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Fachbereich Mathematik

# Sobolev Spaces

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# Chapter 1

# Introduction

In order to have classical solutions to partial differential equations (PDEs), it is often necessary that parameter functions in the PDE are regular enough or the domain where the PDE is considered has a regular boundary (e.g. no edge). However, in applications or in nature, these regularity assumptions are often not satisfied. Hence, a fundamental concept in the theory of PDEs is the concept of weak solutions which is also the basis for important numerical methods (e.g. the finite element method). The definition of these weak solutions is based on a concept of generalized derivatives of functions, the so called *weak derivatives*. *Sobolev spaces* are Banach spaces consisting of functions with weak derivatives. Important properties of these spaces will be studied in this lecture and will be a basis to study weak solutions of PDEs afterwards. Let us start by illustrating the idea behind weak solutions with an example.

**Example 1.1.** Let $\Omega = (0,1) \subseteq \mathbb{R}$ and $f \in \mathrm{C}^0(\overline{\Omega})$ be given. We look for a solution $u \in \mathrm{C}^2(\overline{\Omega})$ to the Poisson equation in one dimension,

$$\begin{cases} -u'' = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \{0,1\} = \partial\Omega \end{cases} \tag{1.1}$$

$u$ e.g. describes the displacement of a rod which is fixed at $x = 0$ and $x = 1$, where $f$ is a force acting on the string. Of course the force $f$ is not necessarily continuous in $\overline{\Omega}$ and could have jumps.

In order to get a weaker solution concept, let $\varphi \in \mathrm{C}_0^\infty(\Omega)$ (infinitely often differentiable with compact support in $\Omega$). Then integration by parts shows

$$\int_0^1 -u''(x)\varphi(x)\,\mathrm{d}x = -u'(1)\varphi(1) + u'(0)\varphi(0) + \int_0^1 u'(x)\varphi'(x)\,\mathrm{d}x.$$

Hence, in view of (1.1)

$$\int_\Omega u'(x)\varphi'(x)\,\mathrm{d}x = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{1.2}$$

For (1.2) to be meaningful we do not need a second derivative of $u$. Moreover, $f$ only has to be integrable instead of continuous. (1.2) will even make sense if $u'$ is only a weak derivative of $u$ as we will see soon. □

In order to motivate the definition of weak derivatives, we note the following identity.

**Lemma 1.2.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and $u \in \mathrm{C}^1(\Omega)$. Then for $i \in \{1, \ldots, n\}$ we have*

$$\int_\Omega \frac{\partial u}{\partial x_i} \varphi \,\mathrm{d}x = - \int_\Omega u \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x \quad \textit{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{1.3}$$

*Proof (Exercise).* As in general neither $\partial\Omega$ nor the boundary of the support of $\varphi$ need to be regular, the proof is not immediate. Define

$$w(x) := \begin{cases} u(x)\varphi(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

As $w \in \mathrm{C}^1(\Omega)$ and $w = 0$ in a neighborhood of $\partial\Omega$, we conclude that $w \in \mathrm{C}^1(\mathbb{R}^n)$. Take an open ball B large enough such that it contains the support of $\varphi$. Then by Gauß' (or Green's formula) we have

$$\int_\Omega w_{x_i} \,\mathrm{d}x = \int_{\partial B} w \cdot \nu_i \,\mathrm{d}\sigma = 0,$$

where $\nu$ is the outward unit normal on $\partial B$. Hence, the product rule implies (1.3). □

(1.3) makes sense even if $u, \frac{\partial u}{\partial x_i} \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ (integrable on any bounded set $V$ with $\overline{V} \subset \Omega$). Hence, we define weak derivatives by:

**Definition 1.3.** Let $\Omega \subseteq \mathbb{R}^n$ be open, $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and $i \in \{1, \ldots, n\}$. $u$ has the *weak partial derivative* $\frac{\partial u}{\partial x_i}$ if there is $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega u(x) \frac{\partial u}{\partial x_i}(x) \,\mathrm{d}x = - \int_\Omega v(x)\varphi(x) \,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{1.4}$$

Then $\frac{\partial u}{\partial x_i} := v$.

Let us see in an example which functions have weak derivatives and how to calculate them.

**Example 1.4.** a) If $u \in \mathrm{C}^1(\Omega)$, then by Lemma 1.2 (1.4) is satisfied with $v := \frac{\partial u}{\partial x_i}$. Hence, $u$ is weakly differentiable and the weak derivative $\frac{\partial u}{\partial x_i}$ coincides with the classical derivative.

b) Let $\Omega = (-1, 1) \subset \mathbb{R}$, $u(x) := |x|$ for $x \in \Omega$. Then as $u \in \mathrm{C}^1(\overline{\Omega} \setminus \{0\}) \cap \mathrm{C}^0(\overline{\Omega})$, we may use the fundamental theorem of calculus to obtain for $\varphi \in \mathrm{C}_0^\infty(\Omega)$:

$$\begin{aligned} \int_\Omega u(x)\varphi'(x) \,\mathrm{d}x &= \int_{-1}^0 -x\varphi'(x) \,\mathrm{d}x + \int_0^1 x\varphi'(x) \,\mathrm{d}x \\ &= \int_{-1}^0 \varphi(x) \,\mathrm{d}x + (-x\varphi(x))\Big|_{-1}^0 - \int_0^1 \varphi(x) \,\mathrm{d}x + (x\varphi(x))\Big|_0^1 \\ &= - \int_{-1}^0 -1\varphi(x) \,\mathrm{d}x - \int_0^1 1\varphi(x) \,\mathrm{d}x = - \int_{-1}^1 v(x)\varphi(x) \end{aligned}$$

if we define $v(x) = \begin{cases} 1, & x \in (0, 1), \\ -1, & x \in (-1, 0). \end{cases}$

Then $v \in \mathrm{L}^1(\Omega)$ and since $\{0\}$ is a set of measure zero in $\mathbb{R}$, we could define $v(0)$ arbitrarily. Hence, $u$ is weakly differentiable with derivative $u' = v$. $u'$ coincides with the classical derivative in all $x \in \Omega$ where the latter exists.

c) Defining again $\Omega = (-1,1)$ and $v$ as in b), we have $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and for all $\varphi \in \mathrm{C}_0^\infty(\Omega)$ we have

$$\int_\Omega v(x)\varphi'(x)\,\mathrm{d}x = \int_{-1}^0 -\varphi'(x)\,\mathrm{d}x + \int_0^1 \varphi'(x)\,\mathrm{d}x = -\varphi(0) + \varphi(-1) + \varphi(1) - \varphi(0) = -2\varphi(0).$$

Now if $v$ would be weakly differentiable, there would be $w \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ with

$$-2\varphi(0) = -\int_\Omega w(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{1.5}$$

Fix some $f \in \mathrm{C}_0^\infty((-1,1))$ with $f(0) = 1$ and define $\varphi_n(x) = f(nx)$ for $x \in (-1,1), n \in \mathbb{N}$ (where $f = 0$ on $\mathbb{R}\setminus(-1,1)$). Then $\varphi_n \in \mathrm{C}_0^\infty((-1,1))$ with $\varphi_n(x) = 0$ for all $x \in \mathbb{R}\setminus\left(-\frac{1}{n}, \frac{1}{n}\right)$ with $\varphi_n(0) = 1$ and $\lim_{n\to\infty}\varphi_n(x) = 0$ for all $x \in \Omega \setminus \{0\}$. As $\|\varphi_n\|_{\mathrm{L}^\infty(\Omega)} \le \|f\|_{\mathrm{L}^\infty(\Omega)} < \infty$, we conclude from the dominated convergence theorem that

$$0 = \lim_{n\to\infty}\left(-\int_\Omega w(x)\varphi_n(x)\,\mathrm{d}x\right) \neq -2 = \lim_{n\to\infty} -2\varphi_n(x)$$

which contradicts (1.5). Hence, $v$ is not weakly differentiable in $\Omega$. □

Hence, there are functions which are not classically differentiable everywhere and have weak derivatives, but there are also functions being not weakly differentiable (although $v \in \mathrm{C}^1(\Omega\setminus\{0\})$ in Example 1.4).
If we define the *Sobolev space*

$$\mathrm{W}^{1,p}(\Omega) := \left\{ u \in \mathrm{L}^p(\Omega)\colon \frac{\partial u}{\partial x_i} \text{ exists in the weak sense, } \frac{\partial u}{\partial x_i} \in \mathrm{L}^p(\Omega) \text{ for all } i \in \{1,\dots,n\} \right\}$$

for $p \in [1,\infty]$, this is a Banach space which will turn out to be particularly useful in the context of weak solutions of PDEs. So we will study important properties of these spaces (and its generalisations to higher order derivatives) and finally will show how to use them for obtaining weak solutions of PDEs. We shortly illustrate the latter in an example.

**Example 1.5.** We continue Example 1.1 with $\Omega = (0,1) \subset \mathbb{R}$ and assume that $f \in \mathrm{L}^2(\Omega)$. Then in view of (1.2) we say that $u$ is a *weak solution* to (1.1) if $u \in \mathrm{W}^{1,2}(\Omega)$,

$$\int_\Omega u'(x)\varphi'(x)\,\mathrm{d}x = \int_\Omega f(x)\varphi(x) \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega),$$

where $u'$ is the weak derivative of $u$, and if $u$ satisfies $u = 0$ on $\partial\Omega$ in a certain weak sense. The latter will be specified in a detailed way in Chapter 5 as $u \in \mathrm{W}^{1,2}(\Omega)$ is not necessarily continuous. In Chapter 7, we will study the generalisation of (1.1) for $\Omega \subset \mathbb{R}^n$ being a bounded domain namely the Poisson equation

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega, \\ \quad u(x) = 0, & x \in \partial\Omega. \end{cases}$$

*Notation* 1.6. Let $n \in \mathbb{N} := \{1, 2, \dots\}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and, unless stated differently, $u\colon \Omega \to \mathbb{R}$ with some $\Omega \subset \mathbb{R}^n$.

I) Partial derivatives:

- $\frac{\partial u}{\partial x_i}(x) = \lim_{h \to 0} \frac{u(x+he_i)-u(x)}{h}$, abbreviations $u_{x_i} := \frac{\partial u}{\partial x_i}$, $u_{x_i x_j} := \frac{\partial^2 u}{\partial x_i \partial x_j}$, etc.
- $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is a multiindex of order $|\alpha| = \sum_{i=1}^n \alpha_i$ with $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$
  Then
  $$\mathrm{D}^\alpha u(x) = \frac{\partial^| \alpha | u(x)}{\partial^{\alpha_1} x_1 \cdot \ldots \partial^{\alpha_n}_{x_n}}.$$
  For $k \in \mathbb{N}$: $\mathrm{D}^k u(x) = \{\, \mathrm{D}^\alpha u(x)\colon \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| = k\}$ is identified with an element of $\mathbb{R}^{n^k}$ and $\mathrm{D}^0 u(x) := u(x)$.

II) Differential Operators:

- $\nabla u = (u_{x_1}, \dots, u_{x_n})$, *gradient* of $u$.
- $\Delta u = \sum_{i=1}^n u_{x_i x_i}$, *Laplacian* of $u$.
- $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$, directional derivative for $\nu \in \mathbb{R}^n \setminus \{0\}$.
- $\nabla \cdot \overline{u} = \operatorname{div} \overline{u} = \sum_{i=1}^n (\overline{u}_i)_{x_i}$, *divergence* of $\overline{u}\colon \mathbb{R} \to \mathbb{R}^n$.

III) Function spaces:

- $\mathrm{C}^0(\Omega) \;= \{u\colon \Omega \to \mathbb{R}\colon u \text{ continuous}\}$.
- $\mathrm{C}^k(\Omega) \;= \{u\colon \Omega \to \mathbb{R}\colon u\ k\text{-times continuously differentiable}\}$ .
  $\{u\colon \Omega \to \mathbb{R}\colon\ \mathrm{D}^\alpha u \in \mathrm{C}^0(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k\}, k \in \mathbb{N}$.
- $\mathrm{C}^\infty(\Omega) = \bigcap_{k \in \mathbb{N}} \mathrm{C}^k(\Omega) = \{u\colon \Omega \to \mathbb{R}\colon u$ *smooth*, i.e. infinitely often differentiable$\}$.
- $\operatorname{supp}(u) = \overline{\{x \in \Omega\colon u(x) \neq 0\}}$.
- $A \Subset \Omega$ if $\overline{A} \subset \Omega$ and $\overline{A}$ is compact, i.e. bounded.
- $\mathrm{C}_0^\infty(\Omega) = \{u \in \mathrm{C}^\infty(\Omega)\colon\ \operatorname{supp}(u) \Subset \Omega\}$,
  $\mathrm{C}_0^k(\Omega) \;= \{u \in \mathrm{C}^k(\Omega)\ :\ \operatorname{supp}(u) \Subset \Omega\}$.

IV)
- $\mathrm{B}_r(x) = \{y \in \mathbb{R}^n\colon |y - x| < r\}$, *open ball for* $x \in \mathbb{R}^n$, $r > 0$.
- $|\Omega|$ = *$n$-dimensional Lebesgue measure of* $\Omega \subset \mathbb{R}^n$ measurable.

# Chapter 2

# Some Facts about Lebesgue Spaces $\mathrm{L}^p(\Omega)$

Here, we recall some facts about Lebesgue spaces which should be known from previous lectures. Throughout this lecture, a set $\Omega \subset \mathbb{R}^n$ is called *measurable* if it is measurable w.r.t. the Lebesgue measure on $\mathbb{R}^n$. Unless otherwise stated, we always assume in this chapter that $\Omega \subset \mathbb{R}^n$ is measurable.
Then $u\colon \Omega \to [-\infty, \infty]$ is measurable *on* $\Omega$ if $\{x \in \Omega \colon u(x) > \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$.

## 2.1 $\mathrm{L}^p(\Omega)$: Definition and Basic Properties

i) If $u, v\colon \Omega \to [-\infty, \infty]$ are measurable on $\Omega$, they are equivalent if $u = v$ a.e. in $\Omega$. $[u]$ is the equivalence class of $u$. We always identify a function $u$ with its equivalence class.

ii) For $p \in [1, \infty]$, we define the Lebesgue space

$$\mathrm{L}^p(\Omega) \coloneqq \{u\colon \Omega \to [-\infty, \infty] \colon u \text{ measurable }, \|u\|_{\mathrm{L}^p(\Omega)} < \infty\},$$

where

$$\begin{aligned} \|u\|_{\mathrm{L}^p(\Omega)} &= \left(\int_\Omega |u(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \quad \text{if } p \in [1, \infty), \\ \|u\|_{\mathrm{L}^\infty(\Omega)} &= \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|. \end{aligned}$$

With the convention from i), $u = 0$ in $\mathrm{L}^p(\Omega)$ if $u = 0$ a.e. in $\Omega$. If $[u]$ contains a continuous function, we assume that $u$ is chosen to be continuous.

iii) $\mathrm{L}^p(\Omega)$ is a Banach space for $p \in [1, \infty]$, i.e. a complete and normed vector space.

iv) $\mathrm{L}^p$-convergence and a.e.-convergence: Let $p \in [1, \infty]$, $(u_n)_{n \in \mathbb{N}} \subset \mathrm{L}^p(\Omega)$ and $u \in \mathrm{L}^p(\Omega)$, such that $u_n \to u$ in $\mathrm{L}^p(\Omega)$, i.e. $\|u_n - u\|_{\mathrm{L}^p(\Omega)} \to 0$ as $n \to \infty$. Then there is a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ and a function $h \in \mathrm{L}^p(\Omega)$ such that $u_{n_k}(x) \to u(x)$ a.e. in $\Omega$ as $k \to \infty$ and $|u_{n_k}(x)| \leq h(x)$ a.e. in $\Omega$ for all $k \in \mathbb{N}$.

v) Minkowski's inequality: Let $1 \leq p \leq \infty$ and $u, v \in \mathrm{L}^p(\Omega)$. Then

$$\|u+v\|_{\mathrm{L}^p(\Omega)} \leq \|u\|_{\mathrm{L}^p(\Omega)} + \|v\|_{\mathrm{L}^p(\Omega)}.$$

vi) Hölder's inequality: Let $p, q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $u \in \mathrm{L}^p(\Omega), v \in \mathrm{L}^q(\Omega)$. Then $uv \in \mathrm{L}^1(\Omega)$ and

$$\|uv\|_{\mathrm{L}^1(\Omega)} \leq \|u\|_{\mathrm{L}^p(\Omega)} \|v\|_{\mathrm{L}^q(\Omega)}.$$

vii) For $x, y \in \mathbb{R}^n$,

$$\begin{aligned} \|x\|_p &= \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \quad \text{if } p \in [1,\infty), \\ \|x\|_\infty &= \max_i |x_i| \end{aligned}$$

the discrete versions of v), vi) are valid:

$$\begin{aligned} \|x+y\|_p &\leq \|x\|_p + \|y\|_p, \\ |x \cdot y| &\leq \|x\|_p \|y\|_q \quad \text{for } p \in [1,\infty], \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

viii) General Hölder inequality: Let $p_k \in [1,\infty]$, $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$, $m \geq 3$, $u_k \in \mathrm{L}^{p_k}(\Omega)$, and $k = 1, \ldots, m$. Then

$$\int_\Omega |u_1 \cdot \ldots \cdot u_m| \, \mathrm{d}x \leq \prod_{k=1}^m \|u_k\|_{\mathrm{L}^{p_k}(\Omega)}.$$

## 2.2 Limit Theorems and Fubini

i) Monotone convergence (Beppo-Levi): Let $(u_n)_{n\in\mathbb{N}}$ be measurable in $\Omega$, non-negative, and point-wise non-decreasing. Then

$$\int_\Omega \left(\lim_{n\to\infty} u_n(x)\right) \mathrm{d}x = \lim_{n\to\infty} \int_\Omega u_n(x) \, \mathrm{d}x.$$

ii) Fatou's lemma: Let $(u_n)_{n\in\mathbb{N}}$ be measurable in $\Omega$ and non-negative. Then

$$\int_\Omega \left(\liminf_{n\to\infty} u_n(x)\right) \mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega u_n(x) \, \mathrm{d}x.$$

iii) Dominated convergence (Lebesgue): Let $(u_n)_{n\in\mathbb{N}}$ and $u$ be measurable on $\Omega$ such that $u_n(x) \to u(x)$ as $n \to \infty$ a.e. in $\Omega$ and $|u_n(x)| \leq h(x)$ a.e. in $\Omega$ for all $n \in \mathbb{N}$ and some $h \in \mathrm{L}^1(\Omega)$. Then $u_n, u \in \mathrm{L}^1(\Omega)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \int_\Omega u_n(x) \, \mathrm{d}x = \int_\Omega u(x) \, \mathrm{d}x$.

iv) Fubini's theorem: Let $u = u(x, y)$ be measurable on $\mathbb{R}^{n+m}$ such that at least one of the following integrals exists and is finite:

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^{n+m}} |u(x,y)| \, \mathrm{d}x \, \mathrm{d}y, \\ I_2 &= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |u(x,y)| \, \mathrm{d}x\right) \mathrm{d}y, \\ I_2 &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |u(x,y)| \, \mathrm{d}y\right) \mathrm{d}x. \end{aligned}$$

Then $u(\cdot, y) \in \mathrm{L}^1(\mathbb{R}^n)$ for a.e. $y \in \mathbb{R}^m$, $\int_{\mathbb{R}^m} u(\cdot, y) \, \mathrm{d}y \in \mathrm{L}^1(\mathbb{R}^n)$, $u(x, \cdot) \in \mathrm{L}^1(\mathbb{R}^m)$ for a.e. $x \in \mathbb{R}^n$, $\int_{\mathbb{R}^n} u(x, \cdot) \, \mathrm{d}x \in \mathrm{L}^1(\mathbb{R}^m)$, and $I_1 = I_2 = I_3$.

## 2.3 Dense Subspaces and Mollifier

In this section let $\Omega \subset \mathbb{R}^n$ be open.

i) $\mathrm{C}_0^\infty(\Omega)$ is dense in $\mathrm{L}^p(\Omega)$ for any $p \in [1, \infty)$, i.e. for any $u \in \mathrm{L}^p(\Omega)$ and $\varepsilon > 0$ there is $\varphi \in \mathrm{C}_0^\infty(\Omega)$ such that $\|\varphi - u\|_{\mathrm{L}^p(\Omega)} < \varepsilon$.

ii) Notation: For $\varepsilon > 0, x \in \mathbb{R}^n$, let

$$\begin{aligned} \mathrm{B}_\varepsilon(x) &:= \{y \in \mathbb{R}^n \colon |y - x| < \varepsilon\}, \\ \Omega_\varepsilon &:= \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > \varepsilon\}, \quad \text{and} \\ \mathrm{L}^p_{\mathrm{loc}}(\Omega) &:= \{u \colon \Omega \to [-\infty, \infty] \colon u \in \mathrm{L}^p(V) \text{ for all } V \Subset \Omega\} \quad \text{for } p \in [1, \infty]. \end{aligned}$$

iii) Standard mollifier: Let

$$\eta(x) := \begin{cases} c \exp\left(\frac{1}{|x|^2 - 1}\right), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where $c > 0$ is chosen such that $\int_{\mathbb{R}^n} \eta(x) \,\mathrm{d}x = 1$. Then $\eta \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ is called *standard mollifier*. For $\varepsilon > 0$, $\eta_\varepsilon(x) := \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right)$, $x \in \mathbb{R}^n$, satisfies $\eta_\varepsilon \in \mathrm{C}_0^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \eta_\varepsilon(x) \,\mathrm{d}x = 1$, and $\operatorname{supp}(\eta_\varepsilon) = \overline{\mathrm{B}_\varepsilon(0)}$.

iv) For $u \in \mathrm{L}^1(\Omega)$, we extend $u$ by $u(x) := 0$ for all $x \in \mathbb{R}^n \setminus \Omega$ to $u \in \mathrm{L}^1(\mathbb{R}^n)$ and define its *mollification* $u_\varepsilon := \eta_\varepsilon * u$ for $\varepsilon > 0$, i.e.

$$u_\varepsilon(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x - y) u(y) \,\mathrm{d}y = \int_\Omega \eta_\varepsilon(x - y) u(y) \,\mathrm{d}y = \int_{\mathrm{B}_\varepsilon(x) \cap \Omega} \eta_\varepsilon(x - y) u(y) \,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

The mollification has the following properties:

**Theorem 2.1.** *Let $u \in \mathrm{L}^1(\Omega)$ and $\varepsilon > 0$. Then the following statements hold true:*

*a) $u_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n)$, $u_\varepsilon(x) \to u(x)$ for a.e. $x \in \Omega$ as $\varepsilon \downarrow 0$.*

*b) If $\operatorname{supp}(u) \Subset \Omega$, then $u_\varepsilon \in \mathrm{C}_0^\infty(\Omega)$ for small enough $\varepsilon$.*

*c) If $u \in \mathrm{C}^0(\Omega)$, $V \Subset \Omega$, then $u_\varepsilon \to u$ uniformly in $V$ as $\varepsilon \downarrow 0$.*

*d) If $u \in \mathrm{L}^p(\Omega)$ for some $p \in [1, \infty)$, then $u_\varepsilon \in \mathrm{L}^p(\Omega)$, $\|u_\varepsilon\|_{\mathrm{L}^p(\Omega)} \leq \|u\|_{\mathrm{L}^p(\Omega)}$ and $u_\varepsilon \to u$ in $\mathrm{L}^p(\Omega)$ as $\varepsilon \downarrow 0$. Moreover, $u_\varepsilon \in \mathrm{C}^\infty(\Omega)$.*

*e) If $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, then $u_\varepsilon \in \mathrm{C}^\infty(\Omega_\varepsilon)$.*

*Proof.* a) For $i \in \{1, \dots, n\}$ and $h \in \mathbb{R} \setminus \{0\}$ let

$$\mathrm{D}_i^h v(x) := \frac{1}{h}(v(x + he_i) - v(x)), \quad x \in \mathbb{R}^n.$$

As $\eta_\varepsilon \in \mathrm{C}_0^\infty(\mathbb{R}^n)$, we have $\nabla\eta_\varepsilon \in \mathrm{L}^\infty(\mathbb{R}^n)^n$. So $\mathrm{D}_i^h\eta_\varepsilon \in \mathrm{L}^\infty(\mathbb{R}^n)$ by the mean value theorem. As moreover $\mathrm{D}_i^h\eta_\varepsilon(z) \to \frac{\partial\eta_\varepsilon}{\partial x_i}(z)$ as $h \to 0$ for any $z \in \mathbb{R}^n$, the dominated convergence theorem implies

$$\begin{aligned}
\mathrm{D}_i^h(u_\varepsilon)(x) &= \int_\Omega \frac{1}{h}(\eta_\varepsilon(x+he_i-y) - \eta_\varepsilon(x-y))u(y)\,\mathrm{d}y = \int_\Omega (\mathrm{D}_i^h\eta_\varepsilon(x-y))u(y)\,\mathrm{d}y \\
&\to \int_\Omega \frac{\partial\eta_\varepsilon}{\partial x_i}(x-y)u(y)\,\mathrm{d}y \quad \text{as } h \downarrow 0.
\end{aligned}$$

Hence, $\frac{\partial}{\partial x_i}u_\varepsilon(x) = \int_\Omega \frac{\partial\eta_\varepsilon}{\partial x_i}(x-y)u(y)\,\mathrm{d}y$. By induction, $u_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n)$. By Lebesgue's differentiation theorem (see [Eva10, Appendix E.4]), we have

$$\lim_{r\downarrow 0}\frac{1}{|\mathrm{B}_r(x)|}\int_{\mathrm{B}_r(x)} |u(y)-u(x)|\,\mathrm{d}y = 0 \quad \text{for a.e. } x \in \Omega. \tag{2.1}$$

For any such $x$ we obtain (by choosing $\varepsilon > 0$ small such that $\overline{\mathrm{B}_\varepsilon(x)} \subset \Omega$)

$$\begin{aligned}
|u_\varepsilon(x) - u(x)| &= \Big|\int_{\mathrm{B}_\varepsilon(x)} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y - u(x)\Big| \\
&= \Big|\int_{\mathrm{B}_\varepsilon(x)} \eta_\varepsilon(x-y)(u(y)-u(x))\,\mathrm{d}y\Big| \\
&\le \frac{1}{\varepsilon^n}\int_{\mathrm{B}_\varepsilon(x)} \|\eta\|_{\mathrm{L}^\infty(\mathbb{R}^n)}|u(y)-u(x)|\,\mathrm{d}y \\
&\le \frac{C}{|\mathrm{B}_\varepsilon(x)|}\int_{\mathrm{B}_\varepsilon(x)} |u(y)-u(x)|\,\mathrm{d}y \quad \to 0 \text{ as } \varepsilon \downarrow 0
\end{aligned} \tag{2.2}$$

due to (2.1).

b) If $\mathrm{supp}(u) \Subset \Omega$, let $\delta := \mathrm{dist}(\mathrm{supp}(u), \partial\Omega) > 0$. Then for any $x \in \Omega \setminus \Omega_{\frac{\delta}{2}}$ and $\varepsilon \le \frac{\delta}{2}$ we have $\mathrm{B}_\varepsilon(x) \cap \mathrm{supp}(u) = \emptyset$ and

$$u_\varepsilon(x) = \int_{\mathrm{B}_\varepsilon(x)\cap\Omega} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y = 0.$$

Hence, $\mathrm{supp}(u_\varepsilon) \subset \overline{\Omega_{\frac{\delta}{2}}} \Subset \Omega$. By a), $u_\varepsilon \in \mathrm{C}_0^\infty(\Omega)$.

c) For $u \in \mathrm{C}^0(\Omega)$ and $V \Subset \Omega$, choose $W$ such that $V \Subset W \Subset \Omega$. Then $u$ is uniformly continuous in $W$ and (2.1) holds uniformly for $x \in V$. Hence, also (2.2) is satisfied uniformly for $x \in V$ and thus $u_\varepsilon \to u$ uniformly in $V$.

d) For $x \in \Omega$, by using Hölder's inequality and $\eta_\varepsilon \ge 0$ along with $\int_{\mathbb{R}^n}\eta_\varepsilon(z)\,\mathrm{d}y = 1$, we get

$$\begin{aligned}
|u_\varepsilon(x)| &= \Big|\int_\Omega (\eta_\varepsilon(x-y))^{1-\frac{1}{p}}(\eta_\varepsilon(x-y))^{\frac{1}{p}}u(y)\,\mathrm{d}y\Big| \\
&\le \underbrace{\Big(\int_\Omega \eta_\varepsilon(x-y)\,\mathrm{d}y\Big)^{\frac{p-1}{p}}}_{\le 1}\Big(\int_\Omega \eta_\varepsilon(x-y)|u(y)|^p\,\mathrm{d}y\Big)^{\frac{1}{p}} \le \Big(\int_\Omega \eta_\varepsilon(x-y)|u(y)|^p\,\mathrm{d}y\Big)^{\frac{1}{p}}.
\end{aligned}$$

Raising this to the power of $p$ and integrating w.r.t $x \in \Omega$, by using Fubini we have

$$\begin{aligned}
\|u_\varepsilon\|^p_{\mathrm{L}^p(\Omega)}\| &\le \int_\Omega\int_\Omega \eta_\varepsilon(x-y)|u(y)|^p\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega |u(y)|^p\underbrace{\Big(\int_\Omega \eta_\varepsilon(x-y)\,\mathrm{d}x\Big)}_{\in[0,1]}\mathrm{d}y \\
&\le \int_\Omega |u(y)|^p\,\mathrm{d}y = \|u\|^p_{\mathrm{L}^p(\Omega)}.
\end{aligned} \tag{2.3}$$

In particular, this implies $u_\varepsilon \in \mathrm{L}^p(\Omega)$.

Given $\mu > 0$, we may choose $\varphi \in \mathrm{C}_0^\infty(\Omega)$ such that $\|u - \varphi\|_{\mathrm{L}^p(\Omega)} < \frac{\mu}{3}$. As $\varphi$ and $\varphi_\varepsilon$ have compact support in $\Omega$ by b), we deduce from c) that $\varphi_\varepsilon \to \varphi$ uniformly in $\Omega$ as $\varepsilon \downarrow 0$. Hence, we may choose $\varepsilon_0 > 0$ small enough such that $\|\varphi_\varepsilon - \varphi\|_{\mathrm{L}^p(\Omega)} < \frac{\mu}{3}$ for all $\varepsilon \in (0, \varepsilon_0)$. But then,

$$\begin{aligned}
\|u_\varepsilon - u\|_{\mathrm{L}^p(\Omega)} &\leq \|u_\varepsilon - \varphi_\varepsilon\|_{\mathrm{L}^p(\Omega)} + \|\varphi_\varepsilon - \varphi\|_{\mathrm{L}^p(\Omega)} + \|\varphi - u\|_{\mathrm{L}^p(\Omega)} \\
&\leq \|\eta_\varepsilon * u - \eta_\varepsilon * \varphi\|_{\mathrm{L}^p(\Omega)} + \frac{2}{3}\mu \\
&= \|\eta_\varepsilon * (u - \varphi)\|_{\mathrm{L}^p(\Omega)} + \frac{2}{3}\mu = \|(u - \varphi)_\varepsilon\|_{\mathrm{L}^p(\Omega)} + \frac{2}{3}\mu \\
&\overset{(2.3)}{\leq} \|u - \varphi\|_{\mathrm{L}^p(\Omega)} + \frac{2}{3}\mu < \mu \quad \text{for all } \varepsilon \in (0, \varepsilon_0).
\end{aligned}$$

We still have $u_\varepsilon \in \mathrm{C}^\infty(\Omega)$ since for $x \in \mathrm{B}_\delta(x_0)$ with $\overline{\mathrm{B}_{2\delta}(x_0)} \subset \Omega$ we have for $x \in K \coloneqq \overline{\mathrm{B}_{2\delta}(x_0)}$ and $\varepsilon \in (0, \delta)$, $u_\varepsilon(x) = \int_K \eta_\varepsilon(x - y) u(y)\,\mathrm{d}y$. A similar argument shows e). ☐

## 2.4 Polar Coordinates

Let $f \in \mathrm{C}^1(\overline{\mathrm{B}_r(x_0)})$ with $x_0 \in \mathbb{R}^n$, $r > 0$. Then by the transformation rule with $x = x_0 + sz$, $s \in (0, r)$, $z \in \partial\mathrm{B}_1(0)$ we have

$$\int_{\mathrm{B}_r(x_0)} f(x)\,\mathrm{d}x = \int_0^r \left( \int_{\partial\mathrm{B}_s(x_0)} f\,\mathrm{d}\sigma(x) \right) \mathrm{d}s = \int_0^r s^{n-1} \int_{\partial\mathrm{B}_1(0)} f(x_0 + sz)\,\mathrm{d}\sigma(z)\,\mathrm{d}s.$$

In particular, if $x_0 = 0$, $f$ is radially symmetric and $\omega_n$ is the surface $|\partial\mathrm{B}_1(0)|$ of $\partial\mathrm{B}_1(0)$, we get

$$\int_{\mathrm{B}_r(0)} f(x)\,\mathrm{d}x = \omega_n \int_0^r f(s) s^{n-1}\,\mathrm{d}s,$$

where $s = |x|$.

