} \sum_{l=2}^{\infty} b_l (2l-2) \, z^{2l-3} + \frac{2\mathrm{i}}{\pi} \sum_{l=0}^{\infty} a_l (2l-1)(2l-2) \, z^{2l-3} \\
&= \frac{2\mathrm{i}}{\pi} \sum_{l=2}^{\infty} d_l \, z^{2l-3} \, C_l + \frac{2\mathrm{i}}{\pi} \sum_{l=2}^{\infty} d_l \, z^{2l-3} \log(z) + \frac{2\mathrm{i}}{\pi} \sum_{l=1}^{\infty} c_l \, z^{2l-3} \\
&\quad + \frac{2\mathrm{i}}{\pi} \sum_{l=2}^{\infty} b_l (2l-2) \, z^{2l-3} + \frac{2\mathrm{i}}{\pi} \sum_{l=0}^{\infty} a_l (2l-1)(2l-2) \, z^{2l-3},
\end{aligned}
$$

where

$$
\begin{aligned}
C_l &:= -\frac{\mathrm{i}\pi}{2} - \log(2) - \psi(l+1) \\
a_l &:= \frac{(-1)^l}{(l!)^2 \, 4^l}, \quad b_l := a_l \cdot 2l, \quad c_l := b_l \cdot (2l-1), \quad d_l := c_l \cdot (2l-2).
\end{aligned}
$$

For $G(x;0) = -\frac{1}{2\pi} \log(|x|)$, we have

$$
\begin{aligned}
\partial_\gamma G(x;0) &= -\frac{1}{2\pi} \frac{x_\gamma}{|x|^2} \\
\partial_\alpha \partial_\gamma G(x;0) &= -\frac{1}{2\pi} \delta_{\alpha\gamma} \cdot \frac{1}{|x|^2} + \frac{1}{\pi} \cdot \frac{x_\alpha x_\gamma}{|x|^4} \\
\partial_\beta \partial_\alpha \partial_\gamma G(x;0) &= \frac{1}{\pi} \frac{\delta_{\beta\gamma} x_\alpha + \delta_{\alpha\gamma} x_\beta + \delta_{\alpha\beta} x_\gamma}{|x|^4} - \frac{4}{\pi} \frac{x_\alpha x_\beta x_\gamma}{|x|^6}.
\end{aligned}
$$

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Darmstadt, XX.04.2018

Fabian Gabel