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Sobolev Spaces

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Contents

Chapter 1

Introduction

In order to have classical solutions to partial differential equations (PDEs), it is often necessary that parameter functions in the PDE are regular enough or the domain where the PDE is considered has a regular boundary (e.g. no edge). However, in applications or in nature, these regularity assumptions are often not satisfied. Hence, a fundamental concept in the theory of PDEs is the concept of weak solutions which is also the basis for important numerical methods (e.g. the finite element method). The definition of these weak solutions is based on a concept of generalized derivatives of functions, the so called *weak derivatives*. *Sobolev spaces* are Banach spaces consisting of functions with weak derivatives. Important properties of these spaces will be studied in this lecture and will be a basis to study weak solutions of PDEs afterwards. Let us start by illustrating the idea behind weak solutions with an example.

Example 1.1. Let $\Omega = (0,1) \subseteq \mathbb{R}$ and $f \in \mathrm{C}^0(\overline{\Omega})$ be given. We look for a solution $u \in \mathrm{C}^2(\overline{\Omega})$ to the Poisson equation in one dimension,

$$\begin{aligned} -u'' &= f(x), \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \{0,1\} = \partial\Omega. \end{aligned} \tag{1.1}$$

u e.g. describes the displacement of a rod which is fixed at $x = 0$ and $x = 1$, where f is a force acting on the string. Of course the force f is not necessarily continuous in $\overline{\Omega}$ and could have jumps.
In order to get a weaker solution concept, let $\varphi \in \mathrm{C}_0^\infty(\Omega)$ (infinitely often differentiable with compact support in Ω). Then integration by parts shows

$$\int_0^1 -u''(x)\varphi(x)\,\mathrm{d}x = -u'(1)\varphi(1) + u'(0)\varphi(0) + \int_0^1 u'(x)\varphi'(x)\,\mathrm{d}x.$$

Hence, in view of (1.1)

$$\int_\Omega u'(x)\varphi'(x)\,\mathrm{d}x = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{1.2}$$

For (1.2) to be meaningful we do not need a second derivative of u. Moreover, f only has to be integrable instead of continuous. (1.2) will even make sense if u' is only a weak derivative of u as we will see soon. □

In order to motivate the definition of weak derivatives, we note the following identity.

Lemma 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be open and $u \in \mathrm{C}^1(\Omega)$. Then for $i \in \{1, \dots, n\}$ we have*

$$\int_\Omega \frac{\partial u}{\partial x_i} \varphi \, \mathrm{d}x = - \int_\Omega u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x \quad \textit{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{1.3}$$

Proof. As in general neither $\partial\Omega$ nor the boundary of the support of φ need to be regular, the proof is not immediate. Define

$$w(x) := \begin{cases} u(x)\varphi(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

As $w \in \mathrm{C}^1(\Omega)$ and $w = 0$ in a neighborhood of $\partial\Omega$, we conclude that $w \in \mathrm{C}^1(\mathbb{R}^n)$. Take an open ball B large enough such that it contains the support of φ. Then by Gauß' (or Green's formula) we have

$$\int_\Omega w_{x_i} \, \mathrm{d}x = \int_{\partial B} w_{x_i} \cdot \nu_i \, \mathrm{d}\sigma = 0,$$

where ν is the outward unit normal on ∂B. Hence, the product rule implies (1.3). $\square$

(1.3) makes sense even if $u, \frac{\partial u}{\partial x_i} \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ (integrable on any bounded set V with $\overline{V} \subset \Omega$). Hence, we define weak derivatives by:

Definition 1.3. Let $\Omega \subseteq \mathbb{R}^n$ be open, $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and $i \in \{1, \dots, n\}$. u has the *weak partial derivative* $\frac{\partial u}{\partial x_i}$ if there is $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega u(x) \frac{\partial u}{\partial x_i}(x) \, \mathrm{d}x = - \int_\Omega v(x)\varphi(x) \, \mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{1.4}$$

Then $\frac{\partial u}{\partial x_i} := v$.

Let us see in an example which functions have weak derivatives and how to calculate them.

Example 1.4. a) If $u \in \mathrm{C}^1(\Omega)$, then by Lemma 1.2 (1.4) is satisfied with $v := \frac{\partial u}{\partial x_i}$. Hence, u is weakly differentiable and the weak derivative $\frac{\partial u}{\partial x_i}$ coincides with the classical derivative.

b) Let $\Omega = (-1, 1) \subset \mathbb{R}$, $u(x) := |x|$ for $x \in \Omega$. Then as $u \in \mathrm{C}^1(\overline{\Omega} \setminus \{0\}) \cap \mathrm{C}^0(\overline{\Omega})$, we may use the fundamental theorem of calculus to obtain for $\varphi \in \mathrm{C}_0^\infty(\Omega)$:

$$\begin{aligned} \int_\Omega u(x)\varphi'(x) \, \mathrm{d}x &= \int_{-1}^0 -x\varphi'(x) \, \mathrm{d}x + \int_0^1 x\varphi'(x) \, \mathrm{d}x \\ &= \int_{-1}^0 \varphi(x) \, \mathrm{d}x + (-x\varphi(x))\Big|_{-1}^0 - \int_0^1 \varphi(x) \, \mathrm{d}x + (x\varphi(x))\Big|_0^1 \\ &= - \int_{-1}^0 -1\varphi(x) \, \mathrm{d}x - \int_0^1 1\varphi(x) \, \mathrm{d}x = - \int_{-1}^1 v(x)\varphi(x) \end{aligned}$$

if we define $v(x) = \begin{cases} 1, & x \in (0, 1), \\ -1, & x \in (-1, 0). \end{cases}$

Then $v \in \mathrm{L}^1(\Omega)$ and since $\{0\}$ is a set of measure zero in $\mathbb{R}$, we could define $v(0)$ arbitrarily. Hence, u is weakly differentiable with derivative $u' = v$. u' coincides with the classical derivative in all $x \in \Omega$ where the latter exists.

c) Defining again $\Omega = (-1,1)$ and v as in b), we have $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and for all $\varphi \in \mathrm{C}_0^\infty(\Omega)$ we have

$$\int_\Omega v(x)\varphi'(x)\,\mathrm{d}x = \int_{-1}^0 -\varphi'(x)\,\mathrm{d}x + \int_0^1 \varphi'(x)\,\mathrm{d}x = -\varphi(0) + \varphi(-1) + \varphi(1) - \varphi(0) = -2\varphi(0).$$

Now if v would be weakly differentiable, there would be $w \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ with

$$-2\varphi(0) = -\int_\Omega w(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{1.5}$$

Fix some $f \in \mathrm{C}_0^\infty((-1,1))$ with $f(0) = 1$ and define $\varphi_n(x) = f(nx)$ for $x \in (-1,1), n \in \mathbb{N}$ (where $f = 0$ on $\mathbb{R}\setminus(-1,1)$). Then $\varphi_n \in \mathrm{C}_0^\infty((-1,1))$ with $\varphi_n(x) = 0$ for all $x \in \mathbb{R}\setminus\left(-\frac{1}{n}, \frac{1}{n}\right)$ with $\varphi_n(0) = 1$ and $\lim_{n\to\infty}\varphi_n(x) = 0$ for all $x \in \Omega \setminus \{0\}$. As $\|\varphi_n\|_{\mathrm{L}^\infty(\Omega)} \le \|f\|_{\mathrm{L}^\infty(\Omega)} < \infty$, we conclude from the dominated convergence theorem that

$$0 = \lim_{n\to\infty}\left(-\int_\Omega w(x)\varphi_n(x)\,\mathrm{d}x\right) \neq -2 = \lim_{n\to\infty} -2\varphi_n(x)$$

which contradicts (1.5). Hence, v is not weakly differentiable in Ω. □

Hence, there are functions which are not classically differentiable everywhere and have weak derivatives, but there are also functions being not weakly differentiable (although $v \in \mathrm{C}^1(\Omega\setminus\{0\})$ in Example 1.4).
If we define the *Sobolev space*

$$\mathrm{W}^{1,p}(\Omega) := \left\{u \in \mathrm{L}^p(\Omega) \colon \frac{\partial u}{\partial x_i} \text{ exists in the weak sense, } \frac{\partial u}{\partial x_i} \in \mathrm{L}^p(\Omega) \text{ for all } i \in \{1,\dots,n\}\right\}$$

for $p \in [1,\infty]$, this is a Banach space which will turn out to be particularly useful in the context of weak solutions of PDEs. So we will study important properties of these spaces (and its generalisations to higher order derivatives) and finally will show how to use them for obtaining weak solutions of PDEs. We shortly illustrate the latter in an example.

Example 1.5. We continue Example 1.1 with $\Omega = (0,1) \subset \mathbb{R}$ and assume that $f \in \mathrm{L}^2(\Omega)$. Then in view of (1.2) we say that u is a *weak solution* to (1.1) if $u \in \mathrm{W}^{1,2}(\Omega)$,

$$\int_\Omega u'(x)\varphi'(x)\,\mathrm{d}x = \int_\Omega f(x)\varphi(x) \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega),$$

where u' is the weak derivative of u, and if u satisfies $u = 0$ on $\partial\Omega$ in a certain weak sense. The latter will be specified in a detailed way in Chapter 7 as $u \in \mathrm{W}^{1,2}(\Omega)$ is not necessarily continuous. In Chapter 7, we will study the generalisation of (1.1) for $\Omega \subset \mathbb{R}^n$ being a bounded domain namely the Poisson equation

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Chapter 2

Some Facts about Lebesgue Spaces $\mathrm{L}^p(\Omega)$

Here, we recall some facts about Lebesgue spaces which should be known from previous lectures. Throughout this lecture, a set $\Omega \subset \mathbb{R}^n$ is called *measurable* if it is measurable w.r.t. the Lebesgue measure on $\mathbb{R}^n$. Unless otherwise stated, we always assume in this chapter that $\Omega \subset \mathbb{R}^n$ is measurable.
Then $u\colon \Omega \to [-\infty, \infty]$ is measurable *on* Ω if $\{x \in \Omega\colon u(x) > \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$.

2.1 $\mathrm{L}^p(\Omega)$: Definition and Basic Properties

i) If $u, v\colon \Omega \to [-\infty, \infty]$ are measurable on Ω, they are equivalent if $u = v$ a.e. in Ω. $[u]$ is the equivalence class of u. We always identify a function u with its equivalence class.

ii) For $p \in [1, \infty]$, we define the Lebesgue space

$$\mathrm{L}^p(\Omega) := \{u\colon \Omega \to [-\infty, \infty]\colon u \text{ measurable }, \|u\|_{\mathrm{L}^p(\Omega)} < \infty\},$$

where

$$\begin{aligned} \|u\|_{\mathrm{L}^p(\Omega)} &= \left(\int_\Omega |u(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \text{ if } p \in [1, \infty), \\ \|u\|_{\mathrm{L}^\infty(\Omega)} &= \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|. \end{aligned}$$

With the convention from i), $u = 0$ in $\mathrm{L}^p(\Omega)$ if $u = 0$ a.e. in Ω. If $[u]$ contains a continuous function, we assume that u is chosen to be continuous.

iii) $\mathrm{L}^p(\Omega)$ is a Banach space for $p \in [1, \infty]$, i.e. a complete and normed vector space.

iv) L^p convergence and a.e. convergence: Let $p \in [1, \infty]$, $(u_n)_{n\in\mathbb{N}} \subset \mathrm{L}^p(\Omega)$ and $u \in \mathrm{L}^p(\Omega)$, such that $u_n \to u$ in $\mathrm{L}^p(\Omega)$, i.e. $\|u_n - u\|_{\mathrm{L}^p(\Omega)} \to 0$ as $n \to \infty$. Then there is a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and a function $h \in \mathrm{L}^p(\Omega)$ such that $u_{n_k}(x) \to u(x)$ a.e. in Ω as $k \to \infty$ and $|u_{n_k}(x)| \leq h(x)$ a.e. in Ω for all $k \in \mathbb{N}$.

v) Minkowski's inequality: Let $1 \leq p \leq \infty$ and $u, v \in \mathrm{L}^p(\Omega)$. Then

$$\|u + v\|_{\mathrm{L}^p(\Omega)} \leq \|u\|_{\mathrm{L}^p(\Omega)} + \|v\|_{\mathrm{L}^p(\Omega)}.$$

vi) Hölder's inequality: Let $p, q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $u \in \mathrm{L}^p(\Omega), v \in \mathrm{L}^q(\Omega)$. Then $uv \in \mathrm{L}^1(\Omega)$ and

$$\|uv\|_{\mathrm{L}^1(\Omega)} \leq \|u\|_{\mathrm{L}^p(\Omega)} \|v\|_{\mathrm{L}^q(\Omega)}.$$

vii) For $x, y \in \mathbb{R}^n$,

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \quad \text{if } p \in [1,\infty),$$
$$\|x\|_\infty = \max_i |x_i|$$

the discrete versions of v), vi) are valid:

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p,$$
$$|x \cdot y| \leq \|x\|_p \|y\|_q \quad \text{for } p \in [1,\infty], \frac{1}{p} + \frac{1}{q} = 1.$$

viii) General Hölder inequality: Let $p_k \in [1,\infty], \frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1, m \geq 3, u_k \in \mathrm{L}^{p_k}(\Omega), k = 1, \ldots, m$. Then

$$\int_\Omega |u_1 \cdot \cdots \cdot u_m| \,\mathrm{d}x \leq \prod_{k=1}^m \|u_k\|_{\mathrm{L}^{p_k}(\Omega)}.$$

2.2 Limit Theorems and Fubini

i) Monotone convergence (Beppo-Levi): Let $(u_n)_{n\in\mathbb{N}}$ be measurable in Ω, non-negative, and point-wise non-decreasing. Then

$$\int_\Omega \left(\lim_{n\to\infty} u_n(x)\right) \mathrm{d}x = \lim_{n\to\infty} \int_\Omega u_n(x) \,\mathrm{d}x.$$

ii) Fatou's lemma: Let $(u_n)_{n\in\mathbb{N}}$ be measurable in Ω and non-negative. Then

$$\int_\Omega \left(\liminf_{n\to\infty} u_n(x)\right) \mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega u_n(x) \,\mathrm{d}x.$$

iii) Dominated convergence (Lebesgue): Let $(u_n)_{n\in\mathbb{N}}$ and u be measurable on Ω such that $u_n(x) \to u(x)$ as $n \to \infty$ a.e. in Ω and $|u_n(x)| \leq h(x)$ a.e. in Ω for all $n \in \mathbb{N}$ and some $h \in \mathrm{L}^1(\Omega)$. Then $u_n, u \in \mathrm{L}^1(\Omega)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \int_\Omega u_n(x) \,\mathrm{d}x = \int_\Omega u(x) \,\mathrm{d}x$.

iv) Fubini's theorem: Let $u = u(x,y)$ be measurable on $\mathbb{R}^{n+m}$ such that at least one of the following integrals exists and is finite:

$$I_1 = \int_{\mathbb{R}^{n+m}} |u(x,y)| \,\mathrm{d}x \,\mathrm{d}y,$$
$$I_2 = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |u(x,y)| \,\mathrm{d}x\right) \mathrm{d}y,$$
$$I_2 = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |u(x,y)| \,\mathrm{d}y\right) \mathrm{d}x.$$

Then $u(\cdot,y) \in \mathrm{L}^1(\mathbb{R}^n)$ for a.e. $y \in \mathbb{R}^m$, $\int_{\mathbb{R}^m} u(\cdot,y) \,\mathrm{d}y \in \mathrm{L}^1(\mathbb{R}^n)$, $u(x,\cdot) \in \mathrm{L}^1(\mathbb{R}^m)$ for a.e. $x \in \mathbb{R}^n$, $\int_{\mathbb{R}^n} u(x,\cdot) \,\mathrm{d}x \in \mathrm{L}^1(\mathbb{R}^m)$, and $I_1 = I_2 = I_3$.

2.3 Dense Subspaces and Mollifier

In this section let $\Omega \subset \mathbb{R}^n$ be open.

i) $\mathrm{C}_0^\infty(\Omega)$ is dense in $\mathrm{L}^p(\Omega)$ for any $p \in [1,\infty)$, i.e. for any $u \in \mathrm{L}^p(\Omega)$ and $\varepsilon > 0$ there is $\varphi \in \mathrm{C}_0^\infty(\Omega)$ such that $\|\varphi - u\|_{\mathrm{L}^p(\Omega)} < \varepsilon$.

ii) Notation: For $\varepsilon > 0, x \in \mathbb{R}^n$, let

$$\begin{aligned}
\mathrm{B}_\varepsilon(x) &\coloneqq \{y \in \mathbb{R}^n \colon |y-x| < \varepsilon\},\\
\Omega_\varepsilon &\coloneqq \{x \in \Omega \colon \operatorname{dist}(x,\partial\Omega) > \varepsilon\}, \quad \text{and}\\
\mathrm{L}^p_{\mathrm{loc}}(\Omega) &\coloneqq \{u \colon \Omega \to [-\infty,\infty] \colon u \in \mathrm{L}^p(V) \text{ for all } V \Subset \Omega\}, \quad \text{for } p \in [1,\infty].
\end{aligned}$$

iii) Standard mollifier: Let

$$\eta(x) \coloneqq \begin{cases} c\exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1,\\ 0, & |x| \geq 1,\end{cases}$$

where $c > 0$ is chosen such that $\int_{\mathbb{R}^n} \eta(x)\,\mathrm{d}x = 1$. Then $\eta \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ is called *standard mollifier*. For $\varepsilon > 0$, $\eta_\varepsilon(x) \coloneqq \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$, $x \in \mathbb{R}^n$, satisfies $\eta_\varepsilon \in \mathrm{C}_0^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \eta_\varepsilon(x)\,\mathrm{d}x = 1$, and $\operatorname{supp}(\eta_\varepsilon) = \overline{\mathrm{B}_\varepsilon(0)}$.

iv) For $u \in \mathrm{L}^1(\Omega)$, we extend u by $u(x) \coloneqq 0$ for all $x \in \mathbb{R}^n \setminus \Omega$ to $u \in \mathrm{L}^1(\mathbb{R}^n)$ and define its *mollification* $u_\varepsilon \coloneqq \eta_\varepsilon * u$ for $\varepsilon > 0$, i.e.

$$\begin{aligned}
u_\varepsilon(x) &= \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y = \int_\Omega \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y\\
&= \int_{\mathrm{B}_\varepsilon(x)\cap\Omega} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^n.
\end{aligned}$$

The mollification has the following properties:

Theorem 2.1. *Let $u \in \mathrm{L}^1(\Omega)$ and $\varepsilon > 0$. Then the following statements hold true:*

a) $u_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n)$, $u_\varepsilon(x) \to u(x)$ as $\varepsilon \downarrow 0$ for a.e. $x \in \Omega$.

b) If $\operatorname{supp}(u) \Subset \Omega$, then $u_\varepsilon \in \mathrm{C}_0^\infty(\Omega)$ for small enough ε.

c) If $u \in \mathrm{C}^0(\Omega)$, $V \Subset \Omega$, then $u_\varepsilon \to u$ uniformly in V as $\varepsilon \downarrow 0$.

d) If $u \in \mathrm{L}^p(\Omega)$ for some $p \in [1,\infty)$, then $u_\varepsilon \in \mathrm{L}^p(\Omega)$, $\|u_\varepsilon\|_{\mathrm{L}^p(\Omega)} \leq \|u\|_{\mathrm{L}^p(\Omega)}$ and $u_\varepsilon \to u$ in $\mathrm{L}^p(\Omega)$ as $\varepsilon \downarrow 0$. Moreover, $u_\varepsilon \in \mathrm{C}^\infty(\Omega)$.

e) If $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, then $u_\varepsilon \in \mathrm{C}^\infty(\Omega_\varepsilon)$.