*Proof.* See [Eva10, Appendix C.3]. ☐

# Chapter 3

# Weak Derivatives and Definitions of Sobolev Spaces

We already saw in the introduction that for $u \in \mathrm{C}^1(\Omega)$ with $\Omega \subset \mathbb{R}^n$ open we have

$$\int_\Omega u(x)\varphi_{x_i}(x)\,\mathrm{d}x = -\int_\Omega u_{x_i}(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{3.1}$$

by Lemma 1.2.
More generally, for higher order derivatives we have the following result:

**Lemma 3.1.** *Let $\Omega \subset \mathbb{R}^n$ be open, $u \in \mathrm{C}^k(\Omega)$ with $k \in \mathbb{N}$, and $\alpha \in \mathbb{N}_0^n$ be a multiindex with $|\alpha| = k$. Then*

$$\int_\Omega u(x)\,\mathrm{D}^\alpha\varphi(x) = (-1)^{|\alpha|}\int_\Omega \mathrm{D}^\alpha u(x)\varphi(x)\,\mathrm{d}x \quad \textit{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{3.2}$$

*Proof.* For $k = 1$, (3.2) is just (3.1). For $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| = k$ we have

$$\mathrm{D}^\alpha\phi(x) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}(\dots(\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}})\dots)\phi(x)$$

and (3.2) follows by applying (3.1) $k$ times. $\square$

In order to define the weak derivative $\mathrm{D}^\alpha u$, we look for a variant of (3.2) which is satisfied if $u$ has less regularity than being in $\mathrm{C}^k(\Omega)$. As the integrals in (3.2) are meaningful if $u$, $\mathrm{D}^\alpha u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, we define the weak derivative $\mathrm{D}^\alpha u$ of $u$ as follows (see introduction for $|\alpha| = 1$).

**Definition 3.2.** Let $\Omega$ be an open set, $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0^n$ a multiindex. $u$ has the *$\alpha$th weak partial derivative* $\mathrm{D}^\alpha u$ if there is $v \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega u(x)\,\mathrm{D}^\alpha\varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega v(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{3.3}$$

If (3.3) is satisfied, we define $\mathrm{D}^\alpha u \coloneqq v$.

In order to show the uniqueness of the weak derivative, we need the following fundamental lemma.

**Lemma 3.3** (Fundamental lemma of calculus of variations)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$. Then we have the equivalence*

$$\int_\Omega u(x)\varphi(x)\,\mathrm{d}x = 0 \quad \textit{for all } \varphi \in \mathrm{C}_0^\infty(\Omega) \iff u = 0 \textit{ a.e. in } \Omega.$$

*Proof.* "$\Leftarrow$" is obvious.
"$\Rightarrow$": Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ with $\int_\Omega u\varphi\,\mathrm{d}x = 0$ for all $\varphi \in \mathrm{C}_0^\infty(\Omega)$. We fix $K \subset \Omega$ compact and define

$$\operatorname{sign}(u(x)) := \begin{cases} 1 & \text{if } u(x) > 0, \\ -1 & \text{if } u(x) < 0, \\ 0 & \text{if } u(x) \in \{0, -\infty, +\infty\} \end{cases}$$

and

$$f(x) := \begin{cases} \operatorname{sign}(u(x)) & \text{if } x \in K, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus K. \end{cases}$$

As $|u| < \infty$ a.e. in $K$ a.e., we have $u(x)f(x) = |u(x)|$ for a.e. $x \in K$. Since $f \in \mathrm{L}^\infty(\Omega)$ with $\operatorname{supp}(f) \subset K \Subset \Omega$, we define $\varphi_n := f_{\frac{1}{n}} = \eta_{\frac{1}{n}} * f$ and deduce from Theorem 2.1 a), b) that $\varphi_n \in \mathrm{C}_0^\infty(\Omega)$ and $\varphi_{n_k}(x) \to f(x)$ a.e. in $\Omega$ as $k \to \infty$ for some subsequence. As moreover

$$|\varphi_{n_k}(x)| \le \int_\Omega \eta_{\frac{1}{n_k}}(x-y)|f(y)|\,\mathrm{d}y \le \underbrace{\|f\|_{\mathrm{L}^\infty(\Omega)}}_{\le 1} \underbrace{\int_\Omega \eta_{\frac{1}{n_k}}(x-y)\,\mathrm{d}y}_{\le 1} \le 1 \quad \text{for all } x \in \Omega, k \in \mathbb{N},$$

the dominated convergence theorem implies

$$0 = \lim_{k\to\infty} \int_\Omega u(x)\varphi_{n_k}(x)\,\mathrm{d}x = \int_\Omega u(x)f(x)\,\mathrm{d}x = \int_K |u(x)|\,\mathrm{d}x.$$

Hence, $u = 0$ a.e. in $K$. As e.g. $\Omega = \bigcup_{k=1}^\infty K_n$ with $K_n := \overline{\Omega_{\frac{1}{n}}} \cap \overline{\mathrm{B}_n(0)}$ and $u = 0$ a.e. in $K_n$ (as $K_n \subset \Omega$ compact), we have $u = 0$ a.e. in $\Omega$. ☐

With this result we show the uniqueness of the weak derivative and its equality with the classical derivative if $u$ is classically differentiable.

**Lemma 3.4.** *Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0^n$, with $|\alpha| = k \in \mathbb{N}$. If the weak derivative $\mathrm{D}^\alpha u$ exists it is uniquely defined up to a set of measure zero. If $u \in \mathrm{C}^k(\Omega)$, then $\mathrm{D}^\alpha u$ exists and is equal to the classical derivative $\mathrm{D}^\alpha u$. Hence, we use $\mathrm{D}^\alpha$ both for weak and classical partial derivatives.*

*Proof.* If $v$ and $\tilde{v}$ are $\alpha$th weak derivatives of $u$, by (3.3)

$$\int_\Omega (v - \tilde{v})(x)\varphi(x)\,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega).$$

Hence, by Lemma 3.3, $v - \tilde{v} = 0$ a.e. in $\Omega$ and $v = \tilde{v}$ a.e. in $\Omega$. If $u \in \mathrm{C}^k(\Omega)$, then by Lemma 3.1, (3.3) is satisfied with $v = \mathrm{D}^\alpha u$ and hence the classical derivative $\mathrm{D}^\alpha u$ is also a weak derivative. Due to the uniqueness, the claim follows. ☐

As the weak derivative is well-defined, we may now define Sobolev spaces consisting of functions having weak derivatives in $\mathrm{L}^p$-spaces.

**Definition 3.5.** a) Let $k \in \mathbb{N}$, $p \in [1,\infty]$ and $\Omega \subset \mathbb{R}^n$ be open. We define the Sobolev space

$$\mathrm{W}^{k,p}(\Omega) := \Big\{u \in \mathrm{L}^p(\Omega)\colon \text{ weak derivative } \mathrm{D}^\alpha u \text{ ex. with } \mathrm{D}^\alpha u \in \mathrm{L}^p(\Omega) \text{ for all } 0 \le |\alpha| \le k\Big\}$$

with the norm

$$\|u\|_{\mathrm{W}^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha|\le k} \|\,\mathrm{D}^\alpha u\|^p_{\mathrm{L}^p(\Omega)}\right)^{\frac{1}{p}} & \text{if } p \in [1,\infty), \\ \sum_{|\alpha|\le k} \|\,\mathrm{D}^\alpha u\|_{\mathrm{L}^\infty(\Omega)} & \text{if } p = \infty. \end{cases}$$

We further define

$$\mathrm{W}^{k,p}_0(\Omega) := \overline{\mathrm{C}^\infty_0(\Omega)}^{\|\cdot\|_{\mathrm{W}^{k,p}(\Omega)}}$$

to be the closure of $\mathrm{C}^\infty_0(\Omega)$ in $\mathrm{W}^{k,p}(\Omega)$ and

$$\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega) := \bigcap_{V \Subset \Omega} W^{k,p}(V).$$

For $p = 2$, we define $\mathrm{H}^k(\Omega) := \mathrm{W}^{k,2}(\Omega)$ and $\mathrm{H}^k_0(\Omega) := \mathrm{W}^{k,2}_0(\Omega)$.

b) For $(u_m)_{m\in\mathbb{N}} \subset \mathrm{W}^{k,p}(\Omega)$ and $u \in \mathrm{W}^{k,p}(\Omega)$, we say $u_m \to u$ in $W^{k,p}(\Omega)$ if

$$\lim_{n\to\infty} \|u_m - u\|_{\mathrm{W}^{k,p}(\Omega)} = 0.$$

We say $u_m \to u$ in $\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega)$ if $u_m \to u$ in $\mathrm{W}^{k,p}(V)$ for all $V \Subset \Omega$.

*Remark* 3.6. a) For $\alpha = (0,\dots,0)$ we set $\mathrm{D}^\alpha u = \mathrm{D}^0 u = u$. We further identify functions in $\mathrm{W}^{k,p}(\Omega)$ which agree a.e. If for $u \in \mathrm{W}^{k,p}(\Omega)$ the equivalence class $[u]$ contains a continuous representative, the latter is chosen for $u$.

b) $u \in \mathrm{W}^{k,p}_0(\Omega)$ if and only if there exists $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^\infty_0(\Omega)$ such that $u_m \to u$ in $\mathrm{W}^{k,p}(\Omega)$. We interpret $\mathrm{W}^{k,p}_0(\Omega)$ as the set of $u \in \mathrm{W}^{k,p}(\Omega)$ such that " $\mathrm{D}^\alpha u = 0$ on $\partial\Omega$ for any $|\alpha| \le k-1$". This interpretation will be made precise in Chapter 5.

c) The letter "H" in $\mathrm{H}^k(\Omega)$ and $\mathrm{H}^k_0(\Omega)$ is used as those are Hilbert spaces as we will see soon.

**Example 3.7.** Let $\Omega = \mathrm{B}_1(0) \subset \mathbb{R}^n$, $u(x) = |x|^{-a}$ for $x \in \Omega \setminus \{0\}$ with some $a > 0$. Given $p \in [1,\infty)$, for which $a$ do we have $u \in \mathrm{W}^{1,p}(\Omega)$?
Since $u \in C^\infty(\Omega \setminus \{0\})$, we have for $x \neq 0$

$$\begin{aligned} u_{x_i}(x) &= -a|x|^{-a-1}\frac{x_i}{|x|} = -\frac{a x_i}{|x|^{a+2}} \quad \text{and} \\ |\nabla u(x)| &= \frac{a}{|x|^{a+1}}. \end{aligned}$$

For fixed $\varphi \in \mathrm{C}^\infty_0(\Omega)$ and $\varepsilon > 0$, Green's formula ($\nu$ is the outward unit normal on $\Omega \setminus \overline{\mathrm{B}_\varepsilon(0)}$ ) implies

$$\int_{\Omega\setminus\overline{\mathrm{B}_\varepsilon(0)}} u\,\varphi_{x_i}\,\mathrm{d}x = -\int_{\Omega\setminus\overline{\mathrm{B}_\varepsilon(0)}} u_{x_i}\varphi\,\mathrm{d}x + \underbrace{\int_{\partial\Omega} u\,\varphi\,\nu_i\,\mathrm{d}\sigma}_{=0} + \int_{\partial\mathrm{B}_\varepsilon(0)} u\,\varphi\,\nu_i\,\mathrm{d}\sigma. \tag{3.4}$$

We may pass to the limit $\varepsilon \downarrow 0$ in the first two integrals if $u \in \mathrm{L}^1(\Omega)$ and $\nabla u \in \mathrm{L}^1(\Omega)^n$, i.e. $a < n$ and $a+1 < n$. As for $a < n-1$, we further have

$$\Big| \int_{\partial \mathrm{B}_\varepsilon(0)} u\, \varphi\, \nu_i \,\mathrm{d}\sigma \Big| \leq \|\varphi\|_{\mathrm{L}^\infty(\Omega)} \int_{\partial \mathrm{B}_\varepsilon(0)} \varepsilon^{-a} \,\mathrm{d}\sigma \leq C\varepsilon^{n-1-a} \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

Hence, for $a < n-1$ we may pass to the limit $\varepsilon \downarrow 0$ in (3.4) and obtain $\int_\Omega u\varphi_{x_i} \,\mathrm{d}x = -\int_\Omega u_{x_i}\varphi \,\mathrm{d}x$. Hence, the weak derivative $u_{x_i}$ exists for $a < n-1$. Hence, $u \in \mathrm{W}^{1,p}(\Omega)$ if $u \in \mathrm{L}^p(\Omega)$ and $\nabla u = \frac{-ax}{|x|^{a+2}} \in \mathrm{L}^p(\Omega)^n$, i.e. $ap < n$ and $(a+1)p < n$. We conclude that

$$u \in \mathrm{W}^{1,p}(\Omega) \iff a < \frac{n-p}{p} \text{ (and } p < n).$$

Next, we prove some elementary properties of weak derivatives which are well known in the case of classical derivatives.

**Proposition 3.8.** *Let $\Omega$ be open, $k \in \mathbb{N}$, $p \in [1,\infty]$, $u, v \in \mathrm{W}^{k,p}(\Omega)$, and $\alpha \in \mathbb{N}_0^n$ with $1 \leq |\alpha| \leq k$.*

*a) $\mathrm{D}^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ (with $\mathrm{W}^{0,p}(\Omega) = \mathrm{L}^p(\Omega)$) and $\mathrm{D}^\beta(\mathrm{D}^\alpha(u)) = \mathrm{D}^\alpha(\mathrm{D}^\beta(u)) = \mathrm{D}^{\alpha+\beta}(u)$ for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \leq k$.*

*b) For $\lambda, \mu \in \mathbb{R}$ we have $\lambda u + \mu v \in \mathrm{W}^{k,p}(\Omega)$ and $\mathrm{D}^\alpha(\lambda u + \mu v) = \lambda\,\mathrm{D}^\alpha u + \mu\,\mathrm{D}^\alpha v$.*

*c) If $V \subset \Omega$ is open, then $u \in \mathrm{W}^{k,p}(V)$.*

*d) If $\xi \in \mathrm{C}_0^\infty(\Omega)$, then $\xi u \in \mathrm{W}^{k,p}(\Omega)$ and the Leibniz formula*

$$\mathrm{D}^\alpha(\xi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \mathrm{D}^\beta \xi\ \mathrm{D}^{\alpha-\beta} u$$

*holds with*

$$\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha-\beta)!\beta!}, \quad \alpha! = \prod_{i=1}^n \alpha_i!,$$

*and*

$$\beta \leq \alpha \iff \forall i \in \{1,\dots,n\}\colon \beta_i \leq \alpha_i.$$

*Proof.* b) and c) easily follow from Definition 3.2.

a) Let $\varphi \in \mathrm{C}_0^\infty(\Omega)$. Then $\mathrm{D}^\beta\varphi \in \mathrm{C}_0^\infty(\Omega)$ and (3.3) implies

$$\begin{aligned}\int_\Omega \mathrm{D}^\alpha u\,\mathrm{D}^\beta\varphi \,\mathrm{d}x &= (-1)^{|\alpha|}\int_\Omega u\,\mathrm{D}^{\alpha+\beta}\varphi \,\mathrm{d}x \\ &= (-1)^{|\alpha|}(-1)^{|\alpha|+|\beta|}\int_\Omega \mathrm{D}^{\alpha+\beta}u\,\varphi \,\mathrm{d}x \\ &= (-1)^{|\beta|}\int_\Omega \mathrm{D}^{\alpha+\beta}u\,\varphi \,\mathrm{d}x\end{aligned}$$

as $|\alpha| + |\beta| = |\alpha+\beta|$. Hence, $\mathrm{D}^\beta(\mathrm{D}^\alpha u) = \mathrm{D}^{\alpha+\beta}u$, for $|\beta| \leq k - |\alpha|$.

d) Let $\varphi \in \mathrm{C}_0^\infty(\Omega)$. In case of $|\alpha| = 1$, we have

$$\int_\Omega \xi u \,\mathrm{D}^\alpha \varphi \,\mathrm{d}x = \int_\Omega u \,\mathrm{D}^\alpha(\xi\varphi) - u(\,\mathrm{D}^\alpha\xi)\varphi \,\mathrm{d}x \overset{(3.3)}{=} -\int_\Omega (\xi \,\mathrm{D}^\alpha u + u \,\mathrm{D}^\alpha \xi)\,\varphi \,\mathrm{d}x.$$

Hence, $\mathrm{D}^\alpha(\xi u) = \xi \,\mathrm{D}^\alpha u + u \,\mathrm{D}^\alpha \xi \in \mathrm{L}^p(\Omega)$ and the claim is true for $|\alpha| = 1$.

Assume the claim is true for all $|\alpha| \le l$ with some $l \in \{1, \dots, k-1\}$ (IA).

Let $\alpha$ satisfy $|\alpha| = l + 1$. Then $\alpha = \beta + \gamma$ for some $\beta, \gamma \in \mathbb{N}_0^k$ with $|\beta| = l$ and $|\gamma| = 1$. Hence,

$$\begin{aligned}
\int_\Omega \xi u \,\mathrm{D}^\alpha \varphi = \; & \int_\Omega \xi u \,\mathrm{D}^\beta(\,\mathrm{D}^\gamma \varphi)\,\mathrm{d}x \\
\overset{(\mathrm{IA})}{=} & (-1)^{|\beta|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \mathrm{D}^\sigma \xi \;\mathrm{D}^{\beta-\sigma} u \;\mathrm{D}^\gamma \varphi \,\mathrm{d}x \\
\overset{(3.3)}{=} & (-1)^{|\beta|+|\gamma|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \mathrm{D}^\gamma \left( \mathrm{D}^\sigma \xi \;\mathrm{D}^{\beta-\sigma} u \right) \varphi \,\mathrm{d}x \\
\overset{(\mathrm{IA})}{=} & (-1)^{|\alpha|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \left[ \mathrm{D}^{\sigma+\gamma} \xi \;\mathrm{D}^{\beta-\sigma} u + \mathrm{D}^\sigma \xi \;\mathrm{D}^{\alpha-\sigma} u \right] \varphi \,\mathrm{d}x \\
= \;& (-1)^{|\alpha|} \int_\Omega \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \left[ \mathrm{D}^{\sigma+\gamma} \xi \;\mathrm{D}^{\alpha-(\sigma+\gamma)} u + \mathrm{D}^\sigma \xi \;\mathrm{D}^{\alpha-\sigma} u \right] \varphi \,\mathrm{d}x \\
= \;& (-1)^{|\alpha|} \int_\Omega \left[ \sum_{\gamma \le \rho \le \alpha} \binom{\beta}{\rho-\gamma} + \sum_{0 \le \rho \le \beta} \binom{\beta}{\rho} \right] \mathrm{D}^\rho \xi \;\mathrm{D}^{\alpha-\rho} u \,\varphi \,\mathrm{d}x \\
= \;& (-1)^{|\alpha|} \int_\Omega \sum_{\rho \le \alpha} \left[ \binom{\beta}{\rho-\gamma} + \binom{\beta}{\rho} \right] \mathrm{D}^\rho \xi \;\mathrm{D}^{\alpha-\rho} u \,\varphi \,\mathrm{d}x
\end{aligned}$$

with the convention $\binom{\beta}{\tilde\beta} = 0$ if $\beta_i < \tilde\beta_i$ or $\tilde\beta_i < 0$ for some $i \in \{1, \dots, n\}$. As

$$\binom{\beta}{\rho-\gamma} + \binom{\beta}{\rho} = \binom{\beta+\gamma}{\rho} = \binom{\alpha}{\rho},$$

we deduce that the claim holds by induction. □

Finally, we show that $\mathrm{W}^{k,p}(\Omega)$ is a Banach space.

**Theorem 3.9.** *Let $\Omega \subset \mathbb{R}^n$ be open, $k \in \mathbb{N}$, $p \in [1, \infty]$. Then $\mathrm{W}^{k,p}(\Omega)$ is a Banach space. Moreover, $\mathrm{H}^k(\Omega)$ is a Hilbert space with the scalar product*

$$(u, v)_{\mathrm{H}^k(\Omega)} := \sum_{|\alpha| \le k} \int_\Omega \mathrm{D}^\alpha u \;\mathrm{D}^\beta v \,\mathrm{d}x.$$

*Proof.* By Proposition 3.8 b), $\mathrm{W}^{k,p}(\Omega)$ is a vector space. For $p \in [1, \infty)$ and $u, v \in \mathrm{W}^{k,p}(\Omega)$, Minkowski's inequality (see 2.1) on $\mathrm{L}^p(\Omega)$ and for $\|\cdot\|_p$ on $\mathbb{R}^m$ implies

$$\begin{aligned}
\|u+v\|_{W^{k,p}(\Omega)} &= \left( \sum_{|\alpha| \le k} \|\,\mathrm{D}^\alpha u + \mathrm{D}^\alpha v\|_{\mathrm{L}^p(\Omega)}^p \right)^{\frac{1}{p}} \le \left( \sum_{|\alpha| \le k} \left( \|\,\mathrm{D}^\alpha u\|_{\mathrm{L}^p(\Omega)} + \|\,\mathrm{D}^\alpha v\|_{\mathrm{L}^p(\Omega)} \right)^p \right)^{\frac{1}{p}} \\
&\le \left( \sum_{|\alpha| \le k} \|\,\mathrm{D}^\alpha u\|_{\mathrm{L}^p(\Omega)}^p \right)^{\frac{1}{p}} + \left( \sum_{|\alpha| \le k} \|\,\mathrm{D}^\alpha v\|_{\mathrm{L}^p(\Omega)}^p \right)^{\frac{1}{p}} = \|u\|_{\mathrm{W}^{k,p}(\Omega)} + \|v\|_{\mathrm{W}^{k,p}(\Omega)}.
\end{aligned}$$

All other properties of the norm are easily verified for $\|\cdot\|_{\mathrm{W}^{k,p}(\Omega)}$. A similar calculation gives the claim for $p = \infty$.
Now let $p \in [1, \infty]$ and $(u_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{W}^{k,p}(\Omega)$. Then for all $|\alpha| \leq k$, $(\mathrm{D}^\alpha u_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^p(\Omega)$ as $\|\mathrm{D}^\alpha u_m - \mathrm{D}^\alpha u_l\|_{\mathrm{L}^p(\Omega)} \leq \|u_m - u_l\|_{\mathrm{W}^{k,p}(\Omega)}$. Hence, there exists $u_\alpha \in \mathrm{L}^p(\Omega)$ with

$$\mathrm{D}^\alpha u_m \to u_\alpha \text{ in } \mathrm{L}^p(\Omega), |\alpha| \leq k. \tag{3.5}$$

For $\alpha = (0, \ldots, 0)$ we define $u_{(0,\ldots,0)} =: u$ and have

$$u_m \to u \text{ in } \mathrm{L}^p(\Omega). \tag{3.6}$$

To show that $u_\alpha = \mathrm{D}^\alpha u$, we fix $\varphi \in \mathrm{C}_0^\infty(\Omega)$ and obtain

$$\int_\Omega u\,\mathrm{D}^\alpha\varphi\,\mathrm{d}x \overset{(3.6)}{=} \lim_{m\to\infty}\int_\Omega u_m\,\mathrm{D}^\alpha\varphi\,\mathrm{d}x = \lim_{m\to\infty}(-1)^{|\alpha|}\int_\Omega \mathrm{D}^\alpha u_m\varphi\,\mathrm{d}x \overset{(3.5)}{=} (-1)^{|\alpha|}\int_\Omega u_\alpha\varphi\,\mathrm{d}x$$

since $\varphi$, $\mathrm{D}^\alpha\varphi \in \mathrm{L}^q(\Omega)$ with $\frac{1}{p}+\frac{1}{q} = 1$. Hence, $\mathrm{D}^\alpha u = u_\alpha \in \mathrm{L}^p(\Omega)$ for all $|\alpha| \leq k$ and $u \in \mathrm{W}^{k,p}(\Omega)$. But then (3.5) and (3.6) imply $u_m \to u$ in $\mathrm{W}^{k,p}(\Omega)$. Hence, $\mathrm{W}^{k,p}(\Omega)$ is complete and a Banach space.
That $(\,\cdot\,,\,\cdot\,)_{\mathrm{H}^k(\Omega)}$ is a scalar product on $\mathrm{H}^k(\Omega)$ easily follows from the properties of the $\mathrm{L}^2$-scalar product. Hence, $\mathrm{H}^k(\Omega)$ is a Hilbert space. $\square$

In particular, $\mathrm{W}_0^{k,p}(\Omega)$ is a Banach space and a subspace of $\mathrm{W}^{k,p}(\Omega)$.

# Chapter 4

# Approximation by Smooth Functions

As it is often complicated to use the definition of weak derivatives for proving properties of Sobolev spaces, we aim to approximate functions in Sobolev spaces by smooth functions.

## 4.1 Interior Approximation

We prove that mollification from 2.3 provides approximating functions in $\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega)$.

**Theorem 4.1.** *Let $\Omega \subseteq \mathbb{R}^n$ be open, $k \in \mathbb{N}$, $p \in [1, \infty)$, and $u \in \mathrm{W}^{k,p}(\Omega)$. Then the following statements hold:*

*a) $u_\varepsilon \in \mathrm{C}^\infty(\Omega)$ and $\mathrm{D}^\alpha(u_\varepsilon)(x) = (\mathrm{D}^\alpha u)_\varepsilon(x)$ for all $x \in \Omega_\varepsilon$ and all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$.*

*b) $u_\varepsilon \to u$ in $\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega)$ as $\varepsilon \downarrow 0$.*

*Proof.* a) By Theorem 2.1, we have $u_\varepsilon \in \mathrm{C}^\infty(\Omega)$ and for $|\alpha| \leq k$

$$\mathrm{D}^\alpha(u_\varepsilon)(x) = \int_\Omega \mathrm{D}^\alpha(\eta_\varepsilon)(x-y)u(y)\,\mathrm{d}y \quad \text{for all } x \in \Omega,$$

see proof of Theorem 2.1 a), d). For fixed $x \in \Omega_\varepsilon$, $\phi(y) := \eta_\varepsilon(x-y)$ satisfies $\phi \in \mathrm{C}_0^\infty(\Omega)$ since $\operatorname{supp}\phi = \overline{\mathrm{B}_\varepsilon(x)}$ and therefore

$$\begin{aligned}\mathrm{D}^\alpha(u_\varepsilon)(x) &= (-1)^{|\alpha|}\int_\Omega \mathrm{D}_y^\alpha(\eta_\varepsilon(x-y))\,u(y)\,\mathrm{d}y = (-1)^{|\alpha|}\int_\Omega \mathrm{D}^\alpha\phi(y)\,u(y)\,\mathrm{d}y \\ &\overset{(3.3)}{=} (-1)^{|\alpha|+|\alpha|}\int_\Omega \phi(y)\,\mathrm{D}^\alpha u(y)\,\mathrm{d}y = \int_\Omega \eta_\varepsilon(x-y)\,\mathrm{D}^\alpha u(y)\,\mathrm{d}y = (\mathrm{D}^\alpha u)_\varepsilon(x).\end{aligned}$$

Since $x \in \Omega_\varepsilon$ was arbitrary, this proves a).

b) In view of a) and Theorem 2.1 d), for fixed $V \Subset \Omega$ we have $\mathrm{D}^\alpha u_\varepsilon = \eta_\varepsilon * \mathrm{D}^\alpha u$ in $V$ for $\varepsilon \in (0, \varepsilon_0)$, as $V \subset \Omega_{\varepsilon_0} \subseteq \Omega_\varepsilon$ for $\varepsilon_0$ small enough so that $\mathrm{D}^\alpha u_\varepsilon \to \mathrm{D}^\alpha u$ in $\mathrm{L}^p(V)$ as $\varepsilon \downarrow 0$ for any $\alpha \in \mathbb{N}_0^n, |\alpha| \leq k$. Then

$$\|u_\varepsilon - u\|^p_{\mathrm{W}^{k,p}(V)} = \sum_{|\alpha|\leq k} \|\mathrm{D}^\alpha u_\varepsilon - \mathrm{D}^\alpha u\|^p_{\mathrm{L}^p(V)} \to 0 \quad \text{as } \varepsilon \downarrow 0. \qquad \square$$

## 4.2 Approximation by Smooth Functions

In order to show that for any $u \in \mathrm{W}^{k,p}(\Omega)$ there is $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ such that $u_m \to u$ in $\mathrm{W}^{k,p}(\Omega)$ (and not only in $\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega)$), we need the following lemmas to construct a partition of unity.

**Lemma 4.2.** *Let $\Omega \subset \mathbb{R}^n$ be open and $K \subset \Omega$ compact. If $\operatorname{dist}(K, \partial\Omega) \geq \delta > 0$, then there exists a cutoff function $\tau \in \mathrm{C}_0^\infty(\Omega)$ w.r.t. $K, \Omega$ with $0 \leq \tau \leq 1$, $\tau = 1$ in $K$, and*

$$|\,\mathrm{D}^\alpha \tau(x)| \leq c\delta^{-k} \quad \textit{for all } x \in \Omega \setminus K, k \in \mathbb{N}, |\alpha| = k,$$

*where $c > 0$ depends on $k$ and $n$ but not on $\Omega$ or $K$.*

*Proof.* We may choose $\delta > 0$ since $K$ is compact. Hence,

$$\tilde{K} \coloneqq \overline{\bigcup_{x\in K} \mathrm{B}_{\frac{\delta}{2}}(x)}$$

is compact with $\operatorname{dist}(\partial\tilde{K}, \partial K) = \frac{\delta}{2} \leq \operatorname{dist}(\partial\tilde{K}, \partial\Omega)$.
As $\chi_{\tilde{K}} \in \mathrm{L}^1(\Omega)$ with $\operatorname{supp} \chi_{\tilde{K}} = \tilde{K} \Subset \Omega$, we have that $\tau \coloneqq \eta_{\frac{\delta}{4}} * \chi_{\tilde{K}}$ satisfies $\tau \in \mathrm{C}_0^\infty(\Omega)$, $0 \leq \tau \leq 1$, and $\tau = 1$ in $K$ by Theorem 2.1, as

$$\tau(x) = \int_{\mathrm{B}_{\frac{\delta}{4}}(x)} \eta_{\frac{\delta}{4}}(x-y) \underbrace{\chi_{\tilde{K}}(y)}_{=1} \,\mathrm{d}y = 1 \quad \text{for all } x \in K$$

since $\mathrm{B}_{\frac{\delta}{4}}(x) \subset \tilde{K}$. Moreover, for $|\alpha| = k$

$$\mathrm{D}^\alpha \eta_{\frac{\delta}{4}}(x) = \left(\frac{4}{\delta}\right)^n \mathrm{D}^\alpha \left[\eta\left(\frac{4}{\delta}x\right)\right] = \left(\frac{4}{\delta}\right)^{n+k} (\,\mathrm{D}^\alpha \eta)\left(\frac{4}{\delta}x\right).$$

Hence, for $x \in \Omega \setminus K$ we have

$$|\,\mathrm{D}^\alpha \tau(x)| \leq \int_{\mathrm{B}_{\frac{\delta}{4}}(x)} \left(\frac{4}{\delta}\right)^{n+k} \|\,\mathrm{D}^\alpha \eta\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \chi_{\tilde{K}}(y) \,\mathrm{d}y \leq \tilde{c}(n,k)\,\delta^{-n-k}\,|\mathrm{B}_{\frac{\delta}{4}}(x)| \leq c(n,k)\,\delta^{-k}$$

which concludes the proof. □

**Lemma 4.3** (Partition of unity)**.** *Let $K \subset \mathbb{R}^n$ be compact and $\{\Omega_k\}_{k=1,\dots,N}$ be an open covering of $K$. Then, there exist $\psi_k, k = 1, \dots, N$, called* partition of unity *such that $\psi_k \in \mathrm{C}_0^\infty(\Omega_k)$, $0 \leq \psi_k \leq 1$ in $\Omega_k$, and $\sum_{k=1}^N \psi_k(x) = 1$ for all $x \in K$.*

*Proof.* For any $x \in K$ there is $r = r(x) > 0$ and $1 \leq k \leq N$ such that $\mathrm{B}_{x,k} \coloneqq \mathrm{B}_r(x) \Subset \Omega_k$. Hence,

$$\{B_{x,k}\}_{\substack{x\in K\cap\Omega_k,\\ k=1,\dots,N}}$$

is an open covering of $K$ and has a finite subset still covering $K$, called

$$\{B_i^k\}_{\substack{i=1,\dots,N_k,\\ k=1,\dots,N}}.$$

Then

$$K_k \coloneqq \overline{\bigcup_{i=1}^{N_k} B_i^k}$$

satisfies $K_k \Subset \Omega_k$ and $\bigcup_{k=1}^N K_k \supset K$. Let $\tilde{\psi}_k$ denote the cutoff function w.r.t $K_k, \Omega_k$. Hence, $\tilde{\psi}_k \in \mathrm{C}_0^\infty(\Omega_k)$ satisfies $0 \le \tilde{\psi}_k \le 1$ and

$$\psi(x) := \sum_{k=1}^N \tilde{\psi}_k(x) \ge 1 \quad \text{for all } x \in K.$$

Furthermore, we have

$$K \Subset \Omega := \bigcup_{k=1}^N \operatorname{supp}(\tilde{\psi}_k)$$

and there is an open set $\Omega_0$ such that $K \subset \Omega_0 \Subset \Omega$. Let $\tau$ be a cutoff function w.r.t $K, \Omega_0$ and

$$\psi_k(x) := \begin{cases} \frac{\tilde{\psi}_k(x)\tau(x)}{\psi(x)}, & \text{if } x \in \Omega_0, \\ 0, & \text{if } x \notin \Omega_0. \end{cases}$$

Then $\psi_1, \dots, \psi_N$ have the claimed properties. $\square$

Now we prove the announced result that $\mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ is dense in $\mathrm{W}^{k,p}(\Omega)$ without assuming any smoothness of $\partial\Omega$.