Proof. a) For $i \in \{1,\ldots,n\}$ and $h \in \mathbb{R}\setminus\{0\}$ let

$$\mathrm{D}_i^h v(x) \coloneqq \frac{1}{h}(v(x+he_i) - v(x)), \quad x \in \mathbb{R}^n.$$

As $\eta_\varepsilon \in \mathrm{C}_0^\infty(\mathbb{R}^n)$, we have $\nabla\eta_\varepsilon \in \mathrm{L}^\infty(\mathbb{R}^n)^n$. So $\mathrm{D}_i^h\eta_\varepsilon \in \mathrm{L}^\infty(\mathbb{R}^n)$ by the mean value theorem. As moreover $\mathrm{D}_i^h\eta_\varepsilon(z) \to \frac{\partial\eta_\varepsilon}{\partial x_i}(z)$ as $h \to 0$ for any $z \in \mathbb{R}^n$, the dominated convergence theorem implies

$$\begin{aligned}\mathrm{D}_i^h(u_\varepsilon)(x) &= \int_\Omega \frac{1}{h}(\eta_\varepsilon(x+he_i-y) - \eta_\varepsilon(x-y))u(y)\,\mathrm{d}y = \int_\Omega (\,\mathrm{D}_i^h\eta_\varepsilon(x-y))u(y)\,\mathrm{d}y \\ &\to \int_\Omega \frac{\partial\eta_\varepsilon}{\partial x_i}(x-y)u(y)\,\mathrm{d}y \quad \text{as } \varepsilon\downarrow 0.\end{aligned}$$

Hence, $\frac{\partial}{\partial x_i}u_\varepsilon(x) = \int_\Omega \frac{\partial\eta_\varepsilon}{\partial x_i}(x-y)u(y)\,\mathrm{d}y$. By induction, $u_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n)$. By Lebesgue's differentiation theorem (see [Eva10][§E.4]), for a.e. $x \in \Omega$ we have

$$\lim_{r\downarrow 0} \frac{1}{|\mathrm{B}_r(x)|}\int_{\mathrm{B}_r(x)} |u(y)-u(x)|\,\mathrm{d}y = 0. \tag{2.1}$$

For any such x we obtain (by choosing $\varepsilon > 0$ small such that $\overline{\mathrm{B}_\varepsilon(x)} \subset \Omega$)

$$\begin{aligned}|u_\varepsilon(x) - u(x)| &= \Big|\int_{\mathrm{B}_\varepsilon(x)} \eta_\varepsilon(x-y)f(y)\,\mathrm{d}y\Big| = \Big|\int_{\mathrm{B}_\varepsilon(x)} \eta_\varepsilon(x-y)(f(y)-f(x))\,\mathrm{d}y\Big| \\ &\le \frac{1}{\varepsilon^n}\int_{\mathrm{B}_\varepsilon(x)} \|\eta\|_{\mathrm{L}^\infty(\mathbb{R}^n)}|f(y)-f(x)|\,\mathrm{d}y \le \frac{C}{|\mathrm{B}_\varepsilon(x)|}\int_{\mathrm{B}_\varepsilon(x)} |f(y)-f(x)|\,\mathrm{d}y \qquad (2.2)\\ &\to 0 \quad \text{as } \varepsilon\downarrow 0\end{aligned}$$

due to (2.1).

b) If $\mathrm{supp}(u) \Subset \Omega$, let $\delta \coloneqq \mathrm{dist}(\mathrm{supp}(u), \partial\Omega) > 0$. Then for any $x \in \Omega\setminus\Omega_{\frac{\delta}{2}}$ and $\varepsilon \le \frac{\delta}{2}$ we have $\mathrm{B}_\varepsilon(x) \cap \mathrm{supp}(u) = \emptyset$ and

$$u_\varepsilon(x) = \int_{\mathrm{B}_\varepsilon(x)\cap\Omega} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y = 0.$$

Hence, $\mathrm{supp}(u_\varepsilon) \subset \overline{\Omega_{\frac{\delta}{3}}} \Subset \Omega$. By a), $u_\varepsilon \in \mathrm{C}_0^\infty(\Omega)$.

c) For $u \in \mathrm{C}^0(\Omega)$ and $V \Subset \Omega$, choose W such that $V \Subset W \Subset \Omega$. Then u is uniformly continuous in W and (2.1) holds uniformly for $x \in V$. Hence, also (2.2) is satisfied uniformly for $x \in V$ and $u_\varepsilon \to u$ uniformly in V.

d) For $x \in \Omega$, by using Hölder's inequality and $\eta_\varepsilon \ge 0$ along with $\int_{\mathbb{R}^n}\eta_\varepsilon(z)\,\mathrm{d}y = 1$, we get

$$\begin{aligned}|u_\varepsilon(x)| &= \Big|\int_\Omega (\eta_\varepsilon(x-y))^{1-\frac{1}{p}}(\eta_\varepsilon(x-y))^{\frac{1}{p}}u(y)\,\mathrm{d}y\Big| \\ &\le \underbrace{\Big(\int \eta_\varepsilon(x-y)\,\mathrm{d}y\Big)^{\frac{p-1}{p}}}_{\le 1}\Big(\int_\Omega \eta_\varepsilon(x-y)|u(y)|^p\,\mathrm{d}y\Big)^{\frac{1}{p}} \le \Big(\int_\Omega \eta_\varepsilon(x-y)|u(y)|^p\,\mathrm{d}y\Big)^{\frac{1}{p}}.\end{aligned}$$

Raising this to the power of p and integrating w.r.t $x \in \Omega$, by using Fubini we have

$$\begin{aligned}\|u_\varepsilon\|^p_{\mathrm{L}^p(\Omega)}\| &\le \int_\Omega\int_\Omega \eta_\varepsilon(x-y)|u(y)|^p\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega |u(y)|^p\underbrace{\Big(\int_\Omega \eta_\varepsilon(x-y)\,\mathrm{d}x\Big)}_{\in[0,1]}\,\mathrm{d}y \\ &\le \int_\Omega |u(y)|^p\,\mathrm{d}y = \|u\|^p_{\mathrm{L}^p(\Omega)}. \qquad (2.3)\end{aligned}$$

In particular, this implies $u_\varepsilon \in \mathrm{L}^p(\Omega)$.

Given $\mu > 0$, we may choose $\varphi \in \mathrm{C}_0^\infty(\Omega)$ such that $\|u-\varphi\|_{\mathrm{L}^p(\Omega)} < \frac{\mu}{3}$. As φ and φ_ε have compact support in Ω by *b*), we deduce from c) that $\varphi_\varepsilon \to \varphi$ uniformly in Ω as $\varepsilon \downarrow 0$. Hence, we may choose $\varepsilon_0 > 0$ small enough such that $\|\varphi_\varepsilon - \varphi\|_{\mathrm{L}^p(\Omega)} < \frac{\mu}{3}$ for all $\varepsilon \in (0, \varepsilon_0)$. But then,

$$\begin{aligned}
\|u_\varepsilon - u\|_{\mathrm{L}^p(\Omega)} &\leq \|u_\varepsilon - \varphi_\varepsilon\|_{\mathrm{L}^p(\Omega)} + \|\varphi_\varepsilon - \varphi\|_{\mathrm{L}^p(\Omega)} + \|\varphi - u\|_{\mathrm{L}^p(\Omega)} \\
&\leq \|\eta_\varepsilon * u - \eta_\varepsilon * \varphi\|_{\mathrm{L}^p(\Omega)} + \frac{2}{3}\mu \\
&= \|\eta_\varepsilon * (u-\varphi)\|_{\mathrm{L}^p(\Omega)} + \frac{2}{3}\mu = \|(u-\varphi)_\varepsilon\|_{\mathrm{L}^p(\Omega)} + \frac{2}{3}\mu \\
&\overset{(2.3)}{\leq} \|u-\varphi\|_{\mathrm{L}^p(\Omega)} + \frac{2}{3}\mu < \mu \quad \text{for all } \varepsilon \in (0, \varepsilon_0).
\end{aligned}$$

We still have $u_\varepsilon \in \mathrm{C}^\infty(\Omega)$ since for $x \in \mathrm{B}_\delta(x_0)$ with $\overline{\mathrm{B}_{2\delta}(x_0)} \subset \Omega$ we have for $x \in K := \overline{\mathrm{B}_{2\delta}(x_0)}$ and $\varepsilon \in (0, \delta)$, $u_\varepsilon(x) = \int_K \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y$. A similar argument shows e). ☐

2.4 Polar Coordinates

Let $f \in \mathrm{C}^1(\overline{\mathrm{B}_r(x_0)})$ with $x_0 \in \mathbb{R}^n$, $r > 0$. Then by the transformation rule with $x = x_0 + sz$, $s \in (0, r)$, $z \in \partial\mathrm{B}_1(0)$ we have

$$\int_{\mathrm{B}_r(x_0)} f(x)\,\mathrm{d}x = \int_0^r \left(\int_{\partial\mathrm{B}_s(x_0)} f\,\mathrm{d}\sigma(x)\right) \mathrm{d}s = \int_0^r s^{n-1} \int_{\partial\mathrm{B}_1(0)} f(x_0 + sz)\,\mathrm{d}\sigma(z)\,\mathrm{d}s.$$

In particular, if $x_0 = 0$, f is radially symmetric and ω_n is the surface $|\partial\mathrm{B}_1(0)|$ of $\partial\mathrm{B}_1(0)$, we get

$$\int_{\mathrm{B}_r(0)} f(x)\,\mathrm{d}x = \omega_n \int_0^r f(s)s^{n-1}\,\mathrm{d}s,$$

where $s = |x|$.

Proof. Appendix in [Eva10]. ☐

Chapter 3

Weak Derivatives and Definitions of Sobolev Spaces

We already saw in the introduction that for $u \in \mathrm{C}^1(\Omega)$ with $\Omega \subset \mathbb{R}^n$ open we have

$$\int_\Omega u(x)\varphi_{x_i}(x)\,\mathrm{d}x = -\int_\Omega u_{x_i}(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{3.1}$$

More generally, for higher order derivatives we have the following result:

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be open, $u \in \mathrm{C}^k(\Omega)$ with $k \in \mathbb{N}$, and $\alpha \in \mathbb{N}_0^n$ be a multiindex with $|\alpha| = k$. Then*

$$\int_\Omega u(x)\,\mathrm{D}^\alpha\varphi(x) = (-1)^{|\alpha|}\int_\Omega \mathrm{D}^\alpha u(x)\varphi(x)\,\mathrm{d}x \quad \textit{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{3.2}$$

Proof. For $k = 1$, (3.2) is just (3.1) which was verified in the exercise. For $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| = k$ we have

$$\mathrm{D}^\alpha\phi(x) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}(\dots(\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}})\dots)\phi(x)$$

and (3.2) follows by applying (3.1) k times. □

In order to define the weak derivative $\mathrm{D}^\alpha u$, we look for a variant of (3.2) which is satisfied if u has less regularity than being in $\mathrm{C}^k(\Omega)$. As the integrals in (3.2) are meaningful if u, $\mathrm{D}^\alpha u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, we define the weak derivative $\mathrm{D}^\alpha u$ of u as follows (see introduction for $|\alpha| = 1$).

Definition 3.2. Let Ω be an open set, $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0^n$ a multiindex. u has the *αth weak partial derivative* $\mathrm{D}^\alpha u$ if there is $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega u(x)\,\mathrm{D}^\alpha\varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega v(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{3.3}$$

If (3.3) is satisfied, we define $\mathrm{D}^\alpha u := v$.

In order to show the uniqueness of the weak derivative, we need the following fundamental lemma.

Lemma 3.3 (Fundamental lemma of calculus of variations)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$. Then we have the equivalence*

$$\int_\Omega u(x)\varphi(x)\,\mathrm{d}x = 0 \quad \textit{for all } \varphi \in \mathrm{C}_0^\infty(\Omega) \iff u = 0 \textit{ a.e. in } \Omega.$$

Proof. " $\Longleftarrow$ " is obvious.

" $\Longrightarrow$ ": Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ with $\int_\Omega u\varphi \,\mathrm{d}x = 0$ for all $\varphi \in \mathrm{C}_0^\infty(\Omega)$. We fix $K \subset \Omega$ compact and define

$$\operatorname{sign}(u(x)) := \begin{cases} 1 & \text{if } u(x) > 0, \\ -1 & \text{if } u(x) < 0, \\ 0 & \text{if } u(x) \in \{0, -\infty, +\infty\} \end{cases}$$

and

$$f(x) := \begin{cases} \operatorname{sign}(u(x)) & \text{if } x \in K, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus K. \end{cases}$$

As $|u| < \infty$ a.e. in K a.e., we have $u(x)f(x) = |u(x)|$ for a.e. $x \in K$. Since $f \in \mathrm{L}^\infty(\Omega)$ with $\operatorname{supp}(f) \subset K \Subset \Omega$, we define $\varphi_n := f_{\frac{1}{n}} = \eta_{\frac{1}{n}} * f$ and deduce from Theorem 2.1 a), b) that $\varphi_n \in \mathrm{C}_0^\infty(\Omega)$ and $\varphi_{n_k}(x) \to f(x)$ a.e. in Ω as $k \to \infty$ for some subsequence. As moreover

$$|\varphi_{n_k}(x)| \le \int_\Omega \eta_{\frac{1}{n_k}}(x-y)|f(y)| \,\mathrm{d}y \le \underbrace{\|f\|_{\mathrm{L}^\infty(\Omega)}}_{\le 1} \underbrace{\int_\Omega \eta_{\frac{1}{n_k}}(x-y) \,\mathrm{d}y}_{\le 1} \le 1 \quad \text{for all } x \in \Omega, k \in \mathbb{N},$$

the dominated convergence theorem implies

$$0 = \lim_{k \to \infty} \int_\Omega u(x)\varphi_{n_k}(x) \,\mathrm{d}x = \int_\Omega u(x)f(x) \,\mathrm{d}x = \int_K |u(x)| \,\mathrm{d}x.$$

Hence, $u = 0$ a.e. in K. As e.g. $\Omega = \bigcup_{k=1}^\infty K_n$ with $K_n := \overline{\Omega_{\frac{1}{n}}} \cap \overline{\mathrm{B}_n(0)}$ and $u = 0$ a.e. in K_n (as $K_n \subset \Omega$ compact), we have $u = 0$ a.e. in Ω. ☐

With this result we show the uniqueness of the weak derivative and its equality with the classical derivative if u is classically differentiable.

Lemma 3.4. *Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0^n$, with $|\alpha| = k \in \mathbb{N}$. If the weak derivative $\mathrm{D}^\alpha u$ exists it is uniquely defined up to a set of measure zero. If $u \in \mathrm{C}^k(\Omega)$, then $\mathrm{D}^\alpha u$ exists and is equal to the classical derivative $\mathrm{D}^\alpha u$. Hence, we use D^α both for weak and classical partial derivatives.*

Proof. If v and $\tilde{v}$ are αth weak derivatives of u, by (3.3)

$$\int_\Omega (v - \tilde{v})(x)\varphi(x) \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega).$$

Hence, by Lemma 3.3 $v - \tilde{v} = 0$ a.e. in Ω and $v = \tilde{v}$ a.e. in Ω. If $u \in \mathrm{C}^k(\Omega)$, then by Lemma 3.1, (3.3) is satisfied with $v = \mathrm{D}^\alpha u$ and hence the classical derivative $\mathrm{D}^\alpha u$ is also a weak derivative. Due to the uniqueness the claim follows. ☐

As the weak derivative is well-defined, we may now define Sobolev spaces consisting of functions having weak derivatives in L^p spaces.

Definition 3.5. a) Let $k \in \mathbb{N}$, $p \in [1, \infty]$ and $\Omega \subset \mathbb{R}^n$ be open. We define the Sobolev space

$$\mathrm{W}^{k,p}(\Omega) := \Big\{ u \in \mathrm{L}^p(\Omega)\colon \text{ weak derivative } \mathrm{D}^\alpha u \text{ ex. with } \mathrm{D}^\alpha u \in \mathrm{L}^p(\Omega) \text{ for all } 0 \le |\alpha| \le k \Big\}$$

with the norm

$$\|u\|_{\mathrm{W}^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha|\le k} \|\,\mathrm{D}^\alpha u\|^p_{\mathrm{L}^p(\Omega)}\right)^{\frac{1}{p}} & \text{if } p \in [1,\infty), \\ \sum_{|\alpha|\le k} \|\,\mathrm{D}^\alpha u\|_{\mathrm{L}^\infty(\Omega)} & \text{if } p = \infty. \end{cases}$$

We further define

$$\mathrm{W}^{k,p}_0(\Omega) := \overline{\mathrm{C}^\infty_0(\Omega)}^{\|\cdot\|_{\mathrm{W}^{k,p}(\Omega)}}$$

to be the closure of $\mathrm{C}^\infty_0(\Omega)$ in $\mathrm{W}^{k,p}(\Omega)$ and

$$\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega) := \bigcap_{V \Subset \Omega} W^{k,p}(V).$$

For $p = 2$, we define $\mathrm{H}^k(\Omega) := \mathrm{W}^{k,2}(\Omega)$ and $\mathrm{H}^k_0(\Omega) := \mathrm{W}^{k,2}_0(\Omega)$.

b) For $(u_m)_{m\in\mathbb{N}} \subset \mathrm{W}^{k,p}(\Omega)$ and $u \in \mathrm{W}^{k,p}(\Omega)$, we say $u_m \to u$ in $W^{k,p}(\Omega)$ if

$$\lim_{n\to\infty} \|u_m - u\|_{\mathrm{W}^{k,p}(\Omega)} = 0.$$

We say $u_m \to u$ in $\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega)$ if $u_m \to u$ in $\mathrm{W}^{k,p}(V)$ for all $V \Subset \Omega$.