**Theorem 4.4** (Meyers and Serrin). *Let $\Omega \subset \mathbb{R}^n$ be open, $k \in \mathbb{N}$, and $p \in [1, \infty)$. Then $\mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ is dense in $\mathrm{W}^{k,p}(\Omega)$, i.e. for any $u \in \mathrm{W}^{k,p}(\Omega)$ there exists $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ such that $u_m \to u$ in $\mathrm{W}^{k,p}(\Omega)$ as $m \to \infty$.*

*Proof.* i) With

$$U_i := \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \frac{1}{i} \text{ and } |x| < i\}, \quad i \in \mathbb{N},$$

we have $\bigcup_{i=1}^\infty U_i = \Omega$ and $U_i \subset U_{i+1}$. Moreover,

$$V_i := U_{i+4} \setminus \overline{U_{i+1}}, i \in \mathbb{N}, \quad \text{and} \quad V_0 := U_4$$

are all open with $V_i \Subset \Omega$ for all $i \in \mathbb{N}_0$ and $\Omega = \bigcup_{i=0}^\infty V_i$. Defining further

$$W_i := \overline{U_{i+3}} \setminus U_{i+2}, i \in \mathbb{N}, \quad \text{and} \quad W_0 := \overline{U}_3,$$

all $W_i \subset V_i$ are compact and we have $\Omega = \bigcup_{i=0}^\infty W_i$. Let $\psi_i \in \mathrm{C}_0^\infty(V_i)$ denote a cutoff function w.r.t. $W_i, V_i$ with $0 \le \psi_i \le 1$ and $\psi_i = 1$ in $W_i$ for $i \in \mathbb{N}_0$. Since for all $j \ge i+2$

$$W_i \cap V_j = \left(\overline{U_{i+3}} \cap \overline{U_{j+1}}^{\mathrm{c}}\right) \cap \left(U_{i+2}^{\mathrm{c}} \cap U_{j+4}\right) = \emptyset,$$

and for all $j \ge i+3$, $V_i \cap V_j = \emptyset$, for any $x \in \Omega$ we have

$$\sigma(x) := \sum_{i=0}^\infty \psi_i(x) > 0$$

and only finitely many of the $\psi_i(x)$ are non-zero. Hence, $\{\xi_i\}_{i=0}^\infty$, defined by

$$\xi_i(x) := \frac{\psi_i(x)}{\sigma(x)}, \quad x \in \Omega,$$

is a *partition of unity subordinate to* $\{V_i\}_{i=0}^\infty$, i.e. $\xi_i \in \mathrm{C}_0^\infty(V_i)$, $0 \le \xi_i \le 1$, and $\sum_{i=0}^\infty \xi_i = 1$ in $\Omega$ and for any $K \Subset \Omega$, $\xi_i|_K \not\equiv 0$ only for finitely many $i$.

ii) Let $u \in \mathrm{W}^{k,p}(\Omega)$ be arbitrary. Then by Proposition 3.8 d) and i) we have $\xi_i u \in \mathrm{W}^{k,p}(\Omega)$ and $\operatorname{supp}(\xi_i u) \subset V_i$ for all $i \in \mathbb{N}_0$. We fix $\delta > 0$. Then for any $i \in \mathbb{N}_0$ we define

$$Z_i := U_{i+5} \setminus \overline{U_i} \supset V_i, i \in \mathbb{N}, \quad \text{and} \quad Z_0 := U_5 \supset V_0.$$

In view of Theorem 4.1, there is $\varepsilon_i > 0$ small enough such that $u_i := \eta_{\varepsilon_i} * (\xi_i u)$ satisfies $u_i \in \mathrm{C}_0^\infty(Z_i)$ and

$$\|u_i - \xi_i u\|_{W^{k,p}(\Omega)} = \|u_i - \xi_i u\|_{W^{k,p}(Z_i)} \leq \frac{\delta}{2^{i+1}} \tag{4.1}$$

for $i \in \mathbb{N}_0$ as $u_i - \xi_i u \equiv 0$ in $\Omega \setminus Z_i$. Define

$$v(x) := \sum_{i=0}^\infty u_i(x), \quad x \in \Omega.$$

Then for any open set $V \Subset \Omega$ only finitely many $u_i$ satisfy $u_i|_V \not\equiv 0$. Since $u = \sum_{i=0}^\infty \xi_i u$, we obtain $v \in \mathrm{C}^\infty(\Omega)$ and

$$\|v - u\|_{\mathrm{W}^{k,p}(V)} \leq \sum_{i=0}^\infty \|u_i - \xi_i u\|_{\mathrm{W}^{k,p}(V)} \overset{(4.1)}{\leq} \delta \sum_{i=0}^\infty \frac{1}{2^{i+1}} = \delta \quad \text{for all } V \Subset \Omega.$$

Since $U_i \subset U_{i+1}$ for all $i \in \mathbb{N}$, $U_i \Subset \Omega$, and $\Omega = \bigcup_{i=1}^\infty U_i$, we conclude by the monotone convergence theorem

$$\|v - u\|^p_{\mathrm{W}^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|\,\mathrm{D}^\alpha(v-u)\|^p_{\mathrm{L}^p(\Omega)} = \lim_{i \to \infty} \sum_{|\alpha| \leq k} \|\,\mathrm{D}^\alpha(v-u)\|^p_{\mathrm{L}^p(U_i)} \leq \delta^p.$$

As $\delta > 0$ was arbitrary, the claim is proved. □

*Remark* 4.5. Historically, there were two definitions of Sobolev spaces. $\mathrm{W}^{k,p}(\Omega)$ was defined as in Definition 3.5 while $\mathrm{H}^{k,p}(\Omega)$ was defined as the closure of $\mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{k,p}(\Omega)$ w.r.t $\|\cdot\|_{\mathrm{W}^{k,p}(\Omega)}$. Obviously $\mathrm{H}^{k,p}(\Omega) \subseteq \mathrm{W}^{k,p}(\Omega)$ but only after Meyers and Serrin in 1964 [MS64] it was clear that $\mathrm{H}^{k,p}(\Omega) \supseteq \mathrm{W}^{k,p}(\Omega)$ without assuming any smoothness condition of $\partial\Omega$.

We can now prove the chain rule for $\mathrm{W}^{1,p}(\Omega)$ functions.

**Proposition 4.6.** *Let $\Omega$ be open, $p \in [1, \infty)$, and $f \in \mathrm{C}^1(\mathbb{R})$ such that $|f'| \leq M$ on $\mathbb{R}$ for some $M > 0$. Assume further that $f(0) = 0$ or $|\Omega| < \infty$ is satisfied. Then for any $u \in \mathrm{W}^{1,p}(\Omega)$ we have $f(u)$ in $\mathrm{W}^{1,p}(\Omega)$ with*

$$\nabla f(u) = f'(u)\nabla u.$$

*Proof.* As $f'$ is continuous and bounded and $u$ is measurable, we have $f'(u) \in \mathrm{L}^\infty(\Omega)$ and $f'(u)\nabla u \in \mathrm{L}^p(\Omega)$. In view of

$$|f(x)| \leq |f(0)| + M|x| \quad \text{for all } x \in \mathbb{R},$$

the assumption $f(0) = 0$ or $|\Omega| < \infty$ implies $f(u) \in \mathrm{L}^p(\Omega)$. By Theorem 4.4, there exists $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{1,p}(\Omega)$ such that $u_m \to u$ in $\mathrm{W}^{1,p}(\Omega)$ and $u_m \to u$ a.e. in $\Omega$. Hence,

$u_m \to u$ and $(u_m)_{x_i} \to u_{x_i}$ in $\mathrm{L}^p(\Omega)$ for all $i \in \{1, \ldots, n\}$. We fix $i \in \{1, \ldots, n\}$ and $\varphi \in \mathrm{C}_0^\infty(\Omega)$. In view of $f(u_m) \in \mathrm{C}^1(\Omega)$, we deduce from (3.2)

$$\int_\Omega f(u_m)\, \varphi_{x_i} \,\mathrm{d}x = -\int_\Omega f'(u_m)(u_m)_{x_i}\, \varphi \,\mathrm{d}x \quad \text{for all } m \in \mathbb{N}, \tag{4.2}$$

by the classical chain rule. On the one hand, for $q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have by Hölder's inequality

$$\begin{aligned}\Big|\int_\Omega (f(u_m) - f(u))\, \varphi_{x_i} \,\mathrm{d}x\Big| &\le M \int_\Omega |u_m - u|\, |\varphi_{x_i}| \,\mathrm{d}x \\ &\le M\, \|u_m - u\|_{\mathrm{L}^p(\Omega)} \|\varphi_{x_i}\|_{\mathrm{L}^q(\Omega)} \to 0 \quad \text{as } m \to \infty\end{aligned}$$

since $f$ is Lipschitz. On the other hand,

$$|f'(u_m) - f'(u)|\, |u_{x_i}|\, |\varphi| \le 2M\, |u_{x_i}| \|\varphi\|_{\mathrm{L}^\infty(\Omega)} \in \mathrm{L}^1(\mathrm{supp}(\varphi))$$

as $\mathrm{supp}(\varphi)$ is bounded and thus

$$\begin{aligned}&\Big|\int_\Omega (f'(u_m)(u_m)_{x_i} - f'(u)u_{x_i})\, \varphi \,\mathrm{d}x\Big| \\ &\quad \le \int_\Omega |f'(u_m)|\, |(u_m)_{x_i} - u_{x_i}|\, |\varphi| \,\mathrm{d}x + \int_\Omega |f'(u_m) - f'(u)|\, |u_{x_i}|\, |\varphi| \,\mathrm{d}x \\ &\quad \le M\, \|(u_m)_{x_i} - u_{x_i}\|_{\mathrm{L}^p(\Omega)} \|\varphi\|_{\mathrm{L}^q(\Omega)} + \int_{\mathrm{supp}(\varphi)} |f'(u_m) - f'(u)|\, |u_{x_i}|\, |\varphi| \,\mathrm{d}x. \\ &\quad \to 0 \quad \text{as } m \to \infty\end{aligned}$$

by the dominated convergence theorem.
Hence, letting $m \to \infty$ in (4.2) we conclude that $(f(u))_{x_i} = f'(u)u_{x_i}$ in the weak sense. □

Unlike for classical derivatives, now $u \in \mathrm{W}^{1,p}(\Omega)$ implies $|u| \in \mathrm{W}^{1,p}(\Omega)$.

**Corollary 4.7.** *Let $\Omega \subset \mathbb{R}^n$ be open, $p \in [1, \infty)$, and $u \in \mathrm{W}^{1,p}(\Omega)$. Define*

$$u_+(x) := \max\{u(x), 0\} \quad \textit{and} \quad u_-(x) = \max\{-u(x), 0\}.$$

*Then $u_+, u_-, |u| \in \mathrm{W}^{1,p}(\Omega)$ with $\nabla u_+(x) = \nabla u(x)\chi_{\{u>0\}}(x)$, $\nabla u_-(x) = -\nabla u(x)\chi_{\{u<0\}}(x)$, and $\nabla|u|(x) = \nabla u(x)\left(\chi_{\{u>0\}}(x) - \chi_{\{u<0\}}(x)\right)$.*

*Proof (Exercise).* For $\varepsilon > 0$ we define

$$f_\varepsilon(u) := \begin{cases} \sqrt{u^2 + \varepsilon^2} - \varepsilon, & \text{if } u > 0, \\ 0, & \text{if } u \le 0. \end{cases}$$

Then $f_\varepsilon \in \mathrm{C}^1(\mathbb{R})$ satisfies $f_\varepsilon(0) = 0$ and $f'_\varepsilon(u) = \frac{u}{\sqrt{u^2+\varepsilon^2}} \cdot \chi_{(0,\infty)}(u)$ and hence $|f'_\varepsilon(u)| \le 1$ for all $u \in \mathbb{R}$. Hence, by Proposition 4.6 we have $f_\varepsilon(u) \in \mathrm{W}^{1,p}(\Omega)$ with

$$(f_\varepsilon(u))_{x_i} = f'_\varepsilon(u)u_{x_i} = \frac{u u_{x_i}}{\sqrt{u^2 + \varepsilon^2}} \cdot \chi_{\{u>0\}}$$

and

$$\int_\Omega f_\varepsilon(u)\varphi_{x_i} \,\mathrm{d}x = -\int_\Omega f'_\varepsilon(u)u_{x_i}\varphi \,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{4.3}$$

Since $f_\varepsilon(u(x)) \to u_+(x)$ and $(f'_\varepsilon(u)u_{x_i})(x) \to u_{x_i}(x)\chi_{\{u>0\}}(x)$ for a.e. $x \in \Omega$ and $|f_\varepsilon(u)\varphi_{x_i}| \le \|\varphi_{x_i}\|_{\mathrm{L}^\infty(\Omega)}|u|$ as well as $|f'_\varepsilon(u)u_{x_i}\varphi| \le \|\varphi\|_{\mathrm{L}^\infty(\Omega)}|u_{x_i}|$ are satisfied, we may let $\varepsilon \downarrow 0$ in 4.3 and obtain from the dominated convergence theorem

$$\int_\Omega u_+(x)\varphi_{x_i}(x)\,\mathrm{d}x = -\int_\Omega u_{x_i}(x)\chi_{\{u>0\}}(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega).$$

This proves the claim for $u_+$. In view of $u_- = (-u)_+$ and $|u| = u_+ + u_-$, the proof is complete due to Proposition 3.8. □

## 4.3 Approximation by $\mathrm{C}^\infty(\overline{\Omega})$-Functions

We now ask the question whether any $u \in \mathrm{W}^{k,p}(\Omega)$ can also be approximated by functions $u_m \in \mathrm{C}^\infty(\overline{\Omega})$ instead of $u_m \in \mathrm{C}^\infty(\Omega)$. The following example shows that this is not true for all open $\Omega \subset \mathbb{R}^n$.

**Example 4.8** (Exercise)**.** Let $\Omega = \{(x,y) \in \mathbb{R}^2 \colon 0 < |x| < 1, 0 < y < 1\}$ and $p \in [1,\infty)$. Then $u\colon \Omega \to \mathbb{R}$ defined by

$$u(x,y) := \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x < 0, \end{cases}$$

belongs to $\mathrm{W}^{1,p}(\Omega)$, but for $\varepsilon > 0$ sufficiently small, there is no function $\varphi \in \mathrm{C}^1(\overline{\Omega})$ such that $\|\varphi - u\|_{\mathrm{W}^{1,p}(\Omega)} < \varepsilon$.
Indeed, it is obvious that $u \in \mathrm{W}^{1,p}(\Omega)$ with $u_x = u_y = 0$ in $\Omega$ as $u \in \mathrm{C}^1(\Omega)$. Assume that for $\varepsilon > 0$ there exists $\varphi \in \mathrm{C}^1(\overline{\Omega})$ with $\|u - \varphi\|_{\mathrm{W}^{1,p}(\Omega)} < \varepsilon$. Then with

$$L := \{(x,y) \in \mathbb{R}^2 \colon x \in [-1,0]\} \quad \text{and} \quad R := \{(x,y) \in \mathbb{R}^2 \colon x, y \in [0,1]\}$$

we have $\overline{\Omega} = L \cup R$ and $|L| = |R| = 1$. Hence, Hölder's inequality implies

$$\|\varphi\|_{\mathrm{L}^1(L)} \le \|\varphi\|_{\mathrm{L}^p(L)} \le \|\varphi\|_{\mathrm{W}^{1,p}(\Omega)} < \varepsilon$$

and similarly $\|1-\varphi\|_{\mathrm{L}^1(R)} < \varepsilon$. The latter yields $\|\varphi\|_{\mathrm{L}^1(R)} > 1 - \varepsilon$, as $|R| = 1$. Hence, with

$$\psi(x) := \int_0^1 \varphi(x,y)\,\mathrm{d}y,$$

we have $\int_{-1}^0 \psi(x)\,\mathrm{d}x < \varepsilon$ and $\int_0^1 \psi(x)\,\mathrm{d}x > 1-\varepsilon$ so that there exist $a \in [-1,0)$ and $b \in (0,1]$ with $\psi(a) < \varepsilon$ and $\psi(b) > 1 - \varepsilon$. Then we get for $\varepsilon \in (0,\frac{1}{2})$ and $\frac{1}{p} + \frac{1}{q} = 1$

$$\begin{aligned} 1 - 2\varepsilon < \psi(b) - \psi(a) &= \int_a^b \psi'(x)\,\mathrm{d}x = \int_a^b \int_0^1 \psi_x(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\overset{\text{Fubini}}{\le} \int_{\overline{\Omega}} |\psi_x(x,y)|\,\mathrm{d}x\,\mathrm{d}y \overset{\text{Hölder}}{\le} |\overline{\Omega}|^{\frac{1}{q}} \|\varphi_x\|_{\mathrm{L}^p(\Omega)} < 2^{\frac{1}{q}}\varepsilon, \end{aligned}$$

as $\|\varphi_x\|_{\mathrm{L}^p(\Omega)} = \|u_x - \varphi_x\|_{\mathrm{L}^p(\Omega)}\|_{\mathrm{L}^p(\Omega)} \le \|u - \varphi\|_{\mathrm{W}^{1,p}(\Omega)} < \varepsilon$. But then $1 < \varepsilon(2 + 2^{\frac{1}{q}})$ which is not possible for $\varepsilon > 0$ small enough. □

The problem with $\Omega$ in Example 4.8 is that it lies on both sides of the segment

$$\Gamma = \{(0,y) \colon y \in [0,1]\}$$

with $\Gamma \subset \partial\Omega$. The following condition excludes this situation. Moreover, we assume from now on that $\Omega$ is a domain, i.e. open and connected.

**Definition 4.9.** Let $\Omega \subset \mathbb{R}^n$ be a domain. We say that $\Omega$ satisfies the *segment condition* if for any $x \in \partial\Omega$ there exists a neighborhood $U_x \subset \mathbb{R}^n$ of $x$ and $0 \neq y_x \in \mathbb{R}^n$ such that $z + ty_x \in \Omega$ for any $z \in \overline{\Omega} \cap U_x$ and any $t \in (0,1)$.

Another condition on $\partial\Omega$ is that it is locally the graph of a $\mathrm{C}^m$ function.

**Definition 4.10.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $m \in \mathbb{N}$. We say that $\Omega$ *is of class* $\mathrm{C}^m$ or simply $\partial\Omega \in \mathrm{C}^m$ if for any $x^0 \in \partial\Omega$ there exist $r = r(x^0) > 0$ and $\gamma = \gamma_{x^0} \in \mathrm{C}^m(\mathbb{R}^{n-1})$ such that, upon relabelling and reorienting the coordinate axes if necessary, we have

$$\begin{aligned} \Omega \cap \mathrm{B}_r(x^0) &= \{x \in \mathrm{B}_r(x^0) \colon x_n > \gamma(x_1, \dots, x_{n-1})\}, \\ \partial\Omega \cap \mathrm{B}_r(x^0) &= \{x \in \mathrm{B}_r(x^0) \colon x_n = \gamma(x_1, \dots, x_{n-1})\}. \end{aligned}$$

Furthermore, we say $\partial\Omega \in \mathrm{C}^\infty$ if $\gamma \in \mathrm{C}^\infty$ and we say $\partial\Omega$ *is analytic* if $\gamma$ is analytic.

*Remark* 4.11. Let $\Omega$ be a bounded domain with $\partial\Omega \in \mathrm{C}^1$. Then for any $x^0 \in \partial\Omega$ there is a unique outward unit vector $\nu(x^0)$, i.e. $|\nu| = 1$, $\nu(x^0) \perp y$ for all $y \in \mathrm{T}(x^0)$, where $\mathrm{T}(x^0)$ is the tangential space on $\partial\Omega$ in $x^0$ and $x^0 + t\nu(x^0) \notin \overline{\Omega}$ for all $t \in (0, \varepsilon_0)$ for $\varepsilon_0 > 0$ small.
Indeed, as

$$\partial\Omega \cap \mathrm{B}_r(x^0) = \left\{ x \in \mathrm{B}_r(x^0) \colon x = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ \gamma(x_1, \dots, x_{n-1}) \end{pmatrix} \right\}$$

we have that

$$\mathrm{T}(x) = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \gamma_{x_1}(x_1, \dots, x_{n-1}) \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \gamma_{x_{n-1}}(x_1, \dots, x_{n-1}) \end{pmatrix} \right\}$$

is $(n-1)$-dimensional so that $\mathrm{T}(x)^\perp$ is one-dimensional. Hence, $\nu(x)$ is uniquely defined and $\nu \colon \partial\Omega \to \mathbb{R}^n$ is continuous as $\gamma \in \mathrm{C}^1(\mathbb{R}^{n-1})$ and

$$\nu(x) = \frac{1}{\sqrt{1 + |\nabla\gamma|^2}} \left(\gamma_{x_1}, \dots, \gamma_{x_{n-1}}, -1\right)^{\mathrm{T}}.$$

In particular, $\Omega$ satisfies the segment condition with $y_x = -\alpha\nu(x)$ and $U_x = \mathrm{B}_\rho(x)$ with some $\rho \in (0,r)$ and $\alpha > 0$ small enough (as $\nu$ is continuous). $\square$

Next we prove that in fact $\mathrm{C}^\infty(\overline{\Omega})$ is dense in $\mathrm{W}^{k,p}(\Omega)$ if $\Omega$ satisfies the segment condition. As a final prepariation we need the continuity of the translation in $\mathrm{L}^p(\mathbb{R}^n)$.

**Proposition 4.12.** *Let $p \in [1, \infty)$ and $u \in \mathrm{L}^p(\mathbb{R}^n)$. Then the translation is continuous in $\mathrm{L}^p(\mathbb{R}^n)$ in the sense that we have (with $h \in \mathbb{R}^n$)*

$$\lim_{|h| \to 0} \|u(\cdot + h) - u(\cdot)\|_{\mathrm{L}^p(\mathbb{R}^n)} = 0.$$

*Proof.* Given $\delta > 0$, by 2.3 there is $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ such that

$$\|u - \varphi\|_{\mathrm{L}^p(\mathbb{R}^n)} < \frac{\delta}{3}.$$

But then also

$$\|u(\cdot + h) - \varphi(\cdot + h)\|_{\mathrm{L}^p(\mathbb{R}^n)} = \|u - \varphi\|_{\mathrm{L}^p(\mathbb{R}^n)} < \frac{\delta}{3}.$$

Since $\varphi$ has compact support, it is uniformly continuous on $\mathbb{R}^n$. Hence, there is $M > 0$ such that

$$|\varphi(x+h) - \varphi(x) < \frac{\delta}{3|\operatorname{supp}(\varphi)|^{\frac{1}{p}}} \quad \text{for all } x \in \mathbb{R}^n,\ h \in \mathrm{B}_M(0).$$

Hence, for $h \in \mathbb{R}^n$ with $|h| < M$ we have

$$\begin{aligned}
&\|u(\cdot + h) - u(\cdot)\|_{\mathrm{L}^p(\mathbb{R}^n)} \\
&\quad \le \|u(\cdot + h) - \varphi(\cdot + h)\|_{\mathrm{L}^p(\mathbb{R}^n)} + \|\varphi(\cdot + h) - \varphi(\cdot)\|_{\mathrm{L}^p(\mathbb{R}^n)} + \|\varphi - u\|_{\mathrm{L}^p(\mathbb{R}^n)} \\
&\quad \le \frac{2}{3}\delta + \|\varphi(\cdot + h) - \varphi(\cdot)\|_{\mathrm{L}^\infty(\mathbb{R}^n)} |\operatorname{supp}(\varphi)|^{\frac{1}{p}} < \delta
\end{aligned}$$

and the claim follows. □

**Theorem 4.13.** *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the segment condition, $k \in \mathbb{N}$, and $p \in [1, \infty)$. Then the set $\{\varphi|_\Omega \colon \varphi \in \mathrm{C}_0^\infty(\mathbb{R}^n)\}$ is dense in $\mathrm{W}^{k,p}(\Omega)$. In particular, if in addition $\Omega \neq \mathbb{R}^n$, then for any $u \in \mathrm{W}^{k,p}(\Omega)$ there is $(u_m)_{m \in \mathbb{N}} \subset \mathrm{C}^\infty(\overline{\Omega})$ such that $u_m \to u$ in $\mathrm{W}^{k,p}(\Omega)$.*

*Proof.* We fix $u \in \mathrm{W}^{k,p}(\Omega)$ and $\delta > 0$.

i) In a first step, we show that in case $\Omega$ is unbounded there exists $v \in \mathrm{W}^{k,p}(\Omega)$ with $\operatorname{supp}(v)$ bounded and $\|u - v\|_{\mathrm{W}^{k,p}(\Omega)} < \delta$. By Lemma 4.2, there exists $\tau \in \mathrm{C}_0^\infty(\mathrm{B}_2(0))$ such that $0 \le \tau \le 1$, $\tau \equiv 1$ in $\overline{\mathrm{B}_1(0)}$ and there is some $M = M(k) > 0$ such that $|\,\mathrm{D}^\alpha \tau(x)| \le M$ for all $x \in \mathbb{R}^n$ and all $|\alpha| \le k$ (choose $K = \overline{\mathrm{B}_1(0)}, \Omega = \mathrm{B}_2(0), \delta = 1$ in Lemma 4.2). For $\varepsilon \in (0,1)$, we define $\tau_\varepsilon \coloneqq \tau(\varepsilon x), x \in \mathbb{R}^n$. Then $\tau_\varepsilon \equiv 1$ in $\overline{\mathrm{B}_{\frac{1}{\varepsilon}}(0)}$, $\tau_\varepsilon \in \mathrm{C}_0^\infty(\mathrm{B}_{\frac{2}{\varepsilon}}(0))$, and

$$|\,\mathrm{D}^\alpha \tau_\varepsilon(x)| \le M \varepsilon^{|\alpha|} \le M \quad \text{for all } |\alpha| \le k. \tag{4.4}$$

Hence, $v_\varepsilon \coloneqq \tau_\varepsilon u$ has bounded support and belongs to $\mathrm{W}^{k,p}(\Omega)$ by Proposition 3.8 d). It further satisfies for $|\alpha| \le k$

$$|\,\mathrm{D}^\alpha v_\varepsilon(x)| = |\sum_{\beta \le \alpha} \binom{\alpha}{\beta} \mathrm{D}^\beta \tau_\varepsilon(x)\, \mathrm{D}^{\alpha - \beta} u(x)| \le M \sum_{\beta \le \alpha} \binom{\alpha}{\beta} |\,\mathrm{D}^{\alpha-\beta} u(x)| \quad \text{for all } x \in \Omega$$

so that for all $\tilde{\Omega} \subset \Omega$ open we have

$$\|v_\varepsilon\|_{\mathrm{W}^{k,p}(\tilde{\Omega})} \le \sum_{|\alpha| \le k} \|\,\mathrm{D}^\alpha v_\varepsilon\|_{\mathrm{L}^p(\tilde{\Omega})} \le M \left( \sum_{|\alpha| \le k} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \right) \|u\|_{\mathrm{W}^{k,p}(\tilde{\Omega})} \le c(k) M \|u\|_{\mathrm{W}^{k,p}(\tilde{\Omega})}$$

with some constant $c(k) > 0$. Hence,

$$\begin{aligned}\|u - v_\varepsilon\|_{\mathrm{W}^{k,p}(\Omega)} &= \|u - v_\varepsilon\|_{\mathrm{W}^{k,p}(\Omega\setminus\overline{\mathrm{B}_{\frac{1}{\varepsilon}}(0)})} \le \|u\|_{\mathrm{W}^{k,p}(\Omega\setminus\overline{\mathrm{B}_{\frac{1}{\varepsilon}}(0)})} + \|v_\varepsilon\|_{\mathrm{W}^{k,p}(\Omega\setminus\overline{\mathrm{B}_{\frac{1}{\varepsilon}}(0)})}\\ &\le (1 + c(k)M)\|u\|_{\mathrm{W}^{k,p}(\Omega\setminus\overline{\mathrm{B}_{\frac{1}{\varepsilon}}(0)})}\\ &\to 0 \quad \text{as } \varepsilon \to 0\end{aligned}$$

in view of $p < \infty$. Hence, $\|u - v_\varepsilon\|_{\mathrm{W}^{k,p}(\Omega)} < \delta$ for $\varepsilon > 0$ small enough and $v = v_\varepsilon$ has bounded support.

ii) In view of i) we may assume w.l.o.g. that $K := \operatorname{supp}(u)$ is bounded and hence compact (if necessary, we replace $u$ by $v$).

For $x \in \partial\Omega$, let $U_x \subset \mathbb{R}^n$ be the open neighborhood of $x$ and $0 \neq y_x \in \mathbb{R}^n$ like in Definition 4.9. Then

$$F := K \setminus \big(\bigcup_{x\in\partial\Omega} U_x\big)$$

is compact with $F \subset \Omega$. Hence, there is $U_0$ open such that $F \Subset U_0 \Subset \Omega$. As $K$ is compact, there exist finitely many of the sets $U_x$ which we call $U_1, \dots, U_N$ such that $K \subset \bigcup_{i=0}^N U_i$. Moreover, we choose $V_i \Subset U_i$ open sets, $i = 0, \dots, N$, such that $K \subset \bigcup_{i=0}^N V_i$ and $V_i$ is still a neighborhood of $x^i$ belonging to $U_i = U_{x^i}$. By Lemma 4.3 there is a partition of unity $\psi_0, \dots, \psi_N$ such that $\psi_i \in \mathrm{C}_0^\infty(V_i)$, $0 \le \psi_i \le 1$ for $i = 0, \dots, N$, and $\sum_{i=0}^N \psi_i(x) = 1$ for all $x \in K$.

Our aim is to find $\varphi_i \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ such that with $u_i := \psi_i u$ we have

$$\|u_i - \varphi_i\|_{\mathrm{W}^{k,p}(\Omega)} < \frac{\delta}{N+1} \quad \text{for all } i \in \{0, \dots, N\}. \tag{4.5}$$

As $\operatorname{supp}(u_0) \Subset V_0 \Subset \Omega$, by Theorems 4.1 and 2.1 b) there exists $\varphi_0 \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ such that (4.5) holds for $i = 0$. Next, we fix $i \in \{1, \dots, N\}$ and extend $u$ by 0 outside $\Omega$. Let $x^i \in \partial\Omega$ be the point belonging to $U_i$ ($U_i$ is a neighborhood of $x^i$) and

$$\Gamma := \overline{V_i} \cap \partial\Omega.$$

As $\psi_i = 0$ on $\partial\Omega \setminus \Gamma$, $u_i = 0$ on $\mathbb{R}^n \setminus \overline{\Omega}$, and $u_i \in \mathrm{W}^{k,p}(\Omega)$ by Proposition 3.8, we get $u_i \in \mathrm{W}^{k,p}(\mathbb{R}^n \setminus \Gamma)$. Let $y := y_{x_i}$ from the segment condition and

$$\Gamma_t := \{x - ty \colon x \in \Gamma\},$$

where

$$0 < t < \min\{1, \frac{1}{|y|} \operatorname{dist}(\partial V_i, \partial U_i)\}.$$

By the choice of $t$, we have $\Gamma_t \subset U_i$ and $\Gamma_t \cap \overline{\Omega} = \emptyset$. The latter follows from the segment condition: For $z = x - sy$ with $x \in \Gamma$ and $s \in (0, 1)$ we have $z + sy = x \in \Gamma \subset \partial\Omega$. Hence, $z \notin \overline{\Omega}$ so that $\Gamma_t \cap \overline{\Omega} = \emptyset$. Define

$$w_t(x) := u_i(x + ty) \quad \text{for all } x \in \mathbb{R}^n.$$

As $u_i \in \mathrm{W}^{k,p}(\mathbb{R}^n \setminus \Gamma)$, we have $w_t \in \mathrm{W}^{k,p}(\mathbb{R}^n \setminus \Gamma_t)$. Hence, Proposition 4.12 yields that $\mathrm{D}^\alpha w_t \to \mathrm{D}^\alpha u_i$ in $\mathrm{L}^p(\Omega)$ as $t \downarrow 0$ for all $|\alpha| \leq k$ (since $\overline{\Omega} \subset \mathbb{R}^n \setminus \Gamma_t$) and we can choose $t$ small enough such that $\|w_t - u_i\|_{\mathrm{W}^{k,p}(\Omega)} \leq \frac{\delta}{2(N+1)}$.