Remark 3.6. a) For $\alpha = (0,\dots,0)$ we set $\mathrm{D}^\alpha u = \mathrm{D}^0 u = u$. We further identify functions in $\mathrm{W}^{k,p}(\Omega)$ which agree a.e. If for $u \in \mathrm{W}^{k,p}(\Omega)$ the equivalence class $[u]$ contains a continuous representative, the latter is chosen for u.

b) $u \in \mathrm{W}^{k,p}_0(\Omega)$ if and only if there exists $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^\infty_0(\Omega)$ such that $u_m \to u$ in $\mathrm{W}^{k,p}(\Omega)$. We interpret $W^{k,p}_0(\Omega)$ as the set of $u \in W^{k,p}(\Omega)$ such that " $\mathrm{D}^\alpha u = 0$ on $\partial\Omega$ for any $|\alpha| \le k-1$". This interpretation will be made precise in Chapter 5.

c) The letter "H" in $\mathrm{H}^k(\Omega)$ and $\mathrm{H}^k_0(\Omega)$ is used as those are Hilbert spaces as we will see soon.

Example 3.7. Let $\Omega = \mathrm{B}_1(0) \subset \mathbb{R}^n$, $u(x) = |x|^{-a}$ for $x \in \Omega \setminus \{0\}$ with some $a > 0$. Given $p \in [1,\infty)$, for which a do we have $u \in \mathrm{W}^{1,p}(\Omega)$?
Since $u \in C^\infty(\Omega \setminus \{0\})$, we have for $x \neq 0$

$$\begin{aligned} u_{x_i}(x) &= -a|x|^{-a-1}\frac{x_i}{|x|} = -\frac{a x_i}{|x|^{a+2}} \quad \text{and} \\ |\nabla u(x)| &= \frac{a}{|x|^{a+1}}. \end{aligned}$$

For fixed $\varphi \in \mathrm{C}^\infty_0(\Omega)$ and $\varepsilon > 0$, Green's formula (ν is the outward unit normal on $\Omega \setminus \overline{\mathrm{B}_\varepsilon(0)}$) implies

$$\int_{\Omega\setminus\overline{\mathrm{B}_\varepsilon(0)}} u\,\varphi_{x_i}\,\mathrm{d}x = -\int_{\Omega\setminus\overline{\mathrm{B}_\varepsilon(0)}} u_{x_i}\varphi\,\mathrm{d}x + \underbrace{\int_{\partial\Omega} u\,\varphi\,\nu_i\,\mathrm{d}\sigma}_{=0} + \int_{\partial\mathrm{B}_\varepsilon(0)} u\,\varphi\,\nu_i\,\mathrm{d}\sigma. \tag{3.4}$$

We may pass to the limit $\varepsilon \downarrow 0$ in the first two integrals if $u \in \mathrm{L}^1(\Omega)$ and $\nabla u \in \mathrm{L}^1(\Omega)^n$, i.e. $a < n$ and $a+1 < n$. As for $a < n-1$, we further have

$$\Big|\int_{\partial\mathrm{B}_\varepsilon(0)} u\,\varphi\,\nu_i\,\mathrm{d}\sigma\Big| \le \|\varphi\|_{\mathrm{L}^\infty(\Omega)} \int_{\partial\mathrm{B}_\varepsilon(0)} \varepsilon^{-a}\,\mathrm{d}\sigma \le C\varepsilon^{k-1-a} \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

Hence, for $a < n-1$ we may pass to the limit $\varepsilon \downarrow 0$ in (3.4) and obtain $\int_\Omega u\varphi_{x_i}\,\mathrm{d}x = -\int_\Omega u_{x_i}\varphi\,\mathrm{d}x$. Hence, the weak derivative u_{x_i} exists for $a < n-1$. Hence, $u \in \mathrm{W}^{1,p}(\Omega)$ if $u \in \mathrm{L}^p(\Omega)$ and $\nabla u = \frac{-ax}{|x|^{a+2}} \in \mathrm{L}^p(\Omega)^n$, i.e. $ap < p$ and $(a+1)p < n$. We conclude that

$$u \in \mathrm{W}^{1,p}(\Omega) \iff a < \frac{n-p}{p} \text{ (and } p < n).$$

Next, we prove some elementary properties of weak derivatives which are well known in the case of classical derivatives.

Proposition 3.8. *Let Ω be open, $k \in \mathbb{N}, p \in [1,\infty]$, $u, v \in \mathrm{W}^{k,p}(\Omega)$, and $\alpha \in \mathbb{N}_0^n$ with $1 \le |\alpha| \le k$.*

a) $\mathrm{D}^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ (with $\mathrm{W}^{0,p}(\Omega) = \mathrm{L}^p(\Omega)$) and $\mathrm{D}^\beta(\mathrm{D}^\alpha(u)) = \mathrm{D}^\alpha(\mathrm{D}^\beta(u)) = \mathrm{D}^{\alpha+\beta}(u)$ for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \le k$.

b) For $\lambda, \mu \in \mathbb{R}$ we have $\lambda u + \mu v \in \mathrm{W}^{k,p}(\Omega)$ and $\mathrm{D}^\alpha(\lambda u + \mu v) = \lambda \mathrm{D}^\alpha u + \mu \mathrm{D}^\alpha v$.

c) If $V \subset \Omega$ is open, then $u \in \mathrm{W}^{k,p}(V)$.

d) If $\xi \in \mathrm{C}_0^\infty(\Omega)$, then $\xi u \in \mathrm{W}^{k,p}(\Omega)$ and Leibniz's formula

$$\mathrm{D}^\alpha(\xi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \mathrm{D}^\beta \xi \, \mathrm{D}^{\alpha-\beta} u$$

holds with

$$\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha-\beta)!\beta!}, \qquad \alpha! = \prod_{i=1}^n \alpha_i!$$

and

$$\beta \le \alpha \iff \forall i \in \{1,\dots,n\}\colon \beta_i \le \alpha_i.$$

Proof. b) and c) easily follow from Definition 3.2.

a) Let $\varphi \in \mathrm{C}_0^\infty(\Omega)$. Then $\mathrm{D}^\beta \varphi \in \mathrm{C}_0^\infty(\Omega)$ and (3.3) implies

$$\begin{aligned}\int_\Omega \mathrm{D}^\alpha u \, \mathrm{D}^\beta \varphi \,\mathrm{d}x &= (-1)^{|\alpha|} \int_\Omega u \, \mathrm{D}^{\alpha+\beta} \varphi \,\mathrm{d}x \\ &= (-1)^{|\alpha|}(-1)^{|\alpha|+|\beta|} \int_\Omega \mathrm{D}^{\alpha+\beta} u \varphi \,\mathrm{d}x \\ &= (-1)^{|\beta|} \int_\Omega \mathrm{D}^{\alpha+\beta} u \varphi \,\mathrm{d}x\end{aligned}$$

as $|\alpha| + |\beta| = |\alpha + \beta|$. Hence, $\mathrm{D}^\beta(\mathrm{D}^\alpha u) = \mathrm{D}^{\alpha+\beta} u$, for $|\beta| \le k - |\alpha|$.

d) Let $\varphi \in \mathrm{C}_0^\infty(\Omega)$. In case of $|\alpha| = 1$, we have

$$\int_\Omega \xi u \, \mathrm{D}^\alpha \varphi \,\mathrm{d}x = \int_\Omega \left(u \, \mathrm{D}^\alpha(\xi\varphi) - u(\mathrm{D}^\alpha \xi)\varphi\right) \,\mathrm{d}x \overset{(3.3)}{=} -\int_\Omega \left(\xi \, \mathrm{D}^\alpha u + u \, \mathrm{D}^\alpha \xi\right) \varphi \,\mathrm{d}x.$$

Hence, $\mathrm{D}^\alpha(\xi u) = \xi \, \mathrm{D}^\alpha u + u \, \mathrm{D}^\alpha \xi \in \mathrm{L}^p(\Omega)$ and the claim is true for $|\alpha| = 1$.

Assume the claim is true for all $|\alpha| \le l$ with some $l \in \{1,\dots,k-1\}$ (IA).

Let α satisfy $|\alpha| = l+1$. Then $\alpha = \beta + \gamma$ for some $\beta, \gamma \in \mathbb{N}_0^k$ with $|\beta| = l$ and $|\gamma| = 1$. Hence,

$$
\begin{aligned}
\int_\Omega \xi u \,\mathrm{D}^\alpha \varphi &= \int_\Omega \xi u \,\mathrm{D}^\beta (\,\mathrm{D}^\gamma \varphi) \,\mathrm{d}x \\
&\overset{\text{(IA)}}{=} (-1)^{|\beta|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \mathrm{D}^\sigma \xi \; \mathrm{D}^{\beta-\sigma} u \; \mathrm{D}^\gamma \varphi \,\mathrm{d}x \\
&\overset{(3.3)}{=} (-1)^{|\beta|+|\gamma|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \mathrm{D}^\gamma \left(\mathrm{D}^\sigma \xi \; \mathrm{D}^{\beta-\sigma} u \right) \varphi \,\mathrm{d}x \\
&\overset{\text{(IA)}}{=} (-1)^{|\alpha|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \left[\mathrm{D}^{\sigma+\gamma} \xi \; \mathrm{D}^{\beta-\sigma} u + \mathrm{D}^\sigma \xi \; \mathrm{D}^{\alpha-\sigma} u \right] \varphi \,\mathrm{d}x \\
&= (-1)^{|\alpha|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \left[\mathrm{D}^{\sigma+\gamma} \xi \; \mathrm{D}^{\alpha-(\sigma+\gamma)} u + \mathrm{D}^\sigma \xi \; \mathrm{D}^{\alpha-\sigma} u \right] \varphi \,\mathrm{d}x \\
&= (-1)^{|\alpha|} \int_\Omega \left[\sum_{\gamma \le \rho \le \alpha} \binom{\beta}{\rho - \gamma} + \sum_{0 \le \rho \le \beta} \binom{\beta}{\rho} \right] \mathrm{D}^\rho \xi \; \mathrm{D}^{\alpha-\rho} u \, \varphi \,\mathrm{d}x \\
&= (-1)^{|\alpha|} \int_\Omega \sum_{\rho \le \alpha} \left[\binom{\beta}{\rho - \gamma} + \binom{\beta}{\rho} \right] \mathrm{D}^\rho \xi \; \mathrm{D}^{\alpha-\rho} u \, \varphi \,\mathrm{d}x
\end{aligned}
$$

with the convention $\binom{\beta}{\tilde{\beta}} = 0$ if $\beta_i < \tilde{\beta}_i$ or $\tilde{\beta}_i < 0$ for some $i \in \{1, \dots, n\}$. As

$$
\binom{\beta}{\rho - \gamma} + \binom{\beta}{\rho} = \binom{\beta + \gamma}{\rho} = \binom{\alpha}{\rho},
$$

we deduce that the claim holds by induction. □

Finally, we show that $\mathrm{W}^{k,p}(\Omega)$ is a Banach space.

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^n$ be open, $k \in \mathbb{N}$, $p \in [1, \infty]$. Then $\mathrm{W}^{k,p}(\Omega)$ is a Banach space. Moreover, $\mathrm{H}^k(\Omega)$ is a Hilbert space with the scalar product*

$$
(u, v)_{\mathrm{H}^k(\Omega)} := \sum_{|\alpha| \le k} \int_\Omega \mathrm{D}^\alpha u \, \overline{\mathrm{D}^\beta v} \,\mathrm{d}x.
$$

Proof. By Proposition 3.8 b), $\mathrm{W}^{k,p}(\Omega)$ is a vector space. For $p \in [1, \infty)$ and $u, v \in \mathrm{W}^{k,p}(\Omega)$, Minkowski's inequality (see 2.1) on $\mathrm{L}^p(\Omega)$ and for $\|\cdot\|_p$ on $\mathbb{R}^m$ implies

$$
\begin{aligned}
\|u + v\|_{W^{k,p}(\Omega)} &= \left(\sum_{|\alpha| \le k} \| \mathrm{D}^\alpha u + \mathrm{D}^\alpha v\|_{\mathrm{L}^p(\Omega)}^p \right)^{\frac{1}{p}} \le \left(\sum_{|\alpha| \le k} \left(\| \mathrm{D}^\alpha u\|_{\mathrm{L}^p(\Omega)} + \| \mathrm{D}^\alpha v\|_{\mathrm{L}^p(\Omega)} \right)^p \right)^{\frac{1}{p}} \\
&\le \left(\sum_{|\alpha| \le k} \| \mathrm{D}^\alpha u\|_{\mathrm{L}^p(\Omega)}^p \right)^{\frac{1}{p}} + \left(\sum_{|\alpha| \le k} \| \mathrm{D}^\alpha v\|_{\mathrm{L}^p(\Omega)}^p \right)^{\frac{1}{p}} = \|u\|_{\mathrm{W}^{k,p}(\Omega)} + \|v\|_{\mathrm{W}^{k,p}(\Omega)}.
\end{aligned}
$$

All other properties of the norm are easily verified for $\|\cdot\|_{\mathrm{W}^{k,p}(\Omega)}$.
Let $(u_m)_{m \in \mathbb{N}}$ be a Cauchy sequence in $\mathrm{W}^{k,p}(\Omega)$. Then for all $|\alpha| \le k$, $(\,\mathrm{D}^\alpha u_m)_{m \in \mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^p(\Omega)$ as $\| \mathrm{D}^\alpha u_m - \mathrm{D}^\alpha u_l\|_{\mathrm{L}^p(\Omega)} \le \|u_m - u_l\|_{\mathrm{W}^{k,p}(\Omega)}$. Hence, there exists $u_\alpha \in \mathrm{L}^p(\Omega)$ with

$$
\mathrm{D}^\alpha u_m \to u_\alpha \text{ in } \mathrm{L}^p(\Omega), |\alpha| \le k. \tag{3.5}
$$

For $\alpha = (0, \ldots, 0)$ we define $u_{(0,\ldots,0)} =: u$ and have

$$u_m \to u \text{ in } \mathrm{L}^p(\Omega). \tag{3.6}$$

To show that $u_\alpha = \mathrm{D}^\alpha u$, we fix $\varphi \in \mathrm{C}_0^\infty(\Omega)$ and obtain

$$\int_\Omega u\, \mathrm{D}^\alpha \varphi \,\mathrm{d}x \overset{(3.6)}{=} \lim_{m\to\infty} \int_\Omega u_m\, \mathrm{D}^\alpha \varphi \,\mathrm{d}x = \lim_{m\to\infty} (-1)^{|\alpha|} \int_\Omega \mathrm{D}^\alpha u_m \varphi \,\mathrm{d}x \overset{(3.5)}{=} (-1)^{|\alpha|} \int_\Omega u_\alpha \varphi \,\mathrm{d}x$$

since φ, $\mathrm{D}^\alpha \varphi \in \mathrm{L}^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Hence, $\mathrm{D}^\alpha u = u_\alpha \in \mathrm{L}^p(\Omega)$ for all $|\alpha| \leq k$ and $u \in \mathrm{W}^{k,p}(\Omega)$. But then (3.5) and (3.6) imply $u_m \to u$ in $\mathrm{W}^{k,p}(\Omega)$. Hence, $\mathrm{W}^{k,p}(\Omega)$ is complete and a Banach space.

That $(\cdot, \cdot)_{\mathrm{H}^k(\Omega)}$ is a scalar product on $\mathrm{H}^k(\Omega)$ easily follows from the L^2-scalar product. Hence, $\mathrm{H}^k(\Omega)$ is a Hilbert space. $\square$

In particular, $\mathrm{W}_0^{k,p}(\Omega)$ is a Banach space and a subspace of $\mathrm{W}^{k,p}(\Omega)$.

Chapter 4

Approximation by Smooth Functions

As it is often complicated to use the definition of weak derivatives for proving properties of Sobolev spaces, we aim to approximate functions in Sobolev spaces by smooth functions.

4.1 Interior Approximation

We prove that mollification from 2.3 provides approximating functions in $\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega)$.

Theorem 4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $k \in \mathbb{N}$, $p \in [1, \infty)$, and $u \in \mathrm{W}^{k,p}(\Omega)$. Then the following statements hold:*

a) $u_\varepsilon \in \mathrm{C}^\infty(\Omega)$ and $\mathrm{D}^\alpha(u_\varepsilon)(x) = (\mathrm{D}^\alpha u)_\varepsilon(x)$ for all $x \in \Omega_\varepsilon$ and all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$.

b) $u_\varepsilon \to u$ in $\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega)$ as $\varepsilon \downarrow 0$.

Proof. a) By Theorem 2.1, we have $u_\varepsilon \in \mathrm{C}^\infty(\Omega)$ and for $|\alpha| \le k$

$$\mathrm{D}^\alpha(u_\varepsilon)(x) = \int_\Omega \mathrm{D}_x^\alpha \eta_\varepsilon(x-y) u(y) \,\mathrm{d}y, \quad x \in \Omega,$$

see proof of Theorem 2.1 a), d). For fixed $x \in \Omega_\varepsilon$, $\phi(y) := \eta_\varepsilon(x-y)$ satisfies $\phi \in \mathrm{C}_0^\infty(\Omega)$ since $\operatorname{supp}\phi = \overline{\mathrm{B}_\varepsilon(x)}$ and therefore

$$\begin{aligned}\mathrm{D}^\alpha(u_\varepsilon)(x) &= (-1)^{|\alpha|} \int_\Omega \mathrm{D}_y^\alpha(\eta_\varepsilon(x-y))\, u(y) \,\mathrm{d}y = (-1)^{|\alpha|} \int_\Omega \mathrm{D}^\alpha \phi(y)\, u(y) \,\mathrm{d}y \\ &\overset{(3.3)}{=} (-1)^{|\alpha|+|\alpha|} \int_\Omega \phi(y)\, \mathrm{D}^\alpha u(y) \,\mathrm{d}y = \int_\Omega \eta_\varepsilon(x-y)\, \mathrm{D}^\alpha u(y) \,\mathrm{d}y = (\mathrm{D}^\alpha u)_\varepsilon(x).\end{aligned}$$

Since $x \in \Omega_\varepsilon$ was arbitrary, this proves a).

b) In view of a) and Theorem 2.1 d), for fixed $V \Subset \Omega$ we have $\mathrm{D}^\alpha u_\varepsilon = \eta_\varepsilon * \mathrm{D}^\alpha u$ in V for $\varepsilon \in (0, \varepsilon_0)$, as $V \subset \Omega_\varepsilon$ for ε small enough so that $\mathrm{D}^\alpha u_\varepsilon \to \mathrm{D}^\alpha u$ in $\mathrm{L}^p(V)$ as $\varepsilon \downarrow 0$ for any $\alpha \in \mathbb{N}_0^n, |\alpha| \le k$. Then

$$\|u_\varepsilon - u\|^p_{\mathrm{W}^{k,p}(V)} = \sum_{|\alpha| \le k} \|\mathrm{D}^\alpha u_\varepsilon - \mathrm{D}^\alpha u\|^p_{\mathrm{L}^p(V)} \to 0 \quad \text{as } \varepsilon \downarrow 0. \qquad \square$$

4.2 Approximation by Smooth Functions

In order to show that for any $u \in \mathrm{W}^{k,p}(\Omega)$ there is $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ such that $u_m \to u$ in $\mathrm{W}^{k,p}(\Omega)$ (and not only in $\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega)$), we need the following lemmas to construct a partition of unity.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open and $K \subset \Omega$ compact. If $\operatorname{dist}(K, \partial\Omega) \geq \delta > 0$, then there exists a cutoff-function $\tau \in \mathrm{C}_0^\infty(\Omega)$ w.r.t. K, Ω with $0 \leq \tau \leq 1$, $\tau = 1$ in K, and*

$$|\,\mathrm{D}^\alpha \tau(x)| \leq c\delta^{-k} \quad \text{for all } x \in \Omega \setminus K, k \in \mathbb{N}, |\alpha| = k,$$

where $c > 0$ depends on k and n but not on Ω or K.