Moreover, since $u \in \mathrm{L}^p(\Omega)$ and $u = 0$ on $\mathbb{R}^n \setminus \Omega$, we have $u \in \mathrm{L}^p(\mathbb{R}^n)$, $u_i = \psi_i u \in \mathrm{L}^p(\mathbb{R}^n)$, and $w_t \in \mathrm{L}^p(\mathbb{R}^n)$. Since $\mathrm{supp}(u_i) \subset \overline{\Omega} \cap V_i$, $w_t$ has compact support in $\mathbb{R}^n$. Hence, by Theorem 2.1 $\varphi_i := \eta_\varepsilon * w_t$ belongs to $\mathrm{C}_0^\infty(\mathbb{R}^n)$ for $\varepsilon > 0$. As $\mathrm{dist}(\Gamma_t, \overline{\Omega}) > 0$, we may choose $\varepsilon > 0$ small enough such that $\|\varphi_i - w_t\|_{\mathrm{W}^{k,p}(\Omega)} < \frac{\delta}{2(N+1)}$ (as $\Omega \cap \mathrm{supp}(w_t) \Subset \mathbb{R}^n \setminus \Gamma_t$ and $w_t \in \mathrm{W}^{k,p}(\mathbb{R}^n \setminus \Gamma_t)$ we may apply Theorem 4.1). Altogether, $\varphi_i \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ satisfies $\|u_i - \varphi_i\| \leq \frac{\delta}{N+1}$ and (4.5) holds for all $i \in \{0, \dots, N\}$. As $u = \sum_{i=1}^n u_i$, the function

$$\varphi := \sum_{i=0}^{N} \varphi_i \in \mathrm{C}_0^\infty(\mathbb{R}^n)$$

satisfies $\|u - \varphi\|_{\mathrm{W}^{k,p}(\Omega)} \leq \delta$.

iii) Combining i) and ii), the claim is proved. $\square$

As a Corollary, we see that $\mathrm{W}_0^{k,p}(\mathbb{R}^n)$ and $\mathrm{W}^{k,p}(\mathbb{R}^n)$ coincide.

**Corollary 4.14.** *For $k \in \mathbb{N}$ and $p \in [1, \infty)$, we have $\mathrm{W}_0^{k,p}(\mathbb{R}^n) = \mathrm{W}^{k,p}(\mathbb{R}^n)$.*

*Proof.* Given $u \in \mathrm{W}^{k,p}(\mathbb{R}^n)$ by part i) of the proof of Theorem 4.13, there exists $v \in \mathrm{W}^{k,p}(\mathbb{R}^n)$ with $K := \mathrm{supp}(v)$ compact and

$$\|u - v\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq \frac{\delta}{2}.$$

But then there is $V_0$ open such that $K \Subset V_0 \Subset \mathbb{R}^n$ and by Theorems 4.1 and 2.1 b) there is $\varepsilon > 0$ small enough such that $v_\varepsilon = \eta_\varepsilon * v \in \mathrm{C}_0^\infty(V_0)$ and

$$\|v - v_\varepsilon\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} = \|v - v_\varepsilon\|_{\mathrm{W}^{k,p}(V_0)} < \frac{\delta}{2}.$$

Hence, $\|u - v_\varepsilon\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} < \delta$ and $v_\varepsilon \in \mathrm{C}_0^\infty(\mathbb{R}^n)$. $\square$

# Chapter 5

# Extension and Traces

Here, we will study how we can extend functions in $\mathrm{W}^{k,p}(\Omega)$ to $\mathrm{W}^{k,p}(\mathbb{R}^n)$. Moreover, we will see how we can define boundary values on $\partial\Omega$ of functions in $\mathrm{W}^{k,p}(\Omega)$.

## 5.1 Flattening the Boundary

We will frequently use that if $\Omega$ is a bounded domain with $\partial\Omega \in \mathrm{C}^m$, we can transform $\Omega \cap \mathrm{B}_r(x^0)$ for $x^0 \in \partial\Omega$ to a domain having a flat boundary, where the transformation is a $\mathrm{C}^m$-diffeomorphism. This will be done in details here.

*Notation* 5.1. We define

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}, \quad \mathbb{R}^n_- := \{x \in \mathbb{R}^n : x_n < 0\},$$

and for $U \subset \mathbb{R}^n$

$$U^+ := U \cap \mathbb{R}^n_+, \quad U^- := U \cap \mathbb{R}^n_-, \quad U^0 := \{x \in U : x_n = 0\}.$$

Moreover, we write $x = (x', x_n)$ for $x \in \mathbb{R}^n$ with $x = (x_1, \dots, x_n)$ and $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

**Definition 5.2.** Let $\Omega$ and $U$ be domains in $\mathbb{R}^n$. A map $g\colon \Omega \to U$ is called $\mathrm{C}^m$*-diffeomorphism* if and only if $g$ is bijective, $g \in \mathrm{C}^m(\overline{\Omega}, \mathbb{R}^n)$, $g^{-1} \in \mathbb{C}^m(\overline{U}, \mathbb{R}^n)$ and $\det(\mathrm{D}g) \neq 0$ in $\overline{\Omega}$.

If $\partial\Omega \in \mathrm{C}^m$, then we can locally transform $\Omega$ to a domain with flat boundary.

**Lemma 5.3.** *Let $m \in \mathbb{N}$ and $\gamma \in \mathbb{C}^m(\mathbb{R}^{n-1})$. Then $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ and $\Psi\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$\begin{aligned} \Phi(x) &= (x_1, \dots, x_{n-1}, x_n - \gamma(x')), \\ \Psi(y) &= (y_1, \dots, y_{n-1}, y_n + \gamma(y')), \quad x, y \in \mathbb{R}^n, \end{aligned}$$

*satisfy $\Phi, \Psi \in \mathrm{C}^m(\mathbb{R}^n, \mathbb{R}^n)$, $\det(\mathrm{D}\Phi) = \det(\mathrm{D}\Psi) = 1$ on $\mathbb{R}^n$, and $\Phi^{-1} = \Psi$. In particular, for any bounded domain $\Omega$, $\Phi|_\Omega\colon \Omega \to \Phi(\Omega)$ is a $\mathrm{C}^m$-diffeomorphism. If $\partial\Omega \in \mathrm{C}^m$ and*

$$\begin{aligned} \Omega \cap \mathrm{B}_r(x^0) &= \{x \in \mathrm{B}_r(x^0) : x_n > \gamma'(x')\}, \\ \partial\Omega \cap \mathrm{B}_r(x^0) &= \{x \in \mathrm{B}_r(x^0) : x_n = \gamma(x')\} \end{aligned}$$

*for some $x^0 \in \partial\Omega$ (see Definition 4.10), then $\Phi\colon \mathrm{B}_r(x^0) \to U := \Phi(\mathrm{B}_r(x^0))$ is a $\mathrm{C}^m$-diffeomorphism with $\Phi(\Omega \cap \mathrm{B}_r(x^0)) = U^+$ and $\Phi(\partial\Omega \cap \mathrm{B}_r(x^0)) = U^0$.*

*Proof.* It is straightforward to see that $\Phi$ and $\Psi$ are $\mathrm{C}^m$-functions with $\Psi = \Phi^{-1}$ and $\det(\mathrm{D}\Phi) = \det(\mathrm{D}\Psi) = 1$. The further claims are immediate consequences of Definition 5.2. $\square$

$\Phi$ now provides a transformation which *flattens* the boundary. A $\mathrm{C}^m$-diffeomorphism also provides a transformation between the corresponding Sobolev spaces.

**Proposition 5.4.** *Let $g\colon U \to \Omega$ be a $\mathrm{C}^m$-diffeomorphism with $m \in \mathbb{N}$, $p \in [1, \infty)$, and $\Omega, U \subset \mathbb{R}^n$ bounded domains. Then the map $T_g\colon \mathrm{W}^{m,p}(\Omega) \to \mathrm{W}^{m,p}(U)$, defined by*

$$(T_g(u))(y) \coloneqq u(g(y)), \quad y \in U, u \in \mathrm{W}^{m,p}(\Omega),$$

*is bijective and there exist $C_1, C_2 > 0$ such that*

$$\|T_g(u)\|_{\mathrm{W}^{m,p}(U)} \le C_1 \|u\|_{\mathrm{W}^{m,p}(\Omega)} \quad \text{and} \quad \|(T_g)^{-1}(v)\|_{\mathrm{W}^{m,p}(\Omega)} \le C_2 \|v\|_{\mathrm{W}^{m,p}(U)}$$

*for all $u \in \mathrm{W}^{m,p}(\Omega)$, $v \in \mathrm{W}^{m,p}(U)$, where $(T_g)^{-1} = T_{g^{-1}}$.*

*Proof.* i) If $T_g$ is well-defined and satisfies the claimed estimate, it is immediate that $(T_g)^{-1} = T_{g^{-1}}$ and hence the estimate for $(T_g)^{-1}$ follows by replacing $g$ by $g^{-1}$. Moreover, we only consider the case $m = 1$ as the general case then follows by induction.

ii) Let $m = 1$ and $T \coloneqq T_g$. Assume first that $u \in \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{1,p}(\Omega)$. Then the chain rule and the transformation rule imply with $M \coloneqq \|\det(\mathrm{D}g^{-1})\|_{\mathrm{C}^0(\overline{\Omega})}$

$$\int_U |T(u)(y)|^p \,\mathrm{d}y = \int_\Omega |u(x)|^p \,|\det(\mathrm{D}g^{-1})(x)| \,\mathrm{d}x \le M \int_\Omega |u(x)|^p \,\mathrm{d}x, \tag{5.1}$$

$$\partial_{y_i}(T(u))(y) = \sum_{j=1}^n u_{x_j}(x) \cdot (g_j)_{y_i}(y) = \sum_{j=1}^n T(u_{x_j})(y) \cdot (g_j)_{y_i}(y), \quad y \in U,$$

where $x = g(y)$ $(x \in \Omega, y \in U)$, and

$$\begin{aligned}\Big(\int_U |(\partial_{y_i}(T(u))(y)|^p \,\mathrm{d}y\Big)^{\frac{1}{p}} &\le \|g\|_{\mathrm{C}^1(\overline{U})} \sum_{j=1}^n \Big(\int_U |T(u_{x_j})(y)|^p \,\mathrm{d}y\Big)^{\frac{1}{p}} \\ &\overset{(5.1)}{\le} \|g\|_{\mathrm{C}^1(\overline{U})}\, M^{\frac{1}{p}} \sum_{j=1}^n \|u_{x_j}\|_{\mathrm{L}^p(\Omega)}.\end{aligned}$$

Hence, $\|T(u)\|_{\mathrm{W}^{1,p}(U)} \le M^{\frac{1}{p}}(1 + n\|g\|_{\mathrm{C}^1(\overline{U})})\, \|u\|_{\mathrm{W}^{1,p}}$ by combining the previous estimate with (5.1) and Minkowski's inequality.

iii) By ii) we have

$$\|T(u)\|_{\mathrm{W}^{1,p}(U)} \le C_1 \|u\|_{\mathrm{W}^{1,p}(\Omega)} \quad \text{for all } u \in \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{1,p}(\Omega).$$

For $u \in \mathrm{W}^{1,p}(\Omega)$ there exists $(u_k)_{k\in\mathbb{N}} \subset \mathrm{C}^\infty(\Omega) \cap \mathrm{W}^{1,p}(\Omega)$ such that $u_k \to u$ in $\mathrm{W}^{1,p}(\Omega)$ and a.e. in $\Omega$. As $T$ is linear, we have

$$\|T(u_k) - T(u_j)\|_{\mathrm{W}^{1,p}(U)} \le C_1 \|u_k - u_j\|_{\mathrm{W}^{1,p}(\Omega)}, \quad \text{for all } k, j \in \mathbb{N}.$$

Hence, $(T(u_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{W}^{1,p}(U)$ and $T(u_k) \to v$ in $\mathrm{W}^{1,p}(U)$. As obviously $T(u_k) \to T(u)$ in $\mathrm{L}^p(U)$ by (5.1) and the dominated convergence theorem, we have $v = T(u) \in \mathrm{W}^{1,p}(U)$ and

$$\|T(u)\|_{\mathrm{W}^{1,p}(U)} = \lim_{k\to\infty} \|T(u_k)\|_{\mathrm{W}^{1,p}(U)} \overset{\text{ii)}}{\leq} C_1 \lim_{k\to\infty} \|u_k\|_{\mathrm{W}^{1,p}(\Omega)} = C_1 \|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

where $C_1$ can be chosen independent of $p$ as $M^{\frac{1}{p}} \leq 1 + M$. □

## 5.2 Extension Theorem

We next extend functions $u \in \mathrm{W}^{k,p}(\Omega)$ to become a function in $\mathrm{W}^{k,p}(\mathbb{R}^n)$. This cannot be done in general by extending $u$ by 0 to $\mathbb{R}^n \setminus \Omega$ as this may create bad discontinuities in some $\mathrm{D}^\alpha u$ on $\partial\Omega$. We need now again boundary regularity of $\partial\Omega$.

**Theorem 5.5.** *Let $m \in \mathbb{N}$, $p \in [1,\infty)$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in \mathrm{C}^m$, and $V \subset \mathbb{R}^n$ be a domain with $\Omega \Subset V$. Then for any $k \in \{1,\ldots,m\}$ there exists a linear Operator*

$$E\colon \mathrm{W}^{k,p}(\Omega) \to \mathrm{W}^{k,p}(\mathbb{R}^n)$$

*such that*

*a)* $E(u) = u$ *a.e. in* $\Omega$,

*b)* $\operatorname{supp}(E(u)) \Subset V$, *i.e.* $E(u) \in \mathrm{W}^{k,p}_0(V)$,

*c)* $\|E(u)\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq C\, \|u\|_{\mathrm{W}^{k,p}(\Omega)}$ *for all* $u \in \mathrm{W}^{k,p}(\Omega)$,

*where $C > 0$ depends on $m$, $p$, $V$, and $\Omega$ but not on $k$ and $u$. Moreover, $E$ does not depend on $k \in \{1,\ldots,m\}$ and on $p \in [1,\infty)$ in the sense that if $u \in \mathrm{W}^{k,p}(\Omega)$ for several $p, k$ then $E(u)$ is uniquely determined in all these spaces.*

*Proof.* We will use the notation from 5.1, Lemma 5.3, and Definition 4.10.

i) We first assume that for some $x^0 \in \partial\Omega$, $\mathrm{B} := \mathrm{B}_r(x^0)$ satisfies $\Omega \cap \mathrm{B} = \mathrm{B}^+$ and $\partial\Omega \cap \mathrm{B} = \mathrm{B}^0$. We further assume that $u \in \mathrm{C}^m(\mathrm{B}^+ \cup \mathrm{B}^0)$ with $\operatorname{supp}(u) \subset \widetilde{\mathrm{B}}^+ := \overline{(\mathrm{B}_s(x^0))^+}$ for some $s \in (0,r)$. As hence $u = 0$ in a neighborhood of $\partial\mathrm{B} \cap \overline{\mathbb{R}^n_+}$, we can extend $u$ by 0 in $\overline{\mathbb{R}^n_+} \setminus \mathrm{B}^+$ and have $u \in \mathrm{C}^m(\overline{\mathbb{R}^n_+})$ with $\operatorname{supp}(u) \subset \widetilde{\mathrm{B}}^+$. Then we extend $u$ from $\overline{\mathbb{R}^n_+}$ to $\mathbb{R}^n$ by a *higher-order reflection.* The $(m+1)\times(m+1)$ system of linear equations

$$\sum_{j=1}^{m+1} (-j)^i \lambda_j = 1, \quad i = 0,\ldots,m, \tag{5.2}$$

has a unique solution $(\lambda_1,\ldots,\lambda_{m+1})$ since the corresponding matrix is of Vandermonde type. We define the extension

$$\tilde{E}(u)(x) := \begin{cases} u(x) & \text{if } x = (x', x_n) \text{ with } x_n \geq 0, \\ \sum_{j=1}^{m+1} \lambda_j u(x', -jx_n) & \text{if } x = (x', x_n) \text{ with } x_n < 0, \end{cases}$$

and for any $\alpha \in \mathbb{N}_0^n$ with $1 \le |\alpha| \le m$

$$\tilde{E}_\alpha(u)(x) \coloneqq \begin{cases} u(x) & \text{if } x_n \ge 0, \\ \sum_{j=1}^{m+1} (-j)^{\alpha_n} \lambda_j u(x', -jx_n) & \text{if } x_n < 0. \end{cases}$$

Then for all $|\alpha| \le m$ we have $\mathrm{D}^\alpha(\tilde{E}(u)) = \tilde{E}_\alpha(\mathrm{D}^\alpha u)$ and hence $\tilde{E}(u) \in \mathrm{C}^n(\mathbb{R}^n)$: This is immediate in $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$, while for $x \in \partial\mathbb{R}^n_+$, namely $x_n = 0$, we have

$$\begin{aligned} \lim_{t\uparrow 0} \mathrm{D}^\alpha(\tilde{E}(u))(x',t) &= \lim_{t\uparrow 0} \tilde{E}_\alpha(\mathrm{D}^\alpha u)(x',t) = \lim_{t\uparrow 0} \sum_{j=1}^{m+1} (-j)^{\alpha_n} \lambda_j \mathrm{D}^\alpha u(x', -jt) \\ &= \Big( \sum_{j=1}^{m+1} (-j)^{\alpha_n} \lambda_j \Big) \mathrm{D}^\alpha u(x',0) \overset{(5.2)}{=} \mathrm{D}^\alpha u(x',0) \\ &= \lim_{t\downarrow 0} \mathrm{D}^\alpha u(x',t) = \lim_{t\downarrow 0} \mathrm{D}^\alpha(\tilde{E}(u))(x',t) \end{aligned}$$

since $u \in \mathrm{C}^m(\overline{\mathbb{R}^n_+})$ by assumption. Moreover,

$$\begin{aligned} \|\mathrm{D}^\alpha(\tilde{E}(u))\|_{\mathrm{L}^p(\mathbb{R}^n)} &\le \Big(1 + \sum_{j=1}^{m+1} j^{\alpha_n} |\lambda_j|\, j^{\frac{1}{p}}\Big) \|\mathrm{D}^\alpha u\|_{\mathrm{L}^p(\mathbb{R}^n_+)} \\ &\overset{p\ge 1}{\le} \Big(1 + \sum_{j=1}^{m+1} j^{\alpha_n+1} |\lambda_j|\Big) \|\mathrm{D}^\alpha u\|_{\mathrm{L}^p(\mathbb{R}^n_+)}, \end{aligned}$$

where we have used the transformation rule with $z = (x', -jx_n)$ and $\operatorname{supp}(u) \subset \widetilde{\mathrm{B}}_+ \subset \overline{\mathrm{B}^+}$. Hence, $\tilde{E}(u) \in \mathrm{C}^m(\mathbb{R}^n)$ with

$$\|\tilde{E}(u)\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \le C_1 \|u\|_{\mathrm{W}^{k,p}(\mathrm{B}^+)} \tag{5.3}$$

with $C_1$ depending on $m$ and $n$.

ii) As $\partial\Omega \in \mathrm{C}^m$ and $\partial\Omega$ is compact, there exists $x^i \in \partial\Omega$ and balls $U_i \coloneqq \mathrm{B}_r(x^i), i = 1, \dots, N$, such that after relabelling and reorienting the coordinate axes (described by a $\mathrm{C}^\infty$-diffeomorphism $\mathcal{T}_i \colon \mathbb{R}^n \to \mathbb{R}^n$), we have

$$\begin{aligned} \Omega \cap U_i &= \{x \in U_i \colon x_n > \gamma_i(x')\} \\ \partial\Omega \cap U_i &= \{x \in U_i \colon x_n = \gamma_i(x')\} \end{aligned}$$

with $\gamma_i \in \mathrm{C}^m(\mathbb{R}^{n-1})$. Choose $U_0 \Subset \Omega$ such that $\overline{\Omega} \subset \bigcup_{i=0}^N U_i$. By Lemma 4.3 there exists a partition of unity $\varphi_i \in \mathrm{C}_0^\infty(\mathcal{T}_i^{-1}(U_i))$ with $0 \le \varphi_i \le 1$ and $\sum_{i=1}^N \varphi_i(x) = 1$ for all $x \in \overline{\Omega}$. Define further $\Phi_i, \Psi_i$ by Lemma 5.3 according to $\gamma = \gamma_i, i = 1, \dots, N$, and set $W_i \coloneqq \Phi_i(U_i)$.

We fix $i \in \{1, \dots, N\}$ and $u \in \mathrm{C}^m(\overline{\Omega})$. Then $\varphi_i u$ and hence also $T_{\mathcal{T}_i^{-1}}(\varphi_i u)$ belong to $\mathrm{C}^m(\overline{U}_i \cap \overline{\Omega})$ with compact support $K_i \Subset U_i, K_i \subset \mathcal{T}_i(\overline{\Omega})$. Then

$$v_i \coloneqq T_{\Psi_i}(T_{\mathcal{T}_i^{-1}}(\varphi_i u)) \in \mathrm{C}^m(\overline{W_i^+})$$

with support in $\Phi(K_i) \Subset W_i$ and $\Phi_i(K_i) \subset W_i^+ \cup W_i^0$. Hence, we may apply $\tilde{E}$ to $v_i$ (by choosing $\widetilde{\mathrm{B}}^+$ and $\mathrm{B}$ appropriately such that $\Phi_i(K_i) \subset \widetilde{\mathrm{B}}^+ \Subset W_i \Subset B$) and obtain

$$\tilde{E}(v_i) \in \mathrm{C}^m(\mathbb{R}^n) \cap \mathrm{W}^{k,p}(\mathbb{R}^n).$$

Transforming back, we finally obtain

$$u_i := T_{\mathcal{T}_i}(T_{\Phi_i}(\tilde{E}(v_i))) \in \mathrm{W}^{k,p}(\mathbb{R}^n).$$

As $\tilde{E}(v_i)$ has compact support in some ball $\mathrm{B}_i \subset \mathbb{R}^n$, we may apply Proposition 5.4 with

$$\Phi_i := \Phi_i^{-1}(\mathrm{B}_i) \to \mathrm{B}_i \quad \text{and} \quad \mathcal{T}_i \colon \mathcal{T}_i^{-1}(\Phi_i^{-1}(\mathrm{B}_i)) \to \Phi_i^{-1}(B_i)$$

and obtain $u_i = \varphi_i u$ in $\Omega$ and

$$\begin{aligned}\|u_i\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq C_2 \|\tilde{E}(v_i)\|_{\mathrm{W}^{k,p}(\mathrm{B}_i)} &\overset{\text{i)}}{\leq} C_3 \|v_i\|_{\mathrm{W}^{k,p}(W_i^+)} \\ \overset{\text{Prop. 5.4}}{\leq} C_4 \|\varphi_i u\|_{\mathrm{W}^{k,p}(\Omega \cap U_i)} &\overset{\substack{\text{Prop. 3.8}\\ \text{Lem. 4.2}}}{\leq} C_5 \|u\|_{\mathrm{W}^{k,p}(\Omega \cap U_i)}.\end{aligned}$$

Defining further $u_0 := \varphi_0 u \in \mathrm{C}^m(U_0)$ with compact support in $U_0$, we have

$$\sum_{i=0}^{n} u_i \in \mathrm{W}^{k,p}(\mathbb{R}^n) \quad \text{with} \quad \| \sum_{i=0}^{N} u_i \|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq C_6 \|u\|_{\mathrm{W}^{k,p}(\Omega)}$$

and

$$\sum_{i=0}^{N} u_i = \sum_{i=0}^{N} \varphi_i u = u \quad \text{in } \Omega.$$

Finally, let $\tau \in \mathrm{C}_0^\infty(V)$ be the cutoff function w.r.t $\overline{\Omega}$ and $V$. Then

$$E(u) := \tau \cdot \sum_{i=0}^{N} u_i \in \mathrm{W}^{k,p}(\mathbb{R}^n) \quad \text{with } \operatorname{supp}(E(u)) \Subset V$$

and

$$\|E(u)\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq C_7 \|u\|_{\mathrm{W}^{k,p}(\Omega)} \quad \text{for all } u \in \mathrm{C}^m(\overline{\Omega}), \tag{5.4}$$

where $C_7 > 0$ does not depend on $k$ or $u$ but on $m, p, \Omega$ and $V$. Moreover, $E(u) = u$ in $\Omega$.

iii) As $\tilde{E}$ is linear in $u$ and $T_g$ is linear in $u$ (see Proposition 5.4), the map

$$E \colon \mathrm{C}^m(\overline{\Omega}) \to \mathrm{W}^{k,p}(\mathbb{R}^n), \quad u \mapsto E(u)$$

is linear. Let $u \in \mathrm{W}^{k,p}(\Omega)$. Then by Theorem 4.13 there exists $(u_j)_{j\in\mathbb{N}} \subset \mathrm{C}^m(\overline{\Omega})$ such that $u_j \to u$ in $\mathrm{W}^{k,p}(\Omega)$. Then by (5.4)

$$\|E(u_j) - E(u_i)\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq C_7 \|u_j - u_i\|_{\mathrm{W}^{k,p}(\Omega)} \quad \text{for all } i, j \in \mathbb{N}$$

so that $(E(u_j))_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{W}^{k,p}(\mathbb{R}^n)$ and has a limit $v \in \mathrm{W}^{k,p}(\mathbb{R}^n)$. As $E(u_j) \to v$ a.e. in $\mathbb{R}^n$ for a subsequence, we still have $\operatorname{supp}(v) \Subset V$ and $u = v$ a.e. in $\Omega$. Defining $E(u) := v$, we have

$$\|E(u)\|_{\mathrm{W}^{k,p}(\mathbb{R}^n)} \leq C_7 \|u\|_{\mathrm{W}^{k,p}(\Omega)}$$

by (5.4) like in the end of the proof of Proposition 5.4. Since (5.4) implies that $E(u) = v$ does not depend on the choice of the approximating sequence $(u_j)_{j\in\mathbb{N}}$ (as $E$ is linear), we thereby have defined $E \colon \mathrm{W}^{k,p}(\Omega) \to \mathrm{W}^{k,p}(\mathbb{R}^n)$ having all claimed properties. $\square$

## 5.3 Trace Operator

A function $u \in \mathrm{W}^{1,p}(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ clearly has boundary values on $\partial\Omega$ in the usual sense. But a general function $u \in \mathrm{W}^{1,p}(\Omega)$ is not continuous and only defined a.e. in $\Omega$. Since $\partial\Omega$ has $n$-dimensional Lebesgue measure 0, we need a trace operator involving the space $\mathrm{L}^p(\partial\Omega)$ for assigning boundary values to $u$.

**Definition 5.6.** Let $\Omega$ be a bounded domain with $\partial\Omega \in \mathrm{C}^1$ and $U_i := \mathrm{B}_{r_i}(x^i), i = 1, \ldots, N$, such that $\Omega \cap U_i = \{x \in U_i \colon x_n > \gamma_i(x')\}$, $\partial\Omega \cap U_i = \{x \in U_i \colon x_n = \gamma_i(x')\}$ with $\gamma_i \in \mathrm{C}^1(\mathbb{R}^{n-1})$ and $x^i \in \partial\Omega$ (where we assume that the coordinate axes are relabeled and reoriented appropriately). Defining

$$U_i' := \{x' \colon x \in \partial\Omega \cap U_i\} \subset \mathbb{R}^{n-1},$$

we have

$$\partial\Omega \cap U_i = \{(x', \gamma_i(x')) \colon x' \in U_i'\}.$$

Let $U$ be a neighborhood of $\partial\Omega$ (in $\mathbb{R}^n$) such that $U \Subset \bigcup_{i=1}^N U_i$ and $\varphi_i, i = 1, \ldots, N$, a corresponding partition of unity such that $\varphi_i \in \mathrm{C}_0^\infty(U_i)$, $\sum_{i=1}^N \varphi_i(x) = 1$ for all $x \in U$, $0 \le \varphi_i \le 1$. Then $v \colon \partial\Omega \to \mathbb{R}$ is called *measurable* on $\partial\Omega$ if

$$v_i \colon U_i' \to \mathbb{R}, \quad v_i(x') := (\varphi_i v)(x', \gamma_i(x'))$$

is measurable in $U_i' \subset \mathbb{R}^{n-1}$.
Defining

$$\int_{\partial\Omega} v_i \,\mathrm{d}\sigma := \int_{\partial\Omega \cap U_i} v_i \,\mathrm{d}\sigma := \int_{U_i'} v_i \sqrt{1 + |\nabla\gamma_i|^2} \,\mathrm{d}x'$$

and

$$\int_{\partial\Omega} v \,\mathrm{d}\sigma := \sum_{i=1}^N \int_{\partial\Omega} v_i \,\mathrm{d}\sigma,$$

then $\mathrm{L}^p(\partial\Omega)$ is the space of functions $v \colon \partial\Omega \to \mathbb{R}$ being measurable on $\partial\Omega$ such that the norm

$$\|v\|_{\mathrm{L}^p(\partial\Omega)} := \left(\int_{\partial\Omega} |v|^p \,\mathrm{d}\sigma\right)^{\frac{1}{p}}$$

is finite where $p \in [1, \infty)$.

Then we have the following theorem:

**Theorem 5.7** (Trace theorem)**.** *Let $\Omega$ be a bounded domain with $\partial\Omega \in \mathrm{C}^1$ and $p \in [1, \infty)$. Then there exists a bounded linear operator $\mathrm{Tr} \colon \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}^p(\partial\Omega)$ such that*

*a)* $\mathrm{Tr}(u) = u|_{\partial\Omega}$ *if $u \in \mathrm{W}^{1,p}(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$,*

*b)* $\|\mathrm{Tr}(u)\|_{\mathrm{L}^p(\partial\Omega)} \le C \|u\|_{\mathrm{W}^{1,p}(\Omega)}$ *for any $u \in \mathrm{W}^{1,p}(\Omega)$, where $C > 0$ only depends on $p$ and $\Omega$.*

$\mathrm{Tr}(u)$ *is called the* trace of $u$ on $\partial\Omega$.

*Proof.* i) Assume like in the proof of Theorem 5.5 that there is $x^0 \in \partial\Omega$ and $\mathrm{B} := \mathrm{B}_r(x^0) \subset \mathbb{R}^n$ such that $\Omega \cap \mathrm{B} = \mathrm{B}^+$ and $\partial\Omega \cap \mathrm{B} = \mathrm{B}^0$. Let $\hat{\mathrm{B}} \Subset \mathrm{B}$ and $\xi \in \mathrm{C}_0^\infty(\mathrm{B})$ such that $\xi > 0$ in $\hat{\mathrm{B}}$ and $0 \le \xi \le 1$ (see Lemma 4.2). Furthermore, assume $p \in (1, \infty)$. Then setting

$$\mathrm{B}' := \{x' : (x', x_n) \in \mathrm{B}\} \subset \mathbb{R}^{n-1},$$

we have for $u \in \mathrm{C}^1(\overline{\Omega})$

$$\begin{aligned}\int_{\partial\Omega\cap\mathrm{B}} |\xi u|^p \,\mathrm{d}\sigma &= \int_{\mathrm{B}'} |\xi u|^p(x', 0)\,\mathrm{d}x' \\ &= \int_{\mathrm{B}'} \left( \Big[ -\int_0^{\tilde{x}} \partial_{x_n}(|\xi u|^p)(x', s)\,\mathrm{d}s \Big] + \underbrace{|\xi u|^p(x', \tilde{x})}_{=0} \,\mathrm{d}x' \right),\end{aligned}$$

where $(x', \tilde{x}) \in \partial\mathrm{B} \cap (\overline{\mathrm{B}^+} \setminus \mathrm{B}^0)$,

$$\begin{aligned} &= -\int_{\mathrm{B}^+} \Big( |u|^p \, p \, \xi^{p-1} \, \xi_{x_n} + \xi^p \, p \, |u|^{p-1} \frac{u}{|u|} \, u_{x_n} \Big)(x)\,\mathrm{d}x \\ &\le C_1 \int_{\mathrm{B}^+} (|u|^p + |u|^{p-1} \, |u_{x_n}|)\,\mathrm{d}x \\ &\le C_2 \int_{\mathrm{B}^+} (|u|^p + |u_{x_n}|^p)\,\mathrm{d}x \\ &\le C_2 \, \|u\|^p_{\mathrm{W}^{1,p}(\mathrm{B}^+)},\end{aligned}$$

where $C_2 > 0$ depends on $p$ and $\xi$ and we used *Young's inequality*

$$|ab| \le \frac{1}{p}|a|^p + \frac{1}{q}|b|^q \quad \text{for } a, b \in \mathbb{R}, p, q \in [1, \infty], \frac{1}{p} + \frac{1}{q} = 1,$$

with $a = u_{x_n}(x)$, $b = |u|^{p-1}(x)$, $q = \frac{p}{p-1}$. Summing up the previous calculation, we have shown

$$\int_{\partial\Omega\cap\mathrm{B}} |\xi u|^p \,\mathrm{d}\sigma \le C_2 \|u\|^p_{\mathrm{W}^{1,p}(\mathrm{B}^+)} \quad \text{for all } u \in \mathrm{C}^1(\overline{\Omega}), p \in (1, \infty). \tag{5.5}$$

A similar calculation shows that (5.5) also holds for $p = 1$.

ii) In the general case of Definition 5.6, let $B^i = B_{s_i}(x^i)$ such that $\Phi_i(U_i) \subset \mathrm{B}^i$ (with $\Phi_i, \Psi_i$ from Lemma 5.3 with $\gamma = \gamma_i$ to flatten the boundary). Then applying (5.5) with $\mathrm{B} = \mathrm{B}^i$, $\xi = T_{\Psi_i}(\varphi_i)$, we obtain with $u_i = \varphi_i u$ for $u \in \mathrm{C}^1(\overline{\Omega})$

$$\begin{aligned}\int_{\partial\Omega} |u_i|^p \,\mathrm{d}\sigma &= \int_{U_i'} |\varphi_i u|^p(x', \gamma_i(x'))\sqrt{1 + |\nabla\gamma_i(x')|^2}\,\mathrm{d}x' \le C_3 \int_{(B^i)'} |T_{\Psi_i}(\varphi_i u)|^p(x', 0)\,\mathrm{d}x' \\ &\overset{(5.5)}{\le} C_4 \, \|T_{\Psi_i} u\|^p_{\mathrm{W}^{1,p}((\mathrm{B}^i)^+)} \overset{\text{Prop. 5.4}}{\le} C_5 \, \|u\|^p_{\mathrm{W}^{1,p}(\Omega)}.\end{aligned}$$

Hence,

$$\|u\|_{\mathrm{L}^p(\partial\Omega)} \le \sum_{i=1}^N \|u_i\|_{\mathrm{L}^p(\partial\Omega)} \le N \cdot C_5^{\frac{1}{p}} \, \|u\|_{\mathrm{W}^{1,p}(\Omega)} \quad \text{for all } u \in \mathrm{C}^1(\overline{\Omega}),$$

with $C_5$ depending on $p$ and $\Omega$.

iii) By defining $\operatorname{Tr}(u) \coloneqq u|_{\partial\Omega}$ for $u \in \mathrm{C}^1(\overline{\Omega})$, we have

$$\|\operatorname{Tr}(u)\|_{\mathrm{L}^p(\partial\Omega)} \leq C_6 \|u\|_{\mathrm{W}^{1,p}(\Omega)} \quad \text{for all } u \in \mathrm{C}^1(\overline{\Omega})$$

by ii). If $u \in \mathrm{W}^{1,p}(\Omega)$ is arbitrary, as Tr is linear in $u$, we can show like in part iii) of the proof of Theorem 5.5 that there exist $(u_l)_{l\in\mathbb{N}} \subset \mathrm{C}^1(\overline{\Omega})$ such that $u_l \to u$ in $\mathrm{W}^{1,p}(\Omega)$ and $\operatorname{Tr}(u_l) \to v$ in $\mathrm{L}^p(\partial\Omega)$ for some $v \in \mathrm{L}^p(\partial\Omega)$. Hence, defining $\operatorname{Tr}(u) \coloneqq v$ the operator $\operatorname{Tr}\colon \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}^p(\partial\Omega)$ is well defined and satisfies b) with $C = C_6$.

iv) If $u \in \mathrm{W}^{1,p}(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ and $(\tilde{u}_l)_{l\in\mathbb{N}}$ in $\mathrm{C}^1(\overline{\Omega})$ such that $\tilde{u}_l \to u$ in $\mathrm{W}^{1,p}(\Omega)$, we remark that we can choose $(\tilde{u}_l)_{l\in\mathbb{N}}$ in the proof of Theorem 4.13 such that $\tilde{u}_l$ converges to $u$ in $\mathrm{C}^0(\overline{\Omega})$. ($u_i$ is uniformly continuous (compact support), $w_t \to u_i$ in $\mathrm{C}^0(\overline{\Omega})$ as $t \downarrow 0$ since $w_t$ is a translation and $\varphi_i = \eta_\varepsilon * w_t \to w_t$ in $\mathrm{C}^0(\overline{\Omega})$ as $\varepsilon \downarrow 0$ by Theorem 2.1 (as $\overline{\Omega} \Subset \mathbb{R}^n \setminus \Gamma_t$)). Hence, $\operatorname{Tr}(u) = \lim_{l\to\infty} \operatorname{Tr}(\tilde{u}_l) = \lim_{l\to\infty} \tilde{u}_l|_{\partial\Omega} = u|_{\partial\Omega}$. ☐

Next, we characterize $\mathrm{W}_0^{k,p}(\Omega)$ functions in terms of trace zero functions.

**Theorem 5.8.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in \mathrm{C}^k$ and assume that $k \in \mathbb{N}$, $p \in [1,\infty)$, and $u \in \mathrm{W}^{k,p}(\Omega)$. Then $u \in \mathrm{W}_0^{k,p}(\Omega)$ if and only if* $\operatorname{Tr}(\mathrm{D}^\alpha u) = 0$ *for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k-1$. In particular,* $\mathrm{W}_0^{1,p}(\Omega) = \{u \in \mathrm{W}^{1,p}(\Omega)\colon \operatorname{Tr}(u) = 0\}$.

*Proof.* i) "⇒" for $k = 1$: Let $u \in \mathrm{W}_0^{1,p}(\Omega)$. Then there are $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}_0^\infty(\Omega)$ with $u_m \to u$ in $\mathrm{W}^{1,p}(\Omega)$. As $\operatorname{Tr}(u_m) = u_m|_{\partial\Omega} = 0$ and Theorem 5.7 shows

$$\begin{aligned}\|\operatorname{Tr}(u)\|_{\mathrm{L}^p(\partial\Omega)} &= \|\operatorname{Tr}(u) - \operatorname{Tr}(u_m)\|_{\mathrm{L}^p(\partial\Omega)} = \|\operatorname{Tr}(u - u_m)\|_{\mathrm{L}^p(\partial\Omega)} \\ &\leq C\|u - u_m\|_{\mathrm{W}^{1,p}(\Omega)} \to 0 \quad \text{as } m \to \infty,\end{aligned}$$

we have $\operatorname{Tr}(u) = 0$ in $\mathrm{L}^p(\partial\Omega)$.

ii) "⇐" for $k = 1$: Assume that $u \in \mathrm{W}^{1,p}(\Omega)$ with $\operatorname{Tr}(u) = 0$. Then by Theorems 4.13 and 5.7 there exists $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^1(\overline{\Omega})$ such that $u_m \to u$ in $\mathrm{W}^{1,p}(\Omega)$ and $\operatorname{Tr}(u_m) \to \operatorname{Tr}(u) = 0$ in $\mathrm{L}^p(\partial\Omega)$. By Definition 5.6 this means $\operatorname{Tr}(\varphi_i u_m) \to \operatorname{Tr}(\varphi_i u) = 0$ in $\mathrm{L}^p(\partial\Omega \cap U_i)$ for $i = 1, \ldots, N$. Hence, fixing $i \in \{1, \ldots, N\}$ and $\Phi_i, \Psi_i$ from Lemma 5.3 (with $\gamma = \gamma_i$) we have $v_m \coloneqq T_{\Psi_i}(\varphi_i u_m) \in \mathrm{C}^1(\mathrm{B}^+ \cup \mathrm{B}^0)$ and $v \coloneqq T_{\Psi_i}(\varphi_i u) \in \mathrm{W}^{1,p}(\mathrm{B}^+)$ with $\operatorname{Tr}(v_m) \to \operatorname{Tr}(v) = 0$ in $\mathrm{L}^p(\mathrm{B}^0)$ for some ball $\mathrm{B} = \mathrm{B}_s(x^i) \subset \mathbb{R}^n$ (see part ii) of the proof of Theorem 5.7), where $v_m$, $v$ have compact supports $K_m$, $K$ in $\mathrm{B} \cap \overline{\mathrm{B}^+}$.

If we find $w_m \in \mathrm{C}_0^\infty(\mathrm{B}^+)$ with $w_m \to v$ in $\mathrm{W}^{1,p}(\mathrm{B}^+)$, by Proposition 5.4 there is $f_i \coloneqq T_{\Phi_i}(w_m) \in \mathrm{C}^1(\Omega)$ with $\operatorname{supp}(f_i) \subset \Omega \cap U_i$ such that $\|f_i - \varphi_i u\|_{\mathrm{W}^{1,p}(\Omega)} \leq \frac{\delta}{2(N+1)}$. Then $\tilde{u} \coloneqq u - \sum_{i=1}^N \varphi_i u$ has compact support in $\Omega$ so that by Theorems 4.1 and 2.1 there is $f_0 \in \mathrm{C}_0^\infty(\Omega)$ such that $\|\tilde{u} - f_0\|_{\mathrm{W}^{1,p}(\Omega)} \leq \frac{\delta}{2(N+1)}$.

Then $\|u - \sum_{i=0}^N f_i\|_{\mathrm{W}^{1,p}(\Omega)} \leq \frac{\delta}{2}$ and $f \coloneqq \sum_{i=0}^N f_i \in \mathrm{C}^1(\Omega)$ with compact support in $\Omega$. Hence, by Theorems 4.1 and 2.1 there is $g \in \mathrm{C}_0^\infty(\Omega)$ such that $\|g - f\|_{\mathrm{W}^{1,p}(\Omega)} \leq \frac{\delta}{2}$ and $\|u - g\|_{\mathrm{W}^{1,p}(\Omega)} \leq \delta$. Hence, $u \in \mathrm{W}_0^{1,p}(\Omega)$ as $\delta > 0$ is arbitrary.

iii) It remains to find $w_m \in \mathrm{C}_0^\infty(\mathrm{B}^+)$ with $w_m \to v$ in $\mathrm{W}^{1,p}(\mathrm{B}^+)$. We have

$$\mathrm{Tr}(v_m) \to \mathrm{Tr}(v) = 0 \text{ in } \mathrm{L}^p(\mathrm{B}^0) \quad \text{and} \quad v_m \to v \text{ in } \mathrm{W}^{1,p}(\mathrm{B}^+) \tag{5.6}$$

by ii) and Proposition 5.4 and 3.8. As $v_m \in \mathrm{C}^1(\mathrm{B}^+ \cup \mathrm{B}^0)$ we have for $x \in \mathrm{B}^+ \cup \mathrm{B}^0$ with $\mathrm{B}' = \{x' \colon (x', x_n) \in \mathrm{B}\}$ and $\frac{1}{q} + \frac{1}{p} = 1$.

$$|v_m(x', x_n)| \leq |v_m(x', 0)| + \int_0^{x_n} |(v_m)_{x_n}(x', t)| \,\mathrm{d}t$$

and

$$\begin{aligned}
&\int_{\mathrm{B}'} |v_m(x', x_n)|^p \,\mathrm{d}x' \\
&\overset{\text{Hölder}}{\leq} 2^p \left( \int_{\mathrm{B}'} |v_m(x', 0)|^p \,\mathrm{d}x' + \int_{\mathrm{B}'} \left[ \Big( \int_0^{x_n} 1^q \,\mathrm{d}t \Big)^{\frac{1}{q}} \Big( \int_0^{x_n} |\nabla v_m(x', t)|^p \,\mathrm{d}t \Big)^{\frac{1}{p}} \right]^p \mathrm{d}x' \right) \\
&\overset{\text{Fubini}}{\leq} 2^p \Big( \|v_m\|_{\mathrm{L}^p(\mathrm{B}^0)}^p + x_n^{p-1} \int_0^{x_n} \int_{\mathrm{B}'} |\nabla v_m(x', t)|^p \,\mathrm{d}x' \,\mathrm{d}t \Big)
\end{aligned}$$

since $\frac{p}{q} = p - 1$ and $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for $a, b \geq 0$.

Hence, by (5.6) and the dominated convergence theorem, we obtain as $m \to \infty$ (for a subsequence)

$$\int_{\mathrm{B}'} |v(x', x_n)|^p \,\mathrm{d}x' \leq 2^p x_n^{p-1} \int_0^{x_n} \int_{\mathrm{B}'} |\nabla v(x', t)|^p \,\mathrm{d}x' \,\mathrm{d}t \quad \text{for a.e. } x_n > 0 \tag{5.7}$$

since we may extend $v, v_m$ by 0 to $\overline{\mathbb{R}^n_+}$ without loosing regularity.

Let $\xi \in \mathrm{C}^\infty([0, \infty))$ such that $\xi \equiv 1$ on $[0, 1]$ and $\xi \equiv 0$ on $[2, \infty)$ and define

$$\xi_m(x) \coloneqq \xi(m x_n), \quad x \in \overline{\mathrm{B}^+}, m \in \mathbb{N},$$

and

$$\tilde{w}_m(x) \coloneqq v(x)(1 - \xi_m(x)).$$

As $(\tilde{w}_m)_{x_n} = v_{x_n}(1 - \xi_m) - m\, v\, \xi'(m x_n)$ and $(\tilde{w}_m)_{x_i} = v_{x_i}(1 - \xi_m)$ for $i = 1, \ldots, n-1$, we have as $\xi_m = 0$ for $x_n \geq \frac{2}{m}$

$$\begin{aligned}
\int_{\mathrm{B}^+} |\nabla \tilde{w}_m - \nabla v|^p \,\mathrm{d}x &\leq C_p \int_{\mathrm{B}^+} (\xi_m)^p |\nabla v|^p \,\mathrm{d}x + C_p m^p \, \|\xi'\|_{\mathrm{L}^\infty([0,\infty))}^p \int_0^{\frac{2}{m}} \int_{\mathrm{B}'} |v|^p \,\mathrm{d}x' \,\mathrm{d}t \\
&=: A_m + B_m.
\end{aligned}$$

As $\xi_m \neq 0$ only for $0 \leq x_n < \frac{2}{m}$, the dominated convergence theorem shows $A_m \to 0$ as $m \to \infty$. Using (5.7), we obtain

$$\begin{aligned}
B_m &\leq C \cdot m^p \int_0^{\frac{2}{m}} \Big[ t^{p-1} \int_0^t \int_{\mathrm{B}'} |\nabla v|^p(x', s) \,\mathrm{d}x' \,\mathrm{d}s \Big] \,\mathrm{d}t \\
&\leq C \cdot m^p \Big( \int_0^{\frac{2}{m}} t^{p-1} \,\mathrm{d}t \Big) \Big( \int_0^{\frac{2}{m}} \int_{\mathrm{B}'} |\nabla v|^p(x', s) \,\mathrm{d}x' \,\mathrm{d}s \Big) \\
&\leq \tilde{C} \int_0^{\frac{2}{m}} \int_{\mathrm{B}'} |\nabla v|^p \,\mathrm{d}x' \,\mathrm{d}s \to 0 \quad \text{as } m \to \infty
\end{aligned}$$

by the dominated convergence theorem. Hence, $\nabla \tilde{w}_m \to \nabla v$ in $\mathrm{L}^p(\mathrm{B}^+)$ and, as obviously $\tilde{w}_m \to v$ in $\mathrm{L}^p(\mathrm{B}^+)$, we have $\tilde{w}_m \to v$ in $\mathrm{W}^{1,p}(\mathrm{B}^+)$. As $\tilde{w}_m = 0$ for $x_m \in [0, \frac{1}{m})$ and $\operatorname{supp} \tilde{w}_m \subset \operatorname{supp} v$, we have $\operatorname{supp} \tilde{w}_m \Subset \mathrm{B}^+$. Hence, $w_m := \eta_{\varepsilon_m} * \tilde{w}_m \in \mathrm{C}_0^\infty(\mathrm{B}^+)$ satisfies $w_m \to v$ in $\mathrm{W}^{1,p}(\mathrm{B}^+)$ for $\varepsilon_m \downarrow 0$ chosen appropriately (by Theorems 4.1 and 2.1). Hence, ii) shows $u \in \mathrm{W}_0^{1,p}(\Omega)$.

iv) We now prove the claim for all $k \in \mathbb{N}$ with $k \geq 2$. If $u \in \mathrm{W}_0^{k,p}(\Omega)$, then $\mathrm{D}^\alpha u \in \mathrm{W}_0^{1,p}(\Omega)$ for all $|\alpha| \leq k-1$ by Proposition 3.8 a) and since $\mathrm{D}^\alpha \varphi \in \mathrm{C}_0^\infty(\Omega)$ for $\varphi \in \mathrm{C}_0^\infty(\Omega)$. Then the case $k = 1$ proves "$\Rightarrow$".

For "$\Leftarrow$", the case $k = 1$ implies $\mathrm{D}^\alpha u \in \mathrm{W}_0^{1,p}(\Omega)$ for all $|\alpha| \leq k-1$. Hence, there exist $(f_m)_{m \in \mathbb{N}}, (g_m)_{m \in \mathbb{N}} \subset \mathrm{C}_0^\infty(\Omega)$ such that $f_m \to u$ and $g_m \to u_{x_n}$ in $\mathrm{W}^{1,p}(\Omega)$ as $m \to \infty$. Assuming $\Omega \cap U_i = \mathrm{B}^+$ and $\partial\Omega \cap U_i = \mathrm{B}^0$ for some Ball B, $g_m$ is 0 in a neighborhood of $\mathrm{B}^0$. Hence,

$$G_m(x) := \int_0^{x_n} g_m(x', t)\, \mathrm{d}t$$

satisfies $(G_m)_{x_n} = g_m$ and $G_m \in \mathrm{C}_0^\infty(\Omega \cap \overline{U}_i) = \mathrm{C}_0^\infty(\overline{\mathrm{B}^+} \setminus \mathrm{B}^0)$. As $f_m \in \mathrm{C}_0^\infty(\Omega)$ we have $f_m(x) = \int_0^{x_n} (f_m)_{x_n}(x', t)\, \mathrm{d}t$ and hence

$$\|f_m - G_m\|_{\mathrm{L}^p(\Omega \cap U_i)} \leq C\, \|(f_m)_{x_n} - g_m\|_{\mathrm{L}^p(\Omega \cap U_i)}$$

with some $C$ depending on $p$ and $U_i$ by Hölder's inequality. Since $g_m \to u_{x_n}$ and $(f_m)_{x_n} \to u_{x_n}$ in $\mathrm{L}^p(\Omega)$, we see that $\mathrm{D}^\alpha G_m \to \mathrm{D}^\alpha u$ in $\mathrm{L}^p(\Omega \cap U_i)$ for all $|\alpha| \leq 2$ with $\alpha_n \geq 1$ and $G_m \to u$ in $\mathrm{L}^p(\Omega \cap U_i)$ as $m \to \infty$. Iterating this argument with $u_{x_i}, i = 1, \ldots, n-1$ and using an induction on $k$ and the arguments of ii), we see that $G_m \to u$ in $\mathrm{W}^{k,p}(\Omega \cap U_i)$ and hence $u \in \mathrm{W}_0^{k,p}(\Omega)$ (like in ii)). $\square$

# Chapter 6

# Embeddings and Sobolev Inequalities

We ask the question if $u \in \mathrm{W}^{k,p}(\Omega)$ automatically belongs to some other function spaces. The answer is *yes*, but we will see that to which spaces $u$ belongs depends on $p$, e.g. for $k = 1$ on $p < n$, $p = n$ or $p > n$.
More precisely, we ask whether $X = \mathrm{W}^{k,p}(\Omega)$ is embedded into a space $Y$ in the following sense.

**Definition 6.1.** Let $X, Y$ be Banach spaces with $X \subset Y$.

a) We say that $X$ is *continuously embedded into* $Y$ if there is $C > 0$ such that

$$\|u\|_Y \leq C \, \|u\|_X \quad \text{for all } u \in X.$$

b) We say that $X$ is *compactly embedded into* $Y$ if it is continuously embedded into $Y$ and any bounded sequence in $X$ is *precompact in* $Y$, i.e. for any $(u_m)_{m\in\mathbb{N}} \subset X$ with $\|u_m\|_X \leq M$ for all $m \in \mathbb{N}$, there is a subsequence $(u_{m_j})_{j\in\mathbb{N}}$ and $u \in Y$ such that $\|u_{m_j} - u\|_Y \to 0$ as $j \to \infty$.

The main tool for studying such embeddings will be Sobolev-type inequalities.

## 6.1 Embeddings into $\mathrm{L}^q(\Omega)$ Spaces

Let us assume $p \in [1, n)$ and ask the question for which $q \in [1, \infty)$ there exists a constant $C > 0$ such that

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^n)} \leq C \, \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)} \quad \text{for all } u \in \mathrm{C}_0^\infty(\mathbb{R}^n). \tag{6.1}$$

*Motivation.* We first demonstrate that if (6.1) holds, then $q$ has to have a specific form: To this end, let $u \in \mathrm{C}_0^\infty(\Omega)$ with $u \not\equiv 0$ and for $\lambda > 0$

$$u_\lambda(x) \coloneqq u(\lambda x) \quad \text{for all } x \in \mathbb{R}^n.$$

Then the transformation rule implies

$$\int_{\mathbb{R}^n} |u_\lambda(x)|^q \,\mathrm{d}x = \int_{\mathbb{R}^n} |u(\lambda x)|^q \,\mathrm{d}x = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u(y)|^q \,\mathrm{d}y,$$
$$\int_{\mathbb{R}^n} |\nabla u_\lambda(x)|^p \,\mathrm{d}x = \lambda^p \int_{\mathbb{R}^n} |(\nabla u)(\lambda x)|^p \,\mathrm{d}x = \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} |\nabla u(y)|^p \,\mathrm{d}y.$$

If (6.1) holds for any $\lambda > 0$, then

$$\lambda^{-\frac{n}{q}} \|u\|_{\mathrm{L}^q(\mathbb{R}^n)} = \|u_\lambda\|_{\mathrm{L}^q(\mathbb{R}^n)} \le C\, \|\nabla u_\lambda\|_{\mathrm{L}^p(\mathbb{R}^n)} = C\,\lambda^{1-\frac{n}{p}} \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)}$$

and therefore

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^n)} \le C\,\lambda^{1-\frac{n}{p}+\frac{n}{q}} \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)}. \tag{6.2}$$

But if $1 - \frac{n}{p} + \frac{n}{q} \neq 0$, then sending $\lambda$ to either 0 or $\infty$ in (6.2) yields $u \equiv 0$, a contradiction. Hence, if (6.1) is satisfied, then necessarily we must have $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and $q = \frac{np}{n-p}$. This motivates the following definition.

**Definition 6.2.** If $p \in [1, n)$, then $p^* := \frac{np}{n-p}$ is the *Sobolev conjugate* of $p$. In particular, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ and $p^* > p$.

We next prove that (6.1) is in fact true for $q = p^*$.

**Theorem 6.3** (Gagliardo-Nirenberg-Sobolev inequality)**.** *Assume that $p \in [1, n)$. Then there exists $C > 0$, depending only on $p$ and $n$, such that*

$$\|u\|_{\mathrm{L}^{p^*}(\mathbb{R}^n)} \le C\, \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)} \quad \textit{for all } u \in \mathrm{C}^1_0(\mathbb{R}^n). \tag{6.3}$$

*Moreover, we have $\mathrm{W}^{1,p}(\mathbb{R}^n) \subset \mathrm{L}^{p^*}(\mathbb{R}^n)$ and (6.3) holds for all $u \in \mathrm{W}^{1,p}(\mathbb{R}^n)$.*

*Remark* 6.4. We need that $u$ in (6.3) has compact support, as $u \equiv 1$ shows, but $C$ does not depend on the size of the support.

*Proof of Theorem 6.3.* i) Let $u \in \mathrm{C}^1_0(\mathbb{R}^n)$ and $p = 1$, whence $p^* = \frac{n}{n-1}$. Since $\mathrm{supp}(u)$ is compact, we have for all $x \in \mathbb{R}^n$ and $i \in \{1, \ldots, n\}$

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)\,\mathrm{d}y_i$$

and

$$|u(x)| \le \int_{-\infty}^{\infty} |\nabla u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)|\,\mathrm{d}y_i.$$

Hence,

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left( \int_{-\infty}^{\infty} |\nabla u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)|\,\mathrm{d}y_i \right)^{\frac{1}{n-1}}.$$

Integrating this with respect to $x_1$ and using the general Hölder inequality (2.1 viii) with $p_k = n-1, k = 1, \ldots, n-1$), we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}(x)\,\mathrm{d}x_1 &\le \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left( \int_{-\infty}^{\infty} |\nabla u|\,\mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\
&= \left( \int_{-\infty}^{\infty} |\nabla u|\,\mathrm{d}y_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left( \int_{-\infty}^{\infty} |\nabla u|\,\mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\
&\le \left( \int_{-\infty}^{\infty} |\nabla u|\,\mathrm{d}y_1 \right)^{\frac{n}{n-1}} \prod_{i=2}^{n} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla u|\,\mathrm{d}y_i\,\mathrm{d}x_1 \right)^{\frac{1}{n-1}}.
\end{aligned}$$

Integrating with respect to $x_2$ and using again the general Hölder inequality, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&\quad \le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u|\,\mathrm{d}y_2\,\mathrm{d}x_1\right)^{\frac{1}{n-1}} \\
&\qquad \cdot \int_{-\infty}^{\infty}\left[\left(\int_{-\infty}^{\infty}|\nabla u|\,\mathrm{d}y_1\right)^{\frac{1}{n-1}}\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u|\,\mathrm{d}x_1\,\mathrm{d}y_i\right)^{\frac{1}{n-1}}\right]\mathrm{d}x_2 \\
&\quad \le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u|\,\mathrm{d}y_2\,\mathrm{d}x_1\right)^{\frac{1}{n-1}} \\
&\qquad \cdot \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u|\,\mathrm{d}y_1\,\mathrm{d}x_2\right)^{\frac{1}{n-1}}\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\nabla u|\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_i\right)^{\frac{1}{n-1}}.
\end{aligned}$$

Continuing like this and integrating with respect to $x_3, \dots, x_n$, we finally have

$$\begin{aligned}
\int_{\mathbb{R}^n}|u(x)|^{\frac{n}{n-1}}\,\mathrm{d}x &\le \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\nabla u|\,\mathrm{d}x_1\ldots\mathrm{d}y_i\ldots\mathrm{d}x_n\right)^{\frac{1}{n-1}} \\
&= \left(\int_{\mathbb{R}^n}|\nabla u(x)|\,\mathrm{d}x\right)^{\frac{n}{n-1}}
\end{aligned} \tag{6.4}$$

which establishes (6.3) for $p = 1$ with $C = 1$.

ii) Let $p \in (1, n)$ and $u \in \mathrm{C}_0^1(\mathbb{R}^n)$. For $\gamma > 1$, $v := |u|^\gamma \in \mathrm{C}_0^1(\mathbb{R}^n)$ and by (6.4) applied to $v$ we have by the Hölder inequality

$$\begin{aligned}
\left(\int_{\mathbb{R}^n}|u|^{\frac{\gamma n}{n-1}}\,\mathrm{d}x\right)^{\frac{n-1}{n}} &\le \int_{\mathbb{R}^n}\left|\nabla(|u|^\gamma)\right|\mathrm{d}x = \int_{\mathbb{R}^n}\gamma|u|^{\gamma-1}|\nabla u|\,\mathrm{d}x \\
&\le \gamma\left(\int_{\mathbb{R}^n}|u|^{(\gamma-1)\frac{p}{p-1}}\,\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n}|\nabla u|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned} \tag{6.5}$$

Choosing $\gamma := \frac{p(n-1)}{n-p} > 1$, we have $\frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1}$ and hence

$$\frac{\gamma n}{n-1} = \frac{(np-p)+(p-n)}{n-p}\cdot\frac{p}{p-1} = \frac{np}{n-p} = p^*.$$

Thus, in view of $\frac{n-1}{n} - \frac{p-1}{p} = \frac{1}{p} - \frac{1}{n} = \frac{1}{p^*}$, (6.5) becomes (6.3) for $C = \gamma$.

iii) As $\mathrm{W}^{1,p}(\mathbb{R}^n) = \mathrm{W}_0^{1,p}(\mathbb{R}^n)$ by Corollary 4.14 for $u \in \mathrm{W}^{1,p}(\mathbb{R}^n)$ there is $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}_0^\infty(\mathbb{R}^n)$ such that $u_m \to u$ in $\mathrm{W}^{1,p}(\mathbb{R}^n)$ and, up to the choice of a subsequence, $u_m \to u$ a.e. Hence, $\nabla u_m \to \nabla u$ in $\mathrm{L}^p(\mathbb{R}^n)$ and, as (6.3) holds for any $u_m$, we conclude by Fatou's lemma that $u$ satisfies (6.3). As (6.3) therefore holds for all $u \in \mathrm{W}^{1,p}(\mathbb{R}^n)$, $\mathrm{W}^{1,p}(\mathbb{R}^n)$ is continuously embedded into $\mathrm{L}^{p^*}(\mathbb{R}^n)$. □

Using the previous result, we get a corresponding result on bounded domains.

**Theorem 6.5** (Embeddings for $\mathrm{W}^{1,p}(\Omega), p < n$)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in \mathrm{C}^1$ and $p \in [1, n)$. Then $\mathrm{W}^{1,p}(\Omega)$ is continuously embedded into $\mathrm{L}^q(\Omega)$ for all $q \in [1, p^*]$. More precisely, for any $q \in [1, p^*]$ there is $C > 0$ depending on $p$, $q$, $n$, and $\Omega$ such that*

$$\|u\|_{\mathrm{L}^q(\Omega)} \le C\,\|u\|_{\mathrm{W}^{1,p}(\Omega)} \quad \textit{for all } u \in \mathrm{W}^{1,p}(\Omega). \tag{6.6}$$

*Proof.* Since $\partial\Omega \in \mathrm{C}^1$, by Theorem 5.5 there exists $v := E(u) \in \mathrm{W}^{1,p}(\mathbb{R}^n)$ such that $v = u$ a.e. in $\Omega$ and $\|v\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)} \leq C_1 \|u\|_{\mathrm{W}^{1,p}(\Omega)}$ with $C_1 > 0$ depending only on $p$ and $\Omega$. Then by Theorem 6.3 we have $v \in \mathrm{L}^{p^*}(\mathbb{R}^n)$ and

$$\|u\|_{\mathrm{L}^{p^*}(\Omega)} = \|v\|_{\mathrm{L}^{p^*}(\Omega)} \leq \|v\|_{\mathrm{L}^{p^*}(\mathbb{R}^n)} \leq C_2 \|\nabla v\|_{\mathrm{L}^p(\mathbb{R}^n)} \leq C_1 C_2 \|u\|_{\mathrm{W}^{1,p}(\Omega)}$$

where $C_2$ is the constant from (6.3) (depending on $p$ and $n$). This shows the claim for $q = p^*$. As $\Omega$ is bounded, Hölder's inequality implies that $u \in \mathrm{L}^q(\Omega)$ for $q \in [1, p^*)$ with

$$\|u\|_{\mathrm{L}^q(\Omega)} \leq \left[\left(\int_\Omega 1^{\frac{p^*}{p^*-q}}\,\mathrm{d}x\right)^{\frac{p^*-q}{p^*}} \cdot \left(\int_\Omega |u|^{q\frac{p^*}{q}}\,\mathrm{d}x\right)^{\frac{q}{p^*}}\right]^{\frac{1}{q}} = |\Omega|^{\frac{p^*-q}{p^*q}} \|u\|_{\mathrm{L}^{p^*}(\Omega)} \tag{6.7}$$

$$\leq |\Omega|^{\frac{p^*-q}{p^*q}} C_1 C_2 \|u\|_{\mathrm{W}^{1,p}(\Omega)}. \qquad \square$$

For $\mathrm{W}_0^{1,p}(\Omega)$, we get these embeddings without assumptions on $\partial\Omega$.

**Theorem 6.6** (Embeddings for $\mathrm{W}_0^{1,p}(\Omega), p < n$)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p \in [1, n)$. Then $\mathrm{W}_0^{1,p}(\Omega)$ is continuously embedded into $\mathrm{L}^q(\Omega)$ for all $q \in [1, p^*]$. More precisely, for any $q \in [1, p^*]$ there is $C > 0$ depending on $p$, $q$, $n$, and $\Omega$ such that*

$$\|u\|_{\mathrm{L}^q(\Omega)} \leq C \|\nabla u\|_{\mathrm{L}^p(\Omega)} \quad \textit{for all } u \in \mathrm{W}_0^{1,p}(\Omega). \tag{6.8}$$

*Proof.* Since $u \in \mathrm{W}_0^{1,p}(\Omega)$, there exist $u_m \in \mathrm{C}_0^\infty(\Omega)$ such that $u_m \to u$ in $\mathrm{W}^{1,p}(\Omega)$. We extend $u_m$ by 0 on $\mathbb{R}^n \setminus \Omega$ and have $u_m \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ so that $u_m$ satisfies (6.3) for all $m \in \mathbb{N}$. In particular, $\|u_m - u_j\|_{\mathrm{L}^{p^*}(\mathbb{R}^n)} \leq C \|\nabla u_m - \nabla u_j\|_{\mathrm{L}^p(\mathbb{R}^n)}$ for all $m, j$ so that $(u_m)_m$ is a Cauchy sequence in $\mathrm{L}^{p^*}(\Omega)$ with limit $v$. As $u_{m_j} \to u$ a.e. in $\Omega$, we conclude that $v = u$, $u_m \to u$ in $\mathrm{L}^{p^*}(\Omega)$, and

$$\|u\|_{\mathrm{L}^{p^*}(\Omega)} = \lim_{m\to\infty} \|u_m\|_{\mathrm{L}^{p^*}(\Omega)} \leq C \lim_{m\to\infty} \|\nabla u_m\|_{\mathrm{L}^p(\Omega)} = C \|\nabla u\|_{\mathrm{L}^p(\Omega)}$$

with $C$ from (6.3). This shows (6.8) for $q = p^*$. For $q \in [1, p^*)$, we have $\|u\|_{\mathrm{L}^q(\Omega)} \leq |\Omega|^{\frac{p^*-q}{p^*q}} \|u\|_{\mathrm{L}^{p^*}(\Omega)}$ (see (6.7)) and the claim follows. $\square$

As $p^* = \frac{np}{n-p} > \frac{np}{n} = p$, we may choose $q = p$ in (6.8) and have the following Corollary.