Proof. We may choose $\delta > 0$ since K is compact. Hence,

$$\tilde{K} := \overline{\bigcup_{x\in K} \mathrm{B}_{\frac{\delta}{2}}(x)}$$

is compact with $\operatorname{dist}(\partial\tilde{K}, \partial K) = \frac{\delta}{2} \leq \operatorname{dist}(\delta\tilde{K}, \partial\Omega)$.
As $\chi_{\tilde{K}} \in \mathrm{L}^1(\Omega)$ with $\operatorname{supp}\chi_{\tilde{K}} = \tilde{K} \Subset \Omega$, we have that $\tau := \eta_{\frac{\delta}{4}} * \chi_{\tilde{K}}$ satisfies $\tau \in \mathrm{C}_0^\infty(\Omega)$, $0 \leq \tau \leq 1$, and $\tau = 1$ in K by Theorem 2.1, as

$$\tau(x) = \int_{\mathrm{B}_{\frac{\delta}{4}}(x)} \eta_{\frac{\delta}{4}}(x-y) \underbrace{\chi_{\tilde{K}}(y)}_{=1} \,\mathrm{d}y = 1 \quad \text{for all } x \in K$$

since $\mathrm{B}_{\frac{\delta}{4}}(x) \subset \tilde{K}$. Moreover, for $|\alpha| = k$

$$\mathrm{D}^\alpha \eta_{\frac{\delta}{4}}(x) = \left(\frac{4}{\delta}\right)^n \mathrm{D}^\alpha \left[\eta\left(\frac{4}{\delta}x\right)\right] = \left(\frac{4}{\delta}\right)^{n+k} (\,\mathrm{D}^\alpha \eta)\left(\frac{4}{\delta}x\right).$$

Hence, for $x \in \Omega \setminus K$ we have

$$|\,\mathrm{D}^\alpha \tau(x)| \leq \int_{\mathrm{B}_{\frac{\delta}{4}}(x)} \left(\frac{4}{\delta}\right)^{n+k} \|\,\mathrm{D}^\alpha \eta\|_{\mathrm{L}^\infty(\mathbb{R}^n)}\, \chi_{\tilde{K}}(y) \,\mathrm{d}y \leq \tilde{c}(n,k)\, \delta^{-n-k}\, |\mathrm{B}_{\frac{\delta}{4}}(x)| \leq c(n,k)\, \delta^{-k}$$

which concludes the proof. ☐

Lemma 4.3 (Partition of unity)**.** *Let $K \subset \mathbb{R}^n$ be compact and $\{\Omega_k\}_{k=1,\ldots,N}$ be an open covering of K. Then, there exist $\psi_k, k = 1, \ldots, N$, called* partition of unity *such that $\psi_k \in \mathrm{C}_0^\infty(\Omega_k)$, $0 \leq \psi_k \leq 1$ in Ω_k, and $\sum_{k=1}^N \psi_k(x) = 1$ for all $x \in K$.*

Proof. For any $x \in K$ there is $r = r(x) > 0$ and $1 \leq k \leq N$ such that $\mathrm{B}_{x,k} := \mathrm{B}_r(x) \Subset \Omega_k$. Hence,

$$\{B_{x,k}\}_{\substack{x\in K\cap\Omega_k,\\ k=1,\ldots,N}}$$

is an open covering of K and has a finite subset still covering K, called

$$\{B_i^k\}_{\substack{i=1,\ldots,N_k\\ k=1,\ldots,N}}.$$

Then

$$K_k := \overline{\bigcup_{i=1}^{N_k} B_i^k}$$

satisfies $K_k \Subset \Omega_k$ and $\bigcup_{k=1}^N K_k \supset K$. Let $\tilde{\psi}_k$ denote the cutoff-function w.r.t K_k, Ω_k. Hence, $\tilde{\psi}_k \in \mathrm{C}_0^\infty(\Omega_k)$ satisfies $0 \le \tilde{\psi}_k \le 1$ and

$$\psi(x) := \sum_{k=1}^N \tilde{\psi}_k(x) \ge 1 \quad \text{for all } x \in K.$$

Furthermore, we have

$$K \Subset \Omega := \bigcup_{k=1}^N \operatorname{supp}(\tilde{\psi}_k)$$

and there is an open set Ω_0 such that $K \subset \Omega_0 \Subset \Omega$. Let τ be a cutoff-function w.r.t K, Ω_0 and

$$\psi_k(x) := \begin{cases} \frac{\tilde{\psi}_k(x)\tau(x)}{\psi(x)} & \text{if } x \in \Omega_0, \\ 0 & \text{if } x \notin \Omega_0. \end{cases}$$

Then $\psi_1, \dots, \psi_N$ have the claimed properties. $\square$

Now we prove the announced result that $\mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ is dense in $\mathrm{W}^{k,p}(\Omega)$ without assuming any smoothness of $\partial\Omega$.

Theorem 4.4 (Meyers and Serrin)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, $k \in \mathbb{N}$, and $p \in [1, \infty)$. Then $\mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ is dense in $\mathrm{W}^{k,p}(\Omega)$, i.e. for any $u \in \mathrm{W}^{k,p}(\Omega)$ there exists $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ such that $u_m \to u$ in $\mathrm{W}^{k,p}(\Omega)$ as $m \to \infty$.*

Proof. i) With

$$U_i := \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \frac{1}{i} \text{ and } |x| < i\}, \quad i \in \mathbb{N},$$

we have $\bigcup_{i=1}^\infty U_i = \Omega$ and $U_i \subset U_{i+1}$. Moreover,

$$V_i := U_{i+4} \setminus \overline{U_{i+1}}, i \in \mathbb{N}, \quad \text{and} \quad V_0 := U_4$$

are all open with $V_i \Subset \Omega$ for all $i \in \mathbb{N}_0$ and $\Omega = \bigcup_{i=0}^\infty V_i$. Defining further

$$W_i := \overline{U_{i+3}} \setminus U_{i+2}, i \in \mathbb{N}, \quad \text{and} \quad W_0 := \overline{U}_3,$$

all $W_i \subset V_i$ are compact and we have $\Omega = \bigcup_{i=0}^\infty W_i$. Let $\psi_i \in \mathrm{C}_0^\infty(V_i)$ denote a cutoff-function w.r.t. W_i, V_i with $0 \le \psi_i \le 1$ and $\psi_i = 1$ in W_i for $i \in \mathbb{N}_0$. Since for all $j \ge i+2$

$$W_i \cap V_j = \left(\overline{U_{i+3}} \cap \overline{U_{j+1}}^{\mathrm{c}}\right) \cap (U_{i+2}^{\mathrm{c}} \cap U_{j+4}) = \emptyset,$$

and for all $j \ge i+3$, $V_i \cap V_j = \emptyset$, for any $x \in \Omega$ we have

$$\sigma(x) := \sum_{i=0}^\infty \psi_i(x) > 0$$

and only finitely many of the $\psi_i(x)$ are non-zero. Hence, $\{\xi_i\}_{i=0}^\infty$, defined by

$$\xi_i(x) := \frac{\psi_i(x)}{\sigma(x)}, \quad x \in \Omega,$$

is a *partition of unity subordinate to* $\{V_i\}_{i=0}^\infty$, i.e. $\xi_i \in \mathrm{C}_0^\infty(V_i)$, $0 \le \xi_i \le 1$, and $\sum_{i=0}^\infty \xi_i = 1$ in Ω and for any $K \Subset \Omega$, $\xi_i|_K \not\equiv 0$ only for finitely many i.

ii) Let $u \in \mathrm{W}^{k,p}(\Omega)$ be arbitrary. Then by Proposition 3.8 d) and i) we have $\xi_i u \in \mathrm{W}^{k,p}(\Omega)$ and $\operatorname{supp}(\xi_i u) \subset V_i$ for all $i \in \mathbb{N}_0$. We fix $\delta > 0$. Then for any $i \in \mathbb{N}_0$ we define

$$Z_i := U_{i+5} \setminus \overline{U_i} \supset V_i, i \in \mathbb{N}, \quad \text{and} \quad Z_0 := U_5 \supset V_0.$$

In view of Theorem 4.1, there is $\varepsilon_i > 0$ small enough such that $u_i := \eta_{\varepsilon_i} * (\xi_i u)$ satisfies $u_i \in \mathrm{C}_0^\infty(Z_i)$ and

$$\|u_i - \xi_i u\|_{W^{k,p}(\Omega)} = \|u_i - \xi_i u\|_{W^{k,p}(Z_i)} \leq \frac{\delta}{2^{i+1}} \tag{4.1}$$

for $i \in \mathbb{N}_0$ as $u_i - \xi_i u \equiv 0$ in $\Omega \setminus Z_i$. Define

$$v(x) := \sum_{i=0}^{\infty} u_i(x), \quad x \in \Omega.$$

Then for any open set $V \Subset \Omega$ only finitely many u_i satisfy $u_i|_V \not\equiv 0$. Since $u = \sum_{i=0}^\infty \xi_i u$, we obtain $v \in \mathrm{C}^\infty(\Omega)$ and

$$\|v - u\|_{\mathrm{W}^{k,p}(V)} \leq \sum_{i=0}^{\infty} \|u_i - \xi_i u\|_{\mathrm{W}^{k,p}(V)} \overset{(4.1)}{\leq} \delta \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = \delta \quad \text{for all } V \Subset \Omega.$$

Since $U_i \subset U_{i+1}$ for all $i \in \mathbb{N}$, $U_i \Subset \Omega$, and $\Omega = \bigcup_{i=1}^\infty U_i$, we conclude by the monotone convergence theorem

$$\|v - u\|^p_{\mathrm{W}^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|\,\mathrm{D}^\alpha(v - u)\|^p_{\mathrm{L}^p(\Omega)} = \lim_{i \to \infty} \sum_{|\alpha| \leq k} \|\,\mathrm{D}^\alpha(v - u)\|^p_{\mathrm{L}^p(U_i)} \leq \delta^p.$$

As $\delta > 0$ was arbitrary, the claim is proved. ☐

Remark 4.5. Historically, there were two definitions of Sobolev spaces. $\mathrm{W}^{k,p}(\Omega)$ was defined as in Definition 3.5 while $\mathrm{H}^{k,p}(\Omega)$ was defined as the closure of $\mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ w.r.t $\|\cdot\|_{\mathrm{W}^{k,p}(\Omega)}$. Obviously $\mathrm{H}^{k,p}(\Omega) \subseteq \mathrm{W}^{k,p}(\Omega)$ but only after Meyers and Serrin in 1964 [MS64] it was clear that $\mathrm{H}^{k,p}(\Omega) \supseteq \mathrm{W}^{k,p}(\Omega)$ without assuming any smoothness condition of $\partial\Omega$.

We can now prove the chain rule for $\mathrm{W}^{1,p}(\Omega)$ functions.

Proposition 4.6. *Let Ω be open, $p \in [1, \infty)$, and $f \in \mathrm{C}^1(\mathbb{R})$ such that $|f'| \leq M$ on $\mathbb{R}$ for some $M > 0$. Assume further that $f(0) = 0$ or $|\Omega| < \infty$ is satisfied. Then for any $u \in \mathrm{W}^{1,p}(\Omega)$ we have $f(u)$ in $\mathrm{W}^{1,p}(\Omega)$ with*

$$\nabla f(u) = f'(u)\nabla u.$$

Proof. As f' is continuous and bounded and u is measurable, we have $f'(u) \in \mathrm{L}^\infty(\Omega)$ and $f'(u)\nabla u \in \mathrm{L}^p(\Omega)$. In view of

$$|f(x)| \leq |f(0)| + M|x| \quad \text{for all } x \in \mathbb{R},$$

the assumption $f(0) = 0$ or $|\Omega| < \infty$ implies $f(u) \in \mathrm{L}^p(\Omega)$. By Theorem 4.4, there exists $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{1,p}(\Omega)$ such that $u_m \to u$ in $\mathrm{W}^{1,p}(\Omega)$. Hence, $u_m \to u$ and $(u_m)_{x_i} \to u_{x_i}$ in $\mathrm{L}^p(\Omega)$ for all $i \in \{1, \ldots, n\}$ and $u_m \to u$ a.e. in Ω. We fix $i \in \{1, \ldots, n\}$ and $\varphi \in \mathrm{C}_0^\infty(\Omega)$. In view of $f(u_m) \in \mathrm{C}^1(\Omega)$, we deduce from (3.2)

$$\int_\Omega f(u_m)\, \varphi_{x_i} \,\mathrm{d}x = -\int_\Omega f'(u_m)(u_m)_{x_i}\, \varphi \,\mathrm{d}x \quad \text{for all } m \in \mathbb{N}, \tag{4.2}$$

by the classical chain rule. On the one hand, for $q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have by Hölder's inequality

$$\begin{aligned}\Big| \int_\Omega (f(u_m) - f(u))\, \varphi_{x_i} \,\mathrm{d}x \Big| &\leq M \int_\Omega |u_m - u|\, |\varphi_{x_i}| \,\mathrm{d}x \\ &\leq M\, \|u_m - u\|_{\mathrm{L}^p(\Omega)} \|\varphi_{x_i}\|_{\mathrm{L}^q(\Omega)} \to 0 \quad \text{as } m \to \infty\end{aligned}$$

since f is Lipschitz. On the other hand,

$$|f'(u_m) - f'(u)|\, |u_{x_i}|\, |\varphi| \leq 2M\, |u_{x_i}| \|\varphi\|_{\mathrm{L}^\infty(\Omega)} \in \mathrm{L}^1(\mathrm{supp}(\varphi))$$

as $\mathrm{supp}(\varphi)$ is bounded and thus

$$\begin{aligned}&\Big| \int_\Omega (f'(u_m)(u_m)_{x_i} - f'(u)u_{x_i})\, \varphi \,\mathrm{d}x \Big| \\ &\leq \int_\Omega |f'(u_m)|\, |(u_m)_{x_i} - u_{x_i}|\, |\varphi| \,\mathrm{d}x + \int_\Omega |f'(u_m) - f'(u)|\, |u_{x_i}|\, |\varphi| \,\mathrm{d}x \\ &\leq M\, \|(u_m)_{x_i} - u_{x_i}\|_{\mathrm{L}^p(\Omega)} \|\varphi\|_{\mathrm{L}^q(\Omega)} + \int_{\mathrm{supp}(\varphi)} |f'(u_m) - f'(u)|\, |u_{x_i}|\, |\varphi| \,\mathrm{d}x. \\ &\to 0 \quad \text{as } m \to \infty\end{aligned}$$

by the dominated convergence theorem.
Hence, letting $m \to \infty$ in (4.2) we conclude that $(f(u))_{x_i} = f'(u)u_{x_i}$ in the weak sense. ☐

Unlike for classical derivatives, now $u \in \mathrm{W}^{1,p}(\Omega)$ implies $|u| \in \mathrm{W}^{1,p}(\Omega)$.

Corollary 4.7. *Let $\Omega \subset \mathbb{R}^n$ be open, $p \in [1, \infty)$, and $u \in \mathrm{W}^{1,p}(\Omega)$. Define*

$$u_+(x) := \max\{u(x), 0\} \quad \textit{and} \quad u_-(x) = \max\{-u(x), 0\}.$$

Then $u_+, u_-, |u| \in \mathrm{W}^{1,p}(\Omega)$ with $\nabla u_+(x) = \nabla u(x) \chi_{\{u>0\}}(x)$, $\nabla u_-(x) = -\nabla u(x) \chi_{\{u<0\}}(x)$, and $\nabla |u|(x) = \nabla u(x) \left(\chi_{\{u>0\}}(x) - \chi_{\{u<0\}}(x) \right)$.

Proof. Exercise. ☐

4.3 Approximation by $\mathrm{C}^\infty(\overline{\Omega})$-Functions

We now ask the question whether any $u \in \mathrm{W}^{k,p}(\Omega)$ can also be approximated by functions $u_m \in \mathrm{C}^\infty(\overline{\Omega})$ instead of $u_m \in \mathrm{C}^\infty(\Omega)$. The following example shows that this is not true for all open $\Omega \subset \mathbb{R}^n$.

Example 4.8. Let $\Omega = \{(x, y) \in \mathbb{R}^2 \colon 0 < |x| < 1, 0 < y < 1\}$ and $p \in [1, \infty)$. Then $u \colon \Omega \to \mathbb{R}$ defined by

$$u(x, y) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0 \end{cases}$$

belongs to $\mathrm{W}^{1,p}(\Omega)$, but for $\varepsilon > 0$ sufficiently small there is no $\varphi \in \mathrm{C}^1(\overline{\Omega})$ such that $\|\varphi - u\|_{\mathrm{W}^{1,p}(\Omega)} < \varepsilon$. (Exercise)

The problem with Ω in the example is that it lies on both sides of the segment $\Gamma = \{(0, y) \colon y \in [0, 1]\}$ with $\gamma \subset \partial\Omega$. The following condition excludes this situation. Moreover, we assume from now on that Ω is a domain, i.e. open and connected.

Definition 4.9. Let $\Omega \subset \mathbb{R}^n$ be a domain. We say that Ω satisfies the *segment condition* if for any $x \in \partial\Omega$ there exists a neighborhood $U_x \subset \mathbb{R}^n$ of x and $0 \neq y_x \in \mathbb{R}^n$ such that $z + ty_x \in \Omega$ for any $z \in \overline{\Omega} \cap U_x$ and any $t \in (0,1)$.

Another condition on $\partial\Omega$ is that it is locally the graph of a C^m function.