**Corollary 6.7** (Poincaré-inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p \in [1, \infty]$. Then we have the Poincaré-inequality*

$$\|u\|_{\mathrm{L}^p(\Omega)} \leq C \|\nabla u\|_{\mathrm{L}^p(\Omega)} \quad \textit{for all } \mathrm{W}_0^{1,p}(\Omega), \tag{6.9}$$

*where $C > 0$ only depends on $p$, $n$, and $\Omega$.*
*In particular, the norm $\|\nabla u\|_{\mathrm{L}^p(\Omega)}$ is equivalent to $\|u\|_{\mathrm{W}^{1,p}(\Omega)}$ in $\mathrm{W}_0^{1,p}(\Omega)$.*

*Proof.* For $p \in [1, n)$, (6.9) immediately follows from (6.8).
For fixed $p \in [n, \infty)$ there is $\tilde{p} \in [1, n)$ such that $\tilde{p}^* = \frac{n\tilde{p}}{n-\tilde{p}} > p$ since $\tilde{p}^* \to \infty$ as $\tilde{p} \uparrow n$. Hence, as $\tilde{p} < n \leq p$, we have by (6.8) (with $p = \tilde{p}$ and $q = p \in [1, \tilde{p}^*]$)

$$\|u\|_{\mathrm{L}^p(\Omega)} \leq C \|\nabla u\|_{\mathrm{L}^{\tilde{p}}(\Omega)} \leq C |\Omega|^{\frac{p-\tilde{p}}{p\tilde{p}}} \|\nabla u\|_{\mathrm{L}^p(\Omega)} \quad \text{for all } u \in \mathrm{W}_0^{1,p}(\Omega). \tag{6.10}$$

Here we used that Hölder's inequality implies (see (6.7)) $\|v\|_{\mathrm{L}^{\tilde{p}}(\Omega)} \leq |\Omega|^{\frac{p-\tilde{p}}{p\tilde{p}}} \|v\|_{\mathrm{L}^p(\Omega)}$ for all $v \in \mathrm{L}^p(\Omega)$ so that $u \in \mathrm{W}^{1,\tilde{p}}(\Omega)$ and, since some $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}_0^\infty(\Omega)$ satisfies $u_m \to u$ in $\mathrm{W}^{1,p}(\Omega)$ it therefore also satisfies $u_m \to u$ in $\mathrm{W}^{1,\tilde{p}}(\Omega)$ so that $u \in \mathrm{W}_0^{1,\tilde{p}}$.
For $p = \infty$ and $u \in \mathrm{W}_0^{1,\infty}(\Omega)$, there is $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}_0^\infty(\Omega)$ such that $u_m \to u$ in $\mathrm{W}^{1,\infty}(\Omega)$. Since there exists $b > 0$ such that $\Omega \subset (-b, b)^n$, for each $m \in \mathbb{N}$ we have

$$|u_m(x)| = |u_m(x', x_n)| = \Big|\underbrace{u_m(x', -b)}_{=0} + \int_{-b}^{x_n} (u_m)_{x_n}(x', t)\,\mathrm{d}t\Big| \leq 2b\,\|\nabla u_m\|_{\mathrm{L}^\infty(\Omega)}.$$

Hence, $\|u_m\|_{\mathrm{L}^\infty(\Omega)} \leq 2b\,\|\nabla u_m\|_{\mathrm{L}^\infty(\Omega)}$ for all $m \in \mathbb{N}$ and taking $m \to \infty$ we conclude that (6.9) holds for $u$ with $C := 2b$ depending only on $\Omega$. $\square$

Next, let us assume $p = n$. As $p^* \to \infty$ for $p \uparrow n$, we might expect $\mathrm{W}^{1,n}(\Omega) \subset \mathrm{L}^\infty(\Omega)$, but this is false in general for $n \geq 2$.

**Proposition 6.8** (Embeddings for $\mathrm{W}^{1,n}(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $q \in [1, \infty)$.*

*a) If $\partial\Omega \in \mathrm{C}^1$, then $\mathrm{W}^{1,n}(\Omega)$ is continuously embedded into $\mathrm{L}^q(\Omega)$ and, with $C > 0$ depending on $q$, $n$, $\Omega$, we have*

$$\|u\|_{\mathrm{L}^q(\Omega)} \leq C\,\|u\|_{\mathrm{W}^{1,n}(\Omega)} \qquad \textit{for all } u \in \mathrm{W}^{1,n}(\Omega).$$

*b) $\mathrm{W}_0^{1,n}(\Omega)$ is continuously embedded to $\mathrm{L}^q(\Omega)$ and, with $C > 0$ depending on $q$, $n$, $\Omega$, we have*

$$\|u\|_{\mathrm{L}^q(\Omega)} \leq C\,\|\nabla u\|_{\mathrm{L}^n(\Omega)} \qquad \textit{for all } u \in \mathrm{W}_0^{1,n}(\Omega).$$

*c) If $n = 1$ and $\Omega = (a, b) \subset \mathbb{R}$ with $-\infty < a < b < +\infty$, then $\mathrm{W}^{1,1}((a, b)) \subset \mathrm{L}^\infty((a, b))$ and there is $C > 0$ depending on $b - a$ such that*

$$\|u\|_{\mathrm{L}^\infty((a,b))} \leq C\,\|u\|_{\mathrm{W}^{1,1}((a,b))} \quad \textit{for all } u \in \mathrm{W}^{1,1}((a, b))$$

*and*

$$\|u\|_{\mathrm{L}^\infty((a,b))} \leq \|u'\|_{\mathrm{L}^1((a,b))} \qquad \textit{for all } u \in \mathrm{W}_0^{1,1}((a, b)).$$

*Proof (Exercise).* a) Let $q \in [1, \infty)$. Then for some $\tilde{p} \in (1, n)$ we have $\tilde{p}^* = \frac{n\tilde{p}}{n-\tilde{p}} > q$ and $\tilde{p} < n$. Hence, by (6.6) and Hölder's inequality we get (similarly to (6.10)

$$\|u\|_{\mathrm{L}^q(\Omega)} \leq C\,\|u\|_{\mathrm{W}^{1,\tilde{p}}(\Omega)} \leq C\,|\Omega|^{\frac{n-\tilde{p}}{n\tilde{p}}}\,\|u\|_{\mathrm{W}^{1,n}(\Omega)} \quad \text{for all } u \in \mathrm{W}^{1,n}(\Omega)$$

and the justification is the same as in the proof of (6.10) in Corollary 6.7.

b) The proof is analogous to a) by using (6.8) instead of (6.6).

c) Let $u \in \mathrm{W}^{1,1}((a, b))$. Then there is $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^1([a, b])$ such that $u_m \to u$ in $\mathrm{W}^{1,1}((a, b))$. Then there is $x_m \in [a, b]$ such that

$$|u_m(x_m)| \leq \frac{\|u_m\|_{\mathrm{L}^1((a,b))}}{b - a}$$

since otherwise

$$\int_a^b |u_m(x)|\,\mathrm{d}x > \int_a^b \frac{\|u_m\|_{\mathrm{L}^1((a,b))}}{b-a}\,\mathrm{d}x,$$

a contradiction. Hence,

$$(6.11)\qquad \begin{aligned} |u_m(x)| &\le |u_m(x_m)| + \Big|\int_{x_m}^x u_m'(s)\,\mathrm{d}s\Big| \\ &\le \frac{1}{b-a}\|u_m\|_{\mathrm{L}^1((a,b))} + \|u_m'\|_{\mathrm{L}^1((a,b))} \quad \text{for all } x \in [a,b]. \end{aligned}$$

As $u_{m_l}(x) \to u(x)$ for a.e. $x \in \Omega$ and some subsequence $(u_{m_l})_l$, we may let $m \to \infty$ in (6.11) and conclude

$$|u(x)| \le \frac{1}{b-a}\|u\|_{\mathrm{L}^1((a,b))} + \|u'\|_{\mathrm{L}^1((a,b))} \quad \text{for a.e. } x \in \Omega.$$

This establishes the first claim concerning $u \in \mathrm{W}^{1,1}((a,b))$ with $C = \frac{1}{b-a} + 1$.

For $u \in \mathrm{W}_0^{1,1}((a,b))$ we have $u_m \in \mathrm{C}_0^1((a,b))$ with $u_m \to u$ in $\mathrm{W}_{1,1}((a,b))$ and may choose $x_m = a$ with $u_m(x_m) = 0$ in (6.11) to conclude the claim concerning $\mathrm{W}_0^{1,1}((a,b))$. □

*Remark* 6.9. If $n = 1$ and $\Omega = (a,b)$ is bounded, in fact $\mathrm{W}^{1,1}((a,b))$ and $\mathrm{W}_0^{1,1}((a,b))$ are both continuously embedded to $\mathrm{C}^0([a,b])$. This can be proved with Proposition 6.8 c) (with the choice of a continuous representative of $u \in \mathrm{W}^{1,1}((a,b))$ ). For $n \ge 2$, $\mathrm{W}^{1,n}(\mathrm{B}_1(0)) \not\subset \mathrm{L}^\infty(\mathrm{B}_1(0))$, as e.g. $u(x) = \ln(\ln(1 + \frac{1}{|x|}))$ satisfies $u \in \mathrm{W}^{1,n}(\mathrm{B}_1(0))$ and $u \notin \mathrm{L}^\infty(\mathrm{B}_1(0))$ (see Exercise).

## 6.2 Embeddings into Hölder Spaces

If $\Omega \subset \mathbb{R}^n$ is open and $u\colon \Omega \to \mathbb{R}$ satisfies

$$(6.12)\qquad |u(x) - u(y)| \le M\,|x-y|^\gamma \quad \text{for all } x, y \in \Omega$$

with some $M > 0$ and $\gamma \in (0,1)$, $u$ is called *Hölder continuous with exponent* $\gamma$. Note that $\gamma = 1$ corresponds to *Lipschitz continuous* functions. We define the corresponding function spaces.

**Definition 6.10.** Let $\Omega \subset \mathbb{R}^n$ be open.

a) For $u \in \mathrm{C}_b^0(\overline{\Omega}) := \mathrm{C}^0(\overline{\Omega}) \cap \mathrm{L}^\infty(\Omega)$, we define $\|u\|_{\mathrm{C}^0(\overline{\Omega})} := \sup_{x\in\Omega} |u(x)|$. For $\Omega$ bounded, we have $\mathrm{C}_b^0(\overline{\Omega}) = \mathrm{C}^0(\overline{\Omega})$.

b) For $\gamma \in (0,1]$ and $u\colon \overline{\Omega} \to \mathbb{R}$, the $\gamma$-*Hölder seminorm* is defined as

$$[u]_{\mathrm{C}^{0,\gamma}(\overline{\Omega})} := \sup_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma}$$

and the $\gamma$-*Hölder norm* is defined as

$$\|u\|_{\mathrm{C}^{0,\gamma}(\overline{\Omega})} := \|u\|_{\mathrm{C}^0(\overline{\Omega})} + [u]_{\mathrm{C}^{0,\gamma}(\overline{\Omega})}.$$

c) For $k \in \mathbb{N}_0$ and $\gamma \in (0,1]$, we define the *Hölder space*

$$\mathrm{C}^{k,\gamma}(\overline{\Omega}) := \{u \in \mathrm{C}^k(\overline{\Omega})\colon \|u\|_{\mathrm{C}^{k,\gamma}(\overline{\Omega})} < \infty\},$$

where

$$\|u\|_{\mathrm{C}^{k,\gamma}(\overline{\Omega})} := \sum_{|\alpha|\le k} \|\,\mathrm{D}^\alpha u\|_{\mathrm{C}^0(\overline{\Omega})} + \sum_{|\alpha|=k} [\,\mathrm{D}^\alpha u]_{\mathrm{C}^{0,\gamma}(\overline{\Omega})}.$$

(In case of $\Omega$ unbounded, one also writes $\mathrm{C}_b^{k,\gamma}(\overline{\Omega})$ instead of $\mathrm{C}^{k,\gamma}(\overline{\Omega})$ to indicate that the functions are in $\mathrm{C}_b^k(\overline{\Omega}) = \{u \in \mathrm{C}^k(\overline{\Omega}) \colon\ \mathrm{D}^\alpha u \in \mathrm{L}^\infty(\Omega),\ \text{for all } |\alpha| \le k\})$

The spaces $\mathrm{C}^{k,\gamma}(\overline{\Omega})$ have a good mathematical structure and are also Banach spaces.

**Proposition 6.11.** *Let $\Omega \subset \mathbb{R}^n$ be open, $k \in \mathbb{N}_0$, and $\gamma \in (0,1]$. Then $\mathrm{C}^{k,\gamma}(\overline{\Omega})$ is a Banach space.*

*Proof.* It is not difficult to verify that $\|\cdot\|_{\mathrm{C}^{k,\gamma}(\overline{\Omega})}$ is a norm on $\mathrm{C}^{k,\gamma}(\overline{\Omega})$. To show that $\mathrm{C}^{k,\gamma}(\overline{\Omega})$ is complete, assume that $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^{k,\gamma}(\overline{\Omega})$ is a Cauchy sequence. Then it is also a Cauchy sequence in $\mathrm{C}_b^k(\overline{\Omega})$ and hence $u_m \to u$ as $m \to \infty$ in $\mathrm{C}_b^k(\overline{\Omega})$ with some $u \in \mathrm{C}_b^k(\overline{\Omega})$. As $\mathrm{D}^\alpha u_m$ satisfy (6.12) for all $m \in \mathbb{N}$ with $M := \sup_{m\in\mathbb{N}} \|u_m\|_{\mathrm{C}^{k,\gamma}(\overline{\Omega})}$ for any $|\alpha| = k$, taking $m \to \infty$, we conclude that $\mathrm{D}^\alpha u$ also satisfies (6.12) with the same $M$. Hence, $u \in \mathrm{C}^{k,\gamma}(\overline{\Omega})$. ☐

We next study the case $u \in \mathrm{W}^{1,p}(\Omega)$ with $p \in (n,\infty)$ and show that $u$ is Hölder continuous after possibly being redefined on a set of measure zero. Again an inequality for $\mathrm{W}^{1,p}(\mathbb{R}^n)$ functions will be the basis.

**Theorem 6.12** (Morrey's inequality)**.** *Assume that $p \in (n,\infty]$ and $\gamma := 1 - \frac{n}{p}$ (with $\gamma = 1$ in case of $p = \infty$). Then there is $C > 0$ depending only on $p$ and $n$ such that*

$$\|u\|_{\mathrm{C}^{0,\gamma}(\mathbb{R}^n)} \le C\,\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)} \quad \textit{for all } u \in \mathrm{C}^1(\mathbb{R}^n) \cap \mathrm{W}^{1,p}(\mathbb{R}^n). \tag{6.13}$$

*Proof.* i) The case $p = \infty$ is immediate. For $u \in \mathrm{C}^1(\mathbb{R}^n) \cap \mathrm{W}^{1,\infty}(\mathbb{R}^n)$, we have

$$|u(x) - u(y)| \le \|\nabla u\|_{\mathrm{L}^\infty(\mathbb{R}^n)} |x-y| \quad \text{for all } x, y \in \mathbb{R}^n$$

by the mean value theorem. Hence,

$$\|u\|_{\mathrm{C}^{0,1}(\mathbb{R}^n)} = \|u\|_{\mathrm{C}^0(\mathbb{R}^n)} + [u]_{\mathrm{C}^{0,1}(\mathbb{R}^n)} \le \|u\|_{\mathrm{L}^\infty(\mathbb{R}^n)} + \|\nabla u\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \le \|u\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^n)}.$$

ii) Let $u \in \mathrm{C}^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $r > 0$, and $\mathrm{B}_r(x) \subset \mathbb{R}^n$ a ball. We want to estimate the difference between $u(x)$ and the mean of $u$ on $\mathrm{B}_r(x)$ such that

$$\begin{aligned} \Big|u(x) - \frac{1}{|\mathrm{B}_r(x)|}\int_{\mathrm{B}_r(x)} u(y)\,\mathrm{d}y\Big| &\le \frac{1}{|\mathrm{B}_r(x)|}\int_{\mathrm{B}_r(x)} |u(y) - u(x)|\,\mathrm{d}y \\ &\le C_1 \int_{\mathrm{B}_r(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\,\mathrm{d}y \end{aligned} \tag{6.14}$$

holds for some $C_1 > 0$ depending only on $n$. In order to prove the second inequality, we fix $w \in \partial\mathrm{B}_1(0)$. Then for $s \in (0,r)$ we have

$$|u(x+sw) - u(x)| = \Big|\int_0^s \frac{\mathrm{d}}{\mathrm{d}t} u(x+tw)\,\mathrm{d}t\Big| = \Big|\int_0^s \nabla u(x+tw)\cdot w\,\mathrm{d}t\Big| \overset{|w|=1}{\le} \int_0^s |\nabla u(x+tw)|\,\mathrm{d}t$$

Hence, integrating and using polar coordinates (see 2.4), we have

$$\begin{aligned}
\int_{\partial \mathrm{B}_1(0)} |u(x+sw)-u(x)|\,\mathrm{d}\sigma(w) &\overset{\text{Fubini}}{\leq} \int_0^s \int_{\partial \mathrm{B}_1(0)} |\nabla u(x+tw)|\,\mathrm{d}\sigma(w)\,\mathrm{d}t \\
&= \int_0^s t^{n-1} \int_{\partial \mathrm{B}_1(0)} \frac{|\nabla u(x+tw)|}{t^{n-1}}\,\mathrm{d}\sigma(w)\,\mathrm{d}t \\
&\overset{|w|=1}{=} \int_0^s t^{n-1} \int_{\partial \mathrm{B}_1(0)} \frac{|\nabla u(x+tw)|}{|x-(x+tw)|^{n-1}}\,\mathrm{d}\sigma(w)\,\mathrm{d}t
\end{aligned}$$

and, by using 2.4 with $x_0 \coloneqq x$ and applying the transformation $y = x + tw$, we get

$$= \int_{\mathrm{B}_s(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y \overset{s<r}{\leq} \int_{\mathrm{B}_r(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y.$$

Multiplying by $s^{n-1}$ and integrating, we obtain from 2.4

$$\begin{aligned}
\int_{\mathrm{B}_r(x)} |u(y)-u(x)|\,\mathrm{d}y &\overset{2.4}{=} \int_0^r s^{n-1} \int_{\partial \mathrm{B}_1(0)} |u(x+sw)-u(x)|\,\mathrm{d}\sigma(w)\,\mathrm{d}s \\
&\leq \frac{r^n}{n} \int_{\mathrm{B}_r(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y.
\end{aligned}$$

Dividing by $|\mathrm{B}_r(x)| = \frac{\omega_n}{n} r^n$, we obtain (6.14) for $C_1 \coloneqq \frac{1}{\omega_n}$.

iii) Now assume that $u \in \mathrm{C}^1(\mathbb{R}^n) \cap \mathrm{W}^{1,p}(\mathbb{R}^n)$ with $p \in (n, \infty)$. Since the choice of $p$ implies $\frac{n-1}{n} = 1 - \frac{1}{n} < \frac{p-1}{p}$, we have $\frac{(n-1)p}{p-1} < n$ and

$$\int_{\mathrm{B}_1(x)} |x-y|^{-\frac{(n-1)p}{p-1}}\,\mathrm{d}y \overset{2.4}{=} \omega_n \int_0^1 r^{-\frac{(n-1)p}{p-1}} \cdot r^{n-1}\,\mathrm{d}r = C_2(n,p) < \infty \quad \text{for all } x \in \mathbb{R}^n.$$

Hence, we obtain for $x \in \mathbb{R}^n$ with $q = \frac{p}{p-1}$ $(\frac{1}{p} + \frac{1}{q} = 1)$

$$\begin{aligned}
|u(x)| &= \frac{1}{|\mathrm{B}_1(x)|} \int_{\mathrm{B}_1(x)} |u(x)|\,\mathrm{d}y \\
&\leq \frac{1}{|\mathrm{B}_1(x)|} \Big( \int_{\mathrm{B}_1(x)} |u(x)-u(y)|\,\mathrm{d}y + \int_{\mathrm{B}_1(x)} |u(y)|\,\mathrm{d}y \Big),
\end{aligned}$$

which by (6.14) and the Hölder inequality can be estimated as

$$\begin{aligned}
&\leq C_1 \int_{\mathrm{B}_1(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y + \frac{1}{|\mathrm{B}_1(x)|} \Big( \int_{\mathrm{B}_1(x)} 1^q\,\mathrm{d}y \Big)^{\frac{1}{q}} \Big( \int_{\mathrm{B}_1(x)} |u(y)|^p\,\mathrm{d}y \Big)^{\frac{1}{p}} \\
&\overset{\text{Hölder}}{\leq} C_1 \Big( \int_{\mathrm{B}_1(x)} |\nabla u(y)|^p\,\mathrm{d}y \Big)^{\frac{1}{p}} \Big( \int_{\mathrm{B}_1(x)} |x-y|^{-\frac{(n-1)p}{p-1}}\,\mathrm{d}y \Big)^{\frac{p-1}{p}} + |\mathrm{B}_1(x)|^{-\frac{1}{p}} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)} \\
&\leq C_1\, C_2(n,p)\, \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)} + |\mathrm{B}_1(x)|^{-\frac{1}{p}} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)} \\
&\leq C_3\, \|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)},
\end{aligned}$$

where $C_3$ only depends on $p$ and $n$. Hence,

$$\|u\|_{\mathrm{C}^0(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |u(x)| \leq C_3\, \|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)}. \tag{6.15}$$

iv) Let $x, y \in \mathbb{R}^n$ with $r \coloneqq |x-y| > 0$. Defining $W \coloneqq \mathrm{B}_r(x) \cap \mathrm{B}_r(y)$, we get

$$\begin{aligned}
|u(x)-u(y)| &= \frac{1}{|W|} \int_W |u(x)-u(y)|\,\mathrm{d}z \\
&\leq \frac{1}{|W|} \Big( \int_W |u(x)-u(z)|\,\mathrm{d}z + \int_W |u(y)-u(z)|\,\mathrm{d}z \Big)
\end{aligned}$$

and, since $\mathrm{B}_{\frac{r}{2}}(\frac{x+y}{2}) \subset W$, we get

$$\begin{aligned}
&\le \frac{1}{|\mathrm{B}_{\frac{r}{2}}(\frac{x+y}{2})|}\Big(\int_{\mathrm{B}_r(x)} |u(x)-u(z)|\,\mathrm{d}z + \int_{\mathrm{B}_r(y)} |u(y)-u(z)|\,\mathrm{d}z\Big)\\
&\le 2^n\Big(\frac{1}{|\mathrm{B}_r(x)|}\int_{\mathrm{B}_r(x)} |u(x)-u(z)|\,\mathrm{d}z + \frac{1}{|\mathrm{B}_r(y)|}\int_{\mathrm{B}_r(y)} |u(y)-u(z)|\,\mathrm{d}z\Big),
\end{aligned}$$

which by (6.14) and the Hölder inequality can be estimated as

$$\begin{aligned}
&\le 2^n C_1\Big[\Big(\int_{\mathrm{B}_r(x)} |\nabla u(z)|^p\,\mathrm{d}z\Big)^{\frac{1}{p}}\Big(\int_{\mathrm{B}_r(x)} |x-z|^{-(n-1)\frac{p}{p-1}}\,\mathrm{d}z\Big)^{\frac{p-1}{p}}\\
&\qquad + \Big(\int_{\mathrm{B}_r(y)} |\nabla u(z)|^p\,\mathrm{d}z\Big)^{\frac{1}{p}}\Big(\int_{\mathrm{B}_r(y)} |y-z|^{-(n-1)\frac{p}{p-1}}\,\mathrm{d}z\Big)^{\frac{p-1}{p}}\Big]\\
&\overset{2.4}{\le} 2\cdot 2^n C_1 \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)}\Big(\omega_n \int_0^r s^{\frac{-(n-1)p}{p-1}} s^{n-1}\,\mathrm{d}s\Big)^{\frac{p-1}{p}}\\
&\le C_4\left(r^{n-\frac{(n-1)p}{p-1}}\right)^{\frac{p-1}{p}} \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)} = C_4\, r^{1-\frac{n}{p}} \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)}
\end{aligned}$$

with $C_4 > 0$ only depending on $n$ and $p$. Hence,

$$[u]_{\mathrm{C}^{0,\gamma}(\mathbb{R}^n)} = [u]_{\mathrm{C}^{0,1-\frac{n}{p}}(\mathbb{R}^n)} = \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}} \le C_4\,\|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)}.$$

Combining this estimate with (6.15), we obtain (6.13) for $p \in (n,\infty)$. ☐

We define a version of a function.

**Definition 6.13.** We say that $u^*\colon U \to \mathbb{R}$ is a *version of* the given function $u\colon U \to \mathbb{R}$ if $u = u^*$ a.e. in $U$.

We next summarize embeddings of $\mathrm{W}^{1,p}(\Omega)$ into $\mathrm{C}^{0,\gamma}(\overline{\Omega})$ for various $p$ and $\Omega$.

**Theorem 6.14** (Embeddings of $\mathrm{W}^{1,p}(\Omega)$ for $p \in (n,\infty]$)**.** *Let $\Omega$ be a bounded domain, $p \in (n,\infty]$ and $\gamma = 1 - \frac{n}{p} \in (0,1]$ (with $\gamma = 1$ in case of $p = \infty$).*

*a) $\mathrm{W}^{1,p}(\mathbb{R}^n)$ is continuously embedded into $\mathrm{C}^{0,\gamma}(\mathbb{R}^n)$ in the sense that for any $u \in \mathrm{W}^{1,p}(\mathbb{R}^n)$ there is a version $u^* \in \mathrm{C}^{0,\gamma}(\mathbb{R}^n)$ and*

$$\|u^*\|_{\mathrm{C}^{0,\gamma}(\mathbb{R}^n)} \le C\,\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)},$$

*where $C > 0$ only depends on $p$ and $n$.*

*b) If $\partial\Omega \in \mathrm{C}^1$, then $\mathrm{W}^{1,p}(\Omega)$ is continuously embedded into $\mathrm{C}^{0,\gamma}(\overline{\Omega})$ in the sense that for any $u \in \mathrm{W}^{1,p}(\Omega)$ there is a version $u^* \in \mathrm{C}^{0,\gamma}(\overline{\Omega})$ and*

$$\|u^*\|_{\mathrm{C}^{0,\gamma}(\overline{\Omega})} \le C\,\|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

*where $C > 0$ depends on $p$, $n$, and $\Omega$.*

*c) $\mathrm{W}^{1,p}_0(\Omega)$ is continuously embedded into $\mathrm{C}^{0,\gamma}(\overline{\Omega})$ in the sense that for any $u \in \mathrm{W}^{1,p}_0(\Omega)$ there is a version $u^* \in \mathrm{C}^{0,\gamma}(\overline{\Omega})$ and*

$$\|u^*\|_{\mathrm{C}^{0,\gamma}(\overline{\Omega})} \le C\,\|\nabla u\|_{\mathrm{L}^p(\Omega)},$$

*where $C > 0$ depends on $p$, $n$, and $\Omega$.*

*Remark* 6.15. In view of Theorem 6.14 b), for any bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in \mathrm{C}^1$ we may define the trace $\mathrm{Tr}\colon \mathrm{W}^{1,\infty}(\Omega) \to \mathrm{C}^0(\partial\Omega), \mathrm{Tr}(u) := u^*|_{\partial\Omega}$ which particularly shows that any $u \in \mathrm{W}_0^{1,\infty}(\Omega)$ satisfies $\mathrm{Tr}(u) = 0$. However, $u(x) := 1 - |x| \in \mathrm{W}^{1,\infty}(\mathrm{B}_1(0))$ with $u = 0$ on $\partial\mathrm{B}_1(0)$ (hence, $\mathrm{Tr}(u) = 0$) but $u \notin \mathrm{W}_0^{1,\infty}(\mathrm{B}_1(0))$, as $\|\nabla u - \nabla\varphi\|_{\mathrm{L}^\infty(\mathrm{B}_1(0))} \geq 1$ for all $\varphi \in \mathrm{C}_0^\infty(\mathrm{B}_1(0))$.

*Proof of Theorem 6.14.* i) *a) for $p \in (n, \infty)$*: For $u \in \mathrm{W}^{1,p}(\mathbb{R}^n)$ there is $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}^1(\mathbb{R}^n) \cap \mathrm{W}^{1,p}(\mathbb{R}^n)$ such that $u_m \to u$ in $\mathrm{W}^{1,p}(\mathbb{R}^n)$. By (6.13), $(u_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{C}^{0,\gamma}(\mathbb{R}^n)$ and hence there is $u^* \in \mathrm{C}^{0,\gamma}(\mathbb{R}^n)$ such that $u_m \to u^*$ in $\mathrm{C}^{0,\gamma}(\mathbb{R}^n)$. As $u_{m_l} \to u$ a.e. in $\mathbb{R}^n$ as $l \to \infty$ for a subsequence, we have $u = u^*$ a.e. in $\mathbb{R}^n$ and

$$\|u^*\|_{\mathrm{C}^{0,\gamma}(\mathbb{R}^n)} = \lim_{m\to\infty} \|u_m\|_{\mathrm{C}^{0,\gamma}(\mathbb{R}^n)} \overset{(6.13)}{\leq} C_1 \lim_{m\to\infty} \|u_m\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)} = C_1 \|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)}.$$

ii) *b) for $p \in (n, \infty)$*: For $u \in \mathrm{W}^{1,p}(\Omega)$ there is $E(u) \in \mathrm{W}^{1,p}(\mathbb{R}^n)$ by Theorem 5.5 which satisfies $(E(u))^*|_\Omega = u$ a.e. in $\Omega$ so that $(E(u))^*|_\Omega =: u^*$ is a version of $u$ satisfying

$$\|u^*\|_{\mathrm{C}^{0,\gamma}(\overline{\Omega})} \leq \|(E(u))^*\|_{\mathrm{C}^{0,\gamma}(\mathbb{R}^n)} \overset{\text{i)}}{\leq} C_1 \|E(u)\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)} \overset{\text{Thm. 5.5}}{\leq} C_1 C_2 \|u\|_{\mathrm{W}^{1,p}(\Omega)},$$

where $C_2 > 0$ is the constant from Theorem 5.5 depending on $p$, $n$, $\Omega$.

iii) *b) for $p = \infty$*: Let $u \in \mathrm{W}^{1,\infty}(\Omega)$. Then $u \in \mathrm{W}^{1,\tilde{p}}(\Omega)$ for some $\tilde{p} \in (n, \infty)$ as $\Omega$ is bounded and by ii) there is a version $u^*$ of $u$ with $u^* \in \mathrm{C}^{0,\tilde{\gamma}}(\overline{\Omega})$, where $\tilde{\gamma} := 1 - \frac{n}{\tilde{p}} \in (0, 1)$. We fix $x, y \in \Omega$ with $x \neq y$. As $\partial\Omega \in \mathrm{C}^1$ there is a domain $V \Subset \Omega$ with $\partial V \in \mathrm{C}^1$ and $x, y \in V$. By Theorem 4.1 $u_\varepsilon := \eta_\varepsilon * u \in \mathrm{C}^\infty(\Omega)$ converges to $u^*$ in $\mathrm{W}^{1,\tilde{p}}(V)$, hence by ii) $u_\varepsilon \to u^*$ in $\mathrm{C}^{0,\tilde{\gamma}}(\overline{V})$ so that $u_\varepsilon \to u^*$ uniformly on $\overline{V}$ as $\varepsilon \downarrow 0$. Furthermore, for $\varepsilon > 0$ small enough such that $V \subset \Omega_\varepsilon$, we have by Theorem 4.1

$$|\nabla u_\varepsilon(z)| \leq \|\nabla u\|_{\mathrm{L}^\infty(\Omega)} \underbrace{\int_{\mathbb{R}^n} \eta_\varepsilon(z - \tilde{y})\,\mathrm{d}\tilde{y}}_{=1} = \|\nabla u\|_{\mathrm{L}^\infty(\Omega)} \quad \text{for all } z \in \Omega_\varepsilon.$$

Hence, the mean value theorem (MVT) implies for $\Omega$ convex, as we may choose $V$ to be convex,

$$|u^*(x) - u^*(y)| = \lim_{\varepsilon\downarrow 0} |u_\varepsilon(x) - u_\varepsilon(y)| \overset{\text{MVT}}{\leq} \liminf_{\varepsilon\downarrow 0} \|\nabla u_\varepsilon\|_{\mathrm{L}^\infty(V)} |x - y| \leq \|\nabla u\|_{\mathrm{L}^\infty(\Omega)} |x - y|.$$

Hence, for $\Omega$ convex, we have

$$\|u^*\|_{\mathrm{C}^{0,1}(\overline{\Omega})} = \|u^*\|_{\mathrm{C}^0(\overline{\Omega})} + [u^*]_{\mathrm{C}^{0,1}(\overline{\Omega})} \leq \|u\|_{\mathrm{L}^\infty(\Omega)} + \|\nabla u\|_{\mathrm{L}^\infty(\Omega)} \leq \|u\|_{\mathrm{W}^{1,\infty}(\Omega)}.$$

iv) *a) for $p = \infty$*: Let $u \in \mathrm{W}^{1,\infty}(\mathbb{R}^n)$ and $\Omega_m := \mathrm{B}_m(0), m \in \mathbb{N}$. Then $u \in \mathrm{W}^{1,\infty}(\Omega_m)$ for any $m \in \mathbb{N}$ so that iii) implies the existence of a version $u^*$ of $u$ such that

$$\|u^*\|_{\mathrm{C}^{0,1}(\overline{\Omega_m})} \leq \|u\|_{\mathrm{W}^{1,\infty}(\Omega_m)} \leq \|u\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^n)} \quad \text{for all } m \in \mathbb{N}.$$

As $\|u^*\|_{\mathrm{C}^{0,1}(\mathbb{R}^n)} = \sup_{m\in\mathbb{N}} \|u^*\|_{\mathrm{C}^{0,1}(\overline{\Omega_m})}$, the claim is proved.

v) *c) for $p \in (n,\infty]$*: For $u \in \mathrm{W}_0^{1,p}(\Omega)$ there is $(u_m)_{m\in\mathbb{N}} \subset \mathrm{C}_0^\infty(\Omega)$ with $u_m \to u$ in $\mathrm{W}^{1,p}(\Omega)$. Extending $u_m$ by 0 on $\mathbb{R}^n \setminus \Omega$, we have $u_m \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ and by a) there exists a function $v \in \mathrm{C}^{0,\gamma}(\mathbb{R}^n) \cap \mathrm{W}^{1,p}(\mathbb{R}^n)$ such that $u_m \to v$ in $\mathrm{C}^{0,\gamma}(\mathbb{R}^n)$ and in $\mathrm{W}^{1,p}(\mathbb{R}^n)$. As $v = 0$ in $\mathbb{R}^n \setminus \Omega$ we conclude with $u^* := v|_{\overline{\Omega}}$ and Poincaré's inequality (Corollary 6.7)

$$\|u^*\|_{\mathrm{C}^{0,\gamma}(\overline{\Omega})} = \|v\|_{\mathrm{C}^{0,\gamma}(\mathbb{R}^n)} \overset{\text{a)}}{\leq} \tilde{C}_4 \|v\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)} = \tilde{C}_4 \|u\|_{\mathrm{W}^{1,p}(\Omega)} \overset{\text{Poincaré}}{\leq} C_4 \|\nabla u\|_{\mathrm{L}^p(\Omega)}$$

as $u = v$ a.e. in $\Omega$.

vi) *End of part iii) for $\Omega$ non-convex*: With the notation from Definition 5.6 and $\Phi_i, \Psi_i$ from Lemma 5.3 (with $\gamma = \gamma_i$), $i = 1, \dots, N$, we may choose $U_i$ small enough such that

$$\Phi_i(\Omega \cap U_i) \subset \mathrm{B}_i^+ \subset \Phi_i(\Omega) \quad \text{for all } i \in \{1, \dots, N\}$$

and some balls $\mathrm{B}_i := B_{s_i}(x^i)$. As $\overline{\Omega}$ is compact, there is $\delta > 0$ such that for $x, y \in \Omega$ with $|x-y| < \delta$, either $x, y \in \overline{\mathrm{B}_\delta(x)} \subset \Omega$ or $x, y \in \Omega \cap U_i$ for some $i \in \{1, \dots, N\}$. Then for $x, y \in \Omega$ we distinguish three cases:

I) *$|x-y| < \delta$ and $x, y \in \overline{\mathrm{B}_\delta(x)}$*: Then we may choose (in iii)) $V = \mathrm{B}_\delta(x)$. As $V$ is convex, the mean value theorem implies

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq \|\nabla u_\varepsilon\|_{\mathrm{L}^\infty(V)} |x-y| \leq \|\nabla u\|_{\mathrm{L}^\infty(\Omega)} |x-y|$$

as in iii).

II) *$|x-y| < \delta$ and $x, y \in \Omega \cap U_i$*: We define the convex hull

$$W := \{(1-t)\,\Phi_i(x) + t\,\Phi_i(y) : t \in [0,1]\} \subset \mathrm{B}_i^+$$

and we may choose $V$ in iii) such that $\Phi_i^{-1}(W) \subset V$. Then

$$\begin{aligned} |u_\varepsilon(x) - u_\varepsilon(y)| &= |u_\varepsilon(\Psi_i(\Phi_i(x))) - u_\varepsilon(\Psi_i(\Phi_i(y)))| \\ &= \Big| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \Big[ (u_\varepsilon \circ \Psi_i)((1-t)\Phi_i(x) + t\Phi_i(y)) \Big] \,\mathrm{d}t \Big| \\ &\leq \|\nabla(u_\varepsilon \circ \Psi_i)\|_{\mathrm{L}^\infty(W)} \cdot |\Phi_i(x) - \Phi_i(y)| \\ &\overset{\text{MVT}}{\leq} \|\nabla u_\varepsilon\|_{\mathrm{L}^\infty(V)} \cdot \|\mathrm{D}\Psi_i\|_{\mathrm{L}^\infty(W)} \cdot \|\mathrm{D}\Phi_i\|_{\mathrm{L}^\infty(\overline{\mathrm{B}})} \cdot |x-y|, \end{aligned}$$

where we used the chain rule and chose a ball $\overline{\mathrm{B}}$ such that $\overline{\Omega} \Subset \mathrm{B}$ which by iii) yields

$$\leq \tilde{C}_3 \|\nabla u\|_{\mathrm{L}^\infty(\Omega)} |x-y|,$$

where $\tilde{C}_3 := \max_{1 \leq i \leq N} \|\mathrm{D}\Psi_i\|_{\mathrm{L}^\infty(\Phi_i(\Omega))} \cdot \|\mathrm{D}\Phi_i\|_{\mathrm{L}^\infty(\overline{\mathrm{B}})}$.

III) *$|x-y| \geq \delta$*: Then $|u^*(x) - u^*(y)| \leq 2\,\|u^*\|_{\mathrm{C}^0(\overline{\Omega})} \cdot \frac{1}{\delta}\,|x-y|$.

Altogether, letting $\varepsilon \downarrow 0$, we have

$$\|u^*\|_{\mathrm{C}^{0,1}(\overline{\Omega})} \leq \|u\|_{\mathrm{L}^\infty(\Omega)} + \Big( \frac{2}{\delta} \|u\|_{\mathrm{L}^\infty(\Omega)} + (1 + \tilde{C}_3) \|\nabla u\|_{\mathrm{L}^\infty(\Omega)} \Big) \leq C_3 \|u\|_{\mathrm{W}^{1,\infty}(\Omega)}. \qquad \square$$

## 6.3 General Embeddings and Sobolev Inequalities

Using the estimates from Section 6.1 and Section 6.2, we get more complicated estimates and embeddings for $\mathrm{W}^{k,p}(\Omega)$ with $k \in \mathbb{N}$. We set

$$\mathrm{W}^{0,q} := \mathrm{L}^q(\Omega)$$

and let

$$[x] := \max\{z \in \mathbb{Z} \colon z \leq x\}$$

for $x \in \mathbb{R}$ denote the *floor function.*

**Theorem 6.16** (Embeddings for $\mathrm{W}^{k,p}(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in \mathrm{C}^1$, $p, q \in [1, \infty]$, and $k \in \mathbb{N}, l \in \mathbb{N}_0$ with $k \geq l$.*

*a) If $k - \frac{n}{p} < l$ and $k - \frac{n}{p} \geq l - \frac{n}{q}$, then $\mathrm{W}^{k,p}(\Omega)$ is continuously embedded into $\mathrm{W}^{l,q}(\Omega)$ and there is $C > 0$ depending only on $p$, $q$, $k$, $l$, $n$, and $\Omega$ such that*

$$\|u\|_{\mathrm{W}^{l,q}(\Omega)} \leq C\, \|u\|_{\mathrm{W}^{k,p}(\Omega)} \quad \textit{for all } u \in \mathrm{W}^{k,p}(\Omega).$$

*b) If $k - \frac{n}{p} = l$ and $q \in [1, \infty)$ (i.e. $k - \frac{n}{p} > l - \frac{n}{q}$), then $\mathrm{W}^{k,p}(\Omega)$ is continuously embedded into $\mathrm{W}^{l,q}(\Omega)$ and there is $C > 0$ depending on $p$, $q$, $k$, $l$, $n$, and $\Omega$ such that*

$$\|u\|_{\mathrm{W}^{l,q}(\Omega)} \leq C\, \|u\|_{\mathrm{W}^{k,p}(\Omega)} \quad \textit{for all } u \in \mathrm{W}^{k,p}(\Omega).$$

*c) If $l < k - \frac{n}{p} \leq l + 1$ (i.e. $l = k - \left[\frac{n}{p}\right] - 1$) and, setting $\frac{n}{p} = 0$ if $p = \infty$,*

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \textit{if } \frac{n}{p} \notin \mathbb{N}, \\ \textit{arbitrary in } (0,1), & \textit{if } \frac{n}{p} \in \mathbb{N}, \end{cases}$$

*then $\mathrm{W}^{k,p}(\Omega)$ is continuously embedded into $\mathrm{C}^{l,\gamma}(\overline{\Omega})$ and there is $C > 0$ depending on $p$, $\gamma$, $k$, $l$, $n$, and $\Omega$ such that*

$$\|u\|_{\mathrm{C}^{l,\gamma}(\overline{\Omega})} \leq C\, \|u\|_{\mathrm{W}^{k,p}(\Omega)}.$$

*This embedding is to be understood up to the choice of a continuous version.*

*Proof.* i) If $u \in \mathrm{W}^{m,r}(\Omega)$ with some $m \in \mathbb{N}$ and $r \in [1, n)$, then $\mathrm{D}^\alpha u \in \mathrm{W}^{1,r}(\Omega)$ for all $|\alpha| \leq m - 1$ by Proposition 3.8 and, by Theorem 6.5, $\mathrm{D}^\alpha u \in \mathrm{L}^q(\Omega)$ for all $q \in [1, r^*]$ which gives

$$\|u\|_{\mathrm{W}^{m-1,q}(\Omega)} \leq C_1 \sum_{|\alpha| \leq m-1} \|\mathrm{D}^\alpha u\|_{\mathrm{L}^q(\Omega)} \leq C_2 \sum_{|\alpha| \leq m-1} \|\mathrm{D}^\alpha u\|_{\mathrm{W}^{1,r}(\Omega)} \leq C_3\, \|u\|_{\mathrm{W}^{m,r}(\Omega)},$$

where $C_3 > 0$ depends only on $n$, $r$, $q$, $m$, and $\Omega$. Hence, defining $p_0 := p$ and $p_j := p_{j-1}^*$ for all $j = 1, \ldots, k - l$, we see that $\frac{1}{p_j} = \frac{1}{p_{j-1}} - \frac{1}{n}$ and therefore $\frac{1}{p_j} = \frac{1}{p} - \frac{j}{n}$ i.e.

$$p_j = \frac{np}{n - p \cdot j} \quad \text{with} \quad p_j > p_{j-1}.$$

If $p_j < n$ (i.e. $\frac{n}{p} > j+1$) we have $p_i < n$ for all $i \in \{0, \ldots, j\}$ and the previous argument yields the continuous embeddings of $\mathrm{W}^{k-(i-1),p_{i-1}}(\Omega)$ into $\mathrm{W}^{k-i,p_i}(\Omega)$ for all $i \in \{1, \ldots, j+1\}$ and $\tilde{C}_i > 0$ depending on $k$, $i$, $p$, $n$, $\Omega$ such that

$$\|u\|_{\mathrm{W}^{k-i,p_i}(\Omega)} \leq \tilde{C}_i \|u\|_{\mathrm{W}^{k-(i-1),p_{i-1}}(\Omega)} \quad \text{for all } u \in \mathrm{W}^{k-(i-1),p_{i-1}}(\Omega).$$

Hence, by combining these estimates for $i = 1, \ldots, j+1$ we have

$$\|u\|_{\mathrm{W}^{k-(j+1),p_{j+1}}(\Omega)} \leq \tilde{C}_1 \cdot \tilde{C}_2 \cdot \ldots \cdot \tilde{C}_{j+1} \|u\|_{\mathrm{W}^{k,p}(\Omega)} \quad \text{for all } u \in \mathrm{W}^{k,p}(\Omega) \tag{6.16}$$

if $\frac{n}{p} > j+1$.

ii) *Proof of a)*: As $k - \frac{n}{p} < l$ implies $\frac{n}{p} > k - l$, we apply (6.16) with $j := k - l - 1$ and obtain

$$\|u\|_{\mathrm{W}^{l,p_{k-l}}(\Omega)} \leq C_4 \|u\|_{\mathrm{W}^{k,p}(\Omega)}.$$

As $\frac{1}{p_{k-l}} = \frac{1}{p} - \frac{k-l}{n}$ implies $k - \frac{n}{p} = l - \frac{n}{p_{k-l}}$ and hence $q \leq p_{k-l}$, Hölders inequality and the boundedness of $\Omega$ imply that $\mathrm{W}^{l,p_{k-l}}(\Omega)$ is continuously embedded into $\mathrm{W}^{l,q}(\Omega)$ and a) is proved ($\|u\|_{\mathrm{W}^{l,q}(\Omega)} \leq C_5 \|u\|_{\mathrm{W}^{l,p_{k-l}}(\Omega)}$).

iii) *Proof of b)*: $k - \frac{n}{p} = l$ implies $\frac{n}{p} > k - l - 1$ and $p_{k-l-1} = n$ so that (6.16) with $j = k - l - 2$ yields

$$\|u\|_{\mathrm{W}^{l+1,n}(\Omega)} \leq C_6 \|u\|_{\mathrm{W}^{k,p}(\Omega)} \quad \text{for all } u \in \mathrm{W}^{k,p}(\Omega) \text{ and } k \geq l+2.$$

Since $k = l+1$ implies $p = n$, the above inequality is trivial for $k = l+1$ if we chose $C_6 \geq 1$. But then, similar to the first part of i), we deduce from Proposition 6.8 that $\mathrm{W}^{l+1,n}(\Omega)$ is continuously embedded into $\mathrm{W}^{l,q}(\Omega)$ for all $q \in [1, \infty)$, which implies b).

iv) *Proof of c)*: For $\frac{n}{p} \notin \mathbb{N}_0$ we have $k - \frac{n}{p} < l + 1$ and (6.16) with $j = k - l - 2$ implies

$$\|u\|_{\mathrm{W}^{l+1,p_{k-l-1}}(\Omega)} \leq C_7 \|u\|_{\mathrm{W}^{k,p}(\Omega)} \quad \text{for all } u \in \mathrm{W}^{k,p}(\Omega).$$

Since we have

$$p_{k-l-1} = \frac{np}{n - p \cdot (k-l-1)} = \frac{np}{p \cdot \left(\frac{n}{p} - k + l + 1\right)} > n,$$

due to $k - \frac{n}{p} > l$, and

$$1 - \frac{n}{p_{k-l-1}} = 1 - \left(\frac{n}{p} - (k-l-1)\right) = 1 - \frac{n}{p} + \left[\frac{n}{p}\right] = \gamma,$$

we conclude from Theorem 6.14 (like in the first part of i)) that $\mathrm{W}^{l+1,p_{k-l-1}}(\Omega)$ is continuously embedded into $\mathrm{C}^{l,\gamma}(\overline{\Omega})$.

For $\frac{n}{p} \in \mathbb{N}$ we have $k - \frac{n}{p} = l + 1$ and $l \leq k - 2$ so that b) implies for any $q \in [1, \infty)$

$$\|u\|_{\mathrm{W}^{l+1,q}(\Omega)} \leq C_8 \|u\|_{\mathrm{W}^{k,p}(\Omega)} \quad \text{for all } u \in \mathrm{W}^{k,p}(\Omega).$$

But then for any $q \in (n, \infty)$, $\mathrm{W}^{l+1,q}(\Omega)$ is continuously embedded into $\mathrm{C}^{l,\gamma}(\Omega)$ with $\gamma = 1 - \frac{n}{q}$. As $q \in (n, \infty)$ is arbitrary we get the claimed embedding for any $\gamma \in (0,1)$.

Finally, if $\frac{n}{p} = 0$, i.e. $p = \infty$, we have $l = k - 1$ and $\gamma = \left[\frac{n}{p}\right] + 1 - \frac{n}{p} = 1 - \frac{n}{p} = 1$. Hence Theorem 6.14 implies the continuous embedding of $\mathrm{W}^{k,\infty}(\Omega)$ into $\mathrm{C}^{k-1,1}(\overline{\Omega}) = \mathrm{C}^{l,\gamma}(\overline{\Omega})$ together with the claimed estimate. Altogether, c) is proved. $\square$

## 6.4 Compact Embeddings

Finally, we will study which compact embeddings we can deduce from the embeddings of Section 6.1 and Section 6.2. Therefore we first need criteria for precompactness of subsets of function spaces.

**Definition 6.17.** Let $X$ be a Banach space. A set $\mathcal{F} \subset X$ is *precompact*, if any sequence $(f_m)_{m\in\mathbb{N}} \subset \mathcal{F}$ has a subsequence which is convergent in $X$.

For $X = \mathrm{C}^0(\overline{\Omega})$ we recall the Theorem of Arzelà-Ascoli.

**Theorem 6.18** (Arzelà-Ascoli)**.** *Let $K \subset \mathbb{R}^n$ be compact. A set $\mathcal{F} \subset \mathrm{C}^0(K)$ is precompact if and only if*

*i)* $\sup_{f\in\mathcal{F}} \|f\|_{\mathrm{C}^0(K)} < \infty$ *and*

*ii)* $\mathcal{F}$ *is* equicontinuous, *i.e. for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $f \in \mathcal{F}$ and for all $x, y \in K$ with $|x - y| < \delta$.*

Next we prove a compactness criterion in $\mathrm{L}^p(\Omega)$.

**Theorem 6.19** (Kolmogoroff-Riesz-Fréchet)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and measurable and $p \in [1, \infty)$. $\mathcal{F} \subset \mathrm{L}^p(\Omega)$ is precompact if and only if*

*i)* $\sup_{f\in\mathcal{F}} \|f\|_{\mathrm{L}^p(\Omega)} < \infty$ *and*

*ii)* $\lim_{|h|\to 0} \left( \sup_{f\in\mathcal{F}} \|\tau_h \tilde{f} - \tilde{f}\|_{\mathrm{L}^p(\Omega)} \right) = 0$, *where $\tilde{f}|_\Omega = f$, $\tilde{f} = 0$ on $\mathbb{R}^n \setminus \Omega$, and $(\tau_h \tilde{f})(x) := \tilde{f}(x + h)$ for all $f \in \mathrm{L}^p(\Omega), x, h \in \mathbb{R}^n$.*

*Remark* 6.20. If $\Omega$ is unbounded, $\mathcal{F} \subset \mathrm{L}^p(\Omega)$ is precompact if and only if i), ii), and

*iii)* $\lim_{R\to\infty} \left( \sup_{f\in\mathcal{F}} \|\tilde{f}\|_{\mathrm{L}^p(\mathbb{R}^n \setminus \mathrm{B}_R(0))} \right) = 0$ hold.

*Proof of Theorem 6.19.* I) Assume that $\mathcal{F} \subset \mathrm{L}^p(\Omega)$ is precompact. If i) would be violated, there was a sequence $(f_m)_{m\in\mathbb{N}} \subset \mathcal{F}$ such that $\|f_m\|_{\mathrm{L}^p(\Omega)} \to \infty$ as $m \to \infty$. This sequence cannot have a convergent subsequence.

If ii) is violated, there exist $(h_m)_{m\in\mathbb{N}}$ with $\lim_{m\to\infty} |h_m| = 0$ and $\varepsilon_0 > 0$ such that

$$\limsup_{m\to\infty} \left( \sup_{f\in\mathcal{F}} \|\tau_{h_m} \tilde{f} - \tilde{f}\|_{\mathrm{L}^p(\Omega)} \right) > \varepsilon_0.$$

Hence, up to the choice of a subsequence, $\sup_{f\in\mathcal{F}} \|\tau_{h_m} \tilde{f} - \tilde{f}\|_{\mathrm{L}^p(\Omega)} > \varepsilon_0$ for all $m \in \mathbb{N}$ which yields a sequence $(f_m)_{m\in\mathbb{N}}$ in $\mathcal{F}$ with

$$\|\tau_{h_m} \tilde{f}_m - \tilde{f}_m\|_{\mathrm{L}^p(\Omega)} > \varepsilon_0 \quad \text{for all } m \in \mathbb{N}.$$

As $\mathcal{F}$ is precompact, there are $f \in \mathrm{L}^p(\Omega)$ and $(f_{m_k})_{k\in\mathbb{N}}$ such that $f_{m_k} \to f$ in $\mathrm{L}^p(\Omega)$ as $k \to \infty$.

Hence,

$$\begin{aligned}
&\liminf_{k\to\infty} \|\tau_{h_{m_k}}\tilde f - \tilde f\|_{\mathrm{L}^p(\Omega)}\\
&\quad\geq \liminf_{k\to\infty} \Big[ \|\tau_{h_{m_k}}\tilde f_{m_k} - \tilde f_{m_k}\|_{\mathrm{L}^p(\Omega)} - \|\tau_{h_{m_k}}\tilde f - \tau_{h_{m_k}}\tilde f_{m_k}\|_{\mathrm{L}^p(\Omega)} - \|\tilde f_{m_k} - \tilde f\|_{\mathrm{L}^p(\Omega)}\Big]\\
&\quad\geq \varepsilon_0 - 0 - 0 = \varepsilon_0 > 0,
\end{aligned}$$

a contradiction to Proposition 4.12 as $h_{m_k} \to 0$.

Hence, i) and ii) have to hold.

II) Assume that i) and ii) hold. For $f \in \mathcal{F}$ and $\varepsilon > 0$ we define $f_\varepsilon := \eta_\varepsilon * f \in \mathrm{C}^\infty(\mathbb{R}^n)$ and let $q \in (1,\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

a) We claim that for all $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that

$$\|f_\varepsilon - f\|_{\mathrm{L}^p(\Omega)} \leq \delta \quad \text{for all } f \in \mathcal{F} \text{ and for all } \varepsilon \in (0,\varepsilon_\delta). \tag{6.17}$$

By Hölder's inequality and $\int_{\mathbb{R}^n} \eta_\varepsilon \,\mathrm{d}x = 1$, we have for $x \in \Omega$

$$\begin{aligned}
|f_\varepsilon(x) - f(x)| &= \Big| \int_{\mathrm{B}_\varepsilon(x)} \eta_\varepsilon(x-y)(\tilde f(y) - \tilde f(x))\,\mathrm{d}y \Big|\\
&= \Big| \int_{\mathrm{B}_\varepsilon(0)} \eta_\varepsilon(z)(\tilde f(x-z) - \tilde f(x))\,\mathrm{d}z \Big|\\
&\leq \underbrace{\Big( \int_{\mathrm{B}_\varepsilon(0)} \eta_\varepsilon(z)\,\mathrm{d}z \Big)}_{=1}{}^{\frac{1}{q}} \Big( \int_{\mathrm{B}_\varepsilon(0)} \eta_\varepsilon(z)\,|\tau_{-z}\tilde f(x) - \tilde f(x)|^p\,\mathrm{d}z \Big)^{\frac{1}{p}}
\end{aligned}$$

so that

$$\begin{aligned}
\|f_\varepsilon - f\|^p_{\mathrm{L}^p(\Omega)} &\leq \int_\Omega \int_{\mathrm{B}_\varepsilon(0)} \eta_\varepsilon(z)|\tau_{-z}\tilde f(x) - \tilde f(x)|^p\,\mathrm{d}z\,\mathrm{d}x\\
&= \int_{\mathrm{B}_\varepsilon(0)} \eta_\varepsilon(z)\Big( \int_\Omega |\tau_{-z}\tilde f(x) - \tilde f(x)|^p\,\mathrm{d}x \Big)\,\mathrm{d}z\\
&\leq \sup_{|z|\leq\varepsilon_\delta} \|\tau_{-z}\tilde f - \tilde f\|^p_{\mathrm{L}^p(\Omega)} \underbrace{\int_{\mathrm{B}_\varepsilon(0)} \eta_\varepsilon(z)\,\mathrm{d}z}_{=1} \leq \delta^p
\end{aligned}$$

by ii) for $\varepsilon_\delta$ small enough. Hence, (6.17) is shown.

b) By Hölder's inequality and Theorem 2.1 (see proof of a)), we have for fixed $\varepsilon > 0$

$$\begin{aligned}
\|f_\varepsilon\|_{\mathrm{L}^\infty(\Omega)} &\leq \|\eta_\varepsilon\|_{\mathrm{L}^q(\mathbb{R}^n)}\|f\|_{\mathrm{L}^p(\Omega)} \leq M\,\|\eta_\varepsilon\|_{\mathrm{L}^q(\mathbb{R}^n)} \quad \text{and}\\
\|\nabla f_\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R}^n)} &= \|\nabla\eta_\varepsilon * f\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \leq \|\nabla\eta_\varepsilon\|_{\mathrm{L}^q(\mathbb{R}^n)}\|f\|_{\mathrm{L}^p(\Omega)} \leq M\|\nabla\eta_\varepsilon\|_{\mathrm{L}^q(\mathbb{R}^n)}
\end{aligned}$$

for all $f \in \mathcal{F}$, where $M := \sup_{f\in\mathcal{F}} \|f\|_{\mathrm{L}^p(\Omega)} < \infty$. Hence, the set

$$\mathcal{F}_\varepsilon := \{f_\varepsilon \colon f \in \mathcal{F}\}$$

is uniformly bounded in $\mathrm{C}^0(\overline{\Omega})$ and equicontinuous (since it is uniformly Lipschitz continuous) and hence precompact in $\mathrm{C}^0(\overline{\Omega})$ by the Arzelà-Ascoli Theorem 6.18. Hence, for any sequence $(g_m)_{m\in\mathbb{N}} \subset \mathcal{F}_\varepsilon$ there exists $g \in \mathrm{C}^0(\overline{\Omega})$ and a subsequence such that $g_{m_k} \to g$ in $\mathrm{C}^0(\overline{\Omega})$. As $\Omega$ is bounded, this implies $g_{m_k} \to g$ in $\mathrm{L}^p(\Omega)$ as $k \to \infty$. Hence, $\mathcal{F}_\varepsilon$ is precompact in $\mathrm{L}^p(\Omega)$ for fixed $\varepsilon > 0$.

c) We show that $\mathcal{F}$ is precompact in $\mathrm{L}^p(\Omega)$. Let $(f_m)_{m\in\mathbb{N}} \subset \mathcal{F}$ be arbitrary. Defining

$$\left(f_m^{(0)}\right)_{m\in\mathbb{N}} := (f_m)_{m\in\mathbb{N}},$$

we claim that for any $k \in \mathbb{N}$ there is a subsequence $\left(f_m^{(k)}\right)_m$ of $\left(f_m^{(k-1)}\right)_m$ such that

$$\|f_m^{(k)} - f_j^{(k)}\|_{\mathrm{L}^p(\Omega)} \leq \frac{1}{k} \quad \text{for all } m, j \geq N_k \tag{6.18}$$

with some $N_k \geq k$. Given $(g_m)_{m\in\mathbb{N}} := \left(f_m^{(k-1)}\right)_{m\in\mathbb{N}}$, by (6.17) there is $\varepsilon_k > 0$ such that

$$\|(g_m)_{\varepsilon_k} - g_m\|_{\mathrm{L}^p(\Omega)} \leq \frac{1}{3k} \quad \text{for all } m \in \mathbb{N}. \tag{6.19}$$

As $\mathcal{F}_{\varepsilon_k}$ is precompact in $\mathrm{L}^p(\Omega)$ by b), there is a subsequence $\left((g_{m_j})_{\varepsilon_k}\right)_{j\in\mathbb{N}}$ which is a Cauchy sequence in $\mathrm{L}^p(\Omega)$, hence for some $N_k \geq k$ we have

$$\|\left(g_{m_j}\right)_{\varepsilon_k} - \left(g_{m_i}\right)_{\varepsilon_k}\|_{\mathrm{L}^p(\Omega)} \leq \frac{1}{3k} \quad \text{for all } i, j \geq N_k. \tag{6.20}$$

Then

$$\begin{aligned} &\|g_{m_j} - g_{m_i}\|_{\mathrm{L}^p(\Omega)} \\ &\quad \leq \|g_{m_j} - (g_{m_j})_{\varepsilon_k}\|_{\mathrm{L}^p(\Omega)} + \|(g_{m_j})_{\varepsilon_k} - (g_{m_i})_{\varepsilon_k}\|_{\mathrm{L}^p(\Omega)} + \|(g_{m_i})_{\varepsilon_k} - g_{m_i}\|_{\mathrm{L}^p(\Omega)} \\ &\quad \overset{(6.19),(6.20)}{\leq} \frac{1}{k} \quad \text{for all } i, j \geq N_k. \end{aligned}$$

Hence, $\left(f_j^{(k)}\right)_{j\in\mathbb{N}} := (g_{m_j})_{j\in\mathbb{N}}$ satisfies (6.18). Then a diagonal argument shows that the subsequence $\left((f_{N_k})^{(k)}\right)_{k\in\mathbb{N}}$ of $(f_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^p(\Omega)$. Hence, $\mathcal{F}$ is precompact in $\mathrm{L}^p(\Omega)$. □

Now we prove the announced compact embeddings of $\mathrm{W}^{1,p}(\Omega)$.