Definition 4.10. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $m \in \mathbb{N}$. We say that Ω *is of class* C^m or simply $\partial\Omega \in \mathrm{C}^m$ if for any $x^0 \in \partial\Omega$ there exist $r = r(x^0) > 0$ and $\gamma = \gamma_{x^0} \in \mathrm{C}^m(\mathbb{R}^{n-1})$ such that, upon relabelling and reorienting the coordinate axes if necessary, we have

$$\begin{aligned} \Omega \cap \mathrm{B}_r(x^0) &= \{x \in \mathrm{B}_r(x^0) \colon x_n > \gamma(x_1, \dots, x_{n-1})\}, \\ \partial\Omega \cap \mathrm{B}_r(x^0) &= \{x \in \mathrm{B}_r(x^0) \colon x_n = \gamma(x_1, \dots, x_{n-1})\}. \end{aligned}$$

Furthermore, we say $\partial\Omega \in \mathrm{C}^\infty$ if $\gamma \in \mathrm{C}^\infty$ and we say $\partial\Omega$ *is analytic* if γ is analytic.

Remark 4.11. Let Ω be a bounded domain with $\partial\Omega \in \mathrm{C}^1$. Then for any $x^0 \in \partial\Omega$ there is a unique outward unit vector $\nu(x^0)$, i.e. $|\nu| = 1$, $\nu(x^0) \perp y$ for all $y \in \mathrm{T}(x^0)$, where $\mathrm{T}(x^0)$ is the tangential space on $\partial\Omega$ in x^0 and $x^0 + t\nu(x^0) \notin \overline{\Omega}$ for all $t \in (0, \varepsilon_0)$ for $\varepsilon_0 > 0$ small.
Indeed, as

$$\partial\Omega \cap \mathrm{B}_r(x^0) = \left\{ x \in \mathrm{B}_r(x^0) \colon x = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ \gamma(x_1, \dots, x_{n-1}) \end{pmatrix} \right\}$$

we have that

$$\mathrm{T}(x) = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \gamma_{x_1}(x_1, \dots, x_{n-1}) \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \gamma_{x_{n-1}}(x_1, \dots, x_{n-1}) \end{pmatrix} \right\}$$

is $(n-1)$-dimensional so that $\mathrm{T}(x)^\perp$ is one-dimensional. Hence, $\nu(x)$ is uniquely defined and $\nu \colon \partial\Omega \to \mathbb{R}^n$ is continuous as $\gamma \in \mathrm{C}^1(\mathbb{R}^{n-1})$ and

$$\nu(x) = \frac{1}{\sqrt{1 + |\nabla\gamma|^2}} (\gamma_{x_1}, \dots, \gamma_{x_{n-1}}, -1)^{\mathrm{T}}.$$

In particular, Ω satisfies the segment condition with $y_x = -\alpha\nu(x)$ and $U_x = \mathrm{B}_\rho(x)$ with some $\rho \in (0, r)$ and $\alpha > 0$ small enough (as ν is continuous). ☐

Next we prove that in fact $\mathrm{C}^\infty(\overline{\Omega})$ is dense in $\mathrm{W}^{k,p}(\Omega)$ if Ω satisfies the segment condition. As a final prepariation we need the continuity of the translation in $\mathrm{L}^p(\mathbb{R}^n)$.

Proposition 4.12. *Let* $p \in [1, \infty)$ *and* $u \in \mathrm{L}^p(\mathbb{R}^n)$. *Then the translation is continuous in* $\mathrm{L}^p(\mathbb{R}^n)$ *in the sense that we have (with* $h \in \mathbb{R}^n$*)*

$$\lim_{|h| \to 0} \|u(\cdot + h) - u(\cdot)\|_{\mathrm{L}^p(\mathbb{R}^n)} = 0.$$

Proof. Given $\delta > 0$, by 2.3 there is $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ such that

$$\|u - \varphi\|_{\mathrm{L}^p(\mathbb{R}^n)} < \frac{\delta}{3}.$$

But then also

$$\|u(\cdot+h)-\varphi(\cdot+h)\|_{\mathrm{L}^p(\mathbb{R}^n)} = \|u-\varphi\|_{\mathrm{L}^p(\mathbb{R}^n)} < \frac{\delta}{3}.$$

Since φ has compact support, it is uniformly continuous on $\mathbb{R}^n$. Hence, there is $M > 0$ such that

$$|\varphi(x+h)-\varphi(x) < \frac{\delta}{3|\operatorname{supp}(\varphi)|^{\frac{1}{p}}} \quad \text{for all } x\in\mathbb{R}^n,\ h\in \mathrm{B}_M(0).$$

Hence, for $h \in \mathbb{R}^n$ with $|h| < M$ we have

$$\begin{aligned}\|u(\cdot+h)-u(\cdot)\|_{\mathrm{L}^p(\mathbb{R}^n)} &\le \|u(\cdot+h)-\varphi(\cdot+h)\|_{\mathrm{L}^p(\mathbb{R}^n)} + \|\varphi(\cdot+h)-\varphi(\cdot)\|_{\mathrm{L}^p(\mathbb{R}^n)} + \|\varphi-u\|_{\mathrm{L}^p(\mathbb{R}^n)} \\ &\le \frac{2}{3}\delta + \|\varphi(\cdot+h)-\varphi(\cdot)\|_{\mathrm{L}^\infty(\mathbb{R}^n)}|\operatorname{supp}(\varphi)|^{\frac{1}{p}} \\ &< \delta\end{aligned}$$

and the claim follows. ☐

Theorem 4.13. *Let $\Omega\subset\mathbb{R}^n$ be a domain satisfying the segment condition, $k\in\mathbb{N}$, and $p\in[1,\infty)$. Then the set $\{\varphi|_\Omega : \varphi\in \mathrm{C}_0^\infty(\mathbb{R}^n)\}$ is dense in $\mathrm{W}^{k,p}(\Omega)$. In particular, if in addition $\Omega\neq\mathbb{R}^n$, then for any $u\in\mathrm{W}^{k,p}(\Omega)$ there is $(u_m)_{m\in\mathbb{N}}\subset \mathrm{C}^\infty(\overline{\Omega})$ such that $u_m\to u$ in $\mathrm{W}^{k,p}(\Omega)$.*

Proof. We fix $u\in\mathrm{W}^{k,p}(\Omega)$ and $\delta>0$.

i) In a first step, we show that in case Ω is unbounded there exists $v\in\mathrm{W}^{k,p}(\Omega)$ with $\operatorname{supp}(v)$ bounded and $\|u-v\|_{\mathrm{W}^{k,p}(\Omega)}<\delta$. By Lemma 4.2, there exists $\tau\in\mathrm{C}_0^\infty(\mathrm{B}_2(0))$ such that $0\le\tau\le 1$, $\tau\equiv 1$ in $\overline{\mathrm{B}_1(0)}$ and there is some $M=M(k)>0$ such that $|\mathrm{D}^\alpha\tau(x)|\le M$ for all $x\in\mathbb{R}^n$ and all $|\alpha|\le k$ (choose $K=\overline{\mathrm{B}_1(0)}, \Omega=\mathrm{B}_2(0), \delta=1$ in Lemma 4.2). For $\varepsilon\in(0,1)$, we define $\tau_\varepsilon := \tau(\varepsilon x), x\in\mathbb{R}^n$. Then $\tau_\varepsilon\equiv 1$ in $\overline{\mathrm{B}_{\frac{1}{\varepsilon}}(0)}$, $\tau_\varepsilon\in\mathrm{C}_0^\infty(\mathrm{B}_{\frac{2}{\varepsilon}}(0))$, and

$$|\mathrm{D}^\alpha\tau_\varepsilon(x)|\le M\varepsilon^{|\alpha|}\le M \quad\text{for all } |\alpha|\le k. \tag{4.4}$$

Hence, $v_\varepsilon := \tau_\varepsilon u$ has bounded support and belongs to $\mathrm{W}^{k,p}(\Omega)$ by Proposition 3.8 d). It further satisfies for $|\alpha|\le k$

$$|\mathrm{D}^\alpha v_\varepsilon(x)| = |\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\mathrm{D}^\beta\tau_\varepsilon(x)\,\mathrm{D}^{\alpha-\beta}u(x)| \le M\sum_{\beta\le\alpha}\binom{\alpha}{\beta}|\mathrm{D}^{\alpha-\beta}u(x)| \quad\text{for all } x\in\Omega$$

so that for all $\tilde\Omega\subset\Omega$ open we have

$$\|v_\varepsilon\|_{\mathrm{W}^{k,p}(\tilde\Omega)} \le \sum_{|\alpha|\le k}\|\mathrm{D}^\alpha v_\varepsilon\|_{\mathrm{L}^p(\tilde\Omega)} \le M\left(\sum_{|\alpha|\le k}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\right)\|u\|_{\mathrm{W}^{k,p}(\tilde\Omega)} \le c(k)M\|u\|_{\mathrm{W}^{k,p}(\tilde\Omega)}$$

with some constant $c(k)>0$. Hence,

$$\begin{aligned}\|u-v_\varepsilon\|_{\mathrm{W}^{k,p}(\Omega)} &= \|u-v_\varepsilon\|_{\mathrm{W}^{k,p}(\Omega\setminus\overline{\mathrm{B}_{\frac{1}{\varepsilon}}(0)})} \le \|u\|_{\mathrm{W}^{k,p}(\Omega\setminus\overline{B_{\frac{1}{\varepsilon}}(0)})} + \|v_\varepsilon\|_{\mathrm{W}^{k,p}(\overline{\Omega\setminus\mathrm{B}_{\frac{1}{\varepsilon}}(0)})} \\ &\le (1+c(k)M)\|u\|_{\mathrm{W}^{k,p}(\Omega\setminus\overline{\mathrm{B}_{\frac{1}{\varepsilon}}(0)})} \\ &\to 0 \quad\text{as } \varepsilon\to 0\end{aligned}$$

in view of $p<\infty$. Hence, $\|u-v_\varepsilon\|_{\mathrm{W}^{k,p}(\Omega)}<\delta$ for $\varepsilon>0$ small enough and $v=v_\varepsilon$ has bounded support.

ii) In view of i) we may assume w.l.o.g. that $K \coloneqq \operatorname{supp}(u)$ is bounded and hence compact (if necessary, we replace u by v).

For $x \in \partial\Omega$, let $U_x \subset \mathbb{R}^n$ be the open neighborhood of x and $0 \neq y_x \in \mathbb{R}^n$ like in Definition 4.9. Then

$$F \coloneqq K \setminus \big(\bigcup_{x \in \partial\Omega} U_x\big)$$

is compact with $F \subset \Omega$. Hence, there is U_0 open such that $F \Subset U_0 \Subset \Omega$. As K is compact, there exist finitely many of the sets U_x which we call $U_1, \dots, U_N$ such that $K \subset \bigcup_{i=0}^N U_i$. Moreover, we choose $V_i \Subset U_i$ open sets, $i = 0, \dots, N$, such that $K \subset \bigcup_{i=0}^N V_i$ and V_i is still a neighborhood of x^i belonging to $U_i = U_{x^i}$. By Lemma 4.3 there is a partition of unity $\psi_0, \dots, \psi_N$ such that $\psi_i \in \mathrm{C}_0^\infty(V_i)$, $0 \le \psi_i \le 1$ for $i = 0, \dots, N$, and $\sum_{i=0}^N \psi_i(x) = 1$ for all $x \in K$.

Our aim is to find $\varphi_i \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ such that with $u_i \coloneqq \psi_i u$ we have

$$\|u_i - \varphi_i\|_{\mathrm{W}^{k,p}(\Omega)} < \frac{\delta}{N+1} \quad \text{for all } i \in \{0, \dots, N\}. \tag{4.5}$$

As $\operatorname{supp}(u_0) \Subset V_0 \Subset \Omega$, by Theorems 4.1 and 2.1 b) there exists $\varphi_0 \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ such that (4.5) holds for $i = 0$. Next, we fix $i \in \{1, \dots, N\}$ and extend u by 0 outside Ω. Let $x^i \in \partial\Omega$ be the point belonging to U_i (U_i is a neighborhood of x^i) and

$$\Gamma \coloneqq \overline{V_i} \cap \partial\Omega.$$

As $\psi_i = 0$ on $\partial\Omega \setminus \Gamma$, $u_i = 0$ on $\mathbb{R}^n \setminus \overline{\Omega}$, and $u_i \in \mathrm{W}^{k,p}(\Omega)$ by Proposition 3.8, we get $u_i \in \mathrm{W}^{k,p}(\mathbb{R}^n \setminus \Gamma)$. Let $y \coloneqq y_{x_i}$ from the segment condition and

$$\Gamma_t \coloneqq \{x - ty \colon x \in \Gamma\},$$

where

$$0 < t < \min\{1, \frac{1}{|y|} \operatorname{dist}(\partial V_i, \partial U_i)\}.$$

By the choice of t, we have $\Gamma_t \subset U_i$ and $\Gamma_t \cap \overline{\Omega} = \emptyset$. The latter follows from the segment condition: For $z = x - sy$ with $x \in \Gamma$ and $s \in (0,1)$ we have $z + sy = x \in \Gamma \subset \partial\Omega$. Hence, $z \notin \overline{\Omega}$ so that $\Gamma_t \cap \overline{\Omega} = \emptyset$. Define

$$w_t(x) \coloneqq u_i(x + ty) \quad \text{for all } x \in \mathbb{R}^n.$$

As $u_i \in \mathrm{W}^{k,p}(\mathbb{R}^n \setminus \Gamma)$, we have $w_t \in \mathrm{W}^{k,p}(\mathbb{R}^n \setminus \Gamma_t)$. Hence, Proposition 4.12 yields that $\mathrm{D}^\alpha w_t \to \mathrm{D}^\alpha u_i$ in $\mathrm{L}^p(\Omega)$ as $t \downarrow 0$ for all $|\alpha| \le k$ (since $\overline{\Omega} \subset \mathbb{R}^n \setminus \Gamma_t$) and we can choose t small enough such that $\|w_t - u_i\|_{\mathrm{W}^{k,p}(\Omega)} \le \frac{\delta}{2(N+1)}$.

Moreover, since $u \in \mathrm{L}^p(\Omega)$ and $u = 0$ on $\mathbb{R}^n \setminus \Omega$, we have $u \in \mathrm{L}^p(\mathbb{R}^n)$, $u_i = \psi_i u \in \mathrm{L}^p(\mathbb{R}^n)$, and $w_t \in \mathrm{L}^p(\mathbb{R}^n)$. Since $\operatorname{supp}(u_i) \subset \overline{\Omega} \cap V_i$, w_t has compact support in $\mathbb{R}^n$. Hence, by Theorem 2.1 $\varphi_i \coloneqq \eta_\varepsilon * w_t$ belongs to $\mathrm{C}_0^\infty(\mathbb{R}^n)$ for $\varepsilon > 0$. As $\operatorname{dist}(\Gamma_t, \overline{\Omega}) > 0$, we may choose $\varepsilon > 0$ small enough such that $\|\varphi_i - w_t\|_{\mathrm{W}^{k,p}(\Omega)} < \frac{\delta}{2(N+1)}$ (as $\Omega \cap \operatorname{supp}(w_t) \Subset \mathbb{R}^n \setminus \Gamma_t$ and $w_t \in \mathrm{W}^{k,p}(\mathbb{R}^n \setminus \Gamma_t)$ we may apply Theorem 4.1). Altogether, $\varphi_i \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ satisfies $\|u_i - \varphi_i\| \le \frac{\delta}{N+1}$ and (4.5) holds for all $i \in \{0, \dots, N\}$. As $u = \sum_{i=1}^n u_i$, the function

$$\varphi \coloneqq \sum_{i=0}^N \varphi_i \in \mathrm{C}_0^\infty(\mathbb{R}^n)$$

satisfies $\|u - \varphi\|_{\mathrm{W}^{k,p}(\Omega)} \le \delta$.

iii) Combining i) and ii), the claim is proved. □

As a Corollary, we see that $\mathrm{W}_0^{k,p}(\mathbb{R}^n)$ and $\mathrm{W}^{k,p}(\mathbb{R}^n)$ coincide.

Corollary 4.14. *For $k \in \mathbb{N}$ and $p \in [1, \infty)$, we have* $\mathrm{W}_0^{k,p}(\mathbb{R}^n) = \mathrm{W}^{k,p}(\mathbb{R}^n)$.

Proof. Given $u \in \mathrm{W}^{k,p}(\mathbb{R}^n)$ by part i) of the proof of Theorem 4.13, there exists $v \in \mathrm{W}^{k,p}(\mathbb{R}^n)$ with $K := \operatorname{supp}(V)$ compact and

$$\|u - v\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq \frac{\delta}{2}.$$

But then there is V_0 open such that $K \Subset V_0 \Subset \mathbb{R}^n$ and by Theorems 4.1 and 2.1 b) there is $\varepsilon > 0$ small enough such that $v_\varepsilon = \eta_\varepsilon * v \in \mathrm{C}_0^\infty(V_0)$ and

$$\|v - v_\varepsilon\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} = \|v - v_\varepsilon\|_{\mathrm{W}^{k,p}(V_0)} < \frac{\delta}{2}.$$

Hence, $\|u - v_\varepsilon\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} < \delta$ and $v_\varepsilon \in \mathrm{C}_0^\infty(\mathbb{R}^n)$. □

Chapter 5

Extension and Traces

Here, we will study how we can extend functions in $\mathrm{W}^{k,p}(\Omega)$ to $\mathrm{W}^{k,p}(\mathbb{R}^n)$. Moreover, we will see how we can define boundary values on $\partial\Omega$ of functions in $\mathrm{W}^{k,p}(\Omega)$.

5.1 Flattening the Boundary

We will frequently use that if Ω is a bounded domain with $\partial\Omega \in \mathrm{C}^m$, we can transform $\Omega \cap \mathrm{B}_r(x^0)$ for $x^0 \in \partial\Omega$ to a domain having a flat boundary, where the transformation is a C^m-diffeomorphism. This will be done in details here.

Notation 5.1. We define

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}, \quad \mathbb{R}^n_- := \{x \in \mathbb{R}^n : x_n < 0\},$$

and for $U \subset \mathbb{R}^n$

$$U^+ := U \cap \mathbb{R}^n_+, \quad U^- := U \cap \mathbb{R}^n_-, \quad U^0 := \{x \in U : x_n = 0\}.$$

Moreover, we write $x = (x', x_n)$ for $x \in \mathbb{R}^n$ with $x = (x_1, \dots, x_n)$ and $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

Definition 5.2. Let Ω and U be domains in $\mathbb{R}^n$. A map $g\colon \Omega \to U$ is called C^m-*diffeomorphism* if and only if g is bijective, $g \in \mathrm{C}^m(\overline{\Omega}, \mathbb{R}^n)$, $g^{-1} \in \mathbb{C}^m(\overline{U}, \mathbb{R}^n)$ and $\det(\mathrm{D}g) \neq 0$ in $\overline{\Omega}$.

If $\partial\Omega \in \mathrm{C}^m$, then we can locally transform Ω to a domain with flat boundary.

Lemma 5.3. *Let $m \in \mathbb{N}$ and $\gamma \in \mathbb{C}^m(\mathbb{R}^{n-1})$. Then $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ and $\Psi\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$\begin{aligned} \Phi(x) &= (x_1, \dots, x_{n-1}, x_n - \gamma(x')), \\ \Psi(y) &= (y_1, \dots, y_{n-1}, y_n + \gamma(y')), \quad x, y \in \mathbb{R}^n, \end{aligned}$$

satisfy $\Phi, \Psi \in \mathrm{C}^m(\mathbb{R}^n, \mathbb{R}^n)$, $\det(\mathrm{D}\Phi) = \det(\mathrm{D}\Psi) = 1$ on $\mathbb{R}^n$, and $\Phi^{-1} = \Psi$. In particular, for any bounded domain Ω, $\Phi|_\Omega\colon \Omega \to \Phi(\Omega)$ is a C^m-diffeomorphism. If $\partial\Omega \in \mathrm{C}^m$ and

$$\begin{aligned} \Omega \cap \mathrm{B}_r(x^0) &= \{x \in \mathrm{B}_r(x^0) : x_n > \gamma'(x')\}, \\ \partial\Omega \cap \mathrm{B}_r(x^0) &= \{x \in \mathrm{B}_r(x^0) : x_n = \gamma(x')\} \end{aligned}$$

for some $x^0 \in \partial\Omega$ (see Definition 4.10), then $\Phi\colon \mathrm{B}_r(x^0) \to U := \Phi(\mathrm{B}_r(x^0))$ is a C^m-diffeomorphism with $\Phi(\Omega \cap \mathrm{B}_r(x^0)) = U^+$ and $\Phi(\partial\Omega \cap \mathrm{B}_r(x^0)) = U^0$.

Proof. It is straightforward to see that Φ and Ψ are C^m-functions with $\Psi = \Phi^{-1}$ and $\det(\mathrm{D}\Phi) = \det(\mathrm{D}\Psi) = 1$. The further claims are immediate consequences of Definition 5.2. ☐

Φ now provides a transformation which *flattens* the boundary. A C^m-diffeomorphism also provides a transformation between the corresponding Sobolev spaces.

Proposition 5.4. *Let $g\colon U \to \Omega$ be a C^m-diffeomorphism with $m \in \mathbb{N}$, $p \in [1, \infty)$, and $\Omega, U \subset \mathbb{R}^n$ bounded domains. Then the map $T_g\colon \mathrm{W}^{m,p}(\Omega) \to \mathrm{W}^{m,p}(U)$, defined by*

$$(T_g(u))(y) \coloneqq u(g(y)), \quad y \in U, u \in \mathrm{W}^{m,p}(\Omega),$$

is bijective and there exist $C_1, C_2 > 0$ such that

$$\|T_g(u)\|_{\mathrm{W}^{m,p}(U)} \leq C_1 \|u\|_{\mathrm{W}^{m,p}(\Omega)} \quad \text{and} \quad \|(T_g)^{-1}(v)\|_{\mathrm{W}^{m,p}(\Omega)} \leq C_2 \|v\|_{\mathrm{W}^{m,p}(U)}$$

for all $u \in \mathrm{W}^{m,p}(\Omega)$, $v \in \mathrm{W}^{m,p}(U)$, where $(T_g)^{-1} = T_{g^{-1}}$.

Proof. i) If T_g is well-defined and satisfies the claimed estimate, it is immediate that $(T_g)^{-1} = T_{g^{-1}}$ and hence the estimate for $(T_g)^{-1}$ follows by replacing g by g^{-1}. Moreover, we only consider the case $m = 1$ as the general case then follows by induction.

ii) Let $m = 1$ and $T \coloneqq T_g$. Assume first that $u \in \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{1,p}(\Omega)$. Then the chain rule and the transformation rule imply with $M \coloneqq \|\det(\mathrm{D}g^{-1})\|_{\mathrm{C}^0(\overline{\Omega})}$

$$\int_U |T(u)(y)|^p \,\mathrm{d}y = \int_\Omega |u(x)|^p \, |\det(\mathrm{D}g^{-1})(x)| \,\mathrm{d}x \leq M \int_\Omega |u(x)|^p \,\mathrm{d}x, \tag{5.1}$$

$$\partial_{y_i}(T(u))(y) = \sum_{j=1}^n u_{x_j}(x) \cdot (g_j)_{y_i}(y) = \sum_{j=1}^n T(u_{x_j})(y) \cdot (g_j)_{y_i}(y), \quad y \in U,$$

where $x = g(y)$ $(x \in \Omega, y \in U)$, and

$$\begin{aligned}\Big(\int_U |(\partial_{y_i}(T(u))(y)|^p \,\mathrm{d}y\Big)^{\frac{1}{p}} &\leq \|g\|_{\mathrm{C}^1(\overline{U})} \sum_{j=1}^n \Big(\int_U |T(u_{x_j})(y)|^p \,\mathrm{d}y\Big)^{\frac{1}{p}} \\ &\overset{(5.1)}{\leq} \|g\|_{\mathrm{C}^1(\overline{U})} \, M^{\frac{1}{p}} \sum_{j=1}^n \|u_{x_j}\|_{\mathrm{L}^p(\Omega)}.\end{aligned}$$

Hence, $\|T(u)\|_{\mathrm{W}^{1,p}(U)} \leq M^{\frac{1}{p}}(1 + n\|g\|_{\mathrm{C}^1(\overline{U})}) \, \|u\|_{\mathrm{W}^{1,p}}$ by combining the previous estimate with (5.1) and Minkowski's inequality.

iii) By ii) we have

$$\|T(u)\|_{\mathrm{W}^{1,p}(U)} \leq C_1 \|u\|_{\mathrm{W}^{1,p}(\Omega)} \quad \text{for all } u \in \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{1,p}(\Omega).$$

For $u \in \mathrm{W}^{1,p}(\Omega)$ there exists $(u_k)_{k\in\mathbb{N}} \subset \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{1,p}(\Omega)$ such that $u_k \to u$ in $\mathrm{W}^{1,p}(\Omega)$ and a.e. in Ω. As T is linear, we have

$$\|T(u_k) - T(u_j)\|_{\mathrm{W}^{1,p}(U)} \leq C_1 \|u_k - u_j\|_{\mathrm{W}^{1,p}(\Omega)}, \quad \text{for all } k, j \in \mathbb{N}.$$

Hence, $(T(u_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{W}^{1,p}(U)$ and $T(u_k) \to v$ in $\mathrm{W}^{1,p}(U)$. As obviously $T(u_k) \to T(u)$ in $\mathrm{L}^p(U)$ by (5.1) and the dominated convergence theorem, we have $v = T(u) \in \mathrm{W}^{1,p}(U)$ and

$$\|T(u)\|_{\mathrm{W}^{1,p}(U)} = \lim_{k\to\infty} \|T(u_k)\|_{\mathrm{W}^{1,p}(U)} \overset{\text{ii)}}{\leq} C_1 \lim_{k\to\infty} \|u_k\|_{\mathrm{W}^{1,p}(\Omega)} = C_1 \|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

where C_1 can be chosen independent of p as $M^{\frac{1}{p}} \leq 1 + M$. ☐

5.2 Extension Theorem

We next extend functions $u \in \mathrm{W}^{k,p}(\Omega)$ to become a function in $\mathrm{W}^{k,p}(\mathbb{R}^n)$. This cannot be done in general by extending u by 0 to $\mathbb{R}^n \setminus \Omega$ as this may create bad discontinuities in some $\mathrm{D}^\alpha u$ on $\partial\Omega$. We need now again boundary regularity of $\partial\Omega$.

Theorem 5.5. *Let $m \in \mathbb{N}$, $p \in [1,\infty)$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in \mathrm{C}^m$, and $V \subset \mathbb{R}^n$ be a domain with $\Omega \Subset V$. Then for any $k \in \{1,\dots,m\}$ there exists a linear Operator*

$$E\colon \mathrm{W}^{k,p}(\Omega) \to \mathrm{W}^{k,p}(\mathbb{R}^n)$$

such that

a) $E(u) = u$ a.e. in Ω,

b) $\operatorname{supp}(E(u)) \Subset V$, i.e. $E(u) \in \mathrm{W}^{k,p}_0(V)$,

c) $\|E(u)\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \le C\|u\|_{\mathrm{W}^{k,p}(\Omega)}$ for all $u \in \mathrm{W}^{k,p}(\Omega)$,

where $C > 0$ depends on m, p, V, and Ω but not on k and u. Moreover, E does not depend on $k \in \{1,\dots,m\}$ and on $p \in [1,\infty)$ in the sense that if $u \in \mathrm{W}^{k,p}(\Omega)$ for several p,k then $E(u)$ is uniquely determined in all these spaces.

Proof. We will use the notation from 5.1, Lemma 5.3, and Definition 4.10.

i) We first assume that for some $x^0 \in \partial\Omega$, $\mathrm{B} := \mathrm{B}_r(x^0)$ satisfies $\Omega \cap \mathrm{B} = \mathrm{B}^+$ and $\partial\Omega \cap \mathrm{B} = \mathrm{B}^0$. We further assume that $u \in \mathrm{C}^m(\mathrm{B}^+ \cup \mathrm{B}^0)$ with $\operatorname{supp}(u) \subset \widetilde{\mathrm{B}}^+ := \overline{(\mathrm{B}_s(x^0))^+}$ for some $s \in (0,r)$. As hence $u = 0$ in a neighborhood of $\partial\mathrm{B} \cap \overline{\mathbb{R}^n_+}$, we can extend u by 0 in $\overline{\mathbb{R}^n_+} \setminus \mathrm{B}^+$ and have $u \in \mathrm{C}^m(\overline{\mathbb{R}^n_+})$ with $\operatorname{supp}(u) \subset \widetilde{\mathrm{B}}^+$. Then we extend u from $\overline{\mathbb{R}^n_+}$ to $\mathbb{R}^n$ by a *higher-order reflection*. The $(m+1)\times(m+1)$ system of linear equations

$$\sum_{j=1}^{m+1} (-j)^i \lambda_j = 1, \quad i = 0,\dots,m, \tag{5.2}$$

has a unique solution $(\lambda_1,\dots,\lambda_{m+1})$ since the corresponding matrix is of Vandermonde type. We define the extension

$$\tilde{E}(u)(x) := \begin{cases} u(x) & \text{if } x = (x', x_n) \text{ with } x_n \ge 0, \\ \sum_{j=1}^{m+1} \lambda_j u(x', -jx_n) & \text{if } x = (x', x_n) \text{ with } x_n < 0, \end{cases}$$

and for any $\alpha \in \mathbb{N}_0^n$ with $1 \le |\alpha| \le m$

$$\tilde{E}_\alpha(u)(x) := \begin{cases} u(x) & \text{if } x_n \ge 0, \\ \sum_{j=1}^{m+1} (-j)^{\alpha_n} \lambda_j u(x', -jx_n) & \text{if } x_n < 0. \end{cases}$$

Then for all $|\alpha| \le m$ we have $\mathrm{D}^\alpha(\tilde{E}(u)) = \tilde{E}_\alpha(\mathrm{D}^\alpha u)$ and hence $\tilde{E}(u) \in \mathrm{C}^n(\mathbb{R}^n)$: This is immediate in $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$, while for $x \in \partial\mathbb{R}^n_+$, namely $x_n = 0$, we have

$$\begin{aligned} \lim_{t\uparrow 0} \mathrm{D}^\alpha(\tilde{E}(u))(x', t) &= \lim_{t\uparrow 0} \tilde{E}_\alpha(\mathrm{D}^\alpha u)(x', t) = \lim_{t\uparrow 0} \sum_{j=1}^{m+1} (-j)^{\alpha_n} \lambda_j \mathrm{D}^\alpha u(x', -jt) \\ &= \Big(\sum_{j=1}^{m+1} (-j)^{\alpha_n} \lambda_j\Big) \mathrm{D}^\alpha u(x', 0) \overset{(5.2)}{=} \mathrm{D}^\alpha u(x', 0) \\ &= \lim_{t\downarrow 0} \mathrm{D}^\alpha u(x', t) = \lim_{t\downarrow 0} \mathrm{D}^\alpha(\tilde{E}(u))(x', t) \end{aligned}$$

since $u \in \mathrm{C}^m(\overline{\mathbb{R}^n_+})$ by assumption. Moreover,

$$\begin{aligned}\|\mathrm{D}^\alpha(\tilde{E}(u))\|_{\mathrm{L}^p(\mathbb{R}^n)} &\leq \Big(1 + \sum_{j=1}^{m+1} j^{\alpha_n} |\lambda_j|\, j^{\frac{1}{p}}\Big) \|\mathrm{D}^\alpha u\|_{\mathrm{L}^p(\mathbb{R}^n_+)} \\ &\overset{p\geq 1}{\leq} \Big(1 + \sum_{j=1}^{m+1} j^{\alpha_n+1} |\lambda_j|\Big) \|\mathrm{D}^\alpha u\|_{\mathrm{L}^p(\mathbb{R}^n_+)},\end{aligned}$$

where we have used the transformation rule with $z = (x', -jx_n)$ and $\operatorname{supp}(u) \subset \widetilde{\mathrm{B}}_+ \subset \overline{\mathrm{B}^+}$. Hence, $\tilde{E}(u) \in \mathrm{C}^m(\mathbb{R}^n)$ with

$$\|\tilde{E}(u)\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq C_1 \|u\|_{\mathrm{W}^{k,p}(\mathrm{B}^+)} \tag{5.3}$$

with C_1 depending on m and n.

ii) As $\partial\Omega \in \mathrm{C}^m$ and $\partial\Omega$ is compact, there exists $x^i \in \partial\Omega$ and balls $U_i := \mathrm{B}_r(x^i), i = 1, \dots, N$, such that after relabelling and reorienting the coordinate axes (described by a C^∞-diffeomorphism $\mathcal{T}_i \colon \mathbb{R}^n \to \mathbb{R}^n$), we have

$$\begin{aligned}\Omega \cap U_i &= \{x \in U_i \colon x_n > \gamma_i(x')\} \\ \partial\Omega \cap U_i &= \{x \in U_i \colon x_n = \gamma_i(x')\}\end{aligned}$$

with $\gamma_i \in \mathrm{C}^m(\mathbb{R}^{n-1})$. Choose $U_0 \Subset \Omega$ such that $\overline{\Omega} \subset \bigcup_{i=0}^N U_i$. By Lemma 4.3 there exists a partition of unity $\varphi_i \in \mathrm{C}_0^\infty(\mathcal{T}_i^{-1}(U_i))$ with $0 \leq \varphi_i \leq 1$ and $\sum_{i=1}^N \varphi_i(x) = 1$ for all $x \in \overline{\Omega}$. Define further Φ_i, Ψ_i by Lemma 5.3 according to $\gamma = \gamma_i, i = 1, \dots, N$, and set $W_i := \Phi_i(U_i)$.

We fix $i \in \{1, \dots, N\}$ and $u \in \mathrm{C}^m(\overline{\Omega})$. Then $\varphi_i u$ and hence also $T_{\mathcal{T}_i^{-1}}(\varphi_i u)$ belong to $\mathrm{C}^m(\overline{U}_i \cap \overline{\Omega})$ with compact support $K_i \Subset U_i, K_i \subset \mathcal{T}_i(\overline{\Omega})$. Then

$$v_i := T_{\Psi_i}(T_{\mathcal{T}_i^{-1}}(\varphi_i u)) \in \mathrm{C}^m(\overline{W_i^+})$$

with support in $\Phi(K_i) \Subset W_i$ and $\Phi_i(K_i) \subset W_i^+ \cup W_i^0$. Hence, we may apply $\tilde{E}$ to v_i (by choosing $\widetilde{\mathrm{B}}^+$ and B appropriately such that $\Phi_i(K_i) \subset \widetilde{\mathrm{B}}^+ \Subset W_i \Subset B$) and obtain

$$\tilde{E}(v_i) \in \mathrm{C}^m(\mathbb{R}^n) \cap \mathrm{W}^{k,p}(\mathbb{R}^n).$$

Transforming back, we finally obtain

$$u_i := T_{\mathcal{T}_i}(T_{\Phi_i}(\tilde{E}(v_i))) \in \mathrm{W}^{k,p}(\mathbb{R}^n).$$

As $\tilde{E}(v_i)$ has compact support in some ball $\mathrm{B}_i \subset \mathbb{R}^n$, we may apply Proposition 5.4 with

$$\Phi_i := \Phi_i^{-1}(\mathrm{B}_i) \to \mathrm{B}_i \quad \text{and} \quad \mathcal{T}_i \colon \mathcal{T}_i^{-1}(\Phi_i^{-1}(\mathrm{B}_i)) \to \Phi_i^{-1}(B_i)$$

and obtain $u_i = \varphi_i u$ in Ω and

$$\begin{aligned}\|u_i\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq C_2 \|\tilde{E}(v_i)\|_{\mathrm{W}^{k,p}(\mathrm{B}_i)} &\overset{\text{i)}}{\leq} C_3 \|v_i\|_{\mathrm{W}^{k,p}(W_i^+)} \\ \overset{\text{Prop. 5.4}}{\leq} C_4 \|\varphi_i u\|_{\mathrm{W}^{k,p}(\Omega\cap U_i)} &\overset{\substack{\text{Prop. 3.8}\\ \text{Lem. 4.2}}}{\leq} C_5 \|u\|_{\mathrm{W}^{k,p}(\Omega\cap U_i)}.\end{aligned}$$

Defining further $u_0 \coloneqq \varphi_0 u \in \mathrm{C}^m(U_0)$ with compact support in U_0, we have

$$\sum_{i=0}^{n} u_i \in \mathrm{W}^{k,p}(\mathbb{R}^n) \quad \text{with} \quad \| \sum_{i=0}^{N} u_i \|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq C_6 \|u\|_{\mathrm{W}^{k,p}(\Omega)}$$

and

$$\sum_{i=0}^{N} u_i = \sum_{i=0}^{N} \varphi_i u = u \quad \text{in } \Omega.$$

Finally, let $\tau \in \mathrm{C}_0^\infty(V)$ be the cutoff-function w.r.t $\overline{\Omega}$ and V. Then

$$E(u) \coloneqq \tau \cdot \sum_{i=0}^{N} u_i \in \mathrm{W}^{k,p}(\mathbb{R}^n) \quad \text{with } \operatorname{supp}(E(u)) \Subset V$$

and

$$\|E(u)\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq C_7 \|u\|_{\mathrm{W}^{k,p}(\Omega} \quad \text{for all } u \in \mathrm{C}^m(\overline{\Omega}), \tag{5.4}$$

where $C_7 > 0$ does not depend on k or u but on m, p, Ω and V. Moreover, $E(u) = u$ in Ω.

iii) As $\tilde{E}$ is linear in u and T_g is linear in u (see Proposition 5.4), the map

$$E \colon \mathrm{C}^m(\overline{\Omega}) \to \mathrm{W}^{k,p}(\mathbb{R}^n), \quad u \mapsto E(u)$$

is linear. Let $u \in \mathrm{W}^{k,p}(\Omega)$. Then by Theorem 4.13 there exists $(u_j)_{j\in\mathbb{N}} \subset \mathrm{C}^m(\overline{\Omega})$ such that $u_j \to u$ in $\mathrm{W}^{k,p}(\Omega)$. Then by (5.4)

$$\|E(u_j) - E(u_i)\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq C_7 \|u_j - u_i\|_{\mathrm{W}^{k,p}(\Omega)} \quad \text{for all } i, j \in \mathbb{N}$$

so that $(E(u_j))_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{W}^{k,p}(\mathbb{R}^n)$ and has a limit $v \in \mathrm{W}^{k,p}(\mathbb{R}^n)$. As $E(u_j) \to v$ a.e. in $\mathbb{R}^n$ for a subsequence, we still have $\operatorname{supp}(v) \Subset V$ and $u = v$ a.e. in Ω. Defining $E(u) \coloneqq v$, we have

$$\|E(u)\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq C_7 \|u\|_{\mathrm{W}^{k,p}(\Omega)}$$

by (5.4) like in the end of the proof of Proposition 5.4. Since (5.4) implies that $E(u) = v$ does not depend on the choice of the approximating sequence $(u_j)_{j\in\mathbb{N}}$ (as E is linear), we thereby have defined $E \colon \mathrm{W}^{k,p}(\Omega) \to \mathrm{W}^{k,p}(\mathbb{R}^n)$ having all claimed properties. $\square$

5.3 Trace Operator

A function $u \in \mathrm{W}^{1,p}(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ clearly has boundary values on $\partial\Omega$ in the usual sense. But a general function $u \in \mathrm{W}^{1,p}(\Omega)$ is not continuous and only defined a.e. in Ω. Since $\partial\Omega$ has n-dimensional Lebesgue measure 0, we need a trace operator involving the space $\mathrm{L}^p(\partial\Omega)$ for assigning boundary values to u.