**Theorem 6.21** (Rellich-Kondrachov)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in \mathrm{C}^1$ and $p \in [1, \infty]$. Then $\mathrm{W}^{1,p}(\Omega)$ is compactly embedded into*

*a) $\mathrm{L}^q(\Omega)$ for any $q \in [1, p^*)$ if $p \in [1, n)$.*

*b) $\mathrm{L}^q(\Omega)$ for any $q \in [1, \infty)$ if $p = n$.*

*c) $\mathrm{C}^0(\overline{\Omega})$ if $p \in (n, \infty]$ up to the choice of a continuous version.*

*Proof.* a) Let $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$. Then

$$|\tau_h\varphi(x) - \varphi(x)| = \Big|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\varphi(x + th)\,\mathrm{d}t\Big| \leq \int_0^1 |\nabla\varphi(x + th)|\,|h|\,\mathrm{d}t \quad \text{for all } x \in \mathbb{R}^n.$$

Hence, with $\frac{1}{p'} + \frac{1}{p} = 1$ we have

$$\|\tau_h\varphi - \varphi\|_{\mathrm{L}^p(\Omega)}^p \overset{\text{Hölder}}{\leq} |h|^p \cdot \int_\Omega \Big[\Big(\int_0^1 1\,\mathrm{d}t\Big)^{\frac{1}{p'}} \Big(\int_0^1 |\nabla\varphi(x+th)|^p\,\mathrm{d}t\Big)^{\frac{1}{p}}\Big]^p \mathrm{d}x \overset{\text{Fubini}}{\leq} |h|^p \|\nabla\varphi\|_{\mathrm{L}^p(\mathbb{R}^n)}^p.$$

As for any $u \in \mathrm{W}^{1,p}(\Omega)$ there is $(\varphi_n)_{n\in\mathbb{N}} \subset \mathrm{C}_0^\infty(\mathbb{R}^n)$ with $\varphi_n \to u$ in $\mathrm{W}^{1,p}(\Omega)$ and

$$\|\varphi_n\|_{\mathrm{W}^{1,p}(\mathbb{R}^n)} \le C_1 \|u\|_{\mathrm{W}^{1,p}(\Omega)} \quad \text{for all } n \in \mathbb{N}$$

with $C_1 > 0$ independent of $u$ (see Theorem 5.5 and Corollary 4.14), we have

$$\|\tau_h \tilde{u} - \tilde{u}\|_{\mathrm{L}^p(\Omega)} \le |h|\, C_1 \|u\|_{\mathrm{W}^{1,p}(\Omega)} \quad \text{for all } u \in \mathrm{W}^{1,p}(\Omega), h \in \mathbb{R}^n. \tag{6.21}$$

Now let $(u_m)_{m\in\mathbb{N}} \subset \mathrm{W}^{1,p}(\Omega)$ be bounded in $\mathrm{W}^{1,p}(\Omega)$ and $\mathcal{F} := \{u_m : m \in \mathbb{N}\}$. Then by (6.6), $\mathcal{F}$ is bounded in $\mathrm{L}^q(\Omega)$, i.e. i) in Theorem 6.19 holds (for $q$), where $q \in [1, p^*)$. Hence, $M := \sup_{m\in\mathbb{N}} \|u_m\|_{\mathrm{W}^{1,p}(\Omega)}$ is finite and, as $q \in [1, p^*)$, there is $\theta \in (0, 1]$ such that $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$. Then, we obtain

$$\begin{aligned}
\|\tau_h \tilde{u}_m - \tilde{u}_m\|_{\mathrm{L}^q(\Omega)} &= \Big(\int_\Omega |\tau_h \tilde{u}_m - \tilde{u}_m|^{q\theta} \cdot |\tau_h \tilde{u}_m - \tilde{u}_m|^{q(1-\theta)}\Big)^{\frac{1}{q}} \\
&\le \Big(\int_\Omega |\tau_h \tilde{u}_m - \tilde{u}_m| \,\mathrm{d}x\Big)^{\theta} \cdot \Big(\int_\Omega |\tau_h \tilde{u}_m - \tilde{u}_m|^{p^*}\Big)^{\frac{(1-\theta)}{p^*}}
\end{aligned}$$

which holds since $1 = \frac{1}{\frac{1}{q\theta}} + \frac{1}{\frac{p^*}{q(1-\theta)}}$ and furthermore by Hölder's inequality

$$\begin{aligned}
&\le \Big(|\Omega|^{\frac{1}{p'}} \|\tau_h \tilde{u}_m - \tilde{u}_m\|_{\mathrm{L}^p(\Omega)}\Big)^{\theta} \cdot \Big(2\,\|u_m\|_{\mathrm{L}^{p^*}(\Omega)}\Big)^{1-\theta} \\
&\overset{(6.6),(6.21)}{\le} |\Omega|^{\frac{\theta}{p'}} |h|^\theta\, C_1^\theta\, \|u_m\|_{\mathrm{W}^{1,p}(\Omega)}^\theta\, 2^{1-\theta} \Big(C_2 \|u_m\|_{\mathrm{W}^{1,p}(\Omega)}\Big)^{1-\theta} \\
&\le |h|^\theta\, M\, C_1^\theta\, |\Omega|^{\frac{\theta}{p'}}\, (2C_1)^{1-\theta} \quad \text{for all } m \in \mathbb{N}.
\end{aligned}$$

As $\theta > 0$, ii) in Theorem 6.19 is satisfied for $\mathcal{F}$. Hence, $\mathcal{F}$ is precompact in $\mathrm{L}^q(\Omega)$ so that the embedding of $\mathrm{W}^{1,p}(\Omega)$ into $\mathrm{L}^q(\Omega)$ is compact by Definition 6.1 and Theorem 6.5.

b) For $q \in [1, \infty)$ there is $\tilde{p} \in [1, n)$ such that $\tilde{p}^* > q$. As $\mathrm{W}^{1,p}(\Omega)$ is continuously embedded into $\mathrm{L}^q(\Omega)$ by Proposition 6.8 and any bounded sequence $(u_m)_{m\in\mathbb{N}} \subset \mathrm{W}^{1,p}(\Omega)$ is also bounded in $\mathrm{W}^{1,\tilde{p}}(\Omega)$ by Hölder's inequality, a) for $\tilde{p}$ shows that $(u_m)_{m\in\mathbb{N}}$ has a convergent subsequence in $\mathrm{L}^q(\Omega)$. This proves b).

c) The continuous embedding of $\mathrm{W}^{1,p}(\Omega)$ into $\mathrm{C}^{0,\gamma}(\overline{\Omega})$ with some $\gamma \in (0, 1]$ by Theorem 6.14 implies that any bounded sequence in $\mathrm{W}^{1,p}(\Omega)$ is equicontinuous and bounded in $\mathrm{C}^0(\overline{\Omega})$. Hence, by Arzelà-Ascoli it has a convergent subsequence in $\mathrm{C}^0(\overline{\Omega})$. This proves part c). □

*Remark* 6.22. a) In particular, $\mathrm{W}^{1,p}(\Omega)$ is compactly embedded into $\mathrm{L}^p(\Omega)$ for all $p \in [1, \infty]$ if $\Omega$ is a bounded domain with $\partial\Omega \in \mathrm{C}^1$. This is in general not true for $\Omega$ unbounded (see [AF03]).

b) All embeddings in Theorem 6.21 are also valid for $\mathrm{W}_0^{1,p}(\Omega)$ for any bounded domain $\Omega \subset \mathbb{R}^n$.

c) The embedding of $\mathrm{W}^{1,p}(\Omega)$ into $\mathrm{L}^{p^*}(\Omega)$ for $p \in [1, n)$ is *not* compact.

# Chapter 7

# Applications to PDEs

As a prototype of so called *elliptic PDEs*, we will study the *Poisson equation with Dirichlet boundary conditions*, i.e.

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega, \\ \quad u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{7.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $f\colon \Omega \to \mathbb{R}$ is given, and $u\colon \overline{\Omega} \to \mathbb{R}$ is the unknown. The Laplacian $\Delta u$ is given by $\Delta u(x) = \sum_{i=1}^n u_{x_i x_i}(x)$.

**Definition 7.1.** A *classical solution* to (7.1) is a function $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ satisfying (7.1) pointwise (in the usual sense). In particular, this requires $f \in \mathrm{C}^0(\Omega)$.

The Poisson equation appears in different contexts in natural and engineering sciences. For instance, $u$ is the displacement of a membrane localized in $\Omega \subset \mathbb{R}^2$, where $f$ describes a force and $u = 0$ on $\partial\Omega$ that the membrane is fixed on $\partial\Omega$. $u$ can also describe the temperature distribution in a solid $\Omega$ (stationary in time), where $f$ is a source of energy and the temperature on the boundary $\partial\Omega$ is given. If $u$ is the concentration of a chemical substance, $f$ describes the production of the chemical and there is no chemical on $\partial\Omega$. In the latter two cases $\Delta u$ describes the diffusion or heat conduction with the flux $-\nabla u$.

Often we do not have classical solutions to (7.1) or we cannot show easily the existence of a classical solution. Therefore, different concepts of generalized solutions are used and we will focus here on weak solutions.

## 7.1 The Concept of Weak Solutions

Let $u$ be a classical solution to (7.1). Then integrating by parts, we obtain

$$\int_\Omega \nabla u(x) \cdot \nabla \varphi(x)\,\mathrm{d}x = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \tag{7.2}$$

For this identity we only need $u \in \mathrm{C}^1(\Omega)$ or even $u \in \mathrm{H}^1(\Omega)$ is sufficient. Recall that $\mathrm{H}^k(\Omega) = \mathrm{W}^{k,2}(\Omega)$ and $\mathrm{H}_0^k(\Omega) = \mathrm{W}_0^{k,2}(\Omega)$. In particular, if $u \in \mathrm{H}_0^1(\Omega)$ and $\partial\Omega \in \mathrm{C}^1$, $u$ satisfies $\mathrm{Tr}(u) = 0$ by Theorem 5.8 so that the boundary condition $u = 0$ holds in a weak sense.

Another concept of generalized solutions for (7.1) is the so called strong solution which has the weak derivatives of second order so that $-\Delta u(x) = f(x)$ a.e. in $\Omega$.

**Definition 7.2.** a) A *weak solution* to (7.1) is a function in $\mathrm{H}_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \text{for all } v \in \mathrm{H}_0^1(\Omega). \tag{7.3}$$

b) A *strong solution* to (7.1) is a function $u \in \mathrm{H}^2_{\mathrm{loc}}(\Omega) \cap \mathrm{H}_0^1(\Omega)$ satisfying $-\Delta u = f$ a.e. in $\Omega$.

Note that (7.2) and (7.3) are equivalent as $\mathrm{C}_0^\infty(\Omega)$ is dense in $\mathrm{H}_0^1(\Omega)$. If $u$ is a classical solution to (7.1) with $u \in \mathrm{H}^1(\Omega)$ it satisfies $u \in \mathrm{H}_0^1(\Omega)$ by Theorem 5.8, as $\mathrm{Tr}(u) = u|_{\partial\Omega} = 0$. Hence, $u$ is a weak solution to (7.1) as it satisfies (7.3). $u$ is also a strong solution.
If $u \in \mathrm{H}_0^1(\Omega)$ is a weak solution to (7.1) satisfying $u \in \mathrm{H}^2_{\mathrm{loc}}(\Omega)$, then by (7.3) and the definition of weak derivatives we have

$$\begin{aligned} \int_\Omega f\varphi \,\mathrm{d}x &= \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x = \sum_{i=1}^n \int_\Omega u_{x_i}\varphi_{x_i} \,\mathrm{d}x \\ &= -\sum_{i=1}^n \int_\Omega u_{x_i x_i}\varphi \,\mathrm{d}x = \int_\Omega (-\Delta u)\varphi \,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{C}_0^\infty(\Omega). \end{aligned}$$

Hence, by Lemma 3.3, $-\Delta u = f$ a.e. in $\Omega$ and $u$ is a strong solution. If $u$ satisfies in addition $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ and $f \in \mathrm{C}^0(\Omega)$, then $-\Delta u = f$ a.e. in $\Omega$ holds in the classical sense. By continuity and since $u \in \mathrm{H}_0^1(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ implies $0 = \mathrm{Tr}(u) = u|_{\partial\Omega}$, we deduce that $u$ is a classical solution to (7.1).
We will therefore first show the existence of a unique weak solution to (7.1) and then try to show that $u$ has more regularity to deduce that it is a strong or even classical solution.
For a general boundary condition $u = g$ on $\partial\Omega$ for the Poisson equation $-\Delta u = f$ in $\Omega$, one requires that $g \in \mathrm{L}^2(\partial\Omega)$ is such that $g = \mathrm{Tr}(w)$ for some $w \in \mathrm{H}^1(\Omega)$. Then $u \in \mathrm{H}^1(\Omega)$ is called a weak solution of this problem, if $\tilde{u} := u - w \in \mathrm{H}_0^1(\Omega)$ and

$$\int_\Omega \nabla \tilde{u} \cdot \nabla v \,\mathrm{d}x = \int_\Omega (fv - \nabla w \cdot \nabla w) \,\mathrm{d}x \quad \text{for all } v \in \mathrm{H}_0^1(\Omega).$$

## 7.2 Existence and Uniqueness of Weak Solutions

In order to prove the existence of weak solutions to (7.1), we need to introduce the concept of a bounded linear operator on the dual space. For more details we refer to the functional analysis course or e.g. [Dob10, Sections 2.3, 2.4].

**Definition 7.3.** Let $X$ and $Y$ be real Banach spaces and $H$ be a real Hilbert space.

a) A mapping $A\colon X \to Y$ is called *bounded linear operator* if $A(\lambda u + \mu v) = \lambda A(u) + \mu A(v)$ for all $u, v \in X$, $\lambda, \mu \in \mathbb{R}$ and there is $C > 0$ such that $\|A(u)\|_Y \le C\,\|u\|_X$ for all $u \in X$. Then, $\|A\| := \sup\{\|A(u)\|_Y \colon \|u\|_X \le 1\} < \infty$.

b) A bounded linear operator $u^*\colon X \to \mathbb{R}$ is called *bounded linear functional* with $\|u^*\|_{X^*} := \sup\{u^*(u)\colon \|u\|_X \le 1\}$. Then

$$X^* := \{u^*\colon X \to \mathbb{R}\colon u^* \text{ linear}, \|u^*\|_{X^*} < \infty\}$$

is the *dual space of* $X$. $X^*$ is a Banach space with norm $\|\cdot\|_{X^*}$ and we denote by $\langle u^*, u\rangle_X := u^*(u) \in \mathbb{R}$ for all $u \in X, u^* \in X^*$ the *pairing of* $X^*$ *and* $X$.

c) If $H$ is a Hilbert space, then $H^*$ is a Hilbert space. For $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$*open*, we denote the dual space of $\mathrm{H}_0^k(\Omega)$ by $\mathrm{H}^{-k}(\Omega) := (\mathrm{H}_0^k(\Omega))^*$.

The elements of $H^*$ have a very specific form.

**Theorem 7.4** (Riesz representation theorem)**.** *Let $H$ be a Hilbert space with scalar product $(\,\cdot\,,\,\cdot\,)$. Then for each $u^* \in H^*$ there exists a unique $u \in H$ such that*

$$\langle u^*, v\rangle_H = (u, v) \quad \textit{for all } v \in H.$$

*The mapping $u^* \to u$ is a linear isomorphism of $H^*$ onto $H$.*

*Proof.* See [Dob10, Satz 2.25]. □

The theorem of Lax-Milgram provides the basis for the existence of weak solutions. In the context of (7.3), please note that $B[u,v] := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x$ is bilinear.

**Theorem 7.5** (Lax-Milgram)**.** *Let $H$ be a Hilbert space with inner product $(\,\cdot\,,\,\cdot\,)$. Assume that $B\colon H \times H \to \mathbb{R}$ is a bilinear mapping such that there exist $\alpha, \beta > 0$ with*

$$|B[u,v]| \le \alpha \|u\|_H \|v\|_H \quad \textit{for all } u, v \in H \textit{ and} \tag{7.4}$$

$$\beta \|u\|_H^2 \le B[u,u] \quad \textit{for all } u \in H. \tag{7.5}$$

*Moreover, let $F^* \in H^*$. Then there exists a unique $u \in H$ such that*

$$B[u,v] = \langle F^*, v\rangle_H \quad \textit{for all } v \in H. \tag{7.6}$$

*Proof.* We will only prove the case of $B$ being symmetric. For the general case, we refer to [Eva10, Theorem 1 in Section 6.2.1]. Assume that $B[u,v] = B[v,u]$ for all $u, v \in H$. Then we claim that

$$(\!(\!(u,v)\!)\!) := B[u,v] \quad \text{for all } u, v \in H$$

defines a scalar product on H. Indeed, $(\!(\!(\,\cdot\,,\,\cdot\,)\!)\!)$ is bilinear and symmetric and, by (7.5), $(\!(\!(u,u)\!)\!) \ge 0$ holds for all $u \in H$ and $(\!(\!(u,u)\!)\!) = 0$ if and only if $u = 0$. If we define the associated norm via $|\!|\!|u|\!|\!| := \Big(B[u,u]\Big)^{\frac{1}{2}}$, (7.4) and (7.5) imply

$$\sqrt{\beta}\, \|u\|_H \le |\!|\!|u|\!|\!| \le \sqrt{\alpha}\, \|u\|_H \quad \text{for all } u \in H.$$

Hence, $|\!|\!|\cdot|\!|\!|$ and $\|\cdot\|_H$ are equivalent norms on $H$ so that it is equivalent to use either $(\!(\!(\,\cdot\,,\,\cdot\,)\!)\!)$ or $(\,\cdot\,,\,\cdot\,)$ on $H$ as a scalar product. In particular, $H$ becomes again a Hilbert space with respect to the new scalar product $(\!(\!(\,\cdot\,,\,\cdot\,)\!)\!)$.
Then applying Theorem 7.4 (with $(\!(\!(\,\cdot\,,\,\cdot\,)\!)\!)$), we obtain $u \in H$ such that

$$\langle F^*, v\rangle_H = (\!(\!(u,v)\!)\!).$$

Hence, $u$ satisfies (7.6).
$u$ is unique, because if $u_1, u_2 \in H$ satisfy (7.6), then

$$B[u_1 - u_2, v] = B[u_1, v] - B[u_2, v] = 0 \quad \text{for all } v \in H.$$

But then by (7.5) we have

$$\beta \|u_1 - u_2\|_H^2 \le B[u_1 - u_2, u_1 - u_2] = 0$$

so that $u_1 = u_2$. $\square$

Now we can prove the existence of a unique weak solution to (7.1).

**Theorem 7.6.** *Let $\Omega$ be a bounded domain and $f \in \mathrm{L}^2(\Omega)$. Then there exists a unique weak solution $u \in \mathrm{H}_0^1(\Omega)$ to* (7.1).

*Proof.* We apply the Lax-Milgram Theorem 7.5 to $H := \mathrm{H}_0^1(\Omega)$, $B[u,v] := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x$ for $u, v \in \mathrm{H}_0^1(\Omega)$ and $F^* \colon \mathrm{H}_0^1(\Omega) \to \mathbb{R}$, $\langle F^*, v\rangle_{\mathrm{H}_0^1(\Omega)} := \int_\Omega f v \,\mathrm{d}x$.

i) Obviously, $B$ is bilinear and symmetric with $B \colon H \times H \to \mathbb{R}$. Moreover, by Hölder's inequality and the Poincaré inequality (6.9) (with constant $C_p > 0$) we have

$$\begin{aligned} |B[u,v]| &\le \|\nabla u\|_{\mathrm{L}^2(\Omega)} \|\nabla v\|_{\mathrm{L}^2(\Omega)} \le \|u\|_{\mathrm{H}^1(\Omega)} \|v\|_{\mathrm{H}^1(\Omega)} \quad \text{for all } u, v \in \mathrm{H}_0^1(\Omega) \text{ and} \\ \|u\|_{\mathrm{H}^1(\Omega)}^2 &= \|u\|_{\mathrm{L}^2(\Omega)}^2 + \|\nabla u\|_{\mathrm{L}^2(\Omega)}^2 \\ &\le (C_p^2 + 1)\|\nabla u\|_{\mathrm{L}^2(\Omega)}^2 = (C_p^2+1) B[u,u] \qquad \text{for all } u \in \mathrm{H}_0^1(\Omega) \end{aligned}$$

so that (7.4) and (7.5) hold with $\alpha = 1$ and $\beta = \frac{1}{C_p^2 + 1}$.

ii) Obviously, $\langle F^*, v\rangle_{\mathrm{H}_0^1(\Omega)} = \int_\Omega f v \,\mathrm{d}x$ is linear in $v$ and by Hölder

$$|\langle F^*, v\rangle_{\mathrm{H}_0^1(\Omega)}| \le \|f\|_{\mathrm{L}^2(\Omega)} \|v\|_{\mathrm{L}^2(\Omega)} \le \|f\|_{\mathrm{L}^2(\Omega)} \|v\|_{\mathrm{H}^1(\Omega)} \quad \text{for all } v \in \mathrm{H}_0^1(\Omega).$$

Hence, $F^* \in H^*$ with $\|F^*\|_{H^*} \le \|f\|_{\mathrm{L}^2(\Omega)}$.

In view of i) and ii) the claim follows from the Lax-Milgram theorem. $\square$

## 7.3 Regularity of Weak Solutions

Our aim is to show that the weak solution $u$ to (7.1) has better regularity that just being in $\mathrm{H}_0^1(\Omega)$. To motivate the following result, assume that $u \in \mathrm{C}_0^\infty(\Omega)$ satisfies (7.1). Then integration by parts yields

$$\begin{aligned} \|f\|_{\mathrm{L}^2(\Omega)}^2 &= \int_\Omega f^2 \,\mathrm{d}x = \int_\Omega \Delta u \,\Delta u \,\mathrm{d}x = \sum_{i,j=1}^n \int_\Omega u_{x_i x_i} u_{x_j x_j} \,\mathrm{d}x \\ &= -\sum_{i,j=1}^n \int_\Omega u_{x_i x_i x_j} u_{x_j} \,\mathrm{d}x = \sum_{i,j=1}^n \int_\Omega u_{x_i x_j} u_{x_i x_j} \,\mathrm{d}x = \sum_{|\alpha|=2} \|\mathrm{D}^\alpha u\|_{\mathrm{L}^2(\Omega)}^2. \end{aligned}$$

So $\|\mathrm{D}^\alpha u\|_{\mathrm{L}^2(\Omega)} \le \|f\|_{\mathrm{L}^2(\Omega)}$ for all $|\alpha| = 2$. Similarly, as $-\Delta(u_{x_i}) = f_{x_i}$, the $\mathrm{L}^2$-norm of the $(m+2)$-nd derivatives of $u$ can be estimated by the $\mathrm{L}^2$-norm of the $m$-th derivatives of $f$. Hence, informally "$u$ has two more derivatives in $\mathrm{L}^2$ than $f$". However, there is no solution $u \in \mathrm{C}_0^\infty(\Omega)$ to (7.1) apart from $u \equiv 0$. But results of this kind can be proved, which are of course more difficult to prove.

**Theorem 7.7.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $f \in \mathrm{L}^2(\Omega)$ and $u \in \mathrm{H}^1_0(\Omega)$ be the unique weak solution to* (7.1)*, and $V \Subset \Omega$ be open, $m \in \mathbb{N}$.*

*a) $u \in \mathrm{H}^2_{\mathrm{loc}}(\Omega)$ and $\|u\|_{\mathrm{H}^2(V)} \le C_1 \|f\|_{\mathrm{L}^2(\Omega)}$. If $\partial\Omega \in \mathrm{C}^2$, then $u \in \mathrm{H}^2(\Omega)$ and*

$$\|u\|_{\mathrm{H}^2(\Omega)} \le C_2 \|f\|_{\mathrm{L}^2(\Omega)}.$$

*b) If $f \in \mathrm{H}^m(\Omega)$, then $u \in \mathrm{H}^{m+2}_{\mathrm{loc}}(\Omega)$ and*

$$\|u\|_{\mathrm{H}^{m+2}(V)} \le C_3 \|f\|_{\mathrm{H}^m(\Omega)}.$$

*If $\partial\Omega \in \mathrm{C}^{m+2}$, then $u \in \mathrm{H}^{m+2}(\Omega)$ and*

$$\|u\|_{\mathrm{H}^{m+2}(\Omega)} \le C_4 \|f\|_{\mathrm{H}^m(\Omega)}.$$

*c) If $f \in \mathrm{C}^\infty(\Omega)$, then $u \in \mathrm{C}^\infty(\Omega)$. If in addition $f \in \mathrm{C}^\infty(\overline{\Omega})$ and $\partial\Omega \in \mathrm{C}^\infty$, then $u \in \mathrm{C}^\infty(\overline{\Omega})$.*

*The constants $C_1, C_3$ depend on $\Omega$, $V$ (and $m$), whereas the constants $C_2, C_4$ depend on $\Omega$ (and $m$).*

*Proof.* We refer to [Eva10, Theorems 1-6 in Section 6.3 and Theorem 6 in Section 6.2]. □

In particular, as $u \in \mathrm{H}^2_{\mathrm{loc}}(\Omega)$, $u$ is always a strong solution (see Section 7.1). As Theorem 6.16 implies that $\mathrm{H}^k(\Omega)$ is continuously embedded into $\mathrm{C}^2(\overline{\Omega})$ if $k > \frac{n}{2} + 2$, $u$ is a classical solution if $f \in \mathrm{H}^m(\Omega)$ and $\partial\Omega \in \mathrm{C}^{m+2}$ for some $m \in \mathbb{N}$ with $m > \frac{n}{2}$.
One important ingredient of the proofs of these regularity results is the approximation of weak derivatives by difference quotients. The latter is also of interest in the theory of Sobolev spaces.

**Definition 7.8.** Let $\Omega \subset \mathbb{R}^n$ be open, $V \Subset \Omega$, and $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$. The *i-th difference quotient of size h* is

$$\mathrm{D}_i^h u(x) := \frac{u(x + he_i) - u(x)}{h} \quad \text{for } i \in \{1, \ldots, n\}, x \in V, h \in \mathbb{R} \text{ with } 0 < |h| < \operatorname{dist}(V, \partial\Omega).$$

We define

$$\mathrm{D}^h u := (\,\mathrm{D}_1^h u, \ldots, D_n^h u).$$

In order to prove the connection between difference quotients and weak derivatives we need further results from functional analysis (see e.g. [Dob10] or [Alt12]):

**Proposition 7.9.** *Let $X$ be a Banach space.*

*a) $X$ is* reflexive *if $(X^*)^* = X$ in the sense that for all $u^{**} \in (X^*)^*$ there exists a unique $u \in X$ with*

$$\langle u^{**}, u^* \rangle_{X^*} = \langle u^*, u \rangle_X \quad \textit{for all } u^* \in X^*.$$

*b) $(u_n)_{n\in\mathbb{N}} \subset X$* converges weakly *to $u \in X$ if*

$$\langle u^*, u_k \rangle_X \to \langle u^*, u \rangle_X \quad \textit{as } k \to \infty \textit{ for all } u^* \in X^*, \ \textit{written } u_k \rightharpoonup u \textit{ in } X.$$

c) *If $X$ is reflexive, then any bounded sequence $(u_k)_{k\in\mathbb{N}} \subset X$ has a weakly convergent subsequence.*

*In particular, a Hilbert space $H$ and the Lebesgue spaces $\mathrm{L}^p(\Omega), p \in (1,\infty)$, are reflexive and $u_k \rightharpoonup u$ in $\mathrm{L}^p(\Omega)$ if and only if $\int_\Omega u_k g \,\mathrm{d}x \to \int_\Omega ug\,\mathrm{d}x$ for all $g \in \mathrm{L}^q(\Omega)$ with $\frac{1}{p}+\frac{1}{q}=1$.*

Then we have the following result on difference quotients.

**Theorem 7.10.** *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$.*

a) *If $p \in [1,\infty)$ and $u \in \mathrm{W}^{1,p}(\Omega)$, then there exists $C_1 > 0$ depending on $p$ and $n$ such that for all $V \Subset \Omega$ and all $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial\Omega)$ we have*

$$\|\mathrm{D}^h u\|_{\mathrm{L}^p(V)} \le C_1 \|\nabla u\|_{\mathrm{L}^p(\Omega)} \tag{7.7}$$

b) *Assume that $p \in (1,\infty)$, $u \in \mathrm{L}^p(V)$ and there exists $C_2 > 0$ such that*

$$\|D^h u\|_{\mathrm{L}^p(V)} \le C_2 \quad \text{for all } 0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial\Omega). \tag{7.8}$$

*Then $u \in \mathrm{W}^{1,p}(V)$ and $\|\nabla u\|_{\mathrm{L}^p(V)} \le C_2$.*

*Proof.* a) First assume that $u \in \mathrm{C}^1(\Omega) \cap \mathrm{W}^{1,p}(\Omega)$, $x \in V$, $i \in \{1,\dots,n\}$. Then by the mean value theorem,

$$|u(x+he_i) - u(x)| \le |h| \int_0^1 |\nabla u(x+the_i)|\,\mathrm{d}t.$$

Hence, by Hölder's inequality we have with $\frac{1}{p}+\frac{1}{q}=1$

$$\begin{aligned}
\int_V |\mathrm{D}^h u|^p \,\mathrm{d}x &\le C_3 \sum_{i=1}^n \int_V \Big[|(0,1)|^{\frac{1}{q}} \cdot \Big(\int_0^1 |\nabla u(x+the_i)|\,\mathrm{d}t\Big)^{\frac{1}{p}}\Big]^p \,\mathrm{d}x \\
&= C_3 \sum_{i=1}^n \int_0^1 \int_V |\nabla u(x+the_i)|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\le C_3\, n\, \|\nabla u\|^p_{\mathrm{L}^p(\Omega)}.
\end{aligned}$$

As this estimate holds for all $u \in \mathrm{C}^1(\Omega) \cap \mathrm{W}^{1,p}(\Omega)$ and this space is dense in $\mathrm{W}^{1,p}(\Omega)$ by Theorem 4.4, we conclude by Fatou's lemma that (7.7) with $C_1 = C_3 n$.

b) Assume that (7.8) holds. Let $i \in \{1,\dots,n\}, \varphi \in \mathrm{C}_0^\infty(V)$ and

$$0 < |h| < \frac{1}{2}\min\{\operatorname{dist}(V,\partial\Omega), \operatorname{dist}(\operatorname{supp}\varphi, \partial V)\}.$$

Then, as the choice of $h$ gives $\operatorname{supp}\varphi \subset V \cap \tilde{V}$ with $\tilde{V} := \{x+he_i : x \in V\}$, the transformation rule implies

$$\begin{aligned}
\int_V u(\mathrm{D}_i^h \varphi)\,\mathrm{d}x &= \int_V u(x)\frac{\varphi(x+he_i)-\varphi(x)}{h}\,\mathrm{d}x \\
&= \int_{\tilde{V}} \frac{u(y-he_i)\varphi(y)}{h}\,\mathrm{d}y - \int_V \frac{u(x)\varphi(x)}{h}\,\mathrm{d}x \\
&= \int_V -\frac{u(x-he_i)-u(x)}{-h}\varphi(x)\,\mathrm{d}x \\
&= -\int_V (\mathrm{D}_i^{-h} u)\varphi\,\mathrm{d}x.
\end{aligned}$$

As (7.8) implies

$$\sup_{0<h<\frac{1}{2}\operatorname{dist}(V,\partial\Omega)} \| \mathrm{D}_i^{-h} u\|_{\mathrm{L}^p(V)} < \infty$$

and $p \in (1,\infty)$, Proposition 7.9 c) implies that there are $h_k \to 0$ and $v_i \in \mathrm{L}^p(V)$ such that $\mathrm{D}_i^{-h_k} u \rightharpoonup v_i$ as $k \to \infty$ weakly in $\mathrm{L}^p(V)$, i.e.

$$\int_V (\mathrm{D}_i^{-h_k} u) g \,\mathrm{d}x \to \int_V v_i g \,\mathrm{d}x \quad \text{for all } g \in \mathrm{L}^q(V). \tag{7.9}$$

As $V$ is bounded, we conclude

$$\int_V u\varphi_{x_i} \,\mathrm{d}x = \lim_{h\to\infty} \int_V u(\mathrm{D}_i^{h_k}\varphi) \,\mathrm{d}x = \lim_{k\to\infty} -\int_V (\mathrm{D}_i^{-h_k} u)\varphi \,\mathrm{d}x = -\int_V v_i\varphi \,\mathrm{d}x.$$

Hence, as $\varphi \in \mathrm{C}_0^\infty(V)$ is arbitrary, we have $u_{x_i} = v_i \in \mathrm{L}^p(V)$ in the weak sense. As $u \in \mathrm{L}^p(V)$, we conclude that $u \in \mathrm{W}^{1,p}(V)$ and

$$\|u_{x_i}\|_{\mathrm{L}^p(V)} \le \liminf_{k\to\infty} \| \mathrm{D}_i^{-h_k} u\|_{\mathrm{L}^p(V)} < \infty. \qquad \square$$

Finally, we compare weak and classical partial derivatives.

**Definition 7.11.** $u\colon \Omega \to \mathbb{R}$ is *differentable at* $x \in \Omega$ if there is $a \in \mathbb{R}^n$ such that

$$u(y) = u(x) + a\cdot(y-x) + o(|y-x|) \quad \text{where } \lim_{y\to x} \frac{o(|y-x|)}{|y-x|} = 0.$$

In this case, $\nabla u(x) := a$ is called the *gradient of* $u$.

Then we have the following result.

**Proposition 7.12.** *Let $\Omega \subset \mathbb{R}^n$ be open, $p \in (n,\infty]$ and $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$. Then $u$ is differentiable a.e. in $\Omega$ and its gradient equals its weak gradient a.e. in $\Omega$.*

*Proof.* i) Let $p \in (n,\infty)$, $x, y \in \Omega$ with $|y-x| = r > 0$, and $u \in \mathrm{C}^1(\overline{\mathrm{B}_{2r}(x)})$. Since we have $\mathrm{B}_r(x), \mathrm{B}_r(y) \subset \mathrm{B}_{2r}(x)$, part iv) of the proof of Theorem 6.12 implies that there is $C_1 > 0$ depending only on $n$ and $p$ such that

$$|u(y)-u(x)| \le C_1 r^{1-\frac{n}{p}} \Big(\int_{\mathrm{B}_{2r}(x)} |\nabla u(z)|^p \,\mathrm{d}z\Big)^{\frac{1}{p}} \quad \text{for all } u \in \mathrm{C}^1(\overline{\mathrm{B}_{2r}(x)}). \tag{7.10}$$

Again by approximation, (7.10) holds for all $u \in \mathrm{W}^{1,p}(\mathrm{B}_{2r}(x))$.

ii) Let $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ and $\nabla u$ the weak gradient of $u$. Then by Lebesgue's differentiation theorem (see [Eva10, Section E.4])

$$\frac{1}{|\mathrm{B}_r(x)|} \int_{\mathrm{B}_r(x)} |\nabla u(x) - \nabla u(z)|^p \,\mathrm{d}z \to 0 \quad \text{as } r \downarrow 0 \text{ for a.e. } x \in \Omega.$$

We fix any such $x$ and apply (7.10) to $v(y) := u(y) - u(x) - \nabla u(x)\cdot(y-x)$ with $r = |y-x|$. Then

$$\begin{aligned} |u(y)-u(x)-\nabla u(x)\cdot(y-x)| &\le C_1\, r^{1-\frac{n}{p}} \Big(\int_{\mathrm{B}_{2r}(x)} |\nabla u(z) - \nabla u(x)|^p \,\mathrm{d}z\Big)^{\frac{1}{p}} \\ &\le C_2\, r \Big(\frac{1}{|\mathrm{B}_{2r}(x)|} \int_{\mathrm{B}_{2r}(x)} |\nabla u(x) - \nabla u(z)|^p \,\mathrm{d}z\Big)^{\frac{1}{p}} \\ &= o(r) = o(|y-x|). \end{aligned}$$

Hence, $u$ is differentiable at $x$ and its gradient equals the weak gradient.

iii) As $\mathrm{W}^{1,\infty}_{\mathrm{loc}} \subset \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ for all $p \in [1, \infty)$, the case $p = \infty$ follows from i), ii). $\square$

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