Definition 5.6. Let Ω be a bounded domain with $\partial\Omega \in \mathrm{C}^1$ and $U_i \coloneqq \mathrm{B}_{r_i}(x^i), i = 1, \ldots, N$, such that $\Omega \cap U_i = \{x \in U_i \colon x_n > \gamma_i(x')\}$, $\partial\Omega \cap U_i = \{x \in U_i \colon x_n = \gamma_i(x')\}$ with $\gamma_i \in \mathrm{C}^1(\mathbb{R}^{n-1})$ and $x^i \in \partial\Omega$ (where we assume that the coordinate axes are relabeled and reoriented appropriately). Defining

$$U_i' \coloneqq \{x' \colon x \in \partial\Omega \cap U_i\} \subset \mathbb{R}^{n-1},$$

we have

$$\partial\Omega \cap U_i = \{(x', \gamma_i(x'))\colon x' \in U_i'\}.$$

Let U be a neighborhood of $\partial\Omega$ (in $\mathbb{R}^n$) such that $U \Subset \bigcup_{i=1}^N U_i$ and $\varphi_i, i = 1, \dots, N$, a corresponding partition of unity such that $\varphi_i \in \mathrm{C}_0^\infty(U_i)$, $\sum_{i=1}^N \varphi_i(x) = 1$ for all $x \in U$, $0 \le \varphi_i \le 1$. Then $v\colon \partial\Omega \to \mathbb{R}$ is called *measurable* on $\partial\Omega$ if

$$v_i\colon U_i' \to \mathbb{R}, \quad v_i(x') \coloneqq (\varphi_i v)(x', \gamma_i(x'))$$

is measurable in $U_i' \subset \mathbb{R}^{n-1}$.
Defining

$$\int_{\partial\Omega} v_i \,\mathrm{d}\sigma \coloneqq \int_{\partial\Omega\cap U_i} v_i \,\mathrm{d}\sigma \coloneqq \int_{U_i'} v_i \sqrt{1 + |\nabla\gamma_i|^2}\,\mathrm{d}x'$$

and

$$\int_{\partial\Omega} v \,\mathrm{d}\sigma \coloneqq \sum_{i=1}^N \int_{\partial\Omega} v_i \,\mathrm{d}\sigma,$$

then $\mathrm{L}^p(\partial\Omega)$ is the space of functions $v\colon \partial\Omega \to \mathbb{R}$ being measurable on $\partial\Omega$ such that the norm

$$\|v\|_{\mathrm{L}^p(\partial\Omega)} \coloneqq \left(\int_{\partial\Omega} |v|^p \,\mathrm{d}\sigma\right)^{\frac{1}{p}}$$

is finite where $p \in [1, \infty)$.

Then we have the following theorem:

Theorem 5.7 (Trace theorem)**.** *Let Ω be a bounded domain with $\partial\Omega \in \mathrm{C}^1$ and $p \in [1, \infty)$. Then there exists a bounded linear operator $\mathrm{Tr}\colon \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}^p(\partial\Omega)$ such that*

a) $\mathrm{Tr}(u) = u|_{\partial\Omega}$ *if $u \in \mathrm{W}^{1,p}(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$,*

b) $\|\mathrm{Tr}(u)\|_{\mathrm{L}^p(\partial\Omega)} \le C\,\|u\|_{\mathrm{W}^{1,p}(\Omega)}$ *for any $u \in \mathrm{W}^{1,p}(\Omega)$, where $C > 0$ only depends on p and Ω.*

$\mathrm{Tr}(u)$ *is called the* trace of u on $\partial\Omega$*.*

Proof. i) Assume like in the proof of Theorem 5.5 that there is $x^0 \in \partial\Omega$ and $\mathrm{B} \coloneqq \mathrm{B}_r(x^0) \subset \mathbb{R}^n$ such that $\Omega \cap \mathrm{B} = \mathrm{B}^+$ and $\partial\Omega \cap \mathrm{B} = \mathrm{B}^0$. Let $\hat{\mathrm{B}} \Subset \mathrm{B}$ and $\xi \in \mathrm{C}_0^\infty(\mathrm{B})$ such that $\xi > 0$ in $\hat{\mathrm{B}}$ and $0 \le \xi \le 1$ (see Lemma 4.2). Furthermore, assume $p \in (1, \infty)$. Then setting

$$\mathrm{B}' \coloneqq \{x'\colon (x', x_n) \in \mathrm{B}\} \subset \mathbb{R}^{n-1},$$

we have for $u \in \mathrm{C}^1(\overline{\Omega})$

$$\begin{aligned}\int_{\partial\Omega\cap\mathrm{B}} |\xi u|^p \,\mathrm{d}\sigma &= \int_{\mathrm{B}'} |\xi u|^p(x', 0)\,\mathrm{d}x' \\ &= \int_{\mathrm{B}'} \left(\Big[-\int_0^{\tilde{x}} \partial_{x_n}(|\xi u|^p)(x', s)\,\mathrm{d}s\Big] + \underbrace{|\xi u|^p(x', \tilde{x})}_{=0}\,\mathrm{d}x'\right),\end{aligned}$$

where $(x', \tilde{x}) \in \partial\mathrm{B} \cap (\overline{\mathrm{B}^+} \setminus \mathrm{B}^0)$,

$$\begin{aligned}&= -\int_{\mathrm{B}^+} \Big(|u|^p\, p\, \xi^{p-1}\, \xi_{x_n} + \xi^p\, p\, |u|^{p-1} \frac{u}{|u|}\, u_{x_n}\Big)(x)\,\mathrm{d}x \\ &\le C_1 \int_{\mathrm{B}^+} (|u|^p + |u|^{p-1}\,|u_{x_n}|)\,\mathrm{d}x \\ &\le C_2 \int_{\mathrm{B}^+} (|u|^p + |u_{x_n}|^p)\,\mathrm{d}x \\ &\le C_2\,\|u\|^p_{\mathrm{W}^{1,p}(\mathrm{B}^+)},\end{aligned}$$

where $C_2 > 0$ depends on p and ξ and we used *Young's inequality*

$$|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q \quad \text{for } a, b \in \mathbb{R}, p, q \in [1, \infty], \frac{1}{p} + \frac{1}{q} = 1,$$

with $a = u_{x_n}(x)$, $b = |u|^{p-1}(x)$, $q = \frac{p}{p-1}$. Summing up the previous calculation, we have shown

$$\int_{\partial\Omega\cap \mathrm{B}} |\xi u|^p \,\mathrm{d}\sigma \leq C_2 \|u\|^p_{\mathrm{W}^{1,p}(\mathrm{B}^+)} \quad \text{for all } u \in \mathrm{C}^1(\overline{\Omega}), p \in (1, \infty). \tag{5.5}$$

A similar calculation shows that (5.5) also holds for $p = 1$.

ii) In the general case of Definition 5.6, let $B^i = B_{s_i}(x^i)$ such that $\Phi_i(U_i) \subset \mathrm{B}^i$ (with Φ_i, Ψ_i from Lemma 5.3 with $\gamma = \gamma_i$ to flatten the boundary). Then applying (5.5) with $\mathrm{B} = \mathrm{B}^i$, $\xi = T_{\Psi_i}(\varphi_i)$, we obtain with $u_i = \varphi_i u$ for $u \in \mathrm{C}^1(\overline{\Omega})$

$$\begin{aligned}\int_{\partial\Omega} |u_i|^p \,\mathrm{d}\sigma &= \int_{U_i'} |\varphi_i u|^p(x', \gamma_i(x'))\sqrt{1 + |\nabla\gamma_i(x')|^2} \,\mathrm{d}x' \leq C_3 \int_{(B^i)'} |T_{\Psi_i}(\varphi_i u)|^p(x', 0) \,\mathrm{d}x' \\ &\overset{(5.5)}{\leq} C_4 \,\|T_{\Psi_i} u\|^p_{\mathrm{W}^{1,p}((\mathrm{B}^i)^+)} \overset{\text{Prop. 5.4}}{\leq} C_5 \,\|u\|^p_{\mathrm{W}^{1,p}(\Omega)}.\end{aligned}$$

Hence,

$$\|u\|_{\mathrm{L}^p(\partial\Omega)} \leq \sum_{i=1}^N \|u_i\|_{\mathrm{L}^p(\partial\Omega)} \leq N \cdot C_5^{\frac{1}{p}} \,\|u\|_{\mathrm{W}^{1,p}(\Omega)} \quad \text{for all } u \in \mathrm{C}^1(\overline{\Omega}),$$

with C_5 depending on p and Ω.

iii) By defining $\mathrm{Tr}(u) \coloneqq u|_{\partial\Omega}$ for $u \in \mathrm{C}^1(\overline{\Omega})$, we have

$$\|\,\mathrm{Tr}(u)\|_{\mathrm{L}^p(\partial\Omega)} \leq C_6 \,\|u\|_{\mathrm{W}^{1,p}(\Omega)} \quad \text{for all } u \in \mathrm{C}^1(\overline{\Omega})$$

by ii). If $u \in \mathrm{W}^{1,p}(\Omega)$ is arbitrary, as Tr is linear in u, we can show like in part iii) of the proof of Theorem 5.5 that there exist $(u_l)_{l\in\mathbb{N}} \subset \mathrm{C}^1(\overline{\Omega})$ such that $u_l \to u$ in $\mathrm{W}^{1,p}(\Omega)$ and $\mathrm{Tr}(u_l) \to v$ in $\mathrm{L}^p(\partial\Omega)$ for some $v \in \mathrm{L}^p(\partial\Omega)$. Hence, defining $\mathrm{Tr}(u) \coloneqq v$ the operator $\mathrm{Tr}\colon \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}^p(\partial\Omega)$ is well defined and satisfies b) with $C = C_6$.

iv) If $u \in \mathrm{W}^{1,p}(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ and $(\tilde{u}_l)_{l\in\mathbb{N}}$ in $\mathrm{C}^1(\overline{\Omega})$ such that $\tilde{u}_l \to u$ in $\mathrm{W}^{1,p}(\Omega)$, we remark that we can choose $(\tilde{u}_l)_{l\in\mathbb{N}}$ in the proof of Theorem 4.13 such that $\tilde{u}_l$ converges to u in $\mathrm{C}^0(\overline{\Omega})$. ($u_i$ is uniformly continuous (compact support), $w_t \to u_i$ in $\mathrm{C}^0(\overline{\Omega})$ as $t \downarrow 0$ since w_t is a translation and $\varphi_i = \eta_\varepsilon * w_t \to w_t$ in $\mathrm{C}^0(\overline{\Omega})$ as $\varepsilon \downarrow 0$ by Theorem 2.1 (as $\overline{\Omega} \Subset \mathbb{R}^n \setminus \Gamma_t$)). Hence, $\mathrm{Tr}(u) = \lim_{l\to\infty} \mathrm{Tr}(\tilde{u}_l) = \lim_{l\to\infty} \tilde{u}_l|_{\partial\Omega} = u|_{\partial\Omega}$. ☐

Next, we characterize $\mathrm{W}^{k,p}_0(\Omega)$ functions in terms of trace zero functions.

Theorem 5.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in \mathrm{C}^k$ and assume that $k \in \mathbb{N}$, $p \in [1, \infty)$, and $u \in \mathrm{W}^{k,p}(\Omega)$. Then $u \in \mathrm{W}^{k,p}_0(\Omega)$ if and only if $\mathrm{Tr}(\,\mathrm{D}^\alpha u) = 0$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k - 1$. In particular, $\mathrm{W}^{1,p}_0(\Omega) = \{u \in \mathrm{W}^{1,p}(\Omega)\colon\ \mathrm{Tr}(u) = 0\}$.*

Proof. i) "$\Rightarrow$" for $k = 1$: Let $u \in \mathrm{W}^{1,p}_0(\Omega)$. Then there are $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^\infty_0(\Omega)$ with $u_m \to u$ in $\mathrm{W}^{1,p}(\Omega)$. As $\mathrm{Tr}(u_m) = u_m|_{\partial\Omega} = 0$ and Theorem 5.7 shows

$$\begin{aligned}\|\,\mathrm{Tr}(u)\|_{\mathrm{L}^p(\partial\Omega)} &= \|\,\mathrm{Tr}(u) - \mathrm{Tr}(u_m)\|_{\mathrm{L}^p(\partial\Omega)} = \|\,\mathrm{Tr}(u - u_m)\|_{\mathrm{L}^p(\partial\Omega)} \\ &\leq C\|u - u_m\|_{\mathrm{W}^{1,p}(\Omega)} \to 0 \quad \text{as } m \to \infty,\end{aligned}$$

we have $\mathrm{Tr}(u) = 0$ in $\mathrm{L}^p(\partial\Omega)$.

ii) "$\Leftarrow$" for $k=1$: Assume that $u \in \mathrm{W}^{1,p}(\Omega)$ with $\mathrm{Tr}(u)=0$. Then by Theorems 4.13 and 5.7 there exists $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^1(\overline{\Omega})$ such that $u_m \to u$ in $\mathrm{W}^{1,p}(\Omega)$ and $\mathrm{Tr}(u_m) \to \mathrm{Tr}(u) = 0$ in $\mathrm{L}^p(\partial\Omega)$. By Definition 5.6 this means $\mathrm{Tr}(\varphi_i u_m) \to \mathrm{Tr}(\varphi_i u) = 0$ in $\mathrm{L}^p(\partial\Omega \cap U_i)$ for $i = 1,\dots,N$. Hence, fixing $i \in \{1,\dots,N\}$ and Φ_i, Ψ_i from Lemma 5.3 (with $\gamma = \gamma_i$) we have $v_m := T_{\Psi_i}(\varphi_i u_m) \in \mathrm{C}^1(\mathrm{B}^+ \cup \mathrm{B}^0)$ and $v := T_{\Psi_i}(\varphi_i u) \in \mathrm{W}^{1,p}(\mathrm{B}^+)$ with $\mathrm{Tr}(v_m) \to \mathrm{Tr}(v) = 0$ in $\mathrm{L}^p(\mathrm{B}^0)$ for some ball $\mathrm{B} = \mathrm{B}_s(x^i) \subset \mathbb{R}^n$ (see part ii) of the proof of Theorem 5.7), where v_m, v have compact supports K_m, K in $\mathrm{B} \cap \overline{\mathrm{B}^+}$.

If we find $w_m \in \mathrm{C}_0^\infty(\mathrm{B}^+)$ with $w_m \to v$ in $\mathrm{W}^{1,p}(\mathrm{B}^+)$, by Proposition 5.4 there is $f_i := T_{\Phi_i}(w_m) \in \mathrm{C}^1(\Omega)$ with $\mathrm{supp}(f_i) \subset \Omega \cap U_i$ such that $\|f_i - \varphi_i u\|_{\mathrm{W}^{1,p}(\Omega)} \le \frac{\delta}{2(N+1)}$. Then $\tilde{u} := u - \sum_{i=1}^N \varphi_i u$ has compact support in Ω so that by Theorems 4.1 and 2.1 there is $f_0 \in \mathrm{C}_0^\infty(\Omega)$ such that $\|\tilde{u} - f_0\|_{\mathrm{W}^{1,p}(\Omega)} \le \frac{\delta}{2(N+1)}$.

Then $\|u - \sum_{i=0}^N f_i\|_{\mathrm{W}^{1,p}(\Omega)} \le \frac{\delta}{2}$ and $f := \sum_{i=0}^N f_i \in \mathrm{C}^1(\Omega)$ with compact support in Ω. Hence, by Theorems 4.1 and 2.1 there is $g \in \mathrm{C}_0^\infty(\Omega)$ such that $\|g - f\|_{\mathrm{W}^{1,p}(\Omega)} \le \frac{\delta}{2}$ and $\|u - g\|_{\mathrm{W}^{1,p}(\Omega)} \le \delta$. Hence, $u \in \mathrm{W}_0^{1,p}(\Omega)$ as $\delta > 0$ is arbitrary.

iii) It remains to find $w_m \in \mathrm{C}_0^\infty(\mathrm{B}^+)$ with $w_m \to v$ in $\mathrm{W}^{1,p}(\mathrm{B}^+)$. We have

$$\mathrm{Tr}(v_m) \to \mathrm{Tr}(v) = 0 \text{ in } \mathrm{L}^p(\mathrm{B}^0) \quad \text{and} \quad v_m \to v \text{ in } \mathrm{W}^{1,p}(\mathrm{B}^+) \tag{5.6}$$

by ii) and Proposition 5.4 and 3.8. As $v_m \in \mathrm{C}^1(\mathrm{B}^+ \cup \mathrm{B}^0)$ we have for $x \in \mathrm{B}^+ \cup \mathrm{B}^0$ with $\mathrm{B}' = \{x' \colon (x', x_n) \in \mathrm{B}\}$ and $\frac{1}{q} + \frac{1}{p} = 1$.

$$|v_m(x', x_n)| \le |v_m(x', 0)| + \int_0^{x_n} |(v_m)_{x_n}(x', t)| \,\mathrm{d}t$$

and

$$\begin{aligned}
&\int_{\mathrm{B}'} |v_m(x', x_n)|^p \,\mathrm{d}x' \\
&\overset{\text{Hölder}}{\le} 2^p \left(\int_{\mathrm{B}'} |v_m(x', 0)|^p \,\mathrm{d}x' + \int_{\mathrm{B}'} \left[\Big(\int_0^{x_n} 1^q \,\mathrm{d}t \Big)^{\frac{1}{q}} \Big(\int_0^{x_n} |\nabla v_m(x', t)|^p \,\mathrm{d}t \Big)^{\frac{1}{p}} \right]^p \mathrm{d}x' \right) \\
&\overset{\text{Fubini}}{\le} 2^p \Big(\|v_m\|_{\mathrm{L}^p(\mathrm{B}^0)}^p + x_n^{p-1} \int_0^{x_n} \int_{\mathrm{B}'} |\nabla v_m(x', t)|^p \,\mathrm{d}x' \,\mathrm{d}t \Big)
\end{aligned}$$

since $\frac{p}{q} = p - 1$ and $(a+b)^p \le 2^{p-1}(a^p + b^p)$ for $a, b \ge 0$.

Hence, by (5.6) and the dominated convergence theorem, we obtain as $m \to \infty$ (for a subsequence)

$$\int_{\mathrm{B}'} |v(x', x_n)|^p \,\mathrm{d}x' \le 2^p x_n^{p-1} \int_0^{x_n} \int_{\mathrm{B}'} |\nabla v(x', t)|^p \,\mathrm{d}x' \,\mathrm{d}t \quad \text{for a.e. } x_n > 0 \tag{5.7}$$

since we may extend v, v_m by 0 to $\overline{\mathbb{R}^n_+}$ without loosing regularity.

Let $\xi \in \mathrm{C}^\infty([0,\infty))$ such that $\xi \equiv 1$ on $[0,1]$ and $\xi \equiv 0$ on $[2,\infty)$ and define

$$\xi_m(x) := \xi(m x_n), \quad x \in \overline{\mathrm{B}^+}, m \in \mathbb{N},$$

and

$$\tilde{w}_m(x) := v(x)(1 - \xi_m(x)).$$

As $(\tilde{w}_m)_{x_n} = v_{x_n}(1-\xi_m) - m\, v\, \xi'(mx_n)$ and $(\tilde{w}_m)_{x_i} = v_{x_i}(1-\xi_m)$ for $i = 1,\dots,n-1$, we have as $\xi_m = 0$ for $x_n \geq \frac{2}{m}$

$$\begin{aligned}\int_{\mathrm{B}^+} |\nabla \tilde{w}_m - \nabla v|^p \,\mathrm{d}x &\leq C_p \int_{\mathrm{B}^+} (\xi_m)^p |\nabla v|^p \,\mathrm{d}x + C_p m^p \,\|\xi'\|^p_{\mathrm{L}^\infty([0,\infty))} \int_0^{\frac{2}{m}} \int_{\mathrm{B}'} |v|^p \,\mathrm{d}x' \,\mathrm{d}t \\ &=: A_m + B_m.\end{aligned}$$

As $\xi_m \neq 0$ only for $0 \leq x_n < \frac{2}{m}$, the dominated convergence theorem shows $A_m \to 0$ as $m \to \infty$. Using (5.7), we obtain

$$\begin{aligned}B_m &\leq C \cdot m^p \int_0^{\frac{2}{m}} \Big[t^{p-1} \int_0^t \int_{\mathrm{B}'} |\nabla v|^p(x',s) \,\mathrm{d}x' \,\mathrm{d}s\Big] \,\mathrm{d}t \\ &\leq C \cdot m^p \Big(\int_0^{\frac{2}{m}} t^{p-1} \,\mathrm{d}t\Big)\Big(\int_0^{\frac{2}{m}} \int_{\mathrm{B}'} |\nabla v|^p(x',s) \,\mathrm{d}x' \,\mathrm{d}s\Big) \\ &\leq \tilde{C} \int_0^{\frac{2}{m}} \int_{\mathrm{B}'} |\nabla v|^p \,\mathrm{d}x' \,\mathrm{d}s \to 0 \quad \text{as } m \to \infty\end{aligned}$$

by the dominated convergence theorem. Hence, $\nabla \tilde{w}_m \to \nabla v$ in $\mathrm{L}^p(\mathrm{B}^+)$ and, as obviously $\tilde{w}_m \to v$ in $\mathrm{L}^p(\mathrm{B}^+)$, we have $\tilde{w}_m \to v$ in $\mathrm{W}^{1,p}(\mathrm{B}^+)$. As $\tilde{w}_m = 0$ for $x_m \in [0, \frac{1}{m})$ and $\operatorname{supp} \tilde{w}_m \subset \operatorname{supp} v$, we have $\operatorname{supp} \tilde{w}_m \Subset \mathrm{B}^+$. Hence, $w_m := \eta_{\varepsilon_m} * \tilde{w}_m \in \mathrm{C}_0^\infty(\mathrm{B}^+)$ satisfies $w_m \to v$ in $\mathrm{W}^{1,p}(\mathrm{B}^+)$ for $\varepsilon_m \downarrow 0$ chosen appropriately (by Theorems 4.1 and 2.1). Hence, ii) shows $u \in \mathrm{W}_0^{1,p}(\Omega)$.

iv) We now prove the claim for all $k \in \mathbb{N}$ with $k \geq 2$. If $u \in \mathrm{W}_0^{k,p}(\Omega)$, then $\mathrm{D}^\alpha u \in \mathrm{W}_0^{1,p}(\Omega)$ by Proposition 3.8 a) and since $\mathrm{D}^\alpha \varphi \in \mathrm{C}_0^\infty(\Omega)$ for $\varphi \in \mathrm{C}_0^\infty(\Omega)$. Then the case $k = 1$ proves "$\Rightarrow$".

For "$\Leftarrow$", the case $k = 1$ implies $\mathrm{D}^\alpha u \in \mathrm{W}_0^{1,p}(\Omega)$ for all $|\alpha| \leq k-1$. Hence, there exist $(f_m)_{m\in\mathbb{N}}, (g_m)_{m\in\mathbb{N}} \subset \mathrm{C}_0^\infty(\Omega)$ such that $f_m \to u$ and $g_m \to u_{x_n}$ in $\mathrm{W}^{1,p}(\Omega)$ as $m \to \infty$. Assuming $\Omega \cap U_i = \mathrm{B}^+$ and $\partial\Omega \cap U_i = \mathrm{B}^0$ for some Ball B, g_m is 0 in a neighborhood of B^0. Hence,

$$G_m(x) := \int_0^{x_n} g_m(x',t) \,\mathrm{d}t$$

satisfies $(G_m)_{x_n} = g_m$ and $G_m \in \mathrm{C}_0^\infty(\Omega \cap \overline{U}_i) = \mathrm{C}_0^\infty(\overline{\mathrm{B}^+} \setminus \mathrm{B}^0)$. As $f_m \in \mathrm{C}_0^\infty(\Omega)$ we have $f_m(x) = \int_0^{x_n} (f_m)_{x_n}(x',t) \,\mathrm{d}t$ and hence

$$\|f_m - G_m\|_{\mathrm{L}^p(\Omega\cap U_i)} \leq C \,\|(f_m)_{x_n} - g_m\|_{\mathrm{L}^p(\Omega\cap U_i)}$$

with some C depending on p and U_i by Hölder's inequality. Since $g_m \to u_{x_n}$ and $(f_m)_{x_n} \to u_{x_n}$ in $\mathrm{L}^p(\Omega)$, we see that $\mathrm{D}^\alpha G_m \to \mathrm{D}^\alpha u$ in $\mathrm{L}^p(\Omega \cap U_i)$ for all $|\alpha| \leq 2$ with $\alpha_n \geq 1$ and $G_m \to u$ in $\mathrm{L}^p(\Omega \cap U_i)$ as $m \to \infty$. Iterating this argument with $u_{x_i}, i = 1,\dots,n-1$ and using an induction on k and the arguments of ii), we see that $G_m \to u$ in $\mathrm{W}^{k,p}(\Omega \cap U_i)$ and hence $u \in \mathrm{W}_0^{k,p}(\Omega)$ (like in ii)) □

Chapter 6

Embeddings and Sobolev Inequalities

We ask the question if $u \in \mathrm{W}^{k,p}(\Omega)$ automatically belongs to some other function spaces. The answer is *yes*, but we will see that to which spaces u belongs depends on p, e.g. for $k = 1$ and $p < n$, $p = n$ or $p > n$.
More precisely we ask whether $X = \mathrm{W}^{k,p}(\Omega)$ is embedded into a space Y in the following sense.

Definition 6.1. Let X, Y be Banach spaces with $X \subset Y$.

a) We say that X is *continuously embedded into* Y if there is $C > 0$ such that

$$\|u\|_Y \leq C \|u\|_X \quad \text{for all } u \in X.$$

b) We say that X is *compactly embedded into* Y if it is continuously embedded into Y and any bounded sequence in X is *precompact in* Y, i.e. for any $(u_m)_{m\in\mathbb{N}} \subset X$ with $\|u_m\|_X \leq M$ for all $m \in \mathbb{N}$, there is a subsequence $(u_{m_j})_{j\in\mathbb{N}}$ and $u \in Y$ such that $\|u_{m_j} - u\|_Y \to 0$ as $j \to \infty$.

The main tool for studying such embeddings will be Sobolev-type inequalities.

6.1 Embeddings into $\mathrm{L}^q(\Omega)$ Spaces

Let us assume $p \in [1, n)$ and ask the question for which $q \in [1, \infty)$ there exists a constant $C > 0$ such that

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^n)} \leq C \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)} \quad \text{for all } u \in \mathrm{C}_0^\infty(\mathbb{R}^n). \tag{6.1}$$

Motivation. We first demonstrate that if (6.1) holds then q has to have a specific form: To this end, let $u \in \mathrm{C}_0^\infty(\Omega)$ with $u \not\equiv 0$ and for $\lambda > 0$

$$u_\lambda(x) \coloneqq u(\lambda x) \quad \text{for all } x \in \mathbb{R}^n$$

Then the transformation rule implies

$$\begin{aligned}\int_{\mathbb{R}^n} |u_\lambda(x)|^q \,\mathrm{d}x &= \int_{\mathbb{R}^n} |u(\lambda x)|^q \,\mathrm{d}x = \int_{\mathbb{R}^n} |u(y)|^q \,\mathrm{d}y \\ \int_{\mathbb{R}^n} |\nabla u_\lambda(x)|^p \,\mathrm{d}x &= \lambda^p \int_{\mathbb{R}^n} |(\nabla u)(\lambda x)|^p \,\mathrm{d}x = \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} |\nabla u(y)|^p \,\mathrm{d}y\end{aligned}$$

If (6.1) holds for any $\lambda > 0$, then

$$\lambda^{\frac{n}{q}} \|u\|_{\mathrm{L}^q(\mathbb{R}^n)} = \|u_\lambda\|_{\mathrm{L}^q(\mathbb{R}^n)} \le C\, \|\nabla u_\lambda\|_{\mathrm{L}^p(\mathbb{R}^n)} = C\, \lambda^{1-\frac{n}{p}} \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)}$$

and therefore

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^n)} \le C\, \lambda^{1-\frac{n}{p}+\frac{n}{q}} \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)}. \tag{6.2}$$

But if $1 - \frac{n}{p} + \frac{n}{q} \neq 0$, then sending λ to either 0 or ∞ in (6.2) yields $u \equiv 1$, a contradiction. Hence, if (6.1) is satisfied, then necessarily we must have $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and $q = \frac{np}{n-p}$. This motivates the following definition.

Definition 6.2. If $p \in [1, n)$, then $p^* := \frac{np}{n-p}$ is the *Sobolev conjugate* of p. In particular, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

We next prove that (6.1) is in fact true for $q = p^*$.

Theorem 6.3 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Assume that $p \in [1, n)$. Then there exists $C > 0$, depending only on p and n, such that*

$$\|u\|_{\mathrm{L}^{p^*}(\mathbb{R}^n)} \le C\, \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)}, \quad \textit{for all } u \in \mathrm{C}^1_0(\mathbb{R}^n). \tag{6.3}$$

Moreover, we have $\mathrm{W}^{1,p}(\mathbb{R}^n) \subset \mathrm{L}^{p^*}(\mathbb{R}^n)$ and (6.3) holds for all $u \in \mathrm{W}^{1,p}(\mathbb{R}^n)$

Remark 6.4. We need that u in (6.3) has compact support, as $u \equiv 1$ shows, but C does not depend on the size of the support.

Proof of Theorem 6.3. i) Let $u \in \mathrm{C}^1_0(\mathbb{R}^n)$ and $p = 1$, whence $p^* = \frac{n}{n-1}$. Since $\operatorname{supp}(u)$ is compact, we have for all $x \in \mathbb{R}^n$ and $i \in \{1, \dots, n\}$

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\, \mathrm{d}y_i$$

and

$$|u(x)| \le \int_{-\infty}^{\infty} |\nabla u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\, \mathrm{d}y_i.$$

Hence,

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |\nabla u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)|\, \mathrm{d}y_i \right)^{\frac{1}{n-1}}.$$

Integrating this with respect to x_1 and using the general Hölder inequality (2.1 viii) with $p_k = n-1, k = 1, \dots, n-1$)

$$\begin{aligned}
\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}(x)\, \mathrm{d}x_1 &\le \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |\nabla u|\, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\
&= \left(\int_{-\infty}^{\infty} |\nabla u|\, \mathrm{d}y_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |\nabla u|\, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\
&\le \left(\int_{-\infty}^{\infty} |\nabla u|\, \mathrm{d}y_1 \right)^{\frac{n}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla u|\, \mathrm{d}y_i\, \mathrm{d}x_1 \right)^{\frac{1}{n-1}}.
\end{aligned}$$

Integrating with respect to x_2 and using again the general Hölder inequality, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u(x)|^{\frac{n}{n-1}}\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&\quad\le\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u|\,\mathrm{d}y_2\,\mathrm{d}x_1\right)^{\frac{1}{n-1}}\\
&\qquad\cdot\left[\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|\nabla u|\,\mathrm{d}y_1\right)^{\frac{1}{n-1}}\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u|\,\mathrm{d}x_1\,\mathrm{d}y_i\right)^{\frac{1}{n-1}}\,\mathrm{d}x_2\right]\\
&\quad\le\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u|\,\mathrm{d}y_2\,\mathrm{d}x_1\right)^{\frac{1}{n-1}}\\
&\qquad\cdot\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u|\,\mathrm{d}y_1\,\mathrm{d}x_2\right)^{\frac{1}{n-1}}\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u|\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_i\right)^{\frac{1}{n-1}}
\end{aligned}$$

Continuing like this and integrating with respect to $x_3,\dots,x_n$ we finally have

$$\begin{aligned}
\int_{\mathbb{R}^n}|u(x)|^{\frac{n}{n-1}}\,\mathrm{d}x&\le\prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\nabla u|\,\mathrm{d}x_1\dots\mathrm{d}y_i\dots\mathrm{d}x_n\right)^{\frac{1}{n-1}}\\
&=\left(\int_{\mathbb{R}^n}|\nabla u(x)|\,\mathrm{d}x\right)^{\frac{n}{n-1}}
\end{aligned}\tag{6.4}$$

which establishes (6.3) for $p=1$ with $C=1$.

ii) Let $p\in(1,n)$ and $u\in \mathrm{C}_0^1(\mathbb{R}^n)$. For $\gamma>1$, $v:=|u|^\gamma\in\mathrm{C}_0^1(\mathbb{R}^n)$ and by (6.4) applied to v we have by the Hölder inequality

$$\begin{aligned}
\left(\int_{\mathbb{R}^n}|u|^{\frac{\gamma n}{n-1}}\right)^{\frac{n-1}{n}}&\le\int_{\mathbb{R}^n}\left|\nabla(|u|^\gamma)\right|\mathrm{d}x=\int_{\mathbb{R}^n}\gamma|u|^{\gamma-1}|\nabla u|\,\mathrm{d}x\\
&\le\gamma\left(\int_{R^n}|u|^{(\gamma-1)\frac{p}{p-1}}\,\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n}|\nabla u|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}\tag{6.5}$$

Choosing $\gamma:=\frac{p(n-1)}{n-p}>1$, we have $\frac{\gamma n}{n-1}=(\gamma-1)\frac{p}{p-1}$ and hence

$$\frac{\gamma n}{n-1}=\frac{(np-p)+(p-n)}{n-p}\cdot\frac{p}{p-1}=\frac{np}{n-p}=p^*.$$

Thus, in view of $\frac{n-1}{n}-\frac{p-1}{p}=\frac{1}{p}-\frac{1}{n}=\frac{1}{p^*}$, (6.5) becomes (6.3) for $C=\gamma$.

iii) As $\mathrm{W}^{1,p}(\mathbb{R}^n)=\mathrm{W}_0^{1,p}(\mathbb{R}^n)$ by Corollary 4.14 for $u\in\mathrm{W}^{1,p}(\mathbb{R}^n)$ there is $(u_m)_{m\in\mathbb{N}}\subset\mathbb{C}_0^\infty(\mathbb{R}^n)$ such that $u_m\to u$ in $\mathrm{W}^{1,p}(\mathbb{R}^n)$. Hence, $\nabla u_m\to\nabla u$ in $\mathrm{L}^p(\mathbb{R}^n)$ and, as (6.3) holds for any u_m, in the limit $l\to\infty$ we conclude by Fatou's lemma that u satisfies (6.3) since $u_{m_l}\to u$ a.e. in $\mathbb{R}^n$. Als (6.3) therefore holds for all $u\in\mathrm{W}^{1,p}(\mathbb{R}^n)$, $\mathrm{W}^{1,p}(\mathbb{R}^n)$ is continuously embedded into $\mathrm{L}^{p^*}(\mathbb{R}^n)$. □

Using the previous result, we get a corresponding result on bounded domains.

Theorem 6.5 (Embeddings for $\mathrm{W}^{1,p}(\Omega), p<n$)**.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with $\partial\Omega\in\mathrm{C}^1$ and $p\in[1,n)$. Then $\mathrm{W}^{1,p}(\Omega)$ is continuously embedded into L^q for all $q\in[1,p^*]$. More precisely, for any $q\in[1,p^*]$ there is $C>0$ depending on p,q,n and Ω such that*

$$\|u\|_{\mathrm{L}^q(\Omega)}\le C\,\|\nabla u\|_{\mathrm{W}^{1,p}(\Omega)}\quad\textit{for all } u\in\mathrm{W}^{1,p}(\Omega).\tag{6.6}$$

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6.2 Embeddings into Hölder Spaces

6.3 General Embeddings and Sobolev Inequalities

Chapter 7

Applications to PDEs

